DATA-GUIDE QUICK CHARTS SUBJECTS

Accounting 1, Basic

Accounting 2, Basic

Accounting 3, Basic

Algebra 1, College

Algebra, Elementary (Ninth Year)

Algebra, Intermediate (11th Year)

Calculus, Differential

Calculus, Integral

Chemistry 1, Basic

Chemistry 2, Basic

Economics, Principles of

English: Grammar & Usage, Basic

English: Punctuation, Basic

English: Writing Guide, Basic

Geometry, Analytic

Geometry, Principles of Plane

Parliamentary Procedure

Philosophy, Principles of

Psychology, Principles of

Sociology, Principles of

Statistics, Essentials of

Trigonometry

DATA-GUIDE QUICK CHART

# BASIC ACCOUNTING CHART ONE

SIMPLIFIED SUMMARY ◆ INSTANT REFERENCE ◆ PERMANENT MEMORY-AID

LEARN & REVIEW IN MINUTES

AUTHOR: Prof. S. Tunick, Ph.D., Dept. Chairman, C.C.N.Y.

ALSO SEE CHARTS 2 & 3

SOLID PLASTIC

SPEED STUDY TO SAVE YOU TIME AND WORK.

QUICK CHARTS PUT THE FACTS IN FULL VIEW.

PERMANENTLY FILL IN FOR MISSING NOTES & TEXTS.

QUICK CHARTS HELP YOU REMEMBER:

CONTAINS NO PAPER

## 1 FUNDAMENTAL EQUATION OF ACC'TS.

**1. Assets:** A business owns **assets.** **2. Liabilities** are debts of a business. **3.** The excess of assets owned by a business over its liabilities is called its **capital** or **net worth.** **4.** Assets, liabilities, and capital are recorded in books of account in terms of money values. **5.** The relationship among them may be expressed in equation form as:

**A (Assets) — L (Liabilities) = C (Capital) or A = L + C**

**A. EFFECT OF TRANSACTIONS ON EQUATION. 1. Assets:** In acquiring an **asset,** a business may give up another asset or incur a liability *(debt)* requiring later payment, or both. Thus, if **I** signifies an increase and **D** a decrease:

| | |
|---|---|
| IA (as in purchase of furniture) | = DA (cash payment therefor in full), or = IL (liability therefor to creditor recorded), or = DA (part payment in cash) + IL (balance due creditor recorded). |

**2. Liabilities.** A liability may be decreased by paying cash or by assuming another liability, or both:

| | |
|---|---|
| DL (as in settlement of indebtedness to creditor) | = DA (cash payment thereof in full), or = IL (issuance of a note therefor to creditor), or = DA (part payment in cash) + IL (issuance of a note for balance to creditor). |

**3. Expenses.** An expense results in a *decrease of capital* **(net worth),** whereas income tends to *increase capital.* It is either paid for at once or the obligation to pay at a future time is undertaken, or both.

| | |
|---|---|
| DC (as in advertising expense incurred) | = DA (cash payment thereof in full), or = IL (liability therefor to creditor recorded), or = DA (part payment in cash) + IL (balance due creditor recorded). |

**4. Income.** When income is earned, it results either in the receipt of cash or in an enforceable claim against a debtor who will pay it at a future time, or both.

| | |
|---|---|
| IA (cash receipt, in full settlement), or IA (account receivable from customer), or IA (cash receipt, on account) + IA (balance due from customer). | = IC (as in fee earned for services rendered) |

**B. EXPANDED FUNDAMENTAL EQUATION OF ACCOUNTS.** Formed when all of the foregoing relationships are combined:

**IA + DL + DC = DA + IL + IC**

This result may also be demonstrated as follows:

A (net assets) = L (net liabilities) + C (net capital)

Substituting: (IA — DA) = (IL — DL) + (IC — DC).

Transposing: **IA + DL + DC = DA + IL + IC.**

**NOTE:** Every transaction may be analyzed with respect to its effects on *assets, liabilities,* and *capital,* and involves at least two of the elements appearing in the *expanded fundamental equation,* with at least one of them on each side thereof. **The left side of the equation is known as its debit side; the right side is known as its credit side.**

**C. BALANCE SHEET—EQUATION AT REST.** The *assets* and *liabilities* of a business may be set up in the form of a **balance sheet** to depict the financial condition of a business at any particular date. (See specimen of classified balance sheet in Accounting Chart 2.)

## 2 LEDGER ACCOUNTS. A. USE AND OPERATION.

**1.** To record and analyze information about the *assets, liabilities,* and *capital* of a business, as well as about its *income* and *expenses.* **2.** A **separate account** is maintained for each asset, liability, income, expense, and other capital item. **3.** *Increases* are shown on one side and *decreases* on the other. In keeping with the expanded fundamental equation, increases, decreases and balances are shown as follows:

| NATURE OF ACCOUNTS | INCREASES | DECREASES | BALANCE |
|---|---|---|---|
| ASSETS | Debit side | Credit side | Debit side |
| LIABILITIES | Credit " | Debit " | Credit " |
| PROPRIETOR'S CAPITAL | Credit " | Debit " | Credit " |
| PROPRIETOR'S DRAWINGS (= DECREASE IN CAPITAL) | Debit " | Credit " | Debit " |
| INCOME (= INCREASE IN CAPITAL) | Credit " | Debit " | Credit " |
| EXPENSES (= DECREASE IN CAPITAL) | Debit " | Credit " | Debit " |

A **ledger** is a book in which such accounts are maintained.

**B. FORM OF AN ACCOUNT. 1.** The operation of all *ledger accounts* follows the rationale of the expanded fundamental equation. The *left side* of an account is called its **debit side** and the *right side* its **credit side. 2.** An *excess* of dollar amount entries in debit column over those in credit column results in a **debit balance:** conversely, a **credit balance.** *For example, since a valid claim against a customer represents the ownership of a valuable right, it is reflected on the books in an asset account. A typical customer's account would be set up and operated as follows.*

(--------DEBIT SIDE--------) (-------CREDIT SIDE-------)

CUSTOMER'S NAME ____ Ledger Page No. —

| 19— | | | | | | 19— | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Feb. | 8 | Sale | S14 | a) 876 50 | | Feb. | 10 | Return | SR2 | a) 16 00 | |
| | 24 | Sale | S15 | 791 23 | | | 18 | Note 3/18 | J3 | a) 500 00 | |
| | | | | 1667 73 | | | 18 | Cash | CR6 | a) 360 50 | |
| | | | | • | | | | | | 876 50 | |

* Indicates number of page in book where transaction was first recorded and from which information was transferred. Cross-reference symbols are as follows: CR = Cash Receipts; CD = Cash Disbursements; P = Purchases; PR = Purchase Returns; S = Sales; SR = Sales Returns; J = General Journal.

**3.** The **increases** in the firm's assets resulting from sales made to the customer on open account are recorded in the foregoing account as **debits** on the debit side. **4. Decreases** resulting from returns of merchandise and payments, by cash and note, are recorded as **credits** on the credit side. *The final balance of this asset account is a net debit balance ($791.23), the amount currently due.*

**C. FOOTING AND BALANCING. 1. Footings** (*totals*) of columns and account balances should be in pencil. Items which are *offsets* against others are sometimes **keyed-off** (see those indicated as "a" above). The *balance* of an account reflects items remaining open or not keyed-off. **2.** As additional items are recorded, they are placed immediately underneath previously recorded items without skipping any lines. Pencil footings do not occupy full lines and need not be erased. **3. Example:**

CUSTOMER'S NAME ____ Ledger Page No. —

| 19— | | | | | | 19— | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Feb. | 8 | Sale | S14 | a) 876 50 | | Feb. | 10 | Return | SR2 | a) 16 00 | |
| | 24 | Sale | S15 | 791 23 | | | 18 | Note 3/18 | J3 | a) 500 00 | |
| | | | | 1667 73 | | | 18 | Cash | CR6 | a) 360 50 | |
| Mar. | 6 | Sale | S17 | b) 82 00 | | | | | | 876 50 | |
| | 17 | Sale | S17 | 101 10 | | Mar. | 22 | Cash | CR8 | b) 82 00 | |
| | | | | 1850 83 | | | | | | 958 50 | |

## 3 JOURNALIZING. A. RECORDING.

As the transactions occur, they are initially recorded in chronological order in a book known as a **journal.** They are analyzed as to nature and extent of their effect on *assets, liabilities, capital, income* and *expenses.*

**B. DEBITS AND CREDITS.** Following the expanded fundamental equation of accounts: increases in assets, decreases in liabilities, and decreases in capital (*including expenses*) are shown as **debits;** decreases in assets, increases in liabilities, and increases in capital (*including income items*) are shown as **credits.**

**C. EXPLANATIONS.** An explanation showing important details may be included as part of each entry. (See illustration below.) Principal sources of journalized information are *sales invoices, check book stubs, purchase invoices, record of money received,* etc.

## 4 POSTING.

**1. Posting** is process of transferring information from a **journal** to appropriate **ledger accounts** for purposes of summarizing all related transactions. **2.** The information so transferred includes the **date of the entry,** the **amount** thereof, and the **page** no. of journal where the entry was first recorded. **3.** A notation of *page number of ledger account* to which posting was made, is made in *journal* where original entry appears.

## 5 FORM OF JOURNAL (TWO-COLUMN). *(See below)*

## 6 TRIAL BALANCE.

**1.** Since *debits* and *credits* in journal entries are always *equal* in amount, the **total of all debits posted** to ledger accounts must likewise *be equal* to **total of all posted credits. 2.** It follows that with respect to all accounts to which postings have been made, the **total** of **net debit** balances must be *equal* to the *total* of the **net credit** balances. This equality is tested by means of **trial balance,** which lists all accounts appearing in ledger at a particular time and their respective balances, either debit or credit, as well as numbers of the ledger pages where they appear. **Debit balances** are shown in one column and **credit balances** in another (see above right).

NAME OF FIRM—TRIAL BALANCE—DATE*

| LEDGER PAGE NO. | NAME OF ACCOUNT | DEBIT | CREDIT |
|---|---|---|---|

* Proper form is three lines, one under the other.

**NOTE:** For an illustration of an actual trial balance, see **work sheet** Acc'ting Chart 2. The reader is cautioned that a trial balance may be in balance and yet not necessarily reflect a correct situation. This will be the case, for example, if an entire entry, or series of entries is not posted, or if compensating errors have been made in posting.

## 7 MERCHANDISING. A. NET INCOME.

**1. In a service business,** *net income* is the excess of *total revenue* or income over the *expenses* of conducting the business (such as rent, salaries, light, telephone, shipping, advertising, etc.). **2. In a merchandising business** it is the excess of *gross profit on sales* over such *expenses.*

**B. GROSS PROFIT ON SALES. 1. It is the excess of the revenue from sales over the cost of goods sold.** Cost includes not only the purchase price of the merchandise, but also all items (*such as duty, freight, etc.*) paid to bring the goods into the buyer's place of business or make them available to him. **2.** Any merchandise inventory carried over from a prior accounting period is added to the cost of current purchases in the determination of the total cost of goods available for sale, and **the cost of goods sold is computed by subtracting the value of the unsold goods at the end of the period from the goods available for sale. 3. Example:**

| | | |
|---|---|---|
| SALES | | $9,000 |
| *Less:* Sales Returns and Allowances | $ 400 | |
| Discount on Sales | 100 | |
| Total Deductions | | 500 |
| NET SALES REVENUE | | $8,500 |
| COST OF GOODS SOLD: | | |
| Inventory (beginning of period) | | $1,110 |
| Purchases | $5,000 | |
| Freight In | 50 | |
| Duty | 300 | |
| Total | $5,350 | |
| *Less:* Purch. Rets. & Allow. $350 | | |
| Discount on Purch. 250 | | |
| Total Deductions | 600 | |
| Net Cost of Purchases | | 4,750 |
| COST OF GOODS AVAILABLE FOR SALE | | $5,860 |
| *Less:* Inventory (end of period) | | 3,000 |
| COST OF GOODS SOLD | | 2,860 |
| GROSS PROFIT ON SALES | | $5,640 |

## 8 DISCOUNTS. A. TRADE DISCOUNTS.

Deductions from list or catalogue prices, and merchandise purchased subject thereto, is recorded at its **net price,** after deducting such trade discounts.

**B. CASH DISCOUNTS.** Allowances made for payments within designated periods. Merchandise purchased on this basis is usually recorded at its **gross purchase price.**

## 9 SPECIAL JOURNALS. A. USE AS LABOR-SAVING DEVICE.

**1.** When the same type of transaction recurs frequently, *special journals* are used to simplify the posting procedure, to permit more than one person to work on accounting records at the same time, and to eliminate the repetition of recurring similar explanations. **2.** During the course of each month *similar items* are listed chronologically in appropriate special journals and the **total** thereof is **posted** to the **ledger account** as **one amount** at the end of each month, instead of posting each entry individually as it is made. Of course, offsetting debits or credits to items which are accumulated each month must be posted individually as they are made.

**B. CASH RECEIPTS JOURNAL.** Records cash received. Basic entry calls for *debit* to **Cash account** and *credit* to account recording the source or nature of the cash receipt.

**C. CASH DISBURSEMENTS JOURNAL.** Entry calls for *debit* to account recording nature of cash payment and *credit* to **Cash account.**

**D. SALES JOURNAL.** Basic entry calls for *debit* to **Customer's account** and a *credit* to **Sales account.**

**E. SALES RETURNS AND ALLOWANCES JOURNAL.** Basic entry calls for *debit* to **Sales Returns and Allowances account** and *credit* to **Customer's account.**

**F. PURCHASES JOURNAL.** Basic entry calls for *debit* to **Purchases account** and *credit* to **Creditor's account.**

**G. PURCHASE RETURNS AND ALLOWANCES JOURNAL.** Basic entry calls for *debit* to **Creditor's account** and *credit* to **Purchase Returns and Allowances account.**

**H. GENERAL JOURNAL.** Used for transactions for which special journals have not been set up. Accounts debited and credited vary according to transactions. All postings from this journal must be made immediately after entries are recorded.

**I. ILLUSTRATIVE SPECIAL PURCHASES JOURNAL.**

PURCHASES JOURNAL (Page 4)

| DATE | | NAME AND TERMS | L.F. | AMOUNT |
|---|---|---|---|---|
| 19— | | | # | |
| Feb. | 2 | Arthur & Co.—2/10, n/30 | 21 | 962 75 |
| | 16 | Merit Trading Co.—Net 30 days | 74 | 1,410 20 |
| | 24 | Smith & Frank—3/10, n/30 | 106 | 124 10 |
| | | | 110 | 2,497 05 |

ARTHUR AND CO. (Page 21)

| 19— | | | |
|---|---|---|---|
| Feb. | 2 | P4 • | 962 75 |

MERIT TRADING CO. (Page 74)

| 19— | | | |
|---|---|---|---|
| Feb. | 16 | P4 • | 1,410 20 |

SMITH AND FRANK (Page 106)

| 19— | | | |
|---|---|---|---|
| Feb. | 24 | P4 • | 124 10 |

PURCHASES (Page 110)

| 19— | | | | |
|---|---|---|---|---|
| Feb. | 28 | P4 • | 2,497 | 05 |

# These numbers are cross-references indicating the pages of the accounts in the ledger to which postings were made.

* These symbols are used to identify cross-references to pages in the special (purchases) journal from which postings were made.

**NOTE:** Each **purchase** transaction requires a *debit* to the **Purchases account** and a corresponding *credit* to the **Creditor's liability account.** However, by using this special journal, it becomes possible to post to the Purchases account, as one amount at the end of a particular month, the **total** of purchases made during that month.

### ILLUSTRATION OF 2-COLUMN JOURNAL

JOURNAL (PAGE 17)

| DATE | | ACCOUNTS AND EXPLANATION | L.F. | DEBIT | CREDIT |
|---|---|---|---|---|---|
| 19— | | | | | |
| Mar. | 22 | A. Brown | 14 | 600 00 | |
| | | Income from Services | 61 | | 600 00 |
| | | To record bill for services rendered to Brown. (Debit to "Brown" records increase in claim against him—an asset, and credit to "Income from Services" records an increase in capital.) | | | |
| Apr. | 1 | Notes Receivable | 7 | 400 00 | |
| | | Cash | 1 | 200 00 | |
| | | A. Brown | 14 | | 600 00 |
| | | To record receipt of cash and 60-day non-interest-bearing note in satisfaction of claim against Brown. (Debits to "Notes Receivable" and "Cash" accounts record increases in assets; the credit to Brown's account records a decrease in the claim against him.) | | | |

CASH (Page 1)

| | | | |
|---|---|---|---|
| Apr. 1 | (J17) | 200.00 | |

A. BROWN (Page 14)

| | | | | | |
|---|---|---|---|---|---|
| Mar. 22 | (J17) | 600.00 | Apr. 1 | (J17) | 600.00 |

NOTES RECEIVABLES (Page 7)

| | | |
|---|---|---|
| Apr. 1 | (J17) | 400.00 |

INCOME FROM SERVICES (Page 61)

| | | |
|---|---|---|
| | Mar. 22 | (J17) 600.00 |

### ILLUSTRATIONS OF EXPANDED COLUMNAR JOURNALS

CASH RECEIPTS JOURNAL

| DATE | | NAME | EXPLANATION | L.F. | DEBITS: CASH | DEBITS: SALES DISCT. | CREDITS: SALES | CREDITS: SALES TAX PAYABLE | CREDITS: GENERAL ACCOUNT | CREDITS: GENERAL AMOUNT |
|---|---|---|---|---|---|---|---|---|---|---|
| 19— | | | | | | | | | | |
| Feb. | 8 | F. Richards | Cash Sale | + | 103 00 | | 100 00 | 3 00 | | |
| | 18 | B. Gordon | Inv. 2/8/—, less 2% | * | 353 29 | 7 21 | | | Gordon | 360 50 |
| | 25 | I. Platt | Inv. 2/16/—, less 2% | * | 588 00 | 12 00 | | | Platt | 600 00 |
| | | | | | • 1044 29 | • 19 21 | • 100 00 | • 3 00 | | ♦ 960 50 |

CASH DISBURSEMENTS JOURNAL

| DATE | | NAME | EXPLANATION | L.F. | CREDITS: CASH | CREDITS: PURCH. DISCT. | CREDITS: W-H. TAX PAYABLE | CREDITS: FICA TAX PAYABLE | DEBITS: SALARIES | DEBITS: GENERAL ACCOUNT | DEBITS: GENERAL AMOUNT |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 19— | | | | | | | | | | | |
| Feb. | 5 | Gardner Co. | Rent | * | 150 00 | | | | | Rent | 150 00 |
| | 5 | Cashier | Sal. w/e 2/5 | + | 160 90 | | 33 10 | 6 00 | 200 00 | | |
| | 17 | Ritchie Co. | Inv. 2/7, less 2% | * | 118 79 | 2 42 | | | | Ritchie | 121 21 |
| | | | | | • 429 69 | • 2 42 | • 33 10 | • 6 00 | • 200 00 | | ♦ 271 21 |

* Indicate here the page number in ledger to which items entered in General column have been posted.

\+ Write "✓" here, to indicate that no posting is required when entry is made.

• Post total of each of these columns at end of each month and show in parentheses, under each amount posted, the number of the ledger page to which the posting was made. ♦ Post each item in this column currently.

## J. EXPANDED SPECIAL JOURNALS—COLUMNIZATION PRINCIPLE.

**1.** All entries made in a **cash disbursements journal** involve *cash*. In addition, they may affect other items with a fair degree of regularity. When they do, special columns are introduced for recording similar recurring items. **2.** At the end of each month, after *crossfooting* all column totals to make sure that the sums of the debit column totals and credit column totals agree, the totals of such special columns are posted as *one amount*, thus saving effort and space. **3.** When the occasion arises to record an item for which no special column has been introduced, the entry is made in a "**General**" column and the posting thereof is done immediately. The same principle is applicable to all special journals. *(See Cash Journals, bottom of front side).*

## 10 CONTROLLING ACCOUNTS. A. SUBSIDIARY LEDGERS.

**1.** When many accounts in a ledger represent similar things *(claims against customers, amounts due creditors)* they are generally removed to an individual special ledger. When this is done, a single account is set up to replace them in the principal or **general ledger.** This account in the general ledger is known as a **controlling account,** because it shows in summary form what appears in detailed form in the individual accounts in the special ledger. **2.** By replacing the accounts removed with a single account *whose balance represents the aggregate of the balances of the transferred accounts,* the equality of debits and credits in the general ledger continues to be maintained. It is necessary to provide **special columns in all journals** for recording transactions and accumulating data affecting these special ledgers. **individual postings** are made therefrom currently to the accounts in the special or **subsidiary** ledgers, but **summary postings** of these column totals in the journals are made to the *controlling accounts* in the **general ledger** at the end of each month.

### B. OPERATION OF TYPICAL SPECIAL JOURNALS.

**SALES JOURNAL**

| DATE | | ACCOUNT | L.F. | AMOUNT |
|---|---|---|---|---|
| 19— Sept. | 18 | Jones (terms of sale) | (a) | 1,000.00 |
| | 20 | Brown (terms of sale) | (a) | 600.00 |
| | 28 | Smith (terms of sale) | (a) | 200.00 |
| | | | | 1,800.00 |
| | | | | (b)/(c) |

(a) Post immediately to appropriate account in *customer's* ledger. Show page number here. **(b)** Post as debit at end of month to *Accounts Receivable controlling account* in the general ledger. Show page number here. **(c)** Post corresponding credit at end of month to the *Sales account* in the general ledger. Show page number here, separated as indicated by a diagonal line. **NOTE:** The total effect of all of the individual sales transactions during the month is summarized in the general ledger accounts by a single total debit **(b)** to the **Accounts Receivable** controlling account and a corresponding single total credit **(c)** to the **Sales Account.** The total of the individual debit postings **(a)** to the various customers' accounts in the **subsidiary** Accounts Receivable ledger is equal to the total debit posting to corresponding **controlling** account in the general ledger.

**PURCHASES JOURNAL**

| DATE | | ACCOUNT | L.F. | AMOUNT |
|---|---|---|---|---|
| 19— Oct. | 2 | Gordon (terms of purchase) | (a) | 850.00 |
| | 9 | Still (terms of purchase) | (a) | 650.00 |
| | 12 | Morgan (terms of purchase) | (a) | 450.00 |
| | | | | 1,950.00 |
| | | | | (b)/(c) |

(a) Post immediately to appropriate account in *creditors' ledger.* Show page number here. **(b)** Post debit at end of month to the *Purchases account* in the general ledger. Show page number here. **(c)** Post corresponding credit at end of month to *Accounts Payable controlling account* in the general ledger. Show page number here, separated as indicated by a diagonal line.

**CASH RECEIPTS JOURNAL**

| DATE | | NAME | L.F. | DEBITS: CASH | DEBITS: SALES DISCT. | CREDITS: ACCTS. RECEIV. | CREDITS: MISCELL. ACCT. | CREDITS: MISCELL. AMT. |
|---|---|---|---|---|---|---|---|---|
| 19— Nov. | 7 | Harrington | (a) | 98.00 | 2.00 | 100.00 | | |
| | 14 | Frank-Note Rec. due | (b) (c) | 204.00 | | | NoteRec. Int. Inc. | 200.00 4.00 |
| | | | | 302.00 | 2.00 | 100.00 | | 204.00 |
| | | | | (d) | (e) | (f) | | (g) |

(a) Post $100 credit immediately to Harrington's account in *subsidiary ledger.* Show page number here. **(b)** Post $200 credit immediately to *Notes Receivable* account in general ledger. Show page number here. **(c)** Post $4 credit immediately to *Interest Income account* in general ledger. Show page number here. **(d)** Post as a debit at end of month to *Cash account* in general ledger. Show page number below total. **(e)** Post as a debit at end of month to *Sales Discount account* in general ledger. Show page number below total. **(f)** Post as a credit at end of month to *Accounts Receivable account* in general ledger. Show page number below total. **(g)** Do not post total since individual items have already been posted singly. Show this by check mark (√) below total.

**GENERAL JOURNAL**

| ACCTS. PAY. (Dr.) | GENERAL (Dr.) | L.F. | ACCOUNTS AND EXPLANATIONS | L.F. | GENERAL (Cr.) | ACCTS. RECEIV. (Cr.) |
|---|---|---|---|---|---|---|
| | 1,000.00 | (a) | February 10, 19— Notes Receivable Jones Received 60-day 6% note to apply on account | (b) | | 1,000.00 |
| (d) | (c) | | | | (c) | (d) |

(a) Post immediately as a debit to *Notes Receivable account* in general ledger. Enter general ledger account page number here. **(b)** Post immediately as a credit to Jones' account in *customers' ledger.* Enter subsidiary ledger account page number here. **(c)** Do not post total since individual items have already been posted singly. Show this by check mark (√) below total. **(d)** Post any total to appropriate *controlling account* at end of month. Indicate below total the page number in the general ledger to which posting has been made.

**C. PROOF OF CONTROLLING ACCOUNT.** At the end of each month after all postings have been made, a **schedule** of *subsidiary ledger account balances* should be prepared and the total thereof compared with balance of *corresponding controlling account.* Both should agree.

**D. DOUBLE ("SPLIT") POSTINGS. 1.** When an entry affecting an account in a subsidiary ledger is made in a journal in which a special column for the controlling account has not been provided, a **double or "split" posting** should be made immediately, one to the **subsidiary ledger account** and another to the **controlling account** in the general ledger. **2. Example:**

**CASH DISBURSEMENTS JOURNAL**

| DATE | CHECK NO. | PAYEE AND EXPLANATION | L.F. | CASH (Cr.) | GENERAL ACCT. | GENERAL AMT. (Dr.) |
|---|---|---|---|---|---|---|
| 19— Dec. | 14 | 101 Bank-(check returned-"insufficient funds") | a/b | 102 00 | Warren | 102 00 |

(a) Designates page number of *Accounts Receivable controlling account* in general ledger to which debit of $102 is posted. **(b)** Designates page number of Warren's account in subsidiary ledger to which debit of $102 is posted.

## 11 DEPRECIATION. A. NATURE OF EXPENSE. 1.

In the case of *assets* acquired for use in business *(such as buildings, machinery, furniture, equipment, etc.)* **the cost is charged off to expense over full period of the respective asset's useful life,** instead of being charged entirely to period in which it was acquired. The amount so prorated each year is called **depreciation** and deemed to be an expense of business *(decrease in capital).* **2.** Depreciation may be **physical** *(wear and tear)* or **economic** *(obsolescence).* Depreciation is generally charged off at the end of each business year.

**B. THE STRAIGHT-LINE METHOD.** Most usually encountered, it calls for proration of **asset's cost,** reduced by its expected ultimate **salvage value,** evenly over the estimated number of years of its **useful life.**

$$\text{Annual Depreciation (D)} = \frac{\text{Cost (C)} - \text{Expected Salvage Value (S)}}{\text{Estimated Useful Life, in Years (L)}}$$

**C. THE DECLINING-BALANCE METHOD. 1.** The annual depreciation expense is computed by applying a **uniform** rate *(not exceeding twice the appropriate straight-line rate)* to the **unrecovered basis** of fixed **asset,** i.e., the cost reduced by accumulated depreciation—a continually declining base (hence, the name). The unrecovered basis may not be reduced below **salvage value.**

**2. Example:** *If a machine cost $1,200 and had an expected salvage value of $200 after an expected useful life of 10 years, the depreciation expense for first, second and third years under the double-declining-balance method would be computed as follows:*

*1st year* = 2 x | 10% $1,200 | = $240.00
*2nd year* = 2 x | 10% $1,200—$240 | = $192.00
*3rd year* = 2 x | 10% $1,200—$432 | = $153.60

*Depreciation written off may not, in this case, reduce basis below $200.*

**D. SUM-OF THE YEARS' DIGITS METHOD. 1.** The annual depreciation expense is computed by applying an **annually changing fraction** to **cost** of the fixed asset *minus* its estimated **salvage value.** As the name implies, the *denominator* of the fraction is the sum of *1, 2, 3, . . . . n,* where "*n*" is the number of years in asset's estimated useful life. The *numerator* is the number of years, including current year, remaining in the asset's expected life (*n*, for the first year; *n-1*, for second year; etc.) **2. Example:** *If a machine had an estimated useful life of 5 years, the depreciation allowance in second year would be computed by applying the fraction* $\left(\frac{5-1}{1+2+3+4+5}\right)$ *or 4/15 to its original cost, reduced by the estimated salvage value.*

## 12 VALUATION ACCOUNTS — DEPRECIATION.

**A. ALLOWANCE FOR DEPRECIATION.** In case of assets subject to depreciation, resulting decreases in value are not ordinarily recorded directly in respective *asset accounts.* Instead, separate accounts called **valuation accounts** are generally set up to record accumulated decreases in the asset values. The entry to record current depreciation would be:

Depreciation of Machinery ..... 100.00
(expense—decrease in capital)
Allowance for Depreciation—Machinery ..... 100.00
(valuation account—decrease in asset)
For dep'n. on $1,000 at 10% per ann.

A separate valuation account is used for each type of asset subject to depreciation. This procedure makes it possible for original cost of such assets to be clearly identifiable at all times. The **book value** of such assets is the difference between the **cost** and the **balance** in the related **valuation account.** Thus:

| | |
|---|---|
| Furniture and Fixtures | $1,000.00 |
| *Less:* Allowance for Depreciation | 250.00 |
| Net Book Value (or Depreciated Cost) | $ 750.00 |

**B. GAIN OR LOSS ON SALE OF DEPRECIATED ASSET.** On a sale of an asset, the amount in the Allowance account representing the accumulated depreciation applicable to the cost of the asset sold is closed into the corresponding account:

(a) Allowance for Depreciation—Machinery ..... 300.00
Machinery ..... 300.00

*If machinery had been acquired for $1,000.00 and sold for $650.00 the preceding entry will leave a $700.00 balance in asset account. When the receipt of cash is then recorded:*

(b) Cash ..... 650.00
Machinery ..... 650.00

*The Machinery account will have a debit balance of $50.00. This represents a loss on the sale and is closed out:*

(c) Loss on Sale of Assets ..... 50.00
Machinery ..... 50.00

Above entries may be summarized in following "T" accounts:

| CASH | | MACHINERY | |
|---|---|---|---|
| (b) 650 | | Cost 1,000 | (a) 300 |
| | | | (b) 650 |
| | | | (c) 50 |

| ALLOW. FOR DEPRECIATION OF MACHINERY | | LOSS ON SALE OF ASSETS |
|---|---|---|
| (a) 300 | Bal. 300 | (c) 50 |

**NOTE:** A credit balance in **Machinery account** would represent a gain and would be closed out in reverse fashion to **Gain on Sale of Assets.** Depreciation must always be written off (as **expense**) to the date of sale before the foregoing entries are made.

**C. TRADE-IN OF ASSETS SUBJECT TO DEPRECIATION.** Amount received as an allowance for depreciated asset traded-in for a new replacement is considered to be its selling price, and resulting gain or loss is recorded in usual manner. The new asset is recorded as a purchase. Thus, if machinery with original cost of $4,000 and accumulated depreciation of $2,100 is given trade-in value of $800 in connection with purchase of new machinery costing $4,500, the combined entry would be:

Allowance for Depreciation—Mach. (traded-in) . . 2,100.00
Loss on Sale of Machinery (traded-in) ..... 1,100.00
Machinery (new) ..... 4,500.00
Machinery (traded-in) ..... 4,000.00
Cash ..... 3,700.00

For tax purposes, no gain or loss is recognized, and book value of old asset plus cash paid becomes tax basis of new asset.

## 13 VALUATION ACCOUNTS — BAD DEBTS.

**A. ALLOWANCE FOR BAD DEBTS. 1.** From time to time it may be found that certain amounts arising out of sales made on credit can't be collected, and bad debt losses result. **2.** To charge such losses to the accounting period in which credit sales took place, it is customary at end of each such period to *estimate future bad debt losses* which will result from sales made during period. This estimate is made on basis of prior experience and current conditions. **3.** Since it is usually not possible to know which specific amount due will become uncollectible, no particular account receivable is written off *(credited)* but instead a **valuation account** is set up to record anticipated bad debt losses:

Bad Debts (Expense—Decrease in capital) . . . 175.00
Allowance for Bad Debts ..... 175.00
(Valuation Account—Decrease in asset)

**B. BAD DEBT WRITE-OFFS.** As soon as any customer's indebtedness is subsequently determined to be uncollectible, his account receivable is **written off** *(credited),* and uncollectible balance is **charged** *(debited)* to **Allowance for Bad Debts** account, as follows:

Allowance for Bad Debts ..... 100.00
Accounts Receivable (customer's name) ..... 100.00

At any time, the estimated recovery value of the Accounts Receivable may be stated as follows:

| | |
|---|---|
| Accounts Receivable | $9,000.00 |
| *Less:* Allowance for Bad Debts | 175.00 |
| Estimated Recovery Value | $8,825.00 |

**C. SUBSEQUENT ANNUAL PROCEDURE. 1.** When the *future bad debts* are again estimated at the end of a subsequent period, consideration is given to the then **balance** of the **Allowance for Bad Debts account. 2.** If there is a **credit balance** in the account, the entry is made for the **difference** between the estimated *future bad debts* and the *account balance*; if there is a **debit balance,** the entry is for the **total** of the amount so estimated and the *debit balance.*

**D. ALTERNATIVE METHOD FOR BAD DEBT WRITE-OFF.** Uncollectible accounts receivable are written off only when they are actually determined to be worthless, thus making them an expense of that fiscal period, rather than the one in which the credit decision and sale were actually consummated. The entry would be:

Bad Debts ..... 100.00
Accounts Receivable (Customer's Name) ..... 100.00

**RECOVERY OF BAD DEBTS PREVIOUSLY WRITTEN-OFF.**

**(1)** If an Allowance account is used, the entry is:
Cash ..... 50.00
Allowance for Bad Debts ..... 50.00

**(2)** If an Allowance account is not used, the entry is:
Cash ..... 50.00
Bad Debts Recovered ..... 50.00
(Increase in capital)

## 14 CASH BASIS vs. ACCRUAL BASIS. 1.

When books are maintained on a **cash** basis, **income** is considered as *earned* when it is *received* in cash, and **expenses** as *incurred* when they are *paid.* **2.** When books are maintained on the **accrual** basis, **income** is considered as *earned when work is performed or sales made,* regardless of when payment therefor is received; and **expenses** are charged to the period in which benefits are enjoyed, *regardless of when payment therefore is made.*

## 15 ADJUSTING ENTRIES.

At the end of each accounting period, it is customary to analyze accounts to determine whether they fairly reflect the income and expenses of the period. If they do not, **adjusting entries** are then made to accomplish this purpose. These adjustments will generally fall into one of the following categories:

**A. CLOSING INVENTORY OF MERCHANDISE. 1.** To set up merchandise inventory at end of fiscal period. *Asset recorded and cost factor (cost of goods sold) decreased:*

| | | | | | |
|---|---|---|---|---|---|
| 19— Dec. | 31 | Merchandise Inventory (ending) | 6 | 32,000.00 | |
| | | Profit and Loss | 100 * | | 32,000.00 |
| | | To set up closing inventory at 12/31/— | | | |

* These numbers indicate the page numbers of the respective ledger accounts listed on the accompanying work sheet (See work sheet: Accounting Chart 2) to which the items have been posted.

**2. The Profit and Loss account** is a summary account to which the *balances of all income and expense accounts* are transferred at the *end of each fiscal period.* The *debit* in the foregoing entry sets up the **inventory** (*asset*) as it exists at the end of the period, and the *credit* to the **Profit and Loss** account serves in the determination of the **Cost of Goods Sold** as an offset to the **Cost of Goods Available for Sale.** (See 7 supra for discussion on *merchandise* and see Accounting Chart 2; 4, B, for the transfer to the Profit and Loss account of all components of the Cost of Goods Available for Sale.)

**B. ACCRUED EXPENSES PAYABLE.** To set up expenses incurred for which payment has not yet been made (*expenses and liability are both increased*):

| | | | | | |
|---|---|---|---|---|---|
| 19— Dec. | 31 | Salaries—Salesmen | 60 | 650.00 | |
| | | Salaries—Office | 70 | 300.00 | |
| | | Interest Expense | 80 | 100.00 | |
| | | Accrued Expenses Payable | 23 | | 1,050.00 |
| | | To set up unpaid salaries and accrued interest expense as of 12/31/— | | | |

**C. ACCRUED INCOME RECEIVABLE.** To set up income earned but not yet received (*asset and income are both increased*):

| | | | | | |
|---|---|---|---|---|---|
| 19— Dec. | 31 | Accrued Income Receivable | 7 | 100.00 | |
| | | Interest Income | 90 | | 100.00 |
| | | To set up interest accrued on notes receivable as of 12/31/— | | | |

**D. PREPAID EXPENSES. 1.** To set up as an asset that portion of an item previously recorded as an expense, which has not been fully consumed or whose benefit has not yet been fully enjoyed (*asset increased and expense reduced*):

| | | | | | |
|---|---|---|---|---|---|
| 19— Dec. | 31 | Prepaid Expenses | 8 | 650.00 | |
| | | Insurance | 72 | | 650.00 |
| | | To set up unexpired (asset) portion of insurance premiums allocable to period after 12/31/— | | | |

**2.** If, instead, the **Prepaid Expense** account had been charged when the item was originally entered on the books, then the consumed portion would now be eliminated from the asset account and transferred to expense (*expense increased and asset decreased*):

| | | | | | |
|---|---|---|---|---|---|
| 19— Dec. | 31 | Insurance | | 1,500.00 | |
| | | Prepaid Expenses | | | 1,500.00 |
| | | To charge expired (expense) portion of insurance premiums, allocable to the year 19— | | | |

**E. UNEARNED INCOME.** To set up as a liability that portion of an item previously recorded as income which has not yet been fully earned (*income reduced and liability increased*):

| | | | | | |
|---|---|---|---|---|---|
| 19— Dec. | 31 | Rent Income—(sub-tenant) | 91 | 250.00 | |
| | | Unearned Income | 24 | | 250.00 |
| | | To set up portion (½) of sub-tenant's rent payment ($500) for month ending 1/15/—, not yet earned in this fiscal period (year ended 12/31/—) | | | |

**F. DEPRECIATION.** See prior discussion under Valuation Accounts (expense increased and asset reduced).

| | | | | | |
|---|---|---|---|---|---|
| 19— Dec. | 31 | Depreciation—Furn. & Fixtures | 75 | 2,000.00 | |
| | | Allow. for Depn.—F. & F. | 11 | | 2,000.00 |
| | | To set up annual charge, at rate of 10% on $21,000 (cost) less $1,000 (estimated salvage value). | | | |

**G. BAD DEBTS.** See prior discussion under Valuation Accounts (expense increased and asset reduced).

| | | | | | |
|---|---|---|---|---|---|
| 19— Dec. | 31 | Bad Debts | 74 | 1,800.00 | |
| | | Allowance for Bad Debts | 4 | | 1,800.00 |
| | | To set up new allowance as of 12/31/—: | | | |
| | | Amount required ..... $1,250 | | | |
| | | Add: Debit bal. in Allowance account ..... 550 | | | |
| | | $1,800 | | | |

DATA-GUIDE QUICK CHART

# BASIC ACCOUNTING CHART TWO

LEARN & REVIEW IN MINUTES

**SIMPLIFIED SUMMARY ◆ INSTANT REFERENCE ◆ PERMANENT MEMORY-AID**

AUTHOR: Prof. S. Tunick, Ph.D., Dept. Chairman, C.C.N.Y.

ALSO SEE CHARTS 1 & 3

SOLID PLASTIC

SPEED STUDY TO SAVE YOU TIME AND WORK.

QUICK CHARTS PUT THE FACTS IN FULL VIEW:

PERMANENTLY FILL IN FOR MISSING NOTES & TEXTS.

QUICK CHARTS HELP YOU REMEMBER:

CONTAINS NO PAPER

## 1 WORK SHEET. A. NATURE, USE.

Device to aid **determining net income** for a portion of fiscal period or at its close. **1. Trial balance figures** are used as a starting point; and after they have been listed on the work sheet, the trial balance columns should be added to insure the equality of debits and credits. **2.** All necessary **adjustments** are listed in an **Adjustments column.** If an adjusting entry affects an account appearing in the trial balance, it is entered on the line where that account appears. **3.** Otherwise, the new account title and the applicable money value are entered on the first unoccupied line below that on which trial balance footings appear (*See Basic Acc't. 1, 15, for sample adjusting entries A thru G which appear on work sheet illustration below*). **4.** After all adjustments have been listed and totalled to see that the *debit adjustments equal the credit adjustments,* the **totals** of, or **differences** between, the **trial balance** and applicable **adjustment** figures are **determined** and **analyzed.** **5.** If they represent **income** or **expenses,** they are **listed** in **Income Statement columns;** if **assets, liabilities,** or **proprietor's capital** or **withdrawals,** they are listed in **Balance Sheet columns.**

**B. SUMMARIZING THE WORK SHEET. 1.** The **balancing figure** added at the end of **Income Statement column** must be same as that added for the same purpose in the **Balance Sheet column.** *A balancing debit* in the Income Statement column represents **net income,** while a *balancing credit* therein represents **net loss. 2.** To ascertain proprietor's **net worth** at the end of the period, the **net income** for the period is **added** to the balance of his **Capital account** as it appears on the books, and the balance of his **Drawing account** is **subtracted** from the total thereof. **3.** The amounts appearing in the **Income Statement columns** are listed on the firm's income statement when it is ultimately prepared, and the net income shown thereon must be the same as the item designated as **Net income** on the work sheet. The amounts appearing in the **Balance Sheet columns** are listed on the firm's balance sheet.

**C. BREAK-EVEN STATEMENT.** Sets forth all items which appear on an income statement except **final inventory.** Such inventory cannot always be taken when such a statement is prepared. In lieu thereof, the statement shows the final inventory required which will result in neither a *profit nor a loss.* The proprietor's knowledge of his inventory on hand will enable him to determine whether there is enough inventory to have made operations profitable.

**JOHN ALLEN WORK SHEET** — **YEAR ENDED DECEMBER 31, 19__**

| L.F. | NAME OF ACCOUNT | TRIAL BALANCE DR. | TRIAL BALANCE CR. | ADJUSTMENTS DR. | ADJUSTMENTS CR. | INCOME STATEMENT DR. | INCOME STATEMENT CR. | BALANCE SHEET DR. | BALANCE SHEET CR. |
|---|---|---|---|---|---|---|---|---|---|
| 1 | Cash | 7 500 | | | | | | 7 500 | |
| 2 | Notes Receivable | 20 000 | | | | | | 20 000 | |
| 3 | Accounts Receivable | 62 500 | | | | | | 62 500 | |
| 4 | Allowance for Bad Debts | 550 | | | (G) 1 800 | | | | 1 250 |
| 6 | Mdse. Inventory (at beginning) | 30 850 | | | | 30 850 | | | |
| 10 | Furniture and Fixtures | 21 000 | | | | | | 21 000 | |
| 11 | Allow. for Depn.—Furn. & Fixt. | | 2 000 | | (F) 2 000 | | | | 4 000 |
| 20 | Notes Payable | | 10 000 | | | | | | 10 000 |
| 21 | Accounts Payable | | 25 000 | | | | | | 25 000 |
| 22 | Taxes Payable | | 1 100 | | | | | | 1 100 |
| 30 | John Allen, Drawing | 12 000 | | | | | | 12 000 | |
| 31 | John Allen, Capital | | 89 600 | | | | | | 89 600 |
| 40 | Sales | | 370 500 | | | | 370 500 | | |
| 41 | Sales Returns & Allowances | 15 500 | | | | 15 500 | | | |
| 42 | Discount on Sales | 5 250 | | | | 5 250 | | | |
| 50 | Purchases | 242 900 | | | | 242 900 | | | |
| 51 | Freight In | 2 100 | | | | 2 100 | | | |
| 52 | Purchase Returns & Allowances | | 14 700 | | | | 14 700 | | |
| 53 | Discount on Purchases | | 4 150 | | | | 4 150 | | |
| 60 | Salaries—Salesmen | 32 250 | | (B) 650 | | 32 900 | | | |
| 61 | Advertising | 11 500 | | | | 11 500 | | | |
| 62 | Misc. Selling Expenses | 5 200 | | | | 5 200 | | | |
| 70 | Salaries—Office | 18 350 | | (B) 300 | | 18 650 | | | |
| 71 | Rent | 24 000 | | | | 24 000 | | | |
| 72 | Insurance | 2 150 | | | (D) 650 | 1 500 | | | |
| 73 | Taxes | 2 500 | | | | 2 500 | | | |
| 76 | Misc. Administrative Expenses | 7 800 | | | | 7 800 | | | |
| 80 | Interest Expense | 800 | | (B) 100 | | 900 | | | |
| 90 | Interest Income | | 1 400 | | (C) 100 | | 1 500 | | |
| 91 | Rent Income—(sub-tenant) | | 6 250 | (E) 250 | | | 6 000 | | |
| | | 524 700 | 524 700 | | | | | | |
| 100 | Profit and Loss | | | | (A) 32 000 | | 32 000 | | |
| 6 | Mdse. Inventory (at end) | | | (A) 32 000 | | | | 32 000 | |
| 7 | Accrued Income Receivable | | | (C) 100 | | | | 100 | |
| 8 | Prepaid Expenses | | | (D) 650 | | | | 650 | |
| 23 | Accrued Expenses Payable | | | | (B) 1 050 | | | | 1 050 |
| 24 | Unearned Income | | | | (E) 250 | | | | 250 |
| 74 | Bad Debts | | | (G) 1 800 | | 1 800 | | | |
| 75 | Depreciation—Furn. & Fixt. | | | (F) 2 000 | | 2 000 | | | |
| | | | | 37 850 | 37 850 | 405 350 | 428 850 | 155 750 | 132 250 |
| | NET INCOME | | | | | 23 500 | | | 23 500 |
| | | | | | | 428 850 | 428 850 | 155 750 | 155 750 |

## 2 STATEMENT TERMINOLOGY.

**1. Current Assets:** Cash and other assets which normally will be converted into cash or sold or consumed during a business operating cycle, or within a year from the balance sheet date, *whichever period is longer.* Include prepaid expense items. **2. Operating Cycle:** Period involved between time cash is invested in merchandise or materials, through the time the latter are sold, and the receivables resulting therefrom are collected. **3. Fixed Assets:** Acquired for use in a business (*real estate, machinery, furniture, equipment, etc.*). **4. Other Assets:** Those neither current nor fixed. **5. Current Liabilities:** Generally those due within a year. Include unearned income items. **6. Long-term Liabilities:** Those maturing more than a year from date of the balance sheet. **7. Selling Expenses:** Incurred in the sale of merchandise and services (*advertising, sales salaries, freight out, packing, depreciation of salesroom equipment, etc.*). **8. Administrative Expenses:** Incurred in administering and managing a business (*rent, office salaries, telephone, stationery, depreciation of office furniture, bad debts, etc.*). **9. Additions to Income:** Include investment and interest income and other income not concerned with business operations. **10. Deductions from Income:** Include interest cost and expenses not concerned with business operations as such.

## 3 CLASSIFIED FINANCIAL STATEMENTS. A. CLASSIFIED BALANCE SHEET.

(Prepared from foregoing work sheet.)

**JOHN ALLEN**
**BALANCE SHEET DECEMBER 31, 19__**

**ASSETS**

| | | |
|---|---|---|
| **CURRENT ASSETS:** | | |
| Cash | | $ 7,500 |
| Notes Receivable | | 20,000 |
| Accounts Receivable | $62,500 | |
| *Less:* Allowance for Bad Debts | 1,250 | |
| Estimated Recovery Value | | 61,250 |
| Merchandise Inventory | | 32,000 |
| Accrued Income Receivable | | 100 |
| Prepaid Expenses | | 650 |
| TOTAL CURRENT ASSETS | | $121,500 |
| **FIXED ASSETS:** | | |
| Furniture and Fixtures | | $21,000 |
| *Less:* Allowance for Depreciation | | 4,000 |
| DEPRECIATED COST | | 17,000 |
| TOTAL ASSETS | | $138,500 |

**LIABILITIES**

| | | |
|---|---|---|
| **CURRENT LIABILITIES:** | | |
| Notes Payable | | $10,000 |
| Accounts Payable | | 25,000 |
| Taxes Payable | | 1,100 |
| Accrued Expenses Payable | | 1,050 |
| Unearned Income | | 250 |
| TOTAL CURRENT LIABILITIES | | $ 37,400 |
| **CAPITAL** | | |
| **CAPITAL:** | | |
| John Allen, Capital—January 1, 19__ | | $89,600 |
| Net Income for 19__ | $23,500 | |
| *Less:* Withdrawals in 19__ | 12,000 | |
| Net Increase in Capital | | 11,500 |
| John Allen, Capital—December 31, 19__ | | 101,100 |
| TOTAL LIABILITIES AND CAPITAL | | $138,500 |

**B. CLASSIFIED INCOME STATEMENT.** (Prepared from work sheet.)

**JOHN ALLEN INCOME STATEMENT**
**FOR THE YEAR ENDED DECEMBER 31, 19__**

| | | |
|---|---|---|
| **SALES REVENUE:** | | |
| Sales | | $370,500 |
| *Less:* Sales Returns and Allowances | $ 15,500 | |
| Discount on Sales | 5,250 | |
| Total Deductions | | 20,750 |
| NET SALES REVENUE | | $349,750 |
| **COST OF GOODS SOLD:** | | |
| Merchandise Inventory—Jan. 1, 19__ | | $ 30,850 |
| Purchases | $242,900 | |
| Freight In | 2,100 | |
| Total | $245,000 | |
| *Less:* Purch. Ret. & Allow. $14,700 | | |
| Discount on Purch. 4,150 | | |
| Total Deductions | 18,850 | |
| Net Cost of Purchases | | 226,150 |
| Cost of Goods Available for Sale | | $257,000 |
| *Less:* Merchandise Inv.—Dec. 31, 19__ | | 32,000 |
| COST OF GOODS SOLD | | 225,000 |
| GROSS PROFIT ON SALES | | $124,750 |
| **OPERATING EXPENSES:** | | |
| **SELLING EXPENSES:** | | |
| Salaries—Salesmen | $ 32,900 | |
| Advertising | 11,500 | |
| Miscellaneous Selling Expenses | 5,200 | |
| Total Selling Expenses | | $ 49,600 |
| **ADMINISTRATIVE EXPENSES:** | | |
| Salaries—Office | $ 18,650 | |
| Rent ($24,000, less income, $6,000) | 18,000 | |
| Insurance | 1,500 | |
| Taxes | 2,500 | |
| Depreciation—Furn. & Fixtures | 2,000 | |
| Bad Debts | 1,800 | |
| Misc. Administrative Expenses | 7,800 | |
| Total Administrative Expenses | | 52,250 |
| TOTAL OPERATING EXPENSES | | 101,850 |
| OPERATING INCOME | | $ 22,900 |
| **ADDITIONS TO INCOME:** | | |
| Interest Income | | $ 1,500 |
| **DEDUCTIONS FROM INCOME:** | | |
| Interest Expense | | 900 |
| NET ADDITIONS TO INCOME | | 600 |
| NET INCOME | | $ 23,500 |

## 4 CLOSING ENTRIES. A. CLOSING THE BOOKS.

**1.** Various individual income and expense accounts are merely separate *special subdivisions* of the enterprise's capital account, temporarily maintained to furnish valuable, detailed managerial information with respect to the nature and extent of the changes in capital which have occurred during a particular fiscal period. They will have completely fulfilled this function at end thereof. **2.** Therefore, each year, after the **net income** has been ascertained, it is customary to **close the books.** This involves: **(a)** Transferring all balances in income and expense accounts (*after adjusting entries have been posted*) to **Profit and Loss account. (b)** Transferring the balance of that account (*representing net income or loss*) to the proprietor's **Drawing account,** where it is combined with withdrawal made during the period to determine the *net increase or decrea* in the proprietor's capital for the period. **(c)** Transferring the *ne balance* in the Drawing account to the proprietor's **Capital account** where it is combined with the existing balance of that account.

**B. ILLUSTRATIVE ENTRIES.** For the work sheet illustration given these entries, recorded in the general journal, would be:

**GENERAL JOURNAL**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 19__ Dec. | 31 | Profit and Loss | 100 | 405,350 | 00 | | |
| | | Inventory (beginning) | 6 | | | 30,850 | 00 |
| | | Sales Returns and Allow. | 41 | | | 15,500 | 00 |
| | | Discount on Sales | 42 | | | 5,250 | 00 |
| | | Purchases | 50 | | | 242,900 | 00 |
| | | Freight In | 51 | | | 2,100 | 00 |
| | | Salaries—Salesmen | 60 | | | 32,900 | 00 |
| | | Advertising | 61 | | | 11,500 | 00 |
| | | Misc. Selling Expenses | 62 | | | 5,200 | 00 |
| | | Salaries—Office | 70 | | | 18,650 | 00 |
| | | Rent | 71 | | | 24,000 | 00 |
| | | Insurance | 72 | | | 1,500 | 00 |
| | | Taxes | 73 | | | 2,500 | 00 |
| | | Misc. Administ. Exp. | 76 | | | 7,800 | 00 |
| | | Interest Expense | 80 | | | 900 | 00 |
| | | Bad Debts | 74 | | | 1,800 | 00 |
| | | Depreciation—F. & F. | 75 | | | 2,000 | 00 |
| | | To close to Profit and Loss | | | | | |

**NOTE:** The debit of $405,350 to the **Profit and Loss** account is equa to the total of the *debit* **Income Statement column** on work shee

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 19__ Dec. | 31 | Sales | 40 | 370,500 | 00 | | |
| | | Purchase Ret. & Allow. | 52 | 14,700 | 00 | | |
| | | Discount on Purchases | 53 | 4,150 | 00 | | |
| | | Interest Income | 90 | 1,500 | 00 | | |
| | | Rent Income—sub-tenant | 91 | 6,000 | 00 | | |
| | | Profit and Loss | 100 | | | 396,850 | 0 |
| | | To close to Profit and Loss | | | | | |

**NOTE:** Credit of $396,850 to **Profit and Loss** account is equal to tot of *credit* **Income Statement column** on the work sheet, $428,85 reduced by $32,000 credit already in this account as a result adjusting entry #A, setting up the closing merchandise inventor

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 19__ Dec. | 31 | Profit and Loss | 100 | 23,500 | 00 | | |
| | | John Allen, Drawing | 30 | | | 23,500 | 0 |
| | | To transfer net income. | | | | | |

**NOTE:** The net income to be transferred to the **Drawing accoun** $23,500, is ascertained by adding the two credits to the **Profit an Loss account,** $32,000 and $396,850, and subtracting therefrom th debit to the account of $405,350.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 19__ Dec. | 31 | John Allen, Drawing | 30 | 11,500 | 00 | | |
| | | John Allen, Capital | 31 | | | 11,500 | 0 |
| | | To transfer net increase in capital to latter account. | | | | | |

**NOTE:** The net increase in capital, $11,500, is the net income fo the period, $23,500, reduced by proprietor's drawings of $12,000.

## 5 POST-CLOSING (REVERSAL) ENTRIES.

**1.** These a entries made on the *first day of the new accounting period* to rever certain adjusting entries made at the end of the preceding fisc period. To illustrate, assume that when the books were previous closed, a $300 accrued office salary item had been set up, the li bility for which now appears as part of credit balance in **Accrue Expenses Payable** account. Upon payment of the first full offi payroll of the new period, inclusive of this accrual, entry would b

Salaries—Office ..... 500.00
Cash ..... 500.0

**2.** By means of the following reversal entry, the **Salaries—Offi** account is adjusted to show that only $200 of the amount so pa is properly applicable to the new period:

Accrued Expenses Payable ..... 300.00
Salaries—Office ..... 300.0

**3.** The posting of this entry cancels the liability for the relate accrual and adjusts the **Salaries—Office** account to show the actu amount of office payroll applicable to the new fiscal period.

**ACCRUED EXPENSES PAYABLE**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 19__ Dec. | 31 | Sales Sal. | | 650.00 | 19__ Jan. | 1 | Reversal | a) | 300.0 |
| | " | Off. Salaries | a) | 300.00 | | | | | |
| | " | Interest Exp. | | 100.00 | | | | | |

**SALARIES—OFFICE**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 19__ Jan. | 1 | Reversal | | 300.00 | 19__ Jan. | 3 | w/e1/3/__ | | 500.0 |

**4.** Post-closing (*reversal*) entries are made for each of the followin types of adjusting entries, if they were set up prior to the close the preceding period: **(a) Accrued Expenses Payable; (b) Accrue Income Receivable; (c) Prepaid Expenses** (if recorded as an adjus ment by debit to Prepaid Expenses and credit to an Expense a count); **(d) Unearned Income** (if recorded as an adjustment by deb to an Income account and credit to Unearned Income account).

## 6 BANK RECONCILIATION. A. PROCEDURE.

Because t balance appearing on the **statements** received from a bank ea month rarely agrees with the balance in the general ledger **Cas account** (*and the check book*), it becomes necessary to trace diffe ences and thus **reconcile** the respective balances. This involves t following steps: **1.** Arranging checks returned by the bank in **n merical order. 2. Checking** to ascertain whether all **checks** enter on the books, including those issued previously but outstandi at the beginning of this period, have been **returned** following pa ment thereof by drawee-bank, and determining those still **outstan ing** (*not charged by drawee-bank*). **3.** Comparing deposits listed on t **bank statement** with those recorded in the **Cash Receipts journa 4.** Checking to see if all **charges and credits** appearing on bank stat ment were recorded on books of account, and vice versa, and a counting for those not so recorded.

**B. SPECIMEN BANK RECONCILIATION STATEMENT.**

| NAME OF BUSINESS | | BANK RECONCILIATION | | DATE | |
|---|---|---|---|---|---|
| **BALANCE**—*Books* | | $3,240 | **BALANCE**—*Bank* | | $3,72 |
| *Add:* Deposit credited on books for less than correct amt.: (Subtract, if more) | 90 | | *Subtract:* Outstanding checks | | |
| | $3,330 | | #126 | $ 60 | |
| | | | #140 | 70 | |
| | | | #141 (certified) | 100 | |
| | | | | $230 | |
| Items collected by bank but not entered on books | 820 | | *Less:* #141 (cert.) | 100 | 13 |
| | $4,150 | | | | $3,59 |
| *Subtract:* Bank charges not entered on books | 10 | | Amount credited by bank in error (Add, if charged) | | 15 |
| | | | | | $3,44 |
| | | | *Add:* Recorded deposit, not yet listed on statement | | 70 |
| **ADJUSTED BALANCE*** | | $4,140 | **ADJUSTED BALANCE*** | | $4,14 |

*These should be identical in amount.

## 7 IMPREST PETTY CASH FUND.

**1.** This type of fund is set up to provide for all **minor disbursements** that are likely to be made within a short period (usually not more than a month) and that cannot conveniently be made by check. When fund is originally set up, or subsequently increased, the entry is:

| | | |
|---|---|---|
| Petty Cash | 50.00 | |
| Cash | | 50.00 |

**2.** A reverse entry is made if the fund is decreased or eliminated. No other entries are made in the **Petty Cash account.** **3.** A memorandum record of petty cash disbursements is generally maintained so that each time the fund is replenished, an analysis of the expenditures may be made to support the typical entry in the **Cash Disbursements journal**, such as:

| | | |
|---|---|---|
| Stationery | 6.20 | |
| Postage | 4.72 | |
| Carfares | 5.40 | |
| Miscellaneous Expense | 2.18 | |
| Cash | | 18.50 |

**4.** The established amount of an **imprest fund** should always be on hand as cash and/or unreimbursed expense vouchers.

## 8 ACCOUNTING FOR NEGOTIABLE INSTRUMENTS.

### A. CUSTOMER'S NOTE RECEIVABLE

**1. RECEIPT OF A NOTE**

| | | |
|---|---|---|
| Notes Receivable | 1,000.00 | |
| Accounts Receivable—Customer | | 1,000.00 |

(The same entry is made whether interest-or non-interest-bearing.)

**2. COLLECTION OF A NOTE AT MATURITY**

| | | |
|---|---|---|
| Cash | 1,010.00 | |
| Notes Receivable | | 1,000.00 |
| Interest Income | | 10.00 |

(In this case, the note was for 60 days at 6%)

**3. DISHONOR OF NOTE RECEIVABLE AT MATURITY**

| | | |
|---|---|---|
| Accounts Receivable—Maker | 1,010.00 | |
| Notes Receivable | | 1,000.00 |
| Interest Income | | 10.00 |

(Maker is charged with face of note plus—in this case—6% interest accrued thereon for 60 days.)

**4. DISCOUNTING OF A NOTE BEFORE MATURITY**

| | | |
|---|---|---|
| Cash | 1,004.95 | |
| Discount Expense (for disct. period) | 5.05 | |
| Notes Receivable Discounted | | 1,000.00 |
| Interest Income (for term of note) | | 10.00 |

The **discount** is computed on the **maturity value**, i.e., the face of the note, plus interest for the full term thereof. In the instant case, a 6%, 60-day note with a maturity value of $1,010 was discounted 30 days before maturity at 6%. The person discounting the note obligates himself to pay it at maturity in the event the maker does not. The **Notes Receivable Discounted** account is set up in which to record this **contingent liability.**

**5. PAYMENT OF DISCOUNTED NOTE BY MAKER AT MATURITY**

| | | |
|---|---|---|
| Notes Receivable Discounted | 1,000.00 | |
| Notes Receivable | | 1,000.00 |

(Upon payment of the note at maturity, the **contingent liability** is eliminated and the note is returned to the maker by the bank. Since the bank gave the **net proceeds** to the business when it discounted the note, it retains the payment made to it by the maker at maturity.)

**6. DISHONOR OF DISCOUNTED NOTE AT MATURITY**

| | | |
|---|---|---|
| Accounts Receivable—Maker | 1,012.00 | |
| Cash | | 1,012.00 |

(This charges the maker for the payment to the bank at maturity of the face of the dishonored note ($1,000), plus the interest thereon ($10), and a possible fee ($2) for protesting its non-payment.)

| | | |
|---|---|---|
| Notes Receivable Discounted | 1,000.00 | |
| Notes Receivable | | 1,000.00 |

(This eliminates the **contingent liability** and removes the note from the accounts.)

### B. ISSUANCE OF OWN NOTE TO CREDITOR.

**1. ISSUANCE OF A NOTE**

| | | |
|---|---|---|
| Accounts Payable—Creditor | 2,000.00 | |
| Notes Payable | | 2,000.00 |

(The same entry is made whether interest-or non-interest-bearing.)

**2. PAYMENT OF NOTE AT MATURITY**

| | | |
|---|---|---|
| Notes Payable | 2,000.00 | |
| Interest Expense | 10.00 | |
| Cash | | 2,010.00 |

(The note was for 30 days at 6%.)

### C. DISCOUNTING OF OWN NOTE AT BANK.

**1. ISSUANCE AND DISCOUNTING OF NOTE**

| | | |
|---|---|---|
| Cash | 1,485.00 | |
| Discount Expense | 15.00 | |
| Notes Payable | | 1,500.00 |

(The borrower receives the face of the **non-interest-bearing note**, less—in this case—the 6% discount for the 60-day period thereof computed on maturity value.)

**2. PAYMENT OF DISCOUNTED OWN NOTE AT MATURITY**

| | | |
|---|---|---|
| Notes Payable | 1,500.00 | |
| Cash | | 1,500.00 |

## 9 VOUCHER SYSTEM.

**1.** Used as a *labor-saving device* in recording and paying creditors' invoices. **2.** Involves **internal control** (automatic detection of errors and prevention of **fraud**) which requires at least two persons to perform unduplicated work of each transaction. **3.** In connection with **purchasing**, the *ordering, receiving, storing, recording, paying*, etc., are handled by different persons, and fraud is largely eliminated as a result. **4. Approval for entry** is based upon *receipt* of goods after proper order, *verification* of price charged, of discounts allowed, and of invoice extensions and additions.

**A. VOUCHER JACKETS AND VOUCHER.** **Jackets** are used for filing invoices, purchase orders, receiving records, etc. Sometimes show details of transaction on face. Provide space for signature of those who approve for entry and for payment. The jacket, with contents above referred to, constitute a **voucher.**

**B. VOUCHER CHECK.** Check in payment of transaction is sometimes prepared simultaneously with **voucher** and is filed away according to date when payment is due. At that time it is signed and mailed.

## 10 VOUCHER REGISTER.

This is used for recording vouchers applicable to all items requiring payment, such as **purchases** and **expense** items (except those paid through *petty cash*), and items for which invoices are not received (*salaries, expense advances, maturing notes and other obligations, reimbursement of petty cash, etc*). Recorded vouchers represent **liabilities**, and, as payments are made, these liabilities are decreased.

**A. ACCOUNT CHARGEABLE IN THE VOUCHER REGISTER.** This is determined by the one approving the voucher for entry. A **chart of accounts** is frequently used to indicate by *account number* the account to be charged.

**B. VOUCHER REGISTER COMPOSITION.** Columns (left to right) are provided for: **1. DETAILS OF VOUCHER:** date of voucher, voucher number, payee's name, explanation. **2. DETAILS OF PAYMENT:** date thereof, check number. **3. CREDITS:** vouchers payable, income tax withheld, F.I.C.A. taxes withheld, other deductions. **4. DEBITS:** purchases, salaries (separate column for each type), expenses (separate column for each expense item recurring regularly, such as commissions, advertising, delivery expenses, etc.), and a sundries column for recording items which occur infrequently.

**C. FORM OF THE VOUCHER REGISTER.** (See bottom of chart)

**D. FUNCTION OF VOUCHER REGISTER.** **1.** The voucher register serves as *combination* of the **Purchases Journal** and the **Accounts Payable Ledger.** Vouchers are entered directly in the register and distributed to the columns appropriate to accounts that must be charged. **2.** The **net amounts payable** are shown in the **Vouchers Payable column.** **3.** *Totals of special columns are posted to appropriate general ledger accounts at the end of each month; items entered in Sundries column are posted immediately to accounts* shown. Postings are *not* made to **individual creditor** accounts in manner followed in posting from a purchases journal to accounts payable ledger. **No record** is maintained of sums due **individual creditors.** **NOTE:** On occasion, extra copies of vouchers are prepared which are filed alphabetically to help in determining how much business was transacted with individual creditors. **4.** When vouchers are paid, these copies are sometimes refiled in **"paid" file.** Upon payment of voucher, notation to that effect is made in the *Paid column* of the register on **same line** where the voucher was first entered (See notations for vouchers # 102,103 and 106.) Where no such notation appears, it follows that the item is still unpaid. **5.** The **total** of individual unpaid vouchers at end of any month should be **equal** to the **balance** of the Vouchers Payable account in **general ledger** at that time, after all postings have been made thereto. Thus, *the voucher register serves as a journal* (to record vouchers prepared) *and as a subsidiary record* (to account for the balance of the Vouchers Payable account in the general ledger).

## 11 CHECK REGISTER.

**A. PURPOSE.** Used for recording checks issued. **1.** *Entries* recorded in this register are *posted to the Paid column in voucher register.* **2. Column totals** *are posted at the end of the month to the appropriate general ledger* accounts.

**B. FORM OF CHECK REGISTER.**

| CHECK NO. | DATE 19— | | VO. NO. | CREDITS: CASH | CREDITS: PURCH. DISCT. | DEBIT: VOUCHER PAYABLE |
|---|---|---|---|---|---|---|
| 201 | Jan. 15 | M. Superior | 103 | 345.25 | | 345.25 |
| 202 | 17 | R. Andrews | 102 | 96.53 | 1.97 | 98.50 |
| 203 | 31 | M. Superior | 106 | 362.50 | | 362.50 |
| | | | | 804.28 | 1.97 | 806.25 |

## 12 VOUCHERS—SPECIAL PROBLEMS.

**A. TREATMENT OF PURCHASE DISCOUNTS.** **1.** When voucher is paid which had been entered in the **voucher register** at its face amount, the purchase discount deducted is recorded in the **check register** and the **Cash account** is credited with the **actual** amount paid. **2.** Sometimes, voucher is prepared for the **net amount due** (*after discount*), and this amount is then recorded in the **Vouchers Payable** column of voucher register. When this is done, the **purchase discount** allowable for prompt payment is entered in **Purchase Discount** column of the register and the full invoice amount is entered in *appropriate distributive debit column.* **3.** In rare cases, invoice is entered at its **net amount** (*after discount*) in both the **Vouchers Payable** and the appropriate **debit columns** and no record is kept of the discount deducted.

**B. PURCHASE RETURNS AND ALLOWANCES.** The most effective method for recording a return requires the **cancellation** of the **original voucher** entry (by reversal thereof) and the preparation and **recording** of a **new voucher** for *net amount* (original amount, less the return.) **Cancellation** is **recorded in red ink** in the *same columns* in which the entry cancelled was first recorded.

**C. PARTIAL PAYMENTS.** **1.** When these are arranged in advance, **separate vouchers** should be prepared for each payment. **2.** When arranged after a voucher has been prepared and recorded, the **original voucher** should be **cancelled** (before payments are made thereon) and **new individual vouchers** prepared for each of the installment payments. **3.** The cancellation of *first voucher* is best effected by **debiting Vouchers Payable** for the total thereof in the **Sundries** (debit) column and **crediting Vouchers Payable** in **Vouchers Payable** column for the amount of each of the new vouchers corresponding to the partial payments agreed upon. *The debit offsetting original credit to Vouchers Payable is not affected by the new entry* but the original voucher is marked **"cancelled"** in the Paid column with reference made there to new vouchers.

**D. PAYMENT BY A NOTE.** **1.** Issuance is generally recorded in the **general journal** as follows:

| | | |
|---|---|---|
| Vouchers Payable | 1,000.00 | |
| Notes Payable | | 1,000.00 |

**2. Notation** should be made *in the Paid column* of voucher register to indicate that the note has been substituted for the **voucher payable.** **3.** When the note **matures**, a **new voucher** is prepared and entered in the voucher register for amount of the *face of the note, plus interest*, with offsetting debits to *Notes Payable* and *Interest Expense.*

**E. VOUCHERS FOR SOME ACCRUALS.** When the amount of an **accrued portion of an expense** is ascertainable with certainty at end of period, voucher may be prepared therefore and properly recorded. Thus, a voucher may be prepared for the portion of wages accrued to the end of period. Subsequently, on the first payroll date of the new year, another voucher is prepared for remaining portion of payroll which is applicable to the new year.

**F. BANK CHARGES.** When a bank in which an account is maintained notifies depositor of a charge that it has made, an *entry* must be made in the **check register** to record a reduction in balance of **Cash account.** Since all **credits** to *Cash* in the check register are offset by **debits** to *Vouchers Payable*, it is necessary to prepare and record a **voucher** for amount of bank charge.

**G. CHANGE-OVER TO A VOUCHER SYSTEM.** Vouchers must be prepared for each **unpaid invoice** and are then entered in the register. The entries call for **debits** (*in Sundries column*) to the **Accounts Payable** account (*to cancel outstanding indebtedness*) with offsetting **credits** to **vouchers payable.** Accounts which were debited when invoices were first entered (with offsets to Accounts Payable), are not affected. Vouchers must be prepared for all **unentered invoices** (including those for expenses and services.)

## 13 RECORDING OF TAXES.

Business firms frequently act as collecting agents for taxes which are imposed, and it is necessary that records clearly **show amounts collected** and the basis used in determining taxes that may become due. On the other hand, *some income may be exempt* from tax and *some expenses not deductible* for tax purposes, and these should be clearly shown.

**A. SALES TAX.** **1.** May be imposed on **purchasers** and/or **sellers.** Study of *applicable laws* is required to ascertain which items and persons are subject to and exempt from tax. **2.** *Use of a columnar sales journal can help to analyze sales as taxable and non-taxable*, and record the sales tax charged, if any. A **monthly summary** of the *Sales Journal* would then be:

| | | |
|---|---|---|
| Accounts Receivable | 10,566.00 | |
| Sales-Taxable | | 9,150.00 |
| Sales-Non Taxable | | 1,050.00 |
| Sales-Tax Liability | | 366.00 |

If the community which imposes the tax requires a more detailed classification, **additional columns** can be utilized in the **sales journal** to provide this information. **3.** Occasionally no tax is collectible on sales (of taxable merchandise) less than a certain amount, but the vendor in such cases is generally required to *remit the larger of the amount collected* or a *stated percentage* of taxable sales. If the amount remitted is larger than that collected, the *seller absorbs the excess*, and the entry is:

| | | |
|---|---|---|
| Sales Tax Liability | 366.00 | |
| Sales Tax Expense | 5.70 | |
| Cash | | 371.70 |

**B. STATE UNEMPLOYMENT INSURANCE TAX.** **1.** Most employment is subject to this tax, but familiarity with **local exemptions** is necessary in order to determine whether any exist and to what extent taxes must be paid. **2.** This tax is **levied on the employer** in all states. In a few states it is also *levied on the employee and collected from him* by **withholding** part of his salary. **3.** The rates *vary* in the same state according to the **regularity of employment** provided by the employer and the **amount of unemployment relief claimed** by and paid to former employees. **4.** When salaries are paid, the entry to record the payments and the various taxes withheld is:

| | | |
|---|---|---|
| Salaries | 400.00 | |
| State Unemployment Ins. Tax Payable | | 4.00 |
| F.I.C.A. Tax Payable | | 14.50 |
| Withheld Income Tax Payable | | 60.00 |
| Cash | | 321.50 |

When remittance of state unemployment insurance tax is made (generally on a quarterly basis), the entry is:

| | | |
|---|---|---|
| State Unemploy. Ins. Tax Pay. (Employee Withholding) | 4.00 | |
| State Unemploy. Ins. Tax (Tax on Employer) | 10.80 | |
| Cash | | 14.80 |

**C. FEDERAL UNEMPLOYMENT INS. TAX.** **1.** This tax is payable **annually** not later than **Jan. 31** of each year by firms which employed *four or more persons on 20 different days* of the preceding calendar year *with each such day being in a different calendar week.* **2.** The tax is **3.1%** of *the first $3,000 of salary* paid during the preceding calendar year to each person in taxable employment. **3.** The amount of **state unemployment insurance tax** *applicable to the same year and paid on or prior to the due date* for filing the federal unemployment insurance tax return may be **credited** *against federal unemployment insurance tax* due. *Such credit may not exceed* 2.7% of the salaries subject to the federal unemployment insurance tax. **4.** When, because of regularity of employment, the rate of state tax is below 2.7%, the **credit** allowed against the *federal tax* is the amount that the *state tax* would have been (*but not exceeding 2.7%*) if the **merit rate** had not been assigned. The entry for payment is:

| | | |
|---|---|---|
| Federal Unemployment Ins. Tax | 200.00 | |
| Cash | | 200.00 |

3.1% of $50,000, less 2.7% of $50,000 for state taxes.

**D. FEDERAL INSURANCE CONTRIBUTIONS ACT TAXES.** **1.** Levied on both employer and employee to *provide retirement benefits to insured persons and dependents and certain other benefits* because of the disability or death of insured persons. **2.** Rates charged to employer and employee are 4.8 % *of the first $7,800 paid* to employees in taxable employment in 1969-70 (5.2% 1971-1972.) **3.** When an individual is **employed by two firms** during the same year, **each** firm is required to deduct 4.8 % of the first $7,800 paid to that individual. Tax deductions on aggregate salaries in excess of $7,800 may be applied by the employee against any income tax which he may owe, or refund thereof may be requested by him. **4.** When a firm is taken over by another, the latter is required to deduct and pay a tax for each employee on only the **difference** between $7,800 and the smaller amount paid to him before the date of takeover, but the *Internal Revenue Service* must be informed of the take-over. **5.** The entry to record the withholding of taxes (both F.I.C.A. and federal income) is:

| | | |
|---|---|---|
| Salaries (monthly) | 1,000.00 | |
| F.I.C.A. Tax Payable | | 48.00 |
| Withheld Income Tax Payable | | 160.00 |
| Cash | | 792.00 |

**6. Quarterly returns** for reporting *F.I.C.A.* and *federal income taxes* withheld *must be filed with the* **Internal Revenue Service.** However, if the taxes withheld plus the F.I.C.A. tax imposed on employer are in excess of $100 monthly, *deposits are required to be made with the Federal Reserve Bank not later than the 15th day of the month* following the 1st and 2nd month of each quarter. When quarterly returns are filed, **credit** is taken for such monthly deposits. When remittances are made for tax collected, the entry is:

| | | |
|---|---|---|
| F.I.C.A. Tax Payable | 48.00 | |
| F.I.C.A. Tax Expense | 48.00 | |
| Withheld Income Tax Payable | 160.00 | |
| Cash | | 256.00 |

## 14 SPECIAL TAX RECORDS.

**A. INCOME TAX REPORTING.** **1.** Much detailed information is required to be furnished, and records must be so maintained that this information is readily available. The problem sometimes becomes involved because certain income is not subject to tax and certain expenses are not deductible in computing net income for income tax purposes. **2. Example:** *interest income* received on state and municipal bonds owned is not subject to income tax. In order to identify the amount which should be ignored, the **Interest Income account** should be set up in **columnar** form with *taxable income* recorded in one column and *non-taxable interest* recorded in another. Still a *third column* is then used to record *interest from all sources.* **3.** Whenever any other income or other expense item is treated differently for tax purposes than for ordinary accounting, *the use of columnar ledger sheets can be put to good advantage.*

**B. PAYROLL RECORDS.** **1.** Because of the different payroll taxes collected and imposed by the **federal** and **state** governments and the fact that *different basic amounts* are involved in each, it is important that appropriate records are maintained so that it will be possible to ascertain when the maximum applicable to each employee has been reached. **2.** Employees must be furnished with statements at the end of each year which indicate the **F.I.C.A.** wages paid during the year which were subject to the tax, the F.I.C.A. tax deducted, the total wages paid during the year, and the federal income tax withheld. A **separate record** *should therefore be maintained for* **each employee** to show this information. **3.** Since exemptions for state and federal taxes are not always identical, **cumulative totals for each tax** measured by payrolls should be available at all times for filing of required payroll reports.

**FORM OF THE VOUCHER REGISTER** (See 10:B.)

| DATE | VO. NO. | PAYEE | EXPLANATION | PAID: DATE | PAID: CHECK NO. | CREDITS: VOUCH. PAY. | CREDITS: W. H. TAX PAY. | CREDITS: F.I.C.A. TAX PAY. | DEBITS: PURCH. | DEBITS: FREIGHT INS. | DEBITS: SALES SALARIES | DEBITS: ADVERTG. | DEBITS: DELIVERY EXPENSE | DEBITS: OFFICE SALARIES | DEBITS: OFFICE SUPPLIES | DEBITS: SUNDRIES ACCOUNT | DEBITS: SUNDRIES L. F. | DEBITS: SUNDRIES AMOUNT |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 19— | | | | | | | | | | | | | | | | | | |
| Jan. 2 | 101 | A. Richard | Inv. of 1/2 | | | 150 00 | | | 150 00 | | | | | | | | | |
| 4 | 102 | R. Andrews | Advertising | 1/17 | 202 | 98 50 | | | | | | 98 50 | | | | | | |
| 15 | 103 | M. Superior | Payroll to 1/15 | 1/15 | 201 | 338 95 | 19 00 | 18 05 | | | 250 00 | | | 126 00 | | | | |
| 20 | 104 | J. Adams | Inv. of 1/8 | | | 249 30 | | | 249 30 | | | | | | | | | |
| 25 | 105 | Hancock Express | On Adams | | | 12 00 | | | | 12 00 | | | | | | | | |
| 31 | 106 | M. Superior | Payroll to 1/31 | 1/31 | 203 | 355 80 | 25 00 | 19 20 | | | 260 00 | | | 140 00 | | | | |
| 31 | 107 | Jones Furn. Co. | Desk | | | 131 00 | | | | | | | | | | Furn. & Fixt. | | 131 00 |
| | | | | | | 1,335 55 | 44 00 | 37 25 | 399 30 | 12 00 | 510 00 | 98 50 | | 266 00 | | | | 131 00 |



DO NOT PLACE ON HOT SURFACES

KNOW THE FACTS ON THIS CHART AND YOU WILL GET HIGHER MARKS IN CLASSWORK, HOMEWORK, TESTS AND EXAMS.

THERE ARE QUICK CHARTS IN MATH • SCIENCE • ENGLISH • ECO • PSYCH • SOC • PHILO • MUSIC • HISTORY • LANGUAGES

DATA-GUIDE EDITOR: JOSEPH L. LEON

PUBLISHED & MFD BY DATA-GUIDE, INC. FLUSHING, N.Y. 11355

DATA-GUIDE QUICK CHART

# BASIC ACCOUNTING CHART THREE

LEARN & REVIEW IN MINUTES

SIMPLIFIED SUMMARY ◆ INSTANT REFERENCE ◆ PERMANENT MEMORY-AID

AUTHOR: Prof. S. Tunick, Ph.D., Dept. Chairman, C.C.N.Y.

SEE ALSO CHARTS 1 & 2

SOLID PLASTIC

SPEED STUDY TO SAVE YOU TIME AND WORK.

QUICK CHARTS PUT THE FACTS IN FULL VIEW:

PERMANENTLY FILL IN FOR MISSING NOTES & TEXTS.

QUICK CHARTS HELP YOU REMEMBER:

CONTAINS NO PAPER

## 1 PARTNERSHIP RECORDS.

**A. CAPITAL AND DRAWING ACCOUNTS.** Are similar to those of a single proprietorship, except: (1) as many *capital* and *drawing accounts* are required as there are partners; (2) at the end of each accounting period the *income or loss is distributed* among the partners in accordance with the agreement among them. **If no agreement exists, the division is equal.**

**B. BALANCE SHEET.** (1) **PARTNERS' CAPITAL INTERESTS** are listed underneath one another. (2) **PARTNERS' SHARE OF NET INCOME** includes **salaries, interest**, and other similar sums paid to partners during the course of an accounting period for services rendered and capital investments made by them.

**C. METHODS OF DIVIDING PARTNERSHIP INCOME AND LOSSES.** (1) **EQUALLY:** when there is no agreement. (2) **BY AGREEMENT:** **(a)** Based on an **income-ratio** satisfactory to all. **(b)** Based on the **ratio of capital balances**—either at the beginning of a period, at the end of period, or employed during the period (*average*).

**D. ACCOUNTING FOR PARTNERSHIP INCOME.** The entry made for crediting partners with their respective **shares of income** is:

| | | | |
|---|---|---|---|
| Profit and Loss | | 10,000.00 | |
| Partner A, Drawing | | | 6,000.00 |
| Partner B, Drawing | | | 4,000.00 |
| Division of income per agreement: | 60% to A | $ 6,000.00 | |
| | 40% to B | 4,000.00 | |
| | | $10,000.00 | |

## 2 ADMISSION OF NEW PARTNER.

**A. PURCHASE OF PARTNER'S INTEREST.** This generally results in *no change* in partnership **assets.** Regardless of how much is paid, the *entry* made merely serves to *transfer* to another the *portion of capital disposed of* as it appears on the books of account. Thus, if A sells to B one-half of his $12,000 capital interest for $8,000, the entry is made, as at right. The amount ($8,000) is **paid privately** and does not pass through the partnership books. When the purchase is made from several partners, **each** is first paid an amount equal to the *book value of interest transferred by him*, and the **balance** of the purchase price is **divided** between them in their *profit- and loss-sharing ratio.*

| | | |
|---|---|---|
| A, Capital | 6,000.00 | |
| B, Capital | | 6,000.00 |

**B. BY INVESTMENT ON THE PART OF A NEW PARTNER.** (1) **INVESTMENT BASED ON BOOK VALUE.** If the new partner (X) invests enough to give him a one-fourth interest, then the amount he must invest will be one-third of the existing capital. Assuming this to be $9,000, it will in turn be three-quarters of the new capital after X has made his investment. The entry made to record the investment is shown above.

| | | |
|---|---|---|
| Cash | 3,000.00 | |
| X, Capital | | 3,000.00 |

The ¼ interest entitles X to ¼ of the net income. (2) **INVESTMENT INVOLVING NEW PARTNER'S GOODWILL.** If S and T have capital balances of $4,000 and $5,000, respectively, and W invests $2,000 to get a one-quarter interest, then S's and T's combined capital of $9,000 will be ¾ of a total capital of $12,000, and W's admission is shown as:

| | | |
|---|---|---|
| Cash | 2,000.00 | |
| Goodwill | 1,000.00 | |
| W, Capital | | 3,000.00 |

*If goodwill is not recorded,* then W's $2,000 will increase the total capital to $11,000, and one-quarter thereof will be $2,750. W's admission is then recorded as:

| | | |
|---|---|---|
| Cash | 2,000.00 | |
| S, Capital | 375.00 | |
| T, Capital | 375.00 | |
| W, Capital | | 2,750.00 |

The excess credited to W must be charged to S and T in their *profit-sharing ratio* (equal since nothing to the contrary is said about profit-sharing). (3) **INVESTMENT INVOLVING FIRM'S GOODWILL.** If M and N have capital balances of $9,000 and $8,000, respectively, and P invests $10,000 to acquire a ⅓ interest then, if $10,000 is the value of a ⅓ interest, $20,000 is the value of the ⅔ interest of M and N.

| | | |
|---|---|---|
| Goodwill | 3,000.00 | |
| M, Capital | | 1,500.00 |
| N, Capital | | 1,500.00 |

M's and N's interests are carried at $17,000, understated by $3,000 and P's admission is recorded as:

| | | |
|---|---|---|
| Cash | 10,000.00 | |
| P, Capital | | 10,000.00 |

If goodwill is not recorded, P is credited with one-third of the combined assets as shown:

| | | |
|---|---|---|
| Cash | 10,000.00 | |
| P, Capital | | 9,000.00 |
| M, Capital | | 500.00 |
| N, Capital | | 500.00 |

M and N are credited with the excess invested by P over that credited to him. (4) **INVESTMENT INVOLVING PRIOR REVALUATION OF ASSETS.** Any **net increase or decrease** in assets is credited or debited to the old partners' capitals in *their profit- and loss-sharing ratio.* (5) **ADMISSION WITHOUT INVESTMENT.** For promising employee brought in as partner.

## 3 PARTNERSHIP DISSOLUTION.

**A. LIQUIDATION.** (1) **PROCEDURE FOR SALE & DISPOSITION OF ASSETS.** **(a)** Payment of liabilities. **(b)** Distribution among the partners of cash and remaining assets on hand. **(c)** Before any such distribution to partners can be made, **loans** payable to partners must first be paid. Payments to the partners in **liquidation** are made according to their *capital account balances* and not according to *profit-sharing ratios.* (2) **LOSSES AND GAINS** in liquidation must first be divided among partners in **profit-and-loss ratios** *before* any payments can be made to them, either on account of loans or in liquidation. Thus, before making payments to partners on account of loans to them, *loan account credit balances* must be applied against their **capital account debit balances.** In fact, provision must also be made for losses in liquidation from the sale of assets still on hand. In that connection, it is generally assumed that *assets not yet sold will not be sold at all* and that a complete **loss** will result therefrom.

(3) **ENTRIES TO RECORD A PARTNERSHIP LIQUIDATION:** (Assume assets of $22,000; outside liabilities of $3,000; loans of $2,000 payable to A, $2,000 to B; capital bal. of $10,000 for A, $5,000 for B. Assets sold for $10,000.)

| | | | |
|---|---|---|---|
| Cash | | 10,000.00 | |
| Realization Profit and Loss | | 12,000.00 | |
| Assets | | | 22,000.00 |
| (Sale of assets) | | | |
| A, Capital | | 6,000.00 | |
| B, Capital | | 6,000.00 | |
| Realization Profit and Loss | | | 12,000.00 |
| (Division of Losses) | | | |
| B, Loan | | 1,000.00 | |
| B, Capital | | | 1,000.00 |
| (Apply capital account debit balance to loan account balance) | | | |
| A, Loan | | 2,000.00 | |
| B, Loan | | 1,000.00 | |
| Liabilities | | 3,000.00 | |
| Cash | | | 6,000.00 |
| (Payment of partners' loans, liabilities) | | | |
| A, Capital | | 4,000.00 | |
| Cash | | | 4,000.00 |
| (Liquidation of capital a/cs) | | | |

**B. RETIREMENT OF A PARTNER.** Find value of all assets; adjust books thereto. If the partner is paid the amount due him, the entry is shown, as at right:

| | | |
|---|---|---|
| X, Capital | 12,500.00 | |
| Cash | | 12,500.00 |

If payment is not made *in full* at once, a **liability account** is substituted for the partner's capital account as shown at right:

| | | |
|---|---|---|
| X, Capital | 12,500.00 | |
| Cash | | 2,500.00 |
| Due Retiring Partner | | 10,000.00 |

**C. DEATH OF A PARTNER.** Find value of all assets; adjust books thereto. Then the balance in the **deceased partner's capital account** is closed into a *Liability to Estate of Deceased Partner account.*

## 4 CORPORATE CAPITAL.

Capital stock issued, retained earnings, and capital surplus. **A. CAPITAL STOCK.** Issue *limited by amount authorized in a charter.* **Shares of stock** are units into which capital stock authorized in a charter are divided. **B. RETAINED EARNINGS.** Separate capital accounts for stockholders are not maintained in general ledger, as in case of partners, although separate accounts showing shares owned are maintained in a stockholders' ledger. At end of each year, **earnings** of a corporation are closed from the *Profit and Loss account* into an *Earned Surplus account.* From time to time distributions of earnings are made among stockholders in form of **dividends** and, when made, these are charged against the *Earned Surplus account.* Accumulated amount not distributed represents retained earnings. **C. CAPITAL SURPLUS** is capital contributed or created *in excess of the par or stated value of shares of stock sold.* It may arise from sale of stock, donation, appreciation of assets.

## 5 CORPORATE DIVIDENDS.

In partnership, earnings may be withdrawn by the partners at will, but earnings are distributed in a corporation only when **directors vote dividends** affirmatively. Dividends are generally payable in **cash,** but are sometimes paid in **capital stock** of corporation, and on *rare* occasions **in property.** Dividends are payable only on outstanding stock — not **treasury stock** (stock once issued and later reacquired by corp. through purchase or donation) or on **unissued stock** (stock authorized for sale but still unsold). Treasury stock can be *resold* or *cancelled.* Cancellation has the effect of reducing authorized capital stock and once paid for, is free of liability to creditors regardless of resale price. However from buyers of original stock issue, creditors can get difference between *par value and lower price* if corp. debts are unpaid.

## 6 CORPORATION ACCOUNTING.

**A. ORGANIZATION.** Pro forma statement indicating state where corporation was organized and number and kind of shares authorized, and/or entry, as follows:

| | | |
|---|---|---|
| Capital Stock Unissued | 10,000.00 | |
| Capital Stock Authorized | | 10,000.00 |
| (Entry made only for par value stock.) | | |
| **B. ISSUE OF STOCK.** | | |
| Cash | 6,000.00 | |
| Capital Stock | | 6,000.00 |
| (Entry is made for no par value stock when sold, or par value stock when only a pro forma statement is shown at time of organization.) Capital stock accounts are set up for each class of stock. | | |
| Cash | 6,000.00 | |
| Capital Stock Unissued | | 6,000.00 |
| (Entry is made when authorized stock is recorded at time of organization.) | | |
| **C. SALE OF STOCK ABOVE PAR VALUE.** | | |
| Cash | 6,600.00 | |
| Capital Stock | | 6,000.00 |
| Premium on Capital Stock | | 600.00 |
| (Premium is part of paid-in capital or capital surplus.) | | |
| **D. SALE OF STOCK BELOW PAR VALUE.** | | |
| Cash | 5,400.00 | |
| Discount on Capital Stock | 600.00 | |
| Capital Stock | | 6,000.00 |
| (Discount appears on balance sheet as a deduction in Stockholders' Equity section.) | | |
| **E. SUBSCRIPTIONS RECEIVED.** | | |
| Subscriptions Receivable | 5,000.00 | |
| Capital Stock Subscribed | | 5,000.00 |
| (The Subscriptions Receivable account represents the balance due from subscribers on subscriptions; the Capital Stock Subscribed account represents stock subscribed for but not issued.) | | |
| **F. RECEIPT OF PAYMENT FOR SUBSCRIPTIONS.** | | |
| Cash | 5,000.00 | |
| Subscriptions Receivable | | 5,000.00 |
| **G. ISSUANCE OF STOCK TO SUBSCRIBERS ON PAYMENT FOR STOCK.** | | |
| Capital Stock Subscribed | 5,000.00 | |
| Capital Stock | | 5,000.00 |
| **H. SUBSCRIPTIONS OF STOCK AT A PREMIUM.** | | |
| Subscriptions Receivable | 5,500.00 | |
| Capital Stock Subscribed | | 5,000.00 |
| Premium on Capital Stock Subscribed | | 500.00 |
| (Entry, when subscription is received.) | | |
| Cash | 5,500.00 | |
| Subscriptions Receivable | | 5,500.00 |
| (Entry, when subscription is paid.) | | |
| Capital Stock Subscribed | 5,000.00 | |
| Capital Stock | | 5,000.00 |
| (Entry, when stock is issued.) | | |
| Premium on Capital Stock Subscribed | 500.00 | |
| Premium on Capital Stock | | 500.00 |
| (Entry, after stock has been paid in full.) | | |
| **I. EARNED SURPLUS OR RETAINED EARNINGS.** | | |
| Profit and Loss | 7,250.00 | |
| Earned Surplus | | 7,250.00 |
| (Transfers net income at end of year.) | | |

**J. PAID-IN SURPLUS.** This represents the excess received in the sale of stock over that credited to a Capital Stock account. When premium on capital stock has been set up, it is closed out by:

| | | |
|---|---|---|
| Premium on Capital Stock | 500.00 | |
| Paid-in Surplus | | 500.00 |
| (Premium is sometimes retained on books to show source of paid-in surplus.) | | |

**K. DONATED SURPLUS.** Created as result of gifts to the corp. The account for asset received is affected and surplus credited, as shown:

| | | |
|---|---|---|
| Machinery | 1,000.00 | |
| Donated Surplus | | 1,000.00 |

**L. SURPLUS FROM APPRECIATION OF ASSETS.** This is created when book value of assets is increased to their actual value:

| | | |
|---|---|---|
| Land | 30,000.00 | |
| Surplus from Apprec. of Assets | | 30,000.00 |
| (In some states dividends may be paid out of such surplus if it represents an actual increase in value.) | | |

**M. SURPLUS ADJUSTMENTS.** Presently trend is to treat correction of income for prior years as adjustments of current net income. Otherwise, adjustments can be recorded through *Earned Surplus account;* over-depreciation of building in prior year is adjusted as:

| | | |
|---|---|---|
| Allowance—Dep'n—Building | 1,250.00 | |
| Earned Surplus | | 1,250.00 |

**N. DECLARATION OF DIVIDENDS.**

| | | |
|---|---|---|
| Earned Surplus | 3,000.00 | |
| Dividend Payable—Common Stock | | 2,000.00 |
| Dividend Payable—Preferred Stock | | 1,000.00 |

**O. PAYMENT OF CASH DIVIDEND.**

| | | |
|---|---|---|
| Dividend Payable—Common Stock | 2,000.00 | |
| Dividend Payable—Preferred Stock | 1,000.00 | |
| Cash | | 3,000.00 |

**P. STOCK DIVIDEND DECLARATION AND PAYMENT.**

| | | |
|---|---|---|
| Earned Surplus | 4,000.00 | |
| Stock Dividend Payable | | 4,000.00 |
| Stock Dividend Payable | 4,000.00 | |
| Capital Stock | | 4,000.00 |

When fair value of share distributed is greater than its par value, excess am't transferred from Earned Surplus to Capital Surplus.

**Q. STOCK SPLIT.** Exchange of ea. outstanding share for larger number of shares, usually of smaller par value. Total capital is not affected.

| | | |
|---|---|---|
| Capital Stock—$50 par value | 20,000.00 | |
| Capital Stock—$25 par value | | 20,000.00 |
| (For issue of 2 shares of $25 par value for each share of $50 par value.) | | |

**R. TREASURY STOCK ACQUIRED BY DONATION.**

| | | |
|---|---|---|
| Treasury Stock ($100 par value) | 5,000.00 | |
| Donated Surplus | | 5,000.00 |

**S. SALE OF DONATED TREASURY STOCK (BELOW PAR).**

| | | |
|---|---|---|
| Cash | 4,000.00 | |
| Donated Surplus | 1,000.00 | |
| Treasury Stock | | 5,000.00 |
| (For sale at $80) | | |

**T. TREASURY STOCK ACQUIRED BY PURCHASE.**

| | | |
|---|---|---|
| Treasury Stock ($100 par value) | 3,200.00 | |
| Cash | | 3,200.00 |
| (40 shares at $80) | | |
| Earned Surplus | 3,200.00 | |
| Restricted Earned Surplus | | 3,200.00 |
| (To reduce surplus available for dividends by amount of stock purchase.) | | |

**U. SALE OF PURCHASED TREASURY STOCK.**

| | | |
|---|---|---|
| Cash | 3,600.00 | |
| Treasury Stock | | 3,200.00 |
| Paid-In Capital | | 400.00 |
| (Sale of 40 shares at $90 a share with resultant profit of $10 a share.) | | |
| Restricted Earned Surplus | 3,200.00 | |
| Earned Surplus | | 3,200.00 |
| (Removes restriction on sale of stock.) | | |

**V. TREASURY STOCK CANCELLED AFTER ACQUISITION.**

| | | |
|---|---|---|
| Capital Stock | 5,000.00 | |
| Capital Surplus | 500.00 | |
| Treasury Stock (50 shares of $100 par value stock purchased at $110.) | | 5,500.00 |
| Restricted Earned Surplus | 5,500.00 | |
| Earned Surplus | | 5,500.00 |
| (Removes restriction upon stock cancellation.) | | |

## 7 CORPORATE BONDS.

**A. DEFINITION.** Bonds are **written promises under seal** to pay a *stated sum* at a designated future time to a named person or to bearer, and with interest payable at a specified rate at stated dates. Sometimes property is pledged to secure the payment thereof. **B. BONDS VS. PREFERRED STOCK.** (1) Bond interest (an expense) is payable regardless of corporate profits; dividends are declared at the will of directors only when income is earned. (2) Bond interest is a *deductible expense;* dividends are not. (3) Bond principal is payable at **liquidation** before any stockholders receive anything.

**C. TYPES OF BONDS.** (1) **COUPON:** Coupons representing *specific interest payments* are affixed to the bond and are detached as each matures. (2) **REGISTERED:** When registered as to *principal,* the owners' names are recorded on both the *corporate bonds* and the *corporate records,* and the face values are paid to the registered owners at maturity. Such bonds are *coupon in form.* When registered as to *principal and interest,* the interest is sent to the registered owners at each interest date. When registered bonds are sold, the corp. must be notified. (3) **SECURED: Property** is pledged to secure the payment of principal and interest. (4) **DEBENTURE:** These are **unsecured** and bond owners are *general creditors* of the corporation. (5) **INCOME:** *Interest* is paid only when *earnings* permit. (6) **SINKING FUND:** Require funds to be set aside from time to time to *assure sufficient funds* on hand at **maturity.** (7) **SERIAL** Bonds mature on a series of dates. (8) **CALLABLE:** The issuing corporation may, if it so decides, pay the bonds *before maturity* according to terms set forth on issue. (9) **CONVERTIBLE:** The **holders** may decide to *exchange the bonds for stock* of the issuing corporation according to terms set forth on issue. (10) **GUARANTEED:** The guaranty runs from a *third person* with respect to the principal amount or interest, or both.

**D. BOND PRICES.** (1) **GENERAL.** Prices are quoted at a **percentage of par value.** On sales between interest dates, interest from the date when last payable up to the date of sale is added to the purchase price:

| | | |
|---|---|---|
| Cash | 1,010.00 | |
| Bonds Payable | | 1,000.00 |
| Bond Interest Expense | | 10.00 |
| (6% bond sold 4/1 at par, int. payable 2/1, 8/1.) | | |

At interest dates, the interest on outstanding bonds is paid for the entire interest period to registered owners or to holders of coupons. The credit to *Bond Interest Expense* on April 1, 1968 is an offset against the *full* payment of $30 interest on August 1, 1968 and reduces the interest expense for the period to $20. (2) **BONDS SOLD "FLAT."** When a bond on which interest is in default is disposed of, no adjustment is made for *accrued interest* thereon from the date when interest was *last paid.* (3) **BOND PREMIUM AND DISCOUNT.** Interest rates are fixed, but money markets fluctuate. Bonds carrying a high interest rate in a low interest market are likely to sell at a price in *excess of par value.* The opposite is likewise true. *The excess over par at which a sale is made is called* **premium;** *the sum less than par is called* **discount.** Both are written off over the remaining period of the bond's existence. This may be done on an actuarial basis or pro-rata (straight-line). *Premium is* **credited** *to Bond Interest Expense and discount is* **debited** *to Bond Interest Expense.* As this is done each period, the effect of crediting *Bond Interest Expense* is to reduce the **interest expense** of the period, and the opposite is applicable when this account is debited. Thus, when bonds are sold at a premium on 2/1/68:

| | | |
|---|---|---|
| Cash | 102,000.00 | |
| Bonds Payable—6%, due 2/1/73 | | 100,000.00 |
| Premium on Bonds | | 2,000.00 |

## 8 AMORTIZATION OF PREMIUM & DISCOUNT.

**A. PREMIUM.** Each year a proportionate share of premium is **written off as an offset against interest expense.** Thus, if books are closed on 12/31, the following entries (straight line method) are made during the first year:

| | | | |
|---|---|---|---|
| 8/1/68 | Bond Interest Expense | 3,000.00 | |
| | Cash | | 3,000.00 |
| 12/31/68 | Bond Interest Expense | 2,500.00 | |
| | Accrued Interest Payable | | 2,500.00 |
| | (For 5 months' interest.) | | |
| | Premium on Bonds | 366.67 | |
| | Bond Interest Expense | | 366.67 |
| | 11/12 of 1/5 of $2,000) | | |
| | Profit and Loss | 5,133.33 | |
| | Bond Interest Expense | | 5,133.33 |
| | (To close to Profit and Loss) | | |

During each year thereafter $400 of premium is written off.

**B. DISCOUNT.** When Bonds are sold at a discount on 4/1/68:

| | | |
|---|---|---|
| Cash | 98,000.00 | |
| Discount on Bonds | 2,000.00 | |
| Bonds Payable—4%, due 4/1/73 | | 100,000.00 |

Each year a proportionate share of discount is **written off as an additional cost of obtaining funds.** Thus, if books are closed on 12/31, the following entries (straight line method) are made during the first year:

| | | | |
|---|---|---|---|
| 10/1/68 | Bond Interest Expense | 2,000.00 | |
| | Cash | | 2,000.00 |
| 12/31/68 | Bond Interest Expense | 1,000.00 | |
| | Accrued Interest Payable | | 1,000.00 |
| | (For 3 months' interest.) | | |
| | Bond Interest Expense | 300.00 | |
| | Discount on Bonds | | 300.00 |
| | (3/4 of 1/5 of $2,000.) | | |
| | Profit and Loss | 3,300.00 | |
| | Bond Interest Expense | | 3,300.00 |

**NOTE: COST OF ISSUING BONDS.** This includes legal fees, costs of preparing and selling bonds, stamp tax expenses, etc. The total is written off over the life of the bond in same manner as **bond discount.**

**9 REACQUIRING BONDS.** Bonds may be reacquired for resale or for cancellation. *If cancelled*, the *Treasury Bonds* account, which is debited when bonds are reacquired, is closed into the *Bonds Payable* account and the bonds may never be sold again. In the case of acquisition at par, the entry on 4/1/68 is:

| | | |
|---|---|---|
| Treasury Bonds Payable (5%, due 1/1/77) | 10,000.00 | |
| Bond Interest Expense | 125.00 | |
| Cash | | 10,125.00 |
| (Interest accrued from 1/1/68.) | | |

If bonds are cancelled, the unamortized portion of the premium or discount on bonds applicable thereto is closed out.

**10 BONDS ON BALANCE SHEET. A. GENERAL.** The bonds should be described as to type, maturity, interest rate, and as to whether secured. They appear under heading **"Long-term Liabilities"** to extent that their maturity is *longer than a year* from balance sheet date and under **"Current Liabilities"** for any that mature *within a year.*

| | | |
|---|---|---|
| Equipment Trust Bonds—5%, due 4/1/73 | | |
| Authorized | $500,000.00 | |
| Less: Unissued | 100,000.00 | |
| Issued | $400,000.00 | |
| In Treasury | 40,000.00 | |
| Outstanding | | $360,000.00 |

**B. DISCOUNT AND PREMIUM ON BONDS ON THE BALANCE SHEET. (1) DISCOUNT:** Listed under **Deferred Charges.** This is not a current asset since it would not require the use of current assets, if not prepaid, during the current operating cycle. **(2) PREMIUM:** listed under **Deferred Credits.**

**C. PURCHASE OF BONDS AS AN INVESTMENT. (1) BONDS ARE RECORDED AT AMOUNT PAID INCLUDING ALL COMMISSIONS AND COSTS.** Any discount or premium at which bonds are acquired is used as the basis for an adjustment of the **interest income account** during period between the purchase date and the date of maturity.

Thus in the case of $1,200 premium on bonds maturing in 8 years after purchase, the yearly adjustment is shown above. The effect of this entry is to *reduce interest income* for the period. In bond discount, the entry above *increases interest income* for period.

| | | |
|---|---|---|
| Bond Interest Income | 150.00 | |
| Investment—Bonds (For 1/8 of $1,200 premium.) | | 150.00 |

| | | |
|---|---|---|
| Investment—Bonds | 150.00 | |
| Bond Interest Income | | 150.00 |

**(2) IF BONDS ARE HELD UNTIL MATURITY,** the **Investment-Bonds account** will have been adjusted by the annual entries in each case to the face value of the bonds. **(3) IF BONDS ARE BOUGHT BETWEEN INTEREST DATES,** the **Bond Interest Income account** *is charged with the interest accrued to the date of purchase* so that it will serve as an offset to the interest which will be received on the next interest date for the full interest period and of which only the portion applicable to the period after the purchase date will have been earned. **(4) IF BONDS PREVIOUSLY PURCHASED ARE LATER SOLD,** the **gain or loss** is the difference between the selling price and the original purchase price adjusted to the date of sale by the amortization of bond premium or discount.

**11 TYPICAL STOCKHOLDER'S EQUITY SECTION ON BALANCE SHEET.** (See **ACCOUNTING #2,** Sec.3 for remainder of *Balance Sheet.*)

| | | |
|---|---|---|
| **CAPITAL CONTRIBUTED FOR SHARES** | | |
| **COMMON STOCK—$100 PAR VALUE** | | |
| Authorized | 2,000 shares | $200,000.00 |
| Unissued | 380 shares | 38,000.00 |
| Issued | 1,620 shares | $162,000.00 |
| Subscribed—Not Issued | 200 shares | 20,000.00 |
| Issued and Subscribed | 1,820 shares | $182,000.00 |
| Stock Dividend Payable To be issued—3/6/68 | 81 shares | 8,100.00 |
| Issued and to be Issued | 1,901 shares | $190,100.00 |
| Amount Received In Excess of Par Value | | 15,000.00 |
| Total Capital Contributed for Shares | | $205,100.00 |
| **RETAINED EARNINGS:** | | |
| Free | $86,800.00 | |
| Restricted for Treasury Stock Cost | 25,000.00 | |
| Total Retained Earnings | | 111,800.00 |
| TOTAL | | $316,900.00 |
| Less: Cost of Treasury Stock (200 shares) | | 25,000.00 |
| TOTAL STOCKHOLDERS' EQUITY | | $291,900.00 |

**12 THE SALE OF BUSINESS. A. GENERAL.** This may be effected by **(1) book value, (2) book value, after adjustment of values,** or **(3) a lump sum.** In all cases it may either involve all or only part of its assets and may or may not require the assumption of existing liabilities by the purchaser. A sale may result in either a gain or loss to the seller and *may be effected for cash, corporate stock, or by any method agreed upon.* When in the sale of a partnership's business a **gain or loss** results, this is first divided among the partners in their *profit-and-loss-sharing ratio* and cash on hand is then distributed in accordance with the *adjusted capital amount balances.* In the sale of a **single proprietorship** the procedure is the same as that for a partnership except that the sole proprietor is credited or charged with *entire* gain or loss.

**B. ACCOUNTING FOR THE SALE OF A BUSINESS.** If the balance sheet of a business about to be sold shows the following:

| | | | |
|---|---|---|---|
| Cash | $ 5,000 | Notes Payable | $10,000 |
| Accounts Receivable | 50,000 | Accounts Payable | 35,000 |
| Allow. for Bad Debts | ( 3,000) | Jones, Capital | 30,000 |
| Merchandise Inventory | 30,000 | Brown, Capital | 16,000 |
| Furn. & Fixtures | 10,000 | | |
| Allow. for Depreciation | ( 1,000) | | |
| | $91,000 | | $91,000 |

**(1) THEN, FOR A SALE TO REYNOLDS AT BOOK VALUE, EXCLUDING CASH:**

| | | |
|---|---|---|
| Reynolds, Vendee | 41,000.00 | |
| Notes Payable | | 10,000.00 |
| Accounts Payable | | 35,000.00 |
| Allow. for Bad Debts | 3,000.00 | |
| Allow. for Depn.—Furn. & Fixtures | 1,000.00 | |
| Accounts Receivable | | 50,000.00 |
| Merchandise Inventory | | 30,000.00 |
| Furniture & Fixtures | | 10,000.00 |
| Cash | 41,000.00 | |
| Reynolds, Vendee | | 41,000.00 |
| Jones, Capital | 30,000.00 | |
| Brown, Capital | 16,000.00 | |
| Cash | | 46,000.00 |

**(2) FOR A SALE AFTER ASSET REVALUATION, EXCLUDING CASH:** If the merchandise is deemed to be worth only $25,000, then effect thereto is first given on the books by charging the capital accounts appropriately. (Whatever adjustments are required in any case are **charged or credited,** as the situation requires, **to existing capital accounts.**)

The sale may also be recorded as follows:

| | | |
|---|---|---|
| Reynolds, Vendee | 36,000.00 | |
| Notes Payable | | 10,000.00 |
| Amounts Payable | | 35,000.00 |
| Allow. for Bad Debts | 3,000.00 | |
| Allow. for Depn.—Furn. & Fixtures | 1,000.00 | |
| Brown, Capital | 2,500.00 | |
| Jones, Capital | 2,500.00 | |
| Merchandise Inventory | | 30,000.00 |
| Accounts Receivable | | 50,000.00 |
| Furniture & Fixtures | | 10,000.00 |
| Cash | 36,000.00 | |
| Reynolds, Vendee | | 36,000.00 |
| Jones, Capital | 27,500.00 | |
| Brown, Capital | 13,500.00 | |
| Cash | | 41,000.00 |

**(3) FOR A SALE AT A LUMP SUM.** The entry is same as for a sale after asset revaluation, provided goodwill is not first recorded. *If goodwill is recorded, it is first necessary to credit partners in their profit-and-loss-sharing ratios with the amount of goodwill* agreed upon, but this goodwill is immediately closed out together with other assets by a debit to **Vendee's account.**

**13 INCORPORATION OF A BUSINESS.** First, all *income* and *expense accounts* are closed out. Then, if same set of books is used, organization of the corporation is **recorded,** and the *Capital Stock* accounts are substituted for the *Capital* accounts of sole proprietor or partners. When a new set of books is used, the existing firm records the transaction as a **sale** to the corporation, *and the latter opens its books by recording its own organization and issue of stock in exchange for the assets received, less the liabilities assumed.* This method keeps each entity separate. Instead of receiving cash from vendee, the following is recorded on the books of **M** and **N,** Vendors:

| | | |
|---|---|---|
| Stock of XYZ Corp. | 80,000.00 | |
| XYZ Corp., Vendee | | 80,000.00 |
| M, Capital | 50,000.00 | |
| N, Capital | 30,000.00 | |
| Stock of XYZ Corp. | | 80,000.00 |
| (For distribution of stock among partners.) | | |

The corporation records acquisition of assets, as shown below but **previously existing Allowance accounts for depreciated assets are not usually set up.** Instead, assets to which these valuation accounts refer are set up at their **"net value"** amounts. This is because the vendee corporation is in effect buying assets for which, if they were acquired in the usual course of events, no allowance accounts would be set up. Thus, in all the situations described, purchase by any vendee is recorded in each case at net amounts. However, **in the case of accounts receivable, the Allowance account is used by vendee** since the latter is *not able to determine which individual accounts will become uncollectible.*

| | | |
|---|---|---|
| Accounts Receivable | 16,000.00 | |
| Merchandise Inventory | 12,000.00 | |
| Machinery | 7,000.00 | |
| Accounts Payable | | 8,000.00 |
| Allowance for Bad Debts | | 1,000.00 |
| Capital Stock | | 26,000.00 |

**14 MANUFACTURING ACCOUNTING. A. ELEMENTS OF MANUFACTURING COSTS.** These comprise **materials** (which become part of a product), **direct labor** (directly used to convert materials into finished products), and **manufacturing expenses,** comprising costs incident to processing, such as *the depreciation of machinery, power, rent, indirect labor* (that is not directly used on the material, but performed in the factory by foremen, engineers, porters, etc.), and manufacturing supplies.

**B. COST OF GOODS SOLD.** In a manufacturing firm, this is computed to be the **difference** between *goods available for sale* and **inventory of finished goods** *on hand* at the end of period. The goods available for sale constitute the finished goods on hand at the beginning of the period plus the cost of goods m'f'd. In other words, the **goods available for sale** represent the *finished goods on hand at the beginning* of the period **plus** the *goods which have gone through all the steps of processing.* This also presents a problem of valuation in that consideration must be given to the raw materials, the direct labor, and the manufacturing expenses applicable thereto.

**C. COST OF GOODS MANUFACTURED.** This represents the total of: **(1)** Work-in-process at the beginning of period. **(2)** Raw materials put into production: **(a)** those on hand at the beginning of the period, plus **(b)** those acquired during the period, less **(c)** those still on hand at the end of the period. **(3)** Direct Labor put into process. **(4)** Manufacturing expenses charged during the period. **(5)** Subtract from the sum of (1) thru (4), work-in-process (in all stages of production) at end of the period. With only one product, cost of each unit is determined by dividing total cost by number of units produced. For different products, detailed cost records are needed.

**15 GROSS PROFIT ON SALES.** *The difference between* **sales revenue and cost of goods sold** set in statement form, top right:

**ANDREWS MFG. CO., INC. YEAR ENDED 12/31/68**
**STATEMENT OF GROSS PROFIT ON SALES**

| | | | |
|---|---|---|---|
| NET SALES | | | $245,000 |
| COST OF GROSS SOLD: | | | |
| Inventory—Finished Goods (1/1/68) | | $ 50,000 | |
| Inventory—Work-in-Process (1/1/68) | | $ 20,000 | |
| Raw Materials Consumed: | | | |
| Inventory—Raw Materials (1/1/68) | $17,000 | | |
| Net Purchases | 60,000 | | |
| Freight In | 5,000 | | |
| | $82,000 | | |
| Inventory—Raw Materials (12/31/68) | 27,000 | 55,000 | |
| DIRECT LABOR | | 70,000 | |
| MANUFACTURING EXPENSES: | | | |
| Rent | $10,000 | | |
| Power | 7,000 | | |
| Indirect Labor | 25,000 | | |
| Insurance | 4,000 | | |
| Factory Supplies | 3,000 | | |
| Depreciation of Machinery | 7,000 | 56,000 | |
| Total Manufacturing Charges | | $201,000 | |
| Inventory—Work-in-Process (12/31/68) | | 38,000 | |
| Cost of Goods Manufactured | | 163,000 | |
| Goods Available for Sale | | $213,000 | |
| Inventory—Finished Goods (12/31/68) | | 43,000 | |
| Cost of Goods Sold | | | 170,000 |
| GROSS PROFIT ON SALES | | | $ 75,000 |

**16 ADJUSTING ENTRIES.**

| | | |
|---|---|---|
| Inventory—Raw Materials | 27,000.00 | |
| Manufacturing | | 27,000.00 |
| To set up inventory at 12/31/68 | | |
| Inventory—Work-in-Process | 38,000.00 | |
| Manufacturing | | 38,000.00 |
| To set up inventory at 12/31/68 | | |
| Inventory—Finished Goods | 43,000.00 | |
| Profit and Loss | | 43,000.00 |
| To set up inventory at 12/31/68 | | |
| Dep'n—Machinery & Equipment | 7,000.00 | |
| Allow. for Dep'n—Mach. & Equip. | | 7,000.00 |
| Cost | $72,000.00 | |
| Salvage Value | 2,000.00 | |
| | $70,000.00 | |
| 10% | $ 7,000.00 | |
| Bad Debts | 1,000.00 | |
| Allow. for Bad Debts | | 1,000.00 |
| Estimated amount required | $ 3,000.00 | |
| In allowance account | 2,000.00 | |
| | $ 1,000.00 | |
| Salesmen's Salaries | 1,000.00 | |
| Indirect Labor | 2,000.00 | |
| Accrued Expenses Payable | | 3,000.00 |
| To set up accrued salaries at 12/31/68 | | |
| Prepaid Expenses | 1,000.00 | |
| Insurance | | 1,000.00 |
| Unexpired at 12/31/68. | | |

**17 CLOSING ENTRIES.** The manufacturing account is used for computing the cost of finished goods manufactured and the balance of this account is closed into the **Profit and Loss account.** The items tabulated in the **Income Statement** are also closed out in the usual manner to the Profit and Loss account.

**18 PROVISION FOR INCOME TAXES. A. CORPORATION.** The taxes which are measured by *income* are charges against the **income earned.** The entry to set up these taxes, whether they be income taxes, franchise taxes, or any others based on income, are closed into the **Profit and Loss account. B. SINGLE PROPRIETORSHIPS OR PARTNERSHIPS.** Business income may be increased or decreased by transactions outside the business. These taxes are not charged to the **Profit and Loss account,** but are shown as charges to the **Capital accounts** when taxes are paid out of business funds. Not recorded when paid out of *personal funds.*

**19 WORK SHEET.** The same as that for a trading company except that a set of columns for Cost of Goods Manufactured is newly introduced, the balance of which is transferred to the Income Statement columns for the purpose of computing net income.

**A. FORM.**

**ANDREWS MFG. CO., INC. WORK SHEET YEAR ENDED DECEMBER 31, 1968**

| L.F. | ACCOUNT NAME | TRIAL BALANCE | | ADJUSTMENTS | | MANUFACTURING | | INCOME STATEMENT | | BALANCE SHEET | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | Cash | 15,000 | | | | | | | | 15,000 | |
| | Accounts Receivable | 56,000 | | | | | | | | 56,000 | |
| | Inventory—Raw Mat. 1/1/68 | 17,000 | | | | 17,000 | | | | | |
| | Inventory—W.-in-P. 1/1/68 | 20,000 | | | | 20,000 | | | | | |
| | Inventory—Fin. Goods 1/1/68 | 50,000 | | | | | | 50,000 | | | |
| | Machinery and Equipment | 72,000 | | | | | | | | 72,000 | |
| | Allow. Depn. Mach. and Equip. | | 12,000 | | 7,000 | | | | | | 19,000 |
| | Allow. Bad Debts | | 2,000 | | 1,000 | | | | | | 3,000 |
| | Accounts Payable | | 42,000 | | | | | | | | 42,000 |
| | Withholding Taxes Payable | | 2,000 | | | | | | | | 2,000 |
| | F.I.C.A. Taxes Payable | | 1,000 | | | | | | | | 1,000 |
| | Capital Stock | | 100,000 | | | | | | | | 100,000 |
| | Surplus—Earned | | 59,000 | | | | | | | | 59,000 |
| | Sales | | 245,000 | | | | | | 245,000 | | |
| | Purchases—Raw Materials | 60,000 | | | | 60,000 | | | | | |
| | Freight In—Raw Materials | 5,000 | | | | 5,000 | | | | | |
| | Rent | 12,000 | | | | 10,000 | | A 1,500 | | | |
| | | | | | | | | S 500 | | | |
| | Direct Labor | 70,000 | | | | 70,000 | | | | | |
| | Power & Light | 9,000 | | | | 7,000 | | A 1,000 | | | |
| | | | | | | | | S 1,000 | | | |
| | Indirect Labor | 23,000 | | 2,000 | | 25,000 | | | | | |
| | Insurance | 6,000 | | | 1,000 | 4,000 | | S 600 | | | |
| | | | | | | | | A 400 | | | |
| | Factory Supplies | 3,000 | | | | 3,000 | | | | | |
| | Salesmen's Salaries | 18,000 | | 1,000 | | | | S 19,000 | | | |
| | Shipping Expense | 2,000 | | | | | | S 2,000 | | | |
| | Office Salaries | 12,000 | | | | | | A 12,000 | | | |
| | Office Expense | 9,000 | | | | | | A 9,000 | | | |
| | Taxes | 3,000 | | | | | | A 3,000 | | | |
| | Interest Expense | 1,000 | | | | | | OD 1,000 | | | |
| | | 463,000 | 463,000 | | | | | | | | |
| | Inventory—Raw Mat. 12/31/68 | | | 27,000 | | | 27,000 | | | 27,000 | |
| | Manufacturing | | | | 27,000 | 27,000 | | | | | |
| | Inventory—W.-in-P. 12/31/68 | | | 38,000 | | | 38,000 | | | 38,000 | |
| | Manufacturing | | | | 38,000 | 38,000 | | | | | |
| | Inventory—Fin. Gds. 12/31/68 | | | 43,000 | | | | | 43,000 | 43,000 | |
| | Profit and Loss | | | | 43,000 | | | 43,000 | | | |
| | Depreciation—Mach. & Equip. | | | 7,000 | | 7,000 | | | | | |
| | Bad Debts | | | 1,000 | | | | A 1,000 | | | |
| | Accrued Expenses Payable | | | | 2,000 | | | | | | 3,000 |
| | Accrued Expenses Payable | | | | 1,000 | | | | | | |
| | Prepaid Expenses | | | 1,000 | | | | | | 1,000 | |
| | | | | 120,000 | 120,000 | 228,000 | 65,000 | | | | |
| | Cost of Goods Manufactured | | | | | | 163,000 | 163,000 | | | |
| | | | | | | 228,000 | 228,000 | 265,000 | 288,000 | | |
| | Income Tax—22% of $23,000 ($288,000, less $265,000) | | | | | | | 5,060 | | | 5,060 |
| | Net Income | | | | | | | 270,060 | 288,000 | 252,000 | 234,060 |
| | | | | | | | | 17,940 | | | 17,940 |
| | | | | | | | | 288,000 | 288,000 | 252,000 | 252,000 |

**B. WORK SHEET DISTRIBUTION.** All items that must be considered in determination of the *cost of goods manufactured* are extended to **Manufacturing column,** and all others required in computation of the *net income* are extended to **Income Statement column.** Since some expenses are shared by several departments and only one column is generally available for tabulating expenses applicable to all of them, it is customary to designate **selling expenses** by letter **S, administrative expenses** by letter **A, other deductions** by letters **OD, other income** by letters **OI,** etc. *Allocations to departments are made on the basis of space occupied, actual use by departments,* etc.

DATA-GUIDE QUICK CHART®

# COLLEGE ALGEBRA 1

MODERN ADVANCED ALGEBRA

A SIMPLIFIED SUMMARY & INSTANT REFERENCE

LEARN & REVIEW IN MINUTES

AUTHOR: Prof. J. Chiccarelli, Chairman, Bridgewater State College.
Consultant: M. Cohen, Chairman, Thomas Jefferson H. S.

SOLID PLASTIC

SPEED STUDY TO SAVE YOU TIME AND WORK.

QUICK CHARTS PUT THE FACTS IN FULL VIEW:

PERMANENTLY FILL IN FOR MISSING NOTES & TEXTS.

QUICK CHARTS HELP YOU REMEMBER:

CONTAINS NO PAPER

## 1 REAL NUMBER SYSTEM. A. RATIONAL. Numbers (+, −, 0) expressible as the quotient of two integers $a/b$, where $b \neq 0$. E: .1 = 1/10; 12 = 12/1; −3.25 = −3¼ = −13/4. B. IRRATIONAL. Numbers (+ and −) not expressible as the quotient of two integers. E: $\sqrt{2} \approx 1.4$; $\pi \approx 3.14159 \approx 355/113$. C. PROPERTIES. Because of the eleven properties shown below, the real (likewise the rational) numbers form a field. If $a$, $b$, and $c$ are arbitrary real numbers:

| LAW | ADDITION | MULTIPLICATION |
|---|---|---|
| Closure | $a+b$ is a unique real no. | $a \cdot b$ is a unique real no. |
| Associative | $(a+b)+c = a+(b+c)$ | $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ |
| Identity | $a+0 = 0+a = a$ | $a \cdot 1 = 1 \cdot a = a$ |
| Inverse | $a+(-a) = (-a)+a = 0$ | $a \cdot (1/a) = (1/a) \cdot a = 1$ for $a \neq 0$ |
| Commutative | $a+b = b+a$ | $a \cdot b = b \cdot a$ |
| Distributive | $a \cdot (b+c) = (a \cdot b)+(a \cdot c)$ | |

NOTE: 1. Zero may never appear in the denom. of a fraction; but $0/a$ for $a \neq 0$ is equal to zero. 2. $ab = 0$ if, and only if, $a = 0$ or $b = 0$.

D. INEQUALITIES. 1. DEF. For any two real numbers:

**a > b** if $a-b$ is a positive no. $(a-b>0)$ — **a ≥ b** implies $a>b$ or $a=b$

**a < b** if $a-b$ is a negative no. $(a-b<0)$ — **a ≤ b** implies $a<b$ or $a=b$

2. PROPERTIES If a > b, then

- $a \pm c > b \pm c$, for any real number $c$.
- $ac > bc$ and $a/c > b/c$, when $c>0$
- $ac < bc$ and $a/c < b/c$, when $c<0$
- $a^m > b^m$ and $a^{-m} < b^{-m}$, $a$, $b$, m positive numbers
- $\sqrt[n]{a} > \sqrt[n]{b}$, for $a>0$, $b>0$ and $n$ an integer $>1$.

E: Solve $-3x+2 > x-4$ Sol. Transposing +2 and x gives $-4x > -6$. Dividing both members by −4 gives $x < 3/2$.

E. ABSOLUTE VALUE. 1. DEF. If $a$ is any nonzero real number, the absolute value of $a$ (symbol $|a|$) is the positive of the two numbers $a$ and $-a$. (*Note:* $|0| = 0$.) E: $|-3| = 3$; $|\sqrt[3]{7}| = \sqrt[3]{7} \approx 1.91+$.

2. PROPERTIES. $|ab| = |a| \cdot |b|$ — $|a \pm b| \leq |a| + |b|$ — $|a \pm b| \geq |a| - |b|$

$|x-a| \leq b$ (b pos.) if and only if $-b \leq x-a \leq b$, or $a-b \leq x \leq a+b$.

E: Solve $|5x+7| < 12$. Sol. $-12 < 5x+7 < 12$; $-19 < 5x < 5$; $-19/5 < x < 1$

## 2 MATHEMATICAL INDUCTION. A. PRINCIPLE. Let $A_n$ be an assertion (true or false) about $n$, where $n$ is a positive integer. If it can be shown: (1) That $A_1$ is true, that is, that the assertion is true for $n = 1$; (2) that $A_k$ implies $A_{k+1}$ for all k, where argument is independent of value of k; then $A_n$ is true for all pos. integral values of $n$.

E: Prove by induction that $A_n = 1+3+5+\cdots+(2n-1) = n^2$ Sol: $A_1 = 1 = 1^2$ so that (1) is true. $A_k = 1+3+5+\cdots+(2k-1) = k^2$ $A_{k+1} = 1+3+5+\cdots+(2k-1)+(2k+1) = (k+1)^2$. Assume that $A_k$ is true, we now must show that $A_{k+1}$ is true. Sum of first k+1 terms $= A_n + (2k+1) = k^2+(2k+1) = (k+1)^2 = A_{k+1}$. Thus (2) is true. Therefore $A_n$ is true for all n.

## 3 ALGEBRA OF SETS. A. A SET is a collection of objects called *the elements of the set*. E: |2, 7, 16, 36| is a finite set of four elements. E1: $X = |x|$ x is a real number|, the set of all real numbers; read: The set $X$ is the set of numbers x such that x is a real number. (The vertical line stands for **such that.**) E2: $XxY = |(x,y)|$ x and y are real numbers|, the set of all ordered pairs (x,y) of real numbers. E3: |a,b| = |x|x is a number in the (closed) interval |a,b|, a and b real. The set of all x such that $a \leq x \leq b$.

B. THE EMPTY OR NULL SET ($\phi$). The set which has no elements.

C. EQUAL SETS. Two sets $A$ and $B$ are called "equal" (written $A = B$) if (1) every element of $A$ is an element of $B$ and (2) every element of $B$ is an element of $A$. D. SUBSET. 1. If every element of set $A$ is an element of set $B$, set $A$ is a subset of set $B$, written $A \subset B$. By convention, $\phi \subset A$ for every set $A$. 2. If $A \subset B$ and if $B$ includes at least one element which is not an element of $A$, then $A$ is a proper subset of $B$. $A$ is an improper subset of $B$ means that $A = B$. A set with $n$ elements has $2^n$ subsets. E. UNIVERSAL SET. If all the elements under consideration are elements of a universal set $I$, then for all sets $A$, $A \subset I$.

E: Let universal set be $X$, the set of all real numbers: |x|x²−5x+6 = 0|, is set of real solutions of x²−5x+6 = 0, namely the set |2, 3|.

F. COMPLEMENTARY SET. Set $A'$, called the complementary set of $A$ (relative to $I$), is the set which contains all the elements of $I$ which are not elements of $A$. G. DIFFERENCE SET. The difference set between $A$ and $B$, $(A-B)$, is the set of all elements of $A$ which are not elements of $B$. |5, 7, 8, 11, 15|−|7, 8, 15| = |5, 11| H. UNION. The union of $A$ and $B$, written $A \cup B$, is the set of all elements which belong to either $A$ or $B$ or to both $A$ and $B$. E: |1, 2, 4, 6, 7| ∪ |2, 3, 6| = |1, 2, 3, 4, 6, 7| I. INTERSECTION. The intersection of $A$ and $B$, written $A \cap B$, is the set of all elements which are elements of both $A$ and $B$. E: |1, 2, 4, 5| ∩ |2, 3, 6| = |2|. J. DISJOINT SETS. If $A$ and $B$ are two *non-empty* sets such that $A \cap B = \phi$, the empty set, then $A$ and $B$ are disjoint sets. E: |1, 3, 4,| and |2, 6, 8, 9|.

K. PROPERTIES. For all sets A, B, C, in a universal set I:

| PROPERTY | UNION OF SETS | INTERSECTION OF SETS | SUBSET PROPERTIES | COMPLEMENTARY SET PROPERTIES |
|---|---|---|---|---|
| Closure | unique set $A \cup B$ | unique set $A \cap B$ | $A \subset (A \cup B)$ | For every set A, there is a unique set $A'$ |
| Commutative | $A \cup B = B \cup A$ | $A \cap B = B \cap A$ | $(A \cap B) \subset A$ | $A \cup A' = I$ |
| Associative | $(A \cup B) \cup C = A \cup (B \cup C)$ | $(A \cap B) \cap C = A \cap (B \cap C)$ | $A \subset I$; $\phi \subset A$ | $A \cap A' = \phi$ |
| Idempotent | $A \cup A = A$ | $A \cap A = A$ | If $A \subset B$, $A \cup B = B$ | $(A \cup B)' = A' \cap B'$ |
| Property of I | $A \cup I = I$ | $A \cap I = A$ | If $A \subset B$, $A \cap B = A$ | $(A \cap B)' = A' \cup B'$ |
| Property of $\phi$ | $A \cup \phi = A$ | $A \cap \phi = \phi$ | | |
| Distributive | $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$<br>$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ | | | |

L. DUALITY. Interchanging { ∪ and ∩ / φ and I / ⊂ and ⊃ } in any correct formula we obtain another correct formula.

## 4 RELATIONS. A relation is a subset of the set of ordered pairs of real numbers, $X \times Y$. A. METHODS OF DEFINING RELATIONS. 1. LIST OF ORDERED PAIRS. E: |(0, 3); (2, −1); (2, 5); (1, 3); (0, 0)| 2. EQUATIONS. |(x, y)| (x, y) satisfy a given equation|. E: $|(x,y)| x^2+y^2-25=0|$ 3. INEQUALITIES. E: $|(x,y)| x+4y>3|$. 4. SIMULTANEOUS SYSTEMS. E: |(x, y)| 2x+y = 4 and x>1|, |(x, y)|2x+3y = 8 and x+y = 4|, |(x, y)|2x+y<0 and x+y≥1|. 5. GRAPHS. The graph of a relation whose ordered pairs are $(x, y)$ is the set of points in the $XY$ plane whose coordinates are the given pairs. B. DOMAIN AND RANGE. The **domain** of a relation $R$ is the subset of $X$: |x|x is the first element of at least one of the pairs $(x, y)$ of $R$|. The **range** of a relation is the subset of $Y$: |y|y is 2nd element of at least one of the pairs $(x, y)$ of $R$|. E: For $|(x,y)| 9x^2+16y^2 = 144|$, x must lie in the interval $-4 \leq x \leq 4$ if y is to be real; domain is $|-4,4| \subset X$. Similarly, range is $|-3,3| \subset Y$.

## 5 FUNCTIONS. A. DEF. 1. A function f is a relation in which no two ordered pairs have same first element. 2. The value f(x) of the function f at x (in domain of $f$) is second element of that unique ordered pair $(x, y)$ belonging to $f$ which has $x$ as its first element. E1: The relation $|(x,y)| 3x+2y-4 = 0|$ is a function since there is a unique $y = f(x) = (4-3x)/2$ associated with each x; domain is $X$, range is $Y$. E2: The relation $|(x,y)| x^2+y^2 = 4|$ is not a function, for to each x in open interval $-2 < x < 2$ there are associated two values of y, namely $y = \pm\sqrt{4-x^2}$. E3: If $f(x) = x^2+5x$, find $f(a-1)$. [This means find value of f(x) when a−1 is substituted for x.] Sol. $f(a-1) = (a-1)^2+5(a-1) = a^2+3a-4$.

B. ALGEBRA OF FUNCTIONS. The values of $f$ and $g$ are f(x) and g(x).

| ALGEBRAIC COMBINATIONS | RULE OF CORRESPONDENCE | DOMAINS d |
|---|---|---|
| SUM: f+g | $(f+g)(x) = f(x)+g(x)$ | $d_{f+g} = d_f \cap d_g$ |
| DIFFERENCE: f−g | $(f-g)(x) = f(x)-g(x)$ | $d_{f-g} = d_f \cap d_g$ |
| PRODUCT: fg | $(fg)(x) = f(x) \cdot g(x)$ | $d_{fg} = d_f \cap d_g$ |
| QUOTIENT: f/g | $(f/g)(x) = f(x)/g(x)$ | $d_{f/g} = d_f \cap d_g$* |

* Except for those x's for which $g(x) = 0$.

E: Let $f(x) = x+5$ and $g(x) = x^3$. Then $(f+g)(x) = x+5+x^3$; $(f-g)(x) = x+5-x^3$; $(fg)(x) = (x+5)x^3 = x^4+5x^3$; $(f/g)(x) = (x+5)/x^3$. Note that x = 0 is not in domain of f/g since g(0) = 0.

C. COMPOSITION. The **composition** of $f$ with $g$ (denoted by $fog$) is function whose domain consists of elements $x$ in the domain of $g$ such that $g(x)$ is in domain of $f$. The rule of correspondence is $(fog)(x) = f(g(x))$.

E: For $f(x) = x^2-7$ and $g(x) = x+2$, $(fog)(x) = f(g(x)) = f(x+2)$ [which means substitute x+2 for x in f(x)]. $(fog)(x) = (x+2)^2-7$.

D. INVERSE FUNCTION. 1. DEF. If a function f does not have same second element in any of its ordered pairs, then inverse function $f^{-1}$ consists of set of ordered pairs derived from f by interchanging the first and second elements in each of ordered pairs. Thus range of f is the domain of $f^{-1}$, and the domain of f is the range of $f^{-1}$, that is $f^{-1} = |(f(x), x)|$ **x is in the domain of f**|. E: Let f be defined by equation $y = 6-2x$. First exchange variables, giving $x = 6-2y$. Solving this for y gives $y = (6-x)/2$, which is $f^{-1}$.

E. IDENTITY FUNCTION. The function I, whose elements are ordered pairs $(x, x)$. The domain and range are same set $X$; $f^{-1} \circ f = f \circ f^{-1} = I$.

## 6 BINOMIAL THEOREM. A. FACTORIALS. For n a positive integer, $n! = |n = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n$. E: $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$. NOTE. $0! = 1! = 1$. For $n$ and $r$ positive integers and $n \geq r$, we define $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \binom{n}{n-r}$; $\binom{n}{1} = n$, $\binom{n}{0} = \binom{n}{n} = 1$.

B. THEOREM. If n is a positive integer,

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

Contains $n+1$ terms. Sum of the exponents in each term is $n$. Coefficients are symmetrical since $\binom{n}{0} = \binom{n}{n}$, $\binom{n}{1} = \binom{n}{n-1}$, etc.

E: Expand $(a-2b)^3$. Sol: Let $x = a$ and $y = -2b$. Thus $(a-2b)^3 = a^3+3a^2(-2b)+3a(-2b)^2+(-2b)^3 = a^3-6a^2b+12ab^2-8b^3$.

C. ALTERNATE COEFFICIENT METHOD. Simplify each coefficient $\frac{\text{(coefficient)(first exponent)}}{\text{term number}} = \text{next coefficient}$. before obtaining the next one.

E: In $(x+y)^4 = 1x^4+$ etc., the second coefficient $= (1)(4)/1 = 4$; therefore second term is $4x^3y$. $(4)(3)/2 = 6$, the next coefficient; therefore the third term is $6x^2y^2$; and $(6)(2)/3 = 4$, therefore fourth term is $4xy^3$. The last term is $(4)(1)/4\ y^4$, or $y^4$.

D. INFINITE SERIES. If n is a positive or negative fraction, or a negative integer, the expansion will lead to an infinite series. Use the alternate method of obtaining coefficients. E:

$$(x+5)^{-4/3} = x^{-4/3} + \frac{(1)(-4/3)}{1}x^{-7/3}5^1 + \frac{(-4/3)(-7/3)}{2}x^{-10/3}5^2 + \cdots$$

E. $(r+1)^{st}$ TERM (TERM WITH $y^r$): $\binom{n}{r}x^{n-r}y^r$. E: Find fifth term of $(a^2-3b)^{10}$. Sol: $x = a^2$, $y = -3b$; $\binom{10}{4}(a^2)^6(-3b)^4 = 17{,}010a^{12}b^4$

## 7 COMPLEX NUMBER. A. DEF. 1. Complex number form is a+bi, where a and b are real numbers and $i = \sqrt{-1}$ with $i^2 = -1$. 2. A real number is of form a+0i or a. 3. A pure imaginary number is of the form 0+bi, or bi. NOTE: complex numbers form a field (*see* 1:C).

B. ALTERNATE DEFINITIONS. 1. A complex number is an ordered pair of real numbers $(a, b)$. 2. A real number is of form $(a, 0)$. 3. A pure imaginary number is of the form $(0, b)$.

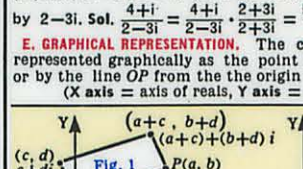
NOTE: $(0, 1) = i = \sqrt{-1}$ with $i^2 = -1$.

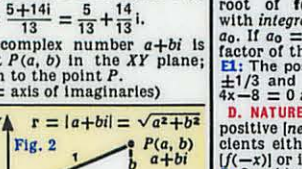
| C. OPERATIONS | NOTATION (a, b) | NOTATION (a+bi) |
|---|---|---|
| Equality | $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$ | $a+bi = c+di$ if and only if $a = c$ and $b = d$ |
| Add., Subt. | $(a, b) \pm (c, d) = (a \pm c, b \pm d)$ | $(a+bi) \pm (c+di) = (a \pm c)+(b \pm d)\,i$ |
| Mult. | $(a, b) \cdot (c, d) = (ac-bd, ad+bc)$ | $(a+bi) \cdot (c+di) = (ac-bd)+(ad+bc)\,i$ |
| Div. | $\frac{(a, b)}{(c, d)} = \left(\frac{ac+bd}{c^2+d^2}, \frac{bc-ad}{c^2+d^2}\right)$, $(c, d) \neq (0, 0)$ | $\frac{a+bi}{c+di} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\,i$, $c+di \neq 0$ |

D. RELATION OF SPECIAL COMPLEX NUMBERS TO REAL NUMBERS. 1. The product of two pure imaginaries is real: $(0, b) \cdot (0, d) = (-bd, 0)$ or $bi \cdot di = -bd$. 2. Conjugates. The complex numbers a+bi and a−bi are called *conjugates*. The sum or product of two conjugate complex numbers is real: $(a+bi)+(a-bi) = 2a$; $(a+bi)(a-bi) = a^2+b^2$. E: Divide 4+i by 2−3i. Sol. $\frac{4+i}{2-3i} = \frac{4+i}{2-3i} \cdot \frac{2+3i}{2+3i} = \frac{5+14i}{13} = \frac{5}{13} + \frac{14}{13}i$.

E. GRAPHICAL REPRESENTATION. The complex number $a+bi$ is represented graphically as the point $P(a, b)$ in the $XY$ plane; or by the line $OP$ from the the origin to the point $P$.

(X axis = axis of reals, Y axis = axis of imaginaries)




The sum of two complex numbers (a, b) and (c, d) is fourth vertex of the parallelogram as constructed in Fig. 1.

F. POWERS OF I: Repeat each other in cycles of four: $i, -1, -i, 1$:

$i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i^4 \cdot i = i$, $i^6 = i^2 = -1$, $i^7 = -i$, $i^8 = 1$, etc.

E: Express $(1+i)^3$ in a+bi form. Sol: Expanding gives $1+3i+3i^2+i^3 = 1+3i-3-i = -2+2i$.

G. POLAR FORM. In Fig. 2, $a = r\cos\theta$, $b = r\sin\theta$, thus $a+bi = r(\cos\theta+i\sin\theta)$, or $r[\cos(\theta+k \cdot 360°)+i\sin(\theta+k \cdot 360°)]$ where k is any integer. NOTE: a+bi is called the rectangular form.

1. MODULUS (ABSOLUTE VALUE): $r = |a+bi| = \sqrt{a^2+b^2}$

2. AMPLITUDE (ARGUMENT): Angle $\theta$ such that $\tan\theta = b/a$.

E: For −2+2i in quadrant II, $r = \sqrt{(-2)^2+(2)^2} = 2\sqrt{2}$ and $\tan\theta = -1$ so that $\theta = 135°$. Thus the polar form of −2+2i is $2\sqrt{2}\,(\cos 135° + i \sin 135°)$.

## H. OPERATIONS USING POLAR FORM.

| OPERATION | RULE | EXAMPLE |
|---|---|---|
| Multiplication | Multiply Moduli and Add Amplitudes | $2(\cos 100° + i \sin 100°) \times 3(\cos 40° + i \sin 40°) = 6(\cos 140° + i \sin 140°)$ |
| Division | Divide Moduli and Subtract Amplitudes | $\frac{2(\cos 100° + i \sin 100°)}{3(\cos 40° + i \sin 40°)} = (2/3)(\cos 60° + i \sin 60°)$ |
| Power (For any real $n$) | De Moivre's thm: $[r(\cos\theta+i\sin\theta)]^n = r^n(\cos n\theta + i \sin n\theta)$. | $[3(\cos 40° + i \sin 40°)]^3 = 3^3(\cos 3 \cdot 40° + i \sin 3 \cdot 40°) = 27(\cos 120° + i \sin 120°)$. |

I. ROOTS. Every complex number except zero, has exactly $n$ distinct $n^{th}$ roots. $(a+bi)^{1/n} = [r(\cos\theta+i\sin\theta)]^{1/n} = r^{1/n}\left[\cos\frac{\theta+k \cdot 360°}{n} + i \sin\frac{\theta+k \cdot 360°}{n}\right]$ where k takes values 0, 1, 2, . . . , (n−1)

E: Find the three cube roots of 1. Sol: Since $1 = 1+0i = 1[\cos(0°+k \cdot 360°)+i \sin(0°+k \cdot 360°)]$, the cube roots are given by $1^{1/3}\left[\cos\left(\frac{k \cdot 360°}{3}\right) + i \sin\left(\frac{k \cdot 360°}{3}\right)\right] = \cos(k \cdot 120°)+i\sin(k \cdot 120°)$ for k = 0, 1, 2. In polar and rectangular form: $\cos 0° + i \sin 0° = 1$; $\cos 120° + i \sin 120° = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$; $\cos 240° + i \sin 240° = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$. These three cube roots of 1 are designated respectively as 1, $\omega$, $\omega^2$

## 8 POLYNOMIAL OPERATIONS. A. RATIONAL INTEGRAL (POLYNOMINAL) EQUATION of degree n is of the form

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0,\ a_0 \neq 0,$$

where n is a positive integer and $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are complex, real, or rational constants. E: $f(x) = 3x^5 - \sqrt{2}x^4 + x^3/3 - 11i = 0$.

B. ROOT. Any value of x which makes $f(x)$ "vanish" in the equation $f(x) = 0$. E: 2 is a root of $x^3-2x^2+6x-12 = 0$, since $f(2) = (2)^3-2(2)^2+6(2)-12 = 0$.

C. REMAINDER THEOREM. If a poly. f(x) is divided by x−r until a remainder independent of x is obtained, this remainder equals f(r). E: $\frac{3x^3-4x^2-12x+10}{x-2} = 3x^2+2x-8+\frac{-6}{x-2}$; since $r = 2$ we also have $f(2) = 3(2)^3-4(2)^2-12(2)+10 = -6$.

D. FACTOR THEOREM. If r is a root of $f(x) = 0$ (i.e., $f(r) = 0$), then x−r is a factor of f(x). Conversely, if x−r is a factor of $f(x)$, then r is a root of $f(x) = 0$. E1: −2 is a root of $f(x) = x^4-2x^2+3x-2 = 0$ since $f(-2) = 0$; hence x+2 is a factor of f(x). E2: For what value of h will x−3 be a factor of $x^3+hx-10x+12$? Sol: Let f(3) = 0, and solve 27+3h−30+12 = 0, h = −3.

E. SYNTHETIC DIVISION. (Used to divide a polynomial $f(x)$ by a divisor of the form x−r). 1. Write $f(x)$ in descending powers of $x$. 2. Arrange the detached coefficients $a_0, a_1, \ldots, a_{n-1}, a_n$ in order in a first line, supplying any missing power of x with a zero coefficient, and write r at the right. 3. Bring down $a_0$ in the first place in the *third* line. 4. Multiply $a_0$ by r, writing the product in the second line under $a_1$, and their sum in the third line; multiply this sum by r, add the product to $a_2$, and write the sum in the third line, and so on, until finally a product is added to the last coefficient of $f(x)$. 5. The last sum in the third line is the remainder $f(r)$. The preceding sums are the coefficients of the powers of x in the quotient, beginning with $x^{n-1}$ and arranged in descending order.

E: Divide $2x^4-x^2+3x-5$ by x+3.

| | 2 | 0 | −1 | +3 | −5 \| −3 |
|---|---|---|---|---|---|
| | | −6 | +18 | −51 | +144 |
| | 2 | −6 | +17 | −48 | +139 |

Sol: r = −3 Quotient is $2x^3-6x^2+17x-48$, remainder is 139 = f(−3).

## 9 THEORY OF EQUATIONS. A. FUNDAMENTAL ALGEBRA THEOREM. 1. Every polynomial equation $f(x) = 0$ has at least one root, real or imaginary. 2. Every poly. $f(x)$ of degree $n$ is the product of $n$ linear factors real and/or imaginary. E1: $x^3+3x^2-4x-12 = (x+2)(x-2)(x+3)$. E2: $x^3+x^2-7x-15 = (x-3)(x+2-i)(x+2+i)$.

3. A poly. equation of degree $n$ has exactly $n$ roots. Some of the roots may be equal (multiple roots). E: $(x-2)^3(x-5)^2(x+4) = 0$ has six roots: 2 is a triple root, 5 a double root, and −4 a simple root.

4. If two polys. of degree $n$ in $x$ are equal for more than $n$ distinct values of $x$, then the coefficients of like powers of $x$ are equal, and the two polys. are identically equal.

E: For $7-x = A(x-1)+B(x+2) = (A+B)x+(2B-A)$ will be an identity if $A+B = -1$ and $-A+2B = 7$, thus $A = -3$, $B = 2$.

5. To form a poly. equation in $x$ whose roots are $r_1, r_2, \ldots r_n$, equate to zero the product of corresponding factors $(x-r_1)(x-r_2)\ldots(x-r_n)$.

E: Find a poly. equation with 3 and −4 as simple roots, −2 as a double root. Sol: $(x-3)(x+4)(x+2)^2 = 0$; $x^4+5x^3-4x^2-44x-48 = 0$.

B. CONJUGATE ROOTS. 1. If a complex number a+bi is a root of the poly. equation $f(x) = 0$ with *real* coefficients, then its conjugate a−bi is also a root. If $f(x)$ is of odd degree it must have at least one real root. E1: $x^3-x^2+x+3 = 0$ has roots $-1, 1+\sqrt{2}i, 1-\sqrt{2}i$.

E2: $x^2-2\sqrt{3}x+7 = 0$ has roots $\sqrt{3}+2i$, $\sqrt{3}-2i$.

2. If the poly. equation $f(x) = 0$ with *rational* coefficients has $a+\sqrt{b}$ as a root, where $a$ and $b$ are rational but $\sqrt{b}$ is irrational, then $a-\sqrt{b}$ is also a root.

E1: $x^3-3x^2-3x+5 = 0$ has roots $1, 1+\sqrt{6}, 1-\sqrt{6}$. E2: Form an equation of lowest possible degree and with rational coefficients, having as roots 2, 1−i, and $3+\sqrt{5}$. Sol: 1+i and $3-\sqrt{5}$ must also be roots. Form the factors $(x-2)(x-1+i)(x-1-i)(x-3-\sqrt{5})(x-3+\sqrt{5}) = 0$ and multiply. $x^5-10x^4+34x^3-56x^2+48x-16 = 0$. NOTE. The quadratic factor for the roots 1−i and 1+i (or $3+\sqrt{5}$ and $3-\sqrt{5}$) could have been formed using $x^2-(r_1+r_2)x+r_1r_2$.

C. RATIONAL ROOTS. If b/c, a rational fraction in lowest terms, is a root of $f(x) = a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_{n-1}x+a_n = 0$, $a_0 \neq 0$, with *integral* coefficients, then $b$ is a factor of $a_n$ and $c$ is a factor of $a_0$. If $a_0 = 1$ then any rational root of $f(x) = 0$ is an integer and a factor of the constant term $a_n$.

E1: The possible rational roots of $3x^3+4x^2-7x+2 = 0$ are $\pm1, \pm2, \pm1/3$ and $\pm2/3$. E2: The possible rational roots of $x^4-x^3+2x^2-4x-8 = 0$ are $\pm1, \pm2, \pm4$, and $\pm8$.

D. NATURE OF ROOTS. 1. Descartes' Rule of Signs. The number of positive [*negative*] roots of a poly. equation $f(x) = 0$ with real coefficients either is equal to the number of variations of sign in $f(x)$ [$f(-x)$] or is less than that number by a positive even integer.

E: Consider $f(x) = 2x^5+4x^3+3x^2-1 = 0$. Sol: f(x) has one variation in sign, thus f(x) = 0 has one positive root. $f(-x) = -2x^5-4x^3+3x^2-1$ has two variations in signs, hence f(x) = 0 has two or no negative roots. Since f(x) = 0 has five roots, we have: one positive, two negative and two imaginary; or one positive and four imaginary.

2. Upper and Lower Limits for the roots. Divide $f(x) = 0$, with $a_0 > 0$, synthetically by x−r; if

| SIGN OF DIVISOR | COEFFICIENTS OF QUOTIENT AND REMAINDER | NEITHER THE DIVISOR NOR ANY REAL NUMBER |
|---|---|---|
| r is positive | all positive or zero | greater than it may be a root. |
| r is negative | alternately positive and negative (or zero) | less than it may be a root. |

E: Consider $x^4+2x^3+x^2-8x-5 = 0$. Since

| | 1 | +2 | +1 | −8 | −5 \| 2 |
|---|---|---|---|---|---|
| | | +2 | +8 | +18 | +20 |
| | 1 | +4 | +9 | +10 | +15 |

| | 1 | +2 | +1 | −8 | −5 \| −3 |
|---|---|---|---|---|---|
| | | −3 | +3 | −12 | +60 |
| | 1 | −1 | +4 | −20 | +55 |

we see that there is no real root greater than 2 nor less than −3.

**E. LOCATION OF ROOTS.** If $f(x)$ is a poly. with *real* coefficients (graph is a continuous curve), and if, for real numbers **a** and **b**, $f(a)$ and $f(b)$ have **opposite signs**, the equation $f(x)=0$ has an **odd number** of real roots between **a** and **b**. If $f(a)$ and $f(b)$ have **like signs** the number of real roots between **a** and **b** is **zero or an even number.**

**NOTE:** A root of multiplicity $m$ must be counted as $m$ roots.

**E:** Consider $f(x)=4x^4-8x^3-3x^2+7x-2=0$. Since $f(0)=-2$, $f(1)=-2$, $f(3)=100$, there are no roots or an even number of roots between 0 and 1, and an odd number of roots between 1 and 3. The roots are 1/2 as a double root, 2 as a simple root; fourth root is $-1$.

**F. RELATION BETWEEN ROOTS AND COEFFICIENTS.** In the poly. equation $f(x)=0$: $-a_1/a_0$ = sum of the roots;

$a_2/a_0$ = sum of products of roots taken two at a time;

$-a_3/a_0$ = sum of products of roots taken three at a time, etc.;

$(-1)^n\frac{a_n}{a_0}$ = product of all the roots.

**E:** Consider $2x^3+x^2-3x-5=0$ with roots $r_1$, $r_2$, $r_3$. Thus $r_1+r_2+r_3=-1/2$; $r_1r_2+r_1r_3+r_2r_3=-3/2$; $r_1r_2r_3=5/2$.

**G. TRANSFORMATIONS OF EQUATIONS. 1. Multiplying each root by a constant.** To form an equation each of whose roots is **k** times a corresponding root of a given equation, first supply each missing power of $x$ with a zero coefficient, then multiply the coefficient of the next to highest degree term by **k**, the next by $\mathbf{k^2}$, and so on. **E:** Form an equation whose roots are three times the corresponding roots of $5x^4-4x^3+2x-6=0$. **Sol:** $5x^4-3\cdot4x^3+3^2\cdot0x^2+3^3\cdot2x-3^4\cdot6=0$, which simplified becomes $5x^4-12x^3+54x-486=0$.

**2. Changing the sign of each root.** To form an equation each of whose roots is the negative of a corresponding root of a given equation, change the signs of the **odd degree** terms of the given equation. (Or use method **G**:1 with $-1$ as the multiplier.)

**E:** The equation $2x^4-3x^3-5x^2+4x-1=0$ has for its roots the negatives of those of $2x^4+3x^3-5x^2-4x-1=0$.

**3. Diminishing each root by a constant.** To form an equation each of whose roots **is less by h** than a corresponding root of a given equation $f(x)=0$, of degree $n$, divide $f(x)$ and each successive resulting quotient synthetically by **h** until $n$ divisions have been performed. The remainder obtained in the first division is the constant term in the required equation, and the successive remainders are the coefficients of the successive terms of ascending degree.

| | | | | |
|---|---|---|---|---|
| 2 | −10 | 0 | −5 | \|2 |
| | +4 | −12 | −24 | |
| 2 | −6 | −12 | −29 | |
| | 4 | −4 | | |
| 2 | −2 | −16 | | |
| | 4 | | | |
| 2 | 2 | | | |

**NOTE.** To form a new equation whose roots are *increased by h*, **diminish the roots by −h.**

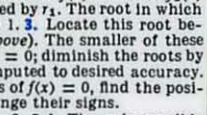
**E:** Find an equation with roots 2 less than those of $2x^3-10x^2-5=0$. **Sol:** $2x^3+2x^2-16x-29=0$.

**H. APPROXIMATING IRRATIONAL ROOTS—HORNER'S METHOD. 1.** Locate a root of $f(x)=0$ between two consecutive integers by synthetic division and **E** above. **2.** If a positive root lies between $r_1$ (which is the integral part of the root) and $r_1+1$, transform $f(x)=0$ into $f_1(x)$ whose roots are those of $f(x)=0$ diminished by $r_1$. The root in which we are interested now lies between 0 and 1. **3.** Locate this root between two consecutive tenths (i.e. *by E above*). The smaller of these tenths is the tenths part of the root of $f(x)=0$; diminish the roots by this amount. **4.** Continue until root is computed to desired accuracy.

**NOTE:** To find the negative irrational roots of $f(x)=0$, find the positive irrational roots of $f(-x)=0$ and change their signs.

**E:** Find the positive root of $x^3+7x-11=0$. **Sol:** The only possible positive rational roots are 1 and 11, neither of which satisfies the equation; therefore the root is irrational. Since $f(1)=-3$ and $f(2)=11$, the root lies between 1 and 2. Diminishing the roots of the equation by 1 gives $f_1(x)=x^3+3x^2+10x-3=0$. Since $f_1(.2)=-.872$ and $f_1(.3)=.297$ the root lies between .2 and .3. Diminishing the roots of $f_1(x)=0$ by .2 gives $f_2(x)=x^3+3.6x^2+11.32x-.872=0$. Since $f_2(.07)=-.061617$ and $f_2(.08)=.057152$, the root lies between .07 and .08; etc. Thus far we have approximated the root as 1.27*.

## 10 EXPONENTIAL FUNCTION (BASE a).




**A. DEF.** The function **f** defined by $\mathbf{y=a^x}$ ($a>0$, and $a\neq1$). Its domain is the set of all real numbers, its range is $0<y<\infty$.

**E:** $y=3^x$; $y=(1/2)^x$. **B. PROPERTIES.**

| For $a>0$, $b>0$ and $x$, $y$ real | For $x$ real and $>0$ | For $x$, $y$ real and $x<y$ |
|---|---|---|
| $a^x\cdot a^y=a^{x+y}$ | $a^x>1$ for $a>1$ | $a^x<a^y$ for $a>1$ |
| $(a^x)^y=a^{xy}$ | $a^x=1$ for $a=1$ | $a^x=a^y$ for $a=1$ |
| $(ab)^x=a^xb^x$ | $a^x<1$ for $0<a<1$ | $a^x>a^y$ for $0<a<1$ |

**C. THE IRRATIONAL NUMBER** $\mathbf{e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=2.71828}$ approximately; such that $\mathbf{y=e^x}$ is called *the* exponential function.

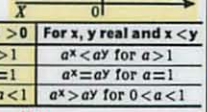
**D. APPLICATIONS. 1. COMPOUND INTEREST** is **interest** (**I**) paid not only on an original **principal** (**P**) but on the accumulated interest as well. The **amount** (**A**) accumulated by a principal placed at compound interest is $\mathbf{A=P(1+r)^n}$, where **n** is number of interest periods and **r** the rate of interest per period. The compound interest is $\mathbf{I=A-P}$.

**E:** A man invests $1000 at 6% compounded semiannually. Find A and I after two years. **Sol:** $P=\$1000$, $r=1/2(6\%)=3\%=.03$, $n=4$ (since each interest period is 1/2 year and there are 4 such periods in 2 years) Thus $A=1000(1+.03)^4=1000(1.03)^4=\$1125.51$, and $I=A-P=\$1125.51-\$1000=\$125.51$.

**2. ANNUITY.** An *annuity* is a sequence of equal periodic payments. The **amount of an annuity** is the sum **S** to which periodic payments accumulate at compound interest. If the periodic payment is **R** dollars: $\mathbf{S=R\frac{(1+r)^n-1}{r}}$.

**E:** $150 is placed in a savings account at the end of each 6 months. Find the amount in the account at the end of 5 years, if interest is at rate of 4% compounded semiannually. **Sol:** $S=(150)\frac{(1+.02)^{10}-1}{.02}=\$1642.46$

## 11 LOGARITHMIC FUNCTION (BASE a).




**A. DEF.** The function inverse to that given by $\mathbf{y=a^x}$, ($a>0$ and $a\neq1$), written $\mathbf{y=\log_a x}$. Its domain is the set of *positive* real numbers, its range is the set of *all* real numbers.

**1. Common logarithms:** base 10, i.e. $\log_{10}x$ or $\log x$. **2. Natural logarithms:** base $e$, i.e. $\log_e x$ or $\ln x$.

**B. PROPERTIES**

1. $\log_a xy=\log_a x+\log_a y$
2. $\log_a\frac{y}{x}=\log_a y-\log_a x$
3. $\log_a\frac{1}{x}=-\log_a x$
4. $\log_a x^y=y\log_a x$
5. $\log_b a=\frac{1}{\log_a b}$
6. $\log_a x=\frac{\log_b x}{\log_b a}$

**E:** Find $\log_7 12$. **Sol:** Let $x=\log_7 12$. Thus $7^x=12$ and $\log_{10}7^x=\log_{10}12$ or $x\log_{10}7=\log_{10}12$. Hence $x=\log_{10}12\div\log_{10}7=1.0792\div0.8451=1.28=\log_7 12$. By property 6, $\log_7 12=\log_{10}12\div\log_{10}7$, etc.

## 12 PERMUTATIONS.

**A. FUNDAMENTAL PRINCIPLE.** If an act can be performed in **m** different ways, and if after it has been performed in any one of these ways a second act is performed in **n** different ways, after which a third act can be performed in **p** ways, and so on, then the number of ways in which all these acts can be performed successively is the product **mnp** · · ·.

**E1:** If there are 6 candidates for governor and 4 for mayor, then the two offices may be filled in $6\cdot4=24$ ways. **E2:** How many even numbers of three digits each (with none repeated) can be formed from the digits 2, 3, 6, 8, 9? **Sol:** There are three choices for the last digit of the number, 2, 6 or 8. After that has been chosen, there are 4 choices for the first digit and 3 choices for the second. Therefore $4\cdot3\cdot3=36$.

**B. DEF.** A *permutation* is a definite order or arrangement of all or part of the elements of a given set. **E:** The permutations of three letters a, b, c taken all at a time are abc, acb, bca, bac, cba, cab.

**C. nPr. THE NUMBER OF PERMUTATIONS OF n DIFFERENT THINGS TAKEN r AT A TIME:** $\mathbf{{}_nP_r\left(\text{or }P^n_r\right)=n(n-1)(n-2)\cdots(n-r+1)=\frac{n!}{(n-r)!}}$

When $r=n$, ${}_nP_n=n!$. **E:** Find the number of ways a student with 9 different books can arrange any five of them on a shelf.

**Sol:** ${}_9P_5=9!/4!=9\cdot8\cdot7\cdot6\cdot5=15{,}120$.

**D. PERMUTATIONS WITH SOME THINGS ALIKE, TAKEN ALL AT A TIME.** The permutations **P** of a set of **n things n at a time** in which $n_1$ things are alike, $n_2$ others alike, $n_3$ others alike, and so on, is $P=\frac{n!}{n_1!n_2!n_3!\cdots}$.

**E:** How many different permutations can be made of letters of word *benzene*? **Sol:** There seven letters of which three are alike (three e's) and two others alike (two n's), then $P=7!/(3!\,2!)=420$.

**E. CIRCULAR PERMUTATIONS.** The number of ways of arranging **n** different objects around a circle is $(n-1)!$

**E:** 8 persons may be seated at a round table in $(8-1)!=7!$ ways.

## 13 COMBINATIONS.

**A. DEF.** A group formed by part or all of a given set of things without regard to arrangement of these things *within the group*. **E:** The combinations of three letters a, b, c taken two at a time are ab, ac, bc.

**B. nCr. NUMBER OF COMBINATIONS OF n DIFFERENT THINGS TAKEN r AT A TIME IS:**

$$\mathbf{{}_nC_r\left(\text{or }C^n_r\right)=\frac{{}_nP_r}{r!}=\frac{n(n-1)\cdots(n-r+1)}{r!}=\frac{n!}{r!(n-r)!}=\binom{n}{r}}$$

Also ${}_nC_r={}_nC_{n-r}$.

**E:** How many straight lines will be determined by eight points no three of which lie in same line? **Sol:** A line will contain two and only two points, ${}_nC_r={}_8C_2=\frac{8!}{2!(8-2)!}=\frac{8\cdot7}{2\cdot1}=28$.

**C. DIFFERENT THINGS TAKEN ANY NUMBER AT A TIME.** The *total* number of combinations **C** of **n** different things taken 1, 2, 3, . . . $n$ at a time is $\mathbf{C=2^n-1}$. **E:** The total number of ways a person can invite one or more of four friends is $2^4-1=15$.

## 14 PROBABILITY.

**A. DEF.** If **m** is number of ways in which an event **can occur** (success) and **n** number of ways in which it can **fail to occur** (failure), each regarded as equally likely, then

$$\text{PROBABILITY OF SUCCESS}=P=\frac{m}{m+n}\qquad\text{PROBABILITY OF FAILURE}=Q=\frac{n}{m+n}$$

It follows that $P+Q=1$; $P=1-Q$; $Q=1-P$.

**E:** If 4 balls are drawn at 1 trial from bag containing 6 white and 4 black balls, what is probability that 2 white and 2 black will be drawn? **Sol:** The number of ways of drawing 2 white balls from 6 is ${}_6C_2$ and for black it is ${}_4C_2$. The total number of ways in which 4 balls can be drawn from 10 is ${}_{10}C_4$. $P=\frac{{}_6C_2\cdot{}_4C_2}{{}_{10}C_4}=\frac{90}{210}=\frac{3}{7}$.

**B. ODDS. 1.** The odds in favor of occurrence of the event are **m/n**, odds against its happening are **n/m**. **2.** If P is probability that an event will occur, the **odds in favor** of its happening are $P/Q=P/(1-P)$; **odds against** its happening are $Q/P=(1-P)/P$.

**E:** Find odds against getting 5 or more with one throw of a die. **Sol:** The probability of getting 5 or more with one throw is $2/6=1/3=P$. Since $P+Q=1$, $Q=2/3$. Thus odds against getting 5 or more are 2/3 to 1/3, or 2 to 1. Or, 5 or more can be obtained in 2 ways and cannot be obtained in 4 ways. The odds are 4 to 2 or 2 to 1.

**C. INDEPENDENT EVENTS. 1. DEF.** Events of a set are independent if occurrence of any one does not affect occurrence of any other event of the set. **2.** The probability that two or more independent events will occur equals **product** of their respective probabilities.

**E:** Three coins are tossed together. What is probability that all will fall heads up? **Sol:** Since each coin can fall in only two ways, probability that one will fall heads up is 1/2. Thus $P=(1/2)^3=1/8$.

**D. DEPENDENT EVENTS. 1. DEF.** The events of a set are dependent if occurrence of any one affects occurrence of any other of the set. **2.** Let $E_1, E_2, \ldots E_n$ be events of a dependent set $E$. Let $P_1$ be the probability that $E_1$ will occur first and $P_2$ the probability that $E_2$ will occur next after the occurrence of $E_1$, and so on. The events separated in this manner can be treated as independent events thus $P=P_1\cdot P_2\cdots P_n$.

**E:** A box contains 3 black balls and 2 white balls. If a ball is drawn at random, probability that it is black is $3/(3+2)=3/5$. If this ball is not replaced and a second ball is drawn, probability that it is also black is $2/(2+2)=1/2$. Thus probability that both will be black is $(3/5)\cdot(1/2)=3/10$.

**E. MUTUALLY EXCLUSIVE EVENTS. 1. DEF.** When in a set of two or more events the occurrence of any one of the set excludes occurrence of any other one at the same trial. **2.** The probability that one of two or more given events of a set of mutually exclusive events will occur is **sum** of probabilities of occurrence of events.

**E:** A box contains 8 white and 6 black balls. What is probability that if three balls are drawn at a trial these will be all white or 2 white and 1 black? **Sol:** Number of ways in which all 3 can be white is ${}_8C_3$, and 2 white and 1 black can occur in ${}_8C_2\cdot{}_6C_1$ ways. The total number of ways that 3 balls can be chosen from 14 is ${}_{14}C_3$. The probability is $P=\frac{{}_8C_3}{{}_{14}C_3}+\frac{{}_8C_2\cdot{}_6C_1}{{}_{14}C_3}=\frac{2}{13}+\frac{6}{13}=\frac{8}{13}$.

**F. EXPECTATION.** If **P** is the probability of success in a single trial of an event, the *expected number* of successes in **n** trials is **nP**. If **P** is the probability of winning money **a**, the *expectation* is $\mathbf{P\cdot a}$.

**E:** Find the expectation of a person who is to win $1. provided he is successful in drawing a heart from a deck of cards. **Sol:** The probability of obtaining a heart on a single draw is $13/52=1/4$, thus his expectation is $1/4\cdot\$1.00=\$0.25$.

**G. REPEATED TRIALS. 1.** If **P** is probability that an event **will occur** and **Q** is probability that it **will fail** in any trial, the probability that it will happen *exactly* **r** times in **n** trials is ${}_nC_rP^rQ^{n-r}$ or ${}_nC_{n-r}P^rQ^{n-r}$. **2.** The probability that event will occur *at least* **r** times in **n** trials is $P^n+{}_nC_1P^{n-1}Q+{}_nC_2P^{n-2}Q^2+\ldots+{}_nC_rP^rQ^{n-r}$.

**E:** Find probability of throwing at least 3 aces in 5 throws of a die. **Sol:** This is the probability of throwing 3, 4, or 5 aces and is thus:

${}_5C_3P^3Q^2+{}_5C_4P^4Q+P^5=10\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2+5\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)+\left(\frac{1}{6}\right)^5=\frac{23}{648}$

## 15 DETERMINANTS.

A numerical value associated with a square array of numbers.

**A. DETERMINANTS OF SECOND ORDER (ORDER TWO).**

$$D_2=\begin{vmatrix}a_1&b_1\\a_2&b_2\end{vmatrix}=a_1b_2-a_2b_1$$

**E:** $\begin{vmatrix}4&2\\-3&1\end{vmatrix}=(4)(-1)-(-3)(2)=2$

**NOTE:** A determinant of order one is the number itself.

**B. SYSTEM OF 2 LINEAR EQUATION IN TWO UNKNOWNS.**

The system $a_1x+b_1y=c_1$, $a_2x+b_2y=c_2$, where $a_1b_2-a_2b_1\neq0$ has sol: $x=\frac{\begin{vmatrix}c_1&b_1\\c_2&b_2\end{vmatrix}=D_x}{\begin{vmatrix}a_1&b_1\\a_2&b_2\end{vmatrix}=D}$; $y=\frac{\begin{vmatrix}a_1&c_1\\a_2&c_2\end{vmatrix}=D_y}{\begin{vmatrix}a_1&b_1\\a_2&b_2\end{vmatrix}=D}$

| D | $D_x$ | $D_y$ | NO. OF SOLUTIONS | SYSTEM |
|---|---|---|---|---|
| $\neq0$ | any value | any value | one | consistent |
| $=0$ | $\neq0$ | $\neq0$ | no solution | inconsistent |
| $=0$ | $=0$ | $=0$ | infinite number | dependent |

**E:** Solve: $2x-3y=16$, $5x+y=6$. **Sol:** Since $\begin{vmatrix}2&-3\\5&1\end{vmatrix}=17$, we have

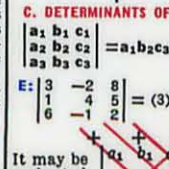
$x=\frac{\begin{vmatrix}16&-3\\6&1\end{vmatrix}}{17}=\frac{34}{17}=2$; $y=\frac{\begin{vmatrix}2&16\\5&6\end{vmatrix}}{17}=\frac{-68}{17}=-4$

**C. DETERMINANTS OF THIRD ORDER (ORDER THREE).**

$$D_3=\begin{vmatrix}a_1&b_1&c_1\\a_2&b_2&c_2\\a_3&b_3&c_3\end{vmatrix}=a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_3b_2c_1-a_2b_1c_3-a_1b_3c_2$$

**E:** $\begin{vmatrix}3&-2&8\\1&4&5\\6&-1&2\end{vmatrix}=(3)(4)(2)+(1)(-1)(8)+(6)(-2)(5)-(6)(4)(8)-(1)(-2)(2)-(3)(-1)(5)=-217$

It may be evaluated by remembering:




**NOTE:** This method does *not* apply to determinants of higher order. For an alternate method of evaluation (which applies to a determinate of any order) see **16**:**F**.

**D. SYSTEM OF 3 LINEAR EQUATIONS IN THREE UNKNOWNS.**

Consider $a_1x+b_1y+c_1z=d_1$, $a_2x+b_2y+c_2z=d_2$ with $D=\begin{vmatrix}a_1&b_1&c_1\\a_2&b_2&c_2\\a_3&b_3&c_3\end{vmatrix}\neq0$

Solution is $x=\frac{\begin{vmatrix}d_1&b_1&c_1\\d_2&b_2&c_2\\d_3&b_3&c_3\end{vmatrix}}{D}$; $y=\frac{\begin{vmatrix}a_1&d_1&c_1\\a_2&d_2&c_2\\a_3&d_3&c_3\end{vmatrix}}{D}$; $z=\frac{\begin{vmatrix}a_1&b_1&d_1\\a_2&b_2&d_2\\a_3&b_3&d_3\end{vmatrix}}{D}$

**E:** Solve: $x+2y-z=6$, $2x-y+3z=-13$, $3x-2y+3z=-16$. **Sol:** Since $D=\begin{vmatrix}1&2&-1\\2&-1&3\\3&-2&3\end{vmatrix}=10$,

$x=\frac{\begin{vmatrix}6&2&-1\\-13&-1&3\\-16&-2&3\end{vmatrix}}{10}=-1$; $y=\frac{\begin{vmatrix}1&6&-1\\2&-13&3\\3&-16&3\end{vmatrix}}{10}=2$; $z=\frac{\begin{vmatrix}1&2&6\\2&-1&-13\\3&-2&-16\end{vmatrix}}{10}=-3$

**E. APPLICATIONS TO ANALYTIC GEOMETRY.**

**1.** Area of triangle with vertices $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ equals the absolute value of: $K=\frac{1}{2}\begin{vmatrix}x_1&y_1&1\\x_2&y_2&1\\x_3&y_3&1\end{vmatrix}$

**NOTE:** If $K=0$, the three points are collinear.

**2.** Three lines $a_1x+b_1y=c_1$, $a_2x+b_2y=c_2$, $a_3x+b_3y=c_3$ meet in one point if and only if: $\begin{vmatrix}a_1&b_1&c_1\\a_2&b_2&c_2\\a_3&b_3&c_3\end{vmatrix}=0$

## 16 THEORY OF DETERMINANTS OF ORDER n.

**A. DEF. 1.** Symbol $D=\begin{vmatrix}a_{11}&a_{12}&a_{13}&\cdots&a_{1n}\\a_{21}&a_{22}&a_{23}&\cdots&a_{2n}\\a_{31}&a_{32}&a_{33}&\cdots&a_{3n}\\\cdots&\cdots&\cdots&\cdots&\cdots\\a_{n1}&a_{n2}&a_{n3}&\cdots&a_{nn}\end{vmatrix}$ consisting of $n^2$ numbers (elements) arranged in $n$ rows (first subscript) and $n$ columns (second subscript).

**E:** $a_{32}$ is the element in the 3rd row, 2nd column. **2.** An **inversion** of the arrangement of the positive integers occurs whenever one integer precedes a smaller integer. **E:** There are 3 inversions in 321. **B. THE VALUE OF D** is defined to be the sum $\Sigma\pm a_{1\alpha}a_{2\beta}a_{3\gamma}\cdots a_{n\nu}$ of **n!** terms, the sign in a given term being taken plus or minus according as the number of inversions (of the numbers 1, 2, 3, . . . , $n$) in the corresponding sequence $\alpha, \beta, \gamma, \ldots, \nu$ is even or odd. **C. PROPERTIES. 1.** If the corresponding rows and columns of D be interchanged, D is unchanged. **2.** If any two rows (or columns) of D be interchanged, D is changed to $-D$. **3.** If all the elements of a row (column) are 0, then $D=0$. **4.** If any two rows (columns) are identical, then $D=0$. **5.** If each element of a row (column) of D be multiplied by $m$, the new determinant is equal to $mD$. **6.** If the elements of two rows (columns) are proportional, then $D=0$. **7.** If each element of a row (column) is expressed as the sum of two (or more) terms, then D can be expressed as the sum of two (or more) determinants. **8.** If to each element of a row (column) is added $m$ times the corresponding element in another row (column), D is unchanged.

**E:** Given $\begin{vmatrix}-2&1&6\\4&0&8\\5&3&9\end{vmatrix}$ Multiply each element of first column by 2, subtract from third column giving the equal determinant: $\begin{vmatrix}-2&1&10\\4&0&0\\5&3&-1\end{vmatrix}$

**D. MINOR.** The minor of an element in D is the determinant of order **n−1** obtained by removing row and column containing given element.

**E:** In $\begin{vmatrix}a_{11}&a_{12}&a_{13}&a_{14}\\a_{21}&a_{22}&a_{23}&a_{24}\\a_{31}&a_{32}&a_{33}&a_{34}\\a_{41}&a_{42}&a_{43}&a_{44}\end{vmatrix}$ the minor of $a_{23}$ is $\begin{vmatrix}a_{11}&a_{12}&a_{14}\\a_{31}&a_{32}&a_{34}\\a_{41}&a_{42}&a_{44}\end{vmatrix}$

**E. COFACTOR.** The cofactor $A_{ij}$ of the element $a_{ij}$ is the product of $(-1)^{i+j}$ and the minor of $a_{ij}$.

**F. VALUE.** The value of D obtained in terms of cofactors as follows: $D=a_{1j}A_{1j}+a_{2j}A_{2j}+\cdots+a_{nj}A_{nj}$, $j=1, 2, 3, \ldots,$ or **n**.

**E:** $\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}=a_{11}\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}-a_{12}\begin{vmatrix}a_{21}&a_{23}\\a_{31}&a_{33}\end{vmatrix}+a_{13}\begin{vmatrix}a_{21}&a_{22}\\a_{31}&a_{32}\end{vmatrix}=$

$a_{11}(a_{22}a_{33}-a_{32}a_{23})-a_{12}(a_{21}a_{33}-a_{31}a_{23})+a_{13}(a_{21}a_{32}-a_{31}a_{22})$

**G. SYSTEMS OF LINEAR EQUATIONS. 1. n LINEAR EQUATIONS IN n UNKNOWNS.** The solution of the system of equations $a_{i1}x_1+a_{i2}x_2+\cdots+a_{in}x_n=c_i$, $i=1, 2, \ldots, n$ is unique if $D\neq0$. The solution is given by the equations $Dx_1=C_1$, $Dx_2=C_2, \ldots, Dx_n=C_n$, where $C_k$ is what D becomes when the elements of its $k$th column are replaced by $c_1, c_2, \ldots, c_n$ respectively. (See Secs. **15**: B, D for the cases when $n=2, 3$). If $D=0$ and at least one of the determinants $C_k\neq0$, ($k=1, \ldots, n$) the given system has no simultaneous solution. If $D=0$ and all $C_k=0$, there may or may not be solutions.

**2. HOMOGENEOUS LINEAR EQUATIONS.** When all $c_i=0$. If $D\neq0$, the only solution is $x_1=x_2=\ldots=x_n=0$. If $D=0$, there are infinitely many non-zero solutions. **3. m EQUATIONS IN n UNKNOWNS. If m>n**, the unknowns in $n$ of the given equations may be obtained. If these values satisfy the remaining $m-n$ equations the system has solutions, otherwise there are none. **If m<n** then $m$ of the unknowns may be determined in terms of the remaining $n-m$ unknowns.

## 17 ELEMENTARY ANALYTIC GEOMETRY.

**A. STRAIGHT LINE. 1. SLOPE.** For points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ the slope of the line joining them is: $\mathbf{m=\frac{y_2-y_1}{x_2-x_1}}$. **2. SPECIAL FORMS OF EQUATION OF A STRAIGHT LINE.**

| | |
|---|---|
| $y=mx+b$ | Line determined by slope $m$, $y$-intercept $b$. |
| $x=a$ | Line parallel to $y$-axis; $m$ undefined; $x$-int. $=a$. |
| $y=b$ | Line parallel to $x$-axis; $m=0$; $y$-intercept $=b$. |
| $y-y_1=m(x-x_1)$ | Line determined by point $(x_1, y_1)$, slope $m$. |
| $x/a+y/b=1$ | Two intercept form; ($a\neq0$, $b\neq0$). |

**3.** If two straight lines are parallel, they have equal slopes; and conversely. If two straight lines are perpendicular, their slopes are negative reciprocals, that is $\mathbf{m_1m_2=-1}$; and conversely.

**E:** Write an equation of the straight line passing through $(-4,1)$ and perpendicular to $5x-3y+1=0$. **Sol:** Rewrite $5x-3y+1=0$ as $y=\frac{5}{3}x+\frac{1}{3}$; its slope is 5/3. The slope of the unknown line is $-3/5$. Using point slope form $y-1=-3/5(x+4)$ or $3x+5y+7=0$.

**B. CIRCLE. 1.** Center at origin, radius $r$: $x^2+y^2=r^2$. **2.** Center at $(h, k)$, radius $r$: $(x-h)^2+(y-k)^2=r^2$.

**E:** Find the center and radius of the circle $x^2+y^2+10x-4y-7=0$. **Sol:** By completing the square in $x$ and $y$: $x^2+10x+25+y^2-4y+4=7+25+4$. Then $(x+5)^2+(y-2)^2=36$; center is at $(-5,2)$, radius is 6.

## 18 ELEMENTARY DIFFERENTIAL CALCULUS (of a polynomial function $y=f(x)$).

**A. AVERAGE RATE OF CHANGE, $\Delta y/\Delta x$,** between two points of a polynomial is the slope of the straight line joining them.

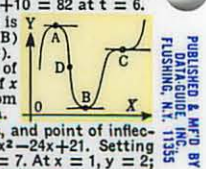
**B. INSTANTANEOUS RATE OF CHANGE, $dy/dx$ or $y'$ or $f'(x)$,** of a polynomial at some point on it, is the slope of the tangent to the curve at that point; $dy/dx$ is the **derivative** of the function. **C. RULE:** If $y=x^n$, derivative $dy/dx=nx^{n-1}$. The derivative of a constant is zero, and the derivative of a sum is the sum of the derivatives.

**E1:** For $y=7x^3-5x+4$, $dy/dx=21x^2-5$. **E2:** Find an equation of the tangent to $y=x^2-3x-1$ at $(2,-3)$. **Sol:** Slope of tangent, $dy/dx=2x-3$, which equals 1 at $x=2$. Using point slope form of straight line, $y+3=1(x-2)$ or $y=x-5$.

**D. THE SECOND DERIVATIVE, $d^2y/dx^2$ or $y''$ or $f''(x)$,** is the derivative of the first derivative and gives the rate of change of the slope.

**NOTE:** When $s=f(t)$, where $s=$ distance (from a point), $t=$ time; $ds/dt=$ velocity, $d^2s/dt^2=$ acceleration. **E:** $s=2t^3+5t^2$, find accel. at $t=6$. **Sol:** $ds/dt=6t^2+10t$; $d^2s/dt^2=12t+10=82$ at $t=6$.

**E. AT $dy/dx=0$ WHERE $d^2y/dx^2$ EXISTS,** $y$-value is a relative maximum or minimum (pt. A or pt. B) or there is a horizontal point of inflection (pt C).




**F. AT VALUE OF x WHERE $d^2y/dx^2=0$,** graph of $y=f(x)$ has a point of inflection if that value of $x$ separates a concave upward portion of graph from a concave downward portion (pt. D), or vice versa.

**E:** Find relative maximum and minimum points, and point of inflection of $y=x^3-12x^2+21x-8$. **Sol:** $dy/dx=3x^2-24x+21$. Setting this equal to zero and solving gives $x=1$ and $x=7$. At $x=1$, $y=2$; at $x=7$, $y=-106$. By checking values of $y$ just below and just above $x=1$, $y=2$ is a relative maximum value (other tests exist). Likewise $(7,-106)$ is a relative minimum point. Now $d^2y/dx^2=6x-24$. Setting equal to zero gives $x=4$ and $(4,-52)$ is a point of inflection.



DO NOT PLACE ON HOT SURFACES

KNOW THE FACTS ON THIS CHART AND YOU WILL GET HIGHER MARKS IN CLASSWORK, HOMEWORK, TESTS AND EXAMS.

THERE ARE QUICK CHARTS IN MATH • SCIENCE • ENGLISH • ECO • HISTORY • LANGUAGES • PSYCH • SOC • PHILO • MUSIC

DATA-GUIDE EDITOR: JOSEPH L. LEON

PUBLISHED & MFD BY DATA GUIDE, INC. FLUSHING, N.Y. 11355

DATA-GUIDE QUICK CHART

# NINTH YEAR ELEMENTARY ALGEBRA

## A SIMPLIFIED SUMMARY & INSTANT REFERENCE

LEARN & REVIEW IN MINUTES

**AUTHOR:** Martin Cohen, Instructor at Bayside H.S. and Queens College.
**CONSULTANT:** Professor Joseph Chicarelli, Fordham University

SOLID PLASTIC

QUICK CHARTS SPEED STUDY: PUT THE IMPORTANT FACTS IN FULL VIEW TO SAVE YOU TIME AND WORK.

QUICK CHARTS IMPROVE MEMORY: PERMANENTLY FILL IN FOR MISSING NOTES & TEXTS IN BASIC & ADVANCED COURSES.

CONTAINS NO PAPER

## I. THE LANGUAGE OF ALGEBRA. A. FUNDAMENTAL OPERATIONS.

| OPERATIONS | SYMBOLS | EXAMPLE |
|---|---|---|
| ADDITION | + | The sum of $x$ and $y$ is $x+y$ |
| SUBTRACTION | − | Difference between 3 and $a$ is $3-a$ |
| MULTIPLICATION | × or · or no symbol | The product of 7 and 5 is $7\times5$; $36=6\cdot6$<br>$cd$ means $c\cdot d$; $6x$ means $6\cdot x$.<br>$1/2\,(x+1)$ means $1/2\cdot(x+1)$ |
| DIVISION | / or ÷ or — | The quotient of $s$ and $r$ is $s/r$ or $s\div r$;<br>$a+5$ divided by $a$ is $(a+5)\div a$ or $\frac{a+5}{a}$ |

**B. COEFFICIENTS AND EXPONENTS.** In $3y^2$, **3** is the *(numerical) coefficient*, **y** the *base* and **2** the *exponent* (or *power*). An exponent is an abbreviation: $y^2=y\cdot y$. **NOTE:** $8x$ means $8x^1$; $-r^3s^1$ means $-1r^3s^1$, etc.

**C. TERMS.** A term (a **monomial**) is an expression containing only *multiplication* or *division* signs: $4ac^2$, $xy$, $7(x-y)$. Two or more terms separated by + or − signs form a **polynomial.**

**D. SPECIAL POLYNOMIALS.** A **binomial** contains two terms: $6a-3b$. A **trinomial** contains three terms: $x^2-3+2(x-1)$.

**E. FACTORS.** Factors are the parts of a **product.** E: The factors of $4ac^2$ are $4,a,c$, and $c$. The factors of $7(x-y)$ are 7 and $(x-y)$. The parentheses have the effect of grouping $x-y$ into a single factor.

**F. ORDER OF OPERATIONS FOR EVALUATION. 1.** Take powers and roots. **2.** Multiply and divide *(from left to right)*. **3.** Add and subtract *(from left to right)*. E: Evaluate $3x^2-2x$ if $x=5$. **Sol:** Substituting 5 for $x$ gives $3\cdot5^2-2\cdot5=3\cdot25-2\cdot5=75-10=65$. **NOTE:** Usually evaluate quantities within ( ) or under $\sqrt{\ }$ signs first then the resultant *nums* and *denoms*. E: Evaluate $4(x+2)/3+(2x)^2$ if $x=3$. **Sol:** $4(3+2)/3+(2\cdot3)^2=4(5)/3+(6)^2=20/3+36=42\,2/3$.

## II. THE LAWS OF ALGEBRA. A. COMMUTATIVE. 1.

If all quantities are separated by + signs, they may be rearranged in any order. E: $3+a+4=a+4+3=4+a+3$, etc.; $7-3\neq3-7$. **2.** If all quantities are separated by × signs, they may be rearranged in any order. E: $5sr=5rs$; $(x-2)(x+3)=(x+3)(x-2)$; but $8\div2\neq2\div8$.

**B. DISTRIBUTIVE. 1.** Multiplication (or division) is distributive over addition or subtraction; that is, multiply (or divide) term by term.

E: $5(x-3)=5\cdot x-5\cdot3=5x-15$. E: $\frac{16x^2-12x}{4x}=\frac{16x^2}{4x}-\frac{12x}{4x}=4x-3$.

**2. Extended distributive law:** Powers (or roots) are distributive over mult or div. E: $(5x)^2=5^2x^2=25x^2$. E: $(x+3)^2\neq x^2+9$. (See $X$:$A$)

## III. SIGNED NUMBERS. A. ABSOLUTE VALUE.

Value of a number without its sign. E: $|-5|=5$; $|+4|=4$. **B. TWO SIGNED NUMBERS.**

| | SIGNS | RULE | EXAMPLES |
|---|---|---|---|
| TO ADD | LIKE | Add absolute values of numbers and use common sign. | $(+7)+(+3)=+10$<br>$(-6)+(-1)=-7$ |
| | UNLIKE | Find difference between absolute values and use sign of no. with greater abs. value. | $(-8)+(+3)=-5$<br>$(-2)+(+6)=+4$<br>$(0)+(-6)=-6$ |
| TO SUBTRACT | LIKE OR UNLIKE | Change sign of subtrahend (number after subtraction sign), then add as above. | $(-4)-(+7)=(-4)+(-7)=-11$<br>$(-4)-(-7)=+3$ |
| TO MULTIPLY | LIKE | Product is positive. | $(-1/2)(-8)=+4$ |
| | UNLIKE | Product is negative. | $(-2)(+3.5)=-7$ |
| | | (zero)(any number) = zero. | $(0)(-3)=0$ |
| TO DIVIDE | LIKE | Quotient is positive. | $(-15)\div(-3)=+5$ |
| | UNLIKE | Quotient is negative. | $+21/-3=-7$ |
| | | **Note: Division by 0 is impossible.** | 6/0 is impossible |

## IV. OPERATIONS ON MONOMIALS. Like terms have identical literal (letter) factors. E: $6ax^2$ and $-2ax^2$ but not $6x^3$ and $6x^2$.

| | CONDITIONS | RULES | EXAMPLES |
|---|---|---|---|
| TO ADD | Like Monomials | Use like part unchanged in answer; add coefficients. | $-6x^3-4x^3=-10x^3$;<br>$5(a+b)-7(a+b)=-2(a+b)$; but $2x^3+5x^2$ cannot be combined. |
| TO SUBTRACT | | Use like part unchanged in ans.; subtract coeffs. | $7ab-(-3ab)=10ab$; but $-6x^3$ minus $x^2=-6x^3-x^2$ |
| TO MULTIPLY | Exponential quantities with same base | Use same base unchanged in ans.; add exponents. | $x^5\cdot x^2=x^7$; $(10^4)(10^2)=10^6$; $(-7a^2)(3a^3)=-21a^5$; but $(x^3)(y^2)=x^3y^2$ |
| TO DIVIDE | | Use same base unchanged in the answer; subtract exponents. | $a^8\div a^2=a^6$; $8^7\div8^2=8^5$; $(-15x^5)\div5x^3=-3x^2$; but $x^3/x^3=1$, and $(-20a^2b^2c^6)\div5ab^2c^2=-4ac^4$ |

**SPECIAL CASE. The power of a power:** Keep the base and multiply the exponents. E: $(x^2)^3=x^6$; $(3^4)^2=3^8$; $(2a^3b^2)^4=16a^{12}b^8$

## V. OPERATIONS WITH POLYNOMIALS. A. ARRANGEMENT.

**Descending Powers** means that terms are arranged with the exponents of a letter in numerical order starting with the largest. E: $-5x^3+4x^2+10x-1$. **Ascending Powers** means that terms are arranged with the exponents of a letter in numerical order starting with the smallest. E: $8-xy^2+6x^2y+x^3$ (ascending powers of $x$).

**B. ADDITION.** To add polynomials, arrange the like terms in columns and then add each column. E: Add $4x^2-5x+2$, $3x-x^2$, $2x-7+3x^2$

**Sol:**
$$\begin{array}{r} -x^2+3x \\ 3x^2+2x-7 \\ \hline 6x^2 \quad -5 \end{array}$$

**C. SUBTRACTION.** To subtract polynomials, arrange like terms in columns and subtract in each column. E: From $a+7b$ subtract $3a-4c$

**Sol:**
$$\begin{array}{r} a+7b \\ 3a \quad -4c \\ \hline -2a+7b+4c \end{array}$$

**D. MULTIPLICATION. Polynomial × Monomial.** Multiply each of the terms of the polynomial by the monomial *(Distributive Law)*. E: $4(3x-2y+5)=12x-8y+20$; $-3x^2(x^3-y^3)=-3x^5+3x^2y^3$.

**Polynomial × Polynomial.** Multiply (usually from *left* to *right*) the multiplicand by each term of the multiplier; arrange these partial products in columns of like terms; add each column.

E: $(4x^2+x-2)(3x-5)$. **Sol:**

$$\begin{array}{r} 4x^2+x-2 \\ 3x \quad -5 \\ \hline 12x^3+3x^2-6x \\ -20x^2-5x+10 \\ \hline 12x^3-17x^2-11x+10 \end{array}$$

The product of $4x^2+x-2$ and $3x$ → $12x^3+3x^2-6x$
The product of $4x^2+x-2$ and $-5$ → $-20x^2-5x+10$

**E. DIVISION. Polynomial ÷ Monomial.** Divide each of the terms of the polynomial by the monomial *(Distributive Law)*.

E: $(16x^2+8y^2)\div8=2x^2+y^2$; $-2a\overline{)6a^2-4a+8ab}$; $\frac{-3a+2-4b}{2a}$; $\frac{5x-4}{3}=\frac{5x}{3}-\frac{4}{3}$

**Polynomial ÷ Polynomial.** E: Divide $15x^3-38x^2+35$ by $3x-4$.

**1.** Arrange both the dividend and the divisor in descending order. **2.** Divide the first term of the dividend ($15x^3$) by the first term of the divisor ($3x$) to obtain the first term of the quotient ($5x^2$). **3.** Multiply the entire divisor by this term of the quotient (giving $15x^3-20x^2$). **4.** Subtract; bring down the next term of the dividend; etc.

$$\begin{array}{r} 5x^2-6x-8 \\ 3x-4\overline{)15x^3-38x^2+0x+35} \\ 15x^3-20x^2 \\ \hline -18x^2+0x \\ -18x^2+24x \\ \hline -24x+35 \\ -24x+32 \\ \hline \text{Remainder} \rightarrow 3 \end{array}$$

The product of $3x-4$ and $5x^2$ → $15x^3-20x^2$
The product of $3x-4$ and $-6x$ → $-18x^2+24x$
The product of $3x-4$ and $-8$ → $-24x+32$

## F. REMOVING PARENTHESES

| CONDITION | RULE | EXAMPLE |
|---|---|---|
| MONOMIAL MULTIPLIER | Multiply each term inside the ( ) by the monomial. | $-3(x-4)=-3x+12$ |
| "+" PRECEDING ( ) | Remove ( ) with no changes of signs. | $8+(x-2)=8+x-2=x+6$ |
| "−" PRECEDING ( ) | Remove inner ( ) and change sign of each term within. | $7x^2-(-x+4)=7x^2+x-4$ |
| ( ) WITHIN [ ] | Remove inner ( ) first, then outer [ ] applying the three basic rules at each step. | $6[4a-(3b-c)]=6[4a-3b+c]=24a-18b+6c$ |

## VI. FIRST DEGREE EQUATIONS. A. DEFINITIONS.

An **equation** states that one quantity equals another. If $5x-12=2x$, then $x$ is called the **unknown**; $5x-12$ is the **left member**, and $2x$ is the **right member. A root** is a value which, when substituted for the unknown, makes both members equal. **To solve an equation means to find its root(s). B. PERMISSIBLE OPERATIONS ON BOTH MEMBERS:**

| OPERATIONS | EXAMPLE | APPLICATION | RESULT |
|---|---|---|---|
| 1. Divide by same quantity (except zero). | $7x=5$ | Divide by 7 | $x=5/7$ |
| 2. Mult. by same quantity | $x/2=6$ | Mult. by 2 | $x=12$ |
| 3. Add same quantity | $2x-3=7$ | Add 3 | $2x=10$, then $x=5$ |
| 4. Subtract same quantity | $\frac{x}{3}+4=9$ | Subtract 4 | $\frac{x}{3}=5$, then $x=15$ |

**C. TRANSPOSITION.** (A short way to carry out the addition and subtraction operations.) To transpose a *term*, move it to the other side of the = sign, change its sign. E: $7x=20-5x$, $7x+5x=20$, etc.

**D. ORDER OF OPERATIONS FOR SOLVING 1ST DEGREE EQUATIONS. 1.** Multiply both members by the L.C.D. (*Lowest Common Denominator*) to clear fractions. **2.** Remove parentheses. **3.** Transpose to collect all unknown terms on one side. **4.** Combine like terms. **5.** Divide each member by the coefficient of the unknown.

| E: $1/2\,(3x-16)=4-1/3\,(66-7x)$ **Sol:** | $6\cdot1/2(3x-16)=6\cdot4-6\cdot1/3(66-7x)$ gives |
|---|---|
| 1. Mult. by L.C.D. =6, to clear fractions: | $3(3x-16)=24-2(66-7x)$ |
| 2. Remove parentheses: | $9x-48=24-132+14x$ |
| 3. Transpose the −48 and the +14x: | $9x-14x=24-132+48$ |
| 4. Combine like terms: | $-5x=-60$ |
| 5. Divide both members by −5: | $x=12$ **(Root)** |

E: $.8x-3=.05x$. **Sol:** Multiply both sides by L.C.D., 100, to clear decimal fractions: $80x-300=5x$; $80x-5x=300$; $75x=300$; $x=4$.

**E. CHECKING.** Substitute root for unknown in *original* equation; evaluate; check for equality of both members.

## VII. FIRST DEGREE LITERAL EQUATIONS. A. DEF.

A literal equation contains two or more letters, one designated as the unknown. **B. TO SOLVE.** Use the same order of operations as in $VI$:$D$. above. E: **Solve for p:** $(p/2)-a=b/2$. **Sol:** Multiply both members by 2: $p-2a=b$. Transpose the $-2a$: $p=2a+b$. (Also see $X$:$F$.)

## VIII. FORMULAS. A. DEF.

**A formula is an equation which expresses a rule by means of algebraic symbols.** E: The area A of a triangle equals 1/2 the product of its base **b** and its altitude **h**: **$A=1/2\,bh$. A variable** is a letter whose value can change. **A constant** is a letter or quantity whose value does not change. E: In $C=2\pi r$, **C** and **r** are *variables*, but 2 and $\pi$ are *constants*.

**B. EVALUATION. 1.** Substitute into the formula the values given for the letters. **2.** Find value of remaining letter. (Use I,F or VI, D.) E: In $A=.5h(b+c)$, find $A$ if $h=12$, $b=7$, $c=3$. **Sol:** Substituting, gives $A=.5\cdot12(7+3)=6(10)=60$. E: In $F=(9C/5)+32$, find $C$ if $F=68$. **Sol:** Substituting, gives $68=(9C/5)+32$. Multiplying both members by 5 gives $340=9C+160$. Transposing +160 gives $180=9C$. Divide both members by 9 to get $20=C$.

**C. TRANSFORMING FORMULAS.** Use order of operations in VI, D above. E: **Solve for h:** $V=(\pi r^2h)\div3$. Clearing fractions, $3V=\pi r^2h$. Divide both members by $\pi r^2$ to get $3V/\pi r^2=h$.

## IX. FIRST DEGREE EQUATION PROBLEMS. A. TRANSLATIONS.

| ENGLISH | ALGEBRA | ENGLISH | ALGEBRA |
|---|---|---|---|
| the sum of $x$ and $y$ | $x+y$ | the difference bet. $x$ and $y$ | $x-y$ |
| $x$ increased by $y$ | $x+y$ | $x$ decreased by $y$ | $x-y$ |
| $x$ added to $y$ | $y+x$ | $x$ subtracted from $y$ | $y-x$ |
| $x$ more than $y$ | $y+x$ | $x$ less than $y$ | $y-x$ |
| $x$ exceeds $y$ by $c$ | $x=y+c$ | Two quantities whose sum is $c$ | $x$ and $(c-x)$ |
| $x$ exceeds $y$ by 6 | $x=y+6$ | | |

**B. NUMBER PROBLEMS.**

E: There are three numbers such that the second is 3 times the first, and the third is 2 less than the first. If their sum is 18, find the three numbers.
**Sol:** Let $x$ = the first no.
then $3x$ = the second no.
and $x-2$ = the third no.
$x+3x+x-2=18$
*Solving:* $x=4$ (first no.)
$3x=3\cdot4=12$ (second no.)
$x-2=4-2=2$ (third no.)

E: Separate 22 into two parts such that the larger divided by the smaller gives a quotient of 3 and a remainder of 2.
**Sol:** Let $x$ = the smaller part
then $22-x$ = the larger part

Division Law:
**Dividend = Quotient × Divisor + Remainder**
Larger = 3 (smaller) +2
$22-x=3x+2$
*Solving:* $x=5$ (smaller part)
$22-x=17$ (larger part)

**C. CONSECUTIVE INTEGER (WHOLE NUMBER) PROBLEMS.**

| TO REPRESENT | Use | Num. Exs. |
|---|---|---|
| CONSECUTIVE INTEGERS | $x, x+1, x+2, x+3, \ldots$ | E: 5, 6, 7, 8, ... |
| CONSEC. EVEN INTEGERS | $x, x+2, x+4, x+6, \ldots$ | E: 4, 6, 8, 10, ... |
| CONSEC. ODD INTEGERS | $x, x+2, x+4, x+6, \ldots$ | E: 3, 5, 7, 9, ... |

E: Find three consecutive odd integers such that the sum of the first two is 25 more than the third. **Sol:** Let $x$= the first consecutive odd integer; then $x+2$= the second; $x+4$= the third. To form the equation, we use **first + second = third + 25.** Therefore $x+x+2=x+4+25$. *Solve:* $x=27$, $x+2=29$, and $x+4=31$.

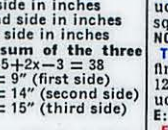
**D. TRIANGLE PROBLEMS.**

E: The second angle of a triangle is twice the first. The third angle exceeds the sum of the first two angles by 12°. Find the three angles. **Sol:**
Let $x$ = degrees in 1st angle
$2x$ = degrees in 2nd angle
$3x+12$ = degrees in 3rd angle
**The sum of the three angles of any triangle is 180°.**
Thus $x+2x+3x+12=180°$
*Solve:* $x=28°$ (1st ∠), and $2x=56°$ (2nd ∠), and $3x+12=96°$ (3rd ∠)

E: The second side of a triangle is 5" more than the first. The third side is 3" less than twice the first side. The perimeter of the triangle is 38". Find the sides. **Sol:**
Let $x$ = first side in inches
$x+5$ = second side in inches
$2x-3$ = third side in inches
**Perimeter is sum of the three sides.** $x+x+5+2x-3=38$
*Solve:* $x=9$" (first side)
$x+5=14$" (second side)
$2x-3=15$" (third side)

**E. RECTANGLE PROBLEMS.**

E: The length of a rectangle exceeds 3 times the width by 5. If the perimeter equals 58, find the length and the width. **Sol:** Let $x$ = width, $3x+5$ = length. **The perimeter equals the sum of all four sides.** $3x+5+x+3x+5+x=58$, $8x+10=58$, $8x=48$, $x=6$ (width) and $3x+5=23$ (length).




**F. AGE PROBLEMS.** E: Mrs. Smith is 24 years older than her daughter. In 3 years, she will be 4 times as old as her daughter is then. Find their present ages. **Sol:** Let $x$ = daughter's age in yrs., and $x+24$ = Mrs. Smith's age in yrs. In 3 years, $x+3$ = daughter's age, and $x+27$ = Mrs. Smith's age. **Mrs. Smith's age in 3 years = 4 (daughter's age in 3 years).** $x+27=4(x+3)$. *Solve:* $x=5$ yrs. (daughter's age) and, $x+24=29$ yrs. (Mrs. Smith's age).

**G. COIN PROBLEMS.** E: A purse contains nickels, dimes and quarters. The number of nickels is 1/3 the number of dimes, and there are 7 more quarters than dimes. The total value of all the coins is $8.35. How many of each kind are there? **Sol:** (To avoid fractions let $3x$ = number of dimes.)

| | No. of coins | × | ¢ per coin | = | Value in cents |
|---|---|---|---|---|---|
| NICKELS | $x$ | | 5 | | $5x$ |
| DIMES | $3x$ | | 10 | | $30x$ |
| QUARTERS | $3x+7$ | | 25 | | $25(3x+7)$ |

**Total value of all coins = 835**
$5x+30x+25(3x+7)=835$
$x=6$ Nickels;
$3x=18$ Dimes; $3x+7=25$ Quarters

**H. INVESTMENT PROBLEMS.** E: $6000 is to be invested, part at 6% and the rest at 3%, to give a total annual income of $294. How much should be invested at each rate? **Sol:** If interest is computed annually, the formula $PRT=I$ becomes $PR\,(1)=I$, or $PR=I$

| | Principal in $ | × | Rate of Int. | = | Interest or Income in $ |
|---|---|---|---|---|---|
| AT 6% | $x$ | | .06 | | $.06x$ |
| AT 3% | $6000-x$ | | .03 | | $.03(6000-x)$ |

**Income at 6%+Inc. at 3% = 294**
$.06x+.03(6000-x)=294$
Mult. by L.C.D. 100;
$6x+3(6000-x)=29400$
Then $x=\$3800$ at 6%, and $6000-x=\$2200$ at 3%

**I. MIXTURE PROBLEMS.** E: How many pounds each of nuts worth 72¢ a lb. and nuts worth 84¢ a lb. should be used to obtain a 40 lb mixture worth 75¢ a lb.? **Sol:**

| | No. of lb. | × | Price ¢ per lb. | = | Value in cents |
|---|---|---|---|---|---|
| LESS EXPENSIVE | $x$ | | 72 | | $72x$ |
| MORE EXPENSIVE | $40-x$ | | 84 | | $84(40-x)$ |
| MIXTURE | 40 | | 75 | | $75(40)$ |

**Value of less exp. nuts + value of more exp nuts = value of mixture.**
$72x+84(40-x)=75(40$
$x=30$ lb. of the 72¢ nut
$40-x=10$ lb. of the 84¢ nuts.

**J. MOTION PROBLEMS.**

E: A train goes from Main City to Harris at the rate of 80 mph; a second train goes from Harris to Main City at 60 mph. If both start at 11 A.M. and the two cities are 455 miles apart, at what time should they pass each other? **Sol:**

| | R (mph) | × | T (hr) | = | D (mi) |
|---|---|---|---|---|---|
| M.C. TO HARRIS | 80 | | $x$ | | $80x$ |
| HARRIS TO M.C. | 60 | | $x$ | | $60x$ |

**The total distance traveled by both trains is 455 miles.**
$80x+60x=455$
$x=3\,1/4$ hr. Ans. 2:15 P.M.

E: A patrol pilot can go east in his plane at 150 mph, and can return at 100 mph. If the plane has 5 hr. of flying time, how far east can he go?
**Sol:** (Distance going = distance returning.)

| | R (mph) | × | T (hr) | = | D (mi) |
|---|---|---|---|---|---|
| GOING | 150 | | $\frac{x}{150}$ | | $x$ |
| RETURNING | 100 | | $\frac{x}{100}$ | | $x$ |

(T Column obtained from $T=\frac{D}{R}$)
**Time going + time ret. = 5 hr.**
$\frac{x}{150}+\frac{x}{100}=5$; $x=300$ mi

E: Mary starts from home on her bicycle at 15 mph. One hour later, her father starts after her in his car going 40 mph. How soon will he overtake her? **Sol:** (Since Mary had a 1 hr. head start her traveling time is 1 hr more than her father's.)

| | R (mph) | × | T (hr) | = | D (mi) |
|---|---|---|---|---|---|
| MARY | 15 | | $x+1$ | | $15(x+1)$ |
| FATHER | 40 | | $x$ | | $40x$ |

**Father's distance = Mary's distance.** $40x=15(x+1)$
$x=3/5$ hr. or 36 min.

E: To find the rate of the wind a pilot whose plane has an airspeed of 220 mph goes with the wind for 10 min. and returns over the same distance in 12 min. What is the rate of the wind?
**Sol:** (Rate against wind = airspeed − wind rate. Rate with wind = airspeed + wind rate.

| | R (mph) | × | T (hr) | = | D (mi) |
|---|---|---|---|---|---|
| AGAINST WIND | $220-x$ | | 1/5 | | $\frac{1/5}{(220-x)}$ |
| WITH WIND | $220+x$ | | 1/6 | | $\frac{1/6}{(220+x)}$ |

(Convert 12 min. and 10 min. to hr.) **Distances are equal.**
$1/5(220-x)=1/6(220+x)$
$x=20$ mph

**K. WORK PROBLEMS.** E: Larry can mow a lawn in 36 min; Jay can in 30 min. If both boys work together, how long will the job take

**Sol:**

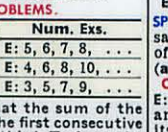
| | No. of min. to do job alone | Part of job done in 1 min. | × | No. of min. actually worked | = | Part of job done |
|---|---|---|---|---|---|---|
| LARRY | 36 | 1/36 | | $x$ | | $x/36$ |
| JAY | 30 | 1/30 | | $x$ | | $x/30$ |

**Part of Job done by Larry + part of job done by Jay = whole job**
$x/36+x/30=1$ (We represent the whole job by 1.)
Multiply by 180; etc. Ans: $x=16\,4/11$ min.

## X. SPECIAL PRODUCTS AND FACTORING. A. THE SQUARE OF A BINOMIAL:

The square of the first term + twice the product of the two terms + the square of the second term. $(x+a)^2=x^2+2ax+a^2$. E: $(x-5)^2=(x)^2+2(-5x)+(-5)^2=x^2-10x+25$.

**B. THE PRODUCT OF TWO BINOMIALS.** Find the product of the First terms, Outer terms, Inner terms, Last terms. (FOIL method




RULE: $(x+a)\quad(y+c)=x(y)+c(x)+a(y)+a(c)=xy+cx+ay+ac$
F L F O I L

E: $(3x+4)(x-5)=3x(x)+3x(-5)+4(x)+4(-5)=3x^2-11x-20$

**SPECIAL CASE:** If two binomials are the sum and difference of the same two terms, the product of the inner terms and the product of the outer terms will total zero. (Answer contains only two terms.) $(a+b)(a-b)=a^2-b^2$. E: $(2x+5)(2x-5)=(2x)^2-5^2=4x^2-25$

**C. PRIMES.** A prime number is divisible only by **itself and 1.** E: 2, 3, 5, 7, etc. However, 6 is not prime because 6 is divisible by 3 and 3 as well as 6 and 1. A prime polynomial is divisible only by itself and 1 such as $x-4$; $a^2+b$.

**D. FACTORING.** To factor an expression is to find the quantities which when multiplied together will give the original expression. All polynomial factors must be **prime.** Check ans. by multiplying.

**TYPE I: Removing the Highest Common Factor.** E: $7a+7b$. Since 7 is the common factor in each term, it is removed, giving $7(a+b)$. E: $p+prt=p(1+rt)$. E: In $12x^2y^2-15x^2y^3+18xy^4$, highest common factor of all the terms is $3xy^2$; factors are $3xy^2$ and $(4x-5xy+6y^2)$.

**TYPE II: The difference of two squares.** The factors are the product of the sum and difference of the square roots of the given squares. $a^2-b^2=(a+b)(a-b)$. E: $1/9-16x^2y^2=(1/3+4xy)(1/3-4xy)$. **NOTE:** The *sum* of two squares is prime. $x^2+9$ is prime.

**TYPE III: Trinomial of the form $ax^2+bx+c$.** By trial and error find the two binomial factors; check by the FOIL method. E: $x^2-7x+12=(x-4)(x-3)$. Note that $(x-6)(x-2)$ is incorrect since its product leads to a middle term of $-8x$. E: $x^2+2x-24=(x+6)(x-4)$. E: $2x^2+7x+3=(2x+1)(x+3)$.

**E. COMPLETE FACTORING.** In some cases, the factors in parentheses are not prime and have to be factored further. E: $cx^2-cy^2=c(x^2-y^2)=c(x+y)(x-y)$ using *Types I* and *II*. E: $a^4-81=(a^2+9)(a^2-9)=(a^2+9)(a+3)(a-3)$ using *Type II* twice. E: $2x^2+10x+12=2(x^2+5x+6)=2(x+3)(x+2)$ using *Types I* and *III*.

**F. MORE DIFFICULT LITERAL EQUATIONS.** Use the same order of operations as in *VI, D*, except that after transposing, if the terms containing the unknown cannot be combined, the unknown should be removed as a common factor. E: **Solve for r:** $kr=6(a+r)$

**Sol:**
1. Remove parentheses: $kr=6a+6r$
2. Transpose +6r to collect terms containing r: $kr-6r=6a$
3. Factor: $r(k-6)=6a$
4. Divide both sides by k−6: $r=6a/(k-6)$

## XI. FRACTIONS. A. FUNDAMENTAL LAWS.

The appearance of a fraction may be changed without changing its value: **1. If both numerator and denominator are multiplied (or divided) by the same quantity (except 0):** $\frac{am}{an} = \frac{am \div a}{an \div a} = \frac{m}{n}$

E: $\frac{6}{8} = \frac{6 \div 2}{8 \div 2} = \frac{3}{4}$; $\frac{5}{y} = \frac{5 \cdot (a-b)}{y \cdot (a-b)} = \frac{5(a-b)}{y(a-b)}$; but $\frac{1}{2} \neq \frac{1+2}{2+2}$ (that is $\frac{1}{2} \neq \frac{3}{4}$)

**2. If any two of the three signs of the fraction are changed.** $\frac{a-b}{a} = \frac{-(a-b)}{-a}$ or $\frac{b-a}{-a}$ or $\frac{b-a}{a}$. E: $\frac{2}{-3} = \frac{-2}{+3} = +\frac{-2}{+3}$ or $\frac{-2}{-3}$, etc.

### B. REDUCTION. METHOD 1.

Divide each term of the num and denom by the same quantity (*by Distributive Law*). E: $\frac{5x-15}{5x} = \frac{x-3}{x}$

(by division by 5). E: $\frac{4x-17}{4x}$ is irreducible because 17 cannot be divided exactly by 4. **METHOD 2.** (*Special case of Method I when num and denom are both monomials.*) Cancel any factor common to the num and denom. (*Cancel means divide num and denom by same quantity.*) E: $\frac{5r^2s}{5s^2} = \frac{r^2}{s}$ by cancelling (*dividing by*) 5 and $s$. But in $\frac{5a+7}{5a}$ we cannot cancel because num is not a monomial. E: Reduce $\frac{a^2-25}{4a+20}$

Sol: Factor to convert num and denom to monomials: $\frac{(a+5)(a-5)}{4(a+5)}$; cancel $(a+5)$. Ans: $\frac{a-5}{4}$

### C. MULTIPLICATION.

Factor polynomials whenever possible. Try to cancel; then place product of nums over product of denoms.

E: $\frac{4xy}{x^2-y^2} \times \frac{5(x+y)^2}{4x+4y} = \frac{\overset{1}{\cancel{4}}\overset{1}{\cancel{x}}\overset{1}{\cancel{y}}}{\underset{1}{\cancel{(x+y)}}(x-y)} \times \frac{5\overset{1}{\cancel{(x+y)}}\overset{1}{\cancel{(x+y)}}}{\underset{1}{\cancel{4}}\underset{1}{\cancel{(x+y)}}} = \frac{5xy}{x-y}$

### D. DIVISION.

Invert the divisor (second fraction); then proceed as in multiplication. E: $x \div \frac{x^2}{y^2} = \frac{x}{1} \cdot \frac{y^2}{x^2} = \frac{y^2}{x}$

### E. ADDITION AND SUBTRACTION. 1.

Only like fractions (*with the same denom*) can be combined. Add the nums algebraically and place the result over the denom: **a/c−b/c = (a−b)/c.** E: 2/11 + 7/11 = 9/11. E: $\frac{x}{x-y} + \frac{y}{x-y} = \frac{x+y}{x-y}$. **2.** If denoms are different, first find L.C.D.; then multiply nums and denoms by same quantity to make the denoms equal to L.C.D. E: $\frac{5}{x} + \frac{5}{y}$. Sol: L.C.D. is $xy$:

$\frac{5 \cdot y}{x \cdot y} + \frac{5 \cdot x}{y \cdot x} = \frac{5y}{xy} + \frac{5x}{xy} = \frac{5y+5x}{xy}$. E: $\frac{x-2}{6} - \frac{x+4}{8} = \frac{4x-8}{24} - \frac{3x+12}{24} = \frac{x-20}{24}$. E: $\frac{7}{x-2} + \frac{4}{3(x-2)} = \frac{21}{3(x-2)} + \frac{4}{3(x-2)} = \frac{25}{3(x-2)}$

### F. COMPLEX FRACTION.

A fraction containing one or more fractions in its num or denom, or both. To simplify, find the L.C.D. of all the fractions within the complex fraction. Multiply each term of num and denom by the L.C.D. Reduce if possible.

E: Simplify $\frac{\frac{4}{x} - \frac{1}{5}}{\frac{7}{x^2} + \frac{1}{x} - 3}$

Sol: L.C.D. is $5x^2$. Multiply each term of num and denom by $5x^2$.

$$\frac{\frac{4}{x} \cdot 5x^2 - \frac{1}{5} \cdot 5x^2}{\frac{7}{x^2} \cdot 5x^2 + \frac{1}{x} \cdot 5x^2 - 3 \cdot 5x^2} = \frac{20x-x^2}{35+5x-15x^2} \text{ or } \frac{x(20-x)}{5(7+x-3x^2)}$$

**Alternate method:** Combine the num into a single fraction; then the denom into a separate single fraction. Divide num by denom.

E: $\left(\frac{4}{x} - \frac{1}{5}\right) \div \left(\frac{7}{x^2} + \frac{1}{x} - 3\right)$ Sol: $\frac{20-x}{5x} \cdot \frac{x^2}{7+x-3x^2} = \frac{x(20-x)}{5(7+x-3x^2)}$

## XII. GRAPHS. A. THE AXES.

The **origin** is 0; horizontal line XOX′ is **X-axis**, vertical line YOY′ is **Y-axis.** (Fig. 1)

### B. DIRECTIONS FROM ORIGIN.

To right, $x$ is positive; to left, $x$ negative; up, $y$ positive; down, $y$ negative. Point $A$ has $x$-coordinate +6 (**abscissa**); $y$-coordinate +2 (**ordinate**). Together (+6, +2) are **coordinates** of $A$: (−1, 3) coordinates of $B$; (0−3), of $C$.

### C. GRAPHING STRAIGHT LINES.

An equation that can be put into the form **ax+by = c**, where $a$ and $b$ are not both zero, is a first degree linear equation and has a **straight line graph.** Plot 3 points.

E: (a) $y = 2x-6$. If we let $x = 0$, then $y = -6$; etc. (Fig. 2)

Therefore

| $x$ | 0 | 2 | 5 |
|---|---|---|---|
| $y$ | −6 | −2 | 4 |

(b) $x+2y = 8$

| $x$ | 0 | 8 | 3 |
|---|---|---|---|
| $y$ | 4 | 0 | 2.5 |

(c) $x = -4$ (line parallel to Y-axis and 4 units left of it).

(d) $y = 6$ (line parallel to X-axis and 6 units above it).

**NOTE: 1.** If a pt. lies on a line, its coords will satisfy the equation of the line. E: Pt. (1, −4) is on line (a). If we substitute 1 for $x$ and −4 for $y$ in $y = 2x-6$, we get $-4 = 2(1)-6$ or $-4 = -4$. **2.** If the coords of a pt. satisfy an equation, that pt. will lie on the graph of the equation. E: Since $x = 2$, $y = 3$ satisfies equation (**b**), the pt. (2, 3) will lie on line (**b**).

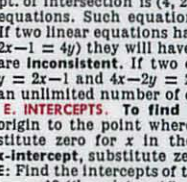


Fig. 1

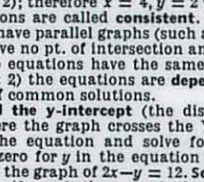


Fig. 2

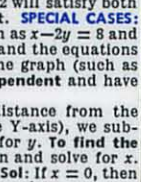


Fig. 3

### D. A SYSTEM OF LINEAR EQUATIONS.

A pair of equations containing two unknowns are called **simultaneous equations** if there is only one set of values of the unknowns that satisfies both. **To solve graphically,** draw both graphs on the same set of axes and note the pt. of intersection. E: $y = 2x-6$, $x+2y = 8$. **Sol:** (See Fig. 2) The pt. of intersection is (4, 2); therefore $x = 4$, $y = 2$ will satisfy both equations. Such equations are called **consistent. SPECIAL CASES:** If two linear equations have parallel graphs (such as $x-2y = 8$ and $2x-1 = 4y$) they will have no pt. of intersection and the equations are **inconsistent.** If two equations have the same graph (such as $y = 2x-1$ and $4x-2y = 2$) the equations are **dependent** and have an unlimited number of common solutions.

### E. INTERCEPTS.

**To find the y-intercept** (the distance from the origin to the point where the graph crosses the Y-axis), we substitute zero for $x$ in the equation and solve for $y$. **To find the x-intercept,** substitute zero for $y$ in the equation and solve for $x$. E: Find the intercepts of the graph of $2x-y = 12$. **Sol:** If $x = 0$, then $y = -12$ (*the y-intercept*); if $y = 0$, then $x = 6$ (*the x-intercept*).

### F. SLOPE

is a measure of steepness. A **positive** slope means the line **rises** to the right (↗); if a **negative** slope, the line **falls** to the right (↘). E: (Fig. 3) Line (**f**) has a + slope; line (**g**) has a − slope. Between any two points on a straight line, slope **m = (change in y) ÷(change in x) = Δy/Δx.** Slope of line (**f**) between points $t$ and $n$ is $m = \Delta y/\Delta x = 6/3 = 2$; between $n$ and $r$, is $m = 2/1 = 2$. The slope of a straight line is constant. Slope of line (**g**), between pts. $v$ and $p$ is $m = \Delta y/\Delta x = -2/3$. Slope of a straight line can also be found from its equation. If a linear equation is written in the form **y = mx+b, m** is the **slope** and **b** is the **y-intercept** of the line. E: (Fig. 3) Line (**f**) has the equation $y = 2x-3$; slope = 2, and $y$-intercept (found at pt. **t**) = −3. E: Find slope and $y$-intercept of line (**g**), equation $3y+2x = 11$. **Sol:** Solve equation for $y$: $y = -2/3\,x + 3\,2/3$. Then $m = -2/3$ and $b = 3\,2/3$.

## XIII. SOLVING A SYSTEM OF LINEAR EQUATIONS ALGEBRAICALLY.

(See XII,D.) **A. ADDITION OR SUBTRACTION. 1.** By proper multiplication, make the absolute value of the coeffs. of one unknown the same in both equations. **2.** Add or subtract to eliminate that unknown, and solve resulting equation for other unknown. **3.** Use this value in either original equation to solve for remaining unknown. **4.** Check in *both* original equations.

E: *Solve*: $2x+5y = -7$ and $3x-2y = 18$. **Sol:**

(a) $2x+5y = -7$ (mult. by 2)
(b) $3x-2y = 18$ (mult. by 5)
(c) $4x+10y = -14$
(d) $15x-10y = 90$

Add (c) and (d): $19x = 76$, then $x = 4$
Substitute 4 for $x$ in (a): $8+5y = -7$
Solv'g gives $y = -3$. Ans: $x = 4$, $y = -3$
Check these values in both (a) and (b)

E: *Solve* (a) $x/3 - y = 1/2$ and (b) $6x-2 = -3(1-y)$. **Sol:** Multiply both sides of (a) by 6; then (c) $2x-6y = 3$. In (b), remove ( ), $6x-2 = -3+3y$; this gives (d) $6x-3y = -1$. Now multiply (**d**) by 2; subtract from (c), etc. Ans: $x = -1/2$, $y = -2/3$.

### B. BY SUBSTITUTION. 1.

Solve either equation for one of the unknowns (giving the *substitution equation*). **2.** Substitute this expression for that unknown in the other equation. **3.** Solve and then apply this value to the *sub. eq.* to find value of other unknown. **4.** Check in *both* original equations. E: *Solve* (a) $5x = 3y-6$, and (b) $2x+y = 13$. **Sol:** *Solve* (**b**) for $y$; then $y = 13-2x$; subst. $13-2x$ for $y$ in (a): $5x = 3(13-2x)-6$. *Solve*: $x = 3$. Subst. 3 for $x$ in $y = 13-2x$ and get $y = 13-2(3) = 7$. Ans: $x = 3$, $y = 7$.

## XIV. PROBLEMS LEADING TO SIMULTANEOUS LINEAR EQUATIONS.

### A. BUSINESS.

E: A football coach bought 5 footballs and 4 helmets for $57. At the same prices, another coach bought 3 footballs and 8 helmets for $65. Find the cost of each football and each helmet. **Sol:** Let $x$ = $ cost of 1 football; let $y$ = $ cost of 1 helmet.

When we have 2 unknowns, we must write 2 equations.)

$\begin{cases} 5x+4y = 57 \\ 3x+8y = 65 \end{cases}$ *Solve*: $x$ = $7 for each football; $y$ = $5.50 for each helmet.

### B. STAMP; MIXTURE; COIN.

E: Forty stamps, some 5¢ stamps and the rest 10¢ stamps, cost $2.85. How many stamps of each kind were bought? **Sol:** $x$ = no. of 5¢ stamps; $y$ = no. of 10¢ stamps. Then

$\begin{cases} x+y = 40 \\ .05x+.10y = 2.85 \end{cases}$ Multiply 2nd equation by 100 to clear decimals.

*Solve*: $x = 23$ (5¢ stamps), $y = 17$ (10¢ stamps.)

### C. INVESTMENT.

E: $4000 is invested, part in 7% stocks and the remainder in 3% bonds, to yield a total annual income of $220. How much is invested in each? **Sol:**

| | $ Principal | × Rate of Interest | = $ Interest or $ Income |
|---|---|---|---|
| STOCKS | x | .07 | .07x |
| BONDS | y | .03 | .03y |

$x+y = 4000$
$.07x+.03y = 220$

*Solve*: $x = \$2500$ in stocks; $y = \$1500$ in bonds

### D. DIGIT.

In no. 86, 8 is the tens' digit, 6 is the units' digit.

| | NUMERICAL | LITERAL |
|---|---|---|
| TENS' DIGIT | 8 | t |
| UNITS' DIGIT | 6 | u |
| NUMBER | 86 | 10t+u |
| NUMBER, DIGITS REVERSED | 68 | 10u+t |
| SUM OF DIGITS | 14 | t+u |

E: The sum of the digits of a two digit number is 11. The number formed by reversing the digits exceeds the original number by 45. Find the number. **Sol:** Let t = the tens' digit and u = the units' digit. **The tens' digit + the units' digit = 11:** $t+u = 11$. **The reversed number = the original number +45:** $10u+t = 10t+u+45$. *Solve*: $t = 3$, and $u = 8$; the original number is 38.

## XV. SQUARE ROOTS AND RADICALS. A. DEFINITIONS.

The square root of a number is one of two equal factors which when multiplied together give the number. The square root of 16 is +4 because (+4)(+4) = 16, or is −4 because (−4)(−4) = 16. The **principal square root** is the positive square root and is always used in evaluating expressions. Thus, $\sqrt{16} = 4$. The number under the radical sign is the **radicand. NOTE: A rational number** is one that can be put into the form **a/b,** where **a** and **b** are **integers.** E: 3/5, −2 (equals −2/1, 3.7 (equals 37/10), $\sqrt{9} = 3/1$. An **irrational number** cannot be expressed as a **quotient** of two integers. E: $\pi$, $\sqrt{2}$, $\sqrt{3}$.

### B. COMPUTING THE SQUARE ROOT OF A NUMBER.

E: Find $\sqrt{123201}$. **Sol:** *Group the radicand in intervals of two digits, left and right from the decimal point.* The largest perfect square contained in first interval 12 is 9. Place $\sqrt{9}$ or 3 in answer above the 12 and subtract 9 from 12. *Bring down the second interval* (32). *A trial divisor is found* by doubling the 3 in the answer and affixing a zero, giving 60. Divide 60 into 332. The quotient 5 is placed as the second digit in the answer. The complete divisor becomes 60+5 or 65; 65×5 = 325, which is subtracted from 332. *Bring down the next interval* (01). *The next trial divisor* is 35×2 or 70. Affix a zero, and divide 700 into 701. The quotient is 1 and 701 is multiplied by 1. To check, square 351 to get 123201.

| | 3 | 5 | 1 |
|---|---|---|---|
| $\sqrt{12'}$ | 32' | 01 | |
| 9 | | | |
| 60 | 3 | 32 | |
| 65 | 3 | 25 | |
| 700 | 7 | 01 | |
| 701 | 7 | 01 | |

E: Find $\sqrt{7}$, correct to the nearest tenth. **Sol:** Since this is irrational, we can approximate the answer to two decimal places by annexing four zeros after the decimal pt. of the radicand. Put a decimal pt. in the answer directly above the decimal pt. of the radicand. Proceed as in the example above. Round off the answer to the nearest tenth. To check, square 2.64 and add remainder .0304 to get 7. Ans: 2.6

| | 2. | 6 | 4 |
|---|---|---|---|
| $\sqrt{7.}$ | 00' | 00 | |
| 4 | | | |
| 40 | 3 | 00 | |
| 46 | 2 | 76 | |
| 520 | 24 | 00 | |
| 524 | 20 | 96 | |
| | 3 | 04 | |

### C. SIMPLIFICATION.

By the extended distributive law: $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$. Factor the radicand so that one of the factors is the largest perfect square contained in the radicand; take that square root from the radical. E: $\sqrt{18} = \sqrt{9} \cdot \sqrt{2} = 3\sqrt{2}$. E: $\sqrt{72} = \sqrt{9} \cdot \sqrt{8} = 3\sqrt{8} = 3\sqrt{4} \cdot \sqrt{2} = 3 \cdot 2\sqrt{2} = 6\sqrt{2}$. Better method: $\sqrt{72} = \sqrt{36} \cdot \sqrt{2} = 6\sqrt{2}$. E: $\sqrt{16a^2b^7} = \sqrt{16a^2b^6} \cdot \sqrt{b} = 4ab^3 \cdot \sqrt{b}$

### D. ADDITION AND SUBTRACTION.

Only like radicals (*with identical radicands*) can be combined. Use the like radical unchanged in the answer and combine the coefficients. E: $7\sqrt{5} + 3\sqrt{5} = 10\sqrt{5}$. E: $\sqrt{6} + \sqrt{5}$ cannot be combined. **NOTE:** Some unlike radicals become like radicals upon simplification and then can be combined. E: $\sqrt{45} - \sqrt{12} - 7\sqrt{5} = \sqrt{9} \cdot \sqrt{5} - \sqrt{4} \cdot \sqrt{3} - 7\sqrt{5} = 3\sqrt{5} - 2\sqrt{3} - 7\sqrt{5} = -4\sqrt{5} - 2\sqrt{3}$

### E. MULTIPLICATION.

$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$. E: $\sqrt{5} \cdot \sqrt{7} = \sqrt{35}$; $\sqrt{3} \cdot \sqrt{3} = \sqrt{9} = 3$; $\sqrt{37} \cdot \sqrt{37} = 37$; $\sqrt{x} \cdot \sqrt{x} = x$ **NOTE:** The coefs. of radicals are multiplied separately. E: $3\sqrt{5} \cdot 7\sqrt{2} = 21\sqrt{10}$. **MONOMIAL × POLYNOMIAL:** E: $\sqrt{3}(2\sqrt{5} + \sqrt{7} - 6\sqrt{3}) = 2\sqrt{15} + \sqrt{21} - 6 \cdot 3 = 2\sqrt{15} + \sqrt{21} - 18$. **BINOMIAL × BINOMIAL** (Use FOIL method of section X, B): $(2+\sqrt{5})(9-\sqrt{5}) = 18 + 7\sqrt{5} - 5 = 13 + 7\sqrt{5}$.

### F. DIVISION.

$\sqrt{a} \div \sqrt{b} = \sqrt{a/b}$. E: $\sqrt{15} \div \sqrt{3} = \sqrt{5}$. The coefs. are divided separately. E: $27\sqrt{24} \div 3\sqrt{6} = 9\sqrt{4} = 9 \cdot 2 = 18$. E: $(\sqrt{21} + 5\sqrt{33}) \div (\sqrt{3}) = \sqrt{7} + 5\sqrt{11}$

### G. RATIONALIZING THE DENOMINATOR.

Multiply num and denom by the smallest radical that will lead to a perfect square radicand in the denom.

E: $\frac{7}{\sqrt{2}} = \frac{7 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{7\sqrt{2}}{2}$; E: $\frac{\sqrt{3}}{\sqrt{5}} = \frac{\sqrt{3} \cdot \sqrt{5}}{\sqrt{5} \cdot \sqrt{5}} = \frac{\sqrt{15}}{5}$; E: $\frac{8}{\sqrt{12}} = \frac{8 \times \sqrt{3}}{\sqrt{12} \times \sqrt{3}} = \frac{8\sqrt{3}}{\sqrt{36}} = \frac{8\sqrt{3}}{6} = \frac{4\sqrt{3}}{3}$; E: Evaluate $\frac{6}{\sqrt{2}}$ to nearest hundredth. **Sol:** $\frac{6 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2} = 3(1.414) = 4.242 = 4.24$

### H. SIMPLIFYING THE SQUARE ROOT OF A FRACTION.

E: $\sqrt{\frac{3}{7}} = \frac{\sqrt{3} \cdot \sqrt{7}}{\sqrt{7} \cdot \sqrt{7}} = \frac{\sqrt{21}}{7}$; E: $10\sqrt{\frac{1}{2}} = \frac{10\sqrt{1} \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2}$

### I. RADICAL EQUATIONS. 1.

Transpose, if necessary, to isolate radical term. **2.** Square both members of the equation. **3.** Solve. **4.** Check root(s) in original equation. E: **Solve for y:** $\sqrt{5y} = 10$. **Sol:** Square both sides, $5y = 100$ (because $\sqrt{5y} \cdot \sqrt{5y} = 5y$), then $y = 20$. E: **Solve for x:** $2\sqrt{x-1} + 5 = 8$. Transpose the +5 to isolate the radical term $2\sqrt{x-1} = 3$. Square both sides: $(2\sqrt{x-1})^2 = (3)^2$; $4(x-1) = 9$. Then: $4x-4 = 9$, $4x = 13$, $x = 13/4$. E: **Solve for A:** $r = \sqrt{A/\pi}$. **Sol:** $r^2 = A/\pi$, then $A = \pi r^2$.

## XVI. QUADRATIC (2ND DEGREE) EQUATIONS.

Represented by $ax^2+bx+c = 0$ where **a**, **b** and **c** can be any numbers, except **a** is not 0. Every second degree equation has **two roots.**

### A. FACTORABLE COMPLETE QUADRATIC.

(Where **a** and **b** are not 0, but **c** may be 0.) **1.** Transpose if necessary to make one member equal to 0. **2.** Factor. **3.** Set each factor equal to 0, because if $(m)(n) = 0$, then either $m = 0$ or $n = 0$, or both $= 0$. **4.** Solve each first degree equation; each gives a root. E: **Solve for x:** $x^2+10 = -7x$. **Sol: 1.** Transpose $-7x$: $x^2+7x+10 = 0$. **2.** Factor: $(x+5)(x+2) = 0$. **3.** Set each factor equal to 0: $x+5 = 0$ or $x+2 = 0$. **4.** Solve each: $x = -5$, or $x = -2$. E: **Solve for s:** $s^2 = 6s$. **Sol:** $s^2-6s = 0$; $s(s-6) = 0$; $s = 0$ or $s-6 = 0$. Roots: $s = 0$ or $s = 6$.

### B. PURE QUADRATIC.

(The middle coeff. **b** is 0.) **METHOD 1.** E: **Solve for x:** $x^2-9 = 0$. **Sol:** Transpose the −9: $x^2 = 9$. Extract sq. root of each side: $x = \pm 3$ (meaning +3 or −3). E: **Solve for w:** $w^2+4w^2 = 30$. **Sol:** $5w^2 = 30$; $w^2 = 6$; $w = \pm\sqrt{6}$ or $\pm 2.449$. **METHOD 2.** (By factoring, if possible.) E: **Solve for x:** $x^2-9 = 0$. **Sol:** $(x+3)(x-3) = 0$; $x+3 = 0$ or $x-3 = 0$; $x = -3$ or $x = 3$.

### C. QUADRATIC FORMULA.

By comparing given equation with $ax^2+bx+c = 0$, obtain values for **a**, **b** and **c**; substitute into formula $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ E: **Solve for x, and leave answers in radical form:** $3x^2-7x+1 = 0$; therefore $a = 3$, $b = -7$, $c = 1$.

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-(-7) \pm \sqrt{(-7)^2-4(3)(1)}}{2(3)} = \frac{7 \pm \sqrt{49-12}}{6} = \frac{7 \pm \sqrt{37}}{6}$. Roots are $\frac{7+\sqrt{37}}{6}$ and $\frac{7-\sqrt{37}}{6}$.

## XVII. PROBLEMS FORMING QUADRATIC EQUATIONS. A. NUMBER.

E: Find two consecutive even integers, the sum of whose squares is 52. **Sol:** $x$ = 1st consecutive even integer and $x+2$ = 2nd integer. Then $x^2+(x+2)^2 = 52$; $x^2+x^2+4x+4 = 52$; $2x^2+4x-48 = 0$. Divide each member by 2: $x^2+2x-24 = 0$; $(x+6)(x-4) = 0$.

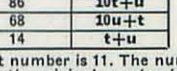
Therefore

| $x = -6$ | $x = 4$ |
|---|---|
| $x+2 = -4$ | $x+2 = 6$ |

(2 sets of answers)

### B. AREA.

E: The length of a rectangle is 7 ft. greater than its width. Find the dimensions if the area is 78 sq. ft. **Sol:** Let $x$ = width in ft. and $x+7$ = length in ft. Then $x(x+7) = 78$; $x^2+7x-78 = 0$; $(x+13)(x-6) = 0$; $x = -13$ ft. or $x = 6$ ft.; We reject the answer −13 ft.; therefore width $x = 6$ ft., length $x+7 = 13$ ft.




## XVIII. RATIO. PROPORTION. VARIATION. A. RATIO

is the quotient of two numbers of the same kind. E: Find the ratio of 4 ft. to 2 yd. **Sol:** 4 ft. to 6 ft. gives ratio 2 : 3 or 2/3. E: Two numbers are in the ratio 5 : 2. If their difference is 12, find them. **Sol:** Let 5x=1st number, and 2x=2nd number. **1st no.−2nd no.=12.** Then $5x-2x = 12$; $3x = 12$; $x = 4$. Therefore, numbers are $5x = 20$ and $2x = 8$.

### B. PROPORTIONS.

A proportion states that two **ratios are equal:** $\frac{12}{18} = \frac{16}{24}$ *or* 12:18 = 16:24, read 12 is to 18 as 16 is to 24. The first and last numbers, 12 and 24, are the **extremes**; the second and third numbers, 18 and 16 are the **means.** In fractional form, $\frac{\text{extreme}}{\text{mean}} = \frac{\text{mean}}{\text{extreme}}$; in horizontal form, **extreme: mean = mean: extreme.** In any proportion, product of the means equals product of the extremes. From above $12 \times 24 = 18 \times 16$; each product is 288. E: **Solve for x:** $2/x = 5/18$ or $2 : x = 5 : 18$. **Sol:** $5x = 36$; $x = 7\,1/5$. E: **Solve for b:** $3 : (b+1) = 4 : (3b-7)$. **Sol:** $3 \cdot (3b-7) = 4 \cdot (b+1)$ because prod. of means = prod. of ext.; $9b-21 = 4b+4$; $b = 5$. E: If a car needs 10 gal. of gasoline for 192 mi, at the same rate, how many gal. would it need for 480 mi.? **Sol:** Let $x$ = no. of gal. needed for 480 mi. The ratio of the number of gal. to the number of miles is constant.

| | 1st case | 2nd case |
|---|---|---|
| GAL. | 10 | x |
| MI. | 192 | 480 |

$10/192 = x/480$; $192x = 4800$; $x = 25$ gal.

### C. DIRECT VARIATION.

If two quantities **y** and **x** vary so that their ratio is constant, **y/x = k** (where **k** is called the constant of variation), they vary directly or are directly proportional. When **x** increases, **y** increases; when **x** decreases, **y** decreases. (Graph is a straight line through origin.)

E: A varies directly as b. If A = 48, when b = 15, find A when b = 35. **Sol:** (Method 1) $\frac{A}{b} = k$; $\frac{48}{15} = k$; $k = \frac{16}{5}$; thus $\frac{A}{b} = \frac{16}{5}$ and $A = 112$ when $b = 35$. **Sol** (Method 2): The ratio of A and b is constant; $A_1/b_1 = A_2/b_2$; $48/15 = A_2/35$; $A_2 = 112$.

### D. INVERSE VARIATION.

If two quantities, $x$ and $y$, vary so that their product is constant (**xy = k**), they vary inversely or are inversely proportional. When **x** increases, **y** decreases; when **x** is doubled, **y** is halved, etc. (Graph is curved line.) E: $R$ varies inversely as $T$. If $R = 30$ when $T = 6$, find $T$ when $R = 40$. **Sol:** Prod. of $R$ and $T$ is constant; $R_1T_1 = R_2T_2$; $(30)(6) = (40)(T)$; $180 = 40T$; $T = 4\,1/2$.

## XIX. GEOMETRY. A. SUPPLEMENTARY, COMPLEMENTARY ANGLES.

If two angles total **180°**, they are **supplementary** ($\angle x$ and $\angle y$); if they total **90°**, they are **complementary** ($\angle v$ and $\angle w$). (Fig. 4) E: Find the sup. of 57°. **Sol:** 180°−57° = 123°. E: The larger of two comp. angles contains 38° more than the smaller. Find the number of degrees in each. **Sol:** Let x = smaller $\angle$ and x+38 = larger $\angle$. Then x+x+38 = 90; smaller $\angle$ x = 26°; larger $\angle$ x+38° = 64°.

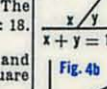
### B. RIGHT TRIANGLES AND PYTHAGOREAN THEOREM.

(See Fig. 5a) A right triangle (rt. $\triangle$ $ABC$) contains one right angle ($\angle C$) and two complementary acute angles ($\angle A$ and $\angle B$). The **hypotenuse** $c$, lies opposite the rt. $\angle C$. Side $a$, (always opp. $\angle A$), and side $b$ (always opp. $\angle B$) are the **legs. PYTHAGOREAN THEOREM: In a right triangle, the sum of the squares of the legs is equal to the square of the hypotenuse:** $a^2+b^2 = c^2$.

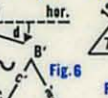
E: Find the hypotenuse of a right triangle whose legs are 6" and 8". **Sol:** $a^2+b^2 = c^2$; $6^2+8^2 = c^2$; $36+64 = c^2$; $100 = c^2$; $c = 10$". E: Find altitude of a rectangle whose base is 2" and diagonal is 3". **Sol:** $a^2+b^2 = c^2$; $a^2+2^2 = 3^2$; $a^2+4 = 9$; $a^2 = 5$; $a = \sqrt{5}$ or 2.2". (Fig. 5b)

### C. SIMILAR TRIANGLES.

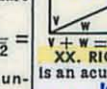
**If two triangles are similar, their corr. angles are equal and their corr. sides are in proportion:** $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, $a/a' = b/b' = c/c'$. (Fig. 6) To prove that two triangles are similar, show that two pairs of angles are equal. E: In similar triangles $ABC$ and $A'B'C'$, $a = 12$, $a' = 8$, $b = 21$. Find $b'$. **Sol:** $a : a' = b : b'$, $12 : 8 = 21 : b'$, $12b' = 168$, $b' = 14$.

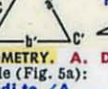











## XX. RIGHT TRIANGLE TRIGONOMETRY. A. DEFINITIONS.

If $A$ is an acute angle of a right triangle (Fig. 5a):

$\sin A = \frac{\text{leg opp } \angle A}{\text{hypotenuse}}$; $\cos A = \frac{\text{leg adj to } \angle A}{\text{hypotenuse}}$; $\tan A = \frac{\text{leg opp } \angle A}{\text{leg adj to } \angle A}$

### B. ANGLE OF ELEVATION (*DEPRESSION*).

The angle between the horizontal and the line of sight when an object is higher (*lower*) than the observer. (Fig. 7). **C. EXAMPLES.**

E: How far (to nearest ft.) from the foot of a building should the foot of a 12' ladder be placed so that it makes an angle of 76° with the ground? (Fig. 8). **Sol:** From the 76° angle, x is adj. leg and 12' is the hyp.; $\cos 76° = \text{adj. leg} \div \text{hyp.}$; $.2419 = x/12$; $x = 3'$ to nearest ft.

E: (Fig. 9) From the top of a lighthouse, an observer 100' above sea level finds the angle of depression of a boat at sea is 16°. Find the distance from the boat to the foot of the lighthouse. **Sol:** To find a usable angle inside the rt. triangle, 90°−16° = 74°. From 74° angle, x is opp. leg, 100 is adj. leg; thus $\tan 74° = x/100$; $x = 349'$

E: (Fig. 10) Find to the nearest degree, the angle of elevation of the sun when a 16' vertical pole casts a 24' shadow. **Sol:** From angle x, 16 is opp. leg, 24 is adj. leg; thus $\tan x = 16/24 = 2/3 = .6667$; $x = 34°$.

DATA-GUIDE QUICK CHART

# 11TH YEAR INTERMEDIATE ALGEBRA

A SIMPLIFIED SUMMARY & INSTANT REFERENCE

LEARN & REVIEW IN MINUTES

AUTHOR: M. Cohen, Instructor, Bayside H.S. and Queens College, N. Y.
Consultant: W. Manheimer, Chairman, Franklin K. Lane H.S., N.Y.

SOLID PLASTIC

QUICK CHARTS SPEED STUDY: PUT THE IMPORTANT FACTS IN FULL VIEW TO SAVE YOU TIME AND WORK.

QUICK CHARTS HELP YOU REMEMBER: PERMANENTLY FILL IN FOR MISSING NOTES & TEXTS IN BASIC & ADVANCED COURSES.

CONTAINS NO PAPER

## I

| BASIC LAWS | ADDITION | SUBTRACTION | MULTIPLICATION | DIVISION |
|---|---|---|---|---|
| A. COMMUTATIVE (Change of order) | $a+b=b+a$<br>$5+3=3+5$ | (Non Commutative)<br>$a-b\neq b-a$<br>$5-3\neq 3-5$ | $ab=ba$<br>$(8)(7)=(7)(8)$ | (Non Commutative)<br>$\frac{a}{b}\neq\frac{b}{a}$, $\frac{12}{4}\neq\frac{4}{12}$ |
| B. ASSOCIATIVE (Change of pairing) | $a+b+c=a+(b+c)=(a+b)+c$<br>$7+(2+6)=(7+2)+6$, or $7+8=9+6$ | (Non Associative)<br>$a-(b-c)\neq(a-b)-c$<br>$12-(9-2)\neq(12-9)-2$, or $12-7\neq 3-2$ | $abc=a(bc)=(ab)c$<br>$3(2\times4)=(3\times2)\times4$, or $3(8)=(6)4$ | (Non Associative)<br>$a\div(b\div c)\neq(a\div b)\div c$<br>$128\div(8\div4)\neq(128\div8)\div4$, or $128\div2\neq16\div4$ |
| C. DISTRIBUTIVE (Multiplication or division of sum or difference) | $a(b+c)=ab+ac$<br>$5(2+7)=5(2)+5(7)$<br>$5(9)=10+35$ | | $(b+c)\div a=(b\div a)+(c\div a)$<br>$\frac{10x+6}{5}=\frac{10x}{5}+\frac{6}{5}=2x+\frac{6}{5}$<br>$\frac{8}{6+2}\neq\frac{8}{6}+\frac{8}{2}$ | |

NOTE: $c-d$ also means $c$ plus $(-d)$, thus $c-d=-d+c$ See *VI:D.*

**D. ORDER OF OPERATIONS.** To evaluate an expression: (1) take powers and roots; (2) multiply and divide (from left to right); (3) add algebraically. First evaluate nums., denoms , quantities inside parens., and quantities under radical signs using steps (1), (2), (3) within each. **E:** Find the value of $-x^2+9x$ when $x=4$.
**Sol:** $-4^2+9\cdot4=-16+36=20$. **E:** Evaluate $\sqrt{12-2(1-r)+7r^2-(2r)^3}$ when $r=3$. **Sol:** $\sqrt{12-2(1-3)+7\cdot3^2-(2\cdot3)^3}=\sqrt{12-2(-2)+7\cdot9-6^3}=\sqrt{12+4+63-216}=4+63-216=-149$

## II

**FRACTIONS. A. TRANSFORMATIONS. 1. MULT. OR DIV.** The appearance of a fraction may be changed without changing its value by multiplying (or dividing) both num and denom by same quantity, except zero. **E:** $\frac{a}{b-2}=\frac{a\cdot3}{(b-2)\cdot3}=\frac{3a}{3b-6}$ **2. CHANGE OF TWO SIGNS.** Any two of the three signs of a fraction (*num, denom, or fraction itself*) may be changed without changing the value of the fraction.
**E:** $\frac{x}{-y}\left(=+\frac{+x}{-y}\right)=\frac{-x}{y}=-\frac{-x}{-y}$. **E:** Rewrite $\frac{a}{c-b}$ so that its denom is b−c. **Sol:** $\frac{a}{c-b}=\frac{a}{(c-b)}=\frac{-a}{-(c-b)}=\frac{-a}{b-c}$ or $-\frac{a}{b-c}$

**B. REDUCTION. 1. BY DIVISION.** Divide each term of the num and denom by the same quantity. **E:** $\frac{6x-9}{6x}=\frac{(6x-9)\div3}{6x\div3}=\frac{2x-3}{2x}$

**2. BY CANCELLATION.** (*Special case of Method 1.*) When a fraction is in form monomial ÷ monomial, any common factor may be cancelled. **E1:** Reduce $\frac{5x+15}{x^2-x-12}$ **Sol:** Factor to get $\frac{\text{monomial}}{\text{monomial}}$: $\frac{5(x+3)}{(x+3)(x-4)}=\frac{5}{x-4}$ **E2:** Reduce $\frac{x^2-9}{12-4x}$. **Sol:** $\frac{(x+3)(x-3)}{4(3-x)}=\frac{-x-3}{4}$

NOTE: When quantities in ( ) are opposite in sign, they may be cancelled provided they are replaced by −1 and 1, or 1 and −1.

**C. MULTIPLICATION.** Factor polynomials wherever possible. Try to cancel; then place product of the nums over product of denoms.
**E:** $\frac{4(x-y)^2}{cx+dx}\cdot\frac{5rd+5rc}{5x-5y}=\frac{4(x-y)^2}{x(c+d)}\cdot\frac{5r(d+c)}{5(x-y)}=\frac{4r(x-y)^2}{x}$

**D. DIVISION.** Invert divisor (second fraction); then multiply.
**E:** $\frac{a-b}{(a+b)^2}\div\frac{a+b}{a^2+b^2}$ **Sol:** $\frac{a-b}{(a+b)^2}\cdot\frac{a^2+b^2}{a+b}=\frac{(a-b)(a^2+b^2)}{(a+b)^3}$

**E. ADDITION, SUBTRACTION.** Only fractions with a common denom can be combined. Place algebraic sum of nums over common denom.
**E1:** $\frac{8}{x-4}+\frac{5}{4-x}$. **Sol:** (Change two signs of second fraction) $\frac{8}{x-4}-\frac{5}{x-4}=\frac{3}{x-4}$
**E2:** $\frac{x-3}{4}-\frac{x+1}{6}$. **Sol:** L.C.D. = 12; thus $\frac{3x-9}{12}-\frac{2x+2}{12}=\frac{(3x-9)-(2x+2)}{12}=\frac{x-11}{12}$
**E3:** $\frac{c}{ab}+\frac{b}{ac}=\frac{c^2}{abc}+\frac{b^2}{abc}=\frac{c^2+b^2}{abc}$
**E4:** $\frac{x}{x^2-4}+\frac{4}{3x+6}$
**Sol:** Factor denoms.
$\frac{x}{(x+2)(x-2)}+\frac{4}{3(x+2)}$,
L.C.D. is $3(x+2)(x-2)$; thus
$\frac{3x}{3(x+2)(x-2)}+\frac{4x-8}{3(x+2)(x-2)}=\frac{7x-8}{3(x+2)(x-2)}$ or $\frac{7x-8}{3x^2-12}$

**F. COMPLEX FRACTION.** Contains one or more fractions inside its num and/or denom. To simplify, multiply each term of the num and denom by the L.C.D. of all the internal denoms.
**E:** $\frac{\frac{1}{5a}-\frac{1}{ab}}{\frac{1}{2b}-\frac{1}{10}}=\frac{10ab\cdot\frac{1}{5a}-10ab\cdot\frac{1}{ab}}{10ab\cdot\frac{1}{2b}-10ab\cdot\frac{1}{10}}=\frac{2b-10}{5a-ab}=\frac{2(b-5)}{a(5-b)}=-\frac{2}{a}$

**ALTERNATE SOLUTION:** Combine fractions in num into a single fraction, those of denom into another single fraction; divide num by denom.
$\frac{\frac{1}{5a}-\frac{1}{ab}}{\frac{1}{2b}-\frac{1}{10}}=\frac{\frac{b}{5ab}-\frac{5}{5ab}}{\frac{5}{10b}-\frac{b}{10b}}=\frac{\frac{b-5}{5ab}}{\frac{5-b}{10b}}=\frac{-1}{a}\cdot\frac{2}{1}=-\frac{2}{a}$

## III

**SOLVING FIRST DEGREE EQUATIONS AND FORMULAS.**

**A. ORDER OF OPERATIONS.** **E:** Solve $\frac{14x+30}{3x-6}=\frac{5x}{x-2}+2$ **Sol:**

1. Factor any polynomial denominators if possible: $\frac{14x+30}{3(x-2)}=\frac{5x}{x-2}+2$
2. Mult. each term of both members by L.C.D., 3(x−2), to clear fractions: $14x+30=15x+2\cdot3(x-2)$ or $14x+30=15x+6(x-2)$
3. Remove parentheses: $14x+30=15x+6x-12$
4. Transpose to isolate unknown terms: $14x-15x-6x=-30-12$
5. Combine terms: $-7x=-42$
6. Divide by coef. of unknown: $x=6$
7. Check by substituting root (6) for x in original equation.

**B. EVALUATING FORMULAS.** Substitute the numerical values given for the letters; then use the proper order of operations (*I:D* or *III:A*).
**E:** $A=P(1+rt)$. Find A if $P=1000$, $r=6\%$, $t=2$. **Sol:** $A=1000(1+.06\cdot2)$; $A=1000(1+.12)=1000(1.12)=1120$.

**C. LITERAL EQUATIONS, TRANSFORMING FORMULAS.** (Use III,A.) **E: Solve for r:** $\frac{6}{c}=\frac{a}{r}-1$. **Sol:**

1. Mult. by L.C.D., cr: $6r=ac-cr$
2. Transpose terms to isolate unknown terms: $6r+cr=ac$
3. Factor to obtain coef. of the unknown: $r(6+c)=ac$
4. Div. both members by coef. of desired unknown: $r=ac/(6+c)$

## IV

**RADICALS. A. DEFINITIONS. 1. RADICAL:** If a quantity "a" is composed of **n equal factors**, any one of them is the **nth root of "a"**, indicated by the **radical** $\sqrt[n]{a}$, where **n** is the **index** and "a" is the **radicand**. The index indicates the **order** of a radical. (When n = 2, it is omitted.) **2. THE PRINCIPAL SQUARE ROOT** of a number is its positive square root and is always used when evaluating. Although the square root of 9 is +3 or −3, with the $\sqrt{\ }$ symbol use only the principal square root; thus $\sqrt{9}=3$. **3. A RATIONAL NUMBER** can be put into the form $a/b$, where $a$ and $b$ are integers. **E:** 2/3, 1.7 = 17/10, $-4=-4/1$, $\sqrt{25}=5/1$, $\sqrt[3]{-8}=-2/1$, .3333 . . . = 1/3.
**4. AN IRRATIONAL NUMBER** cannot be expressed as quotient of two integers. **E:** $\sqrt{2}$, $\sqrt[3]{4}$, $\sqrt{7}$, $\pi$, sin 16°, log 5. **5. A SURD** is an irrational number of the form $\sqrt[n]{a}$, where **a** is rational. **E:** $\sqrt{2}$, $\sqrt[3]{2/5}$
**6. A QUADRATIC SURD** is a surd with index of 2.

**B. MULTIPLICATION OF RADICALS OF SAME ORDER.** $(c\sqrt[n]{a})(d\sqrt[n]{b})=cd\sqrt[n]{ab}$. **E:** $3\sqrt{2}\cdot5\sqrt{7}=15\sqrt{14}$; $\sqrt{3}\cdot7=7\sqrt{3}$; $\sqrt{6}\cdot\sqrt[3]{2/3}=\sqrt[3]{4}$; $\sqrt{17}\cdot\sqrt{17}=17$; $(\sqrt{x})^2=x$; $(\sqrt[3]{a-b})^3=a-b$. **E:** $(4-\sqrt{3})(5+2\sqrt{3})$ [by FOIL method of Elem. Algebra] $=20+3\sqrt{3}-2\cdot3=14+3\sqrt{3}$.

**C. DIVISION OF RADICALS OF THE SAME ORDER.** $\frac{c\sqrt[n]{a}}{d\sqrt[n]{b}}=\frac{c}{d}\sqrt[n]{\frac{a}{b}}$
**E:** $10\sqrt{21}\div2\sqrt{7}=5\sqrt{3}$; $\sqrt{10}\div2=\sqrt{10}/2$; $\sqrt[3]{32x^5}\div\sqrt[3]{4x^2}=\sqrt[3]{8x^3}=2x$

**D. SIMPLIFICATION BY REMOVING A RATIONAL FACTOR.** **E1:** $\sqrt{18}=\sqrt{9}\cdot\sqrt{2}=3\sqrt{2}$. **E2:** $6\sqrt{28x^3}=6\sqrt{4x^2}\cdot\sqrt{7x}=6\cdot2x\cdot\sqrt{7x}=12x\sqrt{7x}$. **E3.** $\sqrt[3]{24x^3}=\sqrt[3]{8x^3}\cdot\sqrt[3]{3}=2x\cdot\sqrt[3]{3}$.

**E. RATIONALIZING A MONOMIAL DENOMINATOR.** Multiply num and denom by smallest quantity which will make denom rational.
**E1:** $\frac{10}{\sqrt{18}}=\frac{10\cdot\sqrt{2}}{\sqrt{18}\cdot\sqrt{2}}=\frac{10\sqrt{2}}{\sqrt{36}}=\frac{10\sqrt{2}}{6}=\frac{5\sqrt{2}}{3}$ **E2:** $\frac{a}{\sqrt[4]{2}}=\frac{a\cdot\sqrt[4]{4}}{\sqrt[4]{2}\cdot\sqrt[4]{4}}=\frac{a\sqrt[4]{4}}{\sqrt[4]{8}}=\frac{a\sqrt[4]{4}}{2}$ **E3:** $\frac{\sqrt{7}-\sqrt{6}}{\sqrt{2}}=\frac{(\sqrt{7}-\sqrt{6})\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}}=\frac{\sqrt{14}-\sqrt{12}}{2}=\frac{\sqrt{14}-2\sqrt{3}}{2}$

**F. SIMPLIFYING A FRACTIONAL RADICAND.** By *Extended Distributive Law* (VI:D), separate into two radicals; then rationalize denom.
**E1:** $\sqrt{\frac{2}{3}}=\frac{\sqrt{2}}{\sqrt{3}}=\frac{\sqrt{2}\cdot\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}}=\frac{\sqrt{6}}{3}$ **E2:** $\sqrt[3]{\frac{c}{d}}=\frac{\sqrt[3]{c}\cdot\sqrt[3]{d^2}}{\sqrt[3]{d}\cdot\sqrt[3]{d^2}}=\frac{\sqrt[3]{cd^2}}{\sqrt[3]{d^3}}=\frac{\sqrt[3]{cd^2}}{d}$

**G. RATIONALIZING A BINOMIAL QUADRATIC SURD DENOMINATOR.** Conjugate binomial surds differ only in their middle signs and their product is rational: $(a\sqrt{x}+b\sqrt{y})(a\sqrt{x}-b\sqrt{y})=a^2x-b^2y$. To rationalize, multiply num and denom by the conjugate of the denom.
**E:** $\frac{12}{4-\sqrt{7}}=\frac{12(4+\sqrt{7})}{(4-\sqrt{7})(4+\sqrt{7})}=\frac{12(4+\sqrt{7})}{(16-7)\text{ or }9}=\frac{4(4+\sqrt{7})}{3}$ or $\frac{16+4\sqrt{7}}{3}$

**H. ADDITION AND SUBTRACTION.** Only like radicals (with same *index* and same *radicand*) can be combined. Use the *"like radical"* unchanged in the answer and combine the coefficients.
**E1:** $6\sqrt[3]{x}+2\sqrt[3]{x}-\sqrt[3]{x}=7\sqrt[3]{x}$ **E2:** $\sqrt{12}-\frac{15}{\sqrt{5}}-\sqrt{\frac{1}{3}}$ when simplified becomes $2\sqrt{3}-3\sqrt{5}-\frac{1}{3}\sqrt{3}=\frac{5}{3}\sqrt{3}-3\sqrt{5}$

**I. RADICAL EQUATIONS.** Contain the unknown under a radical sign. 1. Isolate radical term if possible. 2. Raise both members to the power equal to the index. 3. Solve resulting equation.
**E1:** $4+2\sqrt{x}=10$
$2\sqrt{x}=6$ ← Isolate radical term
$(2\sqrt{x})^2=(6)^2$ ← Raise to power
$4x=36$ ← Solve
$x=9$ (Check ans. in orig. equation)
**E2:** $\sqrt[3]{a-2}+3=0$ → $\sqrt[3]{a-2}=-3$ → $(\sqrt[3]{a-2})^3=(-3)^3$ → $a-2=-27$ → $a=-25$

## V

**IMAGINARY NUMBERS. A. DEFINITIONS. 1.** An imaginary number is the indicated *even root* of a negative number. **E:** $\sqrt{-25}$, $\sqrt{-3}$, $\sqrt[4]{-16}$, $\sqrt[6]{-11}$; but $\sqrt[3]{-8}$ is real, since it equals −2. **2.** The imaginary unit is $i=\sqrt{-1}$ and $i^2=-1$. **3.** A complex number has the form $a+bi$; where $a$, $b$, are real no's. **E:** $3+2i$; $x-i\sqrt{2}$. **B. CHANGING $\sqrt{-a}$ TO "I" FORM.** Remove the factor $\sqrt{-1}$, then simplify $\sqrt{a}$ if possible.
**E:** $\sqrt{-9}=\sqrt{9}\sqrt{-1}=3i$. **E:** $\sqrt{-12}=\sqrt{12}\sqrt{-1}=\sqrt{4}\sqrt{3}\sqrt{-1}=2i\sqrt{3}$. **C. ADDITION, SUBTRACTION.** Change to "I" form; combine like terms algebraically. **E:** $\sqrt{-49}-\sqrt{-9}=7i-3i=4i$. **E:** $\sqrt{25}+\sqrt{-16}=5+4i$. **D. MULTIPLICATION.** **E:** $(5-i)(2+6i)=10+28i-6i^2=10+28i-6(-1)=10+28i+6=16+28i$

## VI

**EXPONENTS. A. DEFINITIONS.** $x^m$, where **m** is a positive integer, is read the **m**th power of x and is equal to $x\cdot x\cdot x\ldots x$ ($m$ factors). **m is the exponent and x is the base.** **E:** $x^3=x\cdot x\cdot x$; $7^4=7\cdot7\cdot7\cdot7$

**B. MULTIPLICATION AND DIVISION LAWS.** If bases are the same:
**1.** $x^m\cdot x^n=x^{m+n}$. **E:** $x^5\cdot x^6=x^{11}$; $b^2a\cdot b^2a=b^{2a}$; $c^m\cdot c=c^{m+1}$; $y^r\cdot z^3=yrz^3$; $10^4\cdot10^5=10^9$. **2.** $x^m\div x^n=x^{m-n}$, $m>n$. **E:** $a^8\div a^2=a^6$; $x^{3a-2}\div x^a=x^{2a-2}$; $7^y/7=7^{y-1}$. **E:** Factor $x^{5a+1}-x$ **Sol:** $x(x^{5a}-1)$

**C. POWER OF A POWER.** $(x^m)^n=x^{mn}$. **E:** $(x^5)^3=x^{15}$; $(a^3)^{4x}=a^{12x}$

**D. EXTENDED DISTRIBUTIVE LAW FOR POWERS AND ROOTS.** To obtain the power (*or root*) of an expression with mult. and div. signs only, raise to that power (*or extract that root from*) each quantity:
$(xy)^m=x^my^m$; $\left(\frac{x}{y}\right)^m=\frac{x^m}{y^m}$; $\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$; $\sqrt[n]{ab}=\sqrt[n]{a}\cdot\sqrt[n]{b}$
**E:** $(5a)^2=5^2a^2=25a^2$; $\sqrt{9x^6}=\sqrt{9}\cdot\sqrt{x^6}=3x^3$;
$(b^2c^3)^m=(b^2)^m\cdot(c^3)^m=b^{2m}c^{3m}$; $\sqrt[3]{\frac{10x^{12}}{27}}=\frac{\sqrt[3]{10}\sqrt[3]{x^{12}}}{\sqrt[3]{27}}=\frac{x^4\sqrt[3]{10}}{3}$

NOTE: $\sqrt{x^2+9}\neq x+3$; $(5-a)^2\neq25+a^2$, $(5-a)^2=25-10a+a^2$

**E. SPECIAL EXPONENTS**

| | | EXAMPLES |
|---|---|---|
| ZERO | $x^0=1$, except when x=0. ($0^0$ is indeterminate) | $6^0=1$; $(2x)^0=1$<br>$2x^0=2\cdot1=2$ |
| FRACTIONAL | $x^{\frac{1}{n}}=\sqrt[n]{x}$; $x^{\frac{m}{n}}=\sqrt[n]{x^m}$ or $(\sqrt[n]{x})^m$ | $7^{1/2}=\sqrt{7}$; $(-8)^{1/3}=\sqrt[3]{-8}=-2$<br>$25^{3/2}=(\sqrt{25})^3=5^3=125$ |
| NEGATIVE | $x^{-m}=\frac{1}{x^m}$; $\frac{1}{x^{-n}}=x^n$ | $4^{-2}=\frac{1}{4^2}=\frac{1}{16}$; $9^{-3/2}=\frac{1}{9^{3/2}}=\frac{1}{27}$ |

**E:** Simplify $\frac{a+2b^{-1}}{c^{-1}}$ **Sol:** $\frac{a+\frac{2}{b}}{1/c}=\frac{\left(a+\frac{2}{b}\right)\cdot bc}{1/c\cdot bc}=\frac{abc+2c}{b}$

NOTE: The exponential laws in B, C and D also hold for *zero, fractional* and *negative* exponents. **E:** Express $43^x\cdot16^{-1/2x}$ as a power of 2. **Sol:** $(2^2)^{3x}\cdot(2^4)^{-1/2x}=2^{6x}\cdot2^{-2x}=2^{4x}$ **E:** $\sqrt{a}\cdot\sqrt[3]{a^4}=a^{1/2}\cdot a^{4/3}=a^{3/6}\cdot a^{8/6}=a^{11/6}$ or $\sqrt[6]{a^{11}}$ or $a\sqrt[6]{a^5}$

**F. SCIENTIFIC (STANDARD) NOTATION.** Any number can be expressed as a number between 1 and 10, multiplied by the appropriate power of 10 **E:** $346{,}000{,}000=3.46\times10^8$; $.000029=2.9\times10^{-5}$

## VII

**QUADRATIC EQUATIONS. A. STANDARD FORM:** $ax^2+bx+c=0$; $a\neq0$. Every quadratic equation has two roots.
**B. RELATION OF ROOTS TO COEFFS. 1. SUM OF ROOTS:** $r_1+r_2=-b/a$.
**2. PRODUCT OF THE ROOTS:** $r_1\cdot r_2=c/a$. **E:** $2x^2+9x-5=0$; $r_1+r_2=-b/a=-9/2$; $r_1\cdot r_2=c/a=-5/2$. **NOTE:** actual roots are $r_1=-5$ and $r_2=\frac{1}{2}$; sum is $-5+\frac{1}{2}=-9/2$; product is $(-5)(\frac{1}{2})=-5/2$.
**C. SOLVING PURE QUADRATICS:** $ax^2+c=0$. **E:** $4x^2-15=x^2$. **Sol:**
1. Solve for $x^2$: $x^2=5.2$. Extract square root of each side: $x=\pm\sqrt{5}$.

**D. SOLVING QUADRATICS BY FACTORING. 1.** Put into standard form. **2.** Factor. **3.** Set each factor equal to zero. **4.** Solve each of the two first degree equations thus formed. **5.** Check roots.

**E1:** Solve $3x^2-2=5x$. **Sol:** $3x^2-5x-2=0$; factor,
$(3x+1)\quad(x-2)=0$
$3x+1=0$ | $x-2=0$
$3x=-1$ | $x=2$
$x=-1/3$

**E2:** Solve $\sqrt{2x^2-14}=-x-1$. **Sol:** Sq. both sides: $2x^2-14=x^2+2x+1$
$x^2-2x-15=0$
$(x-5)\quad(x+3)=0$
$x=5$ | $x=-3$
−3 checks; extraneous root, 5, doesn't since $\sqrt{36}=+6$ only.

**E. SOLVING QUADRATICS BY COMPLETING SQUARE.** **E:** $2x^2-9x+3=0$.
1. Isolate unknown terms: $2x^2-9x=-3$
2. Make coefficient of $x^2$ unity: $x^2-\frac{9}{2}x=-3/2$
3. Complete square: add $\left(\frac{\text{coef. of } x}{2}\right)^2$ to each side: $x^2-\frac{9}{2}x+\left(\frac{9}{4}\right)^2=\frac{-3}{2}+\frac{81}{16}$
4. Factor one side; combine terms on other side: $(x-9/4)^2=57/16$
5. Extract square root of each side; solve $x-\frac{9}{4}=\frac{\pm\sqrt{57}}{4}$, $x=\frac{9\pm\sqrt{57}}{4}$

**F. SOLVING QUADRATICS BY FORMULA:** $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
**E:** Find roots of $3x^2+2=8x$.
1. Put into standard form: $3x^2-8x+2=0$
2. Compare with standard form: $a=3$, $b=-8$, $c=2$
3. Substitute into formula: $x=\frac{-(-8)\pm\sqrt{(-8)^2-4(3)(2)}}{2(3)}$
4. Evaluate: $x=\frac{8\pm\sqrt{40}}{6}=\frac{8\pm6.32}{6}=\frac{14.32}{6}$ or $\frac{1.68}{6}=2.38+$ or $.28$; $x=2.4$ or $.3$

| 5. Check roots: | SUM OF ROOTS | PRODUCT OF ROOTS |
|---|---|---|
| FROM EQUATION | $r_1+r_2=\frac{-b}{a}=\frac{8}{3}=2.67$ | $r_1r_2=\frac{c}{a}=\frac{2}{3}=.67$ |
| FROM ROOTS | $r_1+r_2=2.4+.3=2.7$ | $r_1r_2=(2.4)(.3)=.72$ |

These results are considered sufficiently close because the roots 2.4 and .3 are approximations of the roots to the nearest tenth

**G. FORMATION OF QUADRATIC EQUATION. METHOD 1.** Form factors $(x-r_1)(x-r_2)=0$; then multiply. **E:** Write a quadratic eq. whose roots will be −5 and 3. **Sol:** $(x+5)(x-3)=0$; $x^2+2x-15=0$.
**METHOD 2.** Find sum and product of given roots; substitute into $x^2-(r_1+r_2)x+r_1r_2=0$. **E1:** Find quadratic eq. whose roots are $-2+\sqrt{5}$ and $-2-\sqrt{5}$. **Sol:** $r_1+r_2=-4$; $r_1r_2=4-5=-1$; $x^2-(-4)x+(-1)=0$; $x^2+4x-1=0$. **E2:** Write a quad. eq. whose roots have sum 6 and product 11. **Sol:** $x^2-(6)x+(11)=0$, $x^2-6x+11=0$.

**H. NATURE OR CHARACTER OF ROOTS (a,b,c, rational).**

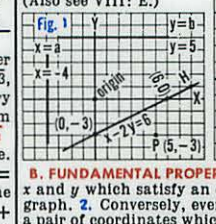
| DISCRIMINANT $b^2-4ac$ | NATURE OF ROOTS | EXAMPLES: EQUATION | DISC. | ROOTS |
|---|---|---|---|---|
| − | imaginary, ≠ | $x^2-6x+10=0$ | −4 | 3+i, 3−i |
| 0 | real, =, rational | $4x^2+12x+9=0$ | 0 | −3/2, −3/2 |
| + (perf. sq.) | real, ≠, rational | $3x^2-x-4=0$ | 49 | 4/3, −1 |
| + (not perf. sq.) | real, ≠, irrational | $x^2-8x+13=0$ | 12 | $4+\sqrt{3}$, $4-\sqrt{3}$ |

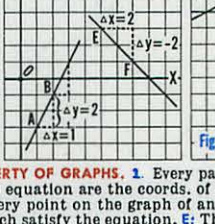
## VIII

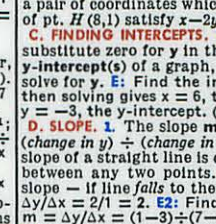
**GRAPHS: STRAIGHT LINES. A. TERMINOLOGY. 1.** Coordinates of a pt. P consist of the **abscissa** or **x-coord.** and the **ordinate** or **y-coord.** The graph of every first degree (*linear*) equation is a straight line. **2.** $x=a$ is parallel to $y$-axis. **3.** $y=b$ is parallel to $x$-axis. **4.** Graph other straight lines by three-pt. method. **E:** $x-2y=6$.

| x | 0 | 6 | 2 |
|---|---|---|---|
| y | −3 | 0 | −2 |

(Fig. 1) (Also see VIII: E.)




**B. FUNDAMENTAL PROPERTY OF GRAPHS. 1.** Every pair of values of $x$ and $y$ which satisfy an equation are the coords. of a point on its graph. **2.** Conversely, every point on the graph of an equation has a pair of coordinates which satisfy the equation. **E:** The coordinates of pt. H (8,1) satisfy $x-2y=6$.

**C. FINDING INTERCEPTS. 1.** To find the **x-intercept(s)** of a graph, substitute zero for **y** in the equation and solve for **x**. **2.** To find the **y-intercept(s)** of a graph, substitute zero for **x** in the equation and solve for **y**. **E:** Find the intercepts of $x-2y=6$. **Sol:** **(a)** Let $y=0$; then solving gives $x=6$, the x-int. **(b)** Let $x=0$; then solving gives $y=-3$, the y-intercept. (*Fig. 1*)

**D. SLOPE. 1.** The slope **m** of a straight line between two points = (*change in y*) ÷ (*change in x*) $=\Delta y/\Delta x=(y_2-y_1)\div(x_2-x_1)$. **2** The slope of a straight line is constant everywhere and will be the same between any two points. **3.** Slope is + if line *rises* to the right; slope − if line *falls* to the right. **E1:** On graph, slope of line AB $=\Delta y/\Delta x=2/1$ **2. E2:** Find slope of line between (5,3) and (7,1). **Sol:** $m=\Delta y/\Delta x=(1-3)\div(7-5)=-2/2=-1$ (Check line EF).(*Fig. 2*)
**4.** If equation is in **slope-intercept form**, $y=mx+b$, then **m** = slope and **b** = $y$-intercept. **E:** Find slope and y-intercept of $2y-x=4$. **Sol: Solve for y;** $y=\frac{1}{2}x+2$; $m=\frac{1}{2}$, $b=2$ (line c). (*Fig. 3*)

**E. DRAWING A GRAPH.** Use the slope, $y$-intercept method. **E:** $y=-3x-5$. **Sol:** $m=-3$, $b=-5$. Start at −5 on y-axis; from here x changes +1 to the right and y changes −3 down (line d). (*Fig. 3*)

**F. PARALLELISM.** Two straight lines are parallel if their slopes are equal (*but their y-intercepts are unequal*). **E:** Lines **(1)** $y=4x-1$ and **(2)** $8x-2y=5$. **(1)** has $m=4$, $b=-1$. Change form of **(2)** to $y=4x-2\frac{1}{2}$; now $m=4$, $b=-2\frac{1}{2}$. Therefore **(1)** and **(2)** are parallel.

**G. DISCOVERING EQUATION OF STRAIGHT LINE GIVEN 2 POINTS.**
**1.** Find $m$ from $\Delta y/\Delta x$. **2.** Substitute in $y=mx+b$ to find $b$.
**E:** $\frac{x}{-1}\ \frac{1}{5}$ / $\frac{y}{11}$ **Sol:** $m=\frac{\Delta y}{\Delta x}=\frac{11-(-1)}{5-1}=\frac{12}{4}=3$. Use $m=3$ and (5,11) in $y=mx+b$, then $11=3(5)+b$ and $b=-4$. **Ans:** $y=3x-4$.

## IX

**SOLUTION OF SYSTEMS OF LINEAR (FIRST DEGREE) EQUATIONS.**

**A. METHOD 1: GRAPHICALLY. 1.** Draw graph of each. **2.** Find common point (*point of intersection*). **E:** $\left\{\begin{array}{l}2y-x=4\\ y+3x+5=0\end{array}\right.$ **Sol:** These are lines c and d of previous graph, and meet at pt. R (−2,1). Therefore $x=-2$, $y=1$. (Check in both equations.) (*Fig. 3*)

**B. METHOD 2: BY ADDITION OR SUBTRACTION. 1** If necessary, by proper multiplication, make coefs. of one unknown numerically equal. **2.** Add or subtract equations to eliminate that unknown. **3.** Solve resulting equation. **4** Substitute this value into either orig. eq. to solve for 2nd unknown. **5.** Check in both orig. equations.
**E: (a)** $3x-2y=10$ [x2] = **(a)** $6x-4y=20$ ‖ Add **(a)** and **(b)**: ‖ Subst. 3 for x
**(b)** $x+4y=1$ **(b)** $x+4y=1$ ‖ $7x=21$, $x=3$. ‖ in **(b)**: $y=-1\frac{1}{2}$

**C. METHOD 3: BY SUBSTITUTION. 1** Solve the simpler equation for one unknown. (This is the *substitution equation.*) **2.** Substitute this expression for the *same* unknown in the other equation. **3.** Solve the equation thus formed for the *single* unknown. **4.** Substitute *resulting value* into *substitution equation* and solve for 2nd unknown. **5.** Check in original equations. **E:** **(a)** $2x+7y=11$; **(b)** $y-3x=18$. **Sol:**
1. Solve **(b)** for y: $y=3x+18$ Subst. Eq.
2. Substitute $3x+18$ for y in **(a)**: $2x+7(3x+18)=11$. **3.** Solve: $x=-5$
4. Substitute −5 for x in Substitution Eq.: $y=3(-5)+18$; $y=3$
5. To check: Use −5 for x and 3 for y in **(a)** and **(b)**.

**D. RELATIONSHIP BETWEEN LINEAR EQUATIONS IN A SYSTEM.**
**1. Consistent** or **independent equations** lead to *one set* of solutions. Their graphs are two *intersecting straight lines.* (*See A, B, C above.*)
**2. Inconsistent equations** lead to *no sets* of solutions. Their graphs are *parallel lines.* **E:** $4x-y=6$ and $2y=8x+1$. **3. Dependent** or **indeterminate equations** lead to an *infinite number of solutions.* Their graphs have the same slope and $y$-intercept, thus are *identical lines.* **E:** $y-2x=5$ and $6x+15=3y$.

**E. SOLVING SYSTEMS CONTAINING RECIPROCALS OF VARIABLES.** Without clearing fractions, use *Method 2 (Add.-or Subt. IX:B)* to eliminate one unknown. Then solve as usual. **E: (a)** $\frac{5}{x}+\frac{1}{y}=\frac{17}{6}$ **(b)** $\frac{6}{x}-\frac{3}{y}=2$. **Sol:** Multiply **(a)** by 3, and add to **(b)**. Thus $21/x=63/6$ and $x=2$. Substitute 2 for x in **(a)** or **(b)** and obtain $y=3$.

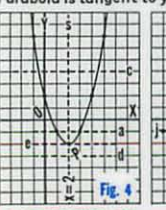
**F. SYSTEMS OF 3 LINEAR EQUATIONS IN 3 UNKNOWNS. 1.** By Method 2 (*Add. and Subt.*) or Method 3 (*Substitution*), take two equations at a time and eliminate the same unknown from each. This produces 2 new equations in the same 2 unknowns. **2.** Solve this pair of equations by Method 2 or 3 above. **3.** Substitute the values of the 2 unknowns into any one of the 3 original equations to solve for the third unknown **E: (a)** $2x-3y-5z=12$, **(b)** $4x+y+2z=-10$, **(c)** $x+2y-3z=13$. **Sol: 1.** Eliminate $y$: Multiply **(b)** by 3 and add to **(c)** to get **(d)** $14x+z=-18$. Multiply **(b)** by 2 and subtract from **(c)** to get **(e)** $-5x-7z=33$. **2.** Solve **(d)** and **(e)**: $x=-1$, $z=-4$. **3.** Substitute $-1$ for $x$, $-4$ for $z$ in **(b)**: $4(-1)+y+2(-4)=-10$; $y=2$. Check $x=-1$, $y=2$, $z=-4$ by substituting in all 3 orig. eqs.

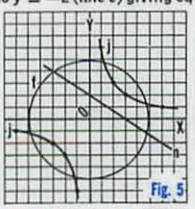
## X

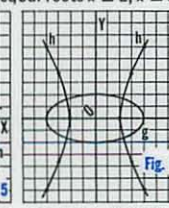
**2ND DEGREE GRAPHS. A. PARABOLA $y=ax^2+bx+c$ ($a\neq 0$) AND RELATED QUADRATIC EQUATIONS.** (Fig. 3)

| $x$ | $x^2-4x+2$ | $y$ |
|---|---|---|
| −1 | 1+ 4+2 | 7 |
| 0 | 0− 0+2 | 2 |
| 1 | 1− 4+2 | −1 |
| 2 | 4− 8+2 | −2 |
| 3 | 9−12+2 | −1 |
| 4 | 16−16+2 | 2 |
| 5 | 25−20+2 | 7 |

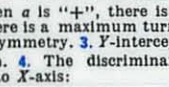
**E1:** Graph $y=x^2-4x+2$ from $x=-1$ to $x=5$. **Sol:** Turning (*minimum*) point is P(2,−2). Axis of symmetry is $x=2$ (*line s*). **E2:** Solve $x^2-4x+2=-1$. **Sol:** Parabola $y=x^2-4x+2$ intersects $y=-1$ (*line a*) at $x=1$ or $x=3$. **E3:** Solve $x^2-4x+2=0$. **Sol:** $y=x^2-4x+2$ intersects $y=0$ (*x*-axis) at $x=.6$ or $x=3.4$. **E4:** Solve $x^2-4x+3=5$. **Sol:** Subtract 1 from each side: $x^2-4x+2=4$. Now see where parabola meets $y=4$ (*line c*). Ans: $x=-.4$ or $x=4.4$ **E5:** Solve $x^2-4x+2=-3$. **Sol:** Parabola does not meet $y=-3$ (*line d*) indicating two imaginary roots. **E6:** Solve $x^2-4x+2=-2$. **Sol:** Parabola is tangent to $y=-2$ (*line e*) giving equal roots $x=2$, $x=2$.




**B. PROPERTIES OF $y=ax^2+bx+c$. 1.** When $a$ is "+", there is a minimum turning pt.; when $a$ is "−", there is a maximum turning pt. **2.** Formula $x=-b/2a$ gives axis of symmetry. **3.** $Y$-intercept $=c$. $X$-intercepts $=(-b\pm\sqrt{b^2-4ac})\div 2a$. **4.** The discriminant $b^2-4ac$ gives position of parabola relative to $X$-axis:

| If $b^2-4ac$ is | Parabola | Graphs |
|---|---|---|
| POSITIVE | Intersects X-axis in 2 distinct pts. (curve a) |  |
| ZERO | is tangent to X-axis (curve b) | |
| NEGATIVE | has no pts. in common with X-axis (curve c). | |

**C. CIRCLE $x^2+y^2=r^2$.** (*Fig. 5*) **E:** $x^2+y^2=25$ is circle, center at origin, radius $=5$ (*graph f*).

**D. ELLIPSE $ax^2+by^2=c$ (a,b,c have same sign, $a\neq b$).** (*Fig. 6*)

**E:** $x^2+4y^2=16$ (*graph g*).

| $x$ | 0 | ±4 | 2 | −2 | +3 | −3 | 5 | −5 |
|---|---|---|---|---|---|---|---|---|
| $y$ | ±2 | 0 | ±1.7 | ±1.7 | ±1.3 | ±1.3 | imag | imag |

For intercepts, first let $x=0$; then let $y=0$.

**E. HYPERBOLA $ax^2+by^2=c$ (a and b have opposite signs).** (*Fig. 6*)

**E:** $3x^2-y^2=12$ (*graph h*).

| $x$ | 0 | ±2 | ±1 | 3 | −3 | ±2.3 | ±2.3 |
|---|---|---|---|---|---|---|---|
| $y$ | imag | 0 | imag | ±3.9 | ±3.9 | 2 | −2 |

Let $x=0$, then let $y=0$.

**F. (EQUILATERAL) HYPERBOLA $xy=k$** (*Fig. 5*)

| $x$ | 0 | imposs. | 1 | 2 | 3 | 6 | −1 | −2 | −3 | −6 |
|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | imposs. | 0 | 6 | 3 | 2 | 1 | −6 | −3 | −2 | −1 |

**E:** $xy=6$ (*graph j*)

**G. GRAPHIC SOLUTION OF SYSTEMS CONTAINING 2nd DEGREE EQUATIONS.** Draw graphs of both equations on same set of axes and read coordinates of common points. (*Fig. 5, 6*)

**E1:** $x^2+y^2=25$, $3y+2x=6$ — Intersection of graphs *f* and *n*:

| $x$ | −3 | 4.8 |
|---|---|---|
| $y$ | 4 | −1.2 |

**E2:** $x^2+y^2=25$, $xy=6$ — Intersection of graphs *f* and *j*:

| $x$ | 4.8 | −4.8 | 1.2 | −1.2 |
|---|---|---|---|---|
| $y$ | 1.2 | −1.2 | 4.8 | −4.8 |

**E3:** $x^2+4y^2=16$, $3x^2-y^2=12$ — Intersection of graphs *g* and *h*:

| $x$ | 2.2 | 2.2 | −2.2 | −2.2 |
|---|---|---|---|---|
| $y$ | 1.7 | −1.7 | 1.7 | −1.7 |

**NOTES: 1. The maximum number of points of intersection of two graphs = (degree of first eq.) × (degree of second eq.).** 2. Fewer pts. of intersection indicate the presence of *imaginary roots*, which occur two at a time. (In some cases, fewer pts. indicate "intersection at infinity"—similar to parallel straight lines.) **3.** When dealing with first and second degree equations, **tangency** gives two identical sets of solutions at the **point of contact.**

| GRAPH | | | | | |
|---|---|---|---|---|---|
| NO. OF REAL, ≠ SOL. | | 2 | 2 | 2 | 2 |
| NO. OF REAL, = SOL. | 2 | | | 2 | |
| NO. OF IMAG. SOL. | | | 2 | 2 | 4 |
| TOTAL NO. OF SOL. | 2 | 2 | 2 | 4 | 4 |

## XI

**SOLVING SYSTEMS OF QUADRATIC EQUATIONS ALGEBRAICALLY.**

**A. 2ND AND 1ST DEGREE EQUATION.** Solve by substitution.

**E: (a)** $2x^2+y^2=27$, **(b)** $y-2x=3$

**(b)** $y=2x+3$ (*Subst. Eq.*)

**(a)** $2x^2+(2x+3)^2=27$

$2x^2+4x^2+12x+9=27$

$6x^2+12x-18=0$

Div. both sides by 6.

$x^2+2x-3=0$

$(x-1)(x+3)=0$

| $x=1$ | $x=-3$ |
|---|---|
| $y=2x+3$ | $y=2x+3$ |
| $y=5$ | $y=-3$ |

**Sol:**

| $x$ | 1 | −3 |
|---|---|---|
| $y$ | 5 | −3 |

Check each set of answers in both original equations.

**B. TWO SECOND DEGREE EQUATIONS.** Use method of addition or subtraction in *IX:B* (or method of substitution).

**E: (a)** $3x^2-y^2=46$, **(b)** $x^2+2y^2=20$

Multiply eq. **(a)** by 2 and add to **(b)**; then $7x^2=112$

$x^2=16$

$x=\pm 4$

| If $x=4$ | If $x=-4$ |
|---|---|
| $16+2y^2=20$ | $16+2y^2=20$ |

$y^2=2$

$y=\pm\sqrt{2}$

$y^2=2$

$y=\pm\sqrt{2}$

**Solutions:**

| $x$ | 4 | 4 | −4 | −4 |
|---|---|---|---|---|
| $y$ | $\sqrt{2}$ | $-\sqrt{2}$ | $\sqrt{2}$ | $-\sqrt{2}$ |

## XII

**LOGARITHMS. A. TERMINOLOGY. 1.** A logarithm is an exponent (used for computation). **2.** In the exponential form, $b^n=n$ (like $3^4=81$) the **base** is b (or 3), the **number** is n (or 81) and the **logarithm** or **exponent** is e (or 4). **3.** The equivalent logarithmic form is $\log_b n=e$ (or $\log_3 81=4$, read log of 81 to the base 3 equals 4). The base $b$ is always "+" and $\neq 1$, $n$ is always "+". **4.** The common base is 10 and is usually omitted: $\log 1000=3$ means $\log_{10}1000=3$. **E:** Solve for $x$: $\log_3 x=5$. **Sol:** $2^5=x$; $x=32$. **E:** Solve for $x$: $\log_7 49=x$. **Sol:** $7^x=49$; $x=2$. **5. CHARACTERISTIC AND MANTISSA:** If $\log 7360=3.8669$ (meaning $10^{3.8669}=7360$), the *integral part* (3) of the logarithm is the **characteristic**; the *decimal part*, which must be positive (.8669), is the **mantissa.**

**B. FINDING CHARACTERISTIC. Rule 1. Number > 1:** char. = one less than number of digits preceding decimal pt. **Number < 1:** char. $=-(1+$ number of zeros directly after decimal pt.$)$. **Rule 2.** If a number is written in standard form, the exponent of 10 equals the characteristic. **E:** $7360=7.36\times 10^3$; characteristic is 3. (See *VI:F*)

**C. USING THE LOGARITHM TABLE**

| N | 0 | 1 | 2 | 3 | 4 | .... | .... | 9 |
|---|---|---|---|---|---|---|---|---|
| 63 | 7993 | 8000 | 8007 | 8014 | 8021 | .... | .... | 8055 |
| 64 | 8062 | 8069 | 8075 | 8082 | 8089 | .... | .... | 8122 |

**1. FINDING THE LOGARITHM OF A 3-DIGIT NUMBER: E1:** log 634. **Sol: (a)** By inspection, char. is 2. **(b)** Find first two digits 63 in N column; find column of third digit 4, giving mantissa 8021. Therefore log 634 = 2.8021. **E2:** log 63 = log 63.0 = 1.7993; log .064 = 8.8062−10.

**2. FINDING THE 3-DIGIT ANTILOGARITHM (OR NUMBER): E1:** If log N = 1.8089, find N. **Sol:** Look up mantissa (*8089*) inside table, then place decimal point according to given char. (*1*). **Ans:** N = 64.4. **E2:** If log x = 9.8000−10. **Sol:** x = .631. **NOTE:** Mantissa is independent of location of decimal point in the number.

**3. FINDING THE LOG OF A 4-DIGIT NUMBER BY INTERPOLATION: E:** Find log 6.418. **Sol:** Mantissa is between those of N = 6.41 and N = 6.42.

| NUMBER | MANTISSA | |
|---|---|---|
| 6420 | 8075 | 10 |
| 6418 | | 8 |
| 6410 | ·8069 | x |

$\frac{8}{10}=\frac{x}{6}$, $x=4.8$ or 5

Mantissa = 8069+5 = 8074

Therefore, log 6.418 = 0.8074

**4. FINDING A 4-DIGIT ANTILOGARITHM FROM A MANTISSA NOT IN TABLE:**

| NUMBER | MANTISSA | |
|---|---|---|
| 6330 | 8014 | 10 |
| x | 8011 | 7 |
| . 6320 | 8007 | 4 |

**E:** If log n = 7.8011−10, find n. **Sol:** $\frac{x}{10}=\frac{4}{7}$, $x=5\frac{5}{7}$ or 6.

Number = 6320+6. Therefore, antilog (7.8011−10) = .006326.

**D. LAWS OF LOGS. NOTE:** $\log 1=0$, $\log 10=1$, $\log 100=2$, $\log .1=-1$.

| LAWS | Using log 2 = .30103, log 3 = .47712, find |
|---|---|
| MULT.: $\log MN=\log M+\log N$ | $\log 6=\log(2\cdot 3)=\log 2+\log 3=$ .30103+.47712 = .77815 = .7782 |
| DIV.: $\log\frac{M}{N}=\log M-\log N$ | $\log 5=\log\frac{10}{2}=\log 10-\log 2=$ 1−.30103 = .69897 = .6990 |
| POWER: $\log M^p=p\log M$ | $\log 16=\log 2^4=4\log 2=$ 4(.30103) = 1.20412 = 1.2041 |
| ROOT: $\log\sqrt[r]{M}=\frac{\log M}{r}$ | $\log\sqrt[3]{2}=\frac{1}{3}\log 2=\frac{1}{3}(.30103)$ = .10034 = .1003 |

**NOTE:** Colog N = log (1/N) = log 1 − log N = 0 − log N. **E:** colog 64 = 0 − log 64 = (10.0000−10) − (1.8062) = 8.1938−10.

**E. APPLICATION OF LOGARITHMS.** According to laws of logs:

| ARITHMETIC OPERATIONS | Add | Subt. | Mult. | Div. | Power | Root |
|---|---|---|---|---|---|---|
| BUT WHEN USING LOGS | | | Add | Subt. | Mult. | Div. |

**E1:** $S=4\pi r^2$. Find S to the nearest tenth if r = 5.36 (use $\pi=3.14$). **Sol:** $S=(4)(3.14)(5.36)^2$; log S = log 4 + log 3.14 + 2 log 5.36.

log 4 = 0.6021

log 3.14 = 0.4969

2 log 5.36 = 1.4584 ←

log S = 2.5574; S = 360.9 (by interpolation)

log 5.36 = 0.7292

× 2

2 log 5.36 = 1.4584

$\sqrt[3]{.0823}$

$\frac{\sqrt[3]{.0823}}{(3.96)(.19)}$

**E2:** Find, correct to the nearest hundredth, $n=\frac{\sqrt[3]{.0823}}{(3.96)(.19)}$.

**Sol:** log n = (1/3) log .0823 − (log 3.96 + log .19).

(1/3) log .0823 = 9.6385−10 ←

To be able to subtract, rewrite as

(1/3) log .0823 = 19.6385−20

− log denom. = 9.8765−10 ←

log x = 9.7620−10

(no interp.) x = .578

x = .58

log .0823 = 8.9154−10

To divide by 3, use 28.9154−30:

3 | 28.9154−30

9.6385−10

log 3.96 = 0.5977

+ log .19 = 9.2788−10

−log denom. = 9.8765−10

**F. EXPONENTIAL EQUATIONS.** An exponential equation has the unknown appearing in one or more exponents.

**CASE 1. BOTH SIDES EXPRESSIBLE AS POWERS OF THE SAME BASE:**

**E:** Solve $8^{2x}=4^{x-1}$ **Sol:**

Express each side as a power of 2: $(2^3)^{2x}=(2^2)^{x-1}$

For power of a power, multiply exponents: $2^{6x}=2^{2x-2}$

If powers of same base are =, exponents are =: $6x=2x-2$

Solving gives: $x=-1/2$

**CASE 2. BOTH SIDES NOT EXPRESSIBLE AS POWERS OF SAME BASE:**

**E:** Solve to the nearest tenth: $3^x=22$. **Sol:** Find log of both sides: $\log 3^x=\log 22$; $x\log 3=\log 22$; $x(.4771)=1.3424$; $x=2.8$ (by long div).

**G. APPLICATIONS TO TRIGONOMETRY. E:** Find to nearest tenth $x=17.4/\sin 28°$. **Sol:** $\log x=\log 17.4-\log\sin 28°=(11.2405-10)-(9.6716-10)=1.5689$; then $x=37.1$. **NOTE:** For log sin 28°, table gives 9.6716, then add −10.

## XIII

**PROGRESSIONS, SEQUENCES, SERIES. A. DEFINITIONS. 1.** An **arithmetic progression** (A.P.) is a sequence of numbers in which the **difference** between any term and the preceding term is *constant* throughout. This difference is called the **common difference, d. 2.** A **geometric progression** (G.P.) is a sequence of numbers in which the **ratio** between any term and the preceding term is *constant* throughout. This ratio is called the **common ratio, r. 3.** In an *A.P.* or a *G.P.*, **a** is the first term, and $l$ is the last or nth term.

**B. ARITHMETIC PROGRESSION**

**d = 2nd term − 1st term = any term − preceding term**

$l=a+(n-1)d$

$S=\frac{n}{2}(a+l)$, or

$S=\frac{n}{2}[2a+(n-1)d]$

Between a and b, the arith. mean $M=(a+b)/2$

If a,b,c form an A.P., then $b-a=c-b$.

**C. GEOMETRIC PROGRESSION**

$r=\frac{\text{2nd term}}{\text{1st term}}=\frac{\text{any term}}{\text{preceding term}}$

$l=ar^{n-1}$

$S=\frac{a-ar^n}{1-r}$ or $\frac{ar^n-a}{r-1}$ $(r\neq 1)$

$S=\frac{a-rl}{1-r}$ or $\frac{rl-a}{r-1}$

Between a and b, the geom. mean $G=\pm\sqrt{ab}$

If a,b,c form a G.P., then $b/a=c/b$.

**NOTE:** The sum of an infinite number of terms of a G.P. with $|r|<1$ is $S_\infty=a/(1-r)$ ($|r|$ represents absolute value of $r$.)

**E1:** Find the 47th term of 2, $5\frac{1}{2}$, 9, . . . **Sol:** $l=?$; $a=2$; $n=47$; $d=3.5$

$l=a+(47-1)\,3.5$

$l=2+161=163$

**E2:** Find the sum of 100 terms of 17, 14, 11, . . . **Sol:**

$S=\frac{n}{2}[2a+(n-1)d]$

$S=\frac{100}{2}[2(17)+(100-1)(-3)]$

$S=50(34-297)=-13{,}150$

**S=?**; **a=17**; **n=100**; **d=−3**

**E3:** Insert 4 arith. means between −8 and 27. **Sol:**

$-8\ \_\ \_\ \_\ \_\ 27$

$l=27$, $a=-8$, $n=6$, $d=?$

Using $l=a+(n-1)d$, find $d=7$.

Means are −1, 6, 13, 20.

**E4:** Find the arith. mean between 11 and −3. **Sol:**

$M=\frac{a+b}{2}=\frac{11-3}{2}=4$

**E5:** Find the 10th term of $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \ldots$. **Sol:** $l=?$; $a=\frac{1}{8}$; $n=10$; $r=\frac{1}{4}/\frac{1}{8}=2$

$l=ar^{n-1}=\frac{1}{8}\cdot 2^9$

$l=\frac{1}{8}\cdot 512=64$

**E6:** Find the sum of the infinite series: $6, 2, \frac{2}{3}, \ldots$.

**Sol:** $S=\frac{a}{1-r}=\frac{6}{1-\frac{1}{3}}=\frac{6}{2/3}=9$

**E7:** The geom. mean between 2 and 18 is $\pm\sqrt{(2)(18)}=\pm\sqrt{36}=\pm 6$.

**D. INFINITE REPEATING DECIMALS. E:** Express .181818 . . . as a common fraction. **Sol:** .181818 . . . is equal to the infinite series .18+.0018+.000018+. . . . $a=.18$, $r=.0018/.18=.01$:

$S=\frac{a}{1-r}$; $S=\frac{.18}{1-.01}=\frac{.18}{.99}=\frac{18}{99}=\frac{2}{11}$.

**NOTE:** All repeating decimals are rational numbers.

## XIV

**BINOMIAL THEOREM. A. FORMULA (n is positive integer):**

$$(x+y)^n=x^n+\frac{n}{1}x^{n-1}y+\frac{n(n-1)}{1\cdot 2}x^{n-2}y^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}y^3+\ldots+\frac{n}{1}xy^{n-1}+y^n$$

**B. PROPERTIES. 1.** There are **(n+1)** terms in the expansion. **2.** In any term, the sum of the exponents equals **n. 3.** *To find succeeding coefficients:* In any term, **(coef.) (exp. of x)/number of term = next coef. 4.** If x and y are both "+", all terms are "+". If x is "+" and y is "−," terms are alternately "+" and "−," starting with "+."

**E1:** $(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$.

**E2:** $(2x-1)^3=(2x)^3-3(2x)^2(1)+3(2x)(1)^2-(1)^3$

$=8x^3-12x^2+6x-1$

**E3:** If the 7th term of the expansion of $(r+s)^{11}$ is $462r^5s^6$, find the coef. of the 8th term = $[(462)(5)]\div 7=330$.

## XV

**VARIATION. If k is a constant of variation and quantities x, y:**

| | A. VARY DIRECTLY | B. VARY INVERSELY |
|---|---|---|
| DEFINITION | Their ratio is constant | Their product is constant |
| FORMULAS | $x/y=k$, $x=ky$; $x_1/y_1=x_2/y_2$ | $xy=k$, $x=k/y$; $x_1y_1=x_2y_2$, $x_1/x_2=y_2/y_1$. |
| CHANGES | 1. x increases, y increases 2. x decreases, y decreases 3. If x is mult. (div.) by a number, y is mult. (div.) by same number. | 1. x increases, y decreases 2. x decreases, y increases 3. If x is mult. (div.) by a number, y is div. (mult.) by same number. |
| GRAPH | Straight Line | Hyperbola |

**E1:** If m varies directly as n and m = 6 when n = 15, find m when n = 55. **Sol:** By proportion method, $m_1/n_1=m_2/n_2$; $6/15=m/55$; $m=22$.

**E2:** r and t vary inversely. $r=8$ when $t=6$. Find t when $r=12$. **Sol:** By "k" method, $rt=k$, $(8)(6)=k$, $k=48$. Therefore formula is $rt=48$. Now when $r=12$, $12t=48$; $t=4$.

**C. JOINT.** $x$ varies jointly as $y$ and $z$ if $x$ varies directly as the product of $y$ and $z$, that is, $x=kyz$ and $x_1/y_1z_1=x_2/y_2z_2$. **E:** c varies jointly as m and r and inversely as the square of d. **Sol:** $c=kmr/d^2$.

## XVI

**VERBAL PROBLEMS.** *NOTE: When using a chart with a formula:* **(1)** *Fill in any two columns with knowns and unknowns;* **(2)** *use the formula to fill in remaining column (shown by heavy bordered box) to obtain equation(s).*

**A. CONSECUTIVE INTEGER. E:** Find 3 consec. odd numbers such that the sum of the squares of the first two is 9 more than square of the third. **Sol:** Let x = 1st odd no., x+2 = 2nd odd no., x+4 = 3rd odd no.

(first)² + (second)² = (third)² + 9

$x^2+(x+2)^2=(x+4)^2+9$

$x^2-4x-21=0$

$(x-7)(x+3)=0$ (Check both solutions in original problem.)

| $x=7$ | $x=-3$ |
|---|---|
| $x+2=9$ | $x+2=-1$ |
| $x+4=11$ | $x+4=1$ |

**B. RECIPROCAL. E:** Separate 20 into two parts such that the reciprocal of the larger is 1/24 less than the reciprocal of the smaller. **Sol:** Let x = smaller part; 20 − x = larger part

larger recip. = smaller recip. − 1/24

$\frac{1}{20-x}=\frac{1}{x}-\frac{1}{24}$ [Mult. by 24x(20−x)]

$24x=24(20-x)-x(20-x)$

$0=x^2-68x+480$

x = 8; 20 − x = 12 (Checks) | x = 60; 20 − x = −40 (Does not check.)

**C. MIXTURE. E1:** How many pints of acid should be added to 10 pt. of a 40% solution of acid to change it to a 50% solution.? **Sol:**

| | No. of pints | × Part = acid | Pints of acid |
|---|---|---|---|
| START | 10 | .40 | .40(10) |
| ADD ACID | x | 1.00 | x |
| RESULT | 10+x | .50 | .50(10+x) |

.40(10)+x = .50(10+x); x = 2 pt.

**E2:** 2 lb. of salt are mixed with 24 lb. of water. How many lb. of water should be evaporated to make it a 9% solution of salt? **Sol:**

| | No. of lb. | × Part = salt | lb. of salt |
|---|---|---|---|
| START | 26 | . | 2 |
| EVAP. WATER | x | 0 | 0 |
| RESULT | 26−x | .09 | .09(26−x) |

2 − 0 = .09(26 − x); x = 3 7/9 lb.

**D. INVESTMENT. E:** $6000 is invested, part at 3.5% and part at 6%, to give an average return of 5% on whole investment. How much is invested at each rate? **Sol:**

| | P | × R = | I |
|---|---|---|---|
| AT 3.5% | x | .035 | .035x |
| AT 6% | 6000−x | .06 | .06(6000−x) |

**Int. from 3.5% invest. + int. from 6% invest. = 5% of $6000.**

.035x + .06(6000 − x) = .05(6000).

Multiply each member by 1000:

35x + 60(6000 − x) = 50(6000);

x = $2400 at 3.5%;

6000 − x = $3600 at 6%.

**E. BUSINESS. E:** A man bought a certain number of shares of stock for $3300. If the price per share had been $55 less, he could have bought 2 more shares. How many shares did he buy? **Sol:**

| | No. of Shares | × Price/share = | Total Cost |
|---|---|---|---|
| ACTUAL | x | 3300/x | 3300 |
| POSSIBLE | x+2 | 3300/(x+2) | 3300 |

**Actual price per share = Possible price per share + $55.**

$\frac{3300}{x}=\frac{3300}{x+2}+55$; x = 10 shares.

**F. UNIFORM MOTION. E1:** A man drove 440 miles. If he had decreased his rate 15 mi. per hr., it would have taken 3 hr. longer. Find his rate. **Sol:**

| | R | × T | = D |
|---|---|---|---|
| ACTUAL | x | y | xy |
| POSS. | x−15 | y+3 | (x−15)(y+3) |

**Distance in both cases = 440 mi.**

(a) $xy=440$

(b) $(x-15)(y+3)=440$

(b) $xy-15y+3x-45=440$

(b) $440-15y+3x-45=440$

(b) $-15y+3x-45=0$

Solving with eq. (a) gives: x = 55 mph.; y = 8 hr. (XVI: E.)

**E2:** An airplane whose airspeed is 190 mph can go 10 mi. with the wind in the same time it can return 9 mi. against the wind. Find the rate of the wind. **Sol:**

Rate against wind = airspeed − wind. Rate with wind = airspeed + wind. Let x = rate of wind.

| | R | × T | = D |
|---|---|---|---|
| WITH WIND | 190+x | 10/(190+x) | 10 |
| AGAINST WIND | 190−x | 9/(190−x) | 9 |

**Time one way = Time back**

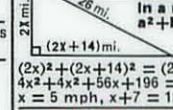
$\frac{10}{190+x}=\frac{9}{190-x}$; x = 10 mi. per hr.

**FOR BOATS: Rate upstream = still water rate − rate of stream**

**Rate downstream = still water rate + rate of stream**

**E3:** Florence and Ronni start at the same time from the intersection of two roads meeting at right angles. Florence's rate is 7 mph greater than Ronni's. If they are 26 mi. apart after 2 hr, find rates. **Sol:**

| | R | × T | = D |
|---|---|---|---|
| FLORENCE | x+7 | 2 | 2x+14 |
| RONNI | x | 2 | 2x |




In a rt. triangle, $a^2+b^2=c^2$.

$(2x)^2+(2x+14)^2=(26)^2$

$4x^2+4x^2+56x+196=676$

x = 5 mph, x+7 = 12 mph.

**G. DIGIT. E:** When a two-digit number is divided by the sum of its digits, the quotient is 6 and the remainder is 8. The number formed by reversing the digits is 27 less than the original number. Find the orig. no. **Sol:**

Let t = tens' digit, u = units' dig.

**(a) Number = 6(Sum of dig)+8**

**(b) Reversed no. = orig. no. − 27**

(a) 10t + u = 6(t + u) + 8

(b) 10u + t = 10t + u − 27

Then t = 7, u = 4. Number = 74

**H. GEOMETRY. E1:** A rectangular lawn is 60 ft. by 40 ft. How wide a uniform strip must be mowed so that 736 sq. ft. will have been mowed? **Sol:**

Let x = width of strip

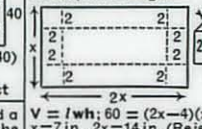
**inside rect. + border = total rect.**

$(60-2x)(40-2x)+736=(60)(40)$

$2400-200x+4x^2+736=2400$

x = 4 ft. | x = 46; reject

**E2:** The length of a rectangular piece of metal is twice its width. A 2-inch square is cut from each corner; then the metal is folded to form an open box whose volume is 60 cu. in. Find dimensions of original metal. **Sol:** Let x = width, 2x = length




V = lwh; 60 = (2x−4)(x−4)(2); x = 7 in., 2x = 14 in. (Reject x = −1)

**I. PROGRESSION. E1:** To build a theater containing 980 seats, the first row is built with 30 seats and each succeeding row contains 2 seats more than the one in front of it. How many rows are needed? **Sol:** Let n = number of rows:

**In an A.P.,** $S=\frac{n}{2}[2a+(n-1)d]$

$S=980$, $a=30$, $d=2$. Thus,

$980=\frac{n}{2}[2(30)+(n-1)2]$

$980=\frac{n}{2}(58+2n)$; $980=29n+n^2$

$n^2+29n-980=0$

$(n+49)(n-20)=0$

n = −49; reject | n = 20 rows

**E2:** Three numbers are in the continued ratio of 1:5:17. When 2 is added to each number, the resulting numbers form a geometric prog. Find numbers. **Sol:**

Let x = 1st no. | 5x = 2nd no. | 17x = 3rd no.

x+2 | 5x+2 | 17x+2 } new nos.

**In a G.P.,** $\frac{\text{2nd term}}{\text{1st term}}=\frac{\text{3rd term}}{\text{2nd term}}$

$\frac{5x+2}{x+2}=\frac{17x+2}{5x+2}$,

$25x^2+20x+4=17x^2+36x+4$;

$8x^2-16x=0$; $8x(x-2)=0$

x = 0 (rej.) | x = 2; 5x = 10, 17x = 34

**J. WORK. E:** Tom and Bill can do a certain job together in 6 hours. Tom alone would need 9 hr. more to do the job than Bill would need alone. How long would it take each alone? **NOTE:** Part of job done per unit time = reciprocal of time needed for job. **Sol:**

| | Hrs. needed alone | Part of job done per hr. | × Hours worked | = Part of job done |
|---|---|---|---|---|
| TOM | x+9 | 1/(x+9) | 6 | 6/(x+9) |
| BILL | x | 1/x | 6 | 6/x |

**Part of job done by Tom + Part of job done by Bill = Whole Job**

6/(x+9) + 6/x = 1; x = 9 hr. for Bill, x+9 = 18 hr. for Tom.

DATA-GUIDE QUICK CHART

# DIFFERENTIAL CALCULUS

A SIMPLIFIED SUMMARY & INSTANT REFERENCE

LEARN & REVIEW IN MINUTES

AUTHOR: Prof. A. Fialkow, Ph.D., Polytechnic Institute of Brooklyn.

SOLID PLASTIC

QUICK CHARTS SPEED STUDY: PUT THE IMPORTANT FACTS IN FULL VIEW TO SAVE YOU TIME AND WORK.

QUICK CHARTS HELP YOU REMEMBER: PERMANENTLY FILL IN FOR MISSING NOTES & TEXTS IN BASIC & ADVANCED COURSES.

CONTAINS NO PAPER

## I

## BASIC CONCEPTS. A. LIMITS. 1. Limit of a sequence.

A sequence $a_1, a_2, \ldots, a_n, \ldots$ is an unending arrangement of numbers. *It has a limit of value* **b** *if the numerical difference between* **b** *and an* **N**th *term sufficiently "far out in the sequence" becomes and thereafter remains less than any preassigned positive number* $\varepsilon$, *however small.* When a value $\varepsilon$ is chosen, a related number $N$ is determined such that the $N$th numbered term $a_N$ and all successive terms $a_{N+1}, a_{N+2}, \ldots$ each differ from the limit $b$ by less than $\varepsilon$; that is, $|a_N - b| < \varepsilon, |a_{N+1} - b| < \varepsilon, \ldots$. Thus the difference $|a_n - b|$ becomes and remains arbitrarily small as $n$ increases without bound; written: $a_n \to b$ or $\lim_{n\to\infty} a_n = b$.
**E:** limit of sequence 1/2, 2/3, 3/4, . . . is 1.

**2. Limit of a function.** *A number* **b** *is the limit of a function* $f(x)$ *as* **x** *approaches the number* **a** *if for every positive value* $\varepsilon$, *however small, there is a related positive number* $\delta$ *such that the value of* $f(x)$ *minus the limit of* $f(x)$ *is numerically less than* $\varepsilon$ *if the difference between the value of* **x** *and its limit* **a** *is less than* $\delta$ *but more than zero,* written: $|f(x) - b| < \varepsilon$ if $0 < |x - a| < \delta$. Thus as the value of $x$ approaches $a$ as its limit, $f(x)$ approaches and remains near its limit $b$: $\lim_{x\to a} f(x) = b$ or $f(x) \to b$ as $x \to a$.

**B. LIMIT THEOREMS. 1. Sum.** $\lim_{x\to a}[f(x) + g(x) - h(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) - \lim_{x\to a} h(x)$. **2. Product.** $\lim_{x\to a}[f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$. **3. Quotient.** If limit of divisor is not zero $[\lim_{x\to a} g(x) \neq 0]$ $\lim_{x\to a}[f(x) \div g(x)] = \lim_{x\to a} f(x) \div \lim_{x\to a} g(x)$.
**E:** $\lim_{x\to 0} \frac{x^2+1}{(x+2)(5-x)} = \lim_{x\to 0}(x^2+1) \div \lim_{x\to 0}(x+2)(5-x) = [\lim_{x\to 0} x \cdot \lim_{x\to 0} x+1] + [\lim_{x\to 0}(x+2) \cdot \lim_{x\to 0}(5-x)] = (0+1) \div (2 \cdot 5) = 1/10$

**C. INFINITY.** If $b$ is the limit of $f(x)$ as $x$ increases without bound, then the limit of $f(x)$ as $x$ becomes infinite is $b$; written $\lim_{x\to\infty} f(x) = b$ or $f(x) \to b$ as $x \to \infty$. Similarly if $f(x)$ grows without bound as $x \to a$, then $f(x)$ becomes infinite as $x$ approaches $a$; written $\lim_{x\to a} f(x) = \infty$ or $f(x) \to \infty$ as $x \to a$.

**D. CONTINUITY.** Continuous functions correspond to unbroken curves. A function $f(x)$ is *continuous* at $x = a$ if the value of $f(x)$ at $x = a$ equals the limit of $f(x)$ as $x \to a$; written: $\lim_{x\to a} f(x) = f(a)$. This implies that $\lim_{x\to a} f(x)$ and $f(a)$ both exist. If $f(x)$ is not continuous at $x = a$, it is *discontinuous* at $x = a$. A function $f(x)$ is continuous in an interval if continuous at each point of interval.

## II

## DIFFERENTIATION OF ALGEBRAIC FUNCTIONS.

**A. INCREMENTS.** If values $(x,y)$ satisfy the equation $y = f(x)$, changing $x$ by the increment $\Delta x$ produces the function increment $\Delta y$ [or $\Delta f(x)$] such that $(x+\Delta x; y+\Delta y)$ also satisfy the equation. Hence $\Delta y = f(x+\Delta x) - f(x)$.
**E:** If $y = x^3$ then $\Delta y = (x+\Delta x)^3 - x^3 = 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3$

**B. THE DERIVATIVE.** The derivative of $f(x)$ with respect to $x$ is the limit approached by the ratio of the increment of $f(x)$ to the increment of $x$ as the increment of $x$ approaches zero as a limit; written: $\lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{d\,f(x)}{dx} = \frac{dy}{dx} = f'(x)$.
Functions for which the $\lim_{\Delta x\to 0} \Delta y/\Delta x$ exists when $x = a$ are called differentiable at $x = a$. If $f(x)$ is differentiable for every value of $x$ in an interval, it is differentiable in the interval.

**C. FINDING THE DERIVATIVE.** ▶ Calculate the ratio $\Delta y/\Delta x = \frac{f(x+\Delta x) - f(x)}{\Delta x}$. ▶ Evaluate the limit of this ratio as $\Delta x \to 0$.
**E:** Find $dy/dx$ if $y = x^3$. **Sol.** $\Delta y = 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3$; $\Delta y/\Delta x = 3x^2 + 3x(\Delta x) + (\Delta x)^2$; $dy/dx = \lim_{\Delta x\to 0}[3x^2 + 3x(\Delta x) + (\Delta x)^2] = 3x^2$

The derivative of a function $f(x)$ is also a function $f'(x)$ of $x$, which in general is differentiable. The derivative of the derivative $f'(x)$ is called the *2nd derivative* of $f(x)$, etc.; written: $d^2y/dx^2$, $f''(x)$; the $n$-th derivative as $d^ny \div dx^n$ or $f^{(n)}(x)$.
**E:** If $y = x^3$ then $\frac{dy}{dx} = 3x^2$; $\frac{d^2y}{dx^2} = \frac{d}{dx}(3x^2) = 6x$; $\frac{d^3x}{dx^3} = \frac{d}{dx}(6x) = 6$; $\frac{d^4y}{dx^4} = \frac{d}{dx}(6) = 0$

**D. DIFFERENTIATION FORMULAS.** In the formulas, $u$ and $v$ are functions of $x$, and $c$ is a constant.

| | |
|---|---|
| $\frac{dc}{dx} = 0$ | $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$ |
| $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$ | $\frac{d}{dx}(x^n) = nx^{n-1}$ |
| $\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}$ | $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ |
| $\frac{d}{dx}(c \cdot v) = c\frac{dv}{dx}$ | |

**E1:** Find $\frac{dy}{dx}$ if $y = \sqrt{\frac{1+x^2}{1-x^2}}$ **Sol:** $y = \left(\frac{1+x^2}{1-x^2}\right)^{1/2}$; $\frac{dy}{dx} = \frac{1}{2}\left(\frac{1+x^2}{1-x^2}\right)^{-1/2} \cdot \frac{d}{dx}\left(\frac{1+x^2}{1-x^2}\right) = \frac{1}{2}\left(\frac{1-x^2}{1+x^2}\right)^{1/2} \cdot \frac{2x(1-x^2) - (-2x)(1+x^2)}{(1-x^2)^2} = \frac{1}{2}\left(\frac{1-x^2}{1+x^2}\right)^{1/2} \cdot \frac{4x}{(1-x^2)^2} = \frac{2x}{(1+x^2)^{1/2}(1-x^2)^{3/2}}$

**E. THE CHAIN RULE.** If $y$ is a function $f(u)$ of a variable $u$ and $u$ is a function $g(x)$ of a variable $x$, then $y = f(u) = f[g(x)]$ is a function of $x$. If $f(u)$ and $g(x)$ are differentiable functions, the *chain rule* gives: $\frac{dy}{dx} = f'(u) \cdot g'(x) = \frac{dy}{du} \cdot \frac{du}{dx}$. **E:** Find $\frac{dy}{dx}$ if $y = u^3 - 3u + 1$, $u = x^2 - x$. **Sol. Method 1:** $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (3u^2 - 3) \cdot (2x - 1) = 3(2x-1)[(x^2-x)^2 - 1]$. **Method 2:** $y = u^3 - 3u + 1 = (x^2-x)^3 - 3(x^2-x) + 1$; $\frac{dy}{dx} = 3(x^2-x)^2(2x-1) - 3(2x-1) = 3(2x-1)[(x^2-x)^2 - 1]$

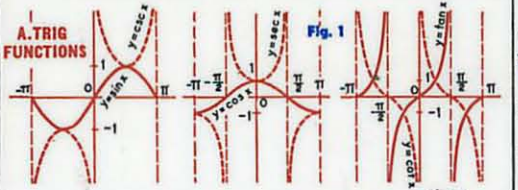
**F. DIFFERENTIATION OF IMPLICIT FUNCTIONS.** From an equation $f(x,y) = 0$, defining $y$ as an *implicit* function of $x$, $dy/dx$ may be obtained by *implicit differentiation* without solving explicitly for $y$. ▶ Differentiate the terms of each member of the implicit equation (considering $y$ as a differentiable function of $x$). ▶ Equate the derivatives of the two members. ▶ Solve for $dy/dx$.
**E:** Find $\frac{dy}{dx}$ if $x^3 + 3x^2y + y^3 = 8$. **Sol:** $\frac{d}{dx}(x^3) + \frac{d}{dx}(3x^2y) + \frac{d}{dx}(y^3) = \frac{d}{dx}(8)$; then $3x^2 + 3\left(2xy + x^2\frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} = 0$; $\frac{dy}{dx}(3x^2 + 3y^2) = -3x^2 - 6xy$; $\frac{dy}{dx} = -\frac{x^2+2xy}{x^2+y^2}$

**G. DIFFERENTIATION OF INVERSE FUNCTIONS.** The equation $h(x,y) = 0$ may be solved for either $y$ or $x$ in the form $y = f(x)$ or $x = g(y)$; here $f(x)$ and $g(y)$ are *inverse functions*. **E:** $y = x^3$, $x = \sqrt[3]{y}$. **NOTE:** The inverse of a single-valued function $f(x)$ may be multiple-valued. ▶ To find the derivative $dx/dy$ of the inverse function $g(y)$ if the derivative $dy/dx$ of $f(x)$ is known, use $\frac{dx}{dy} = \frac{1}{dy/dx}$. **E:** If $y = x^3$, then $\frac{dy}{dx} = 3x^2$ and $\frac{dx}{dy} = \frac{1}{3x^2}$; otherwise, $x = \sqrt[3]{y} = y^{1/3}$ and $\frac{dx}{dy} = \frac{1}{3}y^{-2/3} = \frac{1}{3(\sqrt[3]{y})^2} = \frac{1}{3x^2}$

## III

## TRANSCENDENTAL FUNCTIONS. (Non-algebraic).

**A. TRIG FUNCTIONS**

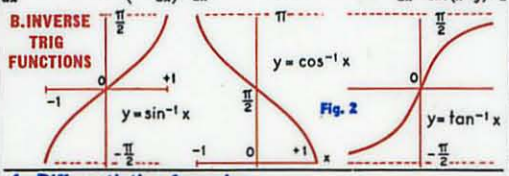



**1. Differentiation formulas** (based on the result $\lim_{x\to 0} \frac{\sin x}{x} = 1$.)
In the formulas, **u** (measured in radians) is a function of **x**.

| | |
|---|---|
| $\frac{d}{dx}(\sin u) = \cos u\frac{du}{dx}$ | $\frac{d}{dx}(\sec u) = \sec u \tan u\frac{du}{dx}$ |
| $\frac{d}{dx}(\cos u) = -\sin u\frac{du}{dx}$ | $\frac{d}{dx}(\csc u) = -\csc u \cot u\frac{du}{dx}$ |
| $\frac{d}{dx}(\tan u) = \sec^2 u\frac{du}{dx}$ | $\frac{d}{dx}(\cot u) = -\csc^2 u\frac{du}{dx}$ |

**E1:** $\frac{d}{dx}(\sin ax \cdot \tan ax) = a\cos ax \cdot \tan ax + \sin ax \cdot a\sec^2 ax = a(\sin ax + \sec ax \cdot \tan ax)$. **E2:** $y = \cos(x - y)$. **Sol.** By implicit differentiation, $\frac{dy}{dx} = -\sin(x-y) \cdot \left(1 - \frac{dy}{dx}\right)$; $\frac{dy}{dx}[1 - \sin(x-y)] = -\sin(x-y)$; $\frac{dy}{dx} = \frac{\sin(x-y)}{\sin(x-y) - 1}$

**B. INVERSE TRIG FUNCTIONS**

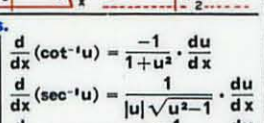



**1. Differentiation formulas.**

| | |
|---|---|
| $\frac{d}{dx}(\sin^{-1}u) = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$ | $\frac{d}{dx}(\cot^{-1}u) = \frac{-1}{1+u^2} \cdot \frac{du}{dx}$ |
| $\frac{d}{dx}(\cos^{-1}u) = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$ | $\frac{d}{dx}(\sec^{-1}u) = \frac{1}{\lvert u\rvert\sqrt{u^2-1}} \cdot \frac{du}{dx}$ |
| $\frac{d}{dx}(\tan^{-1}u) = \frac{1}{1+u^2} \cdot \frac{du}{dx}$ | $\frac{d}{dx}(\csc^{-1}u) = \frac{-1}{\lvert u\rvert\sqrt{u^2-1}} \cdot \frac{du}{dx}$ |

**E:** $\frac{d}{dx}\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{d}{dx}\left(\frac{1-x}{1+x}\right) \div \left[1 + \left(\frac{1-x}{1+x}\right)^2\right] = \frac{-1(1+x) - 1(1-x)}{(1+x)^2} \div \frac{(1+x)^2 + (1-x)^2}{(1+x)^2} = -2 \div [(1+2x+x^2) + (1-2x+x^2)] = -1/(1+x^2)$

**C. EXPONENTIAL AND LOGARITHMIC FUNCTIONS.** ▶ The *exponential function* is defined by $y = a^x$, $a > 0$. ▶ The *logarithmic function* $y = \log_a x$, where $a$ is the base, is the inverse of the exponential function. Logarithms with the base **e** are called *natural logs* (symbol *ln*); $\ln x = \log_e x$ and $e = \lim_{x\to 0}(1+x)^{1/x} = 2.71828$ approx.



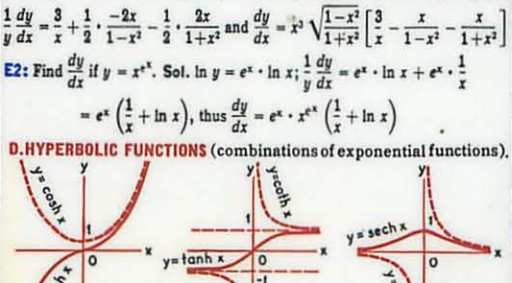

**1. Differentiation formulas.**

| | |
|---|---|
| $\frac{d}{dx}(a^u) = a^u \ln a\frac{du}{dx}$ | $\frac{d}{dx}(\log_a u) = \frac{1}{u} \cdot \log_a e \cdot \frac{du}{dx}$ |
| $\frac{d}{dx}(e^u) = e^u\frac{du}{dx}$ | $\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$ |
| | $\frac{d}{dx}(u^v) = vu^{v-1}\frac{du}{dx} + \ln u \cdot u^v\frac{dv}{dx}$ |

**E:** $\frac{d}{dx}(e^{-x}\ln x) = -e^{-x}\ln x + e^{-x} \cdot \frac{1}{x} = e^{-x}\left(\frac{1}{x} - \ln x\right)$

**2. Logarithmic differentiation.** The derivative of a function which consists of products and quotients of powers may be found by taking its log prior to differentiation.
**E1:** Find $\frac{dy}{dx}$ if $y = x^3\sqrt{\frac{1-x^2}{1+x^2}}$. **Sol.** $\ln y = \ln x^3 + \ln\sqrt{1-x^2} - \ln\sqrt{1+x^2}$, then $\ln y = 3\ln x + 1/2\ln(1-x^2) + 1/2\ln(1+x^2)$. Differentiating each side, $\frac{1}{y}\frac{dy}{dx} = \frac{3}{x} + \frac{1}{2} \cdot \frac{-2x}{1-x^2} - \frac{1}{2} \cdot \frac{2x}{1+x^2}$ and $\frac{dy}{dx} = x^3\sqrt{\frac{1-x^2}{1+x^2}}\left[\frac{3}{x} - \frac{x}{1-x^2} - \frac{x}{1+x^2}\right]$
**E2:** Find $\frac{dy}{dx}$ if $y = x^{x^2}$. **Sol.** $\ln y = e^x \cdot \ln x$; $\frac{1}{y}\frac{dy}{dx} = e^x \cdot \ln x + e^x \cdot \frac{1}{x}$ $= e^x\left(\frac{1}{x} + \ln x\right)$, thus $\frac{dy}{dx} = e^x \cdot x^{e^x}\left(\frac{1}{x} + \ln x\right)$

**D. HYPERBOLIC FUNCTIONS** (combinations of exponential functions).




$\sinh x = \frac{1}{2}(e^x - e^{-x}) = \frac{1}{\operatorname{csch} x}$ | $\cosh x = \frac{1}{2}(e^x + e^{-x}) = \frac{1}{\operatorname{sech} x}$

$\tanh x = \sinh x/\cosh x = 1/\coth x$

**1. Identities.**

| | |
|---|---|
| $\cosh^2 x - \sinh^2 x = 1$ | $\sinh(x+y) = \sinh x\cosh y + \cosh x\sinh y$ |
| $1 - \tanh^2 x = \operatorname{sech}^2 x$ | $\cosh(x+y) = \cosh x\cosh y + \sinh x\sinh y$ |
| $\coth^2 x - 1 = \operatorname{csch}^2 x$ | $\sinh 2x = 2\sinh x\cosh x$ |
| | $\cosh 2x = \cosh^2 x + \sinh^2 x$ |

**2. Differentiation formulas.**

| | |
|---|---|
| $\frac{d}{dx}(\sinh u) = \cosh u\frac{du}{dx}$ | $\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u\frac{du}{dx}$ |
| $\frac{d}{dx}(\cosh u) = \sinh u\frac{du}{dx}$ | $\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u\tanh u\frac{du}{dx}$ |
| $\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u\frac{du}{dx}$ | $\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u\coth u\frac{du}{dx}$ |

**E. INVERSE HYPERBOLIC FUNCTIONS** (written as log functions).
$\sinh^{-1}x = \ln(x + \sqrt{x^2+1}) = \operatorname{csch}^{-1}(1/x)$
$\cosh^{-1}x = \ln(x + \sqrt{x^2-1}) = \operatorname{sech}^{-1}(1/x)$
$\tanh^{-1}x = \frac{1}{2}\ln\frac{1+x}{1-x} = \coth^{-1}\left(\frac{1}{x}\right)$

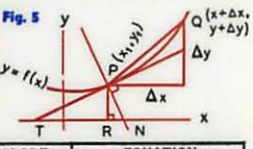
**1. Differentiation formulas.**

| | |
|---|---|
| $\frac{d}{dx}(\sinh^{-1}u) = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$ | $\frac{d}{dx}(\coth^{-1}u) = \frac{1}{1-u^2}\frac{du}{dx}$, $\lvert u\rvert > 1$ |
| $\frac{d}{dx}(\cosh^{-1}u) = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$ | $\frac{d}{dx}(\operatorname{sech}^{-1}u) = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}$ |
| $\frac{d}{dx}(\tanh^{-1}u) = \frac{1}{1-u^2}\frac{du}{dx}$, $\lvert u\rvert < 1$ | $\frac{d}{dx}(\operatorname{csch}^{-1}u) = \frac{-1}{\lvert u\rvert\sqrt{1+u^2}}\frac{du}{dx}$ |

## IV

## GEOMETRIC APPLICATIONS. A. TANGENT AND NORMAL.

If $P(x,y)$ and $Q(x+\Delta x, y+\Delta y)$ are two distinct points of the curve $y = f(x)$, then $PQ$ is a *secant line* drawn at $P$. The *tangent* $PT$ at $P$ is the limiting position of the secant as $Q$ approaches $P$ along the curve. *Normal* is $PN$ perpendicular to tangent at $P$.




| LINE | SLOPE | EQUATION |
|---|---|---|
| SECANT | $\Delta y/\Delta x$ | |
| TANGENT | $f'(x_1) = m$ | $y - y_1 = m(x - x_1)$ |
| NORMAL | $-1/m$ | $y - y_1 = (-1/m)(x - x_1)$ |

**1. Slopes and equations**

Slope $m$ of the tangent at $P$ also called *slope of curve* at $P$.
**E:** Find equation of the tangent and normal to the hyperbola $x^2 - y^2 = 9$ at the point (5, 4). **Sol.** By implicit differentiation, $2x - 2y\frac{dy}{dx} = 0$, $\frac{dy}{dx} = \frac{x}{y}$; $m = \left(\frac{dy}{dx}\right)_P = \frac{5}{4}$. Thus the equation of the tangent is $y - 4 = \frac{5}{4}(x-5)$ or $5x - 4y = 9$. The equation of the normal is $y - 4 = -\frac{4}{5}(x-5)$ or $4x + 5y = 40$.

| SEGMENT | NAME | LENGTH |
|---|---|---|
| RT | Length of *subtangent* | $\lvert y_1/m\rvert$ |
| RN | " " *subnormal* | $\lvert y_1 m\rvert$ |
| PT | " " *tangent* | $\lvert y_1\sqrt{1+m^2}/m\rvert$ |
| PN | " " *normal* | $\lvert y_1\sqrt{1+m^2}\rvert$ |

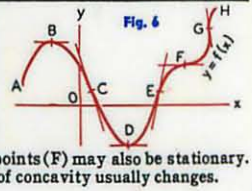
**2. Lengths of line segments**

**3. Angle of intersection.** The *angle of intersection* of two curves is the angle between the tangents at the point of intersection. To determine the *angle of intersection*: ▶ Solve the equations of the curves simultaneously to find the points of intersection. ▶ Find the slopes $m_1$ and $m_2$ of the curves at each point of intersection $P$. ▶ Calculate an *angle of intersection* at $P$ by $\tan\phi = |m_1 - m_2| \div |1 + m_1m_2|$.
**E:** Find angle at which circles $x^2 + y^2 - 4x = 1$ and $x^2 + y^2 - 2y = 9$ intersect in the first quadrant. **Sol.** Solving simultaneously, points of intersection are (3,2) and (1, −2). By implicit differentiation, $2x + 2y\left(\frac{dy}{dx}\right)_1 - 4 = 0$ or $\left(\frac{dy}{dx}\right)_1 = \frac{2-x}{y}$ for first circle and $2x + 2y\left(\frac{dy}{dx}\right)_2 - 2\left(\frac{dy}{dx}\right)_2 = 0$ or $\left(\frac{dy}{dx}\right)_2 = \frac{x}{1-y}$ for second circle. At (3,2), $m_1 = -1/2$; $m_2 = -3$; $\tan\phi = \frac{-1/2+3}{1+3/2} = 1$; $\phi = \pi/4$

**B. CONCAVITY, STATIONARY AND INFLECTION POINTS.**

| NAME | DEFINITION | TEST | FIG. 6 |
|---|---|---|---|
| **Increasing function f(x)** | $f(x)$ increases as $x$ increases | $f'(x) > 0$ | arcs AB, DH |
| **Decreasing function f(x)** | $f(x)$ decreases as $x$ increases | $f'(x) < 0$ | arc BD |
| **Concave upward curve, y = f(x)** | Curve's slope $f'(x)$ is increasing function. | $f''(x) > 0$ | arcs CDE, FG |
| **Concave downward curve, y = f(x)** | Curve's slope $f'(x)$ is a decreasing function | $f''(x) < 0$ | arcs ABC, EF, GH |
| **Stationary point of curve, y = f(x)** | Point where tangent is horizontal | $f'(x) = 0$ | points B, D, F |
| **Critical value of function f(x)** | Value of $f(x)$ at stationary point | | |
| **f(x) has a local or relative maximum at x = x_B** | $f(x_B) \geq f(x_B + h)$ for all sufficiently small values of $h$ | $f'(x_B) = 0$ <br> $f''(x_B) < 0$ | point B |
| **f(x) has a local or relative minimum at x = x_D** | $f(x_D) \leq f(x_D + h)$ for all sufficiently small values of $h$ | $f'(x_D) = 0$ <br> $f''(x_D) > 0$ | point D |
| **Point of inflection of curve, y = f(x)** | Point where slope $f'(x)$ is stationary | $f''(x) = 0$ | points C, E, G |

In exceptional cases, an inflection point may occur when $f''(x)$ does not exist. The local maxima and minima of differentiable functions are always stationary points; that is, they satisfy the equation $f'(x) = 0$. These points separate regions of rise and fall of the function, however other points (F) may also be stationary. At an inflection point, direction of concavity usually changes.



**C. CURVE SKETCHING THE GRAPH OF $y=f(x)$.** ▶ Calculate $f'(x)$ and $f''(x)$. ▶ Determine stationary points from $f'(x)=0$, the corresponding values of $y$, and the ranges where $f'(x)$ is positive or negative. Test stationary points for relative maxima or minima (see D below). ▶ Determine inflection points from $f''(x)=0$, the corresponding values of $y$, and the ranges where $f''(x)$ is positive or negative. ▶ Plot additional points where necessary. ▶ Sketch a curve through the points making use of the rise and fall, the concavity, the stationary points and the points of inflection.

**E:** Sketch the curve $y=x^3-3x^2-9x+10$.
**Sol.** $y'=3x^2-6x-9$, $y''=6x-6$; the stationary points occur at $x=-1,3$ and the inflection point at $x=1$.

| x | y | y' | y'' | Remarks | |
|---|---|---|---|---|---|
| $-3$ | $-17$ | + | − | rising | concave downward |
| $-1$ | 15 | 0 | − | maximum | |
| 1− | 1− | − | 0 | falling | |
| | | | | inflection pt. | |
| 3 | $-17$ | 0 | + | minimum | concave upward |
| 5 | 15 | + | + | rising | |

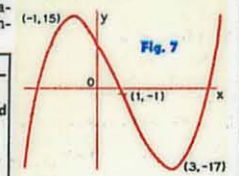


Fig. 7

**D. TO FIND LOCAL MAXIMA AND MINIMA.** Solve $f'(x)=0$. If $x_0$ is any root so that $f'(x_0)=0$, then

| | | | | or | as x increases from $x_0-\delta$ to $x_0+\delta$ | | |
|---|---|---|---|---|---|---|---|
| $f(x_0)$ is | max | if $f''(x_0)$ is | neg | | if either $f(x)-f(x_0)$ | is neg | or if $f'(x)$ changes | +to− |
| | min | | pos | | | is pos | | −to+ |

**E. SOLVING APPLIED PROBLEMS.** ▶ Set up the function whose maximum or minimum value is desired and, if more than one variable is involved, use relationships among them to express the function in terms of one variable. ▶ Then apply above rule. ▶ Draw graph of the function as a check.

**E1:** What are the dimensions of the largest rectangular box with square base and open top which can be made from 192 sq. ft. of material?
**Sol.** Let $x$ be side of base and $y$ the altitude of the box. Then $x^2+4xy=192$, $y=\frac{192-x^2}{4x}$. The volume $V=x^2y=\frac{1}{4}(192x-x^3)$. $V'=\frac{1}{4}(192-3x^2)=0$, $x=+8,-8$. $V''$ (8) is negative so that $x=8$ yields maximum value $V(8)=256$ cu. ft. and $y=4$ ft. Dimensions are 8′x8′x4′.

## V
## POLAR COORDINATES AND PARAMETRIC EQUATIONS. A. POLAR COORDINATES.

The relations between *polar coordinates* $(r,\Theta)$ and rectangular coordinates $(x,y)$ are $x=r\cos\Theta$, $y=r\sin\Theta$; $r=\sqrt{x^2+y^2}$, $\Theta=\tan^{-1}y/x$. In Fig. 8, the angle $\psi$ between the tangent to the curve $r=r(\Theta)$ at $P$ and the radius vector is given by $\tan\psi=r\div dr/d\Theta$. The inclination $\tau$ of the tangent line with the horizontal is then determined by $\tau=\psi+\Theta$. If two curves intersect at $P$, their angle of intersection $\phi$ is given by $\tan\phi=[\tan\psi_1-\tan\psi_2]\div[1+\tan\psi_1\tan\psi_2]$.

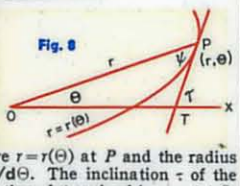


Fig. 8

**E:** Find the angle of intersection of the curves $C_1$: $r=2(1-\cos\Theta)$, $C_2$: $r=2\cos\Theta$. **Sol.** Solve equations simultaneously: $2(1-\cos\Theta)=2\cos\Theta$, $\cos\Theta=\frac{1}{2}$ so that the common points are $P\left(1,\frac{\pi}{3}\right)$ and $Q\left(1,\frac{5\pi}{3}\right)$. For $C_1$: $\frac{dr}{d\Theta}=2\sin\Theta$ and $\left(\frac{dr}{d\Theta}\right)_P=2\cdot\frac{\sqrt{3}}{2}=\sqrt{3}$ so that $\tan\psi_1=\frac{1}{\sqrt{3}}$. For $C_2$: $\frac{dr}{d\Theta}=-2\sin\Theta$ and $\left(\frac{dr}{d\Theta}\right)_P=-2\cdot\frac{\sqrt{3}}{2}=-\sqrt{3}$ so that $\tan\psi_2=-\frac{1}{\sqrt{3}}$. Calculate $\phi$: $\tan\phi=\frac{1/\sqrt{3}+1/\sqrt{3}}{1-1/\sqrt{3}\cdot1/\sqrt{3}}=\sqrt{3}$, $\phi=\frac{\pi}{6}$

**B. PARAMETRIC EQUATIONS OF A CURVE.** Each of the variables $x$ and $y$ is expressed in terms of a third variable, $t$ for example, called the *parameter* in the form $x=f(t)$, $y=g(t)$. By the chain rule:

$$y'=\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt} \qquad y''=\frac{dy'}{dx}=\frac{dy'}{dt}\cdot\frac{dt}{dx}=\frac{dy'}{dt}\Big/\frac{dx}{dt}$$

**E:** Find $y'$, $y''$ for the cycloid whose parametric equations are $x=a(\Theta-\sin\Theta)$, $y=a(1-\cos\Theta)$. **Sol.** $\frac{dy}{d\Theta}=a\sin\Theta$, $\frac{dx}{d\Theta}=a(1-\cos\Theta)$, $y'=\frac{dy}{d\Theta}\Big/\frac{dx}{d\Theta}=\frac{\sin\Theta}{1-\cos\Theta}$

$$\frac{dy'}{d\Theta}=\frac{\cos\Theta(1-\cos\Theta)-\sin\Theta\cdot\sin\Theta}{(1-\cos\Theta)^2}=-\frac{1}{(1-\cos\Theta)^2},\quad y''=\frac{dy'}{d\Theta}\Big/\frac{dx}{d\Theta}=-[1+a(1-\cos\Theta)^2]$$

## VI
## PHYSICAL APPLICATIONS. A. RECTILINEAR MOTION.

Motion in a straight line is described by an equation of the form $s=f(t)$ where $s$ represents *distance* and $t$ *time*. The *average velocity* during the time $t$ to $t+\Delta t$ equals $\Delta s/\Delta t$. The *velocity* $v$ and the *acceleration* $a$ at time $t$ are $v=ds/dt$, $a=d^2s/dt^2$.

**E:** The height $s$ ft. of a particle $t$ sec. after being thrown upward is $s=128t-16t^2$. Find the velocity and acceleration at any time $t$, the initial velocity $v_0$ and maximum height $s_{max}$. **Sol.** $v=ds/dt=128-32t$ ft./sec., $a=-32$ ft./sec.$^2$. When $t=0$, $v_0=128$ ft./sec. At maximum height $v=0$ so that $128-32t=0$, $t=4$ sec., $s_{max}=(128)(4)-(16)(16)=256$ ft.

**B. CURVILINEAR MOTION. 1. Rectangular components.** Motion in a curved path is described by parametric equation like $x=f(t)$, $y=g(t)$. *Velocity vector* $v$, *acceleration vector* $a$ given by:

| | x-component | y-component | magnitude | inclination |
|---|---|---|---|---|
| v | $v_x=dx/dt$ | $v_y=dy/dt$ | $\lvert v\rvert=\sqrt{v_x^2+v_y^2}$ | $\tan\Theta_v=v_y/v_x$ |
| a | $a_x=d^2x/dt^2$ | $a_y=d^2y/dt^2$ | $\lvert a\rvert=\sqrt{a_x^2+a_y^2}$ | $\tan\Theta_a=a_y/a_x$ |

**E:** A particle moves in a circle of radius $b$ with constant angular velocity $\omega$. Find the velocity and acceleration. **Sol.** The parametric equations are $x=b\cos\omega t$, $y=b\sin\omega t$. Then $v_x=-b\omega\sin\omega t$, $v_y=b\omega\cos\omega t$, $a_x=-b\omega^2\cos\omega t=-\omega^2x$, $a_y=-b\omega^2\sin\omega t=-\omega^2y$.

**2. Radial and transverse components.** The components of the velocity vector along the radius vector and perpendicular to it are called the *radial component* $v_r$ and the *transverse component* $v_\theta$ respectively. If the polar equations of motion are $r=f(t)$, $\Theta=g(t)$, then $v_r=dr/dt$, $v_\theta=r\,d\Theta/dt$.

**3. Tangential and normal components.** The direction of the velocity vector is always along the tangent. The components of the acceleration vector along the tangent and normal are called the *tangential component*, $a_t$ and the *normal component*, $a_n$, respectively: $a_t=dv/dt$ and $a_n=v^2K$ where $K$ is the curvature at the point of the curve defined in VIII.

**C. RELATED RATES.** When a variable $z$ is a function of another variable $t$, then the rate of change of $z$ with respect to $t$ equals $dz/dt$. If $t$ is time, $dz/dt$ equals the time rate of change of $z$. If several variables satisfy an equation, the relation between their rates of change can be found by differentiating with respect to $t$. *To find an unknown rate:* ▶ Write an equation among the variables and from it find any variable which is unknown. ▶ Differentiate this equation with respect to $t$ to obtain a rate equation. ▶ Substitute all known variables and rates and solve.

**E1:** A man is walking at the rate of 5 mi./hr. towards a building 120 ft. high. At what rate is he approaching the top when he is 90 ft. from the building? **Sol.** From Fig. 9, $x^2+14400=y^2$. When $x=90$ ft., $y=150$ ft. Differentiate the equation to give $2x\frac{dx}{dt}=2y\frac{dy}{dt}$. Substitute the values of $x$, $y$ and $dx/dt=-5$ mi./hr. to give $dy/dt=\frac{x}{y}\frac{dx}{dt}=\frac{-90}{150}\cdot5$ mi./hr. $=-3$ mi./hr.

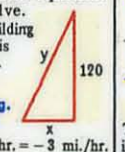


Fig. 9

## VII
## DIFFERENTIALS. A. DEFINITION.

Let $y=f(x)$; then the differential $dx$ of the independent variable $x$ is defined by $dx=\Delta x$. The differential $dy$ of the dependent variable $y$ is defined so that the ratio $dy/dx$ equals the derivative $f'(x)$; that is $dy=f'(x)dx$. Thus $dx$ is any increment of $x$, $\Delta y$ is the corresponding change in $y$ to curve $y=f(x)$; $dy$ is the corresponding change in $y$ to the tangent to the curve at $P(x,y)$ in Fig. 10.

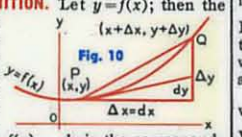


Fig. 10

**B. APPROXIMATIONS; ERRORS. 1. Approximations.** If $\Delta x$ is sufficiently small, $dy$ is an approximation to $\Delta y$.

**E:** Find $\sqrt[3]{65}$ approximately. **Sol.** Let $y=x^{1/3}$. Then $dy=\frac{1}{3}x^{-2/3}dx$. When $x=64$, $dx=1$, $y=4$ and $\Delta y=dy=\frac{1}{3}\cdot\frac{1}{16}\cdot1=\frac{1}{48}=.02$. Hence $\sqrt[3]{65}=4.02$

**2. Small errors.** A small error $\Delta x$ in the independent variable leads to an *error* $\Delta y$ in the dependent variable which may be approximated by $dy$. The *relative error* is $\Delta y/y=dy/y$ and the *percentage error* is $100\,\Delta y/y=100\,dy/y$.

**E:** The side of a cube is measured as 10 in., subject to an error of 1/8 in. What is the relative error in surface area $S$? **Sol.** Let $S=6x^2$. Then $dS=12x\cdot dx$, $\frac{dS}{S}=2\frac{dx}{x}$. When $x=10$, $dx=\frac{1}{8}$; $\frac{dS}{S}=\frac{2(1/8)}{10}=\frac{1}{40}$

**C. DIFFERENTIAL ARC LENGTH.** The arc length of a curve is a function of the independent variable along the curve. The *differential arc length* is equal to $ds=\sqrt{(dx)^2+(dy)^2}$ in rectangular coordinates and $ds=\sqrt{(dr)^2+r^2(d\Theta)^2}$ in polar coordinates.

## VIII
## MISCELLANEOUS APPLICATIONS. A. CURVATURE.

The *curvature of a curve* $K$ is the rate of change of the inclination of its tangent line with respect to arc length. Formulas:

| EQUATION OF CURVE | CURVATURE |
|---|---|
| $y=f(x)$ | $K=y''\div[1+y'^2]^{3/2}$ |
| $r=g(\Theta)$ | $K=[r^2+2r'^2-rr'']\div[r^2+r'^2]^{3/2}$ |
| $x=x(t)$; $y=y(t)$ | $K=[x'y''-y'x'']\div[x'^2+y'^2]^{3/2}$ |

**1. The radius of curvature** $R$ is the absolute value of the reciprocal of the curvature: $R=1/|K|$. **2. The circle of curvature** is the circle which is tangent to a curve at $P$ and has the same radius of curvature as the curve at $P$. It is also the limiting position of the circle through three points of the curve as these points all approach $P$ as a limit. **3. The center of curvature** is the center of the circle of curvature; its rectangular coordinates $(\alpha,\beta)$ are $\alpha=x-\frac{y'(1+y'^2)}{y''}$, $\beta=y+\frac{1+y'^2}{y''}$.

**4. The evolute of a curve** is the locus of the center of curvature of the curve. The above equations are parametric equations of the evolute with $(\alpha,\beta)$ as variable coordinates, $x$ as parameter.

**E:** Find the radius of curvature at any point $(x,y)$ of the curve $y=\ln\sec x$. **Sol.** $y'=\sec x\tan x/\sec x=\tan x$, $y''=\sec^2x$, $K=\sec^2x/[1+\tan^2x]^{3/2}=\sec^2x/\sec^3x=1/\sec x$. $R=|\sec x|$

**B. NEWTON'S METHOD.** After approximate location of a root of $f(x)=0$, *Newton's method* yields successive approximations with increasing accuracy. If $x_1$ is the first approximation, *Newton's formula* for the next value $x_2$ of the root is $x_2=x_1+\Delta x_1$, $\Delta x_1=-[f(x_1)\div f'(x_1)]$. Value $x_2$ in place of $x_1$ yields a third more accurate value $x_3$, etc. This successive root substitution is continued until the desired accuracy is obtained.

**E:** Find the root of $x^3-5x+1=0$ between 0 and 1. **Sol.** Here Newton's formula is $\Delta x_1=-\frac{x_1^3-5x_1+1}{3x_1^2-5}$. Let $x_1=0$. Then $\Delta x_1=-\frac{1}{-5}=0.2$, $x_2=0.2$. Replace $x_1$ by 0.2. Then $\Delta x_2=-\frac{0.008-1+1}{3(0.04)-5}=-\frac{0.008}{-4.88}=0.0016$; $x_3=0.2016$

## IX
## THE MEAN VALUE THEOREM AND INDETERMINATE FORMS. A. MEAN VALUE THEOREM.

The basis of the theorem is that at some point $R$ between $P$ and $Q$, the tangent at $R$ will be parallel to the secant $PQ$: *If* $f(x)$ *is a single-valued differentiable function in the interval* $(a,b)$, *then there is at least one value of* $x_1$ *of* $x$ *such that* $a<x_1<b$ *for which* $f(b)-f(a)=(b-a)f'(x_1)$. If $P$ and $Q$ lie on the $x$-axis, the result says that $f'(x)=0$ and is known as **Rolle's** theorem.

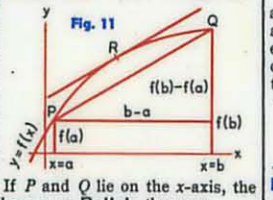


Fig. 11

**B. INDETERMINATE FORMS** $(0/0,\ \infty/\infty,\ 0\cdot\infty;\ 0^0,\ \infty^0,\ 1^\infty;\ \infty-\infty)$. If for a particular value $x=a$, $F(x)$ assumes an indeterminate form, then $F(a)$ is not defined. To find the limit of $F(x)$ as $x$ approaches $a$ (if this limit exists) use the following **Rule:** If $f(x)$ and $g(x)$ are differentiable and either $f(a)=g(a)=0$ or $f(a)=g(a)=\infty$, then $\lim_{x\to a}f(x)/g(x)=\lim_{x\to a}f'(x)/g'(x)$, provided the latter limit exists ($a$ may be finite or infinite). If $f'(x)/g'(x)$ is itself an indeterminate form $0/0$ or $\infty/\infty$, the same rule may be applied to obtain $f''(x)/g''(x)$ and this procedure repeated until the expression is not indeterminate.

**E1:** Evaluate $\lim_{x\to\frac{\pi}{2}}\sec x\cdot\sin2x$. **Sol.** $\lim_{x\to\frac{\pi}{2}}\sec x\cdot\sin2x=\lim_{x\to\frac{\pi}{2}}\frac{\sin2x}{\cos x}=\lim_{x\to\frac{\pi}{2}}\frac{2\cos2x}{-\sin x}=\frac{2(-1)}{(-1)}=2$. **E2:** Evaluate $\lim_{x\to0}(1+ax)^{1/x}$. **Sol.** $y=(1+ax)^{1/x}$, $\ln y=\frac{1}{x}\ln(1+ax)$, $\lim_{x\to0}\ln y=\lim_{x\to0}\frac{\ln(1+ax)}{x}=\lim_{x\to0}\frac{\frac{a}{1+ax}}{1}=a$, $\lim_{x\to0}y=e^a$

| LIMITS | EVALUATION PROCEDURE |
|---|---|
| I. $\lim\frac{f}{g}=\frac{0}{0}$ or $\frac{\infty}{\infty}$ | Apply Rule to $\lim\frac{f}{g}$ |
| II. $\lim f\cdot g=0\cdot\infty$ | Apply Rule to $\lim\frac{f}{1/g}$ or $\lim\frac{g}{1/f}$ |
| III. $\lim fg=\begin{cases}1^\infty\\ \infty^0\\ 0^0\end{cases}$ | If $y=f^g$, $\ln y=g\ln f$ which is of Type II. Apply procedure for Type II to evaluate $\lim\ln y=c$. Then $\lim y=e^c$ |
| IV. $\lim(f-g)=\infty-\infty$ | If possible, transform $f-g$ into Type I, use Type I procedure |

## X
## PARTIAL DIFFERENTIATION. A. PARTIAL DERIVATIVE.

The *partial derivative* of a function $f(x,y)$ with respect to one of its independent variables $x$ is the derivative of $f$ with respect to $x$ when all independent variables except $x$ are held constant. It is written $\frac{\partial f}{\partial x}$ or $f_x$. In symbols, $\frac{\partial f}{\partial x}=\lim_{\Delta x\to0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$.

For the surface $z=f(x,y)$, $f_x$ may be interpreted as the slope of the tangent of the curve which is the intersection of the surface with a plane $y=$ constant. This partial derivative $f_x$, in turn, is also a function of $x$ and $y$ and its higher order partial derivatives, written $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2f}{\partial y\partial x}=f_{yx}$, etc. may be found similarly.

When a function and all its partial derivatives of $n$-th and lower order are continuous, then *the order of differentiation in any partial derivative of order* $n$ *or less is immaterial*, the result depending only on the total number of differentiations with respect to each variable which occurs. For example, $f_{yyxxy}=f_{xxyyy}$

**E:** If $z=e^{2x}\cos y$, find $\frac{\partial^2z}{\partial x^2}$, $\frac{\partial^2z}{\partial x\,\partial y}$ **Sol.** $\frac{\partial z}{\partial x}=2e^{2x}\cos y$, $\frac{\partial^2z}{\partial x\,\partial y}=\frac{\partial^2z}{\partial y\,\partial x}=\frac{\partial}{\partial y}(2e^{2x}\cos y)=-2e^{2x}\sin y$.

**B. TOTAL DIFFERENTIAL; APPROXIMATIONS.** The *total differential* $df$ of a function $f(x,y)$ is defined by $df=f_x\,dx+f_y\,dy$ where $dx=\Delta x$, $dy=\Delta y$. For small increments of the independent variables, $df$ is an approximation to the increment $\Delta f$ of the function (used in approximation and error calculations). For the surface $z=f(x,y)$, $dz$ represents the change in $z$ to the tangent plane to the surface as $x$ and $y$ undergo independent changes while $\Delta z$ represents the change in $z$ to the surface.

**E:** Find the approximate maximum error in the volume of a right circular cylinder of radius 2 ft. and altitude 3 ft. with possible errors of 1/4 in. in each dimension. **Sol.** The volume is given by $V=\pi r^2h$, $\Delta V=dV=\pi(2rh\,dr+r^2dh)$, $r=2$, $h=3$, $dr=dh=1/48$. $\Delta V=\pi(2\cdot2\cdot3\cdot1/48+4\cdot1/48)=\pi/3$ cu. ft.

**C. TOTAL DERIVATIVE; CHANGE OF VARIABLE.** If $x=x(t)$, $y=y(t)$, then the *total derivative* of $f(x,y)$ with respect to $t$ is $\frac{df}{dt}=f_x\frac{dx}{dt}+f_y\frac{dy}{dt}$. It is the ordinary derivative of $f[x(t),y(t)]$ considered as a function of one variable $t$. If $t$ represents time, this may be interpreted as a rate equation, expressing the rate of change of $f(x,y)$ in terms of $x,y$ and their rates of change. If $x$ and $y$ undergo a change of variables, $x=x(u,v)$, $y=y(u,v)$, then $f$ becomes a function of $u$ and $v$. Then $\partial f/\partial u$ is given by $\frac{\partial f}{\partial u}=f_x\frac{\partial x}{\partial u}+f_y\frac{\partial y}{\partial v}$. A similar equation obtains for $\frac{\partial f}{\partial v}$.

**E1:** At what rate is the volume of a rectangular parallelopiped changing when each side is 10 in. long and the dimensions are increasing 1, 2, and 3 in./sec. respectively? **Sol.** The volume $V$ is given by $V=xyz$. $\frac{dV}{dt}=yz\frac{dx}{dt}+xz\frac{dy}{dt}+xy\frac{dz}{dt}$. $x=y=z=10$, $dx=1$, $dy=2$, $dz=3$. $\frac{dV}{dt}=100(1)+100(2)+100(3)=600$ cu. in./sec. **E2:** If $z=y/x$ and $x=r\cos\Theta$, $y=r\sin\Theta$, find $\frac{\partial z}{\partial\Theta}$ **Sol.** $\frac{\partial z}{\partial\Theta}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial\Theta}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial\Theta}=\left(-\frac{y}{x^2}\right)(-r\sin\Theta)+\left(\frac{1}{x}\right)(r\cos\Theta)=\frac{r^2\sin^2\Theta}{r^2\cos^2\Theta}+1=\sec^2\Theta$

**D. DIRECTIONAL DERIVATIVE.** The rate of change at $P(x,y)$ of the function $f(x,y)$ with respect to distance along the line whose direction is fixed by displacements $\Delta x$, $\Delta y$ is given by $\frac{df}{ds}=f_x\frac{dx}{ds}+f_y\frac{dy}{ds}=f_x\cos\alpha+f_y\sin\alpha$. Its value depends upon the direction of angle $\alpha$ as well as point $P(x,y)$. It is called the *directional derivative* of $f$. The maximum value of $\left|\frac{df}{ds}\right|$ at $P$ is $\sqrt{f_x^2+f_y^2}$ and occurs for the direction of the vector whose $x$- and $y$-components are $f_x$ and $f_y$ respectively. This vector is called the *gradient* of $f$ and is orthogonal at $P$ to the curve $f(x,y)=$ constant.

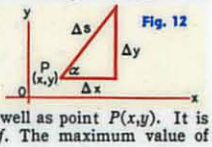


Fig. 12

**E:** Find the directional derivative of $z=\ln(x^2+y^2)$ at (3,4) in the direction $\alpha=45°$. **Sol.** $\frac{dz}{ds}=\frac{2x}{x^2+y^2}\cos\alpha+\frac{2y}{x^2+y^2}\sin\alpha=\frac{6}{25}\cdot\frac{1}{\sqrt{2}}+\frac{8}{25}\cdot\frac{1}{\sqrt{2}}=\frac{7\sqrt{2}}{25}$

**E. MAXIMA AND MINIMA.** A *local maximum* or *local minimum* of a function of two independent variables $z=f(x,y)$ is defined by analogy to the corresponding case for a function of one independent variable. To find local maxima or minima: ▶ Solve simultaneously $f_x=0$, $f_y=0$ to yield all stationary points. ▶ At each stationary point $P(x_0,y_0)$ evaluate $\Delta=f_{xx}f_{yy}-f^2_{xy}$ and $f_{xx}$ (or $f_{yy}$).

| $P(x_0,y_0)$ is a | | |
|---|---|---|
| local maximum if | $\Delta>0$ | and $f_{xx}$ (or $f_{yy}$) $<0$ |
| local minimum if | $\Delta>0$ | and $f_{xx}$ (or $f_{yy}$) $>0$ |
| neither if | $\Delta<0$. | If $\Delta=0$, the test fails. |

**E:** Examine $f(x,y)=x^3-3xy+y^3$ for maximum and minimum values. **Sol.** $f_x=3x^2-3y=0$, $f_y=3y^2-3x=0$. Substitute $y$ from first equation into the second equation to yield $x^4-x=0$ or $x=0,1$. Stationary points are (0,0) and (1,1). Now $f_{xx}=6x$, $f_{xy}=-3$, $f_{yy}=6y$, $\Delta=36xy-9$. At (0,0), $\Delta=-9$, so that (0,0) is neither a max. nor a min. At (1,1), $\Delta>0$, $f_{xx}>0$ so $f(1,1)=-1$ is a local min.

**F. TANGENT PLANE, NORMAL LINE TO A SURFACE.** All lines tangent to a surface $f(x,y,z)=0$ at a point $P(x_1,y_1,z_1)$ lie in a plane called the *tangent plane to the surface* at $P$. Its equation is $(x-x_1)\left(\frac{\partial f}{\partial x}\right)_1+(y-y_1)\left(\frac{\partial f}{\partial y}\right)_1+(z-z_1)\left(\frac{\partial f}{\partial z}\right)_1=0$. Here $\left(\frac{\partial f}{\partial x}\right)_1$ is value of $\partial f/\partial x$ at $P$, etc. The line through $P$ orthogonal to this plane is called the *normal line to the surface* at $P$. Its equation is $[x-x_1]\div\left(\frac{\partial f}{\partial x}\right)_1=[y-y_1]\div\left(\frac{\partial f}{\partial y}\right)_1=[z-z_1]\div\left(\frac{\partial f}{\partial z}\right)_1$.

**E:** Find the equation of the tangent plane and normal line to the surface $x^2-2y^2+z^2=3$ at $P(2,1,-1)$. **Sol.** At $P$, $\left(\frac{\partial f}{\partial x}\right)_1=4$, $\left(\frac{\partial f}{\partial y}\right)_1=-4$, $\left(\frac{\partial f}{\partial z}\right)_1=-2$. The equation tangent plane is $4(x-2)-4(y-1)-2(z+1)=0$ or $2x-2y-z=3$. The equation of the normal line is $(x-2)/2=(y-1)/-2=(z+1)/-1$.

DATA-GUIDE QUICK CHART

# INTEGRAL CALCULUS

## A SIMPLIFIED SUMMARY & INSTANT REFERENCE

LEARN & REVIEW IN MINUTES

**AUTHOR: Prof. J. Chiccarelli, Chairman, Math Dept., Bridgewater State College, Mass.**

SOLID PLASTIC

QUICK CHARTS SPEED STUDY: PUT THE IMPORTANT FACTS IN FULL VIEW TO SAVE YOU TIME AND WORK.

QUICK CHARTS HELP YOU REMEMBER: PERMANENTLY FILL IN FOR MISSING NOTES & TEXTS IN BASIC & ADVANCED COURSES.

CONTAINS NO PAPER

## I

**INDEFINITE INTEGRALS (ANTIDERIVATIVES). A. DEFINITIONS.** **1.** *If* $g$ *is the derivative of a function* $f$ *so that* $f'=g$, *then* $f$ *is said to be an* **antiderivative** *of* $g$. *If* $C$ *is a constant then* $C'=0$, $(f+C)'=g$, *and* $f+C$ *is also an antiderivative of* $g$. **2.** If both $f'=g$ and $h'=g$ then $(h-f)'=0$; there is a constant $C$ such that $h-f=C$ or $h=f+C$. The function $f$ has an antiderivative at each point interior to an interval on which $f$ is continuous.

**E:** $(x^3)'=3x^2$, $x^3$ is an antiderivative of $3x^2$, and each antiderivative of $3x^2$ is of the form $x^3+C$, where $C$ is a constant.

**B. THE INDEFINITE INTEGRAL.** *The collection of all antiderivatives of* $g$ *is called the* **indefinite integral** *of the* **integrand** $g$, *their values are denoted by* $\int g(x)dx$. If $f'=g$, then $\int g(x)dx=f(x)+C$. The constant $C$ is called the **constant of integration.**

**E:** $\int 3x^2dx=x^3+C$; $x^3+5$ is an indefinite integral of $3x^2$, another is $x^3-7$.

**C. DIFFERENTIATION AND INTEGRATION.** Said to be inverse to each other in the sense that, *If* $f$ *is differentiable and if* $\int g(x)dx$ *is an antiderivative of* $g$ *then:*

$$(\int g(x)dx)'=g(x). \qquad \int f'(x)dx=f(x)+C$$

$$(\int 3x^2dx)'=3x^2 \qquad \int 3x^2dx=x^3+C$$

$$(\int \sin x\,dx)'=\sin x \qquad \int \cos x\,dx=\sin x+C.$$

Thus, $\int g(x)dx=f(x)+C$ can always be checked; it will be true if and only if $f'(x)=g(x)$.

**D. PROPERTIES.** **1.** If $k$ is any constant, then $\int k\,[f(x)]dx=k\int f(x)dx$. **2.** The indefinite integral of a sum (or difference) of functions is the sum (or difference) of their indefinite integrals. Thus

$$\int [f(x)\pm g(x)\pm h(x)]dx=\int f(x)dx\pm\int g(x)dx\pm\int h(x)dx$$

**E:** $\int (3x^2-5x+3)\,dx=3\int x^2dx-5\int x\,dx+3\int dx=x^3-5/2\,x^2+3x+C$

**E. STANDARD INDEFINITE INTEGRALS.**

1. $\int u^n du=\frac{u^{n+1}}{n+1}+C\ (n\neq -1)$
2. $\int \frac{du}{u}=\ln|u|+C\ (\ln=\log_e)$
3. $\int a^u du=\frac{a^u}{\ln a}+C\ (a>0, a\neq 1)$
4. $\int e^u du=e^u+C$
5. $\int \sin u\,du=-\cos u+C$
6. $\int \cos u\,du=\sin u+C$
7. $\int \sec^2 u\,du=\tan u+C$
8. $\int csc^2\,u\,du=-\cot u+C$
9. $\int \sec u\tan u\,du=\sec u+C$
10. $\int csc\,u\cot u\,du=-\csc u+C$
11. $\int \tan u\,du=\begin{Bmatrix}-\ln|\cos u|\\ \ln|\sec u|\end{Bmatrix}+C$
12. $\int \cot u\,du=\begin{Bmatrix}\ln|\sin u|\\ -\ln|\csc u|\end{Bmatrix}+C$
13. $\int \sec u\,du=\ln|\sec u+\tan u|+C$
14. $\int \csc u\,du=\ln|\csc u-\cot u|+C$
15. $\int \frac{du}{u^2+a^2}=\frac{1}{a}\operatorname{Arctan}\frac{u}{a}+C$
16. $\int \frac{du}{u^2-a^2}=\frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|+C$
17. $\int \frac{du}{a^2-u^2}=\frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right|+C$
18. $\int \frac{du}{\sqrt{a^2-u^2}}=\operatorname{Arcsin}\frac{u}{a}+C$
19. $\int \frac{du}{\sqrt{u^2\pm a^2}}=\ln|u+\sqrt{u^2\pm a^2}|+C$
20. $\int \frac{du}{u\sqrt{u^2-a^2}}=\frac{1}{a}\operatorname{Arcsec}\frac{u}{a}+C$

## II

**TECHNIQUES OF INTEGRATION.**

**A. CHANGE OF FORM.** Applicable identities (*algebraic, trigonometric,* etc.) may put the integrand into manageable form.

**E1:** $\int x^2(1+\sqrt{x})^2dx=\int (x^2+2x^{3/2}+x^3)dx$. **E2:** $\int \cos ax\,\sin bx\,dx=1/2\int[\sin(a+b)x-\sin(a-b)x]\,dx$. **E3:** $\int \cos^2 ax\,dx=1/2\int(1+\cos 2ax)dx$

**B. INTEGRATION BY PARTS.** Since $(fg)'=f'g+fg'$, integration yields $\int[f'(x)g(x)+f(x)g'(x)]dx=f(x)g(x)+C$

When written $\int f(x)g'(x)dx=f(x)g(x)-\int f'(x)g(x)dx$, it expresses a rule for "integrating by parts."

**E:** If $f(x)=x^2$ and $g'(x)=e^{x/3}$, $g(x)=3e^{x/3}+C$ then $\int x^2e^{x/3}dx=3x^2e^{x/3}-6\int xe^{x/3}dx$. In the last term on the right take $f(x)=x$ and find similarly $\int xe^{x/3}dx=3xe^{x/3}-3\int e^{x/3}dx=3xe^{x/3}-9e^{x/3}+C$

Thus $\int x^2e^{x/3}dx=3x^2e^{x/3}-18xe^{x/3}+54e^{x/3}+C^*$, where $C^*=-6C$

Integration by parts is particularly useful when applied to: products, logarithms, inverse trigonometric functions, odd powers of *sec* and *csc*.

**C. SUBSTITUTION.** *If* $f$ *is a function of* $u$ *and* $u$ *is a function of* $x$, *for which* $\frac{du}{dx}$ *exists, then* $\int f(u)\frac{du}{dx}dx=\int f(u)\,du+C$.

**E1:** Integrate: $J=\int (x^2+1)^3\,2x\,dx$. Here $u=x^2+1$, $f(u)=u^3$; $\frac{du}{dx}=2x$; $du=\frac{du}{dx}dx=2x\,dx$. Thus $J=\int u^3du=\frac{u^4}{3}+C=\frac{(x^2+1)^4}{3}+C$.

**E2:** Integrate: $K=\int \frac{dx}{\sqrt{9-4x^2}}$. Let $u=2x$, thus $K=\frac{1}{2}\int\frac{du}{\sqrt{9-u^2}}=\frac{1}{2}\operatorname{Arcsin}\frac{u}{3}+C=\frac{1}{2}\operatorname{Arcsin}\frac{2x}{3}+C$

**D. ELEMENTARY RATIONALIZING SUBSTITUTION. 1.** If integrand is an algebraic function and involves just one radical $\sqrt[n]{ax+b}$ with a linear radicand as the only irrational feature, substitute $u=\sqrt[n]{ax+b}$.

**E:** Integrate: $J=\int \frac{x\,dx}{\sqrt{3x+1}}$. Let $u=\sqrt{3x+1}$; $u^2=3x+1$; $x=\frac{1}{3}(u^2-1)$; $dx=\frac{2}{3}u\,du$. $J=\frac{2}{9}\int(u^2-1)du=\frac{2}{27}u^3-\frac{2}{9}u+C=\frac{2}{27}(3x+1)^{3/2}-\frac{2}{9}(3x+1)^{1/2}+C$

**2. Integrals of type** $\int\frac{dx}{Ax^2+Bx+C}$ **and** $\int\frac{dx}{\sqrt{Ax^2+Bx+C}}$.

Complete the square for the terms in $x$ to obtain a type which can usually be solved by standard formulas.

**E:** $\int\frac{dx}{\sqrt{16-x^2-6x}}=\int\frac{dx}{\sqrt{16+9-(x^2+6x+9)}}=\int\frac{dx}{\sqrt{25-(x+3)^2}}=\operatorname{Arcsin}\frac{x+3}{5}+C$

**E. RATIONALIZATION BY TRIGONOMETRIC SUBSTITUTIONS.** When integrand contains radicals (with $a>0$) of type:

| | FOR | SUBSTITUTE | TO OBTAIN |
|---|---|---|---|
| 1. | $\sqrt{a^2-u^2}$ | $u=a\sin\theta$ | $\sqrt{a^2-u^2}=a\cos\theta$ |
| 2. | $\sqrt{a^2+u^2}$ | $u=a\tan\theta$ | $\sqrt{a^2+u^2}=a\sec\theta$ |
| 3. | $\sqrt{u^2-a^2}$ | $u=a\sec\theta$ | $\sqrt{u^2-a^2}=a\tan\theta$ |

**E:** Integrate: $J=\int\frac{x^2dx}{(4-x^2)^{3/2}}$. Let $x=2\sin\theta$, with $\cos\theta>0$, $dx=2\cos\theta\,d\theta$; thus $J=\int\frac{\sin^2\theta\cos\theta\,d\theta}{\cos^3\theta}=\int\tan^2\theta\,d\theta=\int(\sec^2\theta-1)d\theta=\tan\theta-\theta+C$

Since $\tan\theta=\frac{x}{\sqrt{4-x^2}}$ and $\theta=\operatorname{Arcsin}\frac{x}{2}$; $J=\frac{x}{\sqrt{4-x^2}}-\operatorname{Arcsin}\frac{x}{2}+C$

**F. RATIONAL FUNCTIONS OF SINES AND COSINES.** The substitution $u=2\operatorname{Arctan}Z$ will replace any rational function of $\sin u$ and $\cos u$ by a rational function of $Z$, since

$$\sin u=\frac{2Z}{1+Z^2};\ \cos u=\frac{1-Z^2}{1+Z^2};\ \text{and } du=\frac{2dZ}{1+Z^2}$$

After integrating, use $Z=\tan\frac{1}{2}u$ to return to the original variable.

**E:** $\int\frac{dx}{3-2\cos x}=\int\frac{\frac{2dZ}{1+Z^2}}{3-2\left(\frac{1-Z^2}{1+Z^2}\right)}=\int\frac{2dZ}{1+5Z^2}=\frac{2\sqrt{5}}{5}\operatorname{Arctan}Z\sqrt{5}+C=\frac{2\sqrt{5}}{5}\operatorname{Arctan}\left(\sqrt{5}\tan\frac{1}{2}x\right)+C$

**G. INTEGRALS OF TYPE** $\int \sin^m u\cos^n u\,du$.

**1. If** $m$ **is a positive odd integer,** use $\sin^{m-1}u\cos^n u\,(\sin u\,du)=-\sin^{m-1}u\cos^n u\,d(\cos u)$ and express $\sin^{m-1}u$ in terms of $\cos u$.

**2. If** $n$ **is a positive odd integer,** use $\sin^m u\cos^{n-1}u\,(\cos u\,du)=\sin^m u\cos^{n-1}u\,d(\sin u)$ and express $\cos^{n-1}u$ in terms of $\sin u$.

**3. If** $m$ **and** $n$ **are nonnegative even integers** transform the integrand by use of double angle identities.

**E:** $\int\sin^2x\cos^2x\,dx=\int\sin^2x\cos^2x\,(\cos x\,dx)=\int\sin^2x\,(1-\sin^2x)\,d(\sin x)=1/3\sin^3x-1/5\sin^5x+C$.

USEFUL IDENTITIES: $\sin^2x+\cos^2x=1$; $\tan^2x+1=\sec^2x$; $\cot^2x+1=\csc^2x$; TRIG. IDENTITIES: $\sin^2x=\frac{1-\cos 2x}{2}$; $\cos^2x=\frac{1+\cos 2x}{2}$

**H. TYPE** $\int\tan^m u\sec^n u\,du$ **AND** $\int\cot^m u\csc^n u\,du$.

**1. If** $m$ **is an integer** $>1$, **use:** $\tan^m u=\tan^{m-2}u\tan^2u=\tan^{m-2}u(\sec^2u-1)$.

$\cot^m u=\cot^{m-2}u\cot^2u=\cot^{m-2}u(\csc^2u-1)$.

**2. If** $n$ **is a positive even integer, use:**

$\tan^m u\sec^n u\,du=\tan^m u\sec^{n-2}u\,(\sec^2u\,du)=\tan^m u\sec^{n-2}u\,d(\tan u)$, and express $\sec^{n-2}u$ in terms of $\tan u$ by $\tan^2\theta+1=\sec^2\theta$. Similarly for $\cot^m u\csc^n u\,du$ (use $\cot^2\theta+1=\csc^2\theta$).

**3. If** $m$ **is a positive odd integer, use:**

$\tan^m u\sec^n u\,du=\tan^{m-1}u\sec^{n-1}u\,(\tan u\sec u\,du)=\tan^{m-1}u\sec^{n-1}u\,d(\sec u)$, and express $\tan^{m-1}u$ in terms of $\sec u$ by $\tan^2+1=\sec^2\theta$. Similarly for $\cot^m u\csc^n u\,du$.

**E:** $\int\cot^5x\csc^3x\,dx=\int\cot^4x\csc^2x\,(\cot x\csc x\,dx)=-\int(\csc^2x-1)^2\csc^2x\,d(\csc x)=-\frac{1}{7}\csc^7x+\frac{2}{5}\csc^5x-\frac{1}{3}\csc^3x+C$

**I. PARTIAL FRACTIONS.** If $P(x)$ and $Q(x)$ are polynomials their ratio $R(x)=P(x)/Q(x)$ is a rational function. In seeking $\int R(x)dx$, first remove common factors of $P(x)$ and $Q(x)$ and then divide, if necessary, until the degree of $P$ is less than the degree of $Q$.

**WHEN THE DENOMINATOR Q(X) HAS:**

**1. Non Repeated Linear Factors:** For each such factor write a term $A/(x-\alpha)$. Each such partial fraction will integrate into a logarithm

**2. Repeated Linear Factors:** For each **r—fold** linear factor write sum: $\frac{A_1}{x-\alpha}+\frac{A_2}{(x-\alpha)^2}+\cdots+\frac{A_r}{(x-\alpha)^r}$. The first of these will integrate into a logarithm; all of the others are of type $\int u^n\,du$.

**3. Non Repeated Quadratic Factors:** For each such factor write $\frac{Ax+B}{x^2+px+q}$. The integral of each such fraction can be accomplished by $\int\frac{du}{u}$ and the standard formulas.

**4. Repeated Quadratic Factors:** For each **r-fold** quadratic factor write the sum: $\frac{A_1x+B_1}{x^2+px+q}+\frac{A_2x+B_2}{(x^2+px+q)^2}+\cdots+\frac{A_rx+B_r}{(x^2+px+q)^r}$.

The first of these is integrated as in 3 above. Each of the others is integrated as follows. Write:

$$\int\frac{(A_kx+B_k)dx}{(x^2+px+q)^k}=\frac{A_k}{2}\int\frac{(2x+p)dx}{(x^2+px+q)^k}+\left(B_k-\frac{pA_k}{2}\right)\int\frac{dx}{(x^2+px+q)^k}.$$

The first integral is of type $\int u^n du$, the second can be reduced to one found in a table of integrals.

**E:** $R(x)=\frac{x^4-8x^3+24x^2-14x+7}{(2x-1)^3(x^2+2)^2}=\frac{a}{2x-1}+\frac{b}{(2x-1)^2}+\frac{cx+d}{x^2+2}+\frac{ex+f}{(x^2+2)^2}$.

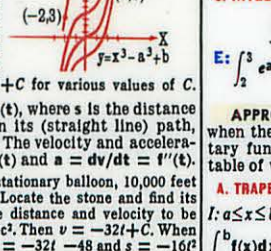
On clearing of fractions we have the identity $x^4-8x^3+24x^2-14x+7=a(2x-1)(x^2+2)^2+b(x^2+2)^2+(cx+d)(2x-1)^2(x^2+2)+(ex+f)(2x-1)^3$. From which $a=0$, $b=1$, $c=0$, $d=0$, $e=-2$, $f=3$.

Thus $R(x)=\frac{1}{(2x-1)^2}-\frac{2x-3}{(x^2+2)^2}$. The integral of $R(x)$ in this form can be accomplished by use of the method above.

## III

**APPLICATIONS OF INDEFINITE INTEGRALS.**

**A. SLOPE OF A CURVE.** If $f(x)$ is differentiable at $x=a$, $f'(a)$ is the slope of the curve $y=f(x)$ at $x=a$, and $f'(a)$ is the instantaneous rate of change of $y$ with respect to $x$ at $x=a$.



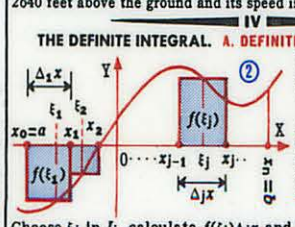

**E:** Find the equation of the curve whose slope at every point is $3x^2$. **Sol:** If the equation of the curve is $y=f(x)$ then $f'(x)=3x^2$, whence $f(x)=x^3+C$ for some value of $C$. If the curve is to pass through point $(-2,3)$ then $C=11$ and $f(x)=x^3+11$. If curve is to pass through $(a,b)$ then $f(x)=x^3-a^3+b$. Fig. 1: Curves $y=x^3+C$ for various values of $C$.

**B. RECTILINEAR MOTION.** An equation $s=f(t)$, where $s$ is the distance at time $t$ of a body from a fixed point in its (straight line) path, completely defines the motion of the body. The velocity and acceleration at time $t$ are given by $v=ds/dt=f'(t)$ and $a=dv/dt=f''(t)$.

**E:** A stone was thrown straight down from a stationary balloon, 10,000 feet above the ground, with a speed of 48 ft/sec. Locate the stone and find its speed 20 seconds later. **Sol:** Assume positive distance and velocity to be directed upward. Thus $a=dv/dt=-32$ ft/sec$^2$. Then $v=-32t+C$. When $t=0$, $v=-48$; hence $C=-48$. Therefore $v=-32t-48$ and $s=-16t^2-48t+C_1$. When $t=0$, $s=10{,}000$; hence $C_1=10{,}000$ and $s=-16t^2-48t+10{,}000$. When $t=20$, $s=2640$, and $v=-688$. After 20 sec. the stone is 2640 feet above the ground and its speed is 688 ft/sec.

## IV

**THE DEFINITE INTEGRAL. A. DEFINITION.** Let $f$ be defined on a closed interval $I$: $a\leq x\leq b$. Choose points $x_j (j=0,1,\ldots,n)$ such that $a=x_0<x_1<\cdots<x_{n-1}<x_n=b$. Let the subintervals $I_j: x_{j-1}\leq x\leq x_j$ $(j=1,\ldots,n)$ have lengths $\Delta_jx=x_j-x_{j-1}$; and denote the length of the largest of these $n$ subintervals by $\|\Delta x\|$.




Choose $\xi_j$ in $I_j$, calculate $f(\xi_j)\Delta_jx$ and form the sum $\sum=\sum_{j=1}^{n}f(\xi_j)\Delta_jx$ (Fig. 2). If $\Sigma$ approaches a limit as $n\to\infty$ and $\|\Delta x\|\to 0$ independently of both the subintervals used and the choice of $\xi_j$, then $f$ is said to be integrable (in the sense of Riemann) on $I$. The limit of $\Sigma$ is denoted by $\int_a^b f(x)dx$ and is said to be the **definite integral** of the integrand $f$ from the **lower** limit $a$ to the **upper** limit $b$. Thus $\int_a^b f(x)dx=\lim_{\|\Delta x\|\to 0}\sum_{j=1}^{n}f(\xi_j)\Delta_jx$.

**If** $I$ **is closed and** $f$ **is continuous on** $I$ **then** $f$ **is integrable on** $I$, **that is** $\int_a^b f(x)dx$ **exists.** The variable of integration in a definite integral can be replaced: $\int_a^b f(x)dx=\int_a^b f(w)dw=\int_a^b f(x)dz=\int_a^b f(\theta)d\theta$.

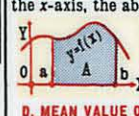
**B. PROPERTIES.** **(1)** $\int_a^b f(x)dx=-\int_b^a f(x)dx$; $\int_a^a f(x)dx=0$

**(2)** $\int_a^b f(x)dx+\int_b^c f(x)dx=\int_a^c f(x)dx$. **(3)** $\int_a^b k\,f(x)dx=k\int_a^b f(x)dx$

**(4)** $\int_a^b|f(x)+g(x)|dx=\int_a^b f(x)dx+\int_a^b g(x)dx$. **(5)** If $f(x)\leq g(x)$ on $I$ then $\int_a^b f(x)dx\leq\int_a^b g(x)dx$. **(6)** $\left|\int_a^b f(x)dx\right|\leq\int_a^b|f(x)|dx$.

**C. AREA.** The area between the curve $y=f(x)$, the $x$-axis and the lines $x=a$ and $x=b$ is given by: $A=\int_a^b y\,dx=\int_a^b f(x)dx$.

This area will be positive if wholly above the $x$-axis, will be negative if wholly below the $x$-axis. If part of the area is above and part below the $x$-axis, the above integral gives the algebraic sum of the positive and negative pieces. Similarly, the area bounded by the curve $x=g(y)$, the lines $y=c$, $y=d$, and the $y$-axis is given by:




③ $A=\int_c^d x\,dy=\int_c^d g(y)dy$

**D. MEAN VALUE OF A FUNCTION.**

If $m\leq f(x)\leq M$ on $I$: $a\leq x\leq b$, then $m(b-a)\leq\int_a^b f(x)dx\leq M(b-a)$. The number $\int_a^b f(x)dx/(b-a)$ is the mean or average value $\bar{y}=\overline{f(x)}$ of $f$ on $I$. Since $(b-a)\,\bar{y}=\int_a^b f(x)dx$, $\bar{y}$ is the altitude of a rectangle with base $I$ (of length $b-a$), and area equal to $\int_a^b f(x)dx$. Evidently $m\leq\bar{y}\leq M$. If $f$ is continuous on $I$, then $I$ has an interior point $\xi$ such that $f(\xi)=\overline{f(x)}$.

## V

**EXACT EVALUATION OF DEFINITE INTEGRALS.**

**A. FUNDAMENTAL THEOREMS. 1.** Let $f$ be integrable on $I$: $a\leq x\leq b$. If a function $g$ is defined on $I$ by the formula $g(x)=\int_a^x f(t)dt$, then $g$ is continuous on $I$ so long as $f$ is integrable.

**2.** If $\alpha$ is a point of $I$ and $f$ is continuous at $x=\alpha$, then $g(x)=\int_a^x f(t)dt$ is differentiable at $x=\alpha$ and $g'(\alpha)=f(\alpha)$. Thus $g$ is an antiderivative of $f$. If $F$ is any antiderivative of $f$, *then* $F(x)=g(x)+C=\int_a^x f(t)dt+C$. When $x=a$, $C=F(a)$; whence

$$F(x)=F(a)+\int_a^x f(t)dt=F(a)+\int_a^x F'(t)dt.$$

**E:** Since $(\sin x)'=\cos x$, $\sin x=\sin a+\int_a^x\cos t\,dt$.

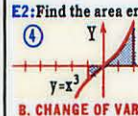
**3. If** $f$ **is continuous in a closed interval and** $F$ **is any antiderivative of** $f$: $\int_a^b f(x)dx=F(b)-F(a)=F(x)\Big|_a^b$ (where **a** and **b** are numbers in the given interval).

**E:** $\int_0^{\pi/2}\cos x\,dx=\sin x\Big|_0^{\pi/2}=\sin\frac{\pi}{2}-\sin 0=1$

The fundamental theorem reduces the calculation $\int_a^b f(x)dx$ to two steps: (**a**) finding an antiderivative $F$ of $f$, (**b**) calculating $F(b)-F(a)$.

**E1:** $\int_{-3}^{-2}\frac{dx}{2+3x}=\frac{1}{3}\ln|2+3x|\Big|_{-3}^{-2}=\frac{1}{3}\left[\ln|2-6|-\ln|2-9|\right]=\frac{1}{3}\ln\frac{4}{7}$

**E2:** Find the area enclosed by $y=x^3$, the lines $x=-1$, $x=2$, and the $x$-axis.




**Sol:** Area $=\left|\int_{-1}^{0}x^3dx\right|+\int_0^2 x^3dx$

$=\left|\frac{x^4}{4}\Big|_{-1}^{0}\right|+\frac{x^4}{4}\Big|_0^2=\left|-\frac{(-1)^4}{4}\right|+\frac{(2)^4}{4}=\frac{17}{4}$

**B. CHANGE OF VARIABLE (SUBSTITUTION).** If $x=g(t)$ with $g'$ continuous and $g'(t)\neq 0$ on $c\leq t\leq d$, $g(c)=a$ and $g(d)=b$, then: $\int_a^b f(x)dx=\int_c^d f(g(t))g'(t)dt$.

**E:** With $x=\alpha\tan t$, $dx=\alpha\sec^2t\,dt$,

$$\int_0^{\alpha}\frac{dx}{\alpha^2+x^2}=\int_0^{\pi/4}\frac{\alpha\sec^2t\,dt}{\alpha^2\sec^2t}=\frac{1}{\alpha}\int_0^{\pi/4}dt=\frac{\pi}{4\alpha}$$

**C. INTEGRATION BY PARTS.** If $f'$ and $g'$ are continuous for $a\leq x\leq b$ then

$$\int_a^b f(x)g'(x)dx=f(b)g(b)-f(a)g(a)-\int_a^b f'(x)g(x)dx$$

**E:** $\int_2^3 e^{ax}\ln x\,dx=\frac{e^{3a}\ln 3-e^{2a}\ln 2}{a}-\frac{1}{a}\int_2^3\frac{e^{ax}}{x}dx$

## VI

**APPROXIMATE INTEGRATION.** Approximation procedures are used when the indefinite integral cannot be expressed in terms of elementary functions, and when the integrand is defined by means of a table of values.

**A. TRAPEZOIDAL RULE.** When $f(x)$ does not change sign on the interval $I$: $a\leq x\leq b$, an approximate value of $\int_a^b f(x)dx$ is:

$$\int_a^b f(x)dx\approx\frac{h}{2}\left[f(a)+2f(a+h)+2f(a+2h)+\cdots+2f(a+[n-1]h)+f(b)\right]$$

where $h=\frac{b-a}{n}$, and $n$ is the number of equal subintervals of $I$.

**E:** Approximate $\int_0^1\sqrt{1+x^3}\,dx$ with $n=5$. **Sol:** $h=0.2$; $a=0$, $a+h=0.2$, $a+2h=0.4$, $a+3h=0.6$, $a+4h=0.8$, $b=1$. $\int_0^1\sqrt{1+x^3}\,dx\approx\frac{0.2}{2}(1+2\sqrt{1.008}+2\sqrt{1.064}+2\sqrt{1.216}+2\sqrt{1.512}+\sqrt{2})=1.115$

**B. SIMPSON'S RULE.** When $f(x)$ does not change sign on the interval $I$: $a\leq x\leq b$, an approximate value of $\int_a^b f(x)dx$ is:

$$\int_a^b f(x)dx\approx\frac{h}{3}\left[f(a)+4f(a+h)+2f(a+2h)+4f(a+3h)+2f(a+4h)+\cdots+2f(a+[2n-2]h)+4f(a+[2n-1]h)+f(b)\right]$$

where $h=\frac{b-a}{2n}$, and $n$ is the number of equal subintervals of $I$.

**NOTE:** Simpson's rule gives an exact value when $f(x)$ is a polynomial of degree three or less.

**E:** Approximate $\int_0^2 x^3\,dx$ with $n=4$. **Sol:** $h=1/2$; $a=0$, $a+h=1/2$, $a+2h=1$, $a+3h=3/2$, $b=2$. **NOTE:** Exact value of $\int_0^2 x^3\,dx = \frac{x^4}{4}\Big|_0^2 = 8/3$.

$$\int_0^2 x^3\,dx \approx \frac{1/2}{3}\left[0+4\left(\frac{1}{4}\right)+2(1)+4\left(\frac{9}{4}\right)+4\right] = 8/3.$$



## IMPROPER INTEGRALS. A. INFINITE LIMITS OF INTEGRATION.

**1.** Suppose that $f$ is continuous for $a \le x < \infty$. We define $\int_a^\infty f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx$, provided the limit exists and is finite.

**E:** If $p \neq 1$, $\int_1^\infty \frac{dx}{x^p} = \lim_{t\to\infty}\int_1^t \frac{dx}{x^p} = \lim_{t\to\infty}\left(\frac{t^{1-p}-1}{1-p}\right) = \frac{1}{p-1}$ if $p>1$; $= +\infty$ if $p<1$.

**2.** Similarly, if $f$ is continuous for $-\infty < x \le b$, we define $\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty}\int_t^b f(x)dx$, provided the limit exists and is finite.

**3.** If $f$ is continuous for $-\infty < x < \infty$, $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$, where $c$ is any number, provided both the integrals on the right exist and are finite.

**E:** Evaluate, if possible, $\int_{-\infty}^\infty x^3dx$. **Sol:** $\int_{-\infty}^\infty x^3dx = \int_{-\infty}^0 x^3\,dx + \int_0^\infty x^3dx$.

Consider $\int_0^\infty x^3dx = \lim_{t\to\infty}\int_0^t x^3dx = \lim_{t\to\infty}\frac{x^4}{4}\Big|_0^t = \lim_{t\to\infty}\frac{t^4}{4} = \infty$.

Hence, $\int_0^\infty x^3dx$ does not exist. Thus, without considering $\int_{-\infty}^0 x^3dx$, the original integral fails to exist.

## B. DISCONTINUOUS INTEGRAND.

**1.** If $f(x)$ is continuous on the interval $a \le x < b$ but is discontinuous at $x=b$, we define $\int_a^b f(x)dx = \lim_{\varepsilon\to0}\int_a^{b-\varepsilon} f(x)dx$, provided the limit exists.

**2.** If $f(x)$ is continuous on $a < x \le b$ but is discontinuous at $x=a$, we define $\int_a^b f(x)dx = \lim_{\varepsilon\to0}\int_{a+\varepsilon}^b f(x)dx$, provided the limit exists.

**3.** If $f(x)$ is continuous for all values of $x$ on $a \le x \le b$ except at $x=c$, where $a<c<b$, $\int_a^b f(x)dx = \lim_{\varepsilon_1\to0}\int_a^{c-\varepsilon_1} f(x)dx + \lim_{\varepsilon_2\to0}\int_{c+\varepsilon_2}^b f(x)dx$ provided both limits exist.

**E:** Evaluate, if possible, $\int_1^2 (2-x)^{-1/3}dx = \lim_{\varepsilon\to0}\int_1^{2-\varepsilon}(2-x)^{-1/3}dx = \lim_{\varepsilon\to0} -\frac{3}{2}(2-x)^{2/3}\Big|_1^{2-\varepsilon} = \lim_{\varepsilon\to0}\left(-\frac{3}{2}\varepsilon^{2/3}+\frac{3}{2}\right) = \frac{3}{2}$



## VIII. MATHEMATICAL APPLICATIONS. A. SLOPE.

**E:** What is ordinate when $x=-5$, for the curve with slope $3x^2$ if ordinate is 3 when $x=-2$.

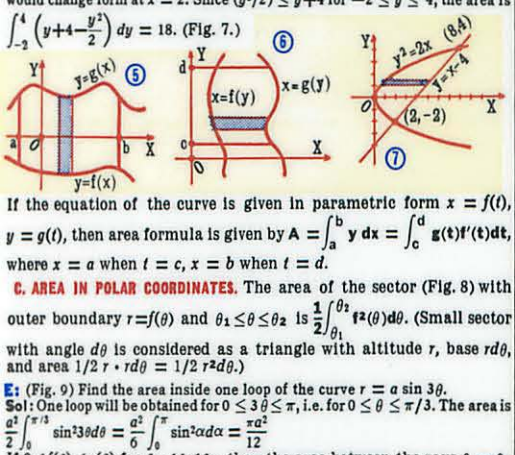
**Sol:** $y = 3 + \int_{-2}^{-5} 3x^2dx = 3+x^3\Big|_{-2}^{-5} = 3-125+8 = -114$

## B. AREA BETWEEN TWO CURVES.

If $f(x) \le g(x)$ for $a \le x \le b$ then the area (Fig. 5) over that interval, above $y=f(x)$ and below $y=g(x)$ is: $A = \int_a^b [g(x)-f(x)]dx$

If the curves (Fig. 6) are given by $x=g(y)$, $x=f(y)$, area is given by: $A = \int_c^d [g(y)-f(y)]dy$

**E:** Find the area bounded by the parabola $y^2=2x$ and the line $y=x-4$. **Sol:** Use $y$ as the independent variable, for if $x$ were to be used, the integral would change form at $x=2$. Since $(y^2/2) \le y+4$ for $-2 \le y \le 4$, the area is $\int_{-2}^4 \left(y+4-\frac{y^2}{2}\right)dy = 18$. (Fig. 7.)




If the equation of the curve is given in parametric form $x=f(t)$, $y=g(t)$, then area formula is given by $A = \int_a^b y\,dx = \int_c^d g(t)f'(t)dt$, where $x=a$ when $t=c$, $x=b$ when $t=d$.

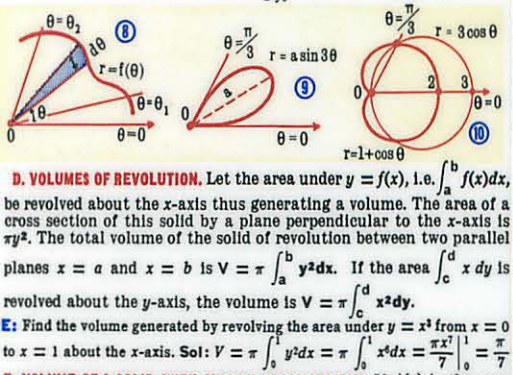
## C. AREA IN POLAR COORDINATES.

The area of the sector (Fig. 8) with outer boundary $r=f(\theta)$ and $\theta_1 \le \theta \le \theta_2$ is $\frac{1}{2}\int_{\theta_1}^{\theta_2} f^2(\theta)d\theta$. (Small sector with angle $d\theta$ is considered as a triangle with altitude $r$, base $rd\theta$, and area $1/2\ r \cdot rd\theta = 1/2\ r^2d\theta$.)

**E:** (Fig. 9) Find the area inside one loop of the curve $r = a\sin 3\theta$. **Sol:** One loop will be obtained for $0 \le 3\theta \le \pi$, i.e. for $0 \le \theta \le \pi/3$. The area is $\frac{a^2}{2}\int_0^{\pi/3}\sin^2 3\theta d\theta = \frac{a^2}{6}\int_0^\pi \sin^2\alpha d\alpha = \frac{\pi a^2}{12}$

If $0 \le f(\theta) \le g(\theta)$ for $\theta_1 \le \theta \le \theta_2$, then the area between the rays $\theta=\theta_1$ and $\theta=\theta_2$, outside $r=f(\theta)$ and inside $r=g(\theta)$ is $\frac{1}{2}\int_{\theta_1}^{\theta_2}[g^2(\theta)-f^2(\theta)]d\theta$.

**E:** (Fig. 10) Find the area inside the circle $r=3\cos\theta$ and outside the cardioid $r=1+\cos\theta$. **Sol:** The area $A$ will be found by symmetry by integrating from 0 to $\pi/3$ and doubling. Hence $A = 2\cdot\frac{1}{2}\int_0^{\pi/3}[9\cos^2\theta-(1+\cos\theta)^2]d\theta = \pi$




## D. VOLUMES OF REVOLUTION.

Let the area under $y=f(x)$, i.e. $\int_a^b f(x)dx$, be revolved about the $x$-axis thus generating a volume. The area of a cross section of this solid by a plane perpendicular to the $x$-axis is $\pi y^2$. The total volume of the solid of revolution between two parallel planes $x=a$ and $x=b$ is $V = \pi\int_a^b y^2dx$. If the area $\int_c^d x\,dy$ is revolved about the $y$-axis, the volume is $V = \pi\int_c^d x^2dy$.

**E:** Find the volume generated by revolving the area under $y=x^3$ from $x=0$ to $x=1$ about the $x$-axis. **Sol:** $V = \pi\int_0^1 y^2dx = \pi\int_0^1 x^6dx = \frac{\pi x^7}{7}\Big|_0^1 = \frac{\pi}{7}$

## E. VOLUME OF A SOLID WITH KNOWN CROSS SECTION.

If $A(x)$ is the area of every section perpendicular to the $x$-axis, then the volume from $x=a$ to $x=b$ of a solid with this cross section is $V = \int_a^b A(x)dx$.

**E:** Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. **Sol:** For each fixed $x$ with $0 \le x < a$ the cross section has area $\pi bc\left(1-\frac{x^2}{a^2}\right)$. The volume is

$$2\pi bc\int_0^a\left(1-\frac{x^2}{a^2}\right)dx = \frac{4}{3}\pi abc$$

## F. LENGTH OF A CURVE.

The length of a curve $L$ is given by integrating the differential of arc length $ds$. **Rectangular Coordinates:**

**1.** For curve $y=f(x)$ between $x=a$ and $x=b$: $L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

**2.** For curve $x=g(y)$ between $y=c$ and $y=d$: $L = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$

**3.** For curve $x=f(t)$, $y=g(t)$ $a \le t \le b$: $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

**Polar Coordinates:**

**4.** For curve $r=f(\theta)$ between $\theta=\theta_1$ and $\theta=\theta_2$: $L = \int_{\theta_1}^{\theta_2}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$

**5.** For curve $\theta=g(r)$ between $r=r_1$ and $r=r_2$: $L = \int_{r_1}^{r_2}\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}\,dr$

**E1:** Find the total length of the circumference $C$ of the circle $x^2+y^2=r^2$.

**Sol:** For $x^2+y^2=r^2$; $\frac{dy}{dx} = -\frac{x}{y}$; $\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{r}{y} = \frac{r}{\sqrt{r^2-x^2}}$.

Thus by symmetry $C = 4\int_0^r \frac{r\,dx}{\sqrt{r^2-x^2}} = 4r\,\text{Arcsin}\frac{x}{r}\Big|_0^r = 2\pi r$.

**E2:** The length of the cardioid $r=1+\cos\theta$ is $L = 2\int_0^\pi \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta = 2\int_0^\pi \sqrt{2+2\cos\theta}\,d\theta = 8$.

## G. AREA OF A SURFACE OF REVOLUTION.

**(1)** When $y=f(x)$ is revolved about the $x$-axis, the area of the surface generated between $x=a$ and $x=b$ is:

$$S = 2\pi\int_a^b y\,ds = 2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx.$$

In parametric form, $x=f(t)$, $y=g(t)$:

$$S = 2\pi\int_{t=a}^{t=b} g(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt.$$

**NOTE:** In all formulas (as well as for F above) all functions and their derivatives are assumed to exist and be continuous.

**(2)** For $x=g(y)$ revolved about $y$-axis between $y=c$ and $y=d$:

$$S = 2\pi\int_c^d x\,ds = 2\pi\int_c^d g(y)\sqrt{1+[g'(y)]^2}\,dy.$$

In parametric form $x=f(t)$, $y=g(t)$: $S = 2\pi\int_{t=a}^{t=b} f(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$.

**(3)** In polar coordinates for $r=f(\theta)$ from $\theta=\theta_1$ to $\theta=\theta_2$ about

the $x$-axis: $S = 2\pi\int_{\theta_1}^{\theta_2} r\sin\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$

the $y$-axis: $S = 2\pi\int_{\theta_1}^{\theta_2} r\cos\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$

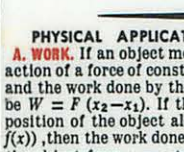
**E:** Find the surface area of a sphere of radius $r$. **Sol:** Can be generated by revolving the upper half of circle $x^2+y^2=r^2$ about $x$-axis.

$$S = 2\pi\int_{-r}^{r} y\sqrt{1+[f'(x)]^2}\,dx = 4\pi\int_0^r\sqrt{r^2-x^2}\left(\frac{r}{y}\right)dx = 4\pi r\int_0^r dx = 4\pi r^2$$



## PHYSICAL APPLICATIONS.

**A. WORK.** If an object moves along the $x$-axis from $x_1$ to $x_2$ under the action of a force of constant magnitude $F$, the displacement is $x_2-x_1$ and the work done by the force during the displacement is defined to be $W = F(x_2-x_1)$. If the force is not constant but varies with the position of the object along the $x$-axis (i.e. is a continuous function $f(x)$), then the work done in moving the object from $x=a$ to $x=b$ is: $W = \int_a^b f(x)dx$




**B. SPRINGS.** Under suitable conditions the force $F$ required to stretch a coiled spring $x$ linear units beyond its unstretched length is proportional to $x$: $f(x)=kx(k>0)$. The work done in increasing the elongation from $x=a$ to $x=b\,(>a)$ is:

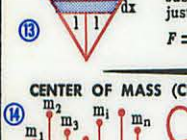
$$W = k\int_a^b x\,dx = \frac{k}{2}(b^2-a^2)$$

**C. KINETIC ENERGY.** Kinetic energy of an object moving on a line. If the force $F$ acts along the $x$-axis, then the motion along the axis of an object of mass $m$ is connected with $F$ by Newton's Law of Motion: $m\frac{d^2x}{dt^2} = F(x)$. Hence $\frac{d}{dt}\left(\frac{m}{2}\left(\frac{dx}{dt}\right)^2\right) = F(x)\frac{dx}{dt}$ whence, if $v=\frac{dx}{dt}$

$$\int_{t_1}^{t_2}\frac{d}{dt}\left(\frac{m}{2}v^2\right)dt = \int_{t_1}^{t_2}F(x)\frac{dx}{dt}dt = \int_{x_1}^{x_2}F(x)dx.\ K = \frac{m}{2}v^2 \text{ is the kinetic}$$

energy of the object. The change $K_2-K_1$ in kinetic energy is $\frac{m}{2}v^2\Big|_{t_1}^{t_2} = \int_{x_1}^{x_2}F(x)dx = W$, where $W$ is the work done by $F$ during the displacement from $x_1$ (when $t=t_1$) to $x_2$ (when $t=t_2$).


**D. FLUID PRESSURE.** Pressure equals force per unit area, $p = wh$, where $w$ is weight per unit volume of the liquid and $h$ is the depth of submersion. The total force $F$ due to liquid pressure on a submerged area of variable depth is $F = \int_a^b \text{whldh} = \int_a^b \text{whdA}$, where $dA = ldh$ is an area element and $a$, $b$ are least and greatest depths of the area.



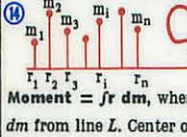

**E:** Find the total force on one side of a plate in the form of an equilateral triangle, of side $2a$, which is submerged vertically in water until one edge is just in the surface of the water. **Sol:**

$$F = \int_0^{\sqrt{3}a} wx\left(a-\frac{\sqrt{3}}{3}x\right)dx = \frac{w}{2}a^3 \text{ lbs.}$$



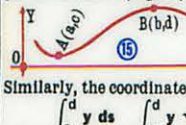
## CENTER OF MASS (CENTROID). A. DEFINITION.

For $n$ particles of point mass $m_i$: **Moment** $= r_1m_1 + r_2m_2 + \cdots + r_nm_n = \sum_{i=1}^n r_i m_i$. For a continuous mass distribution:




**Moment** $= \int r\,dm$, where $r$ equals distance of the element of mass $dm$ from line $L$. Center of mass, measured from $L$ is $\frac{\int r dm}{\int dm}$

**B. CURVE ARC.** Consider a wire of length $s$ and of constant density (mass per unit length), bent to just fit the graph $y=f(x)$ between points A and B (Fig. 15). The centroid $(\bar{x}, \bar{y})$ will be:




$$\bar{x} = \frac{\int_a^b x\,ds}{\int_a^b ds} = \frac{\int_a^b x\sqrt{1+[f'(x)]^2}\,dx}{\int_a^b\sqrt{1+[f'(x)]^2}\,dx}$$

Similarly, the coordinate $\bar{y}$ of $AB$ with equation solved for $x$, $x=g(y)$ is:

$$\bar{y} = \frac{\int_c^d y\,ds}{\int_c^d ds} = \frac{\int_c^d y\sqrt{1+[g'(y)]^2}\,dy}{\int_c^d\sqrt{1+[g'(y)]^2}\,dy}$$

**NOTE:** If a curve has an axis of symmetry, then its centroid is on the axis.

In polar coordinates for $r=f(\theta)$ from $\theta=\theta_1$ to $\theta=\theta_2$:

$$\bar{x} = \frac{\int_{\theta_1}^{\theta_2} r\cos\theta\,ds}{\int_{\theta_1}^{\theta_2} ds} = \frac{\int_{\theta_1}^{\theta_2} r\cos\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta}{\int_{\theta_1}^{\theta_2}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta};\quad \bar{y} = \frac{\int_{\theta_1}^{\theta_2} r\sin\theta\,ds}{\int_{\theta_1}^{\theta_2} ds}.$$

**E:** Find the centroid of a wire of constant density which is in the form of a semicircle. **Sol:** Consider the wire as the upper half of the circle $x^2+y^2=r^2$.

By symmetry $\bar{x}=0$. $\bar{y} = \frac{2\int_0^r \frac{ry}{\sqrt{r^2-y^2}}dy}{\pi r} = \frac{2}{\pi}\int_0^r \frac{y}{\sqrt{r^2-y^2}}\,dy = \frac{2r}{\pi}$

**C. PLANE AREA. 1.** Consider area bounded by $y=f(x)$, [continuous for $I$: $a \le x \le b$], the $x$-axis, and the lines $x=a$ and $x=b$. The $x$-coord. $(\bar{x})$ of the centroid will be:

$$\bar{x} = \frac{\int_a^b x\,f(x)dx}{\text{Area} = \int_a^b f(x)dx}$$

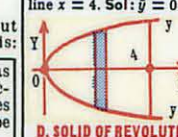
**2.** Similarly, if the area is bounded by $x=g(y)$, the $y$-axis and lines $y=c$ and $y=d$, then the $y$-coord. $(\bar{y})$ of the centroid for this different area will be:

$$\bar{y} = \frac{\int_c^d y\,g(y)dy}{\text{Area} = \int_c^d g(y)dy}$$

**3.** If the area $A$ is such that the expressions for $\bar{x}$ and $\bar{y}$ can be applied then the centroid is uniquely determined.

**NOTE: 1.** If $f(x)$ is non-negative on $I$, then $\bar{y}$ can also be obtained from $\bar{y} = \frac{\frac{1}{2}\int_a^b f^2(x)dx}{\text{Area}}$ **2.** If a plane area is symmetric with respect to a line $L$, then its centroid is on $L$.

**E:** Find the centroid of the area enclosed by the parabola $y^2=4x$ and the line $x=4$. **Sol:** $\bar{y}=0$ since the area is symmetric with respect to the $x$-axis.




Also $f(x) = 2\sqrt{x}$, and the area of a typical rectangle is $2f(x)\Delta x$. Thus

$$\bar{x} = \frac{2\int_0^4 x\,2\sqrt{x}\,dx}{2\int_0^4 2\sqrt{x}\,dx} = \frac{\int_0^4 x^{3/2}dx}{\int_0^4 x^{1/2}dx} = \frac{12}{5}$$

**D. SOLID OF REVOLUTION. 1.** Revolve area bounded by $y=f(x)$; the $x$-axis; lines $x=a$, $x=b$; about the $x$-axis. The centroid of volume is

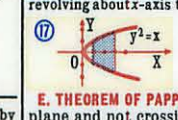
$$\bar{x} = \frac{\int_a^b \pi x[f(x)]^2dx}{\text{Volume} = \pi\int_a^b [f(x)]^2dx};\quad \bar{y}=0$$

**2.** When the area is revolved about the $y$-axis, the centroid $(\bar{x}, \bar{y})$ is on that axis. Thus:

$$\bar{x}=0;\ \bar{y} = \frac{\int_c^d \pi y[g(y)]^2dy}{\text{Volume} = \pi\int_c^d [g(y)]^2dy}$$

**NOTE:** Functions are (assumed to be) continuous and non-negative.

**E:** Find the centroid of the volume of the solid of revolution obtained by revolving about $x$-axis the region bounded by parabola $y^2=x$; $x=1$, $x=3$, $y=0$.




**Sol:** $\bar{x} = \frac{\int_1^3 \pi x\,[\sqrt{x}]^2\,dx}{\pi\int_1^3 x\,dx} = \frac{\int_1^3 x^2\,dx}{\int_1^3 x\,dx} = \frac{13}{6}$; $\bar{y}=0$

**E. THEOREM OF PAPPUS.** If a plane area is revolved about a line in its plane and not crossing the area, the volume of the solid generated is equal to product of area and distance traveled by its centroid.

**E:** Find the centroid of the area of a semicircle, using the fact that the volume of a sphere of radius $r$ is $V = 4/3\pi r^3$. **Sol:** A sphere can be generated by rotating a semicircle about its diameter. Then the axis of revolution does not intersect the area, and since $A = 1/2\,\pi r^2$ while $V = 2\pi\bar{y}A$, we have

**NOTE:** Pappus' Theorem is useful for determining the volume of a solid of revolution when the centroid is known, and conversely.

$$\bar{y} = \frac{V}{2\pi A} = \frac{\frac{4}{3}\pi r^3}{2\pi\cdot\frac{1}{2}\pi r^2} = \frac{4}{3\pi}r$$



## MOMENT OF INERTIA. A. DEF.

For $n$ particles of point mass $m_i$ (fig. 14), the **moment of inertia** is $I = r_1^2m_1 + r_2^2m_2 + \cdots + r_n^2m_n = \sum_{i=1}^n r_i^2m_i$. For a continuous mass distribution (of constant density), $I = \int r^2dm$ where $r$ equals the distance of the element of mass $dm$ from line $L$.

**B. CURVE ARC.** Let the curve $y=f(x)$ (a wire) be given, the moment of inertia $I$ with respect to the $x$-axis is:

$$I_x = \int_a^b y^2ds = \int_a^b [f(x)]^2\sqrt{1+[f'(x)]^2}\,dx.\ \text{Similarly, } I_y = \int_c^d x^2ds.$$

**E:** Find the moment of inertia of a circumference of a circle about a diameter. **Sol:** For $x^2+y^2=r^2$, moment is four times that of the first quadrant arc.

$$I_x = \int_a^b y^2\,ds = 4\int_0^r y^2\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 4\int_0^r ry\,dy = 4r\int_0^r\sqrt{r^2-x^2}\,dx$$
$$= 4r\left[\frac{x}{2}\sqrt{r^2-x^2}+\frac{r^2}{2}\text{Arcsin}\frac{x}{r}\right]_0^r = \pi r^3$$

**C. PLANE AREA.** $I_x = \int_a^b y^2dA$; $I_y = \int_c^d x^2dA$, where $dA$ is an area element so taken that it lies at the same distance from the given axis.

**E:** Find the moment of inertia with respect to the $y$-axis of the plane area between the parabola $y=9-x^2$ and the $x$-axis. **Sol:**

$$I_y = \int_{-3}^3 x^2dA = \int_{-3}^3 x^2y\,dx = 2\int_0^3(9x^2-x^4)\,dx = 2\left[3x^3-\frac{x^5}{5}\right]_0^3 = \frac{324}{5}$$



## MULTIPLE INTEGRATION—DOUBLE INTEGRAL.

**A. DEFINITION.** $\iint_R f(x,y)dA = \int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)dydx$, where the boundary of $R$ consists of two continuous curves $y=g_1(x)$ and $y=g_2(x)$, the boundary of $R$ is met by a line parallel to the $y$-axis in at most two points, and $x=a$, $x=b$ are the extreme values of $x$ on $R$.

Similarly, $\iint_R f(x,y)dA = \int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)dx\,dy$. [$f(x,y)$ is continuous over $R$.]

If $f(x,y)$ has the same sign over $R$, $\iint_R f(x,y)dA$ is the volume of a right cylinder bounded below by the region $R$ in the $xy$ plane and above by the surface $z=f(x,y)$.

When $f(x,y)=1$, $\iint_R f(x,y)dA$ is the area of the region $R$, i.e. $\iint_R dx\,dy$. The double integral is evaluated by means of iterated (successive) integration.

**E:** $\int_1^2\int_y^{3y}(x+y)dx\,dy = \int_1^2\left(\frac{x^2}{2}+xy\right)\Big|_y^{3y}dy = \int_1^2 6y^2dy = 14$



## MULTIPLE INTEGRATION—TRIPLE INTEGRAL.

**A. DEFINITION.** $\iiint_R f(x,y,z)\,dV = \int_a^b\int_{y_1(x)}^{y_2(x)}\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz\,dy\,dx$, etc., where the limits of integration are chosen to cover the region $R$. If $f(x,y,z)=1$, the above triple integral measures the volume of the region $R$. The triple integral is evaluated by iterated integration.

**E:** Find the volume bounded by the paraboloid $z=2x^2+y^2$ and the cylinder $z=4-y^2$.

**Sol:** $V = 4\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}\int_{2x^2+y^2}^{4-y^2} dz\,dy\,dx = 4\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}[(4-y^2)-(2x^2+y^2)]dy\,dx = 4\int_0^{\sqrt{2}}\left(4y-2x^2y-\frac{2y^3}{3}\right)\Big|_0^{\sqrt{2-x^2}}dx = \frac{16}{3}\int_0^{\sqrt{2}}(2-x^2)^{3/2}dx$
$= 4\pi$ cubic units.











PUBLISHED & MF'D BY DATA-GUIDE, INC. FLUSHING, N.Y. 11355

DATA-GUIDE QUICK CHART

# BASIC CHEMISTRY 1

## A SIMPLIFIED SUMMARY & INSTANT REFERENCE

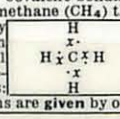
LEARN & REVIEW IN MINUTES

**AUTHOR:** Bernard Kauderer, Ph.D., Chairman, Dept. of Chemistry of the Plainview School, New York. **CONSULTANT:** Prof. John Arents, Ph.D., University of the City of New York.

**1. PROPERTIES.** The characteristics by which substances are identified. **A. PHYSICAL PROPERTIES (P.P.)** (1) **State:** Solid, liquid, gas. (2) **Color:** Red, Orange, Yellow, Green, Blue, Indigo, Violet (ROY G BIV). (3) **Odor.** (4) **Taste:** Sour, sweet, bitter, salty. (5) **Density:** Mass÷volume. (6) **Solubility:** Degree of ability of a substance to dissolve in another substance. **B. CHEMICAL PROPERTIES (C.P.).** (1) Tendency to unite with other elements. (2) Ability to support combustion (help other materials to burn). (3) Ability of the substance itself to burn.

**2. THE ATOM.** Composed of protons (**p**, "+" charges), electrons (**e**, "−" charges), neutrons (**n**, neutral charges). **A. NUCLEUS.** The compact center of the atom; contains all protons and all neutrons. **B. ORBITS.** All electrons move in orbits around the nucleus. No more than 2**e** may occupy the first ring or *K-shell;* no more than 8**e** in any outermost shell. Each orbit can hold up to $2N^2$, where **N** = the number of the orbital shell. **Ex:** Orbit 3 gives $2\times3^2 = 18e$. **C. VALENCE.** The number of **electrons** that an atom *lends, borrows,* or *shares,* in order to complete its own outer orbit. (Atoms tend to approximate the electron structure of the noble gas closest to its own weight. *See 3:A .*) **D. ATOMIC NUMBER (At. No.) OR MOSELEY'S NUMBER.** The number of *protons* in the **nucleus** of the atom. *At. No.* also equals the number of *electrons* in the **orbits.** **E. ATOMIC WEIGHT. (AT. WT.)** (1) **AT. WT. OF AN ISOTOPE.** The *total* number of protons and neutrons in the nucleus. **Ex:** Oxygen = $8p+8n = 16$ *At. Wt.* (2) **AT. WT. OF AN ELEMENT:** Calculated from average weight of the isotopes of an atom in a natural sample referred to oxygen as 16.0000. **Ex:** Hydrogen = 1.0080. **NOTE:** Now atomic weights use the single isotope *Carbon-12* as the standard and not oxygen as 16.0000. Change will in effect reduce all atomic wt. by 4 parts in 100,000; thus hydrogen = 1.0079. **F. DETERMINING NUMBER OF NEUTRONS. Subtract** atomic number from atomic weight. **Ex1:** Draw $_{11}Na^{23}$; Na is symbol of sodium, atomic number = 11, atomic weight = 23. Nucleus contains 11*p*, 12*n*; orbital rings: 2)e 8)e 1)e. A sodium atom tends to give one e away and then has a valence of +1. **Ex2:** Draw $_{17}Cl^{35}$. Cl is symbol of chlorine, atomic number = 17, atomic weight = 35. Nucleus contains 17*p*, 18*n*; orbital rings: 2)e 8)e 7)e. A chlorine atom tends to accept one *e* to fill its outer orbit and then has a valence of −1. **G. ISOTOPES.** Two or more **atoms of one element** *differing only* in number of neutrons or atomic weight. **Ex:** $_1H^1$ = hydrogen, $_1H^2$ heavy hydrogen or deuterium, $_1H^3$ = tritium; each has one *p* and one *e*; they differ only in number of neutrons.

**3. THE MOLECULE.** Composed of one or more atoms. **A. MONATOMIC (ONE ATOM PER MOLECULE).** Noble gases contain the maximum number of electrons in their outermost orbits ($_2$He, $_{10}$Ne, $_{18}$A, $_{36}$Kr, $_{54}$Xe, $_{86}$Rn). Other elements, including metals, may also have one atom to the molecule. **Ex:** Liquid mercury. **B. DIATOMIC.** Elements that have *two atoms* to the molecule at ordinary temperatures include $H_2$, $O_2$, $N_2$, $F_2$, $Cl_2$, $Br_2$, $I_2$. **C. COVALENT BONDING.** Generally applies to gases and liquids with *low boiling pts.* Covalent bonding is the **sharing** of *electrons* between atoms. **Ex:** In methane ($CH_4$) the carbon atom has 4 *e* in its outer shell (shown by dots) and can hold 4 additional electrons. Each hydrogen atom has 1 *e* (shown by *x*) and needs 1 additional *e* to fill its first and only orbit. This results in an electron dot configuration as follows:



**D. ELECTROVALENT BONDING.** Occurs when electrons are **given** by one atom and **accepted** by a second atom. **Ex1: Sodium fluoride** ($Na^{+1}$ $F^{-1}$) Na gives one electron and F accepts it, thus **Na** 2)e 8)e 1)e→7(e 8(e 2(e **Cl** 2)e 8)e 8)e 1)e→7(e 8(e 2(e **Cl.** **Ex2: Calcium chloride** ($Ca^{+11}$ $Cl_2^{-1}$) the Ca gives 2*e* and each Cl accepts one.

**4. SUBSTANCES WITH CONSTANT COMPOSITION. A. ELEMENTS. Cannot be decomposed** by ordinary chemical means. (1) **NATURAL ELEMENTS:** Occur naturally; have atomic numbers from *1-92* (except for elements with atomic numbers 43, 61, 85, 87 which are *unstable* or radioactive). (2) **MAN-MADE ELEMENTS:** Man has created about a dozen heavy elements beyond uranium (U) by bombarding existing elements with atomic particles. **Ex:** Plutonium, 93; Einsteinium, 99. (3) **CHEMICAL SYMBOLS:** Abbreviations of elements established by *international agreement:* **Al**-aluminum; **As**-arsenic; **Ba**-barium; **Bi**-bismuth; **Br**-bromine; **Ca**-calcium; **C**-carbon; **Cl**-chlorine; **Cr**-chromium; **Co**-cobalt; **P**-phosphorus; **Zn**-zinc; etc. **NOTE:** Only *eleven* of the commonly used symbols are unrelated to the American names of their elements: **Ag**-silver; **Au**-gold; **Cu**-copper; **Fe**-iron; **Hg**-mercury; **K**-potassium; **Na**-sodium; **Pb**-lead; **Sb**-antimony; **Sn**-tin; **W**-tungsten. **B. COMPOUNDS.** Can be decomposed into two or more substances by ordinary chemical means. About 2,000,000 compounds are already known. All compounds have a **definite composition by weight. C. RADICAL. A group of elements,** usually either *positively* or *negatively* charged, that acts chemically like a unit. **D. COMMON RADICALS:** $NH_4^+$ ammonium; $SO_4^{--}$ sulfate; $SO_3^{--}$ sulfite; $NO_3^-$ nitrate; $NO_2^-$ nitrite; $OH^-$ hydroxide; $PO_4^{---}$ phosphate; $CrO_4^{--}$ chromate; $CO_3^{--}$ carbonate; $ClO_3^-$ chlorate. **NOTE:** These radicals are *ions.* **E. ION.** An atom or a group of atoms that has given one or more electrons (*net+charge*) or accepted one or more electrons (*net−charge*). **F. HOW TO WRITE FORMULAS. GENERAL RULES:** (1) Place valence in Roman numerals over symbol for the element or radical. (2) **Criss-cross** these numerals to form subscripts. (3) Then reduce subscript numbers so ratio is in simplest terms. **Ex:**

| +I | −I | +II | −I | +III | −II | +I | −II |
|---|---|---|---|---|---|---|---|
| K | Br | Ca | Cl | Al | O | Na | S |
| **KBr** | | **$CaCl_2$** | | **$Al_2O_3$** | | **$Na_2S$** | |

**G. FORMULA:** Writing with Ionic Radicals. Follow same procedure.

| +III | −I | +II | −II | +I | −II | +I | −II |
|---|---|---|---|---|---|---|---|
| Cr | OH | Mg | $SO_4$ | $(NH_4)$ | $SO_3$ | H | $CO_3$ |
| **$Cr(OH)_3$** | | **$Mg_2(SO_4)_2$ $MgSO_4$** | | **$(NH_4)_2SO_3$** | | **$H_2CO_3$** | |

**5. MEASUREMENTS IN CHEMISTRY. A. THERMOMETER CONVERSION RULES.** (1) **TO CONVERT FAHRENHEIT (°F) TO CENTIGRADE (°C or Celsius):** Subtract 32° from °F, then multiply by 5/9, written °C = 5/9(°F−32). **Ex:** Water boils at 212°F; calculate boiling point °C. **Sol:** °C = 5/9(212−32) = 100°C. **NOTE:** Centigrade is now referred to as **Celsius.** (2) **TO CONVERT °C TO °F:** Multiply °C by 9/5 then add 32, written °F = (9/5°C)+32. **Ex:** Convert 45°C to °F. **Sol:** °F = 9/5 45+32 = 113°F. (3) **TO CONVERT °C TO ABSOLUTE OR KELVIN SCALE (°A,°K): Add 273 to °C. Ex:** Convert 18°C to °A. **Sol:** 18+273 = 291°A. (4) **TO CONVERT °F TO °A: First convert °F to °C, then add 273. Ex:** Convert 98.6°F to °A. **Sol:** 98.6−32 = 66.6 then multiply by 5/9 = 37; then 37+273 = 310°A. **B. METRIC SYSTEM.** A decimal system having a simple *factor of 10* relationship between units. (1) **LATIN PREFIXES:** Identify values of *less* than one: deci-- = 0.1, centi-- = 0.01, milli-- = 0.001, micro-- = 0.000001. (2) **GREEK PREFIXES:** Identify values of *more* than one: deka-- = 10., hecto-- = 100., kilo-- = 1000. (3) **UNIT OF LENGTH:** meter (*m.*); 1 m. = 100 centimeters (*cm.*); 1000 *m.* = 1 kilometer; 1 *in.* = 2.54 *cm.* (4) **UNIT OF WEIGHT:** gram (*g.*); 1000 milligrams = 1 *g.*; 1 *kg.* = 2.2 *lb.* or 1 *lb.* = 454 *g.* (5) **UNIT OF VOLUME:** liter (*l.*); 1000 *ml.* = 1 *l.*; 1 *l.* = 1.06 quarts. (6) **METRIC RELATIONSHIPS:** 1 *cm.*³ (cubic centimeter) of water *weighs* 1 *g.* (at 4°C) and has a *volume* of 1 *ml.*; and 1 *l.* of water at 4°C weighs 1 *kg.* **C. EXPONENTIAL EXPRESSIONS.** Large numbers are conveniently expressed as a number times 10 to a *power.* **Ex:** $3\times10^3 = 3\times10\times10\times10 = 3{,}000$; $3.8\times10^{-1} = 3.8\div10 = 0.38$; $4.6\times10^{-5} = 0.000046$. **D. TIME:** Units are seconds, minutes, hours, days. **E. SIGNIFICANT FIGURES.** (1) The number of **digits** in a value (starting with the first non-zero figure is the number of significant figures). The decimal point is ignored in counting significant figures. **Ex1:** 0.00206 has three significant figures. **Ex2:** 20.60 has four significant figures.

**6. ELEMENTARY PROBLEM SOLVING. A. DETERMINING FORMULA OR MOLECULAR WEIGHT [F.W.]. Add** the atomic weights of *all* atoms present in a compound.

| | NaCl | $H_2S$ | $CaCO_3$ | $(NH_4)_2SO_4$ |
|---|---|---|---|---|
| At. Wts. | 23+35 | 2(1)+32 | 40+12+(3)16 | 2[14+(4x1)]+32+4(16) |
| f. w. | 58 | 34 | 100 | 132 |

**B. MOLE.** The amount of a substance whose weight is *equal* to the gram formula weight (*g.f.w.*) which is the **molecular weight of the compound in grams. Ex1:** A mole of oxygen is the amount represented by the formula $O_2$ when O = 16 g.; total weight is 32 g. **Ex2:** Calculate the weight of 3 moles of $H_2SO_4$. **Sol:** 3[2(1)+32+4(16)] = 294 g. **Ex3:** How many moles in 0.34 g of $NH_3$? **Sol:** 14×3(1) = 17 g./ mole; thus g.÷g./mole = moles giving 0.34/17 = .02 moles. **C. DETERMINING PERCENTAGE COMPOSITION.** Divide weight of each element by the formula *weight* of the compound and multiply by 100.

| Ex1: $KNO_3$ | Ex2: $Al_2(SO_4)_3$ |
|---|---|
| At. Wts. = 39+14+3(16) | 2(27)+3[32+4(16)] |
| f.w. = 101 | = 342 |
| %K = 39÷101×100 = 38.61% | %Al = 2(27)÷342×100 = 15.79% |
| %N = 14÷101×100 = 13.86% | %S = 3(32)÷342×100 = 28.07% |
| %O = 3(16)÷101×100 = 47.52% | %O = (3)4(16)÷342×100 = 56.14% |

**D. DETERMINING WATER OF HYDRATION IN CRYSTALS.** Compounds containing a definite proportion of water by weight are called **hydrates. Ex1:** $CuSO_4 \cdot 5H_2O$ (the · 5 means add 5 molecules of water). 64+32+4(16)+5[2(1)+16] = 250 (f.w.); $5H_2O$ = 90; and 90÷250×100 = 36% weight of water crystallized in a molecule of $CuSO_4 \cdot 5H_2O$. **Ex2:** Another common hydrate is **alum** (a double salt) $K_2SO_4 \cdot Al_2(SO_4)_3 \cdot 24H_2O$. **E. EFFLORESCENT COMPOUND.** A **hydrate** which readily **loses** water when its *crystals* (solids having regular geometric shapes) are *exposed* to air. **Ex:** $Na_2SO_4 \cdot 10H_2O$ readily yields much of its 56% water. **F. DELIQUESCENT COMPOUND.** Readily **takes up** water from the air to form a solution. **Ex:** $CaCl_2$, calcium chloride; KOH, potassium hydroxide. **G. DETERMINING EMPIRICAL OR SIMPLEST FORMULA OF A COMPOUND.** (1) **DERIVED FROM PERCENTAGE COMPOSITION:** Divide percentage composition of each element by its atomic weight to find each quotient. Then find smallest whole number *ratio* of atoms in the compound by dividing each of the quotients by the smallest quotient. Clear fractions, if any, by multiplying by a suitable small whole number. **Ex:** Analysis: K = 44.8%, S = 18.4%, O = 36.8%

| % COMPOSITION ÷ ATOMIC WEIGHT | WHOLE NO. RATIO OF ATOMS |
|---|---|
| K = 44.8÷39.1 = 1.14 (quotient); | 1.14÷.57 = 2 or $K_2$ |
| S = 18.4÷32 = .57 (smallest quotient); | .57÷.57 = 1 or S |
| O = 36.8÷16 = 2.30 (quotient); | 2.30÷.57 = 4 or $O_4$ |

In this problem, division by smallest quotient (.57) gave whole numbers. **Answer** = $K_2SO_4$. (2) **DERIVED FROM ACTUAL WEIGHT ANALYSIS OF A COMPOUND:** Calculate percentage composition and proceed as above. **Ex:** Analysis: 50 g. of propane ($C_3H_8$) contains 40.91 g. of C and 9.09 g. of H. What is the empirical formula for propane?

$$\%\,C = \frac{40.91}{50}\times100 = 81.82\%;\quad \%\,H = \frac{9.09}{50}\times100 = 18.18\%$$

| % COMPOSITION | ÷ | ATOMIC WEIGHT | = | QUOTIENT |
|---|---|---|---|---|
| C = 81.82 | ÷ | 12 | = | 6.82; 6.82÷6.82 = 1; 1×3 = 3 |
| H = 18.18 | ÷ | 1 | = | 18.18; 18.18÷6.82 = 2.67 or $2\frac{2}{3}$; $2\frac{2}{3}\times3 = 8$ |

Here since division by smallest quotient did not give whole number multiply by 3. **Answer** = $C_3H_8$.

**7. NAMING COMPOUNDS. A. BINARY COMPOUNDS.** (1) Contain two elements and have names ending in **-ide.** Ex: NaCl = sodium chl*oride;* $Ca_3N_2$ = calcium nitr*ide;* $K_2O$ = potassium ox*ide;* $AlI_3$ = aluminum iod*ide.* (2) If 2 elements form *more* than one compound, *numerical Latin prefixes* (**mono-, di-, tri-, tetra-, penta-**) indicate the number of atoms of the element in *each* molecule of the compound. **Ex:** $N_2O_4$ = *di*nitrogen *tetr*oxide; $SO_2$ = sulfur *di*oxide; $SO_3$ = sulfur *tri*oxide. **B. VALENCE INDICATING SUFFIXES.** If a metal forms two series of compounds, the *higher valence* is indicated by **-ic;** the *lower valence* is indicated by **-ous. Ex:** $Fe^{+II}O^{-II}$ = ferr*ous* oxide; $Fe_2^{+III}O_3^{-II}$ = ferr*ic* oxide. **C. TERNARY COMPOUNDS.** Have names ending in **-ate** or **-ite.** Most common ternary compounds contain *oxygen.* (1) The suffix **-ate,** when used alone, usually denotes the commonest of the ternary compounds of the same three elements. **Ex:** $KNO_3$ = potassium nitr*ate.* (2) Another compound containing less oxygen than the -ate compound will have a name ending in **-ite. Ex:** $KNO_2$ = potassium nitr*ite.* (3) A **hypo-----ite** contains even less oxygen than the *-ite* while a **per-----ate** contains even more oxygen than the *-ate.* **Ex:** KClO = potassium *hypo*chl*orite;* $KClO_2$ = potassium chl*orite;* $KClO_3$ = potassium chl*orate;* $KClO_4$ = potassium *per*chl*orate.* **D. BINARY ACIDS.** Contain hydrogen and one other element and are named as in VII:A. When dissolved in water, are given *prefix* **hydro-** and *suffix* **-ic acid. Ex:** HCl gas = hydrogen chloride; HCl solution = *hydro*chl*oric acid;* HBr solution = *hydro*br*omic acid;* $H_2S$ solution = *hydro*sulf*uric acid.* **E. TERNARY ACIDS.** Are named from the *corresponding salts* changing **-ate to -ic acid** and the **-ite to -ous acid.** Most common ternary acids contain oxygen. Ex: HClO = hypochlorous acid; $HClO_2$ = chlorous acid; $HClO_3$ = chloric acid; $HClO_4$ = perchloric acid. **F. BASES CONTAINING OH (HYDROXIDE).** Named by adding the term *hydroxide* to the name of the metal. **Ex:** NaOH = sodium hydroxide; $Mg(OH)_2$ = magnesium hydroxide.

**8. TYPES OF EQUATIONS. Reactants→Yield→Products. A+B→C+D. A. COMBINATION OR SYNTHESIS.** Involves direct combination of substances. **Ex:** $H_2+S\rightarrow H_2S$; $C+O_2\rightarrow C^{+IV}O_2^{-II}$; $2Hg+O_2\rightarrow 2Hg^{+II}O^{-II}$; $H_2O+SO_3\rightarrow H_2^{+I}SO_4^{-II}$. **B. DECOMPOSITIONS.** The breakdown of a compound into two or more substances. **Ex:** $2H_2^{+I}O^{-II}\rightarrow 2H_2+O_2$; $2Ag^{+I}Cl^{-I}\rightarrow 2Ag+Cl_2\uparrow$. $Ca^{+II}CO_3^{-II}\xrightarrow{\Delta} CaO+CO_2\uparrow$, ($\Delta$ means heat; vertical arrow, ↑ means gas is evolved.) **C. SIMPLE REPLACEMENT.** One chemical element or radical **replaces** another in a substance. (1) **AN ACTIVE METAL** replaces a metal in a compound. **Ex:** $Zn+CuSO_4\rightarrow Cu+ZnSO_4$. (2) **AN ACTIVE NON-METAL** replaces a non-metal in a compound. **Ex:** $3Cl_2+2AlBr_3\rightarrow 2AlCl_3+3Br_2$. (3) **REPLACEMENT OF $H_2$** in water or acids by active metals. **Ex:** $2K+2H_2O\rightarrow 2KOH+H_2\uparrow$; $Sn+2HCl\rightarrow SnCl_2+H_2\uparrow$. **D. DOUBLE REPLACEMENT.** Involves **decomposition** of reactants and **formation** of *new* compounds by the *exchange* of partners. $Ba^{+II}(NO_3)_2^{-I}+H_2^{+I}SO_4^{-II}\rightarrow Ba^{+II}SO_4^{-II}+2H^{+I}NO_3^{-I}$; $Zn^{+II}Cl_2^{-I}+(NH_4)_2^{+I}CO_3^{-II}\rightarrow Zn^{+II}CO_3^{-II}+2NH_4^{+I}Cl^{-I}$.

**9. SOLVING WEIGHT-WEIGHT PROBLEMS.** The weight of one substance is known while the weight of another substance is unknown. **A. SET UP THE PROBLEM.** (1) **Balance** the equation. (2) Write the given weight above the appropriate substance. (3) Place an *x* (*with appropriate units of g., lb., oz.,* etc.) above the substance whose weight is to be determined. (4) Calculate the molecular weight of the substance whose *actual weight* is known. Multiply the mol. wt. by the *coefficient* (the number that precedes the formula of this substance in the balanced equation). Record product under formula. (5) Repeat *Step 4* for substance whose actual weight is unknown.

**Ex:** How many grams of $O_2$ can be produced by thermal decomposition of 100 g. of potassium chlorate **Sol:** Balance equation.

$$2KClO_3 \xrightarrow{\Delta} 2KCl + 3O_2\uparrow$$

100 g. x g.

$$2KClO_3 \rightarrow 2KCl + 3O_2$$

2[39+35.5+3(16)] 3[2(16)]

245 96

**B. SOLVE BY DIRECT PROPORTIONS.** 100÷245 = x÷96; x = 39.2 g. of oxygen. **C. SOLVE BY MOLE METHOD.**

$$2KClO_3 \rightarrow 2KCl + 3O_2$$

2 moles→2 moles+3 moles

Calculate number of moles present in 100 g. of $KClO_3$. 100 g.÷122.5 g./mole = 0.816 moles of $KClO_3$. Since 2 moles of $KClO_3$ yield 3 moles of $O_2$, 1 mole $KClO_3$ yields 1.5 moles $O_2$. Thus, 0.816 mole $KClO_3$ forms 1.5×0.816 or 1.224 moles $O_2$. Grams of $O_2$ in 1.224 moles = 1.224 moles×32.00 g./mole = 39.2 g. oxygen.

**10. SOLVING WEIGHT-VOLUME PROBLEMS.** Weight is given; *find volume* of gas. **A. SET UP THE PROBLEM.** (1) Balance the equation. (2) Write given weight (or volume) above the appropriate substance in the equation. (3) Place an *x* (*with appropriate units of g., l., lb., cm.*³) above the actual volume (*or weight*) to be determined. (4) Since gram-molecular volume (g.m.v.) of any gas is **22.4 liters at S.T.P.** (See XIII:5), record 22.4 under the substance whose volume is involved. (5) Place the value that corresponds to the number of moles (*the coefficient of the formula in the balanced equation*) under the compound. (6) Calculate the molecular weight for the substance whose actual weight is involved.

**Ex:** How many liters of $CO_2$ at S.T.P. (See XIII.5) are produced when 400 g. of calcium carbonate react with hydrochloric acid according to the following equation:

400 g. x l.

$$CaCO_3+2HCl\rightarrow CaCl_2+H_2O+CO_2$$

40+12+3(16) 22.4

100

**B. SOLVE BY PROPORTION. wt.÷f.w. = vol.÷g.m.v.;** 400÷100 = x÷22.4; x = 89.6 liters of $CO_2$.

**11. VOLUME—VOLUME PROBLEMS. (Apply only to gases at constant temperature and pressure.) A. SET UP THE PROBLEM.** (1) Balance the equation. (2) Record the known and unknown given volumes above the appropriate substances in the equation. (3) Place the moles or coefficients (the number that precedes the substance in the balanced equation) under the compounds. **Ex:** How many cubic centimeters (*cm.*³) of $CO_2$ are produced by the total combustion of 30 cm.³ of butane ($C_4H_{10}$)?

**Balance equation:**

$$C_4H_{10}+\frac{13}{2}O_2\rightarrow 4CO_2+5H_2O.$$

30 cm.³ x cm.³

**Remove fractions:**

$$2C_4H_{10}+13O_2 \rightarrow 8CO_2+10H_2O$$

2 moles 8 moles

**B. SOLVE BY PROPORTION.** 30÷2 = x÷8; x = 120 cm.³ of $CO_2$.

**12. EQUIVALENT OR COMBINING WEIGHT OF AN ELEMENT.** Weight of an element which *replaces* or *combines* with **1 gram of hydrogen. A. SET UP THE PROBLEM.** Divide the atomic weight of the element by its valence. **Ex1:** Oxygen. 16÷2 = 8. **Ex2:** In an experiment 6.98 g. of a metal react with 2.00 g. of $O_2$. Calculate equivalent weight of the metal: **Sol:** Weight of $O_2$÷equivalent weight of oxygen = weight of metal÷equivalent weight of metal; thus 2.00÷8.00 = 6.98÷x; x = 27.92. Iron has an equivalent weight of 27.92.

**13. GAS PROPERTIES.** (1) **PRESSURE:** A gas exerts a pressure equally in all directions. (2) **COMPRESSIBILITY AND EXPANSIBILITY:** A gas can fill a container of any size. (3) **DIFFUSIBILITY:** A gas moves freely from one place to another. (4) **PERMEABILITY:** A gas can diffuse into a space already occupied by another gas. (5) **S.T.P. = STANDARD TEMPERATURE AND PRESSURE:** At *S.T.P.,* temperature of a gas is 0° C or 273°*A* and pressure is 1 atmosphere or **760 mm of Hg** (mercury). **NOTE:** Also known as **N.T.P.** (*Normal Temperature and Pressure*). **A. BOYLE'S LAW.** When **temperature (T) is constant,** the volume (V) varies inversely with the pressure (P). $P_1/P_2 = V_2/V_1$. Ex: 500 ml. of a gas is collected at a pressure of 840 mm. At constant temperature, what will the volume be at 760 mm? (1) **SOLUTION BY FORMULA SUBSTITUTION:** 840/760 = $V_2$/500; $V_2$ = 552.6 ml. (2) **SOLUTION BY REASONING:** Since pressure drops, the volume will increase. Multiply original volume by a ratio *greater than one* (larger pressure over smaller pressure): 500×(840/760) = 552.6 ml. **B. GUY-LUSSAC'S LAW.** When **volume (V) is constant,** the pressure (P) of a gas is directly proportional to the temperature (T) on the Kelvin scale: $P_1\div P_2 = T_1\div T_2$. **C. CHARLES' LAW.** When **pressure is constant,** volume (V) of a gas varies directly with the absolute temperature (*T*) in °*K* or °*A*: $V_1/V_2 = T_1/T_2$. **Ex:** At 20°C a gas occupies 500 ml. At constant pressure, what will the volume be at 0°C? **Sol:** First change °*C* to °*K* by adding 273° to each. (1) **SOLUTION BY FORMULA SUBSTITUTION:** 500/$V_2$ = (20+273)°K÷273°K; $V_2$ = 465.8 ml. (2) **SOLUTION BY REASONING:** Since the temperature decreases, the volume will decrease. Multiply original volume by a ratio *smaller than one* (smaller temp. over larger temp.): 500×(273/293) = 465.8 ml. **D. GENERAL GAS-LAW FORMULA.** $P_1V_1\div T_1 = P_2V_2\div T_2$. **Ex:** At a pressure of 800 mm. and a temperature of 30°C a gas has a volume of 500 ml. What will the volume be at S.T.P.? (1) **SOLUTION BY SUBSTITUTION:** (800) (500)÷(273+30) = (760)($V_2$)÷(273); $V_2$ = 474.2 ml. (2) **SOLUTION BY REASONING:** Since temperature decreases, volume decreases (ratio less than 1) or 273÷(273+30); since pressure decreases, volume increases (ratio greater than one or 800÷760):

$$500\times\frac{273}{273+30}\times\frac{800}{760} = 474.2 \text{ ml.}$$

**E. DALTON'S LAW OF PARTIAL PRESSURES. Total pressure** of a *mixture of gases* is equal to the **sum** of the **partial pressures** of the component gases. Partial pressure of each gas is *proportional to the fraction* of its molecules in mixture.

**Ex:** In dry air 20.99% of the molecules are $O_2$, 78.03% are $N_2$, 0.08% are miscellaneous. What is the partial pressure of each when the total atm. pressure is 750 mm. of Hg?

**Sol:** $P_{total} = P_{O_2}+P_{N_2}+P_{misc.}$

$P_t = 20.99P+78.03P+.0008P$

$P_{O_2} = 20.99\%\times750 = 157.42$ mm.

$P_{N_2} = 78.03\%\times750 = 585.23$ mm.

$P_{misc.} = 0.08\%\times750 = 7.35$ mm.

**F. DENSITY OF A GAS.** Weight in grams of 1 cubic centimeter at *S.T.P.* (for convenience may be expressed in grams per liter). **Weight of 1 liter = g.m.w.÷22.4 liters. Ex1:** Under *S.T.P.,* 250 ml. (or cm.³) of a gas weighs 0.41 g. Calculate molecular weight. **Sol:** If 250 ml. weighs 0.41 g., then 1 liter or 1000 ml. weighs 1.64 g. **G. SOLUTION BY SUBSTITUTION.** 1.64 g. = g.m.w./22.4 liters; g.m.w. = 36.7 g. Find the weight of a liter of $O_2$ at S.T.P.? **Sol:** x g./l. = 32/22.4 = 1.43 g./liter (or Density = 0.00143 g./cm.³). **Ex2:** Density of $O_2$ = 0.0014290 g./ml. Calculate molar volume using exact atomic weights to five significant figures. **Sol:** D = W/V; 0.0014290 = 32.00/V; V = 22.393 liters.

**14. METHODS OF EXPRESSING CONCENTRATION. A. MOLARITY OR FORMALITY OF A SOLUTION.** Number of moles (*g.f.w.*) of *solute dissolved* in **one liter of total solution. B. MOLALITY OF A SOLUTION.** Number of moles (*g.f.w.*) of *solute dissolved* in **1000 g. of solvent. C. NORMALITY OF A SOLUTION.** Number of gram-equivalents of solute per liter of solution. **Ex:** Prepare 100 ml. of a 2N solution of $AgNO_3$. **Sol:** *g.f.w.* of $AgNO_3$ = 170 g. = 1 equivalent wt.; 2 equivalent weights = 340 g. in 1000 ml. We need 10 cm.³ therefore 340×100÷1000 = 34.0 g. to which we add water to make final volume 100 ml.

**15. SOLUTIONS. A. DEFINITIONS.** (1) **TRUE SOLUTION:** A homogeneous mixture of two or more substances. The solution will also be *clear* if the solvent itself is clear. (2) **SOLUTE:** Material to be *dissolved.* (3) **SOLVENT:** Substance (*water*) that dissolves solute (*salt*). (4) **DILUTE:** A solution that **contains little solute.** (5) **CONCENTRATED:** A solution that **contains much solute.** (6) **MISCIBLE:** Able to be *mixed in any proportions* to give a true solution (*alcohol and water*). **B. SATURATED, UNSATURATED, SUPERSATURATED.** (1) **SATURATED:** A solution that *has dissolved* as much solute as is possible at the *given temperature* in the presence of some undissolved solute. (2) **UNSATURATED:** A solution in which the *concentration* of solute is **smaller than in the saturated solution** at the same temperature. (3) **SUPERSATURATED:** A solution

in which *concentration* of solute is **greater than in the saturated solution at the same temperature.** NOTE: If a tiny crystal is added to an unsaturated solution, it will dissolve; if it is added to a supersaturated solution, it will grow as the *excess* solute precipitates out; the solution will be saturated when the precipitation stops. NOTE If a tiny crystal is added to a saturated solution, it will appear to neither dissolve nor grow. **C. SUSPENSIONS.** (1) **MECHANICAL SUSPENSION.** *A non-uniform*, **cloudy mixture** of a very finely divided and slowly settling solid or liquid dispersed throughout a less dense medium (*shoe polish or chalk in water*). (2) **COLLOIDAL SUSPENSION OR DISPERSION:** A *uniformly* **opalescent** (*cloudy*) mixture of very small particles of a liquid or solid dispersed throughout a medium (*mayonnaise, smog*). (3) **EMULSION:** A colloidal suspension in which one liquid is **dispersed** in another (*milk*). **D. RULES OF SOLUBILITY IN WATER.** (1) Common **sodium, potassium, and ammonium** compounds are *soluble*. (2) Common **nitrates, acetates, and chlorates** are *soluble*. (3) Common **sulfates** are soluble *except* calcium, barium, strontium, silver, lead, and mercurous. (4) Common **chlorides** are soluble *except* silver, mercurous, and lead. (5) Common **carbonates, phosphates, and silicates** are insoluble *except* group *IA* (**Li, Na, K, Rb, Cs**), barium, radium, strontium, and ammonium. (6) Common **sulfides** are insoluble *except* group *IA*, Group *IIA* (**Mg, Ca, Sr, Ba, Ra**), and ammonium. (7) Common **hydroxides** are insoluble *except* group *IA*, beryllium, calcium, barium, radium, strontium, ammonium.

**16. FACTORS INCREASING RATE OF SOLUTION. A. FOR SOLIDS.** (1) **INCREASING THE TEMPERATURE** makes molecules *move faster* allowing for more collisions. (2) **INCREASING THE SURFACE AREA** by pulverizing solute provides more *reacting* area. (3) **INCREASING THE CONTACT** between solute and solvent particles by *stirring* causes additional collisions. **B. FOR GASES.** (1) **INCREASING THE TEMPERATURE** increases the rate of solution of a gas ($CO_2$ *or* $NH_3$) but decreases the amount that will dissolve. (2) **INCREASING THE PRESSURE** increases the rate of solution and also increases the amount of the gas that will dissolve ($CO_2$ *in soda water*).

**17. AIR, NOBLE GASES, NITROGEN. A. COMPOSITION AND PROPERTIES OF AIR.** (1) **AIR IS A MIXTURE OF GASES OF FAIRLY CONSTANT COMPOSITION:** Nitrogen makes up 78.03% by volume, oxygen 20.99%, argon 0.94%, carbon dioxide 0.035%, plus trace amounts of other noble gases and hydrogen. Air may also contain varying amounts of water vapor and dust. (2) **CHEMICAL SEPARATION OF COMPONENTS:** Magnesium is burned in nitrogen: $3Mg+N_2\rightarrow Mg_3N_2$; $Ca(OH)_2+CO_2\rightarrow CaCO_3\downarrow+H_2O$; $P+O_2\xrightarrow{\Delta}P_2O_5$. **B. INERT GASES.** (1) **HELIUM (He):** Discovered in 1868 by **Lockyer and Frankland** as a result of a *spectrographic analysis* of the sun's atmosphere. It is used as a buoyant agent to lift balloons and dirigibles. It does not burn. $He+O_2$ mixtures prevent *"bends"* in divers working under high pressures and is also used in treating asthma. (2) **ARGON (Ar):** Discovered by **Rayleigh and Ramsay** in 1894. It is used in gas-filled electric light bulbs where it keeps the hot tungsten filament from evaporating; also used with mercury vapor in fluorescent tubes. (3) **NEON (Ne):** Discovered by **Ramsay and Travers** in 1898. It is used in neon signs as it emits a brilliant orange-red glow when an electric current passes through it under low pressure. (4) **KRYPTON (Kr) AND XENON (Xe):** Also isolated by **Ramsay and Travers.** A mixture of Kr and Xe gives a very intense light which lasts only 1/50,000 of a second when electrically agitated. The fast light is ideal for taking high speed photographic pictures. (5) **RADON (Rn):** Discovered by **Dorn** in 1900 as one of the disintegration products of radium. Rn is used in radiotherapy of malignant or cancer growths. (6) All noble gases, having **filled outer orbits,** are nearly completely *non-reactive*. **C. NITROGEN.** (1) **SOURCES:** Air, Chile saltpeter $NaNO_3$, saltpeter $KNO_3$. (2) **PREPARATION:** By fractional distillation of liquid air; by decomposing ammonium nitrite: $NH_4NO_2\xrightarrow{\Delta}N_2\uparrow+2H_2O$. (3) **PHYSICAL PROPERTIES OF $N_2$:** Colorless odorless, tasteless gas, less dense than air, only slightly soluble in water. (4) **CHEMICAL PROPERTIES:** Nitrogen has **5 outermost electrons** and its usual valences are $+5$, $-3$; $N_2$ has a strong *triple covalent bond* causing molecule to be rather inactive; nitrogen does not burn.

**D. OXIDES OF NITROGEN.** (1) **NITROUS OXIDE, $N_2O$:** valence $+1$; a colorless gas, supports combustion. **Prep:** $NH_4NO_3\rightarrow N_2O+2H_2O$. **Gen:** A; **Col:** III. **Iden:** does not react with NO, whereas $O_2+NO\rightarrow 2NO_2$. **Uses:** Anaesthetic (laughing gas). (2) **NITRIC OXIDE, NO:** valence $+2$; a colorless gas, does not support combustion. **Prep:** $3Cu+8HNO_3\rightarrow 3Cu(NO_3)_2+4H_2O+2NO$. **Gen:** A; **Col:** III. **Iden:** $2NO+O_2\rightarrow 2NO_2\uparrow$ brown. **Uses:** catalyst in converting $SO_2$ to $SO_3$ in lead chamber process of $H_2SO_4$ manufacture. (3) **DINITROGEN TRIOXIDE, $N_2O_3$:** valence $+3$, blue liquid (below 2°C), anhydride of nitrous acid ($N_2O_3+H_2O\rightarrow 2HNO_2$). **Prep:** $2NaNO_2+H_2SO_4\rightarrow Na_2SO_4+2HNO_2$. **Iden:** unstable at room temperature. (4) **NITROGEN DIOXIDE, $NO_2$:** valence $+4$; a brown gas; $2NO_2+H_2O\rightarrow HNO_2+HNO_3$. **Prep:** $Cu+4HNO_3\rightarrow Cu(NO_3)_2+2H_2O+NO_2$. **Gen:** A, **Col:** II. **Iden:** turns moist starch-iodide paper blue. **Uses:** vigorous oxidizing agent. (5) **DINITROGEN PENTOXIDE, $N_2O_5$:** valence $+5$; white solid; anhydride of nitric acid ($N_2O_5+H_2O\rightarrow 2HNO_3$). **E. NITROGEN COMPOUNDS.** (1) **Ammonia,** (2) **Nitric acid** (*See 22: K, L*).

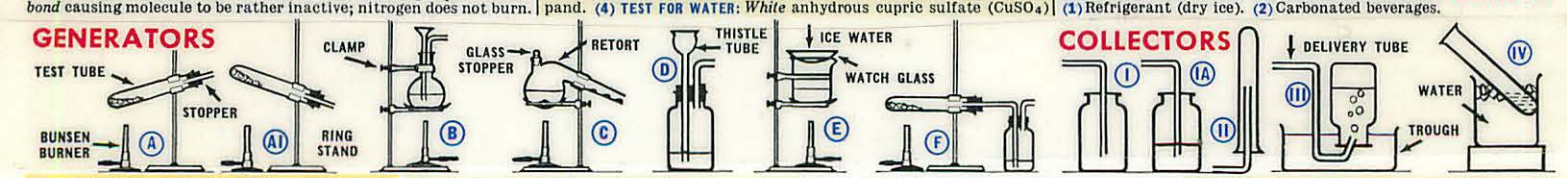
**F. NITROGEN CYCLE.** (1) **NITROGEN FIXATION:** Atmospheric nitrogen is converted into nitrates that can be utilized by plants. **Nitrogen-Fixing-Bacteria** play an important role in this process. (2) **NITRIFICATION:** Certain soil bacteria oxidize nitrites (formed during electrical storms) to nitrates. (3) **DENITRIFICATION: Bacteria of Decay** decompose nitrogen compounds returning nitrogen to the air.

**18. OXYGEN, COMBUSTION, OXIDATION. A. OXYGEN.** (1) Independently discovered by **Priestley** and **Scheele** in 1774: $2HgO\rightarrow 2Hg+O_2\uparrow$. (2) Preparation, Test, Properties: (*See 22:A*). **B. COMBUSTION, BURNING, OR RAPID OXIDATION.** A chemical action producing noticeable *heat* and *light*. (1) **REQUIREMENTS FOR BURNING:** *Fuel, oxygen, kindling temperature* (the lowest temperature at which a substance burns). (2) **TO EXTINGUISH FIRES:** *Remove* combustible material, or *shut off* the oxygen supply, or *cool* the fuel below its kindling temperature. (3) **SPONTANEOUS COMBUSTION:** *Accumulation* of heat (produced by slow oxidation) that reaches the kindling temperature of the fuel (piles of oily rags, soft coal, hay) and starts burning, seemingly by itself. (4) **EXPLOSION:** Instantaneous combustion of a **finely divided fuel** in an *abundant oxygen supply*; the sudden expansion of heated gases causes much damage. **C. OZONE.** (1) **ALLOTROPIC FORM:** *Oxygen* and *ozone* are different forms of the same element exhibiting different properties. (2) **PREPARATION:** Produced in lightning discharges, $3O_2+\text{energy}\rightarrow 2O_3\uparrow$. (2 moles of ozone, each made of 3 atoms). (3) **PHYSICAL PROPERTIES:** Blue gas, irritating odor, 1.5 times as dense as $O_2$, more soluble than $O_2$ in $H_2O$. (4) **CHEMICAL PROPERTIES:** Very *unstable* and changes back to $O_2$; *more active* than $O_2$ as it tarnishes mercury (Hg) and silver (Ag) and destroys the elasticity of rubber. (5) **USES:** Deodorizes and purifies air, kills bacteria, bleaches textile fibers and waxes. **D. SLOW OXIDATION.** Oxygen unites with another substance *without* producing noticeable heat and light. (1) **RUSTING OF IRON:** $4Fe+3O_2+2Fe_2O_3$. (2) **DECAY:** Of plant and animal matter.

**19. HYDROGEN, WATER AND PEROXIDE. A. HYDROGEN.** (1) Discovered by **Cavendish** 1766, named *"hydrogen"* (water producer) by **Lavoisier.** (2) **PREPARATION, TEST, PROPERTIES:** (*See 22:B*). (3) **ADDITIONAL PREPARATION:** From water by replacement with active metals: $2K+2HOH\rightarrow 2KOH+H_2\uparrow$ **B. WATER.** (1) **PHYSICAL PROPERTIES:** Solid $\underset{-\Delta}{\overset{+\Delta}{\rightleftharpoons}}$ liquid $\underset{-\Delta}{\overset{+\Delta}{\rightleftharpoons}}$ gas; colorless (faint blue or blue-green in deep layers only); odorless, tasteless when pure. (2) **POLAR COVALENT** The **positive charge of the oxygen nucleus** is so *much stronger* than the positive charges of the hydrogen nuclei that the shared (*covalent*) electrons are *pulled closer* to the oxygen; since both hydrogen atoms tend to be at one end of the molecule (*about 105° apart*) and are partially bereft of their electrons, the **hydrogen end** becomes the *+ end* of the molecule while the **oxygen end** exhibits *— properties*. (3) **DENSITY OF WATER:** Water is **most dense at 4°C.** The density at 0°C = density at 8°C. As water is heated it tends to expand. (4) **TEST FOR WATER:** *White* anhydrous cupric sulfate ($CuSO_4$) becomes hydrated in the presence of water and turns *blue*. **C. MAKING WATER POTABLE.** Purification of drinking water for household and industrial use. (1) **SEDIMENTATION:** Water is collected and allowed to stand in large reservoirs where most **large particles** (*mud, silt, clay*) **settle out.** (2) **COAGULATION: Lime and aluminum sulfate** are added to the water forming an insoluble, bulky, *gelatinous precipitate* of $Al(OH)_3$, which carries down much suspended matter including bacteria: $Al_2(SO_4)_3+3Ca(OH)_2\rightarrow 2Al(OH)_3\downarrow+3CaSO_4\downarrow$. (3) **FILTRATION:** Water is then passed through beds of *sand and gravel* to remove *floating particles*, then filtered through **charcoal** to remove any *coloring*. (4) **AERATION:** Water is sprayed into the air in some large city systems where organic matter is oxidized, some bacteria are killed, and the taste is improved. (5) **CHLORINATION:** Minute quantities of **chlorine or ozone** are added to *kill* the remaining *bacteria*. **D. $H_2O_2$ OR HYDROGEN PEROXIDE.** (1) **PREPARATION:** Cold dilute $H_2SO_4+BaO_2\rightarrow BaSO_4\downarrow+H_2O_2$. (2) **PHYSICAL PROPERTIES:** A syrupy (*viscous*) liquid that is colorless and odorless; 1.5 × as dense as water, completely miscible in water. (3) **CHEMICAL PROPERTIES:** Unstable compound $2H_2O_2\xrightarrow{\text{light}}2H_2O+O_2\uparrow$; to prevent decomposition, acetanilide, an inhibitor (*negative catalyst*) is added to the $H_2O_2$ and the peroxide is stored in *dark bottles*. (4) **USES:** An oxidizing agent in bleaching hair, wood pulp. Common antiseptic as a 3% solution of $H_2O_2$ in water; a rocket fuel releasing oxygen.

**20. CARBON AND ITS OXIDES. A. ALLOTROPIC FORMS.** (1) **DIAMONDS:** Purest, densest form of carbon; chemically inert; hardest natural substance and can scratch almost anything; colorless and crystalline in form, has high **reflectivity** and **refractivity** when polished. (2) **GRAPHITE: Hexagonal-shaped** crystalline form; soft, good conductor of electricity; used as a lubricant and motor (brush) contact. (3) **LAMPBLACK:** Amorphous form; deposited by smoky kerosine lamps; used in India ink, carbon paper, printing ink. (4) **BONEBLACK:** Produced by **destructive distillation** of bones; porous nature allows **adsorption** (*accumulation of a gas or liquid on the surface of a solid*); used as a decolorizer and sugar refiner. (5) **COKE: Amorphous form;** produced by destructive distillation of soft coal; good reducing agent in metallurgy. (6) **CHARCOAL:** Amorphous form; produced by destructive distillation of wood; good deodorizer, high adsorption; used in gas masks. (7) **CARBON BLACK:** Amorphous form; made by burning natural gas in an insufficient supply of air; used as a filler in auto tires. **B. CHEMICAL PROPERTIES.** (1) Carbon has **4 electrons** in outer orbit and tends to share electrons to form **covalent compounds.** (2) Inactive at ordinary temperatures, strong **reducing agent** at high temp.; used in metallurgy (*reduction of iron oxide to iron*). (3) **CARBORUNDUM:** Hard abrasive made in electric furnace from $SiO_2$ (sand), C, NaCl and sawdust. Furnace reaction: $SiO_2+3C\rightarrow 2CO\uparrow+SiC$. (4) **CALCIUM CARBIDE:** Made in electric furnace: $CaO+3C\rightarrow CaC_2+CO\uparrow$; used to produce acetylene ($C_2H_2$). (5) **CARBON DISULPHIDE:** Made in electric furnace: $C+2S\rightarrow CS_2$; an organic solvent used in the manufacture of viscose rayon and $CCl_4$. [For preparation of $CO_2$ and CO, see 22:C,D.]

**21. USES OF $CO_2$. A. FIRE EXTINGUISHERS.** (1) **SODA-ACID TYPE:** When extinguisher is inverted $H_2SO_4$ pours into a solution of baking soda producing $CO_2$ which forces water under pressure out of extinguisher on to the fire: $2NaHCO_3+H_2SO_4\rightarrow Na_2SO_4+2H_2O+2CO_2\uparrow$. (2) **FOAMITE TYPE:** A licorice extract that stabilizes the foam bubbles is mixed with $NaHCO_3$. When the extinguisher is inverted, aluminum sulfate solution is mixed with the $NaHCO_3$ solution producing $CO_2$. The $CO_2$ is trapped in a frothy foam which acts to cut off the air supply to the fire. (3) **PRESSURIZED LIQUID $CO_2$.** Liquid $CO_2$ vaporizes in air and is directed at fire by nozzle. **B. BAKING POWDER.** Contains: (1) Baking soda ($NaHCO_3$) to produce $CO_2$. (2) Alum [$K_2SO_4\cdot Al_2(SO_4)_3\cdot 24H_2O$] or cream of tartar ($KHC_4H_4O_6$) or monocalcium phosphate [$Ca(H_2PO_4)_2$], any one of which acts on baking soda to make $CO_2$ gas: $NaHCO_3+KHC_4H_4O_6\rightarrow KNaC_4H_4O_6+H_2O+CO_2\uparrow$. (3) Also a diluent or dehydrator, such as starch or flour, to absorb $H_2O$ in the above reaction. **C. OTHER USES.** (1) Refrigerant (dry ice). (2) Carbonated beverages.




**22. COMMON PREPARATIONS, PROPERTIES, USES, AND TESTS.** ABBREVIATIONS. **Lab. Prep.** = Laboratory Preparation. **Com. Prep.** = Commercial Preparation. **P.P.** = Physical Properties. **St.** = State. **C.** = Color. **O.** = Odor. **T.** = Taste. **D.** = Density. **S.** = Solubility. **C.P.** = Chemical Properties.

**A. OXYGEN ($O_2$).** (1) **LAB. PREP.:** Generator *A*. Collector *III*: $2KClO_3+MnO_2\rightarrow 2KCl+3O_2\uparrow+MnO_2$; $MnO_2$ acts as a **catalyst** changing the *speed* (hastening or retarding) of a chemical reaction *without being permanently changed* itself. (2) **COMMERCIAL PREP.:** $2H_2O\xrightarrow{\text{electrolysis}}2H_2+O_2$. Direct current only flows through platinum electrodes. $H_2SO_4$ helps conduct the current. $O_2$ collects at the anode or + terminal. $H_2$ collects at the cathode or — terminal. Also prepared by fractional distillation of liquid air (cooled to —200°C). (3) **P.P.:** St.: gas; C: —; O: —; T: —; D: slightly denser than air; S: slightly soluble in water. (4) **C.P.:** Exceedingly active but does not burn, supports combustion. **Forms oxides with metals:** $4Fe+3O_2\rightarrow 2Fe_2O_3$; **forms oxides with non-metals:** $4P+5O_2\rightarrow 2P_2O_5$ *or* $S+O_2\rightarrow SO_2$. (5) **TEST:** A glowing splint bursts into flames in $O_2$. (6) **USES:** Used in oxyacetylene and oxyhydrogen torches; oxygen tents in hospitals.

**B. HYDROGEN ($H_2$).** (1) **LAB. PREP.:** Generator *D*. Collector *III*: $Zn+2HCl\rightarrow ZnCl_2+H_2\uparrow$. (2) **COMMERCIAL PREP.:** Electrolysis of water (see $O_2$ prep). Steam + hot coke→ water gas mixture: $H_2O+C\rightarrow CO+H_2$. (3) **P.P.:** St: gas; C: —; O: —; T: —; D: lowest density; S: very slightly soluble in water. (4) **C.P.:** Burns to form $H_2O$; does not support combustion. Reacts with non-metals: $H_2+Cl_2\rightarrow 2HCl$; is a *reducing agent* with $CuO+H_2\rightarrow H_2O+Cu$, here **CuO is reduced to Cu** while $H_2$ *is oxidized to* $H_2O$. (Process of taking O from compound is example of *reduction*.) (5) **TEST:** A burning splint pops or burns with a nearly colorless flame and only $H_2O$ is produced. (6) **USES:** Part of common fuel gas mixtures such as coal gas and water gas. Used to convert vegetable oils to solid fats (hydrogenation); oxygen-hydrogen and atomic-hydrogen torches.

**C. CARBON DIOXIDE ($CO_2$).** (1) **LAB. PREP.:** Generator *D*. Collector *III* (rapidly) or *II*: $CaCO_3+2HCl\rightarrow CaCl_2+CO_2\uparrow+H_2O$. (2) **COMMERCIAL PREP.:** $C_6H_{12}O_6\rightarrow 2C_2H_5OH+2CO_2\uparrow$ by *fermentation* of molasses (sugar) resulting in production of *ethyl alcohol*. (3) **P.P.:** St: gas; C: —; O: very faintly pungent; T: slightly pungent; D: 1.5× heavier than air; S: fairly soluble in water. (4) **C.P.:** Stable gas; neither burns nor supports combustion; **anhydride of carbonic acid** (*if it exists*): $H_2CO_3\rightleftarrows H_2O+CO_2$. NOTE: **Acid Anhydride** = non-metallic oxides that react with water to form acids. **Basic Anhydride** = metallic oxides that react with water to form bases. (5) **TEST:** Bubble into lime water to form white precipitate: $Ca(OH)_2+CO_2\rightarrow CaCO_3\downarrow+H_2O$; continue bubbling and precipitate redissolves as a soluble bicarbonate is formed: $CaCO_3+H_2O+CO_2\rightarrow Ca(HCO_3)_2$. (6) **USES:** Making dry ice, leavening bread, fire extinguishers, carbonated soda. **Solvay process** to make $Na_2CO_3$ and $NaHCO_3$.

**D. CARBON MONOXIDE (CO).** (1) **LAB. PREP.:** Generator *B*, modified to regulate flow of formic acid. Collector *III*: $HCOOH+H_2SO_4\rightarrow CO+H_2SO_4\cdot H_2O$. (2) **COMMERCIAL PREP.:** Production of water gas: $H_2O+C\rightarrow H_2+CO$. Producer gas: $2C+\text{air}\,(N_2+O_2)\rightarrow 2CO+N_2$. (3) **P.P.:** St: gas; C: —; O: —; T: —; D: slightly less dense than air; S: slightly soluble in water. (4) **C.P.:** Burns with *blue* flame: $2CO+O_2\rightarrow 2CO_2$. **Reducing agent:** $Fe_2O_3+3CO\rightarrow 2Fe+3CO_2\uparrow$. **Reacts with hydrogen** forming methyl alcohol: $CO+2H_2\rightarrow CH_3OH$. (5) **USES:** Reducing agent in extracting metals from oxide ores.

**E. HYDROGEN CHLORIDE (HCl).** (1) **LAB. PREP.:** Generator *B*. Collector *IA*, modified with glass tubing: $NaCl+H_2SO_4\rightarrow NaHSO_4+HCl\uparrow$. (2) **COMMERCIAL** $2NaCl+H_2SO_4\xrightarrow{\text{high temperature}}Na_2SO_4+2HCl\uparrow$. (3) **P.P.:** St: Gas; C: —; O: sharp, penetrating; T: sour taste in dilute form; D: heavier than air; S: extremely soluble in water. (4) **C.P.:** Covalent *stable* compound; water solution is a strong acid; neutralizes hydroxides, forming salts and water: $HCl+NaOH\rightarrow NaCl+H_2O$. (5) **TEST:** $HCl+AgNO_3\rightarrow AgCl\downarrow(\text{white})+HNO_3$. HCl *fumes* when moistened; turns *wet litmus red*. (6) **USES: Pickling** of metals, the removal of oxide coating of metals by immersion in HCl before *galvanizing* (coating with Zn), *plating*, etc. Used to clean tile and stone, called *muriatic acid;* to prepare $ZnCl_2$, a soldering flux to clean surfaces. Involved in gastric digestion; used in manufacture of glucose and glue.

**F. CHLORINE ($Cl_2$).** (1) **LAB. PREP.:** Generator *B*. Collector *I*: $4HCl+MnO_2\rightarrow MnCl_2+2H_2O+Cl_2\uparrow$ or $2KCl+MnO_2+2H_2SO_4\rightarrow K_2SO_4+MnSO_4+Cl_2\uparrow$. (2) **COMMERCIAL PREP.:** Electrolysis of brine: $2NaCl+2H_2O\rightarrow 2NaOH+H_2\uparrow+Cl_2\uparrow$. (3) **P.P.:** St: gas; C: greenish-yellow; O: irritating; T: sour; D: 2.5× denser than air; S: moderately soluble in water. (4) **C.P.: Action with metals:** $2Sb+3Cl_2\rightarrow 2SbCl_3$. **Action with hydrogen:** $Cl_2+H_2\xrightarrow{\text{light}}2HCl$. **Action with water:** $2H_2O+2Cl_2\rightarrow 2HCl+HClO$. The HClO decomposes into $2HCl+O_2\uparrow$ (powerful oxidizing agent). (5) **TEST:** Filter paper soaked in turpentine flashes into flame releasing a cloud of black smoke in $Cl_2$: $8Cl_2+C_{10}H_{16}\rightarrow 16HCl+10C$. Paste of KI and starch on paper→starch becomes blue: $2KI+Cl_2\rightarrow 2KCl+I_2$. (6) **USES:** Chloroform ($CHCl_3$), an anaesthetic; carbon tetrachloride ($CCl_4$), a fire extinguisher; bleaching agent; water purification; poison gases (mustard gas, phosgene).

**G. BROMINE ($Br_2$).** (1) **LAB. PREP.:** Generator *C*. Collector *IV*: $2KBr+MnO_2+2H_2SO_4\rightarrow K_2SO_4+2H_2O+Br_2$. (2) **COMMERCIAL PREP.:** (sea water) $MgBr_2+Cl_2\rightarrow MgCl_2+Br_2\uparrow$. (3) **P.P.:** St: liquid (only non-metal that is liquid at room temperature); C: deep red; O: choking; T: poisonous; D: 3× water; S: fairly soluble in water; very soluble in $CS_2$. (4) **C.P.: Reacts with hydrogen:** $Br_2+H_2\rightarrow 2HBr$. **Reacts with most metals:** $Cu+Br_2\rightarrow CuBr_2$. **Reacts with water:** $H_2O+Br_2\rightarrow HBr+HBrO$ (weak bleaching agent). (5) **TEST:** Add $CS_2$ or $CCl_4$→orange color. **Test for a bromide ion:** $2NaBr+Cl_2\cdot H_2O\rightarrow 2NaCl+Br_2$, then add $CS_2$ for orange color. (6) **USES:** In photography (*AgBr*); in medicines and dyes.

**H. IODINE ($I_2$).** (1) **LAB. PREP.:** Generator *E*: $2KI+MnO_2+2H_2SO_4\rightarrow K_2SO_4+MnSO_4+2H_2O+I_2$. Collection: *sublimation* (*skips liquid state*): solid→vapor→solid. (2) **COMMERCIAL PREP.:** $2KI+Cl_2\rightarrow 2KCl+I_2$. (3) **P.P.:** St: solid; C: steel gray; O: irritating vapor (chlorine-like); T: —; D: —; S: very slightly soluble in water; very soluble in many organic solvents and alcohol (forms tincture). (4) **C.P.:** $2K+I_2\rightarrow 2KI$; HI most unstable halogen acid, difficult to form. (5) **TEST:** $I_2$+starch→blue color. **Test for iodine ion:** Add $Cl_2+H_2O$ and shake with $CCl_4$ or $CS_2$→violet. (6) **USES:** Tincture of iodine antiseptic (alcohol solution of $I_2$ plus NaI).

**I. HYDROGEN SULFIDE (HYDROSULFURIC ACID—$H_2S$).** (1) **LAB. PREP.:** Generator *D*. Collector *I*: $Fe+2HCl\rightarrow FeCl_2+H_2S\uparrow$. (2) **COMMERCIAL PREP.:** Same as lab. prep. (3) **P.P.:** St: gas; C: —; O: rotten or decayed eggs; T: —; D: denser than air; S: moderately soluble. (4) **C.P.:** Burns completely, $2H_2S+3O_2\rightarrow 2H_2O\uparrow+2SO_2\uparrow$, $2H_2S+O_2\xrightarrow{\text{incomplete}}2H_2O\uparrow+2S\downarrow$; acts on metals by tarnishing them $2Ag+H_2S\rightarrow Ag_2S+H_2$; forms a weak acid in water. (5) **TEST:** $H_2S+Pb(NO_3)_2\rightarrow PbS\downarrow(\text{black})+2HNO_3$. NOTE: $H_2S$ is *poisonous* even in small quantities. (6) **USES:** In chemical analysis (see chart on hydrogen sulfide).

**J. SULFUR DIOXIDE ($SO_2$).** (1) **LAB. PREP.:** Generator *B*. Collector *Ia*: $Na_2SO_3+H_2SO_4\rightarrow Na_2SO_4+H_2O+SO_2\uparrow$. (2) **COMMERCIAL PREP.:** Roasting sulfides (sphalerite): $ZnS+3O_2\rightarrow 2ZnO+2SO_2\uparrow$ or $S+O_2\rightarrow SO_2$. (3) **P.P.:** St: gas; C: —; O: suffocating; T: sour; D: more than 2× air; S: very soluble in $H_2O$. (4) **C.P.:** An acid anhydride: $H_2O+SO_2\rightleftarrows H_2SO_3$ (*if it exists*); stable gas and does not readily burn or support combustion. (5) **TEST:** for sulfite ion and bisulfite ion ($HSO_3$), add dilute $H_2SO_4\rightarrow SO_2\uparrow$ which decolorizes purple potassium permanganate ($KMnO_4$). (6) **USES:** Make $H_2SO_4$; temporary bleaching agent; refrigerant as it is easily liquefied.

**K. AMMONIA ($NH_3$).** (1) **LAB. PREP.** Generator *A1*. Collector *II*: $2NH_4Cl+Ca(OH)_2\rightarrow CaCl_2+2NH_3\uparrow+2H_2O$. (2) **COMMERCIAL PREP.:** (the Haber Process). At 200 atmospheres pressure and 500°C: $N_2+3H_2\rightarrow 2NH_3$ in the presence of the catalyst platinum (*Pt*). Also produced from the destructive dist. of soft coal. (3) **P.P.:** St: gas; C: —; O: pungent; T: —; D: less than air; S: extremely soluble in water. (4) **C.P.:** Unites with water: $NH_3+H_2O\rightleftarrows NH_4OH$ (*if it exists*) or ammonia water solution. $NH_3$ is a *stable* compound; does not burn in air and does not support combustion. **$NH_3$ burns in pure $O_2$:** $4NH_3+3O_2\rightarrow 2N_2+6H_2O$. **Unites with acids to form salts:** $2NH_3+H_2SO_4\rightarrow(NH_4)_2SO_4$; $NH_3+HNO_3\rightarrow NH_4NO_3$. (5) **TEST:** All ammonium compounds, when heated with a soluble base, produce a gas that turns moist red litmus blue: $NH_4Cl+NaOH\rightarrow NaCl+H_2O+NH_3\uparrow$. $NH_3$ fumes in presence of HCl acid→$NH_4Cl$ (*sal ammoniac* or ammonium chloride). (6) **USES:** Refrigerant; manufacture of $HNO_3$ via Ostwald process; explosives manufactured from ammonium nitrate ($NH_4NO_3$); to make fertilizers like $(NH_4)_2SO_4$. Used in Solvay process to produce baking soda ($NaHCO_3$) and washing soda ($Na_2CO_3$).

**L. NITRIC ACID ($HNO_3$).** (1) **LAB. PREP.:** Generator *C*. Collector *IV*: $NaNO_3+\text{concentrated}\,H_2SO_4\rightarrow HNO_3\uparrow+NaHSO_4$. (2) **COMMERCIAL PREP.:** Ostwald process uses platinum as a catalyst: $4NH_3+5O_2\xrightarrow{600°C}4NO+6H_2O$, then $4NO+3O_2+2H_2O\rightarrow 4HNO_3$. (3) **P.P.:** St: liquid; C: —; O: peculiar; T: dangerous; D: 1.5× water; S: miscible in $H_2O$ in all proportions. (4) **C.P.:** Unstable at room temperature, decomposes when heated: $4HNO_3\rightarrow 4NO_2(\text{brown})+2H_2O+O_2$. **Aqua regia** can *dissolve gold* and is 1 part $HNO_3$+3 parts HCl. (5) **TEST: Nitrate ion:** $FeSO_4+NO_3^-+H_2SO_4$ poured down side of tube yields *brown ring* just above $H_2SO_4$ layer. (6) **USES:** Mfg. of guncotton (nitrocellulose), T.N.T. (trinitrotoluene), fertilizers, plastics, rayon, dyes.

**M. DESTRUCTIVE DISTILLATION OF SOFT COAL (BITUMINOUS).** (1) **LAB. PREP.:** Generator *F*. (2) **PRODUCTS: Gas**—Coal gas made of $H_2S$ (rotten egg smell), $NH_3$ (moist litmus turns blue), $CH_4$ (burns). **Liquid**—Coal tar can be separated by distillation into *naphthalene*, $C_{10}H_8$ (kills larvae of clothes moth); *toluene*, $C_7H_8$; *benzene*, $C_6H_6$; *anthracene*, $C_{14}H_{10}$; etc. **Solid**—Coke, a reducing agent for oxides of many metals; also high heat content fuel.

**N. DESTRUCTIVE DISTILLATION OF WOOD.** (1) **LAB PREP.:** Generator *F*. (2) **PRODUCTS: Gas**—Wood gas made of $N_2$ (relatively inactive). CO (burns with a blue flame), $CH_4$ (burns). **Liquid**—Pyroligneous Acid is a mixture of *wood alcohol*, ($CH_3OH$); *acetic acid*, ($CH_3COOH$); *acetone*, ($CH_3COCH_3$). **Solid**—Charcoal, an *adsorbing agent* able to remove odors and colors by attracting and holding impurities onto its surface area only. Used in gas masks and in water purification. NOTE: Not to be confused with "absorption," which is the soaking up of one substance through the entire mass of the other. Ex: Sponges absorb water.

**O. FLAME TESTS**

| METAL | COLOR | METAL | COLOR |
|---|---|---|---|
| Sodium | yellow | Zinc | white |
| Lithium | red (crimson) | Cadmium | yellow |
| Strontium | red (scarlet) | Antimony III | orange |
| Calcium | orange-red | Antimony V | yellow |
| Barium | yellow-green | Copper | black |
| Copper | bright green | Lead | black |
| Potassium | violet | Silver | black |

**P. $H_2S$ TESTS**

**Q. COBALT NITRATE TEST.**

| METAL | COLOR |
|---|---|
| Aluminum | blue |
| Magnesium | pink |
| Zinc | green |

**R. BORAX BEAD TEST**

| METALLIC OXIDE | COLOR |
|---|---|
| Cobalt | blue |
| Iron | yellow |
| Chromium | green |
| Manganese | amethyst |
| Nickel | brown |

DATA-GUIDE QUICK CHART

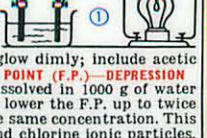
# BASIC CHEMISTRY 2

## A SIMPLIFIED SUMMARY & INSTANT REFERENCE

LEARN & REVIEW IN MINUTES

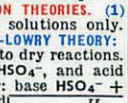
**AUTHOR:** Bernard Kauderer, Ph.D., Chairman, Dept. of Chemistry of the Plainview School, New York. **CONSULTANT:** Prof. John Arents, Ph.D., University of the City of New York.

**1. IONIZATION. A. ELECTROLYTES.** Substances whose water (**aqueous**) solution conducts electricity — acids, bases, salts. **B. NON-ELECTROLYTES.** Aqueous solutions of substances that do not conduct electricity—sugar, alcohol, other organic compounds.



**(1) STRONG ELECTROLYTES:** At moderate concentrations cause the bulb in Fig. 1 to glow brightly; include *NaCl*, $H_2SO_4$, *KOH*. **(2) WEAK ELECTROLYTES:** At moderate concentrations, cause the bulb to glow dimly; include acetic acid, ammonia water. **C. MOLAL FREEZING POINT (F.P.)—DEPRESSION OF WATER.** 1 mole of a non-electrolyte dissolved in 1000 g of water **lowers the F.P. by 1.86°C.** Salt (**NaCl**) can lower the F.P. up to twice as much as would a non-electrolyte at the same concentration. This suggests formation of separate sodium and chlorine ionic particles. $Al(NO_3)_3$ can lower the F.P. up to 4 times as much as a non-electrolyte which suggests four ionic particles. **D. MOLAL BOILING POINT (B.P.)—ELEVATION OF WATER.** 1 mole of a non-electrolyte dissolved in 1000 g of water **raises B.P. by 0.52°C.** Electrolytes can raise B.P. up to 2, 3, or more times as much as non-conductors (e.g., alcohol) at the same concentration. **E. THEORY OF IONIZATION. (1) Arrhenius** in 1887 developed theory to explain B.P. elevation, F.P. depression, and electrolytic conduction. (2) Most acids, bases, salts ionize or *dissociate* in aqueous solution into charged particles called ions. (3) An ion is an atom or **radical** (group of atoms) that carries an *electrical charge* equal to the valence of the ion. **Positive ions** have given up electrons. **Negative ions** have gained electrons. (4) Solutions contain exactly equal numbers of + and − charges. (5) **Atom vs. ion:** Atom of $Cl^0 - 1e \rightarrow K^+$ ion, atom of $S^0 + 2e \rightarrow S^{--}$. **F. DISSOCIATION VS. IONIZATION. (1) DISSOCIATION:** Crystals or molecules of *electrovalent* compounds separate into ions during solution process: $NaCl \rightleftharpoons Na^+ + Cl^-$; $Ba(OH)_2 \rightleftharpoons Ba^{++} + 2OH^-$. **(2) IONIZATION:** Many compounds that are not ionized when pure do form ions when dissolved in water. Ex: liquid hydrogen chloride does not conduct an electric current but bonds by *sharing* electrons. The chlorine tends to pull the shared electrons closer to itself (more electronegative) making the chlorine end of the molecule more negative and the hydrogen end more positive. **HCl** is a polar molecule. When added to water ionization occurs and may be represented as $HCl + H_2O \rightarrow Cl^- + H_3O^+$. The hydronium or oxonium ion $H_3O^+$ is a hydrated proton: $H^+ \cdot H_2O$. **G. MODERN IONIZATION THEORIES. (1) MODIFIED ARRHENIUS THEORY:** Applies to aqueous solutions only. Acids are $H_3O^+$ while bases are $OH^-$. **(2) BRONSTED-LOWRY THEORY:** Applies to aqueous and non-aqueous solutions and to dry reactions. **An acid** is a proton donor: acid $H_2SO_4 \rightarrow H^+ + HSO_4^-$, and acid $HSO_4^- \rightarrow H^+ + SO_4^{=}$. **A base** is a proton acceptor: base $HSO_4^- + H^+ \rightarrow H_2SO_4$. A substance can thus be both *acidic* and *basic*. **(3) LEWIS THEORY:** An acid accepts an electron pair to form a **covalent bond.** Ex 1: $H^+$ is a proton without any electrons and can accept sharing of 2 electrons to form a covalent bond; a base has an electron pair for sharing to form a covalent bond.

Ex 2: BASE ELECTRONS



**2. CHEMICAL EQUILIBRIUM. A. DEF. (1) A state of balance** is attained when two opposing reactions (a, b) occur simultaneously at the same rate $A + B \rightleftharpoons C + D$. **(2) A dynamic state exists** when reactant and product molecules continually break down and re-form. **B. FACTORS AFFECTING SPEED OF REACTION. (1) ACTIVITY OF REACTING ELEMENTS:** Some elements have a greater tendency to react than do others. See *E.M.F.* Table. **(2) PRESENCE OF CATALYSTS:** Alters (increases or decreases) the speed of reaction. **(3) TEMPERATURE:** A rise of 10°C can double reaction speeds by increasing the frequency of *collisions;* increased temperature gives molecules sufficient momentum to realize the necessary *energy of activation.* **(4) SURFACE AREA:** Reactions occur on the surface of substances; increased surface area increases *speed of reaction.* **(5) CONCENTRATION:** Reaction rate is directly proportional to the concentration of each of the reacting substances raised to a power equal to its coefficient in the balanced equation (Law of Mass Action of **Guldberg and Waage,** 1867). Increasing concentration of any reactant increases chances of collision and increases reaction rate. **(6) PRESSURE:** According to **LeChatelier's Principle,** a reaction shifts in the direction that will relieve any stress or strain. Ex: in **Haber process** 3 moles of $H_2$ + 1 mole of $N_2$ yield 2 moles of $NH_3$; by increasing the pressure a stress is established that finds relief by shifting the equilibrium to the right (formation of more ammonia) as four volumes on the left are subjected to a greater stress than the two volumes on the right. **C. EQUILIBRIUM CONSTANT. (1)** When $2A + B \rightleftharpoons 3C + 2D$, the equilibrium constant ($K$) is $K = \frac{[C]^3\,[D]^2}{[A]^2\,[B]}$. **(2)** $K$ = (the product of the concentration of substances produced at equilibrium) divided by (the product of the concentrations of the reacting substances) when each concentration is raised to the power which is its coefficient in the balanced equation.

**3. REACTIONS THAT GO TO COMPLETION.** Reactions can move practically to completion when ions are removed from solution as a **precipitate** forms, a **gas** evolves, or a **non-electrolyte** is produced. **A. FORMATION OF A PRECIPITATE.** An insoluble product removes ions from solution. Sodium chloride and silver nitrate solutions react to form a white precipitate of silver chloride. **MOLECULAR:** $NaCl + AgNO_3 \rightarrow AgCl\downarrow + NaNO_3$. **IONIC:** $Na^+ + Cl^- + Ag^+ + NO_3^- \rightarrow AgCl\downarrow + Na^+ + NO_3^-$. The sodium and nitrate ions are spectator ions (do not react). (2) Some common insoluble compounds are *CuS, BaSO₄, Al(OH)₃, CaCO₃, FeS, HgO.* **B. EVOLUTION OF A GAS.** Volatile products are non-ionized and leave the field of action. **(1) MOLECULAR:** $K_2S + 2HCl \rightarrow KCl + H_2S\uparrow$. **(2) IONIC:** $2K^+ + S^= + 2H^+ + 2Cl^- \rightarrow 2K^+ + 2Cl^- + H_2S\uparrow$. **(3) COMMON GASES:** *HCl, HBr, HF, HI, $H_2S$.* **(4) UNSTABLE COMPOUNDS:** Produce gases that leave the field of reactions: $NH_4OH \rightarrow NH_3\uparrow + H_2O$; $H_2CO_3 \rightarrow H_2O + CO_2\uparrow$; $H_2SO_3 \rightarrow SO_2\uparrow + H_2O$. **C. FORMATION OF A NON-ELECTROLYTE.** Water exists as covalent molecules with a very slight tendency to ionize. **(1) MOLECULAR:** $2KOH + H_2SO_4 \rightarrow K_2SO_4 + 2HOH$. **(2) IONIC:** $2K^+ + 2OH^- + 2H^+ + SO_4^{=} \rightarrow 2K^+ + SO_4^{=} + 2HOH$. **(3)** One molecule of water in $5.55 \times 10^6$ molecules ionizes.

**4. ACIDS. A. PROPERTIES. (1)** Contain hydrogen and yield hydrated hydrogen ions (called **hydronium ions**), in aqueous solution. **(2)** Turns blue litmus to a **red** color. **(3)** Water solutions have a **sour** taste. **B. COMMON ACIDS. (1)** Citric acid (*lemons*), lactic acid (*sour milk*), acetic acid (*vinegar*), malic acid (*apples*), butyric acid (*rancid butter*), tartaric acid (*grapes*). **(2)** Important industrial acids: strong acids ionize almost completely in aqueous solutions: $H_2SO_4$, $HNO_3$, *HCl.* **C. CHEMICAL PROPERTIES. (1) NEUTRALIZATION:** Acid plus base (metallic hydroxides) produces salt and water. **NOTE:** the reverse reaction: salt + water → acid + base, is often called hydrolysis. **(2) ACIDS REACT WITH MANY METALS:** $Zn + H_2SO_4 \rightarrow ZnSO_4 + H_2\uparrow$. **(3) ACIDS REACT WITH CARBONATES:** $MgCO_3 + 2HCl \rightarrow MgCl_2 + H_2O + CO_2\uparrow$.

**5. BASES. A. PROPERTIES. (1)** Yield **hydroxide ions** when dissolved in water. **(2)** Turn red litmus to **blue** **(3)** Aqueous solutions have a **bitter** taste and feel slippery. **B. COMMON BASES.** Caustic potash, *KOH;* lye, *NaOH;* milk of magnesia, $Mg(OH)_2$; limewater, $Ca(OH)_2$; household ammonia, $NH_4OH$. **C. CHEMICAL PROPERTIES. (1) NEUTRALIZATION:** Base + acid → salt + water. **(2) REACT WITH NON-METALLIC OXIDES:** Form salt and water: $CO_2 + 2NaOH \rightarrow Na_2CO_3 + H_2O$. **D. PREPARATION OF BASES. (1) ACTIVE METAL ON WATER:** $2K + 2H_2O \rightarrow I_2\uparrow + 2KOH$. **(2) METALLIC OXIDE ON WATER:** $CaO + H_2O \rightarrow Ca(OH)_2$. **(3) ELECTROLYSIS OF BRINE:** $2NaCl + 2H_2O \rightarrow 2NaOH + H_2\uparrow + Cl_2\uparrow$.

**At anode:** $2Cl^- - 2e \xrightarrow{\text{oxidation}} Cl_2^0\uparrow$.

**At cathode:** $2H^+ + 2e \xrightarrow{\text{reduction}} H_2^0\uparrow$.

**6. SALTS. A. PROPERTIES. (1)** Usually 100% ionized. **(2)** Many salts are nearly neutral to litmus. **(3)** Taste "salty". **B. COMMON SALTS.** Epsom salts, $MgSO_4 \cdot 7H_2O$; gypsum, $CaSO_4 \cdot 2H_2O$; "hypo", $Na_2S_2O_3 \cdot 5H_2O$; plaster of paris, $(CaSO_4)_2 \cdot H_2O$; glauber's salt, $Na_2SO_4 \cdot 10H_2O$. **C. PREPARATION. (1) DIRECT UNION OF ELEMENTS:** $2Na + Cl_2 \rightarrow 2NaCl$. **(2) NEUTRALIZATION:** $Ca(OH)_2 + 2HNO_3 \rightarrow Ca(NO_3)_2 + 2H_2O$. **(3) DOUBLE REPLACEMENT:** $Na_2SO_4 + BaCl_2 \rightarrow BaSO_4\downarrow + 2NaCl$. **(4) ACID AND A BASIC ANHYDRIDE (metallic oxide):** $2HCl + BaO \rightarrow BaCl_2 + H_2O$. **(5) SOLUBLE BASE AND AN ACID ANHYDRIDE (non-metallic oxide):** $2KOH + CO_2 \rightarrow H_2O + K_2CO_3$.

**7. HYDROGEN ION CONCENTRATION OR pH OF A SOLUTION.**

1 2 3 4 5 6 7 8 9 10 11 12 13 14

**A. pH SCALE. (1)** ← acidity increases | neutral | alkalinity increases →

**(2)** Solutions of strong *acids* have *low* pH values: *HCl* (0.1 mole/liter) = pH of 1. **(3)** Solutions of strong *bases* have high pH values. Ex: *KOH* (0.1 mole per liter = pH of 13). **B. FORMULA. pH = log (1 ÷ [H⁺])** or **−log [H⁺].** (1) logarithm of a number (to base 10) [H⁺] = hydrogen-ion concentration (moles per liter). (2) Calculate pH of pure water (0.0000001 mole of $H^+$ per liter): pH = log (1 ÷ 0.0000001), pH = log (1÷10)$^{-7}$, pH = log $10^7$, pH = 7 neutral. **C. COMMON ACID-BASE INDICATORS.** Substances which can be used to detect the presence of *hydronium* and *hydroxyl ions.* **(1) LITMUS:** Acid → red color, base → blue. **(2) PHENOLPHTHALEIN:** Acid → colorless, base → red. **(3) METHYL ORANGE:** Acid → red, base → yellow.

**8. OXIDATION. A. COMBINATION OF AN ELEMENT WITH OXYGEN. (1)** $\overset{+III}{4Fe^0} + \overset{-II}{3O_2^0} \rightarrow \overset{+III\ -II}{2Fe_2O_3}$. **(2)** Electrons are *lost* by the element and taken by the oxygen. **B. COMBINATION OF AN ELEMENT WITH CHLORINE.** $\overset{+I}{H_2^0} + \overset{-I}{O_2^0} + \overset{+I\ -I}{Cl_2^0} \rightarrow \overset{+I\ -I}{2HCl}$. **(1)** Electrons are lost by the element and taken by the chlorine. **C. ELECTRON LOSS IN OXIDATION. (1)** The loss of electrons from an atom, ion, or radical. **(2)** An increase in the positive oxidation rate (valence number) of the atom, ion, or radical. (3) $H_2^0 - 2e \rightarrow 2H^+$, $Fe^0 - 3e \rightarrow Fe^{+++}$, $Fe^{++} - 1e \rightarrow Fe^{+++}$. (4) Oxidation of non-metallic ions to the elemental form: $2Cl^- - 2e \rightarrow Cl_2^0$, $S^{--} - 2e \rightarrow S^0$. **D. OXIDIZING AGENTS. (1)** Have a tendency to take up electrons. Upon gaining electrons, go into a lower oxidation state (valence number). **(2)** They increase the oxidation state of the other substance. Ex: $2Sb^0 + 3Cl_2^0 \rightarrow 2Sb\overset{+III}{Cl_3}$; chlorine is an oxidizing agent; takes electrons, attains a lower oxidation state number, raises oxidation state of antimony (Sb): $Cl_2^0 + 2e \rightarrow 2Cl^-$, $Sb^0 - 3e \rightarrow Sb^{+++}$.

**9. OXIDATION STATES. A. POSITIVE OXIDATION STATE.** Exists when valence electrons have been removed or shifted away from a neutral atom: $Na^0 - 1e \rightarrow Na^+$; actually the $Na^0$ has 11 protons and 11 electrons while the $Na^+$ has 11 protons and 10 electrons, having lost one electron. The + charge indicates the presence of 1 more proton than electrons. **B. NEGATIVE OXIDATION STATE.** Exists when electrons have been gained or shifted toward an atom: $S^0 + 2e \rightarrow S^=$; the $S^=$ ion has 16 protons and 18 electrons or two electrons more than protons. A negative oxidation state exists as the ion has 2 more electrons than protons. **C. TYPICAL OXIDATION STATES. (1) NITROGEN OXIDATION STATES:** +5 ($HNO_3$, $NO_3^-$); +4 ($NO_2$, $N_2O_4$); +3 ($HNO_2$, $NO_2^-$); +2 (*NO*); +1 ($N_2O$); 0($N_2$); −1 ($NH_2OH$); −2 ($N_2H_4$); −3 ($NH_3$, $NH_4^+$). **(2) SULFUR OXIDATION STATES:** +6 ($SO_3$, $H_2SO_4$, $SO_4^=$); +4 ($SO_2$, $H_2SO_3$, $SO_3^=$); +2 ($S_2O_3^=$); 0 ($S_8$); −2 ($H_2S$, $S^=$). **(3) CHLORINE OXIDATION STATES:** +7 ($HClO_4$, $ClO_4^-$); +5 ($HClO_3$, $ClO_3^-$); +4 ($ClO_2$); +3 ($HClO_2$, $ClO_2^-$); +1 ($Cl_2O$, $HClO$, $ClO^-$); 0 ($Cl_2$); −1 ($Cl^-$).

**10. REDUCTION. A. COMBINATION OF AN ELEMENT WITH HYDROGEN.** $\overset{+I\ -II}{2H_2^0} + \overset{-II}{O_2^0} \rightarrow \overset{+I\ -II}{2H_2O}$. Electrons are given to the oxygen by the hydrogen. **B. REDUCTION. (1)** A decrease in the positive oxidation state (valence number) of an atom or ion. **(2)** A *gain* in the number of electrons. **(3)** Reduction with carbon monoxide: $3CO + Fe_2O_3 \rightarrow 3CO_2\uparrow + 2Fe^0$, the oxidation state of oxygen is unchanged, the oxidation state of carbon increases: $C^{++} - 2e \xrightarrow{\text{oxidized}} C^{++++}$; $Fe^{+++} + 3e \rightarrow Fe^0$. **C. REDUCING AGENTS. (1)** Have a tendency to *give up* electrons **(2)** Upon yielding electrons, go into a *higher oxidation state* (are oxidized). **(3)** Decrease the oxidation state of the other substance: $\overset{+II\ -II}{CuO} \rightarrow Cu^0 + \overset{+I\ -II}{H_2O}$; hydrogen is a reducing agent as it gives up electrons, goes into a higher oxidation state and decreases the oxidation state of the copper.

**11. BALANCING OXIDATION—REDUCTION (REDOX) EQUATIONS.** Most combination and decomposition and all single replacement reactions involve oxidation-reduction. **A. DETERMINING OXIDATION NUMBER (VALENCE).** $H_2SO_4$: Hydrogen is +1, oxygen is −2, sulfur is the variable: $O = -2 \times 4 = -8$, $H = +1 \times 2 = +2$, $S = ? \times 1 =$ a number equal in magnitude to $-8 + 2$ but opposite in sign or +6, thus a total charge of zero. $H_2SO_3$: $H = +1 \times 2 = +2$, $O = -2 \times 3 = -6$, $S = ? \times 1 =$ a number equal to $-6 + 2$ but opposite in sign or +4. **B. CONSERVATION OF ELECTRONS.** The total *number* of electrons *gained* must equal the total *number* of electrons *lost.* **C. REDOX EXAMPLES.** (1) Balance $MnO_2 + HCl \rightarrow H_2O + MnCl_2 + Cl_2^0\uparrow$: $Mn^{++++} + 2e \xrightarrow{\text{reduction}} Mn^{++}$ and $2Cl^- - 2e \xrightarrow{\text{oxidation}} Cl_2^0$; one manganese underwent reduction, only two chlorine atoms oxidized to $Cl_2^0$ while the chlorine in $MnCl_2$ remained unchanged. Balance the unchanged atoms in the usual fashion and add the redox atoms to get $MnO_2 + 4HCl \rightarrow 2H_2O + MnCl_2 + Cl_2\uparrow$. (2) Balance $KMnO_4 + HCl \rightarrow Cl_2^0 + H_2O + MnCl_2 + KCl$: $Mn^{+++++++} + 5e \xrightarrow{\text{reduction}} Mn^{++}$ and $2Cl^- - 2e \xrightarrow{\text{oxidation}} Cl_2^0$ however, the +5 does not balance the −2e therefore crossmultiply and add all parts: $(Mn^{+++++++} + 5e \rightarrow Mn^{++}) \cdot 2$ and $(2Cl^- - 2e \rightarrow Cl_2^0) \cdot 5$ or $2Mn^{+++++++} + 10e + 10Cl^- - 10e \rightarrow 2Mn^{++} + 5Cl_2^0$. The +10e and −10e cancel out, proceed to balance the atoms that did not change in the equation: $2KMnO_4 + 16\,HCl \rightarrow 5Cl_2^0 + 8H_2O + 2MnCl_2 + 2KCl$.

**12. RULES FOR BALANCING MORE COMPLEX REDOX EQUATIONS.** Ex: Balance $MnO_2 + H^+ \cdot Cl^- \rightarrow Mn^{++} + Cl_2$. **A. BREAK UP INTO TWO HALF REACTIONS.** $MnO_2 \rightarrow Mn^{++}$ and $Cl^- \rightarrow Cl_2$. **B. BALANCE HALF REACTIONS. (1)** Balance central atom: $MnO_2 \rightarrow Mn^{++}$; $2Cl \rightarrow Cl_2$. (2) Balance other atoms except *O* and *H*, if any. (3) Balance *O* by adding $H_2O$ as needed. (4) Balance *H*, by adding $H^+$ as needed: $MnO_2 + 4H^+ \rightarrow Mn^{++} + 2H_2O$; $2\,Cl^- \rightarrow Cl_2$. (5) Balance charges, by adding *e* as needed: $MnO_2 + 4H^+ + 2e \rightarrow Mn^{++} + 2H_2O$; $2Cl^- \rightarrow Cl_2 + 2e$. **C. MULTIPLY BALANCED EQUATIONS FOR HALF REACTIONS.** $MnO_2 + 4H^+ + 2e \rightarrow Mn^{++} + 2H_2O$; $2Cl^- \rightarrow Cl_2 + 2e$; both already have an equal number of *e;* if they did not, they would be cross multiplied. **D. ADD.** $MnO_2 + 4H^+ + 2Cl^- + 2e \rightarrow Mn^{++} + 2H_2O + Cl_2 + 2e$. **E. CANCEL ON BOTH SIDES.** Cancel any ions or molecules or *e* appearing on both sides: $MnO_2 + 4H^+ + 2Cl^- \rightarrow Mn^{++} + 2H_2O + Cl_2$.

**13. ELECTROCHEMICAL REACTIONS. A. ELECTROLYSIS OF WATER.** $2H_2O \rightarrow 2H_2O^0\uparrow + O_2^0\uparrow$. **(1) WATER DISSOCIATES:** $H_2O \rightarrow H^+ + OH^-$. **(2) HOFFMAN APPARATUS:** $2H^+ + 2e \xrightarrow{\text{reduction}} H_2^0$ at cathode; $4OH^- - 4e \xrightarrow{\text{oxidation}} O_2^0\uparrow + 2H_2O$ at anode (+ terminal). Multiply hydrogen line by two to realize an equal number of electrons: $4H^+ + 4e + 4OH^- - 4e \rightarrow 2H_2^0\uparrow + O_2^0\uparrow + 2H_2O$ or $4HOH \rightarrow 2H_2\uparrow + O_2\uparrow + 2H_2O$. Simplify by subtracting two waters from each side: $2H_2O \rightarrow 2H_2^0\uparrow + O_2^0\uparrow$. **NOTE:** It has been demonstrated that the half reactions taking place at the cathode and anode actually involve molecules of $H_2O$ rather than ions: $2H_2O + 2e \rightarrow H_2 + 2OH^-$ at **cathode;** $2H_2O \rightarrow O_2 + 4H^+ + 4e$ at **anode.** **B. GENERAL RULES FOR ELECTROLYSIS. (1)** The less active an element the more easily will its ion change to the zero-valent form: $I^-$ will change to $I_2$ more easily than $Br^-$ to $Br_2$; $Fe^{++}$ to $Fe^0$ more easily than $Sn^{++}$ to $Sn^0$. **(2)** Single ions will usually change before complex ions. **C. RULES FOR ELECTROPLATING. (1)** Use **D.C.** (direct current) only. **(2)** Object to be plated is connected to the cathode. **(3)** Anode bar made of metal to be deposited onto object. **(4)** Electrolyte in solution is a salt of the metal to be plated. **D. A SILVER ELECTROPLATING CELL. (1) ANODE:** Bar of silver loses electrons and goes into solution: $Ag^0 - 1e \rightarrow Ag^+$. **(2) ELECTROLYTE:** Solution of $Ag^+ + NO_3^-$ (silver cyanide solutions are used commercially). **(3) OBJECT** is connected to cathode. Electrons flow through battery so that silver ions in solution pick up electrons and are deposited as metallic silver $Ag^+ + 1e \rightarrow Ag^0$.

**14. ELEMENTS IN THE HALOGEN ("SALT FORMER") FAMILY. A. FLUORINE.** Discovered by **Moissan** in 1886. **(1) PREPARATION:** electrolysis of potassium hydrogen fluoride dissolved in anhydrous liquid hydrogen fluoride: $2KHF_2 \rightarrow 2KF + H_2\uparrow + F_2\uparrow$ is collected in stainless steel vessels at the anode. **(2) TEST FOR FLUORIDE ION:** *HF* etches glass, $CaSiO_3 + 6HF \rightarrow 3H_2O + SiF_4 + CaF_2$. **(3) IMPORTANT COMPOUNDS:** Freon ($CF_2Cl_2$), a refrigerant; cryolite ($Na_3AlF_6$), solvent in *Al* metallurgy; sodium fluoride (*NaF*), an insecticide and (like stannous fluoride, $SnF_2$) helps prevent tooth decay in children; uranium hexafluoride ($UF_6$), gas used to separate $U^{235}$ from its isotopes. **(4) CHEMICAL CHARACTERISTICS:** Explodes with hydrogen even in absence of light to form highly stable weakly ionized hydrofluoric acid ($H_2 + F_2 \rightarrow H_2F_2$, usually represented as *2HF*). $F_2$ is the most electronegative element, and is therefore a more powerful chemical oxidizing agent than any other element as it removes electrons from all other elements: $Cu + F_2 \rightarrow CuF_2$ or $Cu^0 - 2e$ **oxidation** $Cu^{++}$ as $F_2^0 + 2e \xrightarrow{\text{reduction}} 2F^-$. **B. CHLORINE.** Isolated by **Scheele** in 1774; named and proven to be an element by **Davy** in 1810. **(1) PREPARATION, TEST, PROPERTIES:** See D/G Chem. 1. **(2) IMPORTANT COMPOUNDS:** Sodium hypochlorite (*NaOCl*), a bleaching agent; silver chloride (*AgCl*), used in photographic emulsions; chloroform (*CHCl₃*), an anesthetic; aluminum chloride ($AlCl_3$), a *catalyst* in increasing gasoline-yield by cracking large petroleum hydrocarbons; sulfur chloride ($S_2Cl_2$), used in vulcanizing rubber. **C. BROMINE.** Discovered by **Balard** in 1826. **(1) PREPARATION, TEST, PROPERTIES:** See D/G Chem. 1. **(2) IMPORTANT COMPOUNDS:** Ethylene dibromide ($C_2H_4Br_2$), added to leaded gasoline removes freed lead as lead bromide ($PbBr_2$); silver bromide (*AgBr*) used in photographic enlarging paper. **D. IODINE.** Extracted by **Courtois** in 1812, from sea weeds. **(1) PREPARATION. TEST, PROPERTIES:** See D/G Chem.1. **(2) IMPORTANT COMPOUNDS:** Di-iodine pentoxide ($I_2O_5$), an indicator to detect presence of *carbon monoxide;* iodoquinine sulfate, used in making crystals for Polaroid film. **(4) HYDRIODIC ACID (HI):** Is most unstable and hard-to-form halogen acid; formed by action of phosphorus tri-iodide on water: $PI_3 + 3H_2O \rightarrow H_3PO_3 + 3HI\uparrow$. **E. ASTATINE ($At^{85}$).** A *synthetic* radioactive element that can be made in an atomic accelerator.

**15. GENERAL PROPERTIES OF THE HALOGENS. A. OCCURRENCE.** Halogens are *never* found free and uncombined; they are always found in compounds. **B. RELATIVE ACTIVITY. (1)** Each halogen having *seven* electrons in its outermost orbit has a tendency to gain one electron and become a **halide ion** exhibiting a **−1 valence. (2)** The chemical activities of the halogens *decrease* with increasing atomic number: fluorine, chlorine, bromine, iodine, astatine.

**16. SULFUR AND SULFIDES. A. OCCURRENCE.** Sulfur is abundantly found in the *free* elemental form (in Louisiana and Texas) and in *combined* form (in Sicily and Japan as volcanic rock). **B. EXTRACTION. (1) FRASCH PROCESS:** Three *concentric* pipes are sunk through quicksand into sulfur layers. **Superheated water** (167°C) is forced down outermost pipe under pressure to *melt* sulfur. **Compressed hot air** is forced down innermost pipe forming a froth with the melted sulfur and driving this froth up the middle pipe. The sulfur is about 99.5% pure in this form. **(2) VOLCANIC ROCK PROCESS:** Rock is ignited, melting the sulfur. **C. PURIFICATION.** Sulfur is distilled in large iron retorts. **(1) FLOWERS OF SULFUR:** Powdery condensed vapors. **(2) ROLL SULFUR:** Forms on cooling liquid sulfur in cylindrical molds. **D. ALLOTROPIC FORMS. (1) RHOMBIC SULFUR ($S_8$):** Formed by dissolving roll sulfur in carbon disulfide ($CS_2$) and allowing the solution to evaporate; solid crystalline in form having the shape of an *octahedron* (3-dimensional diamond-shape); **most stable form** at room temperature. **(2) MONOCLINIC OR PRISMATIC SULFUR:** Formed by gently melting roll sulfur and allowing it to cool slowly; crystallized in form of *long needle-like crystals* though still $S_8$; converts to rhombic form below 95°C. **(3) λ—(LAMBDA) SULFUR:** Formed at temperatures slightly above the melting point of *S;* it is a straw-yellow colored cyclic form of $S_8$ that has a fluid form. **(4) μ—(MU) SULFUR:** Formed by heating λ-sulfur to about 200°C; it is a highly viscous (slow-flowing) liquid form that *darkens* to a red to black color. **(5) AMORPHOUS SULFUR:** Formed by pouring boiling *S* into cold water; a sticky, rubbery, dark brown or black *plastic mass;* insoluble in $CS_2$; gradually reverts to rhombic form as it cools. **E. CHEMICAL PROPERTIES. (1) VALENCE:** In sulfides, sulfites, sulfates −2(*ZnS, $H_2S$*) also +4($SO_2$) and +6($SO_3$). **(2) BURNS IN $O_2$:** blue flame, $S + O_2 \rightarrow SO_2\uparrow$. **(3) SUPPORTS COMBUSTION:** an oxidizing agent ( gains electrons) $Cu + S \rightarrow CuS$; $Fe + S \rightarrow FeS$. **F. USES.** Vulcanizes rubber **G. HYDROSULFURIC ACID ($H_2S$).** See Chem 1.

**17. ACIDS AND OXIDES OF SULFUR. A. SULFUR DIOXIDE.** See Chem 1. **B. SULFUROUS ACID.** Unstable: $H_2SO_3 \rightleftharpoons H_2O + SO_2$. **(1) REDUCING AGENT:** Bleaches purple solution of potassium permanganate: $5H_2SO_3 + 2KMnO_4 \rightarrow 2MnSO_4 + K_2SO_4 + 2H_2SO_4 + 3H_2O$. **NOTE:** $Mn^{+7} \xrightarrow{\text{reduction}} Mn^{+2} - 5e$, while $S^{+4} \xrightarrow{\text{oxidation}} S^{+6} + 2e$. **(2) USES:** Preservative; antiseptic. **(3) SULFITE COMPOUNDS:** Calcium bisulfite [$Ca(HSO_3)_2$] used in processing wood pulp into paper. **C. SULFUR TRIOXIDE. (1) PREPARATION:** $2SO_2 + O_2 \rightarrow 2SO_3\uparrow$ at temperatures of 400°C in the presence of the catalyst *vanadium pentoxide* ($V_2O_5$) or finely divided platinum. **(2) PHYSICAL PROPERTIES:** White, crystalline solid at room temperature. **(3) CHEMICAL PROPERTIES:** Reacts vigorously with water: $SO_3 + H_2O \rightarrow H_2SO_4$. Dissolves in concentrated sulfuric acid forming fuming sulfuric acid also known as **oleum** or pyrosulfuric acid: $SO_3 + H_2SO_4 \rightarrow H_2S_2O_7$ or $H_2SO_4 \cdot SO_3$.

**18. SULFURIC ACID OR OIL OF VITRIOL. A. PREPARATION BY CONTACT PROCESS.** $S + O_2 \rightarrow SO_2$; $SO_2 \xrightarrow[\Delta]{Pt} SO_3$; $SO_3 + H_2SO_4 \rightarrow H_2SO_4 \cdot SO_3$; $H_2O + H_2SO_4 \cdot SO_3 \rightarrow 2H_2SO_4$. **B. PHYSICAL PROPERTIES:** A dense, oily liquid that is colorless, odorless, almost twice as dense as water, and miscible with water. **NOTE: Always add acid to water, never the reverse. C. CHEMICAL PROPERTIES. (1) DEHYDRATING AGENT** (removes water from compounds): Chars wood and sugar: $C_{12}H_{22}O_{11} \xrightarrow{H_2SO_4} 12C + 11H_2O$. **(2) ACTION OF DILUTE $H_2SO_4$ ON METALS:** $Zn + H_2SO_4 \rightarrow ZnSO_4 + H_2\uparrow$. **(3) ACTION OF HOT CONCENTRATED $H_2SO_4$:** With *active* metals (*Zn, Fe*) yield $H_2S$: $4Fe + 5H_2SO_4 \rightarrow 4FeSO_4 + 4H_2O + H_2S\uparrow$; with *less active* (*Cu, Ag, Hg*) metals yield $SO_2$: $Cu + H_2SO_4 \rightarrow CuSO_4 + 2H_2O + SO_2\uparrow$. **(4) ACTION ON SALTS:** Forms other acids with boiling points lower than its 338°C point: $2NaCl + H_2SO_4 \rightarrow Na_2SO_4 + 2HCl$. **(5) ACTION WITH METALLIC OXIDES:** Forms salt and water: $MgO + H_2SO_4 \rightarrow MgSO_4 + H_2O$. **D. TEST FOR A SULFATE ION.** A *white precipitate* forms when barium chloride is added. The ppt. is *insoluble* in hydrochloric acid: $H_2SO_4 + BaCl_2 \rightarrow 2HCl + BaSO_4\downarrow$. **E. USES. (1) PREPARE OTHER ACIDS:** $H_2SO_4 + 2NaNO_3 \rightarrow Na_2SO_4 + 2HNO_3$. **(2) FERTILIZERS:** Converts insoluble rock phosphate into soluble mono-calcium phosphate or superphosphate: $Ca_3(PO_4)_2 + 2H_2SO_4 \rightarrow 2CaSO_4 + Ca(H_2PO_4)_2$. **(3) REFINING PETROLEUM:** Removes impurities from gasoline and kerosene. **(4) REFINING COAL TARS:** Production of dyes, disinfectants and drugs. **(5) ELECTROLYTE:** In lead storage battery. **(6) PICKLING OF METALS:** Iron and steel.

**19. METALS.** Elements that have *few electrons* in their outermost shell (all elements with 1 or 2e, all elements with 3 except Boron, and a few exceptions with 4, 5 or 6) are metals. **A. PHYSICAL PROPERTIES. (1) MALLEABILITY:** Ability to be *hammered* or *rolled* into sheets. **(2) CONDUCTIVITY:** Ability to *conduct* heat and electricity. **(3) DUCTILITY:** Ability to be drawn into thin wires. **(4) LUSTER:** Sheen and lack of *transparency.* **(5) STATE:** All are *solids* at room temperature *except* mercury. **(6) CRYSTAL STRUCTURE:** Each metallic atom surrounded by **8 to 12 neighbors. B. CHEMICAL PROPERTIES OF METALS. (1) CHARGE:**

Lose electrons and become positively charged ions: $Al^0 - 3e \rightarrow Al^{+++}$.
(2) **ELECTROPOSITIVE:** All oxidation states are positive ($Fe^{++}$ and $Fe^{+++}$).
) **HYDROXIDES ARE BASIC OR AMPHOTERIC:** Can act as either acids or ases and are soluble in strongly acidic or strongly basic solutions: $l(OH)_3 \downarrow + Na^+ + OH^- \rightarrow Na^+ + Al(OH)_4^-$ or $Al(OH)_3 \downarrow + 3H^+ + Cl^- \rightarrow Al^{+++} + 3Cl^- + 3H_2O$; thus $Al(OH)_3$ may also be written as $_3AlO_3$. (4) **REDUCING AGENTS:** Tendency to lose electrons. **C. OCCURRENCE F METALS.** Ores are materials in which metals or their compounds ccur. (1) **FREE OR NATIVE ORES:** Metals found *uncombined* include pper, silver, gold, platinum. (2) **OXIDE ORES:** Aluminum, iron, tin. ) **CARBONATE ORES:** Iron, lead, copper, zinc, calcium. (4) **SULFIDE RES:** Zinc, mercury, lead, copper, silver. (5) **HALIDES (FROM ALOGENS):** Sodium, potassium, calcium, magnesium. (6) **SULFATES:** alcium, barium, lead.

**20. ALKALI METALS.** Group 1A of Periodic Table—**Sodium Family.** **. LITHIUM.** (1) **DISCOVERY:** 1817 by **Arfvedson** and isolated in 1855 by **unsen and Matthiessen.** (2) **USES:** In ceramics, drugs, welding. **. SODIUM.** (1) **DISCOVERY:** **Davy** prepared *Na* in 1807 by electrolysis: $NaCl \rightarrow 2Na + Cl_2 \uparrow$ (reaction carried out in a **Downs Cell**). (2) **HYSICAL PROPERTIES:** Soft (can be cut with a knife); density less than ater, floats on $H_2O$. (3) **CHEMICAL PROPERTIES:** $2Na + 2HOH \rightarrow NaOH + H_2 \uparrow$; reacts with air at room temperature. **C. IMPORTANT ODIUM COMPOUNDS.** (1) **NaCl:** Found in deposits or rock salt. Used to ake soap, prepare *HCl* acid, preserve foods. (2) **NaOH:** Prepared by lectrolysis of brine: $2NaCl + 2HOH \rightarrow 2NaOH + H_2 \uparrow + Cl_2 \uparrow$, and y adding slaked lime to sodium carbonate: $Ca(OH)_2 + Na_2CO_3 \rightarrow aOH + CaCO_3 \downarrow$. Used to make soap, dissolve silk and wool, make scose rayon. (3) **$NaNO_3$ (Chile Saltpeter):** Used in making $HNO_3$. ) **$Na_2S_2O_3 \cdot 5H_2O$ (hypo):** a fixer in photography. (5) **$Na_2B_4O_7 \cdot 10H_2O$ (borax):** a water softener. (6) **Solvay process** for preparing odium carbonate and sodium hydrogen carbonate: $CaCO_3 \rightarrow aO + CO_2\uparrow$; $CO_2 + NH_3 + H_2O \rightarrow NH_4HCO_3$; $NaCl + NH_4HCO_3 \rightarrow H_4Cl + NaHCO_3 \downarrow$; $2NaHCO_3 \rightarrow Na_2CO_3 + H_2O \uparrow + CO_2 \uparrow$; $a_2CO_3 + H_2O + CO_2 \rightarrow 2NaHCO_3$; $CaO + H_2O \rightarrow Ca(OH)_2$; $a(OH)_2 + 2NH_4Cl \rightarrow CaCl_2 + 2H_2O + 2NH_3 \uparrow$. **D. POTASSIUM.** (1) **DISCOVERY:** By **Davy** in 1807 by electrolysis of fused *KOH*. (2) **ROPERTIES:** Resembles *Na*, but even more active. (3) **IMPORTANT OMPOUNDS:** Saltpeter ($KNO_3$), for making black gun powder; $K_2SO_4$, or fertilizer; caustic potash (*KOH*), used in the Edison battery; $MnO_4$, an oxidizing agent. **E. RUBIDIUM AND CESIUM.** (1) **DISCOVERY:** y **Bunsen** in 1860 via use of spectroscope. (2) **PREPARATION:** Heat *Rb* nd *Cs* compounds with metallic *Ca* or *Mg*. (3) **USE:** In photoelectric ells. **F. FRANCIUM.** (1) **DISCOVERY:** 1937 by **Perey.** (2) **FORMED:** By disintegration of an isotope of actinium.

**21. GENERAL PROPERTIES OF ALKALI METALS. A. OCCURRENCE.** lkali metals are the **most active metals** and are *never* found *free;* hey are always found in compounds. **B. SEPARATION.** All alkali metals re separated from their salts by **electrolysis** of *fused salts.* **. RELATIVE ACTIVITY.** (1) Each metal has 1 electron in its outermost rbit, has a strong tendency to lose this one electron and become a $+$ on. (2) Alkali metals are strong *reducing agents.* The chemical activities of the metals in any family increase with increasing atomic umber: **Na, K, Rb, Cs, Fr.** The heavier the metal, the more easily does lose its one outer electron. (3) The attraction of the positively harged nucleus diminishes with distance from *K-shell* outward and s less able to hold the one electron in the distant orbit. **NOTE:** In queous solutions, *Li* is exceptionally active because of the high nergy of hydration of $Li^+$. **D. PROPERTIES.** (1) Soft, silvery metals have ow melting and boiling points. (2) React even with cold water reeasing hydrogen and forming strongly basic solutions.

**22. GENERAL PROPERTIES OF ALKALINE EARTH METALS OF GROUP II A OF PERIODIC TABLE. A. FAMILY MEMBERS.** Beryllium (*Be*), Magesium (*Mg*), Calcium (*Ca*), Strontium (*Sr*), Barium (*Ba*), Radium *Ra*). **B. PROPERTIES.** (1) React with water releasing hydrogen and orming basic solutions. The alkaline earth metal hydroxides are less oluble than the alkali metal hydroxides. (2) **Group II A** are the second ost active group of metals next to Group I A. (3) Elements are alays found combined in nature. (4) Have two electrons in outermost rbit and tend to lose both of them forming ions with a charge of $+2$. ) Separated from fused salts by electrolysis. **C. RELATIVE ACTIVITY.** hemical activity increases with atomic radius: **Be, Mg, Ca, Sr, Ba, Ra.**

**23. CALCIUM AND ITS COMPOUNDS. A. CALCIUM.** (1) Fifth most undant element in the earth's crust. (2) Reacts with water at a ower rate than either *Na* or *K*. The slightly soluble $Ca(OH)_2$ produced in the reaction coats the surface of the *Ca* and retards oxidaion of more $Ca^0$ to $Ca^{++}$. **B. LIMESTONE AND MARBLE.** (1) **PROPERTIES:** White in pure $CaCO_3$ form; a gray color indicates presence of imurities. (2) **LIMESTONE CAVES:** Formed by loss of $CO_2$ from drops of ater on roof of caves; soluble $Ca(HCO_3)_2 \rightarrow CaCO_3 \downarrow + CO_2 \uparrow + H_2O$; icicle-shaped masses of limestone hanging downward from ceilings re called **stalactites;** $CaCO_3$ masses built upward from ground of ave are called **stalagmites.** (3) **USE:** Building materials; "sweeten" cid soils. **C. QUICKLIME (*CaO*).** (1) Limestone is heated in vertical kilns: $CaCO_3 \rightarrow CaO + CO_2 \uparrow$. (2) $CaO + H_2O \rightarrow Ca(OH)_2$ slaked lime; eaction is **exothermic** (heat is liberated on right side). (3) **USES:** Make calcium carbide in electric furnace: $CaO + 3C \rightarrow CaC_2 + CO \uparrow$, hen $CaC_2 + 2H_2O \rightarrow Ca(OH)_2 + C_2H_2\uparrow$ gives acetylene fuel; also sed for mortar. **D. CALCIUM SULFATE.** (1) **GYPSUM:** Converts to Plaster f Paris when gently heated: $2CaSO_4 \cdot 2H_2O \xrightarrow{\Delta} (CaSO_4)_2H_2O + 3H_2O$. (2) **USES OF GYPSUM:** Manufacture of cement and a fertilizer. 3) **USES OF PLASTER OF PARIS:** wallboards, plaster, surgical casts.

**24. HARD WATER. A. COMPOSITION.** Contains $Ca^{++}$, $Mg^{++}$ or $Fe^{++}$ ions. **B. DISADVANTAGES.** (1) **LOW SUDS: Hard water** forms a **calcium tearate** precipitate and does not lather until all *Ca* ions are removed by precipitation. This wastes much soap. (2) **BOILER SCALE:** Scale of precipitated $CaCO_3$ (from temporary hard water) forms in boilers and pipes reducing efficiency of heating system. **C. TYPES OF HARD WATER.** (1) **TEMPORARY HARD WATER:** Contains soluble bicarbonates ($HCO_3^-$) and may be softened by boiling: $Ca(HCO_3)_2 \xrightarrow{\Delta} CaCO_3 \downarrow + H_2O + CO_2$. This removes the $Ca^{++}$ ($Mg^{++}$ or $Fe^{++}$) ions. (2) **PERMANENT HARD WATER:** Contains soluble sulfates and chlorides and may be softened by addition of *washing soda* ($Na_2CO_3$): $CaSO_4 + Na_2CO_3 \rightarrow CaCO_3 \downarrow + Na_2SO_4$ or $CaCl_2 + Na_2CO_3 \rightarrow CaCO_3 \downarrow + 2NaCl$. **D. WATER SOFTENERS.** (1) **SOFTENERS:** Substances that remove $Ca^{++}$, $Mg^{++}$ and $Fe^{++}$ ions from solution. (2) **ZEOLITES:** Natural minerals containing *sodium silico-aluminate;* exchange $Na^+$ ions for $Ca^{++}$; the newly formed calcium silico-aluminate can be transformed back to the *Na* compound by concentrated *NaCl* solution, and then re-used. (3) **PERMUTIT:** An artificial zeolite that *reacts faster* than the natural form. (4) **ION-EXCHANGE RESINS:** Water passes through granules of a reation *exchange resin* where all $+$ ions are retained in the resin and replaced with $H^+$; the water then passes through granules of an anion exchange resin where all $-$ ions are retained and replaced with $OH^-$. The demineralized water is almost equivalent to distilled (purified) water. $H_2SO_4$ and $Na_2CO_3$ renew ion exchange resins.

**25. METALLURGY.** The process of extracting metals from their ores. **A. METHODS OF ORE CONCENTRATION.** Frequently, the ore is concentrated to remove *impurities* or **gangue.** (1) **FROTH FLOTATION:** Impure ore is ground to a powder, mixed with air, water, and oil. Gangue particles become wetted by water, while oil-covered air bubbles stick to the surface of the ore particles. The ore in oil *floats* on top of the gangue in water. (2) **GRAVITATIONAL SEPARATION:** Running water washes away light gangue particles while the ore settles to the bottom of the pan. (3) **ELECTROSTATIC SEPARATION:** Impure ore is ground and electrically charged. The metallic ore is a conductor and gives away its charge while the gangue retains its charge (being a nonconductor). (4) **MAGNETIC CONCENTRATION:** Magnetic ores ($Fe_3O_4$) are separated from the non-magnetic gangue. **B. REDUCTION OF METALLIC ORES.** Metals in ores are ions that have lost electrons; the metal must be reduced by gaining electrons to return to the pure metal form. (1) **REDUCTION BY ELECTROLYSIS:** All alkali and alkaline earth metals and *Al* are reduced by electrolysis of fused salts. (2) **REDUCTION BY THERMITE REACTION:** Metals less active than *Al* are separated from their oxides by igniting the mixture: $2Al + Fe_2O_3 \rightarrow Al_2O_3 + 2Fe$. (3) **REDUCTION BY MORE ACTIVE ELEMENT:** A more active element (higher on electromotive series) replaces a less active metal from its ore: $Zn + PbO \rightarrow ZnO + Pb$.

**26. IRON AND STEEL. A. IRON.** Fourth most abundant element by weight in earth's crust. (1) **ORES:** Hematite ($Fe_2O_3$), magnetite and taconite ($Fe_3O_4$), pyrite ($FeS_2$), siderite ($FeCO_3$). (2) **TEST:** Formation of a *blue precipitate* when **potassium ferricyanide** is added to a solution containing $Fe^{++}$ ions; formation of a *blue precipitate* when **potassium ferrocyanide** is added to a solution containing $Fe^{+++}$ ions. **B. BLAST FURNACE.** (1) **CHARGE:** Hematite ore, coke, limestone (flux removes impurities), hot air (enters through tuyère ducts). (2) **REACTIONS:** $C + O_2 \rightarrow CO_2\uparrow$; $CO_2 + C \rightarrow 2CO\uparrow$, the reducing agent; $Fe_2O_3 + 3CO \rightarrow 3CO_2\uparrow + 2Fe$, cast into blocks called pig iron; $CaCO_3 \rightarrow CaO + CO_2\uparrow$; $CaO + SiO_2$ (sand impurities from ore) $\rightarrow CaSiO_3$, slag floats on molten iron. **C. TYPES OF IRON.** (1) **CAST OR PIG IRON:** Contains from 2 to 5% carbon; is hard but brittle; used in radiators and stoves. (2) **WROUGHT IRON:** Contains less than 0.1% carbon; resistant to rust and corrosion; used in ornamental furniture, blacksmith iron. (3) **STEEL:** Usually contains from 0.1 to 2% carbon with alloying metals; great tensile strength; used in cars, structural beams.

**27. HOW STEEL IS MADE. A. BESSEMER CONVERTER.** (1) **PROCESS.** Hot air or oxygen is blown up through the pot removing carbon; other *Mn* and *Si* impurities are converted to oxides by oxidation. (2) **CHARGE:** Molten cast iron. (3) **SPIEGELEISEN:** A ferromanganese alloy rich in carbon is added after the oxidation blasts. (4) **PROPERTIES:** Moderately hard, malleable and ductile; can be cast and welded. (5) **USE:** Make rods, wire, sheets. **B. OPEN-HEARTH FURNACE PROCESS.** (1) **CHARGE:** Pig iron, hematite, scrap iron and steel, limestone and a basic lining on fire bed. (2) **HEATING SYSTEM:** Producer gas is reflected down upon the charge as it passes over and across the furnace. The flow is periodically reversed to conserve heat; this is called a **regenerative** heating system. (3) **REACTIONS:** Impurities are oxidized to the desired level: $3C + Fe_2O_3 \rightarrow 2Fe + 3CO_2\uparrow$; $3Mn + Fe_2O_3 \rightarrow 2Fe + 3MnO$; $3Si + 2Fe_2O_3 \rightarrow 4Fe + 3SiO_2$. (4) **USE:** Structural steel for bridges and buildings. **C. ELECTRIC ARC PROCESS.** (1) An electric current passes between three carbon electrodes in a furnace containing little air. (2) High quality, expensive steel is produced. **D. HARDNESS OF STEEL.** (1) **TEMPERING:** Carbon steel is heated and subsequently plunged into cold water, resulting in hard but brittle steel; repeated as necessary.

**28. ALLOYS.** Mixture of two or more metals exhibiting properties different from any component metal. **A. COMMON ALLOYS. Brass:** *Cu + Zn.* **Bronz:** *Cu + Sn.* **Duralumin:** *Al + Cu + Mg.* **Gold Amalgam:** *Hg + Au* (amalgams are alloys of *Hg*). **Magnalium:** *Al + Mg.* **Solder:** *Pb + Sn.* **Nichrome:** *Fe + Ni + Cr.* **B. ALLOY STEELS. Chrome steel:** *Fe + Cr + Si + Mn.* **Silicon steel:** *Fe + Si.* **Stainless steel:** *Fe + Cr.* **Vanadium steel:** *Fe + V.*

**29. RADIOACTIVITY.** The spontaneous *disintegration* of the nucleus of an atom accompanied by emission of particles and rays. **A. NATURAL RADIOACTIVITY.** (1) Discovered by **Becquerel** in 1896 when uranium ores emitted penetrating *rays* that affected a photographic plate. (2) **The Curies** investigated **pitchblende** and discovered *radium* and *polonium* which emit atomic particles and are transmuted to other elements. (3) **Rutherford** discovered two types: charged particles and gamma rays (uncharged). **B. KINDS OF RADIOACTIVE EMISSIONS.** (1) **ALPHA PARTICLES $\alpha$:** ${}_2He^4$ are helium nuclei having a mass (atomic weight) of 4 and a charge (atomic number) of $+2$; may be very fast but have little ability to penetrate—they can be absorbed even by a sheet of paper or thin aluminum foil. (2) **BETA PARTICLES $\beta$:** ${}_{-1}e^0$ are electrons having almost no mass (actually 1/1837 of a hydrogen atom) and a charge of $-1$; are ultra high speed particles with long range (high penetrating power); they can be absorbed by aluminum sheet. (3) **GAMMA RAYS $\gamma$:** Have higher energy than *X*-rays; have neither mass nor charge and are not deflected by magnetic or electric fields; travel at speed of light and have exceedingly high penetrating power; they can pass through 4 inches of lead. **C. NUCLEAR EQUATIONS.** (1) **Subscripts** refer to atomic *numbers;* **superscripts** refer to atomic *weights.* (2) Transmutations affect the nucleus and thus only the nucleus is illustrated. (3) **The sum of the charges** (At. No. or subscripts) on one side of an equation must *equal* the sum of the charges on the other side. (4) **The sum of the masses** (At. Wt. or superscripts) on one side of an equation must *equal* the sum of the masses on the other side. **D. BOMBARDING PARTICLES.** ${}_0n^1$ = neutron, ${}_1H^1$ = proton, ${}_1H^2$ = deuteron, ${}_1H^3$ = triton, ${}_2He^4$ = alpha, ${}_{-1}e^0$ = beta, ${}_{+1}e^0$ = positron. Examples: ${}_7N^{14} \rightarrow {}_6C^{13} + {}_1H^1$; ${}_{48}Cd^{115} + {}_0n^1 \rightarrow {}_{48}Cd^{116} + {}_0\gamma^0$; ${}_7N^{14} + {}_2He^4 \rightarrow {}_8O^{17} + {}_1H^1$; ${}_6C^{10} \rightarrow + {}_1e^0 + {}_5B^{10}$; ${}_{82}Pb^{210} \rightarrow {}_{-1}e^0 + {}_{83}Bi^{210}$; ${}_5B^{10} + {}_2He^4 \rightarrow {}_7N^{13} + {}_0n^1$.

**30. RADIOACTIVE DISINTEGRATION SERIES.** (1) **Thorium$^{232}$ series** ends up as $Pb^{208}$; the 232 is exactly divisible by 4 and is called the **4n + 0 series.** (2) **Uranium$^{238}$ series** ends up as $Pb^{206}$; the 238 divided by 4 leaves 2 left over and is called the **4n + 2 series.** (3) **Actinium$^{231}$ series** ends up as $Pb^{207}$; the 231 divided by 4 leaves 3 left over and is called the **4n + 3 series.** (4) **Neptunium$^{237}$ series** is not found in nature though it may have existed; it can be produced in the laboratory; the 237 divided by 4 leaves 1 left over and is called the **4n + 1 series** and ends in $Bi^{209}$.

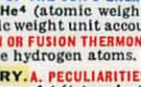
**31. RADIOACTIVE DATING AND DETECTION DEVICES. A. RATE OF DECAY.** (1) **HALF LIFE:** Each radioactive isotope disintegrates at a constant rate described by the **half life** (the length of time required for half of the atoms of the isotope to disintegrate). (2) **TYPICAL HALF-LIVES:** $U^{238}$ = 4.5 billion years, $C^{14}$ = 5,570 years, $Po^{214}$ = $1.5 \times 10^{-4}$ seconds. **B. RADIOACTIVE DATING.** (1) **URANIUM DATING:** Amounts of long-lived parent element (*uranium*) and of a daughter element (*lead*) are determined by analysis. Ratio of these amounts decreases with time at a rate that can be calculated from half-life of the parent. (2) **RADIOCARBON DATING:** Concentration of $C^{14}$ in living matter is constant, but decays at a known rate after death. **Libby** determined ages of fossil samples up to 30,000 years by measuring radioactivity of remaining $C^{14}$. **C. DETECTING AND MEASURING RADIOACTIVITY.** (1) **SPINTHARISCOPE OR SCINTILLATION COUNTER: Alpha particles** produce flashes of light when they strike a *fluorescent zinc sulfide* screen or other suitable material. (2) **ELECTROSCOPE:** A radioactive sample *ionizes the air* around a protruding metal knob removing all or part of the charge on the knob; as this happens, the pair of repelled and diverged gold leaves move together. (3) **CLOUD CHAMBER:** Charged particles moving through air (supersaturated with water vapor in the chamber) form visible fog paths or tracks. (4) **GEIGER COUNTER:** Particles entering the mica window of a **Geiger-Mueller** tube ionize the gas molecules causing a momentary flow of current which is indicated on a counter and by a light flash.

**32. BOMBARDING DEVICES. A. RADIOACTIVE SOURCES.** (1) **Cobalt$^{60}$:** Source of negative beta particles. (2) **Radium:** Source of high energy alpha particles. (3) **Mixture of Beryllium and Radium:** Source of neutrons. **B. PARTICLE ACCELERATORS.** (1) **LINEAR ACCELERATORS:** Produce atomic particles of relatively low energy by Van de Graaff generators, X-ray machines. (2) **CIRCULAR ACCELERATORS:** Produce high-speed, high-energy particles by means of strong magnetic and electrostatic fields in a cyclotron, betatron or synchrotron.

**33. FISSION, FUSION, PILE. A. FISSION.** An atom splits into smaller atoms and particles: $U^{235} + {}_0n^1 \rightarrow Kr^{92} + Ba^{141} + 3{}_0n^1 +$ **tremendous energy. B. NUCLEAR ENERGY.** One mole of $U^{235}$, about 1.2 lb, produces as much energy as the combustion of $4 \times 10^6$ lbs. of coal. According to **Einstein's Conservation of Mass-Energy Law,** mass can be converted to energy and vice versa. The amount of energy is calculated by the formula $E = mc^2$ where $E$ is energy, $m$ is mass, $c$ is velocity of light. **C. CHAIN REACTION.** The particle that initiates a fission reaction is also one of the products, allowing subsequent fissions. The slow ${}_0n^1$ which causes $U^{235}$ to split releases from 2 to 3 other neutrons, each capable of splitting other $U^{235}$ atoms and releasing still other neutrons. **D. CRITICAL SIZE OR MASS.** The minimum amount of fissionable material that will just support a *continuous reaction.* **E. ATOMIC OR FISSION BOMB.** (1) **ISOLATION OF $U^{235}$:** Uranium hexafluoride ($UF_6$) compounds can be *separated* by **gaseous diffusion** through a porous membrane as the lighter $U^{235}$ molecules diffuse faster than the heavier $U^{238}$ molecules. The $U^{235}$ is used as the fuel in the atomic bomb. (2) **IMPLOSION:** Two sub-critical masses are rapidly driven together by small explosions. The large amount of pure fissionable material allows for a fast chain reaction where most of the secondary neutrons are released at high speeds and quickly strike other fissionable nuclei so one splitting nucleus detonates two others. These 2 detonate 4, then 8, 16, 32, 64 and so on in ever increasing profusion, releasing tremendous energy capable of destroying a city. **F. FISSIONABLE FUELS.** $U^{235}$, $U^{233}$, $Pu^{239}$. The latter two are produced in breeder piles where much more fissionable material is produced than is expended: ${}_{92}U^{238} + {}_0n^1 \rightarrow {}_{92}U^{239} \rightarrow {}_{93}Np^{239} + {}_{-1}e^0$. The transuranium element neptunium then becomes plutonium: ${}_{93}Np^{239} \rightarrow {}_{94}Pu^{239} + {}_{-1}e^0$. **G. ATOMIC PILE OR NUCLEAR REACTOR.** (1) **FUEL:** Usually natural uranium enriched $U^{235}$, $Pu^{239}$ or $U^{233}$ sealed in aluminum cans within the pile. (2) **MODERATOR:** Reduces the velocity of the neutrons causing more effective neutron capture; usually are blocks of graphite but may also be heavy water surrounding the pile. (3) **CONTROL RODS:** Absorb neutrons to provide control; may be made of boron or cadmium steel. (4) **COOLANT:** Water, air or liquid sodium (or *Na-K* alloy) may be circulated throughout the pile to absorb the heat that is released. The heat may be converted to electricity or steam. (5) **RADIATION SHIELD:** Pile encased in lead, heavy concrete or steel. **H. FUSION.** (1) **DEFINITION:** The building up of heavier nuclei from lighter nuclei. (2) **SOURCES OF THE SUN'S ENERGY:** ${}_4H^1$ (total atomic weight 4.03032) becomes ${}_2He^4$ (atomic weight 4.003) + energy. The conversion of 0.02732 atomic weight unit accounts for the astronomic energy released. **I. HYDROGEN OR FUSION THERMONUCLEAR BOMB.** A fission bomb explosion triggers the hydrogen atoms.

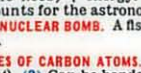
**34. ORGANIC CHEMISTRY. A. PECULIARITIES OF CARBON ATOMS.** (1) Carbon usually has a valence of 4 (*tetravalent*). (2) Can be bonded to other carbon atoms to form chain or ring compounds. (3) Can form *single, double,* or *triple* bonds with other atoms, including other carbon atoms; also form covalent compounds. **B. CHARACTERISTICS OF ORGANIC COMPOUNDS.** (1) **Hydrocarbons** and many other organic compounds are insoluble in water. (2) Reactions usually proceed slowly and require heat or **catalysts.** (3) Generally **covalent** and **non-ionic.** (4) Melt, boil and decompose at relatively low temperatures. (5) Many organic compounds can burn. **C. ISOMERS.** (1) Two or more compounds having the same molecular formula but differing in the arrangement of the atoms within their molecules (2) **Butane** ($C_4H_{10}$) can be a straight or a branched chain; each gives the compound a different property. (3) The number of isomers increases very rapidly with an increase in molecular weight: $C_4H_{10}$ has 2 isomers, $C_8H_{18}$ has 18 isomers, $C_{12}H_{26}$ has 355 isomers, $C_{18}H_{38}$ has 60,523 isomers.

② NORMAL BUTANE



B.P. = −0.5°C

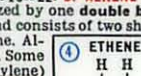
③ ISO-BUTANE

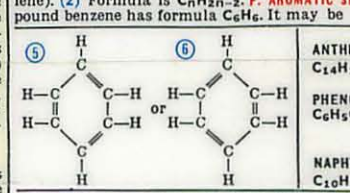


B.P. = −12°C

**35. HYDROCARBONS. A. DEF.** (1) Compounds composed of only *hydrogen* and *carbon.* **B. HOMOLOGOUS SERIES.** Group of compounds that can be represented by a general formula where each compound differs from the next by one carbon and two hydrogen atoms ($-CH_2$). **C. ALKANES OR PARAFFIN OR METHANE SERIES.** (1) Characterized by having only single bonds between carbon atoms. A single bond consists of one pair of shared electrons, represented by a single line. *Alkanes* are saturated compounds. (2) Formula: $C_nH_{(2n+2)}$. The first ten members of the series are: **methane,** $CH_4$; **ethane,** $C_2H_6$; **propane,** $C_3H_8$; **butane,** $C_4H_{10}$; **pentane,** $C_5H_{12}$; **hexane,** $C_6H_{14}$; **heptane,** $C_7H_{16}$; **octane,** $C_8H_{18}$; **nonane,** $C_9H_{20}$; **decane,** $C_{10}H_{22}$. **D. ALKENE OR ETHENE OR ETHYLENE SERIES.** (1) Characterized by one **double bond** between adjacent carbon atoms. A double bond consists of two shared pairs of electrons represented by a double line. Alkenes are unsaturated. (2) Formula is $C_nH_{2n}$. Some representative members are propene (propylene) $C_3H_6$, butene (butylene) $C_4H_8$. **E. ALKYNE OR ETHYNE OR ACETYLENE SERIES.** (1) Characterized by one triple bond between adjacent carbon atoms: $H-C\equiv C-H$ (ethyne or acetylene). (2) Formula is $C_nH_{2n-2}$. **F. AROMATIC SERIES.** (1) Parent compound benzene has formula $C_6H_6$. It may be represented by Fig. 5

④ ETHENE





ANTHRACENE $C_{14}H_{10}$

PHENOL $C_6H_5OH$

NAPHTHALENE $C_{10}H_8$

or Fig. 6 — or a structure intermediate between them. These are resonating structures. All of the carbon-carbon bonds are therefore identical and benzene does not behave like an unsaturated compound such as ethylene. (2) Prepared commercially by distillation of coal tar. (3) Other aromatic compounds (Fig. 7). **G. SUBSTITUTION PRODUCTS.** (1) Replacement of hydrogen atom by another atom or radical: $CH_4$ (methane) + $Cl_2 \rightarrow CH_3Cl + HCl$; $CH_3Cl$ is methyl chloride or chloromethane. (2) Other halogen substitution products: chloroform or trichloromethane ($CHCl_3$), an anaesthetic; freon or dichlorodifluoromethane ($CCl_2F_2$), a refrigerant; iodoform ($CHI_3$), an antiseptic; carbon tetrachloride ($CCl_4$), fire extinguisher; ethylene dibromide ($C_2H_4Br_2$), prevents deposition of *Pb* in cylinder of car; DDT or dichlorodiphenyltrichloroethane [$(C_6H_4Cl)_2CHCl_3$], an insecticide; paradichlorobenzene ($C_6H_4Cl_2$), a moth preventative.

**36. ALCOHOLS.** Contain one or more covalent hydroxyl (OH) groups. *OH* groups have been substituted for one or more hydrogen atoms in a hydrocarbon. **A. METHYL ALCOHOL, METHANOL OR WOOD ALCOHOL ($CH_3OH$).** (1) **PRODUCED** by destructive distillation of wood. Also by synthesis from water gas: $CO + 2H_2 \rightarrow CH_3OH$. (2) **USES:** Antifreeze in cars, making plastics; very poisonous, not used in beverages. **B. ETHYL ALCOHOL: ETHANOL OR GRAIN ALCOHOL, $C_2H_5OH$.** (1) **PRODUCED** by fermentation of sugar by yeast: $C_6H_{12}O_6 \rightarrow 2C_2H_5OH + 2CO_2$. Also by hydration of ethylene obtained from **cracking** of petroleum: $C_2H_4 + H_2O \xrightarrow{H_2SO_4} C_2H_5OH$. (*Cracking* is breaking large molecules into small molecules to increase yield of gasolene.) (2) **USES:** Alcoholic beverages, tinctures, fuel. **C. OTHER ALCOHOLS.** (1) **PROPYL ALCOHOL:** Propanol, $C_3H_7OH$ or $CH_3CH_2CH_2OH$: the hydroxyl group is attached to one of the end carbons. (2) **ISOPROPYL ALCOHOL:** $C_3H_7OH$ or $CH_3CHOHCH_3$; the hydroxyl group is attached to the middle carbon; used as a rubbing alcohol. (3) **GLYCEROL OR GLYCERINE:** $C_3H_5(OH)_3$ or $CH_2OHCHOHCH_2OH$; is a by product of soap making; used in manufacture of nitroglycerine; medicines. (4) **PENTANOL OR AMYL-ALCOHOL:** $C_5H_{11}OH$, used to make lacquers. (5) **ETHYLENE GLYCOL:** $C_2H_4(OH)_2$, is a permanent antifreeze.

**37. ETHERS, ALDEHYDES, KETONES, ORGANIC ACIDS, ESTERS. A. ETHERS.** Ethers are organic compounds in which one oxygen atom is singly bonded to each of two carbon atoms: C—O—C. (1) **PRODUCED** by heating alcohols with a dehydrating agent like $H_2SO_4$: $2C_2H_5OH \xrightarrow{H_2SO_4} C_2H_5-O-C_2H_5 + H_2O$. (2) **NAME:** $(C_2H_5)_2O$ is ethyl ether, $(CH_3)_2O$ is methyl ether, $(C_3H_7)_2O$ is propyl ether. **B. ALDEHYDES.** Obtained from hydrocarbons by replacing one or more hydrogen atoms with $-CHO$ groups. (1) **PRODUCED** by mild oxidation of an alcohol: $2CH_3OH + O_2 \xrightarrow{Cu\ gauze} 2HCHO + 2H_2O$. (2) **NAMES: HCHO** is formaldehyde, $CH_3CHO$ is acetaldehyde. **C. KETONES.** (1) Ketones contain the carbonyl group $>C{=}O$ attached to two hydrocarbon groups. (2) Acetone, the most important ketone is formed by the fermentation of corn and is also a product in the destructive distillation of wood. (3) Acetone, $CH_3COCH_3$, is used to remove nail polish; a solvent for paint, lacquer. **D. ORGANIC ACIDS.** Obtained from hydrocarbons by replacing one or more hydrogen atoms with carboxyl groups $-COOH$. (1) **FORMIC ACID: HCOOH** is found in the sting of ants, bees. (2) **ACETIC ACID: $CH_3COOH$** or $HC_2H_3O_2$ is used in making cellulose acetate. (3) **OXALIC ACID: $(COOH)_2$,** a reducing agent in blueprints. **E. ESTERS** (1) Produced when an acid reacts with an alcohol: ethanol + acetic acid → ethyl acetate + water: $C_2H_5OH + CH_3COOH \xrightarrow{H_2SO_4} CH_3COOC_2H_5 + H_2O$. The $H_2SO_4$ is catalyst and not a dehydrating agent. (2) Glycerol + nitric acid → nitroglycerine + $H_2O$: $C_3H_5(OH)_3 + 3HNO_3 \xrightarrow{H_2SO_4} C_3H_5(NO_3)_3 + 3H_2O$.

DATA-GUIDE QUICK CHART

# PRINCIPLES OF ECONOMICS

SIMPLIFIED SUMMARY ◆ INSTANT REFERENCE ◆ PERMANENT MEMORY-AID

LEARN & REVIEW IN MINUTES

AUTHOR: Prof. E. Lerner, Ph.D., C.C. N.Y.

ALSO SEE ECONOMICS VOCABULARY

SOLID PLASTIC

SPEED STUDY TO SAVE YOU TIME AND WORK.

QUICK CHARTS PUT THE FACTS IN FULL VIEW:

PERMANENTLY FILL IN FOR MISSING NOTES & TEXTS.

QUICK CHARTS HELP YOU REMEMBER:

CONTAINS NO PAPER

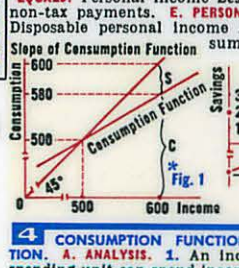
## 1 BASIC CONCEPTS. A. ECONOMICS PROBLEMS.

**1. MACROECONOMICS.** Maintenance of *high* employment and growth. **2. MICROECONOMICS.** The efficient allocation of *scarce goods* among alternative ends. **B. FUNCTIONS OF ALL ECONOMICS SYSTEMS. FUNCTION 1:** What goods should be produced and in what quantities? **FUNCTION 2:** How should the goods be produced? **FUNCTION 3:** For whom should goods be produced, or whose wants should be satisfied and to what extent? **FUNCTION 4:** How can **consumption** be adjusted to the available supply of goods within short periods of time? **FUNCTION 5:** What provisions are made for the maintenance and expansion of the economic system over long periods of time? **C. THE WHEEL OF INCOME.** A free enterprise society performs the **five functions** primarily through a system of **market prices. 1.** Businesses, anticipating a **demand** by the public for a product, hire **resources** (*labor, land, capital*) from their owners (*the public*) and incur production **costs** (*wages, rent, interest*). Business firms seeking to **maximize profits** will try to hire and combine resources to produce the goods the public wants at the **least cost. 2.** Consumers spend the income earned from selling their services to business, for the goods and services produced. The money income of any one consumer depends upon the types and amounts of **productive resources** he owns and the **prices** these services command. **3. High prices** for goods help to meet the problems of inadequate supply by discouraging consumption of available supplies and simultaneously stimulating an increase in production. **4.** If a firm correctly anticipates **consumer demand,** it will reap a **profit;** if not, it will sustain a **loss.** If a firm makes unusually large profits, and there is **free entry,** new firms will be formed to compete. By producing substitute goods, the new firms will make prices lower (*or costs higher*) thus reducing the large profits. **5. Capital** is invested both by business and the public to promote growth and expansion. (*See Investment*; *Sec.* 5.)

## 2 CHARACTERISTICS OF BUSINESS ORGANIZATIONS. A. TYPES (S.P., Sole Proprietorship; Part., Partnership; Corp., Corporation). B. LIABILITY OF OWNERS FOR DEBT. 1. S.P. Unlimited. Personal assets may be attached to pay business debts. 2. Part. Unlimited. (See B:1.) 3. Corp. Limited. Business creditors do not have a claim against property of stockholders. C. PERMANENCE. 1. S.P. Begins and ends at will of owner. 2. Part. Ends by death, withdrawal, bankruptcy, or legal disability of any single partner. 3. Corp. Perpetual life, unless dissolved by stockholders. D. EASE OF TRANSFERRING OWNERSHIP INTEREST. 1. S.P. Owner must find a buyer with enough cash to buy entire enterprise and who is interested in management. 2. Part. Difficult. Usually partnership must be dissolved. 3. Corp. A stockholder can usually sell his shares whenever and to whomever he pleases. The shares are in units of convenient size. The purchaser need not be concerned with management. E. TAXATION. 1. S.P. Profits taxed as personal income. 2. Part. Profits taxed as personal income. 3. Corp. A tax upon incorporation; annual franchise taxes; a tax upon corporate net income in some states; Federal taxes upon corporate income. Dividends to stockholders taxed as personal income. F. EXPENSE AND DIFFICULTY IN STARTING. 1. S.P. No special problem raised by the form of organization. 2. Part. May have a partnership agreement drawn up. 3. Corp. Legal expenses, fees and taxes but ordinarily not prohibitive. G. EASE OF MANAGEMENT CONTROL. 1. S.P. One man is sole arbiter of business decision. 2. Part. Any partner can bind the firm on a contract. 3. Corp. Duties of representatives defined in corporate charter. The corporate form permits control (management) to be separated from ownership (stockholders).

## 3 MEASURING NATIONAL INCOME. A. GROSS NATIONAL PRODUCT (GNP). 1. G.N.P. is the money value of all goods and services produced within a single year or the total expenditures for **final goods and services.** *G.N.P.* can increase because (a) more goods and services are produced or (b) the output sells at higher prices. "**Constant dollar**" or "**real**" G.N.P. may be computed by deflating (dividing) money *G.N.P.* by a **price index. 2.** Expenditures for intermediate products are not included in GNP. It would be double counting, for example, to add the total value of iron ore to the total value of rolled steel and consider the sum as the value of national output. B. NATIONAL INCOME EQUALS: Gross national product *Less* capital consumption allowances, indirect business tax, business transfer payments, statistical discrepancy *Plus* subsidies *Less* current surplus on government enterprises. C. PERSONAL INCOME EQUALS: National income *Less* corporate profits and inventory valuation, contribution for social insurance, excess of wage accruals over disbursements, *Plus* gov't transfer payments, net interest paid by gov't, dividends, business transfer payments. D. DISPOSABLE PERSONAL INCOME EQUALS: Personal Income *Less* personal tax and non-tax payments. E. PERSONAL SAVING EQUALS: Disposable personal income *Less* personal consumption expenditures.

Slope of Consumption Function




## 4 CONSUMPTION FUNCTION; SAVINGS FUNCTION. A. ANALYSIS. 1. An individual, family, or spending unit can spend income (Y) on consumption (C) or save a portion (S). As income rises, people may be expected to consume more, therefore the Consumption Function will slope upward to the right. 2. If Y is greater (*or less*) than the point of intersection of the 45 degree line and the consumption functions, then **savings** (or *dissavings*) will take place. B. THE MARGINAL PROPENSITY TO CONSUME (MPC) is the change in C associated with the change in **income.** C. THE MARGINAL PROPENSITY TO SAVE (MPS) is the change in **saving.** When income rises from 500 to 600 and C rises by 80, the MPC is .8 and the MPS .2. **NOTE:** Some economists believe that the consumption function shifts upward from year to year as national living standards rise, as wealth increases, or as income becomes distributed more equitably. Others believe that permanent consumption is a constant proportion of permanent income.

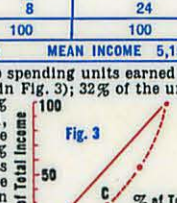
**D. A SPENDING UNIT. Consists of all related persons living together who pool their incomes.** Concern over the distribution of income centers on **spending units** rather than individuals. Spending units frequently have more than one person earning income. **E. LORENZ CURVES. Used to show the distribution of income.** The percentage of spending units is plotted on the horizontal axis; the percentage of income on the vertical axis.

INCOME DISTRIBUTION OF SPENDING UNITS AND OF TOTAL MONEY INCOME BEFORE TAXES

| MONEY INCOME BEFORE TAXES | % Spending units | % Total money income |
|---|---|---|
| Under $1000 | 7 | 1 |
| $1000-2999 | 25 | 10 |
| $3000-4999 | 24 | 18 |
| $5000-7499 | 24 | 27 |
| $7500-9999 | 12 | 19 |
| over $10,000 | 8 | 24 |
| | 100 | 100 |

MEDIAN INCOME $4,400 MEAN INCOME 5,150

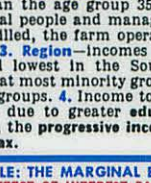
In the table, 7% of the spending units earned 1% of the income (*Point A* in Fig. 3); 32% of the units (7 plus 25) earned 11% of the income (10 plus 1, *Point B*). If income were equally distributed, 10% of the spending units would earn 10% of the money; 20% would earn 20%, etc. The Lorenz curve would lie along the 45 degree line. The more unequal the distribution, the further away from the 45 degree line.




**F. THE DISTRIBUTION OF INCOME IN THE U.S.** is related to: **1. Age**—the young and the old earn less than the age group 35-55. **2. Occupation**—professional people and managers earn more than the unskilled, the farm operator and the retired person. **3. Region**—incomes are highest in the West and lowest in the South. Finer breakdowns show that most minority groups earn less than the major groups. **4.** Income today distributed more equally due to greater **educational opportunity** for all, the **progressive income tax,** and the **inheritance tax.**

## 5 INVESTMENT SCHEDULE: THE MARGINAL EFFICIENCY OF CAPITAL. A. EFFECT OF INTEREST RATE.

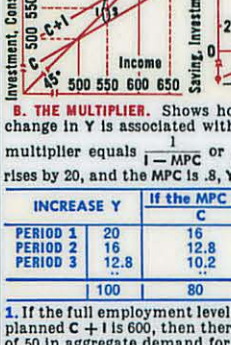
The amount of investment at any moment is related to the interest rate. Along investment schedule II', $OI_0$ investment takes place at interest rate $r_0$; if the rate falls to $r_1$, $OI_1$ investment takes place. Many economists believe that shifts in the investment schedule to the right or to the left have a greater effect on the amount of investment than movements along the schedule caused by changes in interest rates. **B. SHIFTS IN THE SCHEDULE.** Due to (1) new discoveries, (2) changes in technology, (3) population shifts, (4) and changes in incomes. The volume of investment fluctuates sharply over time. Changes in **inventory** are the most volatile component of **total investments. C. SAVINGS AND INVESTMENT. 1.** Ignoring government and foreign expenditures, total expenditures or aggregate demand equal the outlays for C and I. These outlays are the source of income: $Y = C + I$. Income can be spent on consumption or saved. The equation for the use of income is $Y = C + S$. **2. Ex post,** or at the end of the period, $S = I$ since the Y's and C's are identical in both the source and use equations. **3. Ex ante,** or at the beginning of the period, planned I and planned S need not be equal. **4.** If planned I exceeds the saving that will be done out of that current level of income, expenditures will rise from one period to the next; if saving exceeds investment, expenditures and income will fall.




## 6 GOVERNMENT EXPENDITURES AND RECEIPTS.

**Defense expenditures** account for the bulk of federal outlays; **schools and highways** are the principal state and local gov't expenses. **A. TAXES. 1. The personal income tax** and **corporate income tax** are the principal sources of receipts for the federal gov't; the **property and sales tax** are the principal sources for state and local gov'ts. **2.** Taxation results in the transfer of funds from private into public hands. Since all people do not pay the same tax nor receive the benefits from the gov't, the process of taxing and spending may result in a redistribution of income. **Ex:** *A tax levied on a rich man used to finance unemployment comp.* **3.** A **progressive tax** takes a greater proportion of high incomes than low; a **regressive tax** falls more heavily upon the low income groups. **4. The federal income tax** is progressive because additional earned income is taxed at progressively high rates. **5. A general sales tax or property tax** is regressive because food and housing expenditures do not rise in proportion to income. A poor man therefore pays a larger proportion of his total income in taxes under regressive taxation than does a rich man. **B. NATIONAL DEBT. 1.** If expenditures exceed receipts in a given year, governments as well as firms can meet their outlays by issuing **debt instruments.** The Federal debt rose rapidly to finance the expenses of World War II. Since then, it has increased moderately while state and local gov't debt and private debt has risen rapidly. **2.** From the point of view of the economy as a whole "we owe the debt to ourselves." Individuals are the largest holders of the public debt and banks the second largest. **3.** However the size of the national debt is important because (a) **interest payments** on the debt result in a redistribution of income (*from those who pay taxes to those who own the debt and receive interest payments*) and (b) the size of the debt and the frequency of **refinancing** affects the monetary system. **4.** The holders of government debt own **highly liquid instruments** (*bonds that have no reasonable risk of default*). Because banks hold so large a portion of their assets in **short term bonds,** they can assume a greater degree of risk in other investments. Like all fixed income assets, the real purchasing power of bonds fluctuates **inversely** to the price level: *as the price level rises, the real value of bonds declines.* **C. PRINTING MONEY.** The Federal Gov't can and has raised funds by printing money. The Revolutionary and Civil War were financed in large part this way. Since 1914, the government has financed its large war time deficits by borrowing from the **Federal Reserve** and from **commercial banks.** (See Section 5, C). **D. GOV'T REGULATIONS AND OPERATIONS** influence economic activity. **1. Licenses** must be obtained before many types of businesses can be started: *taverns, medical practices, taxi cabs, banks.* **2.** The **CAB** determines what cities airlines can service. **3.** The **ICC** passes on freight rates and passenger fares. **4. The Maritime Administration** determines which members of the merchant marine receive subsidy. **5.** Some businesses are encouraged through patents, protective tariffs, or *SBA* loans; others are retarded by anti-trust actions. **6. Minimum wages, Taft-Hartley** and the **Landrum-Griffen** bill influence working conditions. **7.** The government also acts as entrepreneur supplying certain necessary services: the post office, armed forces, police, fire and sanitation departments. Some government businesses, similar to **TVA,** compete directly with private enterprise. **E. INCOME ANALYSIS.** If gov't expenditures, **G,** are added to the source equation, then $Y = C + I + G$. Adding taxes T to the use equation, $Y = C + S + T$. A rise in G, like a rise in I, will raise Y. If T and planned S exceed $I + G$, Y will fall.

## 7 THEORY OF INCOME DETERMINATION. A. THE EQUILIBRIUM SOLUTIONS. If I equals 20, then the **equilibrium level of income** is 600 for this is the income level that generates 20 in S. At income level 550, I is greater than S. This is not an equilibrium solution because $C + I$ exceeds $C + S$, or planned expenditures exceed planned receipts. Y must therefore continue to rise until S equals I. At income level 650, S exceeds I. Y will fall because **expenditures** are less than **planned receipts.**




**B. THE MULTIPLIER.** Shows how great a change a change in Y is associated with a change in I. The multiplier equals $\frac{1}{1 - MPC}$ or $\frac{1}{MPS}$. If investment rises by 20, and the MPC is .8, Y will rise by 100:

| | INCREASE Y | If the MPC is .8 and the MPS .2 | |
|---|---|---|---|
| | | C | S |
| PERIOD 1 | 20 | 16 | 4 |
| PERIOD 2 | 16 | 12.8 | 3.2 |
| PERIOD 3 | 12.8 | 10.2 | 2.6 |
| | .. | .. | .. |
| | 100 | 80 | 20 |

**1.** If the full employment level of income is 650 and planned $C + I$ is 600, then there will be a deficiency of 50 in aggregate demand for the maintenance of full employment. **2.** If the MPC is .8, the multiplier is 5, and a government expenditure of 10 will give rise to 50 in additional expenditures. **3.** If Y is 700, resources are overemployed (**inflationary gap**). **4.** A rise in taxes will lower expenditures. If the multiplier is 5, then 10 in additional taxes will lower income by 50. **C. CRITICISMS OF THE MULTIPLIER. 1.** Some economists challange the statement that there is a unique relation between government spending and national income. They believe that a rise in government spending may, at times, discourage or replace private investment; while at other times, it may stimulate private investment. If there are these offsetting or compensating changes in private I, then the G or T needs to be greater (*or smaller*) than otherwise to generate the desired level of **aggregate expenditure. 2.** Economists have also questioned the relationship between **full employment** and a prescribed level of **aggregate demand.** They contend that if unions (or other groups) exert an upward pressure on wages (or other prices) **price inflation** may begin before full employment is reached.

## 8 THE QUANTITY THEORY. Aggregate demand can also be analyzed as the total supply of money (M) times the number of times each dollar is spent for goods and services (velocity or V). The **income expenditure** approach stresses change in investment and the multiplier; the **monetary approach** focusses on the determinants of M and V. **A. THE EQUATION OF EXCHANGE.** $MV=PY$ where Y is **real income** and P is **average price.** (*This equation is sometimes written* $MV=PT$ *where* T stands for all transactions). **1.** If Y is assumed constant in the short run, then an increase in either M or V will lead to a rise in prices. M can increase (*decrease*) if **total bank reserves** rise. (See **10.**) V can change if expectations of the price level or the cost of holding money change. **2.** If people expect prices to rise, they will spend their money more rapidly causing money expenditures to rise. A rise in **interest rates,** making money more costly to hold, also leads to a rise in **velocity. 3.** If M and V remain constant, an increase in output (Y) will lead to a decline in P. **B. THE MONEY SUPPLY. 1.** Usually defined as **demand deposits** (*checking account*) and currency outside banks. Both currency and checks can be used to purchase goods and services. **2.** Some economists also include **time deposits** (*savings account*) as part of the money supply. **3. Liquid assets** (*such as government bonds, savings and loan shares, common stocks and usually time deposits*) are not included as part of the money supply for they must first be converted into money before they can be spent. These liquid assets do perform some of the functions of money for they serve as a store of **value.** If liquid assets rise, need to hold money may decline or velocity may rise.

## 9 THE BANKING SYSTEM. A. COMMERCIAL. Each of 13,600 **commercial banks** must hold some percentage of their demand deposits in **reserves.** Both **deposits** at the *F.R.* Banks and some percentage of **vault cash** count as reserves. **B. DEPOSIT EXPANSION. Bank "A"** receives a $100 deposit in cash and must hold 20% of its demand in required reserves (**step 1**):

| ASSETS | LIABILITIES |
|---|---|
| (1) Required res. +20 | (1) Demand deposits +100 |
| (1) Excess reserves +80 | |
| (2) Loans +80 | (2) Demand deposits +80 |
| (3) Excess reserves −80 | (3) Demand deposits −80 |

This bank now has $80 in **excess reserves** and can make a loan up to the amount of its excess reserves. When the loan is made (**step 2**) the bank must expect that the borrower will spend the **demand deposit** he received. Funds will leave the bank—**a check will be written and deposited in another bank**—and reserves will be reduced (**step 3**). Any one bank can therefore only lend amounts equal to their excess reserves. **Bank "B"** receives the $80 in deposits and must also hold 20 per cent in reserves, or $16. The maximum it can loan is $64. If this loan were made, Bank B must expect that checks will be written and deposited in still other banks. When all the changes in loans and deposits resulting from the $100 rise in reserves are combined, loans would rise $400 and deposits $500. **A bank can only lend its excess reserves.** The **banking system as a whole can create money.**

| Bank | DEPOSIT INCREASE | REQUIRED RESERVES | EXCESS RESERVES | MAXIMUM LOAN |
|---|---|---|---|---|
| A | $100 | $20 | $80 | $80 |
| B | 80 | 16 | 64 | 64 |
| C | 64 | ... | ... | ... |
| TOTAL | $500 | $100 | | $400 |

**C. THE EXPANSION RATIO is the reciprocal of the required reserve ratio:** If $100 in new reserves are provided when the reserve ratio is 25%, deposits can rise by a maximum of $400; when the **reserve ratio** is 10%, by a maximum of $1000. The **F.R. Banks** control both the minimum reserve ratio that member banks must hold and the total amount of reserves in the banking system. Through their control over total reserves, the Federal Reserve influences the stock of money and thereby **aggregate demand. D. CENTRAL BANKING.** The *F.R. Banks* can influence economic activity through *quantitative or general* controls and through *qualitative or selective credit controls.* **1. QUANTITATIVE CONTROLS** influence the **total** supply of money these include (a) open market operations, (b) discounting, and (c) changes in reserve requirements. **2. QUALITATIVE CONTROLS** attempt to influence the **velocity** of money: these include (a) control over stock market margin requirements, and in the past included (b) consumer credit and (c) real estate loans which specified both the size of the downpayment and the period of repayment. **E. OPEN MARKET OPERATIONS.** When the Federal Reserve buys a U.S. Government **bond** in the **open market,** it makes payment by giving a **deposit.** Since deposits at the Fed. Res. are legal reserves, **open market buying** enables banks to create additional loans and deposits. It therefore has an **expansionary effect** upon the economy.

| FEDERAL RESERVE BANK | COMMERCIAL BKG SYSTEM |
|---|---|
| (1) Bond: + | (1) Bond: − |
| Member bank reserves: + | (1) Member bank reserves: + |

**Open market selling** (reverse of step 1 above) is **deflationary. F. DISCOUNTING.** When the Federal Reserve makes a loan to a bank or discounts a note, member bank reserves **increase.** Since reserves are higher, the banks can make additional loans and the money supply can increase.

| FEDERAL RESERVE | | COMMERCIAL BANKS | |
|---|---|---|---|
| (1) Discounts, advances: + | (1) Member bank reserves: + | (1) Member bank reserves: + | (1) Advance from the F.R.B.: + |

When banks repay their borrowings from the Federal Reserve, total reserves will **fall** and deposits will contract. **Moral suasion,** or conference between individual commercial bankers and officials of the *F.R.B.* are also used to influence the volume of **discounting** and other bank practices. **G. THE DISCOUNT RATE. The charge that the F.R.B. makes for its loans to member banks.** The *F.R.B.* tries to keep this rate consistent with other short term interest rates. Changes in the discount rate are sometimes considered to have important psychological effects upon the business community for some feel that changes in these rates indicate impending changes in *F.R.B.* policy. **H. TREASURY OPERATIONS.** The Treasury keeps its deposits at the *F.R.B.* When people pay their taxes, the Treasury first redeposits the money in the commercial banking system (**step 1**). When the Treasury "**calls**" the money, it builds up its balance at the *F.R.B.* and lowers member bank reserves.

| FEDERAL RESERVE | COMMERCIAL BANK |
|---|---|
| (2) Member bank reserves: − | (1) Private demand deposits: − |
| Treasury deposits: + | (1) Treasury deposits + (2) Treasury deposits: − |

The payment of taxes by reducing reserves will lead to a **contraction** in the money supply and is **deflationary.** Treasury expenditures, the reverse of tax payments, are inflationary because they lead to an increase in bank reserves. **I. GOLD FLOW** into the country is sold to the Treasury. (**EX:** *An exporter receives payment in gold for his merchandise.*) The Treasury pays for the gold with a check drawn against its account at the Fed. When the check is deposited in a commercial bank and then sent to the *F.R.B.* to be cleared, member bank deposits (reserves) will rise and Treasury deposits will fall. **Gold outflows, the reverse of this process, are deflationary. J. CURRENCY WITHDRAWALS** are **deflationary** for vault cash is included as part of

reserve requirement. When cash is deposited in the bank (**EX:** *a store's receipts on Monday morning*) total currency in circulation falls and total reserves rise. **K. SUMMARY.** The only way the *F.R.B.* can control the money supply is through **(a)** open market operations, **(b)** the discount window, **(c)** changes in reserve requirements. It uses these three tools to offset all other day to day transactions, such as treasury operations, gold flows, and currency withdrawals, as well as to influence the direction of economic activity. The *F.R.* has resisted accepting the responsibility for administering all selective credit controls except those over **margin requirements.** Selective controls are difficult to police and hurt certain industries.

**10 BUSINESS CYCLES. A. ANALYTICAL TOOLS.** Both the quantity **theory of money** and the **income-expenditure** approach are truisms. Both help economists analyze economic fluctuations (*business cycles*) by emphasizing important relations. A change in I times the multiplier leads to a rise in Y. An increase in the money supply leads to a change in PY because **(1)** more bank credit lowers interest rates in the short run and thereby stimulates investment and **(2)** increases the community's holdings of wealth and thereby stimulates consumption. **B. ACCELERATION PRINCIPLE.** The volatility of investment has been stressed (*Section 3, C*) as one source of instability. I may fluctuate widely because a change in sales may produce a greater percentage change in expenditures on **capital goods.** This relationship is called the **accelerator principle.** Assume: **(1)** A machine that costs $5 is necessary to produce $1 in sales. **(2)** Machines last for ten years and they are then replaced on a regular basis (*10% of each year*). Note that a change in sales from 10 to 15 in year 3 made investment rise from 5 to 32.5. When sales stopped rising and only continued at the high level (15) investment fell to a quarter of its previous level. **C. MONETARY THEORY.** Some economists have stressed that business expansions are financed by increases in **bank credit** or an expansion in the **money supply.** This increase in bank credit facilitates **inventory buying** and other investment expenditures. When the volume of reserves becomes restricted, either through **gold outflows** or deliberate Federal Reserve actions, the money supply levels off, restricting expenditures. There has never been a major inflation without an increase in the money supply; there has never been a major depression without a contraction in the money supply. During minor recessions, the money supply usually remains constant or rises at a slower rate than during the previous expansion. **D. BUILT IN STABILIZERS. 1. INCOME TAXES AND UNEMPLOYMENT COMPENSATION.** These automatically generate countercyclical activity. Taxes rise during prosperity because personal income mounts; but the increases in taxes (*under* **progressive** *rates*) reduce consumer **disposable income** below what it would otherwise be and thereby lower consumer expenditures. When the economy begins a downturn and incomes decline, **tax receipts** automatically fall. Consumers are left with a larger percentage of their income, and expenditures are higher than they would be in the absence of reduced tax payments. **Unemployment compensation** enables the jobless to continue their expenditures. As prosperity cumulates, the number of people receiving unemployment compensation dwindles, and expenditures are lower than they would be if this compensation had continued. These automatically reduce some of the cyclical fluctuations in income. **2. THE FDIC.** Insures deposits up to $10,000 and has done much to eliminate the possibility that individuals will lose their money through **bank failure.** Bank failures in the past aggravated **recessions** by wiping out a portion of the money supply. Since deposits are now insured, a "run" on a bank to withdraw funds is unnecessary. Banks need not liquidate their **assets** at depressed prices or call in loans. The FDIC therefore represents an important structural change that has taken place in our economy since the 1930's to make it more stable. **E. ECONOMIC GROWTH.** Important because the improvements in the **standard of living** that can arise through faster growth are greater in the long run than the improvements that can originate through a better allocation of resources. **1. In a free society,** the rate of growth will depend upon the volume of saving and investment. The higher these schedules are, the faster will be the growth rate. **2. The problems of economic growth are: (a)** to improve the productivity of laborers by increasing the capital equipment they work with **(b)** to eliminate unemployment.

| YR. | CURRENT SALES | CAPITAL STOCK NECESSARY FOR THE CURRENT | REPLACEMENT EXPEND. | NEW INVESTMENT | *GROSS INVESTMENT |
|---|---|---|---|---|---|
| 1 | 10 | 50 | 5 | 0 | 5 |
| 2 | 10 | 50 | 5 | 0 | 5 |
| 3 | 15 | 75 | 7.5 | 25 | 32.5 |
| 4 | 15 | 75 | 7.5 | 0 | 7.5 |

*Gross investment equals replacement expenditures plus new expenditures.

**11 DEMAND. A. DEMAND SCHEDULE.** A demand schedule indicates how many units of a specific, good buyers would be willing to purchase at various alternate prices, **ceteris paribus** (*assuming that nothing but the price of that good changes*). At price **OR**, people would be willing to buy quantity **OA** but no more; at a lower price, **OS**, they would be willing to buy **OB**. The demand curve slopes downward and to the right because at lower prices each consumer buys more of the product and new consumers enter the market. **B. SHIFTS IN DEMAND. 1.** Demand increases from **D** to **D'** or decreases from **D** to **D"** if incomes change, tastes change, or the price of substitute or complementary goods change. **2.** If demand increases as income rises, the commodity is called a **superior good;** if the demand decreases as income rises, the commodity is an **inferior good. 3.** Oleo and butter are substitutes; bread and butter are complements. If the price of oleo falls (*rises*) the demand for butter shifts down (up). If the price of bread falls (*rises*) people will want more (*less*) butter or demand for butter will shift up (*down*). **C. ELASTICITY OF DEMAND: percentage change in quantity demand / percentage change in price.** (In computation, always use the larger **p** or **q.**) If the price falls from 5 to 4, and the quantity purchased rises from 3 to 4, the elasticity is:

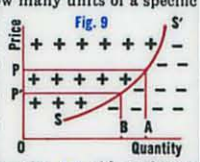
$$\frac{\text{change in q}}{q} \div \frac{\text{change in p}}{p} = \frac{1}{4} \div \frac{5}{5} = \frac{5}{4} = 1.25$$

When the price falls from 4 to 3, the **elasticity** is 0.8. If elasticity is greater than 1, the demand curve is said to be elastic; if equal to 1, it is of unitary elasticity; and if less than 1, inelastic. **D. ELASTICITY AND REVENUE.** Elasticity indicates the change in revenue that will accompany price changes:

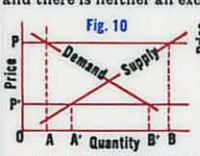
| IF DEMAND IS | AND THE PRICE | REVENUE WILL |
|---|---|---|
| Elastic | *rises* | *fall* |
| | falls | rise |
| Unitary | *rises* | *remain same* |
| | falls | remain same |
| Inelastic | *rises* | *rise* |
| | falls | fall |

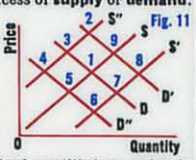
**E. INCOME ELASCITY.** Relates the percentage change in **Q** demanded associated with a percentage change in income, (**ceteris paribus** *assuming that nothing but income changes*). **Cross elasticity** relates the percentage change in quantity of product **A** demanded to the percentage change in the price of product **B. F. NOTES: 1.** The more substitutes there are for a given product, the greater the elasticity of demand; thus if one grocer tries to raise the price of bread, his sales will fall sharply for customers can buy bread in many stores. If a utility raises the price of electricity, it is unlikely that the use of electricity will fall markedly (**inelastic demand**). **2.** Demand curves are more elastic over a long period of time than in the short run. For given time, competition can develop or people can become aware of additional alternatives to the product.

**12 SUPPLY AND MARKET PRICE. A. THE SUPPLY SCHEDULE.** Indicates how many units of a specific good would be offered for sale at various alternate prices, **ceteris paribus. 1.** At price **OP**, firms are willing to sell **OA** but no more. **2.** At a lower price, **OP'**, they will sell only **OB. 3.** The supply curve slopes upward and to the right for two reasons: as prices rise, **(a)** old producers are willing to expand their output, **(b)** new producers are attracted into the industry. **4.** The supply curve can shift to the right or to the left if production costs fall or rise. The **elasticity of supply** relates the percentage change in quantity offered for sale to the percentage change in price.

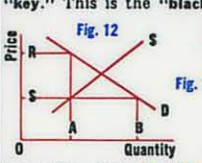



**B. EQUILIBRIUM PRICE. 1.** The intersection of the **S** and **D** curves determines the market price of the product. At price **OP**, consumers are willing to buy **OA** amount but producers are willing to sell **OB** amount. **AB** will remain unsold at price **OP**, and the price will tend to fall. At price **OP'**, consumers would be willing to buy **OB'** amount, but producers are willing to sell only **OA'**. **A'B'** represents the **unsatisfied demand,** and the price will tend to rise. **2.** Only where the curves intersect is the price stable. At that price, buyers can purchase all they want, sellers can sell all they want and there is neither an excess of **supply** or **demand.**



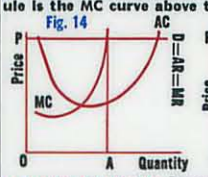



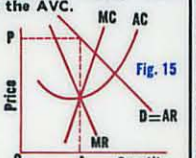

**C. ANALYSIS.** If the original equilibrium point is **1** (*Figure II*), an upward shift in demand, other things remaining the same, will make the new **equilibrium price point 9.** If the supply curve shifts, the amount demanded changes but not the **demand schedule.** If the demand curve shifts, the amount supplied changes but the **supply schedule** remains constant. Starting from **point 1:** Increase in demand, new equilibrium, **point 9.** Decrease in demand, new equilibrium, **point 5.** Increase in supply, new equilibrium, **point 7.** Increase in demand and increase in supply, new equilibrium, **point 8.** Increase in demand and decrease in supply, new equilibrium, **point 2.** Decrease in demand and increase in supply, new equilibrium, **point 6.** Decrease in demand and decrease in supply, new equilibrium, **point 4. D. ADMINISTERED PRICES. 1. RENT CONTROL.** If prices are controlled at price **OS**, only **OA** will be available. **OB** people would like to rent at this price. **AB** unsatisfied demand must be eliminated. One way is by **"squatters rights"**—whoever lives in an apartment can keep it. Another way is to ask the tenant to buy the furniture. Tenants would be willing to pay up to **OR** for the furniture or the **"key."** This is the **"black market"** price of any controlled good. **2. AGRICULTURAL PRICES.** Farm prices are administered above the **equilibrium price.** People are willing to produce, at price **OR**, **OB** amount. But consumers will buy only **OA** amount. To keep the price at **OR**, the government must purchase **AB** amount of output. Sect. of Agriculture Brannan (*under Pres. Truman*) proposed that price **OR** be guaranteed to the farmers but that the price consumers pay should be low enough to clear the market. For output **OB** this price would be **OS.** The government would then pay the farmers **RS** for each bushel sold at price **OS.** The **soil bank program** was designed to curb output—shift the supply curve back to **S'** so **S** and **demand** intersect at price **OR.**



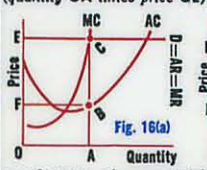


**13 THEORY OF THE FIRM. A. COSTS.** The costs to a firm can either remain constant (**fixed**) or change (**variable**) if output changes. Property taxes, rent and other overhead expenses illustrate short run fixed costs (**FC**); labor and materials illustrate costs that vary (**VC**) directly with output. **Total cost-fixed costs plus variable costs. Marginal costs** are the change in total cost that result from producing an additional unit. **Average costs equal total costs divided by total output. B. DIMINISHING RETURNS. 1.** Output will rise if the quantity of one input (**factor of production**) is increased while the quantity of other inputs remain the same. After a point the extra output (**marginal product**) resulting from equal inputs will decline (columns 2 and 4 below). **2. The law of diminishing returns** (variable proportions) is technological; it refers only to physical inputs and outputs. It is of no direct help in deciding how much product should be produced. Specifically, a firm does not stop producing when diminishing returns set in. It will continue to produce until the most profitable output is reached. **3.** The transition from the law of diminishing returns to **cost analysis** is done by introducing the prices of **productive services. 4.** Each additional variable factor produces more than one unit of output. Total costs rise as output expands; both **average variable** and **marginal costs** first fall then rise. **AC is at a minimum where it equals MC. C. OUTPUT DECISIONS.** Firms must make two output decisions: **whether to produce any amount at all? and if so, how much? 1.** The first decision depends upon whether **fixed costs** are covered: no firm will open if it appears that it will lose money through the foreseeable future. **2.** Once the decision to begin production has been made, fixed costs no longer influence the output decision. For fixed costs remain the same regardless of the level of output, so long as the firm decides to remain in business. Only **variable costs** are a consideration in the determination of the **rate of output,** for these alone can be increased or diminished by varying output. **3.** The rate of output selected depends upon the difference between **total receipts** and **total costs.** So long as an additional unit of output adds to total receipts (**marginal revenue**) more than it adds to total costs (**marginal costs**) it is profitable to produce that unit. When **MC** exceeds **MR**, the extra unit of output will no longer be profitable. **D. COMPETITIVE OUTPUT. 1.** Under competition, the **demand schedule** facing individual producers is **infinitely elastic. 2.** If one firm tries to raise its price, its sales will fall until they approach zero. **3.** If a firm lowers its price, it will capture the entire market. **4.** Each additional unit of output will have a negligible effect upon the price. Each firm is so small that it does not take into consideration the impact of its action on the market price. Since it can sell as much as it wants at the prevailing price, **MR** equals price or average revenue (**AR**). **OA** is the most profitable output because there **MC = MR.** Output **OA** will be sold at price **OP. 5. Under perfect competition, the supply schedule is the MC curve above the AVC.**




**E. MONOPOLY OUTPUT.** Demand facing a monopolist is not infinitely elastic. **1.** For additional sales, the monopolist, like the industry as a whole, must lower his price. The additional revenue received from new sales will therefore be lower than the price presently prevailing in the market. **2.** If extra units are to be sold at lower prices than those presently prevailing, all sales must be made at the new low price. **MR therefore is below AR. 3.** The maximum profit output for the monopolist, as for the competitor, is where **MC = MR.** This is output **OA.** This output can be sold at price **OP'** (*where the amount produced,* **OA** *equals demand*). There is no supply schedule under monopoly. **F. DISTRIBUTION OF RECEIPTS. 1. Total receipts** under both **competition** and **monopoly** equal **OECA** (*quantity* **OA** *times price* **OE**). **2. Total variable costs** equal **OFBA.** (*Average variable costs* **OF** *times output* **OA.**) **3.** Returns to the **fixed factors** (*including the entrepreneur*) are **EFBC,** i.e., *all of the receipts that do not go to the variable factors.* If **EFBC** is large enough to cover the fixed costs—*the rent, property taxes, interest payments*—and if there is a residual, it will go to the entrepreneur. If **EFBC** is not large enough to cover fixed costs, the firm will be operating at the most profitable (minimum loss) point but still losing money. **4.** Unless fixed costs are covered, the firm will cease production. (See **9, C.**)





**14 DISTRIBUTION. A. PRICES OF PRODUCTIVE FACTORS UNDER PERFECT COMPETITION. 1.** The statement on **diminishing returns** (section **9, B**) related the change in output (**product**) arising from adding units of one type of input and holding other inputs constant. Similar tables (**marginal product schedules**) can be constructed for each productive service. **2.** If the costs of two productive services are identical but the **MP's** differ—*say both factor* **A** *and* **B** *each costs $5 but factor* **A** *produces 5 units and factor* **B** *3 units*—then the firm will minimize its costs by buying more of **A** and less of **B.** This process will continue until the ratio of **marginal physical product to price** is equal for each factor: $\frac{MPP_a}{P_a} = \frac{MPP_b}{P_b}$ **3.** Multiplying the price of the output times the marginal physical product yields the **marginal revenue product** (**MRP**). Firms will hire productive resources until the **MRP** equals the **MC**—the extra revenue contributed by the factor equals the extra cost incurred. **B. DERIVED DEMAND. The demand for all factors of production is derived from the demand for the final product.** Therefore: **1.** The less elastic the demand for the final product, the less elastic is the demand for the factor input. **2.** The poorer the substitutes for this factor input, the more inelastic the demand for the input. **3.** The more inelastic the supply of other inputs, the more inelastic the demand for this one input. **NOTE:** If the price of substitute inputs rises rapidly as the demand for them increases, then the incentive to shift in favor of using more of these items in production drops. **4.** The less the cost of the input relative to **total costs,** the more inelastic the demand. (**EX:** *A large percentage change in the price of zippers, for example, say from 50¢ to $1.00, will have a small effect on the price of a $100 suit.*) **C. RENT. 1.** If the **supply curve** of a factor is completely inelastic, then output is fixed and the price of the factor is determined only by demand: *a fall in demand lowers the price, a rise in demand raises price.* **2.** Some economists in the 1890's thought that land had this kind of **supply schedule,** for property continued to be employed even if demand fell. One implication of this view was that a tax placed on land would not reduce the effective supply. More careful analysis indicates that neither the supply schedule of land or property are inelastic in the long run. **Crop support** programs bring new land into use; the **soil bank** programs retire land. Property deteriorates, neighborhoods change and buildings are torn down and rebuilt. **D. WAGES. 1. WAGE RATES,** like all factor prices, are determined by supply and demand. The demand for labor in America is high because the **marginal revenue product** of labor is high: each worker in his job has thousands of dollars of capital at his command. The supply of labor seeking employment depends upon **(a)** *population size*—greater immigration lowers wage rates because it shifts the supply schedule to the right—**(b)** *The proportion of the population in the labor force*—age and sex are the most important determinants of the proportion—and **(c)** *average hours worked.* Wage rates can rise if shorter hours are worked because the **MRP** of labor will be higher. Total earnings will fall if the percentage drop in hours is greater than the percentage change in earnings. **2. WAGE DIFFERENTIALS** can and do persist. These can be explained in terms of the **supply schedule of labor.** Unpleasant jobs require higher wage rates to attract people (**compensating differentials**). If laborers are not **mobile** between regions or skills, different wage rates will persist (**noncompeting group differentials**). **3. UNIONS** can try to raise wage rates by **(a)** shifting the **supply schedule** through restricting entry, **(b)** raise the **demand curve** for the final product by advertising the product as union made, and **(c)** impede the development of substitutes for labor. To the extent that unions can raise wages above the **equilibrium point,** employment is reduced in those trades and some means must be found for rationing the jobs among the workers that seek employment. One such rationing device is the **seniority system. 4.** Different unions have had different effects upon wage rates. There is evidence that some, like *Teamsters,* have raised wages above what they would have been in the absence of the union. Other unions, like the *Garment Workers,* do not appear to have had much effect upon wage rates. **E. INTEREST.** The **rate of interest** is determined by the intersection of the **investment schedule** and the **saving schedule.** The principal determinant of **I** is the **MPP** of capital; the determinant of **S** is income. From the point of view of individual firms, the prevailing interest rate determines which projects to adopt and which to reject, just as higher interest rates elicit more savings on the part of individuals. **F. PROFIT. Profit in the accounting sense is the residual that remains after all costs are met.** This sum may be positive or negative. **1.** In small firms where the owner and family are the laborers, owners of the building, chief supplier of capital, the "profit" for the year includes wages, rent, and interest. The resources employed in this firm could have been gainfully employed elsewhere. **2. Normal profit** is the rate of return that keeps the size of the industry stable; if profits are consistently higher, new firms are attracted, if lower, old firms leave. **3. Pure profits** arise out of uncertainty. The risks that give rise to pure profits cannot be insured against for they cannot be predicted. Pure profit is the difference between what factors expected and were willing to earn and what they did in fact receive.

**15 INTERNATIONAL TRADE. A. ABSOLUTE TRADE ADVANTAGE.** If *Country 1* can produce twice as many units of food with its resources as *Country 2* and the latter twice as many units of clothing as the former, then both countries will be better off if each specializes. **B. COMPARATIVE ADVANTAGE.** *Country 1* can produce food twice as efficiently and clothing 1½ times as efficiently as *Country 2.* Trade will still be beneficial to both countries if each specializes in its area. **C. FREE TRADE & PROTECTION.** A higher standard of living for all is the economic rational for specialization and trade. **1.** The more impediments there are to trade, the lower the standard of living for the groups involved. **Domestic producers** frequently find that the pressure of competition from abroad keeps prices lower than they would be in the absence of trade. **2.** Some of the arguments advanced by special interest groups as to why foreign competition is harmful even though consumers benefit are **(a)** foreign imports will lead to a loss of jobs for Americans. (*Not valid if full employment policies are maintained through proper monetary and fiscal policies.*) **(b)** Foreigners use cheap labor. (*American productivity is high; labor costs per unit of output are lower.*) **(c) Tariffs** should be just high enough to equalize the difference in costs and then competition will be fair. (*If prices were identical there usually would be no reason for any trade at all to ever take place.*) **(d)** Infant industries must be protected. (*Valid under certain limiting restrictions.*) **(e)** Domestic industries must be developed so that we are not dependent upon foreign supplies during time of war. (*A military argument, not economic, that questions the concept of having allies.*) **D. FOREIGN EXCHANGE RATES.** Each country has its own currency and purchases must be consumated in that currency. **1.** Americans buying British goods (importing) must pay in pounds just as Englishmen must pay in dollars for our exports. The only way foreign countries can earn American dollars necessary to buy goods from us is by selling us some of their exports. Similarly, the only way we can earn British pounds to buy goods from her is to sell England some of our exports. **2.** The supply of and demand for foreign exchange are also influenced by investments, gifts and **gold flows.**



DO NOT PLACE ON HOT SURFACES

KNOW THE FACTS ON THIS CHART AND YOU WILL GET HIGHER MARKS IN CLASSWORK, HOMEWORK, TESTS AND EXAMS.

THERE ARE QUICK CHARTS IN MATH • SCIENCE • ENGLISH • ECO • HISTORY • LANGUAGES • PSYCH • SOC • PHILO • MUSIC

DATA-GUIDE EDITOR: JOSEPH L. LEON

PUBLISHED & MFD BY DATA-GUIDE, INC. FLUSHING, N.Y. 11355

DATA-GUIDE QUICK CHART

# BASIC ENGLISH GRAMMAR & USAGE

## A SIMPLIFIED SUMMARY & INSTANT REFERENCE

LEARN & REVIEW IN MINUTES

**AUTHOR:** Prof. John Middendorf, Ph.D., Columbia University.

SOLID PLASTIC

SPEED STUDY TO SAVE YOU TIME AND WORK.

QUICK CHARTS PUT THE FACTS IN FULL VIEW:

PERMANENTLY FILL IN FOR MISSING NOTES & TEXTS.

QUICK CHARTS HELP YOU REMEMBER:

CONTAINS NO PAPER

### GLOSSARY OF TERMS NOT DEFINED IN TEXT

**ANTECEDENT.** The word or group of words to which a pronoun refers. (In "Tom lost *his* book." *Tom* is the antecedent of *his*.)

**APPOSITIVE.** A substantive set beside another substantive and signifying the same thing. (In "Our first president, *Washington*, lived in Virginia." *Washington* is in apposition with *president*.)

**COMPLEMENT.** A word or group of words used to complete the sense of a verb. The complement of a transitive verb is an *object* (He hit the *dog*.) The complement of a copulative verb may be a *predicate noun* (This is *Tom*.), a *predicate pronoun* (This is *he*.), or a *predicate adjective* (This is *beautiful*.).

**ELLIPSIS.** (Elliptical expression). An expression not grammatically complete but still clear and "correct" because the omitted words are easily understood. (While [*he was*] reading, he fell asleep.)

**GENDER.** The classification of substantives according to sex. The four genders are *masculine, feminine, neuter* and *common* (either masculine or feminine).

**IDIOM.** An expression which is in good use even though it sometimes violates grammar or logic or both (*How do you do? aim to prove at home*).

**NUMBER.** A change in the form of pronoun, noun, or verb to show *one* (singular) or *more than one* (plural).

**PERSON.** Changes in the form of pronouns and verbs to show whether the person is *speaking* (first person), is *spoken to* (second person), or is *spoken about* (third person).

**SUBJECT.** A word or group of words about which an assertion is made (The *car* hit the boy).

**SUBSTANTIVE.** A word or group of words used as a noun.

**VERBAL.** Verb forms which function as *nouns, adjectives,* or *adverbs*; which have some features of verbs (i.e., may take complements); but which cannot, together with the subject, stand alone as an independent clause. *Participles* function as *adjectives* (The *running* water); *gerunds* function as *nouns* (*Running* is fun); *infinitives* function as *nouns* (*To run* is fun), as *adjectives* (He had lots of money *to spend*), and as *adverbs* (It was too good *to last*).

## I. PARTS OF SPEECH

Any word falls into one of eight groups depending on its use in a sentence. Thus, words in sentences may **name** (nouns, pronouns), **assert** (verbs), **modify** (adjectives, adverbs), **join** (prepositions, conjunctions), or **express sudden emotion** (interjection). A change in function or use will change the part of speech of a word: He read the **book** (noun). The police **book** (verb) the thief. The **book** (adjective) value of the car is high.

### A. NAMING WORDS

**1. NOUNS.** A noun names a person, place, thing, quality, idea, or action.

*a.* ***Common nouns*** name members of a class of persons, places, or things (man, state, table).

*b.* ***Proper nouns*** name particular persons, places or things (John Adams, New York, Statue of Liberty).

*c.* ***Collective nouns*** name a group as a unit (army, crew, band).

*d.* ***Concrete nouns*** name things perceived by the senses (book, water, house).

*e.* ***Abstract nouns*** name ideas or qualities (love, strength, democracy). Nouns can be used as *subjects of verbs* (**Cain** killed Abel.); *objects of verbs* (The boy hit the **dog**.); *objects of verbals* (He decided to close the **show**.); and *objects of prepositions* (She ran into the **house**.). Nouns may also be used as *predicate nouns* (This is his **chance** to make good.); *appositives* (Jones, our new **teacher**, arrived late.).

**2. PRONOUNS.** A pronoun substitutes for a noun.

*a.* ***Personal pronouns*** refer to the speaker (first person: *I, we*), the person spoken to (second person: *you*), the person spoken of (third person: *he, she, it, they*) (see III:A; CASE).

*b.* ***Relative pronouns*** (**r. p.**) relate an adjective clause to its antecedent (*who, which, that*). **The girl** (antecedent) **who** (r.p.) sold the tickets is Mary Jones.

*c.* ***Interrogative pronouns*** are used to ask a question (*who, which, what*).

*d.* ***Demonstrative pronouns*** point out the person or thing referred to (*this, that, these, those*).

*e.* ***Indefinite pronouns*** imply, but do not have, a specific antecedent (*each, either, one, someone, anyone, nobody, everything, nothing*, etc.).

*f.* ***Reflexive and intensive pronouns*** combine some form of the personal pronoun with *self* or *selves* (*myself, yourselves*, etc.) and are used either to emphasize (He *himself* will go.) or to reflect back to the subject (He talks to *himself*.). Pronouns are used as *subjects of verbs* (**He** passed the course easily.); *objects of verbs* (Place **it** on the table.); *objects of verbals* (Finding **them** was sheer luck.); *predicate pronouns* (This is **she** speaking.); *appositives* (Brown, **he** with the top-hat, arrived late.).

### B. ASSERTING WORDS

**1. VERBS.** A verb expresses *action* (She *scolded* her son.) or *state of being* (He *was* in the room.).

*a.* ***A transitive verb*** needs a direct object to complete its meaning (The girl **cut** her *finger* (direct object).).

*b.* ***An intransitive verb*** does not need a direct object to complete its meaning (The boy *ran* down the stairs.). Frequently misused are the intransitive verbs *lie, sit, rise* and the transitive verbs *lay, set, raise*. (WRONG: The book *sets* on the table. RIGHT: The book *sits* on the table.).

*c.* ***Copulative verbs*** (chiefly: *be, become, appear, seem, taste, smell, sound, look, feel, grow, prove*) express the relationship between subject and complement (Her name *is* Mary. The price *seems* cheap.), but they do not assert *action* of the subject ("He *tasted* the roast.", *tasted* is a transitive verb.).

*d.* ***Auxiliary verbs*** "help" to form other verbs (He *should* go quickly. He *did* arrive on time.). Chief auxiliary verbs: *be, can, do, have, may, must, ought, shall, will, should, would*.

**2. PRINCIPAL PARTS.** The three parts of a verb are **present infinitive, past tense, past participle**, from which all tenses, moods, and voices are formed. Most verbs are *weak* or *regular* verbs whose past tense and past participle are formed by adding *-d, -ed*, or *-t* to the infinitive (walk, walked, walked; build, built, built).

**Strong or irregular verbs** do not form their principal parts systematically. *Bite, bit, bitten* (*bit*); *blow, blew, blown*, etc.

**3. TENSES.** A change in the verb form to indicate the time of the action or state of being.

*a.* ***Present tense*** indicates a present or habitual action or condition (He *is walking* home. She *works* carefully.).

*b.* ***Past tense*** indicates action or condition occurred or existed during some definite past time (He *was walking* home. She *saw* him yesterday.).

*c.* ***Future tense*** indicates action or condition will occur or exist in the future (She *will await* you. We *shall go* next week. NOTE: Distinctions in the use of *shall* and *will, should* and *would* are fast disappearing. *Will* and *would* are generally correct in informal usage. More careful usage demands **(1)** *Shall* (*should*) in the first person, *will* (*would*) in the second and third persons to express simple futurity (I [we] *shall* go. He [you] *will* come.). **(2)** *Will* (*would*) in the first person, *shall* (*should*) in the second and third person expresses determination, promise, intention, etc. (I [we] *will* speak. He [you] *shall* come.). **(3)** *In questions*, use the form expected in the answer (*Shall* you go? I *shall*. *Will* you go? I *will*.). Use *should* in all persons to express obligation (I [you, he, she, we, they] *should* fight for freedom.). **(4)** *The future* may also be expressed by the present tense plus an adverb (I am enrolling *tomorrow*.) or by an expression like "going to" (I *am going to* go on vacation next month.).

*d.* ***Present perfect tense.*** A combination of *has/have* with the past participle, indicates action or condition begun in the past, completed or still going on in the present (I *have been* ill for a week. It *has been snowing* for three hours.).

*e.* ***Past perfect tense.*** A combination of *had* and the past participle, indicates action or condition begun in the past, completed in the past *prior* to some past action (He was annoyed because he *had waited* for three hours.).

*f.* ***Future perfect tense.*** A combination of the future tense of *have* plus the past participle, indicates an action or condition that will be finished in the future before some future time (I *shall have arrived* by then.).

**4. TONE.** *a.* ***The progressive tone*** of a verb is formed by combining the proper tense of *to be* with the present participle (I *shall be seeing* you. He *was walking* quickly.).

*b.* ***Emphatic tone.*** Formed by combining *do, does*, or *did* with the present infinitive; used only in the present and past tenses (I *did wish* to go. She *does intend* to enter.).

**5. MOOD.** The form of the verb indicating the manner in which the action is thought of.

*a.* ***Indicative mood*** asks a question or states a fact (Are you going? I am going tomorrow.).

*b.* ***Subjunctive mood*** indicates a desire, a possibility, regret or condition contrary to fact (I wish he *were* here. If I *were* a member, I would certainly go.). The only common subjunctive forms of *to be* are the present tense *be* (I command that he *be* hanged.) and the past tense *were* (Suppose he *were* to ask you!); both forms are used for all persons, singular and plural. Other verbs use indicative mood forms except in the *third person singular present*, which merely drops the *-s* ending (*I come, you come, he come*). These latter forms are rarely used; instead, the auxiliary verbs (*should, would, can, could*, etc.) are used (RARE: If he *write* me, I'll go. COMMON: If he *should write* me, I will say good-bye.).

*c.* ***Imperative mood*** expresses a command (*Finish* your work, please.). The imperative forms are the same as the present infinitive without *to* (*go, come, walk, talk, sit*, etc.).

**6. VOICE.** A change in verb form to indicate whether the subject *acts* (active voice) or is *acted upon* (passive voice). ACTIVE: The girl *kissed* her mother. PASSIVE: The mother *was kissed* by the girl. Passive forms are combinations of a form of *to be* and the *past participle*.

### C. MODIFYING WORDS

**1. ADJECTIVES.** An adjective modifies (describes, limits, makes more exact) a noun or pronoun (*the green* tree *the sixth* form *an* apple *my* house *those* boys *which* girl.). Adjectives appear close to the words they modify (the *tired* children the children, *tired* and *hungry*).

*a.* ***Predicate adjectives*** modify the subject of the verb after copulative verbs. (The rose smelled *sweet*.). *Adjectives, not adverbs*, should be used to denote the *condition* of the subject or object of a sentence (He held the rope *tight*.). Do not misuse the following adjectives as adverbs: *sure, some, real, good*. (He *surely* [not *sure*] runs fast. The work becomes *somewhat* [not *some*] more difficult as we progress. I was *really* [not *real*] angry. He reads *well* [not *good*].).

WELL as an adjective means "in good health" (He is *well*.).

DUE TO, an adjective phrase, should not be used adverbially (His cold was *due to* a draft. NOT: He became cold *due to* a draft.).

**2. ADVERBS.** An adverb modifies or describes a verb (He sang *sweetly*.), an adjective (the *very* good man), or another adverb (He ran *very* quickly.).

*a.* ***Conjunctive adverbs*** connect independent clauses (q.v.) and thus help form compound sentences (q.v.) (He is ill; *nevertheless*, he'll be there.). **(1) The principal conjunctive adverbs are** *therefore, however, nevertheless, too, hence, then, besides, also, so, further, moreover, still, only, consequently, accordingly*, etc. **(2)** To distinguish conjunctive adverbs from subordinating conjunctions (q.v.), shift the position of the word in its sentence. If, after the shift, the sentence still makes sense, the word is a conjunctive adverb (He failed the course; **however** [conjunctive adverb], he was still as jolly as ever. He failed the course; he was still, *however*, as jolly as ever. NOTE: The following can't be shifted: He studied **until** [subordinate conjunction] he fell asleep.).

*b.* ***A modifier*** following a verb and its direct object is an adverb if it refers to the *manner* of action (He held the rope *tightly*.). Some words may be used as either adjectives or adverbs without changing form (*cheap, deep, far, quick, slow*, etc.).

**3. COMPARISON OF ADJECTIVES AND ADVERBS.**

*a.* ***Comparison*** refers to the change in form of an adjective or adverb to indicate degrees of superiority. If there is no comparison, the degree is *positive;* if two are compared, the degree is *comparative;* if three or more, *superlative*.

*b.* ***The comparative.*** **(1)** The comparative of one and some two-syllable adjectives and adverbs is formed by adding *-er* (the *taller* man he runs *faster*). **(2)** With words of two or more syllables, use *more* or *less* (the *more* [or *less*] *intelligent* boy He runs *more* [or *less*] *quickly*.). **(3)** Compare only what can logically be compared. WRONG: *His eye is keen as a hawk*. RIGHT: *His eye is as keen as a hawk's*. **(4)** Make comparisons clear. WRONG: *I love June as much as Jane*. RIGHT: *I love June as much as I love Jane*. OR: *I love June as much as Jane does*. **(5)** Use *other* or *else* when comparing things in the same class. WRONG: *He is smarter than any man*. RIGHT: *He is smarter than any other man*. **(6)** In the superlative, **do not** use *other* or *else*. WRONG: *He is the greatest of all other presidents*. RIGHT: *He is the greatest of all presidents*.

*c.* ***The superlative*** of one and some two syllable adjectives and adverbs is formed by adding *-est* (the *tallest* man He runs *fastest* of all.). With words of two or more syllables, use the words *most* or *least* (the *most* [or *least*] intelligent boy).

### D. JOINING WORDS

**1. PREPOSITIONS.** A preposition links (or shows the relation of) a noun or pronoun (the object of the preposition) to some other word in the sentence (He walked *into* the house.). Do not omit prepositions necessary to make meanings clear (*On* this side of the house is a tree. *At* eight o'clock I awoke. I had no hope *for*, or faith *in*, the World Congress.). Do not use prepositions that fail to clarify meaning (He entered [not *into*] the room. He left *about* [not *at about*] ten.). Learn the correct prepositions used in idiomatic combinations such as: agree to (a proposition), agree with (a person), agree on (a plan); consist in/of; differ about/from/in/on/with; etc.

**2. CONJUNCTIONS.** A conjunction connects words, phrases, or clauses in a sentence.

*a.* ***Coordinating conjunctions*** (*and, but, or, for, nor* sometimes *so* and *yet*) connect words, phrases, or clauses of equal rank (boys *and* girls across the river *and* into the trees I like her, *but* I'd never marry her.). Use coordinating conjunctions logically (WRONG: She has red hair, *but* she is beautiful. RIGHT: She has red hair, *and* she is beautiful.).

*b.* ***Subordinating conjunctions*** (*if, although, because, since, unless, after, when, while*, etc.) connect subordinate clauses with independent clauses (I shall wait *until* she comes.). Use subordinate conjunctions exactly (The reason is *that* [not *because*] he is ill.). Do not use *like* as a conjunction (He read it *as* [not *like*] you said.).

*c.* ***Correlative conjunctions*** are conjunctions used in pairs (*both . . . and, either . . . or, neither . . . nor, so . . . as, whether . . . or*, etc.).

### E. INTERJECTIONS.

An interjection expresses sudden, strong emotion. It is not closely connected to other elements in the sentence (*Oh*, you don't say so? *Hurrah!* let's go.).

## II. GROUPS OF WORDS

Groups of words can perform the functions of single words. Such groups can be classified as *clauses* (which contain a subject and a verb) or *phrases* (which do not). A clause or phrase may also be classified according to its function as *noun, adjective*, or *adverb*. A phrase may be classified according to its introductory word—either a *preposition* or a *verbal*.

### A. PHRASES

**1. PREPOSITIONAL PHRASES** consist of a preposition plus a noun or pronoun object (*into the house up the tree* that book *of theirs*) and function as adjectives (In "The boy *in the house*" the phrase modifies the noun *boy*.) or as adverbs (In "He walked *into the street*" the phrase modifies the verb *walked*.).

**2. VERBAL PHRASES** consist of a verbal (*participle, gerund*, or *infinitive*) plus a noun or pronoun object.

*a.* ***Participial (or Adjective) Phrases,*** consisting of a participle plus an object, function as adjectives. In "The player *hitting the first home-run* will be rewarded." the phrase modifies the noun *player*.

*b.* ***Gerund (or Noun) Phrases,*** consisting of a gerund plus an object, function as nouns. In "*Passing exams* is tough." the phrase is the subject of the verb *is*. In "He found *passing exams* tough work." the phrase is the object of the verb.

*c.* ***Infinitive (or Noun, Adjective, or Adverb) Phrases,*** consisting of an infinitive plus an object, function as nouns, adjectives, or adverbs. In "He likes *to read books*." the noun phrase is the object of the verb *likes*. In "*To write poems* is difficult." the noun phrase is the subject of the verb *is*. In "It is time *to put the plan into effect*." the adjective phrase modifies the noun *time*. In "He is sure *to want more cake*." the adverb phrase modifies the adjective *sure*.

### B. CLAUSES.

A clause is a group of words containing a subject and a verb and functioning either as a sentence or as a part of a sentence – that is, as a *noun*, an *adjective*, or an *adverb*.

**1. INDEPENDENT (or MAIN) CLAUSES.** A clause which, complete in itself, can stand as a single sentence—that is, a clause *not* introduced by a subordinate conjunction. In "He plans to enter law, but his brother wants him to be a doctor." each clause is independent; each could be punctuated as a separate simple sentence. Independent clauses *may be* introduced by a coordinating conjunction (And *then he told me to go*.).

**2. DEPENDENT (or SUBORDINATE) CLAUSES.** A clause which, not complete in itself, cannot stand as a simple sentence; depending upon an independent clause, it therefore appears as part of a sentence (We cannot start *unless we get money*.).

**a. A Noun Clause.** May serve as the subject of a sentence (In "*That he is intelligent* is clear." the clause is the subject of the verb *is*.), as the object of a verb (In "He wrote *that he would be home soon*." the clause is the object of the verb *wrote*.), as the object of a preposition (In "Begin your examination with *whatever is easiest*." the clause is the object of the preposition *with*.), as the object of a verbal (In "Hoping *that he will come* will never get him here." the clause is the object of the gerund *hoping*.), or as a predicate noun (The reason is *that she is ill*.). Most noun clauses begin with *that, what, whatever, whoever*, etc. Often the introductory word may be omitted (We thought [that] *you should go*.).

**b. An Adjective Clause** modifies a noun or pronoun (In "The boy *who sold the tickets* ran away with the money." the clause modifies the noun *boy*.). Most adjective clauses begin with a relative pronoun (*who, whose, whom, which, that*). Sometimes the clause is introduced by a preposition (The box *in which I carried the books* is very heavy.); sometimes the introductory relative pronoun is omitted (The book [that] *I enjoy most* is Robinson Crusoe.). Occasionally adjective clauses begin with *where* or *when* (In "This is the place *where he saw it*." the clause modifies the noun *place*.).

**c. An Adverb Clause** modifies a verb, an adjective, or another adverb (In "He is smarter *than you think he is*." the clause modifies the adjective *smarter*. In "You can work more quickly *than I can*." the clause modifies the adverb *quickly*.). Most adverb clauses can be recognized because they answer one of the following questions: WHEN? She sang *until she fainted*. WHERE? *Wherever I go* I think of you. WHY? I left *because I felt sick*. HOW? He walked *as if he were drunk*. HOW MUCH? Run *as fast as you can*.

**d. Condition Clause.** An adverb clause explaining the circumstances under which the meaning of the rest of the sentence would be true. Most condition clauses begin with *if* (*If you scream*, I'll run. *If you can't dance* I'll teach you.).

**e. Concession Clause.** Another kind of adverb clause usually beginning with *although* or *though* and stating an idea opposed—but not contradictory—to the idea of the main clause (In "*Although he's intelligent* he wastes his money." the idea of the concession clause is opposed to the implications of the main clause.).

### C. SENTENCES AND FRAGMENTS

**1. A SENTENCE** is a group of words containing a subject and a verb and expressing a complete thought. According to their grammatical structure, sentences may be classified as *simple, compound, complex*, or *compound-complex*.

**a. A Simple Sentence** contains only one main clause (*He read the book. The boys and girls laughed and played.*).

**b. A Compound Sentence** contains two or more main clauses but no subordinate clauses (*I read and she wrote letters. You may not pass, but you should at least try.*)

**c. A Complex Sentence** has one main clause and one or more subordinate clauses (*If the day is clear, we'll go.*).

**d. A Compound-Complex Sentence** has two or more main clauses and one or more subordinate clauses (*If the day is clear, we'll go and they'll stay home.*).

**2. FRAGMENTS.** A fragment is a group of words which is punctuated as a sentence but does not contain a main clause. Writer fails to see either proper relationship between sentences or fails to see the dependent nature of the fragment. Such fragments should be avoided. The following fragmentary constructions **are often incorrectly punctuated as sentences.**

**REMEMBER:** All four examples would be correct if written as single sentences.

**a. Prepositional phrases** (He's pretty shrewd. *Especially in a business deal.*).

**b. Verbal phrases** (I entered the room. *Knowing I'd find him there.*).

**c. Subordinate clauses** (The judge let her go. *Although she had killed three men.*).

**d. Appositives** (He is a snob. *The kind of person I hate.*). Some fragments lack subjects (*Was starting to go home*) or verbs (*The man working hard and getting nowhere*), or objects (*I believe that if teachers would just let me alone*). A very small number of fragments are acceptable in good writing: **Requests and commands** (*Please go. Don't do that!*) do not have stated subjects. **Exclamations** can stand alone (*Ouch! What a party!*).

Subjects and verbs are often omitted in answer to questions (*Yes. No.*) or in answer to rhetorical questions (What are the qualities of leadership? *Intelligence, courage, imagination.*). Avoid fragments in all other instances.

## III. PROPER USAGE

### A. CASE

Case refers to the form taken by a noun or pronoun to indicate its function in the sentence. Since nouns change only in the possessive case, pronouns are the chief source of trouble.

**1. THE NOMINATIVE CASE.** *Includes subjects of verbs and predicate nouns and pronouns* (*I* wrote the book. It is *I* [or *he, she, they*].). **NOTE:** Informal usage accepts "It is *me* (or *him, her*).

**2. THE OBJECTIVE CASE. (a)** *Includes objects of verbs* (He told *us*. He took *Tom* and *me*. *Whom* did you visit?) *or objects of prepositions* (All of *us* went.). **(b)** *The subject, object, or complement of an infinitive* is in the objective case (In "Whom did you think her to be?" *her* is the subject of *to be, whom* the complement. In "He worked to please her." *her* is the object of *to please*.). **(c)** *The case of a relative pronoun* is determined by its function in its own clause (In "Give the book to whoever wants it." *whoever* is the subject of *wants*, not the object of *to*. In "The man who he says is coming is Mr. Smith." *who* is the subject of *is coming*, not the object of *says*. In "The man whom I love is leaving me." *whom* is the object of *love*.). **(d)** *An appositive* is in the same case as its antecedent (We, *you* and *I*, must study. He hit us, *you* and *me*.). **(e)** *Adverbial clauses of comparison* are often elliptical; if such clauses (introduced by *than* or *as*) contain a pronoun, its case is whatever would be correct if the clause were complete (George gave Mary more than [he gave] *me*. He is as fast as *I* [am fast].).

**3. THE POSSESSIVE CASE.** A noun or pronoun preceding a gerund is usually in the possessive case (*His* singing annoys me. He objected to *John's* taking the job.). Occasionally, depending upon the emphasis in the sentence, the possessive is not used (In "Imagine John's taking me for a duke!" the emphasis is on the idea of *taking*. In "Imagine John taking me for a duke!" the emphasis is on John.). It is best not to use the possessive case with inanimate objects (WRONG: *the wall's color, the house's cellar* **RIGHT:** *the color of the wall, the cellar of the house*). Usage allows exceptions to this rule: *day's work, an hour's time, a dollar's worth, boat's length*, etc. Common pronouns whose form changes with case are as follows: **Nominative** (*I, you, he, she, it, we, they, who*), **Objective** (*me, you, him, her, it, us, them, whom*), **Possessive** (*my, your, his, her, its, our, their, whose*).

### B. PRONOUN AGREEMENT

All pronouns must agree with their antecedents in person, number, gender.

**1. SINGULAR PRONOUNS** refer to singular antecedents among which are included *each, anyone, anybody, someone, somebody*, etc. (*Each* of the boys did *his* work well.)

**2. TWO OR MORE ANTECEDENTS** joined by *either . . . or* or *neither . . . nor* are referred to by a singular pronoun if both antecedents are singular (Neither Joe nor Jim will wear *his* uniform.) or by a plural pronoun if both antecedents are plural (Neither the girls nor the boys wore *their* uniforms.). If one antecedent is singular and the other plural, the pronoun agrees with the nearer (Neither *Mr. Smith* nor his *workers* knew of *their* danger.).

**3. COLLECTIVE NOUNS.** Use either singular or plural pronouns with collective nouns, depending on the sense of the sentence (In "The crowd does its best" both the verb *does* and the pronoun *its* show that *crowd* is considered a unit. In "The family do their work well." the verb *do* and the pronoun *their* show that *family* is considered as individuals.).

**4. INDEFINITE PRONOUN.** When an indefinite pronoun (e.g. everybody) refers to men and women, avoid the awkward *his or her* and use the masculine form (Everybody took *his* hat.).

### C. PRONOUN REFERENCE

**1.** Make every pronoun refer unmistakably to a specific antecedent. **2.** Place all pronouns as close as possible to antecedents. **3.** Avoid vague reference to more than one antecedent. (VAGUE: George told John that *he* couldn't play in Saturday's game. **CLEAR:** *George said, "John, you [I] won't play in Saturday's game."*). **4.** Avoid reference to a subordinate construction (WRONG: In Galsworthy's play *he* deals with an explosive theme. **RIGHT:** *In his play, Galsworthy deals with an explosive theme.*). **5.** Avoid reference to an antecedent merely implied (In "I'm going to study law; they make good money" *they* refers to the implied antecedent *lawyers*. **CORRECT:** *I'm going to study law; lawyers make good money.*). **6.** Usually it is best to avoid using *this, which*, and *that* to refer to the general idea of a clause or sentence (In "He spent many hours at his easel, which helped him earn a living." *which* has no specific antecedent. **CORRECT:** *He added to his income by selling his paintings.*). **7.** Avoid indefinite use of *it, they, you* (In "*It* tells about the Korean War in this book." *it* has no antecedent. **CORRECT:** *This book tells about the Korean War.*" In "*They* all eat caviar in Russia." *they* has no antecedent. **CORRECT:** *All Russians eat caviar.* In "Fighting a war does take a lot out of *you*." *you* has no antecedent. **CORRECT:** *Fighting a war does take a lot out of a man.*). **NOTE:** Careful use of the indefinite *we* is allowable (*Whenever we read Shakespeare we're impressed by his genius.*). **8.** *One* (followed by *he*) is usually better than the indefinite *you* (*One* [not *you*] should always be brave.). **9.** The indefinite *it* is permissible in some idioms, such as *it seems, it is warm, it's raining*, etc.

### D. SUBJECT-VERB AGREEMENT

**1.** A verb must agree with its subject in number: a singular subject takes a singular verb (*He plays well.*); a plural subject takes a plural verb (*They play well.*). **2.** Do not confuse the subject with words that appear between subject and verb. (WRONG: The aim of all his efforts *were* to gain peace. **CORRECT:** The aim of all his efforts *was* to gain peace." Here the verb *was* correctly agrees with its subject *aim*, not with *efforts*.). **3.** Two or more subjects connected by *and* take a plural verb (Apples, pears, and oranges *are* good for one.). **4.** Indefinite pronouns are singular and must take singular verbs: (Nobody *dashes* down the street. No one *acts* better than she.) **5.** If two singular subjects are connected by *either . . . or* or *neither . . . nor* the verb is singular (Either Joe or Jim *is going* to go.); if *both subjects are plural*, the verb is plural (Neither the cities nor the towns *wish* to impose taxes.); if *one subject is singular, the other plural*, the verb agrees with the nearer (Neither Mary nor her friends *are planning* to go. Neither the girls nor Mary *is planning* to go.). **6.** The subjects of sentences beginning with *there is* [*are*] always *follow* the verb (There *are* twenty *people* in the room. There *is* an old *man*.). **7.** Some nouns, plural in form (physics, mathematics, mumps, news, politics, etc.), take singular verbs (The news *is* horrifying.). **8.** Learn to distinguish the singular and plural forms of certain words borrowed from Greek and Latin. (Singular: *memorandum, datum, thesis, alumnus*, etc. Plural: *memoranda, data, theses, alumni*, etc.). **9.** A collective noun takes a singular verb when the group is considered a unit (The crowd *was shouting*), a plural verb when the members of the group are thought of separately (The *crowd were tossing* their hats into the air.). **10.** A verb following a relative pronoun used as a subject is either plural or singular according to the antecedent of the pronoun (It is *I who am speaking*. It is *they who are speaking*.) **11.** A verb never agrees with a predicate noun (His chief goal *is* peace and prosperity. Peace and prosperity *are* his chief goal.). **12.** Words and expressions like *as well as, together with, accompanied by*, etc. do not change the number of the subject (Joe, together with his father, *is going* hunting.) **13.** Quantities and sums take a singular verb when they express a single idea. (*Thirty-six inches is one yard. Six and two makes eight. Ten dollars is too high a price.*). **14.** Fractions take singular verbs if followed by singular objects (Three-fifths of the *bottle is* empty.), plural verbs if followed by plural objects (Three-fifths of the *bottles are* empty.).

### E. POSITION OF MODIFIERS

**1.** Every modifying word, phrase, or clause should be clearly and logically connected with the word it modifies. **2.** Be careful of words such as *only, hardly, almost*, etc.; the position of such words can change the meaning of a sentence (*Only* he can row a boat. He *only* can row a boat.). **3.** Verbals, verbal phrases, prepositional phrases, and elliptical clauses, especially if they are introductory, should not be allowed to dangle; that is, they must logically agree with the words they modify. (In "*To read* well, good light is needed." the infinitive illogically modifies *light*. **CORRECT:** *To read well, one needs a good light.* In "*Approaching the mountains*, the trees were seen." the participial phrase illogically modifies *trees*. **CORRECT:** *Approaching the mountains, we saw the trees.* In "*After fixing the carburetor*, the motor will start again." the prepositional phrase illogically modifies *motor*. **CORRECT:** *After fixing the carburetor, we started the motor again.* In "*When four years old*, my grandfather took me to the circus." the elliptical clause illogically modifies *grandfather*. **CORRECT:** *My grandfather took me to the circus when I was four years old.*). **4.** Not only introductory phrases and clauses can be misplaced (WRONG: My uncle's first daughter was born *at the age of thirty-five*. **CORRECT:** *When my uncle was thirty-five his first daughter was born.* **5.** Avoid squinting modifiers; i.e., modifiers which may refer to either of two members of a sentence. (WRONG: The student who works hard, *in nine cases out of ten*, deserves praise. **CORRECT:** *In nine cases out of ten, the student who works hard deserves praise.* OR: *The student who works hard deserves praise in nine cases out of ten.*

### F. PARALLELISM

**1. TWO OR MORE COORDINATE IDEAS** should be expressed in parallel form; that is, one should avoid unnecessarily mixing clauses and phrases, infinitives and gerunds, etc.

**a. Parallel verbals:** He likes *walking* and *swimming*. He was ordered *to report* to the captain and *to hand* in his credentials.

**b. Parallel prepositional phrases:** The Marines are noted *for their courage* and *for their loyalty*.

**c. Parallel clauses:** Mary told him *that she doubted his motives, that she mistrusted his opinions*, and *that she disliked the color of his eyes*.

**2. CORRELATIVE CONJUNCTIONS** should be followed by parallel elements. (She was *neither a borrower nor a lender*.) **3.** Each element of a parallel construction should be properly connected to the rest of the sentence (WRONG: He believes and defends democracy. **RIGHT:** He believes *in* and defends democracy. WRONG: I have and always will be your friend. **RIGHT:** I *have been* and always *will be* your friend.). **4.** Do not use an *and which* or *and who* clause except after a preceding *which* or *who* clause. (WRONG: She is a woman of beauty and who is very rich. **RIGHT:** *She is a woman who is beautiful and who is rich.*).

### G. POINT OF VIEW

**1.** Avoid needless shifts in point of view. **2.** Avoid unnecessary shifting of subject or voice (WEAK: The children rushed [active voice] into the house and dinner was eaten [passive voice]. **BETTER:** *The children rushed* [active] *into the house and ate* [active] *dinner*.). **3.** Avoid unnecessary shifting of **(a) person** (WRONG: *One* should work hard if *you* expect to get ahead. **RIGHT:** *One* should work hard if *he* expects to get ahead.); **(b) number** (WRONG: Be sure to bring your *book* [singular]; you'll need *them* [plural] for the test. **RIGHT:** Be sure to bring your *book* [singular]; you'll need *it* [singular] for the test.); **(c) mood** (WRONG: First *mix* [imperative] the glue, and then you *should spread* [indicative] it. **RIGHT:** First *mix* [imperative] the glue and then *spread* [imperative] it.); **(d) tense** (WRONG: The bull *crashed* [past] through the fence and then *comes* [present] charging up to me. **RIGHT:** The bull *crashed* [past] . . . and then *came* [past] . . . .). **4.** Tenses should be used in logical sequence. Usually the verb in the main clause determines the verb-form of subordinate clauses, e.g. if the main clause verb is past, the subordinate clause is past (He *fainted* when he *saw* us yesterday.). **5.** When statements are permanently true, however, the present tense is preferred (Shakespeare *portrays* Hamlet very subtly.). Be careful to use the present tense in permanently true statements occurring in subordinate clauses which follow verbs in the past tense (He *said* that Shakespeare *is* our greatest dramatist.). **6.** Infinitives should be in the present tense unless they speak of action earlier than that of the main verb (WRONG: I intended to have gone. **RIGHT:** *I intended to go.*). **7.** Past participles state action earlier than that of main verb. **8.** Present participles express action parallel in time to that of the main verb (WRONG: He is old, *being born* in 1875. **RIGHT:** He is old, *having been born* in 1875.).

## IV. MISCELLANEOUS

### A. SPLIT CONSTRUCTION

**1.** Do not aimlessly split verb phrases (WEAK: She *has*, although it's hard to believe, *been buying* yachts at the rate of two a year. **BETTER:** Although it's hard to believe, she *has been buying* . . . . **2.** Avoid aimlessly separating **(a) subject and verb** (WEAK: *She*, despite my warnings to be careful, *fell* over the cliff. **BETTER:** Despite my warnings . . . , *she fell* . . . .); **(b) verb and object** (WEAK: He *grasped*, in a frenzy of anticipation, his lost *child*. **BETTER:** In a frenzy of . . . , he *grasped* . . . .); **(c) preposition and object** (WEAK: He came *to*, though I don't think he had an invitation, the *party*. **BETTER:** Though I don't think . . . , he came *to the party*.). **3.** Avoid split infinitives, unless the split construction is clear and natural (WEAK: She told us *to* no matter what happened *visit* her. **BETTER:** She told us *to visit* her no matter . . . .).

### B. DOUBLE NEGATIVES

**1.** Double negatives are considered to cancel each other ("It don't make no difference." means "It *makes* a difference.") **2.** In addition to *no* and *not*, words like *but, nor, only, hardly, except, scarcely, merely, just, almost, ever, quite*, and *nearly* are "negative." (WRONG: I can't hardly breathe. She isn't but sixteen years old. I haven't scarcely the time. **RIGHT:** *can hardly breathe. She is but sixteen. I scarcely have the time.*). Double negatives like *not unnecessary, not unavoidable*, etc. are acceptable.





PUBLISHED & MFD BY DATA-GUIDE, INC. FLUSHING, N.Y. 11355

DATA-GUIDE QUICK CHART

# BASIC PUNCTUATION

A SIMPLIFIED SUMMARY & INSTANT REFERENCE

LEARN & REVIEW IN MINUTES

AUTHOR: Professor Roger M. Jones, Northwestern U.

SOLID PLASTIC

SPEED STUDY TO SAVE YOU TIME AND WORK.

QUICK CHARTS PUT THE FACTS IN FULL VIEW:

PERMANENTLY FILL IN FOR MISSING NOTES & TEXTS.

QUICK CHARTS HELP YOU REMEMBER:

CONTAINS NO PAPER

## A. USE THE APOSTROPHE

### Possessives

**1** To form possessives. *boy's hat boys' hats bees' nest Philip's books Jones's coat Joneses' coats*

### Contractions

**2** To indicate omission of letters in a contraction and numbers in a date. *it's(it is) who's(who is) mf'g Class of '74*

### Plural of symbols

**3** To form with *s* the plural of letters, signs, symbols, numbers, and abbreviations. *Spell Spenser with two "s's." 8's in the '40's P.O.W.'s*

### Plural of words

**4** To form the plural of a word referred to without regard to its meaning. *There are four "and's" in your closing sentence.* **NOTE:** If there is a meaning connected to the word, use the standard plural forms. *There are five "fives" in twenty-five.*

## B. USE THE BRACKETS

### Clarifying insertions

**1** To enclose a clarifying insertion or correction by anyone other than the original writer. *It was this book ["The Literature of Business"] which he recommended.*

### Parenthetical elements

**2** To enclose a parenthetical element already within parentheses. *Several of those essays are included in Loomis and Clark's book. (See their "Modern English Readings" [5th ed.].)*

### Directions

**3** To enclose expressions that give directions at the beginning and end of chapters and articles. *[To be continued]*

### Corrections

**4** To correct a mistake. *Edmund Spencer [Spenser] was an English poet.*

### Phonetics

**5** To enclose phonetic transcription. *sobriquet [so'bri ka']*

## C. USE THE COLON

### Introducing statements

**1** After a complete statement introducing either a formal or a long quotation, statement, or question. *The President began his talk with these words: "It is a pleasure to be here tonight."* **NOTE:** The colon is not used before a long indirect quotation. *The President in his report said that the . . .*

### Introducing listings

**2** After a complete statement introducing a listing. Words such as *these, the following, as follows,* etc., are usually employed. *He visited the following cities: Paris, London, and Rome. The exercise is as follows: hop, skip, then turn and jump.* **NOTE:** Do not use the colon when a verb precedes the listing *(The supply box contains shoes, jackets, blankets, and gloves.)* or when another sentence follows the introductory statement *(You are required to take the following courses. You will find it advantageous to do so early in your career. French Algebra English History).*

### Introducing additional material

**3** After a word or phrase which serves as a means of introducing additional material. *Verbs: From Mr. Smith's report: For sale: One pair of binoculars*

### Explanations

**4** After a statement followed by a closely related explanation or illustration. *The newspaper tries to serve in a double capacity: the friend of all that is good and the foe of all that is evil.*

### Salutations

**5** After the salutation in a business letter. *Gentlemen: Dear Mr. Brown: Sir:*

### Divisions of plays

**6** Between the act and the scene of a play. *Enobarbus's famous speech is in "Antony and Cleopatra," II:2.*

### Divisions of the Bible

**7** Between the chapter and verse in Biblical passages. *John 3:16*

### Book titles, subtitles

**8** Between the title of a book and its subtitle. *"Dr. Quicksilver: The Life of Charles Lever"*

### References, footnotes

**9** Between the place of publication and the publisher's name in footnotes and bibliographical references. *Richards, I. A. "Principles of Literary Criticism" New York: Harcourt, 1964.* **NOTE:** The comma may be used after each unit in the reference. *Richards, I. A., "Principles of Literary Criticism," New York, Harcourt, 1964.*

### Hours, ratios

**10** Between hours; between the parts of numerical ratios. *4:30 p.m. a:b 3:2*

## D. USE THE COMMA

### Explanatory elements

**1** To set off explanatory words and phrases. *On May 16, Tuesday, the contract will be signed. Einstein, the physicist, was present. Mr. Taylor, who is our lawyer, arrived late.* **NOTE:** Do not separate closely connected (essential) expressions. (**TEST:** The meaning of the sentence is changed when an essential element is omitted.) *The physicist Einstein was present. The man who is our lawyer arrived late.*

### Parenthetical elements

**2** To set off words and phrases such as *first, also, well, furthermore, therefore, to begin with, in addition,* when they are used parenthetically. *To begin with, let us examine the facts. The situation, however, is still critical.* **NOTE:** Do not use a comma when the expression is essential to the meaning of the sentence. *We shall succeed however critical the situation. It is first necessary to examine the facts.*

### Introductory phrases

**3** After a long introductory adverbial phrase. *By the end of August or the first of September, we shall visit Paris.*

**4** To set off introductory and non-essential infinitive phrases. *To arrive on time, leave at six. The suit should have been cleaned, to look presentable.* **NOTE:** Do not set off essential elements. *It is an honor to be chosen chairman.*

**5** To set off introductory and non-essential participial phrases. *Seeing him fall, we ran to his side. Our team, playing skillfully, won the game.* **NOTE:** Do not set off essential elements. *Four of the players returning by car were in an accident.*

### Independent clauses , conjunctions

**6** Before coordinating conjunctions *(and, or, nor, but, for)* joining two independent clauses. *They left New York on Friday night, and they arrived here on Tuesday afternoon.* If the clauses are very short, omit the comma. *I'll scrape but you paint.* **NOTE:** Do not separate **(a)** compound predicates *(They left New York on Friday night and arrived on Tuesday afternoon.),* **(b)** compound subjects *(The shingles on the roof and the screens on the porch are to be repaired.),* **(c)** compound objects of prepositions *(I can put the peg in either the square hole or the round hole.).*

### Subordinate clauses

**7** To set off a subordinate clause that precedes the main clause. *Unless we raise additional capital, we cannot start.*

**8** To set off a subordinate clause that follows the main clause if it is not essential to the meaning of the sentence. *We shall sign the contract, even if it is only a formality.* **NOTE:** Do not separate essential clauses. *We cannot start unless we raise additional capital.*

### Contrasting elements

**9** Between interdependent contrasting expressions. *The sooner we leave, the sooner we shall return.*

### Series of items

**10** After each unit in a series of three or more items when the last member is preceded by a coordinate conjunction. In a series of two items, omit the commas. *For lunch we had ham and eggs, bread and butter, and milk. I told her that I'd arrive on Friday, talk to the group for an hour, but leave immediately thereafter. I laugh, I run, and I leap.* **NOTE:** When the coordinate conjunction is repeated with each unit, omit the comma. *We have a surplus of salt and sugar and tea.*

### Identical words

**11** Between two identical words. *"Whatever is, is right." — Pope*

### Adjectives

**12** Between coordinate adjectives modifying the same noun. (If *and* can be placed between the adjectives without changing the meaning of the sentence, they are coordinate.) *She replied in a soft, low voice.* **NOTE:** Do not separate noncoordinate adjectives. *He wore a brown corduroy shirt.*

### Coordinate phrases

**13** Between coordinate phrases modifying the same noun. *The working hours are shorter, though less convenient, than those in similar firms. He was as heavy as, though taller than, his brother.*

### Direct address

**14** To set off names and words in direct address. *I'm sorry, Mr. Smith, we're out of stock. Yes, sir, I'll do it.*

### Quotations

**15** To set off the explanatory part of a direct quotation. *"Will the campaign," he asked, "begin next week?" The note said, "Class is recessed."* **NOTE:** Indirect quotations require no commas. *The note said that class was recessed.*

### Omitted words

**16** To indicate the omission of an obvious word or phrase. *The first shot was good; the second, poor; the third, excellent. Some men are strong; most, weak.*

### Series of numbers

**17** Between two adjacent numbers and between a series of numbers to indicate that they are not consecutive. *In 1952, 169 people joined our firm. Read pp. 20, 23, 25-27, and 29.*

### Names and titles

**18** After a proper name when a title or its abbreviation follows. *J. Brown, D.D.S. Philip Simms, Professor of English Robert Holmes, Esquire Data-Guide, Inc.*

### With "of"

**19** Before *of* when given with a residence. *Mr. Harold Nevins, of Dallas, Texas.*

### Addresses and dates

**20** After each unit in a date or address. *On Tuesday, December 1, 1963, the lease expired. They will move to 45 Lake Drive, Newark, New Jersey, next month.*

### With parentheses

**21** After the closing parenthesis if the sentence structure, such as separating elements in a series, requires it. *Perhaps we should decide to serve cakes, ice cream (preferably brick), and coffee.*

## E. USE THE DASH

### Summarizing statements

**1** Before a summarizing statement beginning with words such as *these* or *all*. *Newark, Camden, and Trenton—these are the cities he will visit in New Jersey. Friends, neighbors, relatives—all attended the homecoming.*

### Repeated elements

**2** To emphasize repeated expressions. *He is the man—the man the people want.*

### Introductory elements

**3** After an introductory word or phrase implied before each of the listed items. *Use your Data-Guide—to study, to review, to memorize.*

### Defining elements

**4** Before a defining complementary comment. *He rightly believes that seldom are the two faculties found in any one person—the creative and the critical.*

### Appositive elements

**5** To emphasize appositive expressions in a sentence that has other internal punctuation. *"The English Constitution—that indescribable entity—is a living thing, growing with the growth of men. . . ."* —*Strachey*

### Authors and titles

**6** Before the name of an author or title of a book that has been directly quoted (see 5).

### Changes in thought

**7** To indicate an abrupt change in thought or in sentence structure. *He told me—but perhaps I shouldn't report it so soon—that everything may be put up at auction. If you should change your mind—but don't!—let me know at once.*

### With "to"

**8** In place of *to* between dates, numbers, and compound proper names. *He lived in New York, 1950—1951. Philip Roe, Author, 1899—Read pages 25—44. The New York—Boston express is on Track 7.*

### Omitted letters

**9** To indicate missing letters in confidential correspondence. *This message is for Mr. B——.*

## F. USE THE EXCLAMATION POINT

### Strong emotion

**1** To indicate strong emotion or great surprise. *"Never!" he shouted. What! You were the one!*

### Amusement

**2** To indicate sarcasm or amusement. *Look at the big he-man! That's a good one!*

### Commands

**3** To indicate a command or emphasize a decisive point of view. *Return to your room at once! The matter is closed!*

### Irony

**4** In parentheses, after a word or phrase, to indicate an ironical remark. *This is an example of your best(!) writing.*

## G. USE THE PARENTHESIS

### Explanatory elements

**1** Around explanatory, non-essential material used within a sentence. *The exchange student is from Thailand (formerly Siam). The number of applications was 34 (or was it 37?).*

### Clarifying insertions

**2** To enclose references, directions, and sources of information. *The quotation (page 64) is from Thoreau's "Walden." Employment in industry has risen by 15% (U.S. Dept. of Labor).*

### Subdivisions

**3** Around numbers or letters indicating listings or divisions within a sentence. *He offered three reasons for leaving the city: (a) his chance to earn a higher salary, (b) his preference for a warm climate, (c) his desire to own a home.*

### Translations

**4** To enclose a translation of a foreign word, phrase, or statement. *nec amor nec tussis celatur (neither love nor a cough can be hidden)*

### Repeated Figures

**5** Around figures repeated to insure accuracy. *The money order is for thirty-five dollars ($35).*

## H. USE THE PERIOD

### Sentences

**1** At the end of all declarative and imperative sentences. *There will be a sale of dresses today. Come as soon as you can.*

### Outlines

**2** After a number or letter indicating division in an outline.

*A. Traffic control*
*1. Roads*

**NOTE:** Omit the period when the letter or number is enclosed in parentheses. Omit the period after Roman numerals when they are not used in an outline. *Volume II Edward IV*

### Incomplete sentences

**3** To indicate omitted matter. When the omitted matter is within the sentence, use three periods; when it is at the end of the sentence, use four periods. *". . . they contain examples for the student of creative writing . . . a beautifully bound book which lovers of splendid editions will want to have. . . ."*

### Headings and titles

**4** After a heading or title only when it is on the same line as the subject matter. *Traffic Control. This problem has long been a . . . .*

### Abbreviations

**5** With initials and abbreviations. *Ph.D. C.P.A. p.m. Y.M.C.A. N.A.M. C.O.D. f.o.b. ft. Sept.* **NOTE:** Omit the period with technical symbols (*log NaCl*), governmental departments (*FHA FCC*), labor unions (*CIO*), contractions (*sec'y mf'g*), money in dollar denominations (*$20*), abbreviated forms of words in general use (*memo*), ordinals (*5th*), radio stations (*WKXB*).

## I. USE THE QUESTION MARK

### Questions

**1** After every direct question. *How many are coming tonight?*

**2** After a short direct question following a statement. *You will do it, won't you?* **NOTE:** The question mark is not used after the following: an indirect question (*He asked what the message was.*), a polite request (*Will you ask Miss Kent to enter.*), a rhetorical question (one not requiring an answer) (*But that's rather obvious, isn't it.*).

### Doubt

**3** Within parentheses after a word, phrase, or date for indicating doubt of its accuracy or for indicating irony.

*Geoffrey Chaucer, 1340(?)—1400*
*She told us she had bought a valuable(?) ring.*

## J. USE THE QUOTATION MARKS

### Quotations

**1** To enclose a direct quotation. *"Your appointment is for Monday," she told me. "How are you?" he said.* **NOTE:** Do not use quotes with indirect quotations (*He said he felt fine and would return to work soon.*) and quoted thoughts (*I reminded myself: This letter must be mailed today.*).

### Partial quotations

**2** When the exact words of the speaker are repeated in a sentence. *He remarked that he would attend the meeting only if we "get rid of the time-wasters."*

### Quoted poetry

**3** At the beginning of each stanza and at the end of the final stanza when quoting poetry.

### Quoted paragraphs

**4** At the beginning and end of a quoted paragraph or group of sentences. When the quotation totals several paragraphs, use quotes only at the beginning of each paragraph and at the end of the final one.

### Quoted quotations

**5** For a quotation within a quotation (single quotes). *"I wish I could remember," he added, "whoever it was who said 'Practice makes perfect.'"*

### Names and titles

**6** To indicate the names of ships, trains, etc., the titles of publications, articles, stories, poems, musical pieces, works of art, plays, and the specific titles of divisions in books. *U.S.S. "Wasp" the "Pacemaker" The article appeared in "The Saturday Review." He read Frost's "Mending Wall." They played the "New World Symphony." Study all of Chapter IV, "The Art of Living."* **NOTE:** Divisions in books, reports, etc., referred to without specific titles, are capitalized only. *Refer to the Appendix.*

### Slang words

**7** Around slang words and provincialisms. *There "aint" much sense to that. He went to Hot Springs for the "cure."*

### Technical words

**8** Around technical words used in non-technical material. *He began working with "Craftint."*

### Defined words

**9** Around a defined word or phrase. *"Flush" in playing cards means a hand all of one suit; "flush" in printing indicates no indentation.*

### Emphasized words

**10** Around a word referred to without regard to its meaning. *"And" is used too often in your letter.*

**11** Around an emphasized word or phrase. *Did he mean to say "possibly" or "probably"? Please give me the "Handle with care!" parcels.*

### With identifying words

**12** Around expressions that follow words such as *signed, marked, stamped, known as,* etc. *He received a letter that was signed "An Admirer."*

**NOTE:** Place the colon and the semicolon outside the quotation marks; the comma and the period, inside. Place the question mark and the exclamation point inside the quotes when they relate to the quoted extract; if they relate to the entire sentence, place them outside the quotation marks.

## K. USE THE SEMICOLON

### Independent clauses, conjunctions

**1** Between independent clauses not joined by a coordinating conjunction. *Mr. Young can't give us an answer today; he hopes to tell us by the weekend.*

**2** Before a coordinating conjunction joining two independent clauses when one clause (or both clauses) has internal punctuation. *Considered individually, these savings represent only a small amount of money; but taken as a whole for an entire month, they represent several hundred dollars.*

### Series of clauses

**3** After each clause in a series of three or more independent clauses when they are long or when one (or more) has internal punctuation. *"The clock has just struck two; the expiring taper rises and sinks in the socket; the watchman forgets his hour in slumber; the laborious and happy are at rest; and nothing wakes but meditation, guilt, revelry, and despair."* —*Goldsmith*

### Conjunctive adverbs

**4** Before conjunctive adverbs such as *therefore, however, nevertheless,* etc., connecting two independent clauses. *The company is expanding its facilities; consequently, the price of its items will be reduced. Your request will have to be referred to the manager; hence no decision has been reached.*

### Listings

**5** Between elements in a listing when there are commas within the elements. *The presidents of the three competing companies are Rogers, of American Brass; Bart, of United Products; and Simms, of Acme.*

### Explanatory elements

**6** Before expressions such as *namely, for example,* and *for instance* when they precede a listing. *Three of the members were appointed to serve as delegates; namely, Mr. Watts, Mr. Roberts, and Mr. Lewis.*

### References

**7** Between separate references to several parts of the same work. *"Hamlet" I:1,2; IV:3,5*

DATA-GUIDE QUICK CHART

# BASIC WRITING GUIDE

## A SIMPLIFIED SUMMARY & INSTANT REFERENCE

LEARN & REVIEW IN MINUTES

AUTHOR: Professor Roger M. Jones, Northwestern U.

SOLID PLASTIC

SPEED STUDY TO SAVE YOU TIME AND WORK.

QUICK CHARTS PUT THE FACTS IN FULL VIEW:

PERMANENTLY FILL IN FOR MISSING NOTES & TEXTS.

QUICK CHARTS HELP YOU REMEMBER:

CONTAINS NO PAPER

## A. DIVIDING WORDS

**1. DIVIDE WORDS BETWEEN SYLLABLES ONLY.**

**2. WORDS OF ONE SYLLABLE** must not be divided. *search searched*

**3. WORDS OF FIVE OR FEWER LETTERS** must not be divided even when more than one syllable. *idea* **NOTE:** One letter must not precede or follow a hyphen. *abroad*

**4. A VOWEL FORMING A SEPARATE SYLLABLE** must be written before the hyphen. *sepa-rate*

**5. TWO ONE-VOWEL SYLLABLES** that come together may be divided. *medi-ator fluctu-ation*

**6. TWO-LETTER SYLLABLES** must not be carried on to the following line. *like-ly taller*

**7. TWO-LETTER PREFIXES** may be separated from the root. *re-flect*

**8. TWO-LETTER SYLLABLES THAT ARE NOT PREFIXES** must not be separated from the rest of the word. *select*

**9. DOUBLE CONSONANTS ENDING ROOT WORDS** may be divided after the second consonant. *spell-ing*

**10. DOUBLE CONSONANTS NOT ENDING ROOT WORDS** may be divided between consonants. *permit-ted*

**11. PREFIX OF THREE OR MORE LETTERS** may be separated from the letters that follow. *trans-continental*

**12. SUFFIX OF THREE OR MORE LETTERS** may be separated from the preceding letters. *manage-able deduct-ible multi-tude* **NOTE:** Divide after an *a* or *i* that begins the suffix when it is sounded with the preceding syllable. *capa-ble intangi-ble*

**13. DO NOT DIVIDE** proper names, titles with proper names, abbreviations, contractions, and numbers.

## B. CAPITALIZING

**1. CAPITALIZE THE FIRST WORD OF—**

***Sentences***

(a) Every sentence and groups of words used as a sentence. *Tomorrow is another day. No need to go.*

***Quotations***

(b) A direct quotation. *Mr. Hayes replied, "Certainly I will do it," and then sat down.*

***Poetry***

(c) Each line of poetry.

*"Men must be taught as if you taught them not,*
*And things unknown propos'd as things forgot."*
*—Pope*

***Statements following colon***

(d) A long formal statement following a colon. *My suggestion is this: We should appoint a five-member committee which will have the power to. . . .*

***Listings***

(e) Each part of a listing when complete sentences follow a formal introduction. *He offered three reasons for leaving New York: (1) He had been offered a better job. (2) He could earn a higher salary. (3) He liked the country.*

***Outlines***

(f) Each separate item in an outline.

***Questions***

(g) An independent question within a sentence. *These figures (Can you read them?) require close study.* **NOTE:** If the question is more closely connected to the main thought, use the dash and do not capitalize.

*"She had*
*A heart—how shall I say?—too soon made glad,"*
*—Browning*

***Quoted thoughts***

(h) A quoted thought (no quotation marks). *I reminded myself: This letter must be mailed today.*

***Complimentary close***

(i) The complimentary close. *Very truly yours,*

**2. CAPITALIZE THE FIRST WORD AND MAIN WORDS OF—**

***Salutation***

(a) The salutation. *Dear Mr. Smith:*
*My dear Mr. Smith: Sir: Dear Doctor Smith:*

***Subtitles***

(b) A subtitle. *"G. B. Shaw: His Plays"*

***Titles of works***

(c) Titles of books, newspapers, magazines, musical pieces, pictures, etc. *"The Robe" "The Fifth Symphony"* **NOTE:** Capitalize a definite or an indefinite article only when it is the first word of the title. *He enjoys reading "The Saturday Review." This is an editorial from the New York "Times."*

***"Resolved," "Whereas"***

**3.** Every letter in the words *resolved* and *whereas* when used formally and the first word of the statement or resolution. *RESOLVED: That on this day . . .*

***Abbreviations***

**4.** Abbreviations of academic degrees, radio stations, telephone exchanges, divisions of government, etc. *B.A. WXKB ME 1-1111 FHA USN*

***Divisions in sequence***

**5.** A noun or an abbreviation of a noun followed by a number indicating place, position, or major division in a sequence. *Act II Vol. III Room 42 Track 5 Table 3* **NOTE:** Do not capitalize minor subdivisions such as page, line, or paragraph. *see page 6, line 41*

***Abstract nouns***

**6.** Abstract nouns when in formal writing they refer to ideals. *". . . the knowledge of Good and Evil (or moral Good and Evil which are not natural Good and Bad or puritan Right and Wrong)."—Eliot*

***Substitute nouns***

**7.** Common nouns or adjectives when they are used in place of, or in reference to, a specific person, place, or thing. *Traffic was heavy on the Bridge. (Golden Gate) Several new men joined the Company. (Data-Guide, Inc.)*

***Personified nouns***

**8.** Personified nouns. *"Where wasteful Time debateth with Decay,"—Shakespeare*
*The Red and Green won the game.*

***Derived names***

**9.** Names of particular persons, places, objects and adjectives derived from them. *George Washington Washingtonian Washington pie*

***Brand names***

**10.** Brand names and commercial products. *Jell-O*

***Geographical names***

**11.** Geographical names. *the Everglades*
*the Pacific Ocean Red Sea Hudson Bay*

***Religious names***

**12.** Religious names. *God the Bible Genesis*
**NOTE:** Capitalize the personal pronouns *He, His, Him, Thee, Thine, Thou* but not the relative pronouns *who, whose, whom, which, that.*

***Academic titles***

**13.** Academic degrees and departments, honors, fellowships, chairs, and officers. *Doctor of Philosophy*
*the Bursar Department of English*

***Streets, buildings, etc.***

**14.** Avenues, streets, parks, squares, buildings, monuments, etc. *Fifth Avenue Gramercy Park*
*Lincoln Memorial the R. C. A. Building*

***Countries, cities, etc.***

**15.** Countries, states, cities, etc., and their commonly known parts. *the United States of America*
*Houston, Texas the Hub the Left Bank*

***Academic courses***

**16.** Courses of study in education but only the proper nouns or adjectives in the name of a subject. *Elementary Physics II He studied physics.*
*Basic European History American history*

***Governmental divisions***

**17.** Courts, bureaus, agencies, boards, commissions, political divisions, etc. *Probate Court Welfare Bureau Board of Education Seventh Congressional District Ways and Means Committee*

***Days, holidays, etc.***

**18.** Days of the week, months, holidays, and special days and weeks. *the first Friday in May*
*Mother's Day Fire Prevention Week Labor Day*

***Historical events, eras***

**19.** Historical events, eras, periods, etc.
*Battle of the Bulge Victorian Era the Renaissance*

***Laws, documents***

**20.** Laws, documents, bills, etc. *Fulbright Act G. I. Bill of Rights Marshall Plan the Constitution*

***Armed forces***

**21.** Military services, their branches and divisions. *Marines Second Division Squadron A*

***Organizations***

**22.** Organizations and institutions (business, fraternal, social, educational, political, professional, religious, etc.). *Masons Conservative Party Writers Guild*
*Boy Scouts Presbyterian*

***Peoples, languages***

**23.** Peoples, languages, races, tribes, and people identified with definite areas. *the Arabs South American*
*Frenchman Mongoloid Cherokee Northerner*

***Compass points***

**24.** Points of the compass only when they refer to geographical sections, not directions.
*I am going South this winter. the Far East*
*France is south of England.*

***Seasons***

**25.** The seasons only when personified. *It happened last summer. Come, sweet Summer.*

***Titles of people***

**26.** Titles of position and honor when placed immediately before the proper name. *Mayor Smith Senator Johnson the Duke of Edinburgh Chairman of the Board J. P. Wright.* **NOTE:** Only academic titles and titles of high governmental officials following the proper name are capitalized. *John Field, Secretary of State T. I. Spaak, Professor of Latin (or: professor of Latin) J. P. Wright, chairman of the Board*

***Substitute titles***

**27.** Titles used in place of a specific person but not when used independently. *The Governor will speak tonight. The governor is elected for a term of two years. He asked Father for the car.* **NOTE:** Do not capitalize titles referring to specific persons when they are preceded by a possessive pronoun. *He asked my father for the car.*

***Descriptive titles***

**28.** Descriptive titles (epithets) with or without the proper name. *the Lone Star State*
*William the Conqueror Old Hickory*

***Prefixes "vice-," "ex-," etc.***

**29.** The prefix *vice* when the title following begins with a capital letter. *Within the next week we shall know who will succeed Vice-President Robbins.* **NOTE:** Do not capitalize *ex-, former, late, -elect* with titles. *Senator-elect Smith former President Truman*

***The word "the"***

**30.** The word *the* only when it is a definite part of the name or title. *The First National Bank the Great Lakes* **NOTE:** Do not capitalize articles, conjunctions, and short prepositions in titles and names. *A Trip to the Moon William of Orange*

***"city," "state"***

**31.** The words *city* and *state* only when they follow the name or are used in place of specific names. *Garden City the city of New York He is employed by the City. (Chicago)*

***"government," "nation," etc.***

**32.** The words *government, nation, administration, federal, union, commonwealth,* etc., only when they refer to a specific country or political group. *Her Majesty's Government The Administration will conduct the probe.* **NOTE:** *Fight for good government.*

***"school," "college," etc.***

**33.** The words *school, university, academy, college,* etc., only when part of a title. *The School of Mines Lang College I attend college. Central High School He is to be graduated from high school.*

## C. WRITING POSSESSIVES

**1. ABBREVIATIONS.** Place *'s* (singular) or *'* (plural) after the period. *Data-Guide, Inc.'s publications Stephen Bros.' annual sale*

**2. APPOSITION.** Add the possessive to the noun in apposition only. *Here is Mr. Snow, the manager's office. We visited Mr. and Mrs. Hart, our employers' new home.*

**3. COMPOUND NOUNS.** Place the possessive at the end of the compound. *his son-in-law's work someone else's turn*

**4. INANIMATE OBJECTS.** Use an "of" phrase rather than a possessive. *the jacket of the book* (NOT: *the book's jacket*) **NOTE:** Use the possessive with several everyday references to time and measurement and with personifications. *a week's vacation*
*their money's worth two years' experience*
*for conscience' sake the semester's reading list*

**5. GERUNDS** (verbal nouns). Use the possessive form of a noun or pronoun before a gerund. *The time of the manager's leaving is indefinite.*

**6. INDEFINITE PRONOUNS.** Add the possessive to the end of an indefinite pronoun. *one's decision*
*the others' decisions*

**7. IMPLIED NOUNS.** Use the possessive before an implied noun. *John's is the most interesting report.*

**8. OWNERSHIP (*Combined*).** Add the possessive to the end of two or more nouns indicating combined ownership. *Bob and Mary's log cabin*
*Brooks, Carll & Norton's store*

**9. OWNERSHIP (*Separate, by two or more people*).** Add the possessive to each of the nouns.
*the secretary's and the treasurer's reports*

**10. PERSONAL PRONOUNS.** Do not use the possessive mark with personal pronouns in the possessive case or with the possessive form of "who." *yours hers its ours theirs whose*

**11. PROPER NOUNS.** (a) Add *'s* to singular proper nouns not ending in *s*. *Mr. Brown's book* (b) To proper nouns of one syllable ending in *s*, add *'s* (*Mr. Ross's trip*) and to those of two or more syllables, just *'* (*Mr. Nevins' hat the Williamses' party*).

**12. SINGULAR NOUNS.** (a) Add *'s* to a singular noun not ending in *s*. *man's life* (b) Add *'s* to a singular noun ending in *s* or an *s*-sound whenever a new syllable is created by pronouncing the possessive. *the boss's plan the actress's costume* (c) Add *'* to avoid a repetition of *s*-sound. *for goodness' sake*

**13. PLURAL NOUNS.** (a) Add *'* to plural nouns ending in *s*. *lawyers' briefs* (b) Add *'s* to other plural noun endings. *women's hats children's toys*

**14. OFFICIAL NAMES** of companies and organizations. (a) Omit the apostrophe in plural proper nouns that are possessive. *Manufacturers Trust Bankers Association* (b) Use the apostrophe in singular proper nouns that are possessive. *Woman's Day Collier's*

## D. WRITING THE PLURAL NOUN

**1. SINGULAR NOUNS** usually add *s* to form the plural. *alibi alibis book books letter letters*

**2.** Singular nouns with certain endings form the plural as follows: (a) ***f, fe, or ff*** usually add *s*. *belief beliefs safe safes cliff cliffs* **NOTE:** Some nouns ending in *f* or *fe* form the plural by changing to *v* and adding *es*. *half halves life lives shelf shelves* (b) ***o preceded by a vowel*** (a, e, i, o, u) add *s*. *bamboo bamboos duo duos ratio ratios* (c) ***o preceded by a consonant*** usually add *es*. *hero heroes potato potatoes veto vetoes* **NOTE:** ***Musical terms ending in o*** add *s* (*piano pianos solo solos alto altos*). ***Some nouns ending in o preceded by a consonant*** add *s* (*tobacco tobaccos photo photos*); other nouns may add *e* or *es* though *es* is the preferred form (*cargo cargoes cargos buffalo buffaloes buffalos*). (d) ***s, x, ch, sh, add es.*** *glass glasses box boxes birch birches dish dishes* **NOTE:** ***When the final s is silent,*** the singular and plural are identical in spelling. *a corps several corps* (e) ***y preceded by a vowel*** add *s*. *donkey donkeys money moneys* **NOTE:** The irregular plural "monies" is correct in the sense of sums of money.

**(f) *y preceded by a consonant*** change *y* to *i* and add *es*. *company companies ecstasy ecstasies* **NOTE:** Since *u* sometimes serves as the consonant *w*, the rule applies to nouns ending in *quy*. *colloquy colloquies*

**3. PROPER NOUNS ENDING IN S, X, Z, CH, SH** add *es*. *Hayes (all the) Hayeses Murch Murches*

**4. PROPER NOUNS ENDING IN Y PRECEDED BY A VOWEL OR CONSONANT** add *s*. *Shirley (all the) Shirleys Whitney Whitneys Kennedy Kennedys*

**5. IRREGULAR PLURALS** are usually formed by a vowel change inside the word. *man men mouse mice foot feet* **NOTE:** Certain irregular plurals are formed by adding *en*. *ox oxen child children*

**6. HYPHENATED COMPOUNDS** add the proper plural form to the main part of the word. *son-in-law sons-in-law Secretary-General Secretaries-General* **NOTE:** The plural form is added to the end of a solid compound noun. *glassful glassfuls*

**7.** A hyphenated compound formed of a noun and a preposition adds the plural ending to the noun. *passer-by passers-by runner-up runners-up*

**8.** A hyphenated compound adds the plural form to the final word when neither part of the compound is a noun. *lean-to lean-tos run-on run-ons fill-in fill-ins*

**9. ABBREVIATIONS, FIGURES, LETTERS, SIGNS, AND WORDS OUT OF CONTEXT** form their plurals by adding *s*. *chap. chaps. bx. bxs.* **NOTE:** Sometimes *'s* is added though there is a growing tendency to omit the apostrophe (*s's the three R's G I's*). The singular and plural abbreviations may be identical (*mo. = month and months; in. = inch and inches*). Some one-letter abbreviations form the plural by doubling the letter (*p. page pp. pages*).

**10. EXCEPTIONS.** Certain nouns ending in *s* are singular only. *economics ethics mathematics news*
Certain nouns are plural only. *auspices goods headquarters proceeds scissors*
Certain nouns may be both singular and plural. *deer politics sheep trout wheat*
Certain nouns have two plurals with different meanings. *clothes cloths fish fishes*

**11. COLLECTIVE NOUNS** are singular when the group is considered as a body but plural when the members are thought of individually. *The staff is asked to consider the suggestion. The staff are leaving.*

**12. FOREIGN PLURALS. NOTE:** Some foreign nouns have English plural forms; these should be used in preference to the foreign plurals. Foreign singular nouns form the plural by making the following changes or additions. **(a) *a to ae*** *alumna alumnae formula formulae* (Eng. plural: formulas) **(b) *eau plus x*** *bureau bureaux* (Eng. plural: bureaus) *chateau chateaux* **(c) *is to es*** *basis bases crisis crises thesis theses* **(d) *on to a*** *criterion criteria* (Eng. plural: criterions) *phenomenon phenomena* **(e) *um to a*** *bacterium bacteria datum data memorandum memoranda* (Eng. plural: memorandums) **(f) *us to i*** *alumnus alumni syllabus syllabi terminus termini* (Eng. plural: terminuses) **(g) *x to ces*** *appendix appendices* (Eng. plural: appendixes)

**13. PLURAL OF TITLES.** *Mr. Chase Messrs. Chase Dr. Rand Drs. Rand Mrs. Grant Mrs. Grants Miss Grant Misses Grant* (or) *Miss Grants Mesdames Bliss, Hunt, and Lary Messrs. Gray and Hill*

## E. WRITING NUMBERS

**1. AT THE BEGINNING OF A SENTENCE.** Spell out a number beginning a sentence even though figures may be required later in the sentence. *Two of the newspapers are raising their price from 5 to 7 cents.* **NOTE:** Reconstruct the sentence when the spelled-out number totals three or more words. *There were 187 orders filled.* NOT: *One hundred eighty-seven*

**2. AGES.** Use figures only when age is stated in years, months, and days. *The baby is 2 years 8 months 20 days old. Her mother is twenty-four years old. They live in a one-hundred-fifty-year-old house.*

**3. BALLOTING.** Use figures. *73 for; 42 against*

**4. COMPOUND ADJECTIVE.** Use a hyphen to join a compound adjective having a figure. *There is a 10-minute break between periods.* **NOTE:** An "of" phrase makes a hyphen unnecessary. *There is a break of 10 minutes between periods.*

**5. DATES. (a)** Spell out numbers and decades and centuries. *the twentieth century the roaring twenties* **(b)** Spell out dates in legal documents. *the sixteenth day of June, one thousand nine hundred and fifty-three* **(c)** Use figures in letters. *On October 7, 1984, the project will begin. We thank you for your order of June 19.* **NOTE:** Omit *th, st, rd, nd* after the day of the month when the year is given and when the month and day stand alone. *the tenth of the month your letter of the 10th* **(d)** Spell out years ordinarily in the informal and non-legal style. *the sixteenth of June, nineteen hundred and fifty-three*

**6. DECIMALS AND PERCENTAGES.** Use figures for writing decimals and percentages. *36.57 .05 He invested 15 per cent of his salary in bonds.* **NOTE:** Spell out percentages at the beginning of a sentence. *Fifteen per cent is being invested.* **NOTE:** A cipher placed before a decimal fraction gives emphasis to the fraction (*0.75*); a cipher placed after, shows that the sum has been carried to three decimal points (*.750*).

**7. EXACT NUMBERS. (a)** Spell out numbers under ten. *We packed six crates this morning.* **(b)** Use figures in business writing for numbers above ten. *He obtained 64 orders.* **NOTE:** In other writing, numbers under 100 are usually spelled out unless they appear in connected groups. *There were 10, 5, and 15 present at each of the meetings, respectively.* **(c)** Write large even numbers in business letters and reports as follows: *5 million dollars* (or: *$5 million*).

**8. FOUR OR MORE DIGITS.** Use figures and commas to show uneven thousands, millions, and billions. *120,355 $2,555.65 3,507,085*

**9. FRACTIONS.** Spell out simple fractions independent of whole numbers. *one third of the electorate* Use figures for complex fractions. *27/64ths 1¾*

**10. JUXTAPOSED NUMBERS** not closely related. Use figures if the second number can not be spelled out in one or two words. *In 1963, 120 men joined the company. In 1967, seventy men joined us.*

**11. MEASUREMENTS.** Use figures for capacities (*75 volts*), degrees (*45° F. 40° C.*), dimensions (*9 by 12 feet* or *9 x 12*), distances (*73 miles*), measures (*3 quarts*), sizes (*size 14*), stock quotations (*68—*), weights (*20 pounds*). **NOTE:** As quantity is a unit, a comma is not used to separate the elements. *It is 2 feet 7 inches.*

**12. MILITARY GROUPS, POLITICAL DIVISIONS, SESSIONS OF CONGRESS.** Spell out all numbers that refer to such groups. *the twenty-sixth division Ward One the Fourth Congressional District the Eighty-second Congress*

**13. MONEY. (a)** Use figures for definite sums of money. *$10 10 cents* **(b)** In a business letter, omit decimal points and ciphers in even amounts of dollars. *We have your check for $150 which we are crediting to your account.* **(c)** Spell out indefinite sums of money. *He earned a little over a hundred dollars.* **(d)** The dollar sign should be placed before each amount in a series of dollar amounts. *Price of cars range from $2,000–$3,200* **(e)** Spell out the word *cents* for amounts under one dollar. *The ticket costs 75 cents.*

**14. ORDINALS.** Spell out ordinals except with the "of" phrase in dates. *He is the second speaker on the program. I have your letter of the 3rd.*

**15. PAGE NUMBERS.** The word *page* is not capitalized nor preceded by the abbreviation *No. page 63*

**16. PROPORTIONS AND RATIOS.** Use figures. *a ratio of 3 to 1 (3:1) 20-20 vision*

**17. REFERENCES.** In footnotes and bibliographical matter, use an abbreviation for a part that precedes a number. *Chap. V p. 22*

**18. ROMAN NUMERALS**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | I | 20 | XX | 90 | XC | 900 | CM |
| 5 | V | 40 | XL | 100 | C | 1000 | M |
| 9 | IX | 50 | L | 500 | D | 2000 | MM |
| 10 | X | 60 | LX | 600 | DC | 4000 | $M\overline{V}$ |

$5000\ \overline{V}$

1969

MCMLXIX

**19. ROUND NUMBERS.** Spell out. *about one thousand five hundred and twenty dollars Nearly ninety trees were cut down.*

**20. STREET AND HOUSE NUMBERS. (a)** Spell out numbers that name streets and avenues. *Thirty-fourth Street Fifth Avenue* **NOTE:** Use figures when the name of the street represents three or more digits. *West 116 Street* **(b)** Spell out *one* when it numbers houses and buildings. *One Hudson Avenue*

**21. TELEPHONE NUMBERS** and travel directions. Use figures. *Bryant 4-6500 Route 35 Track 3*

**22. TIME. (a)** Spell out the hour when the word *o'clock* is given. *Your appointment is for four o'clock.* **(b)** Use figures with a.m. or A.M., p.m. or P.M. *The class begins at 8:30 a.m.* **NOTE:** Use two ciphers only when an even hour and one having minutes are given in a single sentence. *I will arrive at 9 a.m.* (or) *at nine in the morning. I will be in my office from 8:30 to 9:00 a.m.* **(c)** Spell out numbers indicating periods of time. *He lived in Wichita for twelve years.* **(d)** Use figures for academic years. *the class of '54*

## F. USING THE HYPHEN

***Compound numbers***

**1.** Between compound numbers (from twenty-one to ninety-nine) and fractions used as adjectives. *two hundred fifty-five two and one-half quarts* **NOTE:** When the fraction is not a single adjective, the hyphen is unnecessary. *He is asking one half of the group to return.*

***Compound adjectives***

**2.** Between two words forming a single adjective. *a first-class performance a silver-plated spoon a well-known story* **NOTE:** Do not use a hyphen when the adjective follows the modified noun (*He is a man well known for his honesty.*); when two independent adjectives precede a noun (*She wore an old red coat.*); when an adverb modifies an adjective (*She is a highly recommended nurse.*); when one of the words is a compound modifier ending in "er" or "est" (*a low-priced lot a lower priced lot*); when the compound modifier is a two-word proper noun (*a South American nation a Pulitzer Prize winner*); when one word in a compound modifier has an apostrophe (*the second semester's project*); when a foreign phrase precedes the modified noun (*an a priori statement*).

**3.** Between three or more words that form a compound adjective. *one-act-play contest once-a-year sale up-to-date inventory ship-to-shore movement*

***Hyphenated series***

**4.** After each element in a series of hyphenated words with a single base word. *Will they erect a three-, four-, or five-story building? the hard-and soft-cover editions of the novel*

***Compound verbs***

**5.** Between compound verbs. *The suit will be dry-cleaned.*

***Compound adverbs***

**6.** Between compound adverbs *She replied half-heartedly.*

***Compound nouns***

**7.** Between certain compound nouns. *will-power sea-level story-teller trade-mark end-product*

**8.** With nouns compounded with *mother, father, brother, sister, fellow,* etc. *mother-love fellow-man* **NOTE:** *motherhood* and *fellowship*

**9.** For nouns compounded of a verb and a noun or of a verb and a verb. *do-nothing make-believe*

***Verb compounds***

**10.** Between a present participle and a preposition used as a verb. *The tractor was leveling-off the ground.*

***Participle compounds***

**11.** When an adjective or noun is combined with a present or past participle. *foreign-spending word-ending native-grown*

***"self" compounds***

**12.** In most compounds with the word *self*. *self-interest self-preservation self-taught* **NOTE:** Do not use the hyphen in *selfless* or *selfsame* or in the reflexive and intensive pronouns (*myself, himself, themselves,* etc.).

***Prepositional compounds***

**13.** In compounds that have a prepositional phrase. *mother-of-pearl out-of-town visitors*

***Compound titles, names***

**14.** To separate compound names or titles. *Secretary-Treasurer Smith Henri-Marie-Raymond de Toulouse-Lautrec-Monfa*

***Prefix "re-"***

**15.** Between the prefix *re* when it means *again* and the verb. *Re-form the troops!* **NOTE:** *He'll reform.*

***Prefixes***

**16.** With prefixes ending in the same vowel that begins the root if the diaeresis is not used. *co-operate coöperate co-owner pre-election*

**17.** When a prefix is added to a word that begins with a capital letter. *un-American mid-Pacific pro-Ally*

**18.** In titles formed with *ex, elect,* and *vice*. *ex-Mayor Senator-elect Vice-President Baker*

***"non-" compounds***

**19.** In many words formed with non. *non-contagious non-co-operative non-pros* **NOTE:** There are many exceptions. *nonessential nonconformist nonplus*

***Letter-word compounds***

**20.** To join a letter to a number or a word to form an adjective, or a number to a letter to form a noun. *A-1 condition T-shaped bone 3-D*

***Number-quantity compounds***

**21.** In tabulated lists to link the number with the quantity. *Seven 5-gal. cans Barry House Paint*

***Noun compounds***

**22.** Between certain root words and a suffix to form nouns. *play-off drive-in shake-up*

***Adjectival compounds***

**23.** Between compound nouns and verbs used as adjectives. *a father-son banquet the would-be equestrian a wait-and-see policy*

***"cross-" compounds***

**24.** In some words formed with *cross*. *cross-question cross-purpose cross-town* **NOTE:** *crosswise cross reference*

***Civil and military titles***

**25.** Most civil, military, and naval titles of more than one word are not hyphenated when the title indicates one office. *Secretary of Defense General of the Army Lieutenant Commander General Manager*

## G. SPELLING RULES

**1. DOUBLING A FINAL CONSONANT**

**(a)** Words of one syllable are doubled when they end with a single consonant (*run*) preceded by a single vowel (*run*) and followed by a suffix that begins with a vowel (*running*). **EXCEPTIONS:** *gas gaseous lax laxity*

**(b)** Words of two or more syllables with the accent on the final syllable of the root word are doubled when they end with a single consonant (*refer*) preceded by a single vowel (*refer*) and followed by a suffix that begins with a vowel (*referred*). **NOTE:** When the accent shifts to first syllable (*reference*), final consonant of root word is not doubled. **EXCEPTION:** *excellence excellent*

**2. KEEPING FINAL CONSONANTS**

**(a)** Words that end with double consonants usually keep both letters before a suffix. *enroll enrolled install installing* **(b)** Words that end in *l* keep the letter before *ly*. *accidental accidentally* **(c)** Words that end in *n* keep the letter before *ness*. *thin thinness stern sternness*

**3. IDENTICAL LETTERS JOINING PREFIXES AND ROOT WORDS.** When the same consonant ends the prefix and begins the root word, both letters are kept. *misspell*

**4. FINAL SILENT e**

**(a)** Words that end with a silent *e* usually drop the *e* before a suffix that begins with a vowel. *come coming*

**(b)** The *e* is usually kept before a suffix that begins with a consonant. *hope hopeful* **EXCEPTIONS:** *judgment ninth truly wholly* **(c)** Final silent *e* preceded by either *c* or *g* is kept before a suffix beginning with *a* or *o*. *notice noticeable courage courageous*

**5. FINAL y**

**(a)** Words that end in *y* preceded by a consonant usually change the *y* to *i* before a suffix. *hearty heartily*

**(b)** Words that end in *y* preceded by a vowel usually keep the *y* before a suffix. *sway swayed*
**EXCEPTIONS:** *say said pay paid*

**6. Ei and ie (a)** Usually *i* follows *l* and *e* follows *c* (like: police). *believe relieve deceive conceive*

**(b)** Usually *i* precedes *e*. *brief chief field fierce yield*

**(c)** Usually *e* precedes *i* after *c* and when pronounced like long *a*. *ceiling receipt freight heir neighbor their* **EXCEPTIONS:** *ancient either financier forfeit inveigle leisure seize weird*

DATA-GUIDE QUICK CHART

# ANALYTIC GEOMETRY

SIMPLIFIED SUMMARY ◆ INSTANT REFERENCE ◆ PERMANENT MEMORY-AID

LEARN & REVIEW IN MINUTES

AUTHOR: Prof. Joseph Feld, Ph.D., University of the City of N.Y.
Consultant: Prof. J. Chiccarelli, Chairman, Bridgewater State College.

SOLID PLASTIC

SPEED STUDY TO SAVE YOU TIME AND WORK.

QUICK CHARTS PUT THE FACTS IN FULL VIEW:

PERMANENTLY FILL IN FOR MISSING NOTES & TEXTS.

QUICK CHARTS HELP YOU REMEMBER:

CONTAINS NO PAPER

## 1 BASIC CONCEPTS.

**A. DIRECTED SEGMENTS, LINES.** **1.** If $A$ is the *point initial* and $B$ the *terminal point*, $AB$ becomes the *directed* (or oriented) segment $AB$. Segment $BA$ has the same length as $AB$ but it is traversed in the opposite sense (from $B$ to $A$); thus $AB = -BA$. If $C$ is any third point on $L$, then $AB+BC+CA=0$. **2.** A directed segment on $L$ is *positively* directed when has it *same* sense as $L$; *negatively* directed when of *opposite* sense. (The signed length of $AB$ is $+$ when positively directed, $-$ when negatively directed.

Fig. 1.

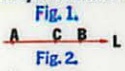
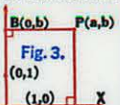



Fig. 2.


Fig. 3.




**B. RECTANGULAR COORDINATE SYSTEM.** (Permits location of point in a plane, Fig. 3.) **1.** The coordinates of point $A$ are $(a, 0)$; of B, $(0, b)$. Value $a$ is the *abscissa* or the *x-coordinate* of $P$; value $b$ is the *ordinate* or *y-coordinate* of $P$; thus the coordinates of $P$ in the plane are $(a,b)$. **2.** The coordinates of the origin are (0,0). **3.** Thus the abscissa of any point to the *right* of the $y$-axis is "+"; to the *left*, "−". The ordinate of any point above the $x$-axis is "+"; below, "−".

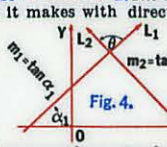
## 2 FUNDAMENTAL RELATIONSHIPS.

**A. ANGLES.** **1.** *The angle of inclination* ($\alpha$) of a directed line is angle it makes with directed $x$-axis, measured in a *ccw* direction ($0 \leq \alpha < 180°$). **2.** *Slope* ($m$) of line of inclination $\alpha$ is $\tan \alpha$.

| INCLINATION $\alpha$ | 0° | acute | obtuse | 90° |
|---|---|---|---|---|
| SLOPE $m$ | 0 | + | − | ∞ |

**3.** *The angle between two oriented lines* is the angle $\theta$ between their *positive* directions. The angle from $L_1$ to $L_2$ is $\theta$ measured counterclockwise (positively); the angle from $L_2$ to $L_1$ is $180° - \theta$.

Fig. 4.




**B. POINTS AND SEGMENTS.** **1.** Projection of a point $P$ on a line $L$ is the foot of the perpendicular dropped from $P$ on to $L$ (Fig. 5). **2.** The projection $P_1P_2$ on the $x$-axis is $AB$, where $A$ is the projection of $P_1$; $B$, the projection of $P_2$. **3.** If the distance between $P_1$ and $P_2$ is $d$ (a positive number) and the angle between $P_1P_2$ and the $x$-axis is $\alpha$, then $AB = P_1C = d \cos \alpha$. **4.** Segment $P_1P_2$ is divided by point $Q$ in ratio $r_1 : r_2$ if $P_1Q : QP_2 = r_1 : r_2$. If $r_1 : r_2$ is $+$, $Q$ is an internal point of division; $-$, then $Q$ is an external pt. of division.

Fig. 5

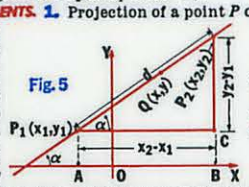
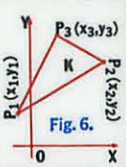



Fig. 6.




$x_2 - x_1$ — Projection of $P_1P_2$ on $x$-axis.

$y_2 - y_1$ — Projection of $P_1P_2$ on $y$-axis.

$m = \tan \alpha = \dfrac{y_2 - y_1}{x_2 - x_1}$ — Slope of line $P_1P_2$. NOTE. When $x_1 = x_2$, the line is perpendicular to $x$-axis and has no slope ($m = \infty$).

$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ — Distance between $P_1$ and $P_2$.

$\left(\dfrac{r_2x_1 + r_1x_2}{r_1 + r_2}, \dfrac{r_2y_1 + r_1y_2}{r_1 + r_2}\right)$ — Point dividing $P_1P_2$ in ratio $r_1 : r_2$.

$\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$ — Midpoint of $P_1P_2$. ($r_1 = r_2$).

$\tan \theta = \dfrac{m_2 - m_1}{1 + m_1m_2}$ — Formula for angle from line $L_1$ having slope $m_1$, to line $L_2$ with slope $m_2$.

$m_1m_2 = -1$ — Condition that $L_1$ be perpendicular to $L_2$.

$m_1 = m_2$ — Condition that $L_1$ be parallel to $L_2$.

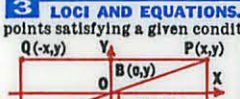
$K = \dfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ — Area of triangle $P_1P_2P_3$ (Fig. 6). | $K = 0$ — Condition that points $P_1, P_2, P_3$ be collinear.

**E:** Given $P_1(-2, 3)$; $P_2$ (4, 6). Projection of $P_1P_2$ on $x$-axis $= 4 - (-2) = 6$; slope $= (6-3)/[4-(-2)] = 1/2$; $d$ between $P_1$, $P_2 = \sqrt{(-2-4)^2 + (3-6)^2} = \sqrt{45}$; midpoint of $P_1P_2 = \left(\dfrac{-2+4}{2}, \dfrac{3+6}{2}\right) = \left(1, \dfrac{9}{2}\right)$.

## 3 LOCI AND EQUATIONS.

**A. DEFINITIONS.** **1.** A locus is a set of points satisfying a given condition or group of conditions. **2.** A curve $C$ is said to be the locus of $f(x,y) = 0$ if every point of $C$ has coordinates that satisfy the equation and if every point whose coordinates satisfy the equation lies on $C$. **E:** The locus of $x^2 = 1$ is the pair of lines parallel to the $y$-axis and one unit away from it.

Fig. 7.




**B. SYMMETRY.** **1. Symmetry with respect to a line:** Two points $P$, $Q$ are symmetric with respect to a line $L$ if $L$ is the perpendicular bisector of segment $PQ$. The points $P(x,y)$ and $Q(-x,y)$ are symmetric in the $y$-axis (*Fig.* 7); so are $S(x, -y)$ and $R(-x-y)$. The pts $P(x,y)$ and $S(x,-y)$ are symmetric in $x$-axis.

Fig. 8.

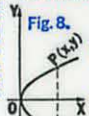



Fig. 9.

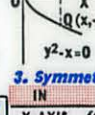
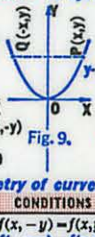



Fig. 10.




**2. Symmetry with respect to a point:** $P(x,y)$ and $R(-x, -y)$ are symmetric with respect to $O$ (the origin) if $O$ bisects $PR$.

**3. Symmetry of curve:**

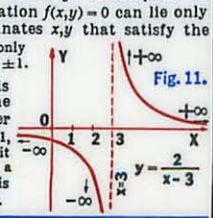
| IN | CONDITIONS | PROPERTY OF EQUATION | EXAMPLE |
|---|---|---|---|
| X-AXIS | $f(x,-y) = f(x,y)$ | Only even powers of $y$ | $y^2 - x = 0$ (Fig. 8) |
| Y-AXIS | $f(-x,y) = f(x,y)$ | Only even powers of $x$ | $y - x^2 = 0$ (Fig. 9) |
| ORIGIN | $f(-x,-y) = \pm f(x,y)$ | All terms of odd (or even) degree in $x$ and $y$ | $y - x^3 = 0$ (Fig. 10) $xy = 4$ |

**C. INTERCEPTS OF A CURVE.** **1.** *The x-intercepts* are the abscissas of the points of intersection of the curve with the $x$-axis. **2.** *The y-intercepts* are the ordinates of the points of intersection with the $y$-axis. **3.** The intercepts are found by letting $y = 0$ (or $x = 0$) and solving for the real values of $x$ (or $y$). **E:** $x$-intercept of $2x - 3y = 6$ is 3; $y$ − intercept is $-2$.

**D. EXTENT OF A LOCUS.** Locus of an equation $f(x,y) = 0$ can lie only on those parts of the plane where coordinates $x,y$ that satisfy the equation are both real. **E:** $y = \log x$ exists only where $x > 0$. **E:** $y = \sin x$ lies between lines $y = \pm 1$.

**E. ASYMPTOTES.** An asymptote to a curve is a line, the points of which eventually come closer to curve than any quantity, however small, without crossing curve. **E:** In Fig. 11, the curve $y = 2/(x-3)$ exists when $x \neq 3$; thus it has branches on both sides of the line $x = 3$, a vertical asymptote since as $x \to 3$, $|y| \to \infty$. $x$-axis is a horizontal asymptote since as $|x| \to \infty$, $y \to 0$.

Fig. 11.




## 4 STRAIGHT LINE.

**A. GENERAL EQUATION OF 1ST DEGREE (in $x, y$).** The locus of $Ax + By + C = 0$, is a straight line and every straight line has an equation of the first degree.

Fig. 12.

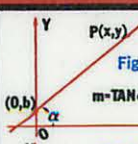



Fig. 13.


Fig. 14.

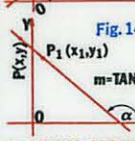



Fig. 15.

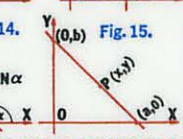



Fig. 16.


**B. SPECIAL FORMS OF THE EQUATION OF A STRAIGHT LINE.**

$y = mx + b$ — Line determined by slope $m$, $y$-intercept $b$. (Fig. 12)

$x = a$ — Line parallel to $y$-axis, $m = \infty$; $x$-intercept $= a$. (Fig. 13)

$y = b$ — Line parallel to $x$-axis; $m = 0$; $y$-intercept $= b$. (Fig. 13)

$y - y_1 = m(x - x_1)$ — Line determined by pt. $(x_1, y_1)$, slope $m$. (Fig. 14)

$(y - y_1) \div (x - x_1) = (y_2 - y_1) \div (x_2 - x_1)$ — Line determined by two pts.

$(x/a) + (y/b) = 1$ — Two-intercept form; $a \neq 0$, $b \neq 0$. (Fig. 15)

$x \cos \theta + y \sin \theta - p = 0$; $p > 0$ — Normal form: Line determined by length ($p$) of normal from origin and angle to $x$-axis. (Fig. 16)

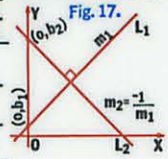
$\dfrac{A}{\pm\sqrt{A^2+B^2}}x + \dfrac{B}{\pm\sqrt{A^2+B^2}}y + \dfrac{C}{\pm\sqrt{A^2+B^2}} = 0$ — Reduction of Gen. Eq. to normal form. (Sign before radical is chosen opposite to $C$. If $C = 0$, use sign of $B$.)

**E:** Line $3x + 4y + 10 = 0$ has normal form $(3/-5)x + (4/-5)y + 10/-5 = 0$. Its distance from the origin is $p = 10/5 = 2$. Since $y = -(3/4)x - 5/2$, the line has the slope $m = -3/4$ and $y$-intercept $b = -5/2$.

**C. PARALLEL AND PERPENDICULAR LINES.** Given lines $L_1$, $L_2$. (Fig. 17)

| | | |
|---|---|---|
| Eq. of $L_1$ | $A_1x + B_1y + C_1 = 0$ | $y = m_1x + b_1$ |
| Eq. of $L_2$ | $A_2x + B_2y + C_2 = 0$ | $y = m_2x + b_2$ |
| $L_1$ and $L_2$ coincide | $\dfrac{A_1}{A_2} = \dfrac{B_1}{B_2} = \dfrac{C_1}{C_2}$ | $m_1 = m_2$ $b_1 = b_2$ |
| $L_1$ and $L_2$ parallel | $\dfrac{A_1}{A_2} = \dfrac{B_1}{B_2} \neq \dfrac{C_1}{C_2}$ | $m_1 = m_2$ $b_1 \neq b_2$ |
| $L_1$ and $L_2$ rt. angles | $A_1A_2 + B_1B_2 = 0$ | $m_1m_2 = -1$ |

Fig. 17.

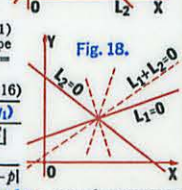



**E:** Equation of line passing thru point $(4, -1)$ and perpendicular to line $y = 2x - 3$. **Sol.** Slope of required line is $-1/2$; equation is $y + 1 = -1/2(x - 4)$ or $x + 2y - 2 = 0$.

**D. DISTANCE OF A POINT TO A LINE.** (Fig. 16)

| EQUATION OF L | DISTANCE OF $P_1(x_1,y_1)$ |
|---|---|
| $Ax + By + C = 0$ | $d = \dfrac{\lvert Ax_1 + By_1 + C \rvert}{\sqrt{A^2 + B^2}}$ |
| $x \cos\theta + y \sin\theta - p = 0$ | $d = \lvert x_1 \cos\theta + y_1 \sin\theta - p \rvert$ |

Fig. 18.




**E. FAMILIES OF STRAIGHT LINES.** **1. Equation $L$:** $y = mx + b$ represents an infinite number of straight lines each having a specific value of $m$ and $b$. **2.** If value of $m$ is held fixed and $b$ is allowed to vary, the equation represents a family of parallel lines of slope $m$. **3.** If $b$ is fixed and $m$ varies, equation is a family of lines passing through $(0,b)$. **4.** Both $b$ and $m$ are *parameters*; to each corresponds one line of the family. **5. Pencil of lines:** Given two lines, $L_1 = A_1x + B_1y + C_1 = 0$ and $L_2 = A_2x + B_2y + C_2 = 0$, a pencil of straight lines is given by $L_1 + kL_2 = 0$. (Fig. 18)

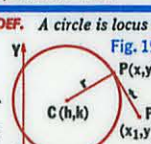
| CONDITION | FAMILY | Parameter k |
|---|---|---|
| $L_1, L_2$ intersect | All lines pass thru pt. of intersection | To every real value of $k$ corresponds 1 line of pencil. |
| $L_1, L_2$ parallel | Pencil of parallel lines. | To $k = 0$, corresponds $L_1 = 0$, to $k = \infty$, corresponds $L_2 = 0$. |

**E:** Find equation of line determined by the point (1,0) and the point of intersection of line $x + y - 3 = 0$, $x - y + 1 = 0$. **Sol.** Required line belongs to pencil $(x + y - 3) + k(x - y + 1) = 0$. Since (1, 0) lies on line, $(1 + 0 - 3) + k(1 - 0 + 1) = 0$, $k = 1$; thus $(x + y - 3) + (x - y + 1) = 0$ or $x - 1 = 0$.

## 5 CONIC SECTIONS — CIRCLE.

**A. DEF.** *A circle is locus of a point* (P) *which moves such that it is always a constant distance* (*the radius*, r) *from a fixed point* [*center*, C at (h, k)].

Fig. 19.




| EQUATION | CENTER | RADIUS |
|---|---|---|
| $(x-h)^2 + (y-k)^2 = r^2$ | $(h,k)$ | $r$ |
| $x^2 + y^2 + dx + ey + f = 0$ | $\left(-\dfrac{d}{2}, -\dfrac{e}{2}\right)$ | $\dfrac{\sqrt{d^2 + e^2 - 4f}}{2}$ |

**NOTE:** When $d^2 + e^2 - 4f = 0$, circle is a point $(-d/2, -e/2)$ and $r = 0$. When $d^2 + e^2 - 4f < 0$, circle is imaginary.

**E:** Find the equation of the circle with center at (2, 1) that is tangent to the line $3x + 4y - 5 = 0$. **Sol.** Radius = distance of (2, 1) from the line or $|3(2) + 4(1) - 5| \div \sqrt{3^2 + 4^2} = 1$. Ans: $(x-2)^2 + (y-1)^2 = 1$.

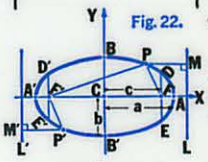
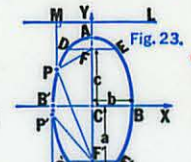
**B. TANGENT TO A CIRCLE.** Length of a tangent from $P_1(x_1, y_1)$ to a circle is $\sqrt{(x_1 - h)^2 + (y_1 - k)^2 - r^2}$. Equation of tangent when $P_1(x_1, y_1)$ is on the circle is $(x_1 - h)(x - h) + (y_1 - k)(y - k) = r^2$.

**C. PENCIL OF CIRCLES.** **1.** Two circles determine a pencil of circles. Let $C_1$ $(x^2 + y^2 + d_1x + e_1y + f_1 = 0)$ and $C_2$ $(x^2 + y^2 + d_2x + e_2y + f_2 = 0)$ be two given circles. Pencil of circles they determine is given by $(x^2 + y^2 + d_1x + e_1y + f_1) + k(x^2 + y^2 + d_2x + e_2y + f_2) = 0$; $C_1 + kC_2 = 0$, where $k$ is parameter. **2.** To each value of $k$ (except to $k = -1$) corresponds a circle of pencil. By convention, $C_2$ corresponds to $k = \infty$ (not a number). To $k = -1$, corresponds a line $L$ called radical axis of pencil. *NOTE:* when $C_1$ and $C_2$ are concentric, no radical axis. **3.** $L$ is locus of all points from which tangent line segments drawn to $C_1$ and $C_2$ have equal length. $L$ is perpendicular to line of centers of $C_1$ and $C_2$. **4. Types:** $C_1$, $C_2$ intersect in two points (*radical axis is the common chord*), are tangent (*radical axis is the common tangent*), do not intersect.

## 6 CONIC SECTIONS—PARABOLA, ELLIPSE, HYPERBOLA

**A. DEF.** **1.** *These conic sections are locus of a point* (P) *which moves such that its distance from a fixed point* (*a focus*, F) *is in a constant ratio* (*eccentricity*, e) *to its distance from a fixed line* (*a directrix*, L). Conic sections are the curves of intersection of a circular cone by a plane. **2. The principal axis** is the line through a focus perpendicular to a directrix. **3. A vertex** (A; A′) is a point at which principal axis intersects curve. **4. The center** (C) is mid-point of the segment which joins the foci of the ellipse or hyperbola. **5. The major axis** (AA′ *in the ellipse*) or **the transverse axis** (A A′ *in the hyperbola*) is segment of the principal axis that connects vertices of the conic. **6. The minor axis** (BB′ *in ellipse*) or **the conjugate axis** (BB′ *in hyperbola*) is the segment of a line through the center, perpendicular to principal axis of the conic section. **7. A latus rectum** (DE; D′E′) is a chord through a focus, perpendicular to principal axis, intersected by curve. **8. A focal radius** (FP; F′P′) is any line segment from a focus to any point on curve.

**B. IDENTIFICATION OF THE PARABOLA, ELLIPSE, AND HYPERBOLA**

| PARABOLA | ELLIPSE | HYPERBOLA |
|---|---|---|
| *The locus of a point* (P) *whose distance from a fixed line* (L) *always equals its distance from a fixed point* (F) *not on L.* | *The locus of a point* (P) *such that the sum of its distances from two fixed points* (F, F′) *is a constant.* | *The locus of a point* (P) *such that the difference of its distances from two fixed points* (F, F′) *is a constant.* |
| Fig. 20.  | Fig. 22.  | Fig. 24.  |
| Fig. 21.  | Fig. 23.  | Fig. 25.  |
| $\dfrac{FP}{MP} = e = 1$; $FP = MP$ | $\dfrac{FP}{MP} = \dfrac{F'P'}{M'P'} = e < 1$ \| $FP + F'P' = 2a$ | $\dfrac{FP}{MP} = \dfrac{F'P'}{M'P'} = e > 1$ \| $FP - F'P' = 2a$ |
| *On the Principal axis:* $AF = p$ | *Major axis:* $AA' = 2a$ *Minor axis:* $BB' = 2b$ $FF' = 2c$ $a^2 = b^2 + c^2$ | *Transverse axis:* $AA' = 2a$ *Conjugate axis:* $BB' = 2b$ $FF' = 2c$ $c^2 = a^2 + b^2$ |

| | PRINCIPAL AXIS x-axis | PRINCIPAL AXIS y-axis | MAJOR AXIS x-axis | MAJOR AXIS y-axis | TRANSVERSE AXIS x-axis | TRANSVERSE AXIS y-axis |
|---|---|---|---|---|---|---|
| EQ | $y^2 = 4px$ | $x^2 = 4py$ | $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ | $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$ | $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ | $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$ |
| C | — | — | (0, 0) | (0, 0) | (0, 0) | (0, 0) |
| F | (p, 0) | (0, p) | (c, 0) | (0, c) | (c, 0) | (0, c) |
| F′ | — | — | (−c, 0) | (0, −c) | (−c, 0) | (0, −c) |
| A | (0, 0) | (0, 0) | (a, 0) | (0, a) | (a, 0) | (0, a) |
| A′ | — | — | (−a, 0) | (0, −a) | (−a, 0) | (0, −a) |
| B | — | — | (0, b) | (b, 0) | (0, b) | (b, 0) |
| B′ | — | — | (0, −b) | (−b, 0) | (0, −b) | (−b, 0) |
| DE D′E′ | \| 4p \| | \| 4p \| | $\dfrac{2b^2}{a}$ | $\dfrac{2b^2}{a}$ | $\dfrac{2b^2}{a}$ | $\dfrac{2b^2}{a}$ |
| e | 1 | 1 | $(c/a) < 1$ | $(c/a) < 1$ | $(c/a) > 1$ | $(c/a) > 1$ |
| L | $x = -p$ | $y = -p$ | $x = a/e$ | $y = a/e$ | $x = a/e$ | $y = a/e$ |
| L′ | — | — | $x = -a/e$ | $y = -a/e$ | $x = -a/e$ | $y = -a/e$ |
| ASY | — | — | — | — | $y = \pm(b/a)x$ | $y = \pm(a/b)x$ |
| TAN AT $x_1, y_1$ | $yy_1 = 2p(x + x_1)$ $xx_1 = 2p(y + y_1)$ | | $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$ | $\dfrac{xx_1}{b^2} + \dfrac{yy_1}{a^2} = 1$ | $\dfrac{xx_1}{a^2} - \dfrac{yy_1}{b^2} = 1$ | $\dfrac{yy_1}{a^2} - \dfrac{xx_1}{b^2} = 1$ |

**E1:** For the ellipse $9x^2 + 25y^2 = 225$, $(x^2/25) + (y^2/9) = 1$; $a^2 = 25$, $b^2 = 9$, $c^2 = a^2 - b^2 = 25 - 9 = 16$; $e = c/a = 4/5$; the foci are $(\pm 4, 0)$; the directrices are $x = \pm a/e = \pm 25/4$.

**E2:** One focus of a hyperbola is at (0, 5), its center is at origin, eccentricity is 5/3. Find the equation. **Sol.** $c = 5$; $e = 5/3 = 5/a$, thus $a = 3$; $b^2 = c^2 - a^2 = 25 - 9 = 16$; the foci are on the $y$-axis: $(y^2/9) - (x^2/16) = 1$.

**E3:** Find equation of parabola that has vertex at the origin and directrix $x = 4$. **Sol.** $p = -4$, thus $y^2 = -16x$.

Fig. 26.


**C. OPTICAL PROPERTIES OF CONIC SECTIONS.** **1.** The focal radius $FP$ of a parabola at any point $P$ of the curve and the parallel line to axis at $P$ make equal angles with the tangent at $P$. **2.** A tangent to the ellipse and hyperbola at any point makes equal angles with the focal radii of that point.

## 7 TRANSFORMATION OF COORDINATES.

**A. TRANSLATION OF AXES.** **1.** Given two sets of axes, the old $xy$-axes with origin at $O$; the new or $x'y'$-axes with origin at $O'$, so that the $x'$- and $y'$-axes are parallel respectively to the $x$- and $y$-axes. Either set of axes is said to be translated with respect to the other. **2.** Let $O'$ have the coordinates $(h,k)$ with respect to the $x,y$-axes; then a point $P$ has two sets of coordinates: $(x,y)$ and $(x',y')$

Fig. 27.




**3. Equations of translation:** $x = x' + h$, $y = y' + k$; $x' = x - h$, $y' = y - k$

| | PARABOLA PRINCIPAL AXIS ∥ TO x-axis | PARABOLA PRINCIPAL AXIS ∥ TO y-axis | ELLIPSE MAJOR AXIS ∥ TO x-axis | ELLIPSE MAJOR AXIS ∥ TO y-axis | HYPERBOLA TRANSVERSE AXIS ∥ TO x-axis | HYPERBOLA TRANSVERSE AXIS ∥ TO y-axis |
|---|---|---|---|---|---|---|
| EQ | $(y-k)^2 = 4p(x-h)$ | $(x-h)^2 = 4p(y-k)$ | $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ | $\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2} = 1$ | $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$ | $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$ |
| C | — | — | $(h, k)$ | $(h, k)$ | $(h, k)$ | $(h, k)$ |
| A | $(h, k)$ | $(h, k)$ | $(h+a, k)$ | $(h, k+a)$ | $(h+a, k)$ | $(h, k+a)$ |
| A′ | — | — | $(h-a, k)$ | $(h, k-a)$ | $(h-a, k)$ | $(h, k-a)$ |
| F | $(h+p, k)$ | $(h, k+p)$ | $(h+c, k)$ | $(h, k+c)$ | $(h+c, k)$ | $(h, k+c)$ |
| F′ | — | — | $(h-c, k)$ | $(h, k-c)$ | $(h-c, k)$ | $(h, k-c)$ |
| B | — | — | $(h, k+b)$ | $(h+b, k)$ | $(h, k+b)$ | $(h+b, k)$ |
| B′ | — | — | $(h, k-b)$ | $(h-b, k)$ | $(h, k-b)$ | $(h-b, k)$ |
| L | $x = h-p$ | $y = k-p$ | $x = h+a/e$ | $y = k+a/e$ | $x = h+a/e$ | $y = k+a/e$ |
| L′ | — | — | $x = h-a/e$ | $y = k-a/e$ | $x = h-a/e$ | $y = k-a/e$ |
| ASY | — | — | — | — | $y - k = \pm\dfrac{b}{a}(x-h)$ | $y - k = \pm\dfrac{a}{b}(x-h)$ |

Fig. 28.


Fig. 30.


Fig. 32.


Fig. 29.


Fig. 31


Fig. 33.

**E1:** Find the equation of curve $y^2-2y-4x+9=0$ referred to parallel axes with the new origin at (2,1). **Sol.** $h=2$, $k=1$; $x=x'+2$, $y=y'+1$. Substitute: $(y'+1)^2-2(y'+1)-4(x'+2)+9=0$. Simplify: $y'^2=4x'$, which is a parabola in standard form with respect to the $x'y'$-axes. **E2:** By means of a translation of axes, transform $9x^2+4y^2+18x-24y+9=0$ into equation without terms of the 1st degree. **Sol.** Substitute $x=x'+h$ and $y=y'+k$, thus $9x'^2+4y'^2+(18h+18)x'+(8k-24)y'+(9h^2+4k^2+18h-24k+9)=0$. Equating to zero the coefficients of $x'$ and $y'$ yields $18h+18=0$ and $8k-24=0$, thus $h=-1$, $k=3$ and the transformed equation is $9x'^2+4y'^2=36$; an ellipse in standard form with respect to the $x'y'$-axes.

Fig. 34. E1

Fig. 35. E2

**B. ROTATION OF AXES AROUND ORIGIN.** **1.** Let axes of coordinates be rotated around the origin $O$ through angle $\theta$. (Fig. 36) Let the point $P$, whose coordinates are $(x,y)$ with respect to the $xy$-axes, have the coordinates $(x',y')$ with respect to the rotated $x'y'$-axes.

Fig. 36.

**2. Equations of rotation:**

$$x=x'\cos\theta-y'\sin\theta \qquad y=x'\sin\theta+y'\cos\theta$$

$$x'=x\cos\theta+y\sin\theta \qquad y'=-x\sin\theta+y\cos\theta$$

**E:** Transform the equation of the equilateral hyperbola $xy=2a^2$ by rotating the axes through an angle of 45°. **Sol.** $x=x'\cos 45°-y'\sin 45°=(x'-y')/\sqrt{2}$; $y=x'\sin 45°+y'\cos 45°=(x'+y')/\sqrt{2}$, thus

$$y=\frac{(x'-y')(x'+y')}{(\sqrt{2})^2}=\frac{x'^2-y'^2}{2}$$

$$=2a^2, \text{ or } x'^2-y'^2=4a^2$$

Fig. 37.

## 8 GENERAL EQUATION OF THE SECOND DEGREE

**A. DEFINITIONS.** **1.** Every conic (section) has an equation of the 2nd degree in rectangular coordinates. General equation of 2nd degree is $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$ ($A,B,C$ not all zero). The locus (when it exists) is always a conic. **2. Proper conics:** circle, parabola, ellipse, hyperbola. **3. Improper conics:** a pair of straight lines (which may coincide) and a single point. **E:** The locus of $x^2-y^2=0$ is the pair of straight lines $x+y=0$ and $x-y=0$. The locus of $x^2=0$ is the $y$-axis counted twice; locus of $2x^2+y^2=0$ is origin; equation $x^2+y^2=-1$ has no locus.

**B. CONICS WITH PRINCIPAL AXIS PARALLEL TO AN AXIS OF COORDINATES.** The $xy$-term in its equation is missing ($B=0$) and vice versa: $Ax^2+Cy^2+Dx+Ey+F=0$.

| | |
|---|---|
| CIRCLE: $A=C\neq 0$ | PARABOLA: $A=0, C\neq 0$; $C=0, A\neq 0$ |
| ELLIPSE: $AC>0$ | HYPERBOLA: $AC<0$ |

**E:** By means of a translation of axes, find the standard form of the equation of the conic $9x^2+4y^2+18x-16y=11$. **Sol.** Collecting separately the terms in $x$ and $y$, $9(x^2+2x+\ )+4(y^2-4y+\ )=11$, then completing the squares within the parentheses gives $9(x^2+2x+1)+4(y^2-4y+4)=11+9+16=36$ or $9(x+1)^2+4(y-2)^2=36$. The translation $x'=x+1$, $y'=y-2$ yields $9x'^2+4y'^2=36$ or $(x'^2/4)+(y'^2/9)=1$; an ellipse with $a=3$ and $b=2$.

Fig. 38.

**C. TO REMOVE THE $xy$-TERM.** Rotate the axes through an angle $\theta$ such that $\cot 2\theta=(A-C)/B$ or $\tan 2\theta=B/(A-C)$. **E:** By means of a rotation remove the $xy$-term from $8x^2+4xy+5y^2=9$. **Sol.** $\tan 2\theta=4/(8-5)=4/3$;

$$\cos 2\theta=\frac{3}{5};\ \sin\theta=\sqrt{\frac{1-\cos 2\theta}{2}}=\sqrt{\frac{1-(3/5)}{2}}=\frac{1}{\sqrt{5}};\ \cos\theta=\sqrt{\frac{1+\cos 2\theta}{2}}=\frac{2}{\sqrt{5}};$$

the required rotation is given by $x=x'\cos\theta-y'\sin\theta=(2x'-y')/\sqrt{5}$; $y=x'\sin\theta+y'\cos\theta=(x'+2y')/\sqrt{5}$. Substituting for $x$ and $y$ in the equation gives

$$\frac{8(2x'-y')^2}{5}+\frac{4(2x'-y')(x'+2y')}{5}+\frac{5(x'+2y')^2}{5}=9x'^2+4y'^2=9 \text{ (ellipse, } a=3/2, b=1).$$

Fig. 39.

## 9 INVARIANTS UNDER DISPLACEMENTS OF THE AXES.

**A. DEFINITIONS.** **1.** When the axes are displaced (*translated, rotated, or both*), the general equation $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$ is transformed into $A'x'^2+B'x'y'+C'y'^2+D'x'+E'y'+F'=0$. **2.** A function of the coefficients, $I=f(A,B,C,\ldots,F)$ such that $f(A,B,C,\ldots,F)=f(A',B',C',\ldots F')$, is an invariant under displacements of the axes.

**B. INVARIANTS.** $B^2-4AC$; $\Delta=4ACF-B^2F-AE^2-CD^2+BDE$

**C. CLASSIFICATION OF THE LOCI OF GENERAL EQUATION OF 2ND DEGREE BY MEANS OF $B^2-4AC$ AND $\Delta$.**

| | $B^2-4AC<0$ | $B^2-4AC=0$ | $B^2-4AC>0$ |
|---|---|---|---|
| $\Delta\neq 0$ | Ellipse if $A\Delta<0$; No locus if $A\Delta>0$ | Parabola | Hyperbola |
| $\Delta=0$ | Point | Two parallel lines, two coincident lines, or no locus | Two intersecting lines |

**E:** Identify the curve $8x^2+4xy+5y^2-9=0$. **Sol.** $B^2-4AC=16-4(8)(5)<0$; $A\Delta=8[4(8)(5)(-9)-(16)(-9)+0+0+0]<0$; curve is an ellipse.

## 10 POLAR COORDINATES. A. DEFINITIONS.

**1.** Position of point **P** is determined by its distance $r$ from a fixed point **0** and the angle $\theta$ that **OP** makes with a fixed indefinite line **OA** (the initial line.) **2.** The ordered pair of numbers $(r,\theta)$ are called the *polar coordinates* of $P$; $r$ is the *radius vector* of $P$ and $\theta$ its *vectorial angle*. The rays eminating from the pole $O$ are described by their vectorial angles; $OC$ is the 90° ray. **NOTE:** $(r,\theta)$, $(r,\theta+360°)$, $(-r,\theta+180°)$ represent same point.

Fig. 40.

**B. TRANSFORMATION OF POLAR AND RECTANGULAR COORDINATES.**

**1.** $x=r\cos\theta \qquad y=r\sin\theta$

**2.** $x^2+y^2=r^2 \qquad y/x=\tan\theta$

**E1:** Transform to polar coordinates the equation of the circle $x^2+y^2-2ax=0$. **Sol.** $x^2+y^2-2ax=r^2-2ar\cos\theta=0$; $r=2a\cos\theta$. **E2:** Transform to rectangular coordinates the equation of the lemniscate $r^2=\sin 2\theta$. **Sol.** $r^2=2\sin\theta\cos\theta$; $r^4=2r\sin\theta\cdot r\cos\theta$ and $(x^2+y^2)^2=2yx$.

Fig. 41.

## 11 PLOTTING CURVES IN POLAR COORDINATES. A. TESTS FOR SYMMETRY.

**1.** $P(r,\theta)$; $Q(r,-\theta)$ are symmetric in *initial line* or *polar axis*. **2.** $P(r,\theta)$; $S(r,\ 180°-\theta)$ are symmetric in the *90° line*. **3.** $(r,\theta)$; $Y(-r,\theta)$ are symmetric in *pole*. **4. To test a curve $r=f(\theta)$ for symmetry in the polar axis,** substitute $-\theta$ for $\theta$; if $f(-\theta)=f(\theta)$ the curve is symmetric. **E:** $r=1+\cos\theta$ is symmetric in the polar axis since $f(-\theta)=1+\cos(-\theta)=1+\cos\theta=f(\theta)$. **5. Test for symmetry in the 90° line:** Substitute $180°-\theta$ for $\theta$. There is symmetry if $f(180°-\theta)=f(\theta)$. $r=1+\sin\theta$; $(180°-\theta)=1+\sin(180°-\theta)=1+\sin\theta=f(\theta)$. **6. Test for symmetry in the pole:** Substitute $-r$ for $r$; if eq. unchanged, there is symmetry.

Fig. 42.

**B. TESTS FOR EXTENT.** **1.** When for all values of $\theta$, $|r(\theta)|<k$ a positive number, a curve $r=f(\theta)$ lies within a circle of radius $k$ and center at pole. **2.** When in the sector $\alpha<\theta<\beta$, $r$ is imaginary, no points of the curve $r=f(\theta)$ lie in that sector.

**E:** Plot lemniscate $r^2=4\cos 2\theta$.

**Sol.** Since $\cos(-2\theta)=\cos 2\theta$, the curve is symmetric in polar axis. Since $(-r)^2=r^2$ it is symmetric in pole. Since $\cos 2\theta$ is negative when $45°<\theta<135°$ and $225°<\theta<315°$ there is no locus in these sectors. Since $|r|\leq 2$, the curve lies in circle of radius $=2$.

Fig. 43.

**C. STRAIGHT LINE (POLAR AND RECTANGULAR COORDINATES).**

GENERAL CASE — Fig. 44.

$$r\cos(\theta-\phi)=p \qquad x\cos\phi+y\sin\phi-p=0$$

SPECIAL CASES — Fig. 45.

$$r\cos\theta=p \qquad x=p$$

Fig. 46.

$$r\sin\theta=p \qquad y=p$$

**D. POLAR EQUATION OF CIRCLE.** **NOTE:** $R$ is radius.

EQUATION — Fig. 47.

$$r^2-2ar\cos(\theta-\alpha)+a^2=R^2$$

SPECIAL CASES — Fig. 48.

$$r=2a\cos\theta \qquad x^2+y^2=2ax$$

Fig. 49.

$$r=2a\sin\theta \qquad x^2+y^2=2ay$$

Fig. 50.

$$r=a\cos\theta+b\sin\theta \qquad x^2+y^2=ax+by$$

**E. POLAR EQUATION OF PARABOLA, ELLIPSE, HYPERBOLA.** **1.** Equation in polar coordinates is $r=ep/(1\pm e\cos\theta)$; $e$ is the eccentricity, $p$ the distance from directrix **L** to focus **F**. The pole is at $F$ and the directrix is perpendicular to the initial ray. **2.** When directrix is parallel to the initial ray, the equation of the conic takes the form $r=ep/(1\pm e\sin\theta)$.

Fig. 51.

## 12 PARAMETRIC EQUATIONS. A. DEFINITION.

**1.** Rectangular coordinates of points on a curve $C$ can be expressed as functions of an auxiliary variable: $x=f(t)$, $y=g(t)$. The variable $t$ is called *parameter*. **2.** When parameter is eliminated, we obtain the single equation $F(x,y)=0$ of $C$. **E:** The parametric equations $x=t+1$, $y=2t$ represent the line $y=2(x-1)$.

**B. PARAMETRIC EQUATIONS.**

| 1. LINE | 2. CIRCLE | 3. ELLIPSE | 4. HYPERBOLA |
|---|---|---|---|
| $\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$ | $x^2+y^2=a^2$ | $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ | $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ |
| $x=x_1+t(x_2-x_1)$ $y=y_1+t(y_2-y_1)$ | $x=a\cos\theta$ $y=a\sin\theta$ | $x=a\cos\theta$ $y=b\sin\theta$ | $x=a\sec\theta$ $y=b\tan\theta$ |

## 13 SOLID ANALYTIC. A. DEFINITIONS.

**1.** If two lines are *coplanar* they either *intersect* or are *parallel;* if not, they are "*skew.*" **2.** The angle between two directed coplanar lines is the angle between their *positive* directions. **3.** The angle between two oriented skew lines $L_1$ and $L_2$ is the angle between two intersecting oriented lines $L'_1$ and $L'_2$ such that $L'_1\parallel L_1$ and $L'_2\parallel L_2$. **4.** The projection of $P$ on $L$ is the foot $A$ of the perpendicular from $P$ onto $L$. $A$ is also intersection of plane through $P$ perpendicular to $L$. The projection of directed segment $PQ$ on $L$ is $AB$. **5.** Sum of projections of $PQ$ and $QR$ equals projection of $PR$. **6.** $AB=PQ\cos\theta$; $AB$ is "+" when $\theta$ is $<90°$; "−" when $\theta>90°$.

Fig. 52.

Fig. 53.

Fig. 54.

**B. COORDINATE SYSTEM.** (Fig. 54) **1.** Formed by 3 directed lines $x,y,z$ through a point $O$ (origin), each line perpendicular to other two. **2.** The coordinates of a point $P$ in space is the ordered triad of numbers $(a,b,c)$ if $a,b,c$ are the projections of $OP$ on $x$-, $y$-, $z$-axes respectively. **3.** *Coordinate planes*, determined by pairs of coordinate axes, divide space into *8 octants* (coordinates are all "+" in the 1st octant).

**C. DIRECTION COSINES AND NUMBERS.** **1.** If oriented line $L$ thru $O$ makes angles $\alpha,\beta,\gamma$ with $x,y,z$ axes (Fig. 55) then $\alpha,\beta,\gamma$, are *direction angles* of $L$ and $\cos\alpha$, $\cos\beta$, $\cos\gamma$, the *direction cosines* of $L$. **2.** Direction angles of a directed line $L'$ (not through $O$) are same as those of the directed line $L$ parallel to $L'$ through $O$. **3.** When sense of $L$ is reversed, its direction angles are changed into their *supplements* and signs of directed cosines of $L$ *reversed* **NOTE:** The direction numbers of $L$ are any three numbers $l,m,n$, proportional to direction cosines of line $L$. Any three numbers (at least one of which $\neq 0$) can be direction numbers. The projections of any directed segment on $L$ are direction numbers of $L$.

Fig. 55.

Fig. 56.

**D. BASIC FORMULAS.** (Fig. 56) Given $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$.

$x_2-x_1$; $y_2-y_1$; $z_2-z_1$ — Projection of $P_1P_2$ on the axes.

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(x_2-x_1)^2}$ — Distance between $P_1$ and $P_2$

$x=\frac{r_2x_1+r_1x_2}{r_1+r_2}$; $y=\frac{r_2y_1+r_1y_2}{r_1+r_2}$; $z=\frac{r_2z_1+r_1z_2}{r_1+r_2}$ — Point that divides $P_1P_2$ in ratio $r_1:r_2$.

$x=\frac{x_1+x_2}{2}$; $y=\frac{y_1+y_2}{2}$; $z=\frac{z_1+z_2}{2}$ — Midpoint formula

$\cos\alpha=\frac{x_2-x_1}{d}$; $\cos\beta=\frac{y_2-y_1}{d}$; $\cos\gamma=\frac{z_2-z_1}{d}$ — Direction cosines of $P_1P_2$.

$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$ Direction cosines formula.

$\cos\alpha=\frac{l}{\pm\sqrt{l^2+m^2+n^2}}$; $\cos\beta=\frac{m}{\pm\sqrt{l^2+m^2+n^2}}$; $\cos\gamma=\frac{n}{\pm\sqrt{l^2+m^2+n^2}}$ — Direction cosines of line with the direction numbers $l,m,n$.

$\cos\theta=\frac{|l_1l_2+m_1m_2+n_1n_2|}{\sqrt{l_1^2+m_1^2+n_1^2}\sqrt{l_2^2+m_2^2+n_2^2}}$ — Acute angle between two lines with direction no's. $l_1,m_1,n_1$ and $l_2,m_2,n_2$.

$\cos\theta=\cos\alpha_1\cos\alpha_2+\cos\beta_1\cos\beta_2+\cos\gamma_1\cos\gamma_2$ Angle between two directed lines.

$l_1:m_1:n_1=l_2:m_2:n_2$ Condition that 2 lines be parallel.

$l_1l_2+m_1m_2+n_1n_2=0$ Condition that 2 lines be perpendicular.

$l:m:n=\left|\frac{m_1n_1}{m_2n_2}\right|:\left|\frac{n_1l_1}{n_2l_2}\right|:\left|\frac{l_1m_1}{l_2m_2}\right|$ Direction numbers of line perpendicular to directions $l_1,m_1,n_1$ and $l_2,m_2,n_2$.

## 14 SURFACES AND CURVES. A. BASIC CONCEPTS.

**1.** A *surface* is locus of an equation in $x,y,z$ if it contains every point whose coordinates satisfy the equation, and only those. **2.** The intersection of a *surface* with a *coordinate plane* is its *trace* on that plane. **3.** The intersection of a *surface* with an *axis* are its *intercepts* on the axis. **E:** The locus of $x+2y=2$ is a plane parallel to the $z$-axis. Its trace on the $xy$-plane is a line whose equations are $x+2y=2$ and $z=0$. Its trace on the $yz$-plane is a line parallel to the $z$-axis. Its intercepts on the $x$-axis is 2; on the $y$-axis it is 1; no $z$-intercept (Fig. 57). **4.** A curve in space can be represented by the equations of *two surfaces* that contain it as their complete intersection. Sometimes *three* equations are necessary: *when two surfaces intersect in more than one curve.* **5.** A curve can also be represented *parametrically:* $x=f(t)$, $y=g(t)$, $z=h(t)$.

**E1:** $x^2+y^2+z^2=25$ is equation of sphere of radius 5 with center at origin.

**E2:** $x^2+y^2+z^2=25$, $z=3$ represent circle formed by intersection of sphere of radius 5 by a plane parallel to the $xy$-plane and 3 units above it.

Fig. 57.

Fig. 58.

**B. FORMULAS FOR PLANES** Fig. 58. **NOTE 1:** Coefficients $A,B,C$ are direction numbers of any normal to the plane; $\cos\alpha$, $\cos\beta$, $\cos\gamma$, are the direction cosines of the normal. **NOTE 2:** Plane $S_1$ is $A_1x+B_1y+C_1z+D_1=0$ and $S_2$ is $A_2x+B_2y+C_2z+D_2=0$.

$Ax+By+Cz+D=0$ — General form (every plane has an equation of the first degree, and vice versa).

$x\cos\alpha+y\cos\beta+z\cos\gamma-p=0$ — Normal form (length of normal from origin is $p$).

$\frac{Ax+By+Cz+D}{\pm\sqrt{A^2+B^2+C^2}}=0$ — Transformation of general equation to normal form.

$l(x-x_1)+m(y-y_1)+n(z-z_1)=0$ — Plane determined by point $(x_1,y_1,z_1)$, normal direction $l,m,n$.

$(x/a)+(y/b)+(z/c)=1$ — Intercept form (3 non-zero intercepts).

$Ax+By+D=0$ — Plane parallel to $z$-axis.

$z=k$ — Plane parallel to $xy$-plane.

$\cos\theta=\frac{|A_1A_2+B_1B_2+C_1C_2|}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}}$ — Acute angle between $S_1$ and $S_2$.

$A_1:B_1:C_1=A_2:B_2:C_2$ — Condition that $S_1$ be parallel to $S_2$.

$A_1:B_1:C_1:D_1=A_2:B_2:C_2:D_2$ — Condition that $S_1,S_2$ coincide.

$A_1A_2+B_1B_2+C_1C_2=0$ — Condition that $S_1$ be perpendicular to $S_2$.

$(A_1x+B_1y+C_1z+D_1)+k(A_2x+B_2y+C_2z+D_2)=0$ — Pencil of planes determined by $S_1$ and $S_2$.

$\frac{|A_1x_1+B_1y_1+C_1z_1+D_1|}{\sqrt{A_1^2+B_1^2+C_1^2}}$ — Distance from $S_1$ to $P_1(x_1,y_1,z_1)$.

$d=|x_1\cos\alpha+y_1\cos\beta+z_1\cos\gamma-p|$ — Dist. from plane to $P_1(x_1,y_1,z_1)$.

**E:** Find the distance of $P(3,1,-2)$ from the plane $2x+y+2z-6=0$. **Sol.** Evaluate the normal form: $(2x+y+2z-6)\div(\sqrt{2^2+1^2+2^2})=|2(3)+1(1)+2(-2)-6|\div 3=1$.

**C. STANDARD FORMS FOR EQ. OF A STRAIGHT LINE** $\left.\begin{matrix}A_1x+B_1y+C_1z+D_1=0\\ A_2x+B_2y+C_2z+D_2=0\end{matrix}\right\}$ Line determined by intersection of two planes.

$x=az+b$; $y=cz+d$; Line not parallel to $xy$-plane (projection form).

$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ Line determined by point $(x_1,y_1,z_1)$; direction numbers $(l,m,n)$ (Symmetric form).

$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$ Two point form for line thru $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$.

$x=a_1t+b_1$; $y=a_2t+b_2$; $x=a_3t+b_3$ Parametric form for line thru pt. $(b_1,b_2,b_3)$ with direction numbers $a_1,a_2,a_3$.

**E:** Find equations of the line thru $P(1,2,-1)$ and perpendicular to plane $4x+2y-3z+1=0$. **Sol.** The direction numbers are 4, 2, −3 giving $(x-1)/4=(y-2)/2=(z+1)/-3$.

## 15 QUADRIC SURFACES. A. GENERAL EQUATION OF THE 2ND DEGREE: $ax^2+by^2+cz^2+dxy+eyz+fxz+gx+hy+iz+m=0$.

**1. A *quadric surface*** has a 2nd deg. equation. **2.** When *left side* is factorable into two linear factors, the quadric is *improper* (degenerate) and is composed of a pair of planes. **E:** $x^2-y^2=0$ represents the planes $x-y=0$ and $x+y=0$. **NOTE:** A 2nd degree equation may have no locus at all ($x^2+y^2+z^2+1=0$) or represent a line ($x^2+y^2=0$).

**B. SPHERES.** A sphere is the locus of a point $(x,y,z)$ at a given distance $r$ from a given center point $(x_0,y_0,z_0)$. (Fig. 59.)

| | EQUATION OF SPHERE | CENTER | RADIUS |
|---|---|---|---|
| 1 | $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2$ | $(x_0,y_0,z_0)$ | $r$ |
| 2 | $x^2+y^2+z^2+ax+by+cz+d=0$ | $(-a/2,-b/2,-c/2)$ | $\frac{\sqrt{a^2+b^2+c^2-4d}}{2}$ |

$a^2+b^2+c^2-4d=0$ — locus is degenerate (a point sphere)

$a^2+b^2+c^2-4d<0$ — no locus (imaginary sphere)

Fig. 59.

Fig. 60.

Fig. 61.

Fig. 62.

**C. CYLINDERS.** **1.** A *cylinder* is the locus of the points on the straight lines that intersect a given plane curve $c$ (other than a straight line) and are perpendicular to the plane of $c$. **2.** The lines are called *rulings* or *generators* of the cylinder and $c$ is its *directrix.* When $c$ is a *conic section,* the cylinder is called *quadric cylinder.* **3.** An equation of the 2nd degree in which one variable is absent represents a quadric cylinder whose generators are parallel to the axis corresponding to the absent variable. (Fig. 60, Fig. 61.)

**D. CONES.** **1.** A *cone* is the locus of the points on lines that join the points of a given plane curve $c$ (not a straight line) to a fixed point O not in plane of $c$. **2.** O is *vertex*, $c$ the *directrix*, and lines joining O to $c$ are the *rulings* or *generators*. The two parts of the surface that meet at O are called the *nappes*. **3.** A *quadric cone* is one whose directrix is a proper conic section. A quadric cone whose vertex is at the origin is represented by a homogeneous equation of the 2nd degree: $ax^2+by^2+cz^2+dxy+eyz+fxz=0$ (Fig. 62.)

**E. SURFACES OF REVOLUTION.** **1.** When a plane curve is rotated about an axis line $L$ in the plane of the curve, a surface of revolution is generated. **2.** Spheres, circular cylinders, circular cones, and surfaces generated by rotating conics about their axes, are quadric surfaces of revolution. **3.** The equation of the surface generated by revolving a curve about a coordinate axis in its plane, is as follows:

| REV. | EQ. | REPLACE | REV. | EQ. | REPLACE | REV. | EQ. | REPLACE |
|---|---|---|---|---|---|---|---|---|
| Z- AXIS | $f(y,z)=0$ | $\|y\|$ by $\sqrt{x^2+y^2}$ | Y- AXIS | $f(x,y)=0$ | $\|x\|$ by $\sqrt{x^2+z^2}$ | X- AXIS | $f(x,y)=0$ | $\|y\|$ by $\sqrt{y^2+z^2}$ |
| | $f(x,z)=0$ | $\|x\|$ | | $f(y,z)=0$ | $\|z\|$ | | $f(x,z)=0$ | $\|z\|$ |

**E:** Find the equation of the surface generated when the ellipse $x^2+2y^2=1$ is rotated around the $y$-axis. **Sol.:** Substitute $x^2+z^2$ for $x^2$; the surface of revolution is $x^2+z^2+2y^2=1$.

DATA-GUIDE QUICK CHART

# PRINCIPLES OF PLANE GEOMETRY

A SIMPLIFIED SUMMARY & INSTANT REFERENCE

LEARN & REVIEW IN MINUTES

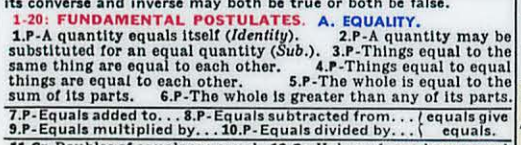
AUTHORS: W. Manheimer, Chairman Math. Dept., Franklin K. Lane School, N. Y. M. Cohen, Instructor in Math. at the Bayside School and Queens College, N. Y.

KEY: P- postulate; Cr- corollary; T- theorem; Cv- converse; D- definition; K- area; C- circumference. All statements are numbered consecutively. The numbers in [brackets] at the end of a statement indicate those P's, Cr's, D's or T's used in its proof.
NOTE: PROPOSITION: If A is true, then B is true. CONVERSE: If B is true, then A is true. INVERSE: If A is not true, then B is not true. CONTRAPOSITIVE: If B is not true, then A is not true. NOTE: If a proposition is true, its contrapositive must also be true; however, its converse and inverse may both be true or both be false.

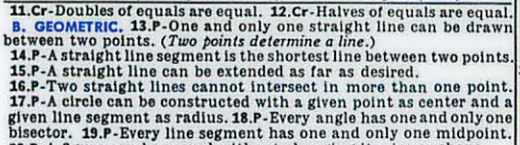
## 1-20: FUNDAMENTAL POSTULATES. A. EQUALITY.

1.P-A quantity equals itself (*Identity*). 2.P-A quantity may be substituted for an equal quantity (*Sub.*). 3.P-Things equal to the same thing are equal to each other. 4.P-Things equal to equal things are equal to each other. 5.P-The whole is equal to the sum of its parts. 6.P-The whole is greater than any of its parts.
7.P-Equals added to... 8.P-Equals subtracted from... } equals give
9.P-Equals multiplied by... 10.P-Equals divided by... } equals.
11.Cr-Doubles of equals are equal. 12.Cr-Halves of equals are equal.

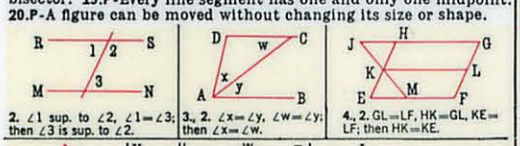
**B. GEOMETRIC.** 13.P-One and only one straight line can be drawn between two points. (*Two points determine a line.*)
14.P-A straight line segment is the shortest line between two points.
15.P-A straight line can be extended as far as desired.
16.P-Two straight lines cannot intersect in more than one point.
17.P-A circle can be constructed with a given point as center and a given line segment as radius. 18.P-Every angle has one and only one bisector. 19.P-Every line segment has one and only one midpoint.
20.P-A figure can be moved without changing its size or shape.



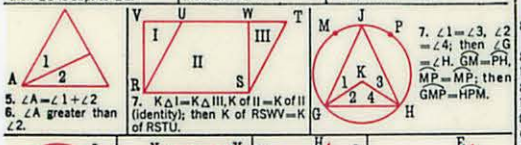

2. ∠1 sup. to ∠2, ∠1=∠3; then ∠3 is sup. to ∠2. 3. 2. ∠x=∠y, ∠w=∠y; then ∠x=∠w. 4. 2. GL=LF, HK=GL, KE=LF; then HK=KE.




5. ∠A=∠1+∠2
6. ∠A greater than ∠2.
7. K△I=K△III, K of II=K of II (identity); then K of RSWV=K of RSTU.
7. ∠1=∠3, ∠2=∠4; then ∠G=∠H, GM=PH, MP=MP; then GMP=HPM.




8. RQT=RST, RQ=RS; then QT=ST.
8. K△VWY=K△VWX, K△III=K△III; then K△I=K△II.
8. ∠GHL=∠HLM, ∠1=∠4; then ∠2=∠3
8. AC=BD, BC=BC; then AB=CD.




9., 11. RM=SP, R and S are midpoints; then LM=NP.
10. AD=AB; F, E, G, H trisection pts.; then AF=AG.
10., 12. AC=CB; D and E are midpoints; then AD=EB.

## 21-30: 1st THEOREMS. A. SUPPLEMENTS, COMPLEMENTS, EQUALITY.

TWO ADJACENT ANGLES WITH EXTERIOR SIDES
21.T-on the same straight line are supplementary.
22.T-forming a right angle are complementary.
23.T-Supplements of the same angle or of equal angles are equal.
24.T-Complements of the same angle or of equal angles are equal.
25.T-All straight angles... 26.T-All right angles... 27.T-Vertical angles... } are equal.
28.T-If two angles are equal and supplementary, they are rt. angles.

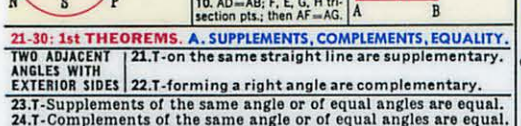
**B. SUM.** 29.T-The sum of the successive angles about a point=360°.
30.T-The sum of the successive angles on one side of a line=180°.

## 31-34: CONGRUENT TRIANGLES. A. CONDITIONS. [See also 42.]

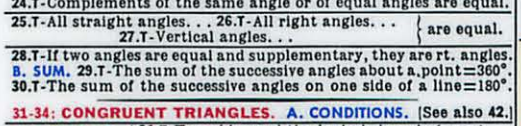
TWO TRIANGLES ARE CONGRUENT IF THEY AGREE RESPECTIVELY IN...
31.T-Two sides and the included angle (s.a.s.).
32.T-Two angles and the included side (a.s.a.).
33.T-Three sides (s.s.s.).
34.T-Two angles and a side opposite one of them (s.a.a.). [NOTE: Proof depends on 63.]

## 35-44: THEOREMS DERIVED FROM CONGRUENT TRIANGLES.

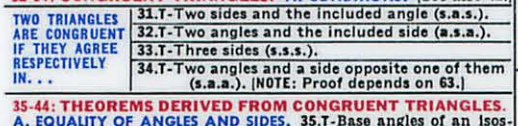
**A. EQUALITY OF ANGLES AND SIDES.** 35.T-Base angles of an isosceles triangle are equal. (Alternate statement: *If two sides of a triangle are equal, the angles opposite them are equal.*) [18, 31].
36.Cr-An equilateral triangle is also equiangular. [35].
37.T-(*Converse of* 35) If two angles of a triangle are equal, the sides opposite them are equal. [18, 34].
38.Cr-An equiangular triangle is also equilateral. [37].




23. ∠1=∠2, ∠x sup. ∠2, ∠y sup. ∠1; then ∠x=∠y. 24. ∠v=∠w, ∠v comp. ∠3, ∠w comp. ∠4; then ∠3=∠4.
35. CA=CB, then ∠A=∠B. 37. ∠A=∠x, then LA=LJ.
40. AX=XB then X lies on ⊥ bis. of AB.




DISTANCE BETWEEN: 2 POINTS is line segment joining them; POINT AND LINE is ⊥ from pt. to line; 2 PARALLEL LINES is any common ⊥; POINT AND CIRCLE is shortest line segment from pt. to circle along a radius (extended if necessary).




41. AX=XB, AY=YB; then XY will be ⊥ bis. of AB.
44. Dist. PX=dist. PY, then bisector of ∠B passes through P.

**B. PERPENDICULAR BISECTOR.** 39.T-Any point on the perpendicular bisector of a line segment is equidistant from segment ends. [31].
40.Cv-Any point equidistant from the ends of a line segment lies on its perpendicular bisector. [19, 33, 28].
41.T-Two points each equidistant from the ends of a line segment determine its perpendicular bisector. [33, 31, 28].

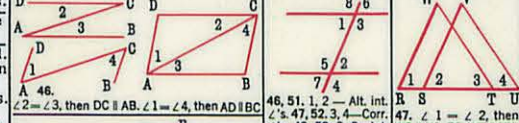
**C. CONGRUENCE.** 42.T-Two right triangles are congruent if the hypotenuse and a leg of one equal respectively the hypotenuse and a leg of the other. [20, 34].

**D. ANGLE BISECTOR.** 43.T-Any point on an angle bisector is equidistant from the sides of the angle. [34]. 44.T-(*Converse*) Any point equidistant from sides of an angle lies on the angle bisector. [42].

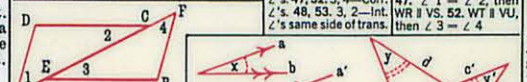
## 45-56: PARALLEL LINES. A. EUCLIDEAN POST.

45.P-Through a point outside a line there exists one and only one parallel line.

| B. TWO LINES ARE PARALLEL IF... | C. IF TWO LINES ARE PARALLEL, THEN... |
|---|---|
| 46.T-A pair of alternate interior angles are equal. [Indirect, 102]. | 51.T-*Alternate interior angles are equal.* [Indirect, 45]. |
| 47.T-A pair of corresponding angles are equal. [46]. | 52.T-*Corresponding angles are equal.* [51]. |
| 48.Cr-A pair of interior angles on the same side of the transversal are supplementary. [46]. | 53.Cr-*Interior angles on the same side of the transversal are supplementary.* [51]. |
| 49.Cr-Both are parallel to the same line. [51, 46 or 47]. | 54.Cr-*A line parallel to one of them is parallel to the other.* [51, 46 or 47]. |
| 50.Cr-Both are perpendicular to the same line. [46]. | 55.Cr-*A line perpendicular to one is perpendicular to the other.* [51]. |




A 46. ∠2=∠3, then DC∥AB. ∠1=∠4, then AD∥BC
46, 51, 1, 2 — Alt. int. ∠'s. 47, 52, 3, 4—Corr. ∠'s. 48, 53, 3, 2—Int. ∠'s same side of trans.
47. ∠1=∠2, then WR∥VS. 52. WT∥VU, then ∠3=∠4




A 51. DC∥EB then DA∥FB then ∠1=∠4.
56. a∥a', b∥b'; then ∠x=∠x'
56. c⊥c', d⊥d'; then ∠y=∠y'

**D. SIDES OF ACUTE ANGLES PARALLEL (OR PERPENDICULAR).** 56.T-If two acute angles have their sides respectively parallel (*or perpendicular*), they are equal. [52 (*63*)].

## 57-64: SUM OF ANGLES OF A TRIANGLE.

57.T-The sum of the angles of a triangle is a straight angle. [51].

**A. COROLLARIES.** 58.Cr-In any triangle, there can be only one right (*or obtuse*) angle. 59.Cr-For any triangle, an exterior angle equals the sum of the two non-adjacent (*remote*) interior angles. 60.Cr-In a right triangle, the acute angles are complementary. 61.Cr-In an equilateral triangle, each angle =60°. 62.Cr-In a quadrilateral, the sum of the angles =360°. 63.Cr-If two angles of one triangle equal two angles of another triangle, the third angles are equal. 64.Cr-[See 34].

## 65-83: PARALLELOGRAMS. A. PROPERTIES

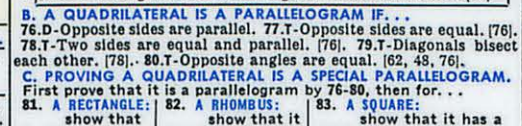
65.D-Opposite sides are parallel.
66.T-A diagonal divides it into two congruent triangles. [32].
67.Cr-Opposite sides are equal. [66].
68.Cr-Opposite angles are equal. [66].
69.T-Diagonals bisect each other. [32].
70.T-Consecutive angles are supplementary. [53].
71.T-All angles are right angles. [70].
72.T-Diagonals are equal. [31].
73.T-All sides are equal. [67].
74.T-Diagonals are perpendicular to each other. [41].
75.T-Each diagonal bisects the angles to which it is drawn. [33].

PARALLELOGRAM 65-70 — RECTANGLE 65-72 — SQUARE 65-75 — RHOMBUS 65-70, 73-75

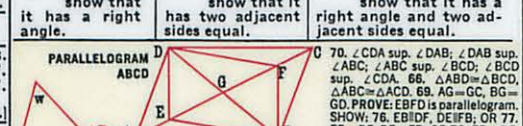
**B. A QUADRILATERAL IS A PARALLELOGRAM IF...** 76.D-Opposite sides are parallel. 77.T-Opposite sides are equal. [76]. 78.T-Two sides are equal and parallel. [76]. 79.T-Diagonals bisect each other. [78]. 80.T-Opposite angles are equal. [62, 48, 76].

**C. PROVING A QUADRILATERAL IS A SPECIAL PARALLELOGRAM.** First prove that it is a parallelogram by 76-80, then for...

| 81. A RECTANGLE: show that it has a right angle. | 82. A RHOMBUS: show that it has two adjacent sides equal. | 83. A SQUARE: show that it has a right angle and two adjacent sides equal. |
|---|---|---|




59. ∠e=∠w+∠z.
Parallelogram ABCD. 65. AB∥DC, AD∥BC. 67. AB=DC, AD=BC. 68. ∠DAB=∠BCD, ∠ABC=∠ADC.
70. ∠CDA sup. ∠DAB; ∠DAB sup. ∠ABC; ∠ABC sup. ∠BCD; ∠BCD sup. ∠CDA. 66. △ABD≅△BCD, △ABC≅△ACD. 69. AG=GC, BG=GD. PROVE: EBFD is parallelogram. SHOW: 76. EB∥DF, DE∥FB; OR 77. EB=DF, DE=FB; OR 78. EB=DF, EB∥DF; OR DE=FB, DE∥FB; OR 79. DG=GB, EG=GF; OR 80. ∠DEB=∠BFD, ∠EBF=∠EDF.




## 84-91: THEOREMS DERIVED FROM PARALLELOGRAMS.

**A. PARALLELS AND EQUAL SEGMENTS.** 84.Cr-Segments of parallels intercepted between parallels are equal. [67]. 85.Cr-Parallel lines are everywhere equidistant. [67]. 86.T-If three or more parallels cut off equal segments on one transversal, they cut off equal segments on any transversal. [67, 34].

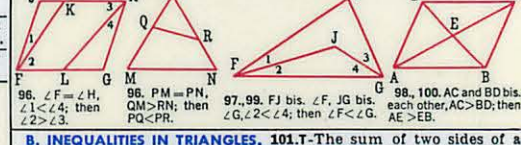
**B. LINES BISECTING SIDES OF A TRIANGLE.** 87.T-If a line is parallel to one side of a triangle and bisects a second side, it also bisects the third side. [86]. 88.T-If a line joins the midpoints of two sides of a triangle, it is parallel to, and equals 1/2, the third side. [78].

**C. MEDIANS.** 89.T-The median to the hypotenuse of a right triangle equals 1/2 the hypotenuse. [79, 81, 72]. 90.T-The medians of a triangle are concurrent at a point that is 2/3 the distance from any vertex to the midpoint of the opposite side. [88, 78]. 91.T-The median of a trapezoid is parallel to the bases and equal to 1/2 their sum. [88]. NOTE: In a triangle, the **perpendicular bisectors** of the sides, the **medians**, the **angle bisectors**, and the **altitudes** are respectively concurrent.

## 92-106: INEQUALITIES. A. POSTULATES. (*See also 6, 14.*)


92. ∠x>∠2, ∠2>∠1; then ∠x>∠1. 2. ∠x=∠y, ∠y>∠1; ∠x>∠1.
93. ∠2=∠4, ∠1<∠3; then ∠A<∠B. AD=BD, DF>DE; AF>BE.
94. LM>LR, MN>RP; then LN>LP.
95. ∠Q>∠S, ∠1=∠2; then ∠3>∠4.

| | |
|---|---|
| 92.P-If the first of three quantities is greater than the second and the second is greater than the third, then first is greater than third. | *If a>b and b>c, then a>c.* |
| 93.P-If unequals are added to equals, the results are unequal in the same order. | *If a=b and c>d, then a+c>b+d.* |
| 94.P-If unequals are added to unequals in the same order, the results are unequal in the same order. | *If a>b and c>d, then a+c>b+d.* |
| 95.P-If equals are subtracted from unequals, the results are unequal in the same order. | *If a>b and c=d, then a−c>b−d.* |
| 96.P-If unequals are subtracted from equals, the results are unequal in the *opposite* order. | *If a=b and c>d, then a−c<b−d.* |
| 97.P-If unequals are multiplied (98.P-or divided) by (*positive*) equals, the results are unequal in the same order. | *If x>y and z=w, then xz>yw; x/z>y/w. (z, w>0).* |
| 99.Cr-Doubles of unequals / 100.Cr-Halves of unequals } are unequal in the same order. | *If a>b, then 2a>2b and a/2>b/2.* |




96. ∠F=∠H, ∠1<∠4; then ∠2>∠3.
96. PM=PN, QM>RN; then PQ<PR.
97.,99. FJ bis. ∠F, JG bis. ∠G, ∠2<∠4; then ∠F<∠G.
98.,100. AC and BD bis. each other, AC>BD; then AE>EB.

**B. INEQUALITIES IN TRIANGLES.** 101.T-The sum of two sides of a triangle is greater than the third side. [14]. 102.T-An exterior angle of a triangle is greater than either non-adjacent (*remote*) interior angle. [31, 6]. (*This theorem normally precedes Section 45-56.*) 103.T-If two sides of a triangle are unequal, the opposite angles are unequal in the same order. [102]. (*Greater angle lies opposite the greater side.*)
104.Cv-If two angles of a triangle are unequal, the opposite sides are unequal in the same order. [101; or indirect by 35, 103]. (*Greater side lies opposite the greater angle.*) 105.T-The perpendicular is the shortest line that can be drawn to a given line from an outside point; therefore the hypotenuse of a right triangle is its longest side. [104]. 106.T-If two triangles have two sides of one equal respectively to two sides of the other and the included angles are unequal, the triangle that has the greater included angle has the greater third side. [20, 18, 31, 101].

## 107-123: CIRCLES. A. FUNDAMENTAL THEOREMS.

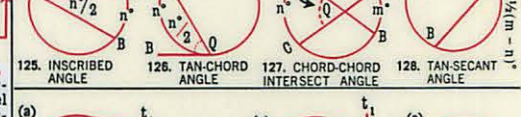
107.T-Radii } of the same or of equal circles are equal.
108.T-Diameters }
109.T-A diameter bisects a circle (*into 2 semicircles*).
110.T-A point is { inside / on / outside } circle if distance from center { < / = / > } radius.

**B. IN A CIRCLE OR IN EQUAL CIRCLES:**
111.T-Equal central angles have equal minor (*or major*) arcs.
112.T-Equal minor (*or major*) arcs have equal central angles.
113.T-Equal minor (*or major*) arcs have equal chords.
114.T-Equal chords have equal minor (*or major*) arcs.

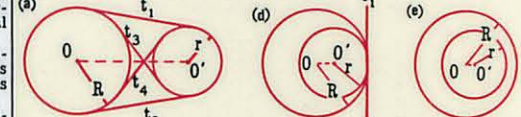
**C. CHORDS.** 115.T-The diameter perpendicular to a chord bisects the chord and its minor and major arcs. [42, 111]. 116.T-(*A converse*) The perpendicular bisector of a chord passes through the center of the circle. [40]. 117.T-In a circle or in equal circles, equal chords are equidistant from the center. [115, 42]. 118.T-(*Converse*) In a circle or in equal circles, chords that are equidistant from the center are equal. [42, 115]. 119.T-In the same circle or in equal circles, the greater of two central angles has the greater arc and the greater chord. [106]. 120.T-In a circle or in equal circles, if two chords are unequal, the greater chord is nearer the center. [117, 104].

**D. TANGENTS.** 121.T-A tangent to a circle is perpendicular to the radius drawn to the point of contact. [105]. 122.T-(*A converse*) A line perpendicular to a radius at its outer extremity is a tangent to the circle. [105]. 123.T-Tangents to a circle from an external point are equal. [121, 42].

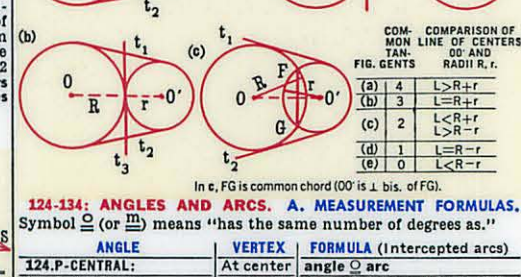

115. OE⊥AB; then AE=EB, AD=DB, AC=CB.
118. OE⊥CD, OF⊥AB, OE=OF; then CD=AB.
123. AB, AC and AD are tangents; then AB=AD, AC=AD.




107.–123. Tangent—RB, Pt. of contact—A; Secants—FC, ED.
107.–123. Radii—OA, OC, OD; Diameter—AC; Chord—FE;
124. CENTRAL ANGLE




125. INSCRIBED ANGLE
126. TAN-CHORD ANGLE
127. CHORD-CHORD INTERSECT ANGLE
128. TAN-SECANT ANGLE




| FIG. | COMMON TANGENTS | COMPARISON OF LINE OF CENTERS 00' AND RADII R, r. |
|---|---|---|
| (a) | 4 | L>R+r |
| (b) | 3 | L=R+r |
| (c) | 2 | L<R+r, L>R−r |
| (d) | 1 | L=R−r |
| (e) | 0 | L<R−r |

In c, FG is common chord (00' is ⊥ bis. of FG).

## 124-134: ANGLES AND ARCS. A. MEASUREMENT FORMULAS.

Symbol ≙ (or $\overset{m}{=}$) means "has the same number of degrees as."

| ANGLE | VERTEX | FORMULA (Intercepted arcs) |
|---|---|---|
| 124.P-CENTRAL: | At center | angle ≙ arc |
| 125.T-INSCRIBED: | On circle | angle ≙ 1/2 arc [35, 59, 124] |
| 126.T-TANGENT-CHORD: | On circle | angle ≙ 1/2 arc [121, 125] |
| 127.T-FORMED BY TWO INTERSECTING CHORDS: | Inside circle | angle ≙ 1/2 (arc + opposite arc) [59, 125] |
| 128.T-FORMED BY TWO SECANTS, TWO TANGENTS, OR A SECANT AND A TANGENT: | Outside circle | angle ≙ 1/2 (larger arc—smaller arc) [59, 125, 126] |
| 129.Cr—FORMED BY TWO TANGENTS (SPECIAL CASE): | Outside circle | angle ≙ 180°—minor arc [128] |

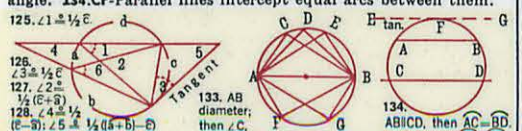
SOLID PLASTIC

QUICK CHARTS SPEED STUDY: PUT THE IMPORTANT FACTS IN FULL VIEW TO SAVE YOU TIME AND WORK.

QUICK CHARTS IMPROVE MEMORY: PERMANENTLY FILL IN FOR MISSING NOTES & TEXTS IN BASIC & ADVANCED COURSES.

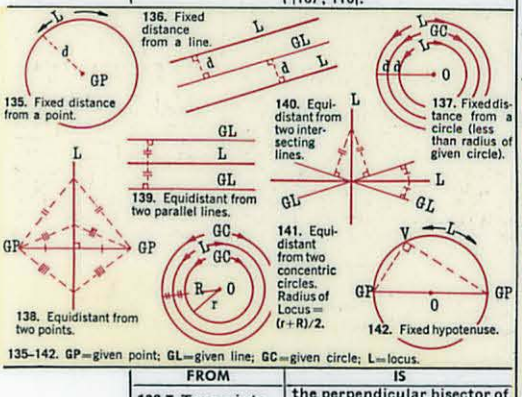
CONTAINS NO PAPER

**B. COROLLARIES OF THE INSCRIBED ANGLE THEOREM.** [125]. **130.Cr**-If two inscribed angles intercept the same arc or equal arcs, they are equal. **131.Cr**-Equal inscribed angles intercept equal arcs. **132.Cr**-The opposite angles of an inscribed quadrilateral are supplementary. **133.Cr**-An angle inscribed in a semicircle is a right angle. **134.Cr**-Parallel lines intercept equal arcs between them.




**135-142: LOCUS. A. PROVING A LOCUS.** ▶Prove that every point on the proposed locus satisfies the given condition; and ▶Prove that every point that satisfies the given condition lies on proposed locus.

| | FROM | IS |
|---|---|---|
| **B. LOCUS OF POINTS AT A GIVEN DISTANCE** | **135.T**-A point | a circle. [107; 110]. |
| | **136.T**-A line | two parallel lines. [85; 78, 45]. |
| | **137.T**-A circle | two concentric circles, if distance is less than the radius. [107; 110]. |




135-142. GP=given point; GL=given line; GC=given circle; L=locus.

| | FROM | IS |
|---|---|---|
| **C. LOCUS OF POINTS EQUIDISTANT** | **138.T**-Two points | the perpendicular bisector of the line joining them. [39; 40]. |
| | **139.T**-Two parallel lines | a third parallel midway between them. [85; 78, 45]. |
| | **140.T**-Two intersecting lines | two (perpendicular) lines bisecting the four angles formed. [43; 44]. |
| | **141.T**-Two concentric circles | a third concentric circle with radius equal to average of the two given radii. [107; 110]. |

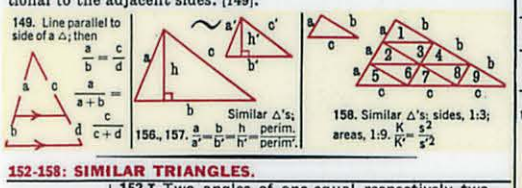
**D. SPECIAL CASE.** **142.T**-The locus of the vertex of a right angle whose sides pass through the ends of a given line segment is the circle whose diameter is that segment. [133; indirect, 127, 128].

**E. INTERSECTION OF LOCI.** The locus of points obeying two conditions is the intersection of loci for each condition. E: Locus of points equidistant from pts. *A* and *B* and at a fixed distance from one of them, *B*. ANS: The pair of points (if any) common to the perpendicular bisector of *AB* and the circle with center at *B*.

**143-151: PROPORTIONS. A. FUNDAMENTAL THEOREMS.**

| | |
|---|---|
| **143.T**-The product of the means equals the product of the extremes. [9]. | *If a/b=c/d, then ad=bc.* |
| **144.T**-(Converse). When product of two quantities equals product of two other quantities, a proportion can be formed by making one pair the means and the other pair the extremes. [10]. | *If rs=pq, then r/p=q/s, r/q=p/s, s/p=q/r, etc. (8 proportions in all.)* |
| **145.T**-If three terms of one proportion equal respectively three terms of another, the 4th terms are equal. [3]. | *If a/b=c/x and a/b=c/y, then x=y.* |
| **146.T**-A proportion may be transformed by inversion. [143, 144]. | *If p/q=r/s, then q/p=s/r.* |
| **147.T**-A proportion may be transformed by alternation. [143, 144]. | *If x/y=z/w, then x/z=y/w.* |
| **148.T**-A proportion may be formed by addition (*or subtraction*). [7, (8)]. | *If a/b=c/d, then (a+b)/b=(c+d)/d and (a−b)/b=(c−d)/d.* |

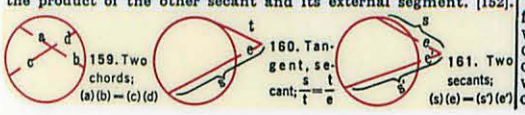
**B. PROPORTIONS OF SEGMENTS AND SIDES OF A TRIANGLE.** **149.T**-A line parallel to a side of a triangle and intersecting other two sides divides those sides proportionally. [86]. **150.T**-(*Converse*) A line that divides two sides of a triangle proportionally is parallel to 3rd side. [indirect, 149]. **151.T**-The bisector of an angle of a triangle divides the opposite side into segments that are proportional to the adjacent sides. [149].




**152-158: SIMILAR TRIANGLES.**

| | |
|---|---|
| **A. TWO TRIANGLES ARE SIMILAR IF...** | **152.T**-Two angles of one equal respectively two angles of the other. [149]. **153.Cr**-If two triangles are similar to the same triangle, they are similar to each other. [152]. |
| | **154.T**-Two sides of one triangle are proportional to two sides of the other and the included angles are equal. [150, 152]. |
| | **155.T**-Their sides are respectively proportional. [154]. |
| **B. IF TWO TRIANGLES ARE SIMILAR...** | **156.T**-Corresponding altitudes (angle bisectors or medians) are proportional to corresponding sides. [152 or 154]. |
| | **157.T**-Perimeters are proportional to corresponding sides. $p/p'=s/s'$ [148]. |
| | **158.T**-Areas are proportional to squares of corr. sides or altitudes: $K/K'=s^2/s'^2$. [169, 156]. |

**159-164: THEOREMS (FORMULAS) DERIVED FROM SIMILAR TRIANGLES. A. LINES IN CIRCLES.** **159.T**-If two chords intersect in a circle, the product of the segments of one equals the product of the segments of the other. [152]. **160.T**-If a tangent and a secant are drawn to a circle from an external point, the tangent is the mean proportional between the secant and its external segment. [152]. **161.T**-If two secants are drawn to a circle from an external point, the product of one secant and its external segment equals the product of the other secant and its external segment. [152].

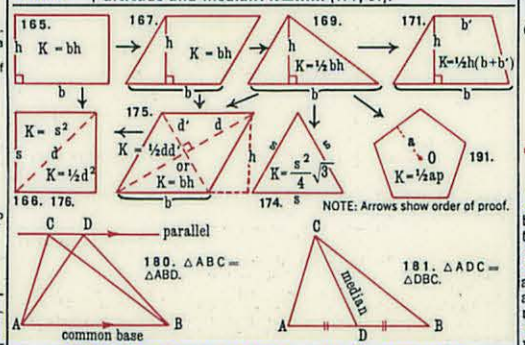


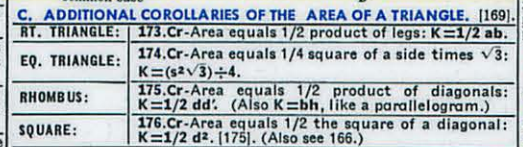

**B. RIGHT TRIANGLES.** **162.T**-If an altitude is drawn to the hypotenuse of a right triangle: (a) The altitude is the mean proportional between the segments of the hypotenuse; (b) Either leg is the mean proportional between the whole hypotenuse and the segment of the hypotenuse adjacent to that leg. [152]. **163.T-Pythagorean Theorem.** The sum of the squares of the legs of a right triangle equals the square of the hypotenuse: $a^2+b^2=c^2$. [162b]. **164.T**-(*Converse*) If the sum of the squares of two sides of a triangle equals the square of the third, it is a right triangle having the third side as the hypotenuse. [163, 33]. (See diagrams in *Section 212-222*.)

**165-181: AREA. A. BASIC CONCEPTS.** Area is amount of surface contained in a closed figure (*polygon, circle*, etc.). Unit of area is area within a square having a side of unit length,e.g.,square inch, square foot. Equal (*equivalent*) figures have equal areas; sides, shapes, need not correspond. NOTE: Congruent figures are equal.

**B. QUADRILATERALS AND TRIANGLES.** (K represents area.)

| | |
|---|---|
| **RECTANGLE:** | **165.P**-Area = product of its base and altitude: $K=bh$. **166.Cr**-Area of a square equals a side squared: $K=s^2$. |
| **PARALLELOGRAM:** | **167.T**-Area equals product of a side (or base) and an altitude to that side: $K=bh$. [165]. **168.Cr**-Area equals product of two adjacent sides and sine of included angle: $K=ab\sin C$. |
| **TRIANGLE:** | **169.T**-Area is 1/2 product of side (or base) and altitude to that side: $K=1/2\ bh$. [167]. **170.Cr**-Area equals 1/2 product of two sides and sine of included angle: $K=1/2\ ab\sin C$. [See also 173-176]. |
| **TRAPEZOID:** | **171.T**-Area equals 1/2 altitude times sum of bases: $K=1/2\ h(b+b')$. [169]. **172.Cr**-Area equals product of altitude and median: $K=hm$. [171, 91]. |




**C. ADDITIONAL COROLLARIES OF THE AREA OF A TRIANGLE.** [169].

| | |
|---|---|
| **RT. TRIANGLE:** | **173.Cr**-Area equals 1/2 product of legs: $K=1/2\ ab$. |
| **EQ. TRIANGLE:** | **174.Cr**-Area equals 1/4 square of a side times $\sqrt{3}$: $K=(s^2\sqrt{3})\div 4$. |
| **RHOMBUS:** | **175.Cr**-Area equals 1/2 product of diagonals: $K=1/2\ dd'$. (Also $K=bh$, like a parallelogram.) |
| **SQUARE:** | **176.Cr**-Area equals 1/2 the square of a diagonal: $K=1/2\ d^2$. [175]. (Also see 166.) |

**D. 2 RECTANGLES, 2 PARALLELOGRAMS, OR 2 TRIANGLES.** **177.Cr**-The areas are to each other as the products of their bases and altitudes. [165, 167 or 169]. **178.Cr**-If two rectangles, parallelograms, or triangles have equal (**a**) altitudes, (**b**) bases, their areas are to each other as their (**a**) bases, (**b**) altitudes. [165, 167 or 169].

| | |
|---|---|
| **E. TWO TRIANGLES ARE EQUAL IN AREA IF...** | **179.T**-They have equal bases and equal altitudes. [169]. |
| | **180.Cr**-They have a common base and their vertices lie on a line parallel to that base. [179]. |

**181.Cr**-Median of triangle divides it into two equal triangles. [179].

**182-191: POLYGONS. A. SUM OF ANGLES.** EXAMPLES.

| | |
|---|---|
| **182.T**-Sum of interior angles of an n-sided polygon is (n-2) str. angles. [57]: $S_{int}=(n-2)180°$. | *Octagon:* $S_{int}=(8-2)180°=1080°$ |
| **183.T**-Sum of exterior angles of n-sided polygon is two str. angles. [182]: $S_{ext}=2(180°)=360°$. | *Octagon:* $S_{ext}=360°$ |

**B. SIMILAR POLYGONS** (Abbreviation: s.p.'s).

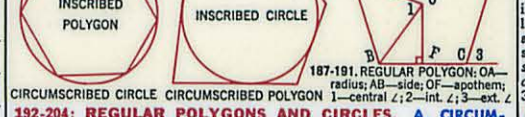
| | |
|---|---|
| **184.T**-Perimeters of s.p.'s are in same ratio as any pair of corresponding sides or lines. [148]: $p/p'=s/s'$. | *Corresponding sides 3, 5; $p/p'=3/5$* |
| **185.T**-Areas of s.p.'s are to each other as the squares of corresponding sides. [158]: $K/K'=s^2/s'^2$. | *From example above: $K/K'=9/25$* |

**186.T**-S.p.'s can be divided into the same number of triangles, similar each to each and similarly placed. [154].

**C. REGULAR POLYGONS** (Abbreviation: r.p.'s).

| | |
|---|---|
| **187.T**-An interior angle of a r.p. of n sides equals $180(n-2)/n$ degrees. [182]. **188.Cr**-Two r.p.'s with the same number of sides are similar. [187, 10]. | *Pentagon:* $A=(3/5)180°=108°$ |
| **189.T**-An exterior angle of a regular polygon of n sides equals 360/n degrees. [183]: $E=360°/n$. | *Hexagon:* $E=360/6=60°$ |
| **190.T**-A central angle of a regular polygon of n sides equals 360/n degrees. [29]. $C=360°/n$. | *Hexagon:* $C=360/6=60°$ |
| **191.T**-The area of a r.p. equals 1/2 the product of its apothem and perimeter. [169]: $K=1/2\ ap$. | *If $p=72, a=6\sqrt{3}$, $K=216\sqrt{3}$* |

NOTE: A short way to find an interior angle of a regular polygon is to determine an exterior angle and take its supplement.


**192-204: REGULAR POLYGONS AND CIRCLES. A. CIRCUMSCRIBED AND INSCRIBED CIRCLES.** **192.T**-A circle can be circumscribed about any regular polygon. [31, 110].

**193.T**-A circle can be inscribed in any regular polygon. [117].

NOTES: ▶The circumscribed and inscribed circles are concentric. ▶The radius of inscribed circle equals apothem of the polygon.

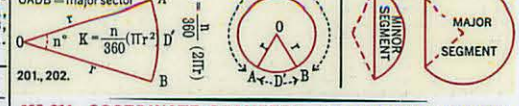
**B. ARCS. IF A CIRCLE IS DIVIDED INTO A NUMBER OF EQUAL ARCS...** **194.T**-The chords of those arcs form a regular polygon. [113, 130]. **195.T**-The tangents at their ends form a regular polygon. [126].

**C. CIRCUMFERENCE AND AREA.** NOTE: Circum. and area of circle found by taking limiting results of inscribed regular polygons with an increasing number of sides. **196.T**-Circumferences of two circles are to each other as their radii. [184]. **197.T**-The ratio of the circumference of a circle to the diameter is constant. [196]. The value of this constant, $\pi$, is 3.14159 approx. **198.T**-The area of a circle equals 1/2 product of its radius and circumference. [191].

**D. CIRCLE FORMULAS.**

| | |
|---|---|
| **199.T-CIRCUMFERENCE:** | $C=2\pi r$, $C=\pi d$ [197] |
| **200.T-AREA:** | $K=\pi r^2$, $K=1/4\ \pi d^2$ [198, 199] |
| **201.T-LENGTH OF ARC:** | $L=n/360\ (2\pi r)$ } n equals degrees in central angle or arc of sector |
| **202.T-AREA OF SECTOR:** | $K=n/360\ (\pi r^2)$ } |
| **203.T-AREA OF MINOR (major) SEGMENT:** | Area of corresponding sector minus (plus) area of triangle. |

**204.T**-Areas of two circles are to each other as the squares of their radii. [200]: $K/K'=r^2/r'^2$.

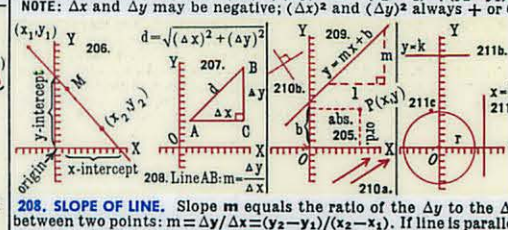



**205-211: COORDINATE GEOMETRY. 205. LOCATING (GRAPHING) POINTS.** $P(x_1, y_1)$ denotes point $P$ with abscissa $x_1$, ordinate $y_1$. If $x=0$, $P$ is on the $y$-axis; if $y=0$, $P$ is on the $x$-axis.

**206. MIDPOINT OF A LINE SEGMENT.** If segment endpoints are $A(x_1, y_1)$ and $B(x_2, y_2)$, the coordinates of the midpoint M are $(x_1+x_2)/2, (y_1+y_2)/2$.

**207. DISTANCE BETWEEN TWO POINTS.** (**a**) If $x$- or $y$-coordinates are equal, **d** equals (positive) difference between the two unequal coordinates: If $x_1=x_2$ and $y_2>y_1$, then $d=y_2-y_1$. (**b**) If $x$- and $y$-coordinates unequal, $d=\sqrt{(\Delta x)^2+(\Delta y)^2}=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.

NOTE: $\Delta x$ and $\Delta y$ may be negative; $(\Delta x)^2$ and $(\Delta y)^2$ always + or 0.




**208. SLOPE OF LINE.** Slope **m** equals the ratio of the $\Delta y$ to the $\Delta x$ between two points: $m=\Delta y/\Delta x=(y_2-y_1)/(x_2-x_1)$. If line is parallel to $x$-axis, $\Delta y=0$ and $m=0$; if parallel to $y$-axis, $\Delta x=0$, $m$ undefined.

**209. SLOPE-INTERCEPT FORM. $y=mx+b$;** slope $m$, $y$-intercept $b$.

**210. SLOPES OF PARALLEL AND PERPENDICULAR LINES.** (**a**) Lines are parallel if slopes are equal ($m_1=m_2$), or are identical if they also have a common point. (**b**) Lines are perpendicular if slopes are negative reciprocals ($m_1m_2=-1$), or if $m_1$ is undefined and $m_2=0$.

**211. GRAPHS OF LOCI.** (**a**) Equation of $x$-axis is $y=0$; of $y$-axis, $x=0$. (**b**) Equation of point moving at distance $k$ from the $x$-axis is $y=k$; from the $y$-axis, $x=k$. (**c**) Locus of point at distance $r$(radius) from origin (center) describes a circle: $x^2+y^2=r^2$.

**212-222: PROPERTIES OF A RIGHT TRIANGLE. A. MEDIAN.** **212.** *Median to hypotenuse* equals 1/2 the hypotenuse and divides the right triangle into two isosceles triangles (equal in area).

**B. BISECTORS.** **213.** The *perpendicular bisectors* of the sides are concurrent at the mid point of the hypotenuse. This point is the center of circumscribed circle. (*Right triangle can be inscribed in a semicircle*).

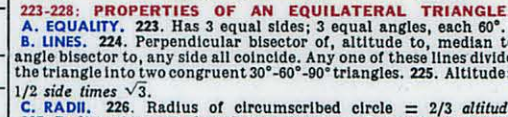
**C. ALTITUDES.** **214.** Altitudes from vertices of acute angles coincide with the legs. **215.** Altitude to the hypotenuse divides triangle into two triangles, similar to each other and to the original. **216.** Altitudes are concurrent at vertex of the right angle. **217.** If altitude is drawn to the hypotenuse: (**a**) *altitude is mean proportional between segments of hypotenuse*; (**b**) *either leg is mean proportional between the whole hypotenuse and segment (of the hypotenuse) adjacent to that leg.*

**D. LENGTHS OF SIDES.** **218.** Pythagorean theorem: $a^2+b^2=c^2$. (*Special cases*—3, 4, 5; 5, 12, 13; 8, 15, 17; etc.) **219.** Length of hypotenuse in a right triangle with a 45° angle *equals either leg times* $\sqrt{2}$.

| | |
|---|---|
| **220. LENGTH OF LEG OPPOSITE A** | 30° angle = 1/2 hypotenuse |
| | 45° angle = 1/2 hypotenuse times $\sqrt{2}$. ($\sqrt{2}=1.414+$) |
| | 60° angle = 1/2 hypotenuse times $\sqrt{3}$. ($\sqrt{3}=1.732+$) |

**E. TRIGONOMETRY.** **221.** If $A$ is an acute angle of a right triangle: **sin** A = leg opposite angle $A$ ÷ hypotenuse; **cos** A = leg adjacent to angle $A$ ÷ hypotenuse; **tan** A = leg opposite angle $A$ ÷ leg adjacent to angle $A$.

**222. $K=1/2\times$ leg $\times$ other leg.**




**223-228: PROPERTIES OF AN EQUILATERAL TRIANGLE. A. EQUALITY.** **223.** Has 3 equal sides; 3 equal angles, each 60°.

**B. LINES.** **224.** Perpendicular bisector of, altitude to, median to, angle bisector to, any side all coincide. Any one of these lines divides the triangle into two congruent 30°-60°-90° triangles. **225.** Altitude = 1/2 *side* $\times\sqrt{3}$.

**C. RADII.** **226.** Radius of circumscribed circle = 2/3 *altitude*. **227.** Radius of inscribed circle (apothem) = 1/3 *altitude* or 1/2 *radius of circumscribed circle*. **D. AREA.** **228.** $K=[(side)^2\times\sqrt{3}]\div 4$.

## ◀ BASIC METHODS OF PROOF ▶

**A. PROVING TRIANGLES CONGRUENT.** See 31-34, 42; 66.

**B. PROVING LINES EQUAL.** **Def.**—*Corresponding sides of congruent triangles are equal.* **37.** If two angles of a triangle are equal, opposite sides are equal. **67.** Opposite sides of a parallelogram are equal. See also 1-5, 7-12, 38, 39, 43, 69, 72, 73, 84-87, 107, 108, 113, 115, 117, 118, 123, 145, 207.

**C. PROVING ANGLES EQUAL.** **Def.**—*Corresponding angles of congruent triangles are equal.* **35.** Base angles of an isosceles triangles are equal. **51.** If two lines are parallel, alternate interior angles are equal. **52.** If two lines are parallel, corresponding angles are equal. **130.** Angles inscribed in the same arc are equal. **Def.**—*Corresponding angles of similar triangles are equal.* See also 1-5, 7-12, 23-27, 36, 56, 63, 68, 71, 75, 112, 124-129.

**D. PROVING LINES PERPENDICULAR.** **28.** If two lines meet forming equal and supplementary angles, they are perpendicular. **41.** Two points each equidistant from the ends of a line segment determine its perpendicular bisector. See also 55, 71, 74, 121, 133, 164, 210b.

**E. PROVING LINES PARALLEL.** See 46-50; 54, 65, 88, 91, 150, 210a.

**F. PROVING A QUADRILATERAL IS A PARALLELOGRAM.** See 76-80.

**G. PROVING ARCS EQUAL.** *In the same or in equal circles*: **111.** Equal central angles have equal arcs. **114.** Equal chords have equal arcs. See also 109, 115, 131, 134, 201.

**H. PROVING CHORDS EQUAL.** *In the same circle or in equal circles*: **113.** Equal arcs have equal chords. **118.** Chords equidistant from the center are equal. **Def.**—*Corresponding sides of congruent triangles are equal.*

**I. PROVING LINES UNEQUAL.** **101.** The sum of two sides of a triangle is greater than the third side. **104.** In a triangle, if two angles are unequal, the side opposite the larger angle is greater than the side opposite the smaller angle. See also 92-100, 105, 106, 119, 120.

**J. PROVING ANGLES UNEQUAL.** **102.** An exterior angle of a triangle is greater than either non-adjacent (*remote*) interior angle. **103.** In a triangle, if two sides are unequal, the angle opposite the larger side is greater than the angle opposite the smaller side. See also 92-100.

**K. PROVING TRIANGLES SIMILAR.** See 152-155, 186.

**L. PROVING LINES PROPORTIONAL.** **149.** A line parallel to one side of a triangle and intersecting the other two sides, divides those sides proportionally. **Def.**—*Corresponding sides of similar triangles are in proportion.* See also 151, 156, 159-162.

**M. PROVING TRIANGLES EQUAL IN AREA.** See 179-181. **Def.**—Congruent figures are equal. **7, 8.** If equals are added to (or subtracted from) equals, the results are equal. See also 2-5, 9-12, 66, 169.

**N. PROVING ANGLES SUPPLEMENTARY.** See 21, 53, 70, 132.



DO NOT PLACE ON HOT SURFACES

KNOW THE FACTS ON THIS QUICK CHART AND YOU WILL GET HIGHER GRADES IN CLASSWORK, HOMEWORK, TESTS AND EXAMS.

THERE ARE QUICK CHARTS IN • MATH • SCIENCE • ENGLISH • HISTORY LANGUAGES • ECO • PSYCH • SOC • PHILO • MUSIC • RELIGION • GAMES

DATA-GUIDE EDITOR: JOSEPH L. LEON

PUBLISHED & MF'D BY DATA-GUIDE, INC. FLUSHING, N.Y. 11355

DATA-GUIDE QUICK CHART

# PARLIAMENTARY PROCEDURE (RULES AGENDA MOTIONS VOTING)

## A SIMPLIFIED SUMMARY & INSTANT REFERENCE

LEARN & REVIEW IN MINUTES

AUTHOR: Prof. David G. Powers, Ph.D., University of City of N.Y. CONSULTANTS: R Lewinson, N.Y. County Law and Bar Assoc.; Prof. H.P. Kerr, Ph.D., Harvard U.

SOLID PLASTIC

QUICK CHARTS SPEED STUDY: PUT THE IMPORTANT FACTS IN FULL VIEW TO SAVE YOU TIME AND WORK.

QUICK CHARTS IMPROVE MEMORY: PERMANENTLY FILL IN FOR MISSING NOTES & TEXTS IN BASIC & ADVANCED COURSES.

CONTAINS NO PAPER

**I. PRINCIPLES. A. Majority rule must prevail. B. The rights of members with a minority opinion must be protected. C. Respect for dignity of members must be assured. D. Logical order of business must be provided.**

**II. CONSTITUTION, BY-LAWS, STANDING RULES. A. CONSTITUTION.** Defines the structure, purpose and organization of the group. Contains: (1) Name of organization. (2) *Purpose and aim of group.* (3) Qualifications of members. (4) *Officers and method of their election.* (5) Time and place of regular meetings. (6) *Means of amending the constitution.*

**B. BY-LAWS.** The constitution contains the policy; the **by-laws** specify the means of carrying out this policy. **THE BY-LAWS DETAIL:** (1) Term of office and authority of officers. (2) *Standing (permanent) Committees and their functions.* (3) Dues, fines, bills and other assessments. (4) *Date and methods of nominations and elections.* (5) Order of business for meetings. (6) *Number of members needed for a quorum.* (7) Means of amending the by-laws.

**C. STANDING RULES ("HOUSE RULES").** Cover matters pertaining to the orderly process of business not significant enough to be in the *by-laws*. Majority vote is sufficient for their establishment and they may be amended or rescinded by a ⅔ vote.

**III. TYPES OF MEETINGS.** A **"meeting"** is the assembly of the members for any length of time; **"Session"** refers to a series of meetings (*constituting a season, a session of Congress, etc.*).

**A. REGULAR MEETING.** Held at specified times and deals with general business of the organization.

**B. "SPECIAL" MEETING.** Convened to treat a specific problem. No other business is in order.

**IV. TYPES OF PROCEDURE. A. FORMAL PROCEDURE.** Does not permit discussion of a subject until it is offered as a **motion**, then *seconded* and *restated* by the Chair. This procedure results in quick, orderly action.

**B. INFORMAL PROCEDURE.** The membership is permitted to discuss the subject prior to framing of motions. The informal procedure forms better motions by permitting the group to arrive at a general opinion (*"the sense of the meeting"*) before the making of a motion.

**C. CONDUCT OF THE MEETING. (1) ADDRESSING THE CHAIR.** Whether formal or informal procedure is used, all discussion is controlled by the Chairman. Members address the Chairman as Mr. (or Madam) Chairman; the Chairman refers to himself in the third person: **"The Chair rules . . . "** or **"The Chair disagrees . . . "** **(2) RECOGNIZING MEMBERS.** The Chairman need not rise when recognizing a member who wishes to speak. The Chair must recognize members in the order in which they raise their hands.

**V. AGENDA: ORDER OF BUSINESS. A. SEQUENCE.**

**(1) Call to order.**
**(2) Minutes of previous meeting.**
**(3) Reports of officers, boards, standing committees.**
**(4) Reports of special committees.**
**(5) Announcements.**
**(6) Unfinished business.**
**(7) New business.**
**(8) Adjournment.**

**B. PURPOSE.** The **Order of Business** provides a logical system for group considerations and the procedure by which the Chair advances from one matter to another. **(1)** It reviews actions taken at the last meeting. **(2)** *Reports actions of the elected officers.* **(3)** Contributes knowledge gained by special committees. **(4)** *Reminds of actions still pending.* **(5)** Furnishes knowledge and facts recently attained. The *Order of Business* can be rearranged at any time by a ⅔ vote. **NOTE:** Written copies of the agenda should be available at the start of each meeting.

**VI. AGENDA: CALL TO ORDER.** The President of an organization usually presides at its meetings as the Chairman. Should he be absent, the Vice-President presides, and next, the Secretary.

**A. QUORUM.** (1) The Chair opens the meeting by inquiring of the secretary if a quorum is present. A **quorum** is a simple majority in legislative bodies, but in social and professional organizations, it can be as low as 25% of the membership. The specific number for a quorum is in the by-laws. (2) To open the meeting, the Chairman stands, raps the gavel, and announces: **"The meeting will please come to order."** When no quorum can be had, he says: **"As there is no quorum, a motion to adjourn is in order."** The motion is made by a member, seconded and the meeting is adjourned by the Chairman.

**B. CALL THE ROLL.** Usually the **roll** is called only if there is a question as to whether there is a proper quorum or to identify all individuals present and not present. **(1)** In cases of extreme urgency, the Chair may conduct a meeting without a quorum. Any actions ratified must be approved at the next legal meeting. *An action undertaken, however, which breaks the faith or inflicts injury by repealing or rescinding is illegal.* **(2)** Should a meeting start with a quorum and lose it (*members leave before adjournment*), discussion may continue but no vote can be taken.

**VII. AGENDA: MINUTES OF PREVIOUS MEETING. Minutes** are the official record of the actions of the group. They are read and approved at the opening of each meeting to establish continuity of action and to check on the group's affairs. Reading the minutes may be postponed by a **majority vote** but then must be read at the beginning of the next meeting. *Chairman:* **"The Secretary will please read the minutes."**

**A. CONTENTS OF THE MINUTES REPORT. (1)** Name of the group. **(2)** *Kind of meeting, i.e., regular or special.* **(3)** Place, date, and time of meeting. **(4)** *Name of presiding officer.* **(5)** Approval of the minutes of previous meeting. **(6)** *List of motions introduced, their proposers, and their final disposition.* **(7)** Time of adjournment of meeting.

**B. SANCTIONING OF MINUTES.** The minutes are read and the Chair announces: **"You have listened to the minutes. Are there any corrections?** (*The Chair waits*). **If there are none, the minutes stand approved as read."** (*Approval is given by silent consent.*) If corrections are suggested, the Chair instructs the Secretary to make them. Should any objection to the correction arise, a vote must be taken. *Form:* **"Shall the proposed correction** (state the correction) **be made? Those in favor say 'aye' those opposed, 'nay.'"** The Chair announces the results. The process is repeated if other corrections arise. The Chair finally announces: **"There being no further corrections, the minutes stand approved as corrected."** **NOTE:** Minutes may be corrected at any time, *but if already approved*, **a two-thirds vote** is required to change them. If notice of the desired correction is posted or distributed for the membership to read prior to the consideration, **a majority vote** is sufficient.

**THE TOOLS OF THE MEETING: THE BASIC PARLIAMENTARY MOTIONS.** The operation of the motions should be known by every participant. Motions are listed in the order in which they have the right of presentation and discussion (**precedence**) over motions in other categories and within groupings. **Example:** *Prior to a vote, discussion on a* **MOTION TO AMEND (23)** *can be stopped by a higher ranked Subsidiary Motion, such as,* **TO LIMIT DEBATE (20).** *Action on this motion can be diverted by a motion from Group B, e.g.,* **DIVIDE THE QUESTION (12).** *Motion* **(12)** *can itself be superseded before the vote by a motion to* **RECESS (3),** *which because it is a Privileged Motion with higher precedence, requires immediate action.* The table also tells the purposes, rules and voting requirements of each motion. **NOTES:** ▶When a motion is **"NOT DEBATABLE,"** it usually requires immediate action. ▶**"TAKES SUBSIDIARY MOTION"** means motions **18-24** are applicable.

### TABLE OF MOTIONS AND THEIR USES

| | MOTIONS BY CATEGORY AND PRECEDENCE | PURPOSE OF MOTION | INTERRUPT SPEAKER | NEEDS SECOND | AMEND-ABLE | DEBAT-ABLE | CAN BE RECON-SIDERED | TAKES SUBSIDIARY MOTION | REQUIRED VOTE | |
|---|---|---|---|---|---|---|---|---|---|---|
| **PRIVILEGED MOTIONS :** Arise from questions of meeting arrangements, comfort, member's rights; requires immediate attention. | | | | | | | | | | |
| 1. | FIX TIME, PLACE FOR NEXT MEETING | CLOSE MEETING | NO | YES | YES | YES (a) | YES | YES | MAJ. | 1 |
| 2. | TO ADJOURN | CLOSE MEETING | NO | YES | NO | NO | NO | NO | MAJ. | 2 |
| 3. | TO RECESS | INTERRUPT MEETING | NO | YES | NO (a) | YES (b) | — | YES | MAJ. | 3 |
| 4. | QUESTION OF PRIVILEGE | ASSERT RIGHTS | YES | NO | NO | NO | NO | NO | CHAIR (a) | 4 |
| 5. | ORDERS OF THE DAY | ASSERT RIGHTS | YES | NO | NO | NO | NO | NO | MAJ. (a) | 5 |
| **INCIDENTAL MOTIONS :** Relate to questions which arise from other motions or business. (No order of precedence within this group.) | | | | | | | | | | |
| 6. | POINT OF ORDER | ASSERT RIGHTS | YES | NO | NO | NO | NO | NO | CHAIR (a) | 6 |
| 7. | APPEAL DECISION OF THE CHAIR | ASSERT RIGHTS | YES | YES | NO | YES (a) | YES | YES | MAJ. | 7 |
| 8. | OBJECT TO CONSIDERATION | PREVENT ACTION | YES | NO | NO | NO | YES (a) | NO | 2/3 | 8 |
| 9. | READING OF PAPERS | PRESENT INFORMATION | NO | YES | NO | NO | YES | NO | MAJ. | 9 |
| 10. | WITHDRAWAL OF A MOTION | PREVENT ACTION | NO | NO | NO | NO | YES (a) | NO | MAJ. (b) | 10 |
| 11. | SUSPEND RULES | SPEED UP ACTION | NO | YES | NO | NO | NO | NO | 2/3 (a) | 11 |
| 12. | DIVIDE QUESTION | SIMPLIFY COMPLEX MOTION | YES | YES (a) | YES | NO | NO | YES | MAJ. | 12 |
| 13. | NOMINATE | ELECTIONS | NO | NO | NO | YES | NO | YES | MAJ. | 13 |
| 14. | PARLIAMENTARY INQUIRY | CLARIFY RULES | YES | NO | NO | NO | NO | NO | CHAIR (a) | 14 |
| 15. | DIVIDE THE ASSEMBLY | COUNT VOTE | YES | NO | NO | NO | NO | NO | — (a) | 15 |
| 16. | POINT OF INFORMATION | REQUEST INFORMATION | YES (a) | NO | NO | NO | NO | — | CHAIR (b) | 16 |
| 17. | COMMITTEE OF THE WHOLE | CONSIDER INFORMALLY | NO | YES | NO | YES | YES | NO | MAJ. | 17 |
| **SUBSIDIARY MOTIONS :** Act upon motions in order to dispose of them; do not amend. | | | | | | | | | | |
| 18. | TO TABLE | DELAY ACTION | NO | YES | NO | NO | NO | NO | MAJ. | 18 |
| 19. | ORDER PREVIOUS QUESTION | CLOSE DEBATE | NO | YES | NO | NO | YES | NO | 2/3 | 19 |
| 20. | EXTEND OR LIMIT DEBATE | SPEED (SUPPRESS) DEBATE | NO | YES | YES (a) | NO | YES | YES | 2/3 | 20 |
| 21. | POSTPONE TO A DEFINITE TIME | DELAY ACTION | NO | YES | YES (a) | YES (b) | YES | YES | MAJ. | 21 |
| 22. | REFER TO COMMITTEE | FURTHER STUDY (DELAY) | NO | YES | YES | YES | YES (a) | YES | MAJ. | 22 |
| 23. | AMEND | MODIFY MOTION | NO | YES | YES (a) | YES (b) | YES | YES | MAJ. (c) | 23 |
| 24. | POSTPONE INDEFINITELY | PREVENT ACTION | NO | YES | NO | YES | NO (a) | NO (b) | MAJ. | 24 |
| **PRINCIPAL MOTIONS:** Directly concerns a proposition, idea or Main Motion (30). (No order of precedence within this group.) | | | | | | | | | | |
| 25. | RECONSIDER | CHANGE A DECISION | YES | YES | NO | YES (a) | NO | YES | MAJ. | 25 |
| 26. | RESCIND (REPEAL) | CHANGE A DECISION | NO | YES | YES | YES | YES (a) | YES | 2/3 (b) | 26 |
| 27. | TAKE FROM THE TABLE | RENEW DISCUSSION | NO | YES | NO | NO | NO | NO | MAJ. | 27 |
| 28. | SPECIAL ORDER OF BUSINESS | SPEED UP ACTION | NO | YES | YES | YES | YES | YES | 2/3 | 28 |
| 29. | DISCHARGE COMMITTEE | SPEED ACTION | NO | YES | NO | YES | YES | YES | 2/3 (a) | 29 |
| | MAIN MOTIONS. (30). | NEW BUSINESS | NO | YES | YES | YES | YES | YES | MAJ. | 30 |

**6. "I rise to a point of order."** (a) *The Chair's decision can be appealed* (*See* **7**). When a member thinks there is a breach of order, he should insist upon correction of any irregularities. **7. "I appeal from the decision of the Chair."** Chair puts question **"Shall the decision of the Chair stand?"** (a) *Undebatable if it concerns a lack of decorum, business priority, any transgression of speaking rules, or if proposed while there is division of the assembly, or if made when the question which is pending is undebatable.* **8. "I object to consideration of the question."** Must be introduced immediately after disputed motion has been stated and before debate begins. Chairman then puts question of consideration to vote. **"Shall this question be considered?"** (a) *If vote is affirmative, "motion to reconsider" cannot be reconsidered again.* **9.** Allowed if member desires information. If another member objects, it must be put to a vote. **10.** Once motion has been made, seconded, and stated by the Chair, it becomes the property of the group and cannot be withdrawn except by general consent. (a) *Affirmative vote cannot be reconsidered.* (b) *Chair may grant withdrawal request. Vote needed only if a member objects.* **11. "I move that the rules concerning _____ be suspended."** It is not permissible to make a blanket motion suspending the rules; the motion must be specific. (a) *Many constitutions require a unanimous vote.* **12.** This motion separates a complex main motion into distinct propositions and specifies the form of the division. (a) *No second is necessary if components are unrelated.* **13. "I nominate _____ for President (or other office)."** **14. "Mr. Chairman, I rise to a parliamentary inquiry."** Chairman may answer the parliamentary inquire. (a) *If information is appealed, a majority vote follows.* **15. "I call for division of the house"** or **"Division."** (a) *Chair must comply.* **16. "I rise to a point of information."** (a) *Speaker decides whether to yield to hear the question.* (b) *See* **7.** **17. "I move we form a Committee of the Whole."** No action can be taken while the group is in this form. It must rise and report before regular meeting can resume and deal with matters discussed by the Committee of the Whole.

**18. "I move that the question be laid on the table."** You cannot include in this motion a time at which the question will be taken from the table. This requires the motion **"TO TAKE FROM THE TABLE."** **19. "I call for the previous question."** Chair says: **"Shall the main question be put?"** If decision is affirmative, vote is taken first on the amendments and then on the main motion. "Previous question" may be limited to amendments only. In such cases, it affects only the amendment to which it applies. This still allows debate on the main motion. **20.** Motion sets hour for closing debate, limits length of debate, sets time for speeches. (a) *Amendable only as to the time limit.* **21. "I move that question be postponed until _____."** Allows time for study. (a) *Amendable only as to time.* (b) *Debatable only on propriety of the motion.* **22. "I move the question be referred to the _____ committee."** (a) *It cannot be reconsidered once the committee has begun its study.* Committee can be discharged by a motion to that purpose on a 2/3 vote. **23. "I move to amend the motion by striking out (inserting, substituting)."** Amendment must be germane. No limit to number of amendments that can be voted upon in turn. (a) *Cannot amend to the third degree.* (b) *Motion is undebatable when motion to which it is applied is itself undebatable.* (c) *Changes in the Constitution, by-laws, etc., require previous notice and 2/3 vote.* **24. "I move the question be postponed indefinitely."** Prevents action on the main motion, opens main question to debate, enables opponents of main motion to sound out the group. (a) *If vote is negative, "postponement" cannot be reconsidered; if affirmative, it can be.* (b) *No subsidiary motions allowed except the motion to "limit debate" or to "extend the limits of debate," or to "order the previous question."*

**25. "I move to reconsider the motion."** One who voted with the prevailing side must make this motion. (a) *If question to be reconsidered is undebatable, motion for reconsideration is undebatable; and vice versa.* **26.** (a) *If vote is affirmative, "repeal" cannot be reconsidered.* (b) *Constitutions need 2/3 vote for repeal of a rule and may require previous notice of desired change.* **27. "I move to take (state proposal) from the table."** **28. "I move this matter (state resolution) be made a special order at the meeting of _____."** **29. "I move that _____ committee considering _____ be discharged," or, "that the _____ being considered by _____ committee be brought to the floor."** This motion cannot be referred to a committee. (a) Majority is sufficient if previous notice has been given. **30.** The Chair states the motion: **"It has been moved and seconded that (states specific action desired). Are there any remarks?"** Once the motion has been stated, it becomes the property of the group and cannot be withdrawn except by general consent or by a motion to withdraw.

**VIII. AGENDA: REPORTS OF OFFICERS, BOARDS, STANDING COMMITTEES.** **A. EXECUTIVE REPORTS.** Formal accounts of the actions of the elected officials.

**B. EXECUTIVE ANNOUNCEMENTS.** Officers, boards and standing committees have vested power to act. Such officers and boards must report to the group to inform them of those actions that are binding on all members.

**C. ORDER AND PROCEDURE OF REPORTS.**

**(1) PRESIDENT.** **"The President wishes to report that . . ."**

**(2) VICE-PRESIDENT.** *Chairman:* **"We will listen to the report of the Vice-President . . ."**

**(3) SECRETARY (OR CORRESPONDING SECRETARY).** *Chairman:* **"Mr. Secretary, is there any correspondence."** The Secretary reads all communications. If action is required on them, the Chairman refers the matter to the proper committee or it may be taken up as new business.

**(4) TREASURER.** *Chairman:* **"We will now have the report of the Treasurer."** The Treasurer speaks standing: **"Mr. Chairman, the balance on hand is . . ., receipts . . ., etc. . . ."**

**(5) BOARD OF DIRECTORS.** *Chairman:* **"We have received the following communication from the Board."** or, **"The Board of Directors wishes to report on its meeting of** (date). **Mr. ——— will speak for the Board."**

**D. STANDING COMMITTEES.** Same category as committees *"empowered to act." Standing* means appointed for the year or entire session (Membership Committee, etc.). Protests made against actions of a committee take the form of resolutions and appeals. (See **TABLE OF MOTIONS**) The Chairman reports: **"The following action has been taken by the committee on . . ."**

**IX. AGENDA: REPORTS OF SPECIAL COMMITTEES.** Special committees are empowered by the group to study a proposal and submit their findings. Their role is advisory and they do not have the *"power to act."* Such committees can be appointed by the Chair or by a majority motion of the group. To avoid tie votes in a committee, staff with an odd number of members (3, 5, 7, etc.). **NOTE:** *"Minority reports"* reflect disagreement in a committee; their recommendations are treated as **amendments** to the main report.

**A. FORM.** Committee reports should contain: **(1)** Problem assigned and reason. **(2)** Approach used in seeking solution. **(3)** Information gathered. **(4)** Recommendations.

**B. GENERAL RULES. (1)** All special committees should keep the President informed of their progress and should report at the meeting that coincides with completion of their work, or when the matter assigned to them requires action. **(2)** Members should be advised prior to the end of a meeting that specific reports will be delivered at the next meeting and a listing should appear in that agenda. **(3)** A copy of every report should be filed with the Secretary prior to the meeting. **(4)** Important reports may be duplicated and distributed to the membership or read aloud at the meeting. **(5)** If a previously accepted report is to be altered or expanded, only those parts related to the change need be read to the membership.

**C. ACTION. (1)** A special report is *advisory* and does not require the voting acceptance of the group. However, the person presenting a report may move: **(a)** that it **"be received and placed on file"** in which case it becomes part of the record of the meeting, **(b)** that it **"be accepted"** in which case a favorable vote indicates the membership approves of the findings and recommendations and will act if necessary. **(2)** Should there be no recommendations, the Chair inquires: **"Are there any specific recommendations?** If recommendations are made, the Chair announces: **"You have heard the report of the committee. What is your pleasure?"**

**X. AGENDA: ANNOUNCEMENTS.** The Chair mentions items of interest and events submitted to him. Should there be none, he inquires: **"Are there any announcements to be made at this time?"** Good announcements contain five elements: *"Who, What, Where, When,* and *Why."* **NOTE:** Informal questions and discussion of the announcements are in order at this time.

**XI. AGENDA: UNFINISHED BUSINESS. A. CONSIDERATION.** Unfinished business includes matters undisposed of, matters postponed to this meeting, and matters set as general orders of the day. When a motion is interrupted by adjournment, it becomes the first order of business after the reading of the minutes at the next meeting—it is as if there had been no adjournment. Such questions are called *"unfinished business."* At the appropriate point in the meeting, the Chairman must state: **"Is there any other old or unfinished business to be acted upon before we move to new business?."**

**B. DELAY.** Should some pressing current business arise, the old question may be postponed—but only temporarily—until the more urgent business is settled. The Chair announces: **"In order to proceed to the more urgent business of** (stating urgent business), **a motion to postpone** (stating old business) **is in order."** If the motion is made, seconded, and receives a majority vote, Chair so acts. Otherwise, the "unfinished business" has priority.

**XII. AGENDA: NEW BUSINESS.** New business means any proposal not previously considered by the group. It is introduced in the form of *motions.* In addition to the membership, the Chair is privileged to suggest new business and to hold informal discussions on a topic before a formal motion is made. *Chair:* **"New business is now in order. What is your pleasure?" NOTE:** If a special program is on the agenda, the Chairman may turn the meeting over to the *Program Chairman* to preside. After the special program, the regular Chairman resumes his role.

**XIII. AGENDA: ADJOURNMENT. A. TYPES.** **A motion to adjourn** has highest precedence. It is in order at any time but it may not interrupt a speaker or voting. There are two types of adjournment motions: **(1)** To adjourn until the next regular meeting; **(2)** to adjourn to specific time, and/or place. The second motion has priority over the first and is amendable concerning time and place (*but not to the issue of adjournment*). The first is not debatable and hence not amendable. A final adjournment dissolves the group and is termed *"adjournment sine die."*

**B. FORM. (1) SIMPLE.** *Chair:* **"There being no other business, the meeting stands adjourned."** Or, **"The motion to adjourn has been moved and seconded. All in favor say "aye"; those opposed "nay."**

**(2) MODIFIED. "I move that we adjourn until Monday at 2 o'clock at Jones Pavillion."** *Chair:* **"It has been moved and seconded that we adjourn until Monday at 2 o'clock at Jones Pavillion. What is your pleasure?"** Discussion follows and a majority vote prevails. The Chair announces the time and place of the next meeting.

**XIV. PRESENTING A MOTION (WITHOUT AMENDMENTS).**

**A. PURPOSE.** *A motion is the means by which action is attained.* Informal discussion helps develop opinions, but to get action, a motion must be introduced, seconded by another member (*except for nominations and privileged motions*), and re-stated by the Chair. All motions should be expressed in the affirmative.

**B. THE ROLE AND INFLUENCE OF THE CHAIRMAN.** The Chair has the authority to close the general discussion and call for a specific motion: **"The Chair will entertain a motion to . . ."** (Only one motion can be entertained at a time.) **(1)** After motion is seconded, the Chairman may explain or give information concerning the effect of the motion. The question is then open for concentrated debate (*See XVI*) by the membership. **(2)** A skilled Chairman tries to anticipate problems and avoid them. When the motion suggested is too complicated or in improper form, he suggests it be reworded. He tactfully explains the difficulty the improper motion would encounter.

**C. PROCEDURE: (1)** Member rises and addresses the Chair. **(2)** Member is recognized by Chair: **"The Chair recognizes Mr. Smith."** If the Chairman does not know the member, he says: **"Will the member please state his name,"** and then recognizes him. **(3)** Member states his proposal: **"Mr. Chairman, I move that . . ."** **(4)** The Chair calls for a second and a member seconds the motion. If no one seconds the motion, the Chair repeats his request: **"Is there a second for the motion?"** If no second is forthcoming, the Chair announces: **"The motion is lost for want of a second."** **(5)** If seconded, the motion is re-stated by the Chair: **"It has been moved and seconded that . . ."** **(6)** The Chair conducts the discussion (See XVI). **(7)** The Chair puts the question to a vote (See XVIII). **(8)** The Chair announces the result.

**XV. AMENDING A MOTION.** There are four basic methods of amending a motion: **(1)** To amend by inserting. **(2)** To amend by adding or placing at the end. **(3)** To amend by striking out and inserting. **(4)** To amend by substituting a paragraph.

**A. FORM: "I move to amend the main motion by** (*inserting*) (*adding*) (*striking out and inserting*) (*substituting*)." The Chairman conducts the discussion and then says: **"All in favor of the amendment to** (stating change) **say 'aye'; all those opposed say 'nay.' The 'ayes' prove it and the amendment is carried"** (or vice versa). The question now is on the total resolution: **"Resolved that . . ."** Chairman reads the resolution as amended.

**B. AMENDING AN AMENDMENT.** The main motion can be amended and the amendment may in turn be amended. *Amendments can only go to the second degree.* For example, the main motion is proposed: *"I move that the Club donate $100 to Boys Town."* A motion can be made to amend the main motion by striking out $100 and inserting $50. An amendment of the amendment is proposed that $50 be striken out and $25 be inserted. This amendment is in order but no amendments relative to the $25 can be offered at this time.

**C. ROLE OF THE CHAIR.** The skillful Chairman can avoid confusion by simplifying the amendments and by using set forms when putting the question to a vote. Using the example above, the Chair would put the following issues before the group: **(1)** Does the group wish the $25 to be the amendment? *"It has been moved and seconded that the amendment be amended by striking out $50 and inserting $25. All in favor of the amendment reading $25 say 'aye'; those opposed, say 'nay.'"* **(2)** If the amendment of the amendment carries, should the $25 become part of the main motion? **(3)** If the substitution of $25 fails in amending the amendment, then the second question would be whether the $50 should become part of the main motion. *"It has been moved and seconded to amend the main motion by striking out $100 and inserting $50. All those in favor of the main motion reading $50 say 'aye'; those opposed, say 'nay.'"* **(4)** The amended motion is then open to discussion and other amendments using the same process.

**D. GERMANE.** There is one restriction on an amendment. It must be **"germane,"** i.e., *it must be relevant to, or intrinsically associated with, the main proposal.* However, an amendment may be *"hostile"* to the main proposal and still be germane. A motion *"to raise the dues"* could be amended by striking out *"to raise"* and inserting *"to lower."* However, the motion *"to go on an outing"* could not be amended by adding *"and to elect a new secretary."* This amendment is not germane and therefore would be out of order. *Chairman:* **"Since this amendment is not germane to the main motion, it is out of order."**

**XVI. DEBATING A MOTION.** A debate ensues only if the motion is classified as **debatable.**

**A. RULES. (1)** No member can speak without first being recognized by the Chair. **(2)** The Chair should first recognize the *original mover* of the motion, if he wishes to speak. The Chair should try to alternate the speakers, pro and con. **(3)** No member should speak a second time until all members who wish to speak have had one chance.

**B. CHAIRMAN'S OPINIONS.** The Chairman cannot discuss the merits of the motion nor enter the debate without first appointing a temporary Chairman and vacating the chair. (After the vote on the motion, the original Chairman can resume his place.)

**C. TERMINATING THE DEBATE. (1)** The Chairman can attempt to terminate debate by asking: **"Is the membership ready to act on the question?"** If there is opposition, he may call for a voice vote or a show of hands. The Chair should not act until he feels that the question has been debated adequately. **(2)** A member may move to close the debate by saying: **"I move the previous question."** Such a motion cannot be debated and must be put to a vote immediately. It requires a 2/3 vote.

**XVII. VOTING PRINCIPLES. A. MAJORITY VOTE.** Usually used in normal procedures with no complications.

**B. TWO-THIRDS VOTE.** Used in all situations where some right of the membership is curtailed and/or a change of law or constitution is proposed.

**C. DEFINING THE VOTE MARGIN.** The by-laws of the group should state the conditions under which one of the following methods of determining the vote applies. **(1) TOTAL VOTES CAST.** Example: Membership 100, members present 80—votes cast 20. *Needed Majority = 11; two-thirds = 14.* **NOTE:** Illegal votes are counted in determining total votes cast. Blank votes and those present and not voting are not counted.

**(2) TOTAL VOTES OF THOSE PRESENT.** Example: Membership 100, members present 80. *Needed Majority = 41; two-thirds = 54.*

**(3) TOTAL VOTES OF THOSE PRESENT AND VOTING.** The situation is similar to that of (1) above, however, the term "present" prohibits *proxy* voting.

**(4) TOTAL VOTES OF TOTAL MEMBERSHIP (OR STOCKHOLDERS).** Example: Membership 100, members present 80. *Needed majority = 51; two-thirds = 67.* In this situation if 60 members were present and all voted for a measure that required a 2/3 vote, the measure, not receiving the required 67 votes, could not be passed.

**D. PRELIMINARIES. (1)** The matter to be voted upon must be restated by the Chairman for the membership. **(2)** The Chairman (or any member) may challenge the right of anyone to vote by checking his name in the roster. **(3)** The membership should be advised by the Chairman whether a majority or 2/3 vote is needed to pass.

**E. CHAIRMAN'S VOTE.** Chairman always has the right to vote but usually does not (*except in case of a tie or if he is a stockholder*).

**XVIII. TAKING THE VOTE. A. VOTE BY SILENT ASSENT (GENERAL CONSENT). (1)** On routine matters and those of minor importance, the Chairman may use silent assent. The Chairman says: **"If there are no objections, we shall consider this matter approved." (2)** If one member objects, there is no longer silent assent and a formal vote must be taken.

**B. VOICE VOTE (VIVA VOCE).** The Chairman calls for the vote by saying: **"All in favor, say aye; all opposed, say no."** The Chairman must always call for the "no" votes. A voice vote is not effective until the result is announced by the Chair.

**C. VOTE BY DIVISION OF THE HOUSE (RAISING OF HANDS OR STANDING).** These methods are effective when a member requests the Chair to check on the accuracy of a voiced vote; the appeal is usually granted. **(1)** Either method can be used at the discretion of the Chairman (*unless the matter requires a secret ballot*). The Chairman says: **"All in favor stand (or raise your hand)."** The vote is counted. **"All opposed stand (or raise your hand)." (2)** In small meetings, both the Chairman and Secretary count each category of votes. In large meetings, the group is divided into sections and *tellers* appointed by the Chair report the vote back to a *chief teller* or the Chairman.

**D. VOTE BY ROLL CALL. (1)** This method is used when it is necessary to know where each member stands on the question. (*Requires a motion, a second, a majority vote to carry, and the motion itself is undebatable.*) **(2)** The Secretary calls each member by name and records his vote. A member can abstain from voting by answering **"present"** or **"not voting"** instead of **"yea"** or **"nay." (3)** At the end of the roll call, Secretary asks if there are any corrections or changes. Members may change their votes prior to the tally (*unless the vote is by ballot*).

**E. VOTE BY WRITTEN (SECRET) BALLOT. (1)** This method is usually used when emotions and personalities are involved (*contested elections, disciplinary action, admitting new members, etc.*). **NOTE:** A member may move that any vote on an issue be taken by written ballot. The motion is undebatable and must be carried by a majority vote (voice or division of the house). **(2)** To record a written vote, the Chairman appoints tellers to distribute uniformly sized sheets of paper. The members write the name(s) or issues to be voted upon and their *"yea"* or *"nay"* mark next to each. Before the tellers collect the ballots the Chairman inquires: **"Has everyone entitled to vote done so?"**

**XIX. AFTER THE VOTE. (1)** As soon as the vote is counted by the tellers and the Secretary, the Chairman addresses the appropriate statement to the members: **"The ayes have it." "The resolution is adopted." "The motion has been defeated." "Mr. Smith has been elected.",** etc. **(2)** The Chairman also announces the exact number of votes cast, number of yeas and nays or, in an election, the number of votes received by each candidate.

**A. CHALLENGING THE VOTE.** If the Chairman or any member questions a vote, he may demand a recount. If there is further dispute, a roll call vote may be used or new ballots distributed.

**B. TIE VOTES.** Tie votes defeat the motion or resolution except in the following: **(1)** A tie vote sustains the Chairman when one of his decisions is challenged by the membership. **(2)** The tie vote endorses an action already taken by an officer.

**C. CHANGE OF VOTE. (1)** A vote in written form can never be withdrawn or changed once it is in the hands of the teller or has been dropped into the ballot box. **(2)** If a tie vote is the result of division of the house, any member may rise and change his vote. **(3)** A member can rise to change his vote after the house has been divided if the results have not yet been announced. However, if the tally is not a tie, a member may be permitted to change his vote only if given *silent assent* by those present. If there is an objection, the member turns his request into a motion. It must be seconded and carried by a majority.

**D. FOLLOW-UP ACTION. (1)** If the motion is carried, the Chairman institutes the necessary action, committees, changes, etc., to carry out the intent of the motion. **(2)** All aspects concerning the matter under consideration are continued until all related motions have been debated and voted upon. Then the Chairman progresses to the next order of business in the agenda.

**XX. NOMINATIONS. A. SPECIAL RULES.** Check the constitution and by-laws for any special instructions covering nomination and election procedure.

**B. NOMINATIONS BY COMMITTEE. (1)** A nominating committee can be appointed by the President or elected by the membership some months prior to the elections. Its function is to present to the membership a slate of nominees for the various offices. **(2)** Names can be suggested to the committee both by committee members and individuals in the organization.

**C. NOMINATIONS BY PETITION.** A certain number of signatures on a petition delivered to the secretary can place names in contention with those of the nominating committee. Both sets of nominees must be made known to the members at election time.

**D. NOMINATIONS FROM THE FLOOR.** When nominations are permitted from the floor, the Chairman announces: **"Nominations from the floor are now in order."** A member raises his hand, is recognized, and says: **"I nominate Mr. —— for the position of . . ."** The nomination need not be seconded but must be accepted by the nominee. If he declines the nomination, his name must be withdrawn.

**E. NOMINATING SPEECHES.** There may be brief nominating speeches (5-10 minutes) by usually no more than 2 members supporting a nominee. Some organizations forbid such speeches in by-laws.

**F. CLOSING NOMINATIONS. (1)** Nominations for each office are listed separately (preferably on a blackboard) and noted whether they are by *committee, by petition, or from the floor.* **(2)** After each list is ready, the Chairman asks: **"Are there any further nominations?"** If not, the Chairman calls for a motion to close the nominations. If carried, the Chairman calls for the election.

**G. REOPENING NOMINATIONS.** A motion to reopen nominations (not debatable, requires a majority vote) is in order: **(1)** If the election was uncontested (*See XXI*) and the meeting has not yet proceeded to other business. **(2)** If the election was contested (*See XXI*) but the ballots have not yet been distributed.

**XXI. ELECTIONS. A. UNCONTESTED ELECTION.** If there is only one candidate for an office, the Chairman can declare him elected without balloting. If the organization permits absentee voting (*write-in votes*), the secret ballot must be used even if there is only one publicly nominated candidate.

**B. CONTESTED ELECTION. (1)** More than one candidate for a position requires the use of the written ballot. Unless otherwise stated, a majority (*over 50%*) elects the candidates. **(2)** If there is a tie vote, additional elections are held until a majority is received by one name. Tie votes can also be resolved if one of the candidates withdraws prior to a new vote.

**C. SEQUENCE OF BALLOTING.** If more than one position is to be filled: **(1)** There can be a separate election for each post with the ballot listing only the candidates concerned and the result announced after each count. **(2)** A single ballot may be distributed with the members voting for all candidates at one time.









DATA-GUIDE EDITOR: JOSEPH L. LEON

PUBLISHED & MFD BY DATA-GUIDE, INC. FLUSHING, N.Y. 11355

DATA-GUIDE QUICK CHART

# PRINCIPLES OF PHILOSOPHY

A SIMPLIFIED SUMMARY & INSTANT REFERENCE

LEARN & REVIEW IN MINUTES

AUTHOR: Prof. A. Hofstadter, Ph.D., Columbia U.

SOLID PLASTIC

SPEED STUDY TO SAVE YOU TIME AND WORK.

QUICK CHARTS PUT THE FACTS IN FULL VIEW:

PERMANENTLY FILL IN FOR MISSING NOTES & TEXTS.

QUICK CHARTS HELP YOU REMEMBER:

CONTAINS NO PAPER

## 1

**METAPHYSICAL SCEPTICISM.** A. HUME. Metaphysical knowledge impossible. B. CARNAP, AYER. Metaphysical statements meaningless. C. WITTGENSTEIN. Metaphysical problems due to misuse of language.

## 2

**METAPHYSICAL DUALISM.** A. THEOLOGICAL DUALISM The two reals are **God** and **nature.** (*Aquinas:* God transcends nature, as Creator transcends created). (1) EXISTENCE OF GOD. (a) **Cosmological Argument:** Existence of God inferred from necessity of assigning a **first or supreme cause:** *Why does world exist?* Neither any part nor the whole can be the cause of its own existence. Hence there must be a cause not identical with the world, namely God. (b) **Ontological Argument:** God's existence follows from the definition or analysis of the **idea of God:** *perfection is an attribute contained in the idea of God* (implies God lacks nothing). To exist is more perfect (complete) than not to exist. Hence God exists. (c) **Teleological Argument:** Existence of God inferred from evidence of **order, design, purpose** in the world: *World exhibits order.* Where there is order there is an orderer. Hence there must be an orderer of the world, i.e., God. B. MIND—MATTER (PSYCHOPHYSICAL) DUALISM. (1) THE TWO REALS. **Mind, Matter**—neither is reducible to the other. *Descartes not a pure dualist,* admits (a) God as the third real and (b) two kinds of substances. (2) THE TWO KINDS OF SUBSTANCES. **Mind** (*res cogitans* = thinking substance) and **body** (*res extensa* = extended substance). *Mind* has thought, including will; *body* has extension, figure, motion, divisibility, etc. (so-called "primary qualities"). Body and mind interact causally; seat of interaction is pineal gland. (3) DESCARTES' PROOF THAT MIND IS INDEPENDENT OF BODY. I can doubt existence of my body, but can't doubt my own existence as a thinking being. If I doubt, I think, hence I exist. **(Cogito ergo sum:** I think, therefore I am.) Hence I am a thinking substance essentially; as a pure thinking being, I do not need body to exist; hence my thinking self is substantially different from my body. C. FORM—MATTER (HYLOMORPHIC) DUALISM. The two reals are **form** (*idea, universal, structure*) and **matter** (*stuff*). Form more real than matter. (1) PLATO'S CONCEPTS. (a) **World of eternal ideas or Forms is superior to changing world of sense-experience.** *True reality* = universal Idea or form of which perceived things are individual instances: *an individual animal is an instance of a universal form or Idea of its species.* (b) **Form is superior to the changing things of the world.** Form is *perfect,* particular things of sense-experience are not. Form is *intelligible;* particulars not completely so. **Particulars** "participate" in forms, are reflections of Ideas, "imitate" them. **Form, Idea** = *archetype* or model on which particulars are patterned. (c) **Highest Idea = Idea of Good.** Other Ideas form a system below it, according to degree of their universality, intelligibility, perfection. (2) ARISTOTLE'S CONCEPTS. (a) **Truly real things are individual substances:** *this tree, this man, God.* (b) **Substances:** (except divine substances, which are pure forms) are analyzable into *form* and *matter,* according to the distinction of actuality and potentiality in them. (c) **Form** = principle of actuality in a substance, in virtue of which a thing is or functions as this or that substance. (d) **Matter** = principle of potentiality, in virtue of which a thing can have form, can be this or that; the substratum in which form is present and actualized: *Form of man is his soul, matter of man is tissues and organs of his body.* In substance, matter's potentialities are actualized in development of form. (3) ARISTOTLE'S FOUR CAUSES. (a) **Formal Cause or Essence:** The form in the thing itself, making it what it is; its structure. (b) **Efficient Cause:** The form in another thing acting on the given thing and inducing change: *the art of Polyclitus is the form in his soul, acting through his skill on marble to induce shape in the marble, resulting in statue.* (c) **Final Cause:** Form, end, goal for the sake of which a thing exists or a change occurs (can be in or out of the thing): *acorn grows into oak, heading toward actualization of form of oak in matter available to it.* God is the final cause of all process in world, as object of love. (d) **Material Cause:** That out of which a thing comes or is made and that remains in the result as substratum: *marble is the material cause of a statue, wood is material cause of a tree.* (4) NEO-PLATONISM (PLOTINUS). (a) **Ultimate Reality** = the Ineffable *One,* source of all being, goodness, beauty, truth. The universe is an "emanation" from the One, a descent down the scale of being, each lower level being an image reflecting (*participating in*) the next higher. (b) **Successive Levels:** (i) *One.* (ii) *Mind* (Nous). (iii) *Soul* (Psyche): Physical world of *nature* = descent of soul into *matter;* a fall, bringing evil. (iv) *Matter* = lowest or outermost limit; "indeterminateness and nothing else;" ultimate darkness into which being radiates. (c) *Man* = soul fallen into matter. His salvation: to rise to ultimate union with the One. (d) *Stages of Ascent.* (i) Moral purification of the soul. (ii) Philosophical thinking moving upwards to **dialectic:** *ascent in mind to comprehension of unity of knower and known.* (iii) Ecstatic identification with the One.

## 3

**IDEALISM.** Reality essentially mental; matter not ultimately real. A. SOLIPSISM. My mind is the only **reality;** all else exists in it as its idea. Things, other persons, are mere **representations** within my own self. No philosopher adopts this position in the merely individual sense of *self.* (1) SCHOPENHAUER. The World is my Idea = Ideas of Mind. (2) FICHTE. Doctrine of Absolute Ego = Absolute Solipsism. Sole true reality is the **Absolute Ego,** or *God,* of which our individual selves are modifications and which produces a phenomenal nature as its own opposite (or *non-ego*) to provide a challenge of overcoming it in achieving self-consciousness, freedom. B. BERKELEY'S IDEALISM. The only realities are immaterial minds, their ideas and volitions. Material substance cannot exist, is a contradiction in terms. Being of non-thinking things lies in their being **perceived** by thinking things; their principle is **Esse est percipi** (*to be is to be perceived*): *A table really exists, but only as collection and sequence of ideas in minds of its perceivers.* Since we do not always perceive all things, to save their existence we must infer a Supreme Mind who always perceives them.

## 4

**KANT'S TRANSCENDENTAL IDEALISM.** A. PRINCIPAL VIEW OF THE CRITIQUE OF PURE REASON. (1) **NATURE** = knowable world of appearance, phenomenal world governed by necessary laws. (2) **REALITY = noumenal** world of things-in-themselves, unknowable by theoretical reason. But since morality presupposes God, freedom, and immortality, we must postulate that reality is essentially **spiritual:** *"I have . . . found it necessary to deny knowledge, in order to make room for faith."* B. KANT'S STATEMENT OF TRANSCENDENTAL IDEALISM. Must distinguish object of experience from **thing-in-itself.** *"Objects of experience . . . are never given in themselves, but only in* **experience,** *and have no existence outside it. The nonsensible cause of these representations is completely unknown to us, and cannot therefore be intuited by us as an object."* C. PROBLEM OF SYNTHETIC A PRIORI KNOWLEDGE. In mathematics and foundations of science we have knowledge which is **synthetic.** The predicate of synthetic judgment adds something to its subject (*7+5 = 12; all bodies have weight*). **Analytic knowledge,** instead, is empty because the predicate of an analytic judgment merely repeats, analyzes, its subject (*7+5 = 7+5; all bodies have extension*). Moreover, such synthetic judgments are **a priori,** hence they are **necessary** and **universal,** and knowable independently of experience. How is such knowledge (*laws of mathematics and fundamental principles of physics*), synthetic *a priori* judgment, possible? Answer given in *Transcendental Logic.* D. KANT'S TRANSCENDENTAL LOGIC. (1) TRANSCENDENTAL AESTHETIC. Studies forms (*space, time*) of **"sensibility."** Mathematical knowledge *a priori* is possible because space and time are forms of **intuition.** Geometry is possible because *space* is a form of the mind imposed on external objects of sense experience. Arithmetic is possible because *time* is the other (internal) form of mind imposed on all experienced events. We can know *a priori* those conditions (*forms of space and time*) which the mind imposes on all things before they can be its objects in experience. (2) TRANSCENDENTAL ANALYTIC. (a) **Scope:** Studies **understanding** or **"spontaneity."** Basic *a priori* principles of Nature (e.g., *law of causality*) are possible because the human mind imposes a coherent structure on objects of experience to make them intelligible to *understanding.* **Understanding = lawgiver of Nature.** (b) **Categories:** Experience is organized by *"schemata"* according to 12 basic concepts—the **"categories"** (e.g., *unity, reality, causality, necessity,* etc.). Unity of total organization of experience by categoreal schematism presupposes **transcendental unity of apperception** (*self-consciousness*). This is the ultimate condition of experience, that it should be a single coherent experience of a unitary conscious self. **Kant's categories** represent 12 different connected ways in which the mind organizes experience. (3) TRANSCENDENTAL DIALECTIC. (a) **Scope:** Studies **reason** (beyond forms of *sense-experience* and categories of *understanding*). A critique of reason's natural illusions such as *the proof of the unity of the world, of God, immortality, and free-will.* (b) **Is a transcendent metaphysics possible?** (*"Transcendent"* = *knowledge professing to pass beyond limits of experience.*) The answer is *no.* We suffer from **"transcendental illusion"** in attempting to apply categories of *understanding* to things-in-themselves in a **non-empirical** way; the categories are legitimate only when applied to objects of *sense-experience.* We go beyond possible experience when we ascribe reality to **Ideas of reason or transcendental Ideas** (e.g., *ideas of Self, World, God*). (c) **Consequences:** Each of these *Ideas of reason* gives rise to a metaphysical science: about the **Soul** (*rational psychology*), the **Universe** as a whole (*rational cosmology*), **God** and his attributes (*transcendental theology*). Such dogmatic sciences suffer from inevitable fallacies due to failure to note that *categories of understanding* have no legitimate use except in relation to experience. (d) **Conclusion:** *Transcendent metaphysics* is impossible as knowledge. Morality, religion, art presuppose God, self, and the world of free spiritual beings—unknowable through theoretical reason.

## 5

**OBJECTIVE IDEALISM.** A. CONTENT. Nature, though an appearance, is not merely the subjective mind's topic of consciousness, but a manifestation in its own right of an underlying **spiritual reality.** Hence Nature is relatively independent of the subjective mind; the underlying reality manifests itself both in Nature and Mind. B. PLURALISTIC OBJECTIVE IDEALISM (MONADISM). LEIBNIZ. (1) MONADS. World consists of invisible, eternal substances, no two alike. All material qualities are only the outward appearances arising from these centers of activity called **"monads."** Each monad is like a unique soul governed by an internal principle of change appearing in **desires.** Monads form a continuous gradation from the least active to the most perceptive minds (*God is governing principle of all monads*). Monads are spiritual substances or individual atoms of being—each complete in itself, simple, indecomposable, immortal; each different (*identity of indiscernibles*). Each "represents" the universe by the activity within itself. (2) PRE-ESTABLISHED HARMONY. No monad really acts externally on another for, as *Leibniz* states, monads have no *"windows and no doors"* through which elements might pass in or out. Instead, there is **pre-established harmony:** the world of monads is like a system of synchronous clocks; each monad follows its own inner law of development, but corresponds with every other monad's development. Hence harmony of **mechanism** (*physical laws*) and **teleology** (*purposiveness*) in world: the **Kingdom of Nature** (*mechanical law*) harmonizes with **Kingdom of Grace** (*freedom and love of God*). Hence **Theodicy** is possible (*the justification of God's way with world and men*). God chooses *"best of all possible worlds,"* and least amount of evil consistent with His plan.

## 6

**ABSOLUTE IDEALISM (MONISTIC OBJECTIVE IDEALISM).** A. CONTENT. Reality a single spiritual **Absolute. Nature** and the subjective mind are modes of actualization of the Absolute: *Schelling, Hegel.* B. PHILOSOPHY OF IDENTITY (SCHELLING). **Reality** = Absolute Reason, the identity of Spirit-Nature, subject-object, ideal and real. The *Philosophy of Identity* is a union of **Philosophy of Nature** and **Transcendental Idealism.** (1) PHILOSOPHY OF NATURE. Starts with object (*Nature*), shows how it leads to subject (*Mind*). Nature is a petrified giant intelligence; organized; the **world-soul** its unconscious creative principle. Nature evolves through a series of *"potencies"* (levels) to **self-consciousness** in human reason. (2) TRANSCENDENTAL IDEALISM. Starts with subject (*Mind*), shows how it leads to object (*Objective World for Mind*). (a) **Theoretical Stage:** Consciousness rises from **sensation** to productive intuition, then to reflection, then to act of will, thereby arriving at a world of *knowledge.* (b) **Practical Stage:** Consciousness arrives at world of **will,** in organized society, state, free,will, history (progressive revelation of God). (c) **Aesthetic Stage:** Art unites the *necessity of Nature with Freedom.* Consciousness arrives at absolute *self-consciousness.* Artistic genius creates like God in history: a work of art is a presentation of the *Infinite* in the *finite* (beauty). **Art** is *instrument* (**organon**) *of philosophy, a model for unified philosophical thought.* In art the self achieves insight into unity of *conscious-unconscious* activity of reason. (3) PHILOSOPHY OF IDENTITY. Union of the realism of the *Philosophy of Nature* with *Transcendental Idealism* through **Absolute Reason.** Here mind and thing-in-itself, subject and object, are united in total "indifference" or identity as the highest law. Everything *finite* is an **appearance** due to departure from the standpoint of the *Absolute.* **Subject-object cleavage** = *the Absolute knowing itself.* Leads to descending **potencies: Nature** (*real series*), **Mind** (*ideal series*); yet nothing is outside the Absolute.

## 7

**ABSOLUTE RATIONAL OR LOGICAL IDEALISM (HEGEL).** A. CONTENT. Reality is the self-unfolding of the **Absolute Idea** from God to His creation (immediacy to otherness), and to return into itself. Object of philosophy is *Knowledge of the unfolding Idea.* **Truth** = *self-becoming—realization by a thing of its ideal concept* (Begriff) *in its concrete actuality* (Wirklichkeit). Principle suggested by ordinary usage: a true friend is one whose actual manner of conduct accords with, and realizes the concept of, friendship. B. TRUTH IS THE WHOLE. It is the essential nature actualizing itself through **self-development.** Truth is not merely substance but also **subject,** the living process of self-development. C. ABSOLUTE IS MIND (*Geist, Spirit*). Its concept is to know itself to be not merely **in-itself** but also to work in history **for-itself;** mind that knows itself to be mind; the truth in its true form. **Philosophy** is the *science* of the Absolute Mind; it thinks the Absolute's self-actualization. D. HEGEL'S DIALECTIC. The 3 stages of the self-unfolding whole: (1) THESIS: Given, *implicit* phase; the whole as *abstract,* universal, immediate, in-itself. (2) ANTITHESIS: Phase of **explication;** *the whole as split into opposites,* infected with negativity, particularity, as for-itself or out-of-itself. (3) SYNTHESIS: The whole as **re-integrated** at a higher level of actuality, in-and-for-itself; the opposites are sublimated, *"aufgehoben,"* i.e., *their contradictions cancelled and the whole preserved by being raised to a higher unity.*

## 8

**VOLUNTARISTIC IDEALISM.** Reality is essentially **will** (*Schopenhauer*). A. THE WORLD IS MY IDEA. It is an object for myself, as subject. The thing of **perception** is the appearance of a *thing-in-itself.* As an intellectual subject, I experience Nature through the category of **causality,** of which space, time, and matter are specific forms. **Human reason,** concerned with phenomena alone, is merely an *instrument of will.* B. THE ABSOLUTE IS WILL. (1) Inner consciousness of **self** is a clue to reality and reveals the real self as *Will.* My body and its movements = objectifications of my will. (2) Similarly, **Nature** is the *objectification of Will,* from unconscious to self-conscious level. The Absolute is a blind, irrational force; no moral intent, mere will to be, live. C. PESSIMISM AND SALVATION. The Absolute Will gives rise to evil in the world. **Salvation** lies in overcoming the will: *the Will must vanquish itself and return to* **nirvana,** *final rest.* D. TEMPORARY ESCAPE FROM WILL. In *art,* as the contemplation of **Platonic Ideas** (*forms of Will's objectifications*). E. ULTIMATE ESCAPE. Achieved through the ethics of compassion to the ascetic negation of Will. Suicide no solution; Will is deathless, life continues.

## 9

**PHENOMENOLOGY (HUSSERL).** A. THE MAIN THEME OF PHENOMENOLOGY. *Consciousness in its structure of subjectivity-objectivity.* **Procedure:** not causal or psychological, but pure description, analysis, understanding by going to the facts themselves, using pure **"eidetic"** vision (*essence-vision*) of structures, acts, objects of consciousness. Hence, *eidetic science of consciousness* = study of essential possibilities of consciousness. B. BASIC CHARACTER OF CONSCIOUSNESS (1) INTENTIONALITY. Consciousness is always consciousness-of; a subjective process, act, *"intending"* an object that transcends the act. (2) OBJECT-ASPECT. The *"noematic"* aspect. (3) SUBJECT-ASPECT. *"Noetic"* aspect; the two are in essential relation. C. "EPOCHE". We can "bracket" all claims of object-subject to exist and restrict attention to what it is as we **intuit** it; thus maintain an attitude of pure vision or intuition of *facts of consciousness.* Thus we observe a "physical thing" as an object for consciousness without raising questions about its actual existence, empirical connections, etc. Can be done at various levels, e.g., for the whole of the *"natural world."* D. CONSCIOUSNESS IS "TRANSCENDENTAL." In so far as the subject or the mind is prior to being in a world or belonging to an existent being. Can use **epoche** to study the self as a *transcendental* subject and the world as its object. This gives universal **eidetic** science: *transcendental phenomenology,* knowledge of essential possibilities of *self-world* structure.

## 10

**PHILO. OF NATURALISM.** A. MATERIALISM. *Democritus, Lucretius, Hobbes, Holbach.* All things are forms or functions of physical matter and its laws. *Frequently associated with materialism are:* (1) ATOMISM. (2) REDUCTIONISM. Explanation of things by reducing them to material elements and physical laws; e.g., *life is a form of physico-chemical interaction.* (3) DETERMINISM. Every event is fully **caused;** *no free will.* (4) MECHANISM. Cause of an effect is a complex of preceding **physical** events; everything behaves like physical *machine.* (5) REJECTION OF TELEOLOGY. Apparently **purposive** behavior, as in living things, can be explained *mechanistically.* B. MONISTIC SUBSTANTIVE NATURALISM (SPINOZA). Reality is a single **substance** = *Nature or God.* All else is an **attribute** and/or a **mode** of the one substance. (1) DEF. OF SUBSTANCE. That which is **self-sufficient,** exists in itself and can be conceived solely through itself, i.e., it needs nothing else either to be or to be conceived. There can be only one such ultimate substance. It is *self-caused, absolutely infinite,* free (working through inner necessity of its own nature), and exists necessarily (its **essence** involves its existence). (2) SUBSTANCE IS NATURE in two senses: **Natura naturans** (Nature as the *indwelling active cause* of world) and **Natura naturata** (Nature as *effect,* or all the modes as they follow from Substance and its attributes). (3) NATURE IS GOD. **God** = *the absolutely infinite being* whose substance consists of infinite attributes, each expressing eternal *infinite essence.* (4) DEF. OF ATTRIBUTE. What mind perceives as constituting the essence of Substance. Of the infinitely many attributes, we know two: **thought, extension.** (5) MODES. **Ideas** and **minds** are modes (i.e., particular modifications) of the *attribute of thought* as **physical bodies** are modes of the *attributes of extension.* (6) ORDER. Order and connection of **ideas** is the same as order and connection of **things.** Hence, there is not **interactionism** (*Descartes*) but **parallelism** between mind and body. Body works on body, mind on mind, both in perfect agreement. (7) NATURE NOT TELEOLOGICAL. Nature not to be interpreted by **human goals,** nature works with *strict necessity.* Teleology confuses effects with causes. Will is not a free, but only a necessary, cause. Nothing is **contingent;** everything is *determined* by God's necessary nature. (8) HUMAN KNOWLEDGE. Varies from inadequate ideas of the imagination to *adequate ideas of the intellect.* With adequate intellectual ideas, we perceive all things **intuitively** via the Idea in God expressing their **essence;** we perceive *sub specie aeternitatis* (under the form of eternity).

## 11

**PHILOSOPHY OF NATURALISM – DIALECTICAL MATERIALISM.** (*Marx, Engels, Lenin.*) **Official philosophy of Soviet Communism.** A. CONTENT. Basic reality is matter. Out of matter evolve all further things according to dialectical laws. B. DIALECTIC. Emphasis on **constant change** and interconnectedness of things. C. LAWS OF DIALECTIC. (1) INTERPENETRATION OF OPPOSITES. No real thing is merely **A.** It is also **not-A.** This conflict generates change. *E.g., a moving thing is not only here, it is also not-here; interpenetration of here and not-here is its motion.* (2) TRANSFORMATION OF QUANTITY INTO QUALITY. Natural changes are not merely quantitative; at certain points a sufficient change in **quantity** is connected with a sudden change in **quality.** *E.g., water when heated, rises quantitatively in temperature until, at boiling point, it suddenly changes to vapor; money accumulated, at certain point turns to capital.* (3) NEGATION OF THE NEGATION. Change is not abstract repetition, but moves forward to a higher level from *A,* to *not-A,* to a *new A* (negation of not-A, i.e., negation of the negation). *E.g., seed (A) grows into plant (not-A) which produces new seeds (new A).* With this dialectical logic of change is a dialectical theory of nature, thought, history, society.

## 12

**CRITICAL NATURALISM.** (*Dewey, Santayana, Alexander, Whitehead.*) A. CONTENT. Nature not merely **material;** it includes everything that exists (while dualisms separate the natural vs. supernatural, body vs. mind). B. ACCOMPANYING DOCTRINES. (1) REJECTION OF REDUCTIONISM, EMPHASIS ON EMERGENCE OF NOVELTY (Alexander). Emergent evolution of Nature develops by rising levels from **space-time** through matter, secondary qualities, life, mind, toward *deity.* (2) ACCEPTANCE OF ALL FORMS OF EXISTENCE AS EQUALLY REAL, INCLUDING VALUES. Values exist as real presences or properties of things existing in **natural contexts:** *Whitehead. (Santayana:* value dependent on human interest.) (3) TENDENCY TO ADOPT REALISM, PRAGMATISM, INSTRUMENTALISM IN THE THEORY OF KNOWLEDGE. (a) **Alexander—Neo-realism:** In consciousness, an organism is **"compresent"** with a thing; does not alter it, and *perceives it in a particular perspective.* (b) **Santayana—Critical Realism:** External thing exists **independently** of the knowing mind; is known *mediately* via essence *intuited by the mind.* (c) **Dewey—Instrumentalism:** Knowing is **transforming** situations *indeterminate* in significance into situations *determinate in significance.* Ideas are **instruments** for effecting the transformation; their validity lies in their *effectiveness* for this function. (4) REJECTION OF CATEGORY OF SUBSTANCE IN FAVOR OF QUALITY, RELATION, ACTIVITY, PROCESS, AS CENTRAL METAPHYSICAL NOTIONS. (*Instance* is treatment of mind.) (a) **Alexander:** Mind is an emergent quality of organisms. (b) **Dewey:** Mind is the functioning of meanings in life-activities of organisms. (c) **Santayana:** *Psyche* is the self-maintaining pattern of an organism; *Spirit,* the awareness emergent in animals, conditioned by material circumstances. (5) INTERPRETATION OF DIVINE AS CONSTITUENT FACTOR IN NATURE OR IN EXPERIENCE. (a) **Alexander:** *God is in the making;* the existing world is the body of God with tendency (*"nisus"*) toward deity. *Religious experience* = feeling the movement of, union with, God. (b) **Whitehead:** God, the principle of individual existence (*concretion*), is involved in every actual event. The entrance of eternal ideals or objects (*universals*), and already actual occasions, into the **becoming** of a new event is restricted by God—the ultimate limitation, the source of restriction, the **"ultimate irrationality."** (c) **Dewey:** *Divine* = the ideal possibilities unified through *imagination. Faith* = the unification of self through willing *allegiance* to the inclusive ideal ends presented by imagination.

## 13

**EXISTENTIALISM.** (*Kierkegaard, Heidegger, Sartre, Jaspers, Marcel.*) A. CONTENT. **Protest** against rationalism (*esp. Hegel*), scientific positivism, anonymous mass existence, impersonality of modern thought and life. B. GENERAL EMPHASIS. *Freedom,* significance of individual human person and personal relationships. Emphasizes the *total self,* not only reason, in man's thinking and relation to the world around him. Hence: **Being** and reality are found not in objects of knowledge alone but in something accessible only to the free *total person.* Man finds **being** only via his *destiny.*

**C. EXISTENTIAL THINKING.** Thinking is done by man as actor *engaged in existence*, not as a mere *spectator*. **(1) KIERKEGAARD.** Thinking by an actual living person, involved in the human situation, is coming to grips with the situation, trying to achieve an authentic relation to reality, and ends in a commitment and leap of faith. **(2) HEIDEGGER.** Man as an individual—a free and responsible *being-in-the-world* (*Dasein*), confronted with the threat of **Nothingness**, death, in dread and guilt—is **Care** (*Sorge*). As **Transcendence**, capable of going beyond itself, Dasein can resolutely seek truth, i.e., seek to emerge into the openness and light of Being, by "letting" the existent be, serving as the "caretaker" of Being. **D. EXISTENZ.** To refer to the **self** in its possibility of being or not being itself, existentialists speak of **EXISTENZ**—man as **radically free**, a **genuine individual**, irreducible to a scheme of metaphysical or scientific knowledge; capable of choosing to be **authentic** or inauthentic. *Jaspers:* **EXISTENZ** = man as freedom, not as the object of knowledge; man as the "unobjective" that he is and of which he becomes aware when genuinely conscious of the self; man is more than he can know of himself; he can experience this only in the origination of **thinking** and **doing**, not in knowing-about himself. **E. EXISTENTIALIST VIEWS OF BEING. (1) HEIDEGGER.** Distinguishes **Being** (*Sein*) from **what-is** (*das Seiende:* the be-ing, that which is, a being, beings). Different beings have different **modes** of being, e.g., physical objects are merely **present-at-hand** (*vorhanden*), tools are **ready-to-hand** (*zuhanden*), man is **being-there** (*Dasein*). Man's being is analyzed as care, more deeply as **temporality**. In *inauthentic* everyday existence, man's true condition and relation to Being are concealed. Truth, following the Greek conception of it as *aletheia*, is **un-concealment.** Only through metaphysical basic moods, like dread, does man face the threat of Nothingness, realizing his **being-for-death**, and through this, move toward truth and the possibility of the revelation of Being. Because man is **Care**, and Care is constituted by *temporality*, to arrive at the revelation of Being and his own authentic existence, man must question the meaning of his own being as historical action relating himself to *what-is in time*. Hence man originates his *historical-spiritual* existence not by studying his history but by articulating his essential relations to *what-is*. **(2) SARTRE.** Two forms of being, **in-itself** (*the merely contingent*, e.g., unconscious things) and **for-itself** (*conscious being*). **Man's being** = *being-for-itself* which introduces the "Nothing" into the world, a source of negation, distinction; never complete, always self-transcending; a *project*, hence free and capable of bringing the self into being through the exercise of freedom. **(3) MARCEL.** Being is a *mystery*, not merely a problem. Can't detach it from self; can deal with it only in personal thought, action. Our **"exigence of being"** = craving for participation in being. Man can find being through respect and love for persons as centers of responsible freedom, in participation with them—not as objects—but as *presences* (**I-Thou** relation of *Buber*), in faith, hope, love of God. **(4) JASPERS.** Being lies in the **Encompassing** which is the source of the entire **subject-object cleavage**, and hence can never be an object. We can conceive the Encompassing metaphysically as Transcendence (God) to which we relate our self as *Existenz* by interpreting things of the world as ciphers and symbols of Transcendence.

## 14

**ETHICS. (1)** Study of good-bad; **axiological ethics** = theory of ethical values. **(2)** Study of right-wrong; **deontology** = theory of ethical obligation, duty.

## 15

**ANALYSES OF ETHICAL CONCEPTS, JUDGMENTS. A. EMOTIVE MEANING THEORY.** Ethical judgments are not entirely factual. Essentially they express emotions, attitudes; hence are incapable of being true or false (*A. J. Ayer, C. L. Stevenson*). **B. INDEFINABILITY OR INTUITIONAL THEORY.** Basic ethical concepts are indefinable, but denote intrinsic, objective qualities or relations apprehended intuitively. Hence ethical judgments are objectively true or false (*G. E. Moore, W. D. Ross*). **C. DEFINABILITY THEORIES.** Basic ethical concepts are definable in terms of: **(1) ATTITUDES.** "Good," "right" refer to someone's *approbation*. **(a) Subjective Approbative Theory.** *X* is good, right = *I* approve *X*. (Part of *Hume's* view.) **(b) Social Approbative Theory.** *X* is good, right = society approves *X*. (*Durkheim, some Sophists*). **(c) Theological Approbative Theory:** *X* is good, right = God approves *X* (*Karl Barth*). **(2) PSYCHOLOGICAL FACTORS** (other than attitudes): **(a) Hedonism:** "Good" = pleasure; "bad" = pain (*Aristippus, Epicurus, Bentham*). **(b) Affective Theory:** *X* is good = *X* satisfies. Right = what contributes best toward fullest satisfaction (*Santayana*). **(c) Interest Theory.** *X* is valuable = interest is taken in *X*. Act is right if it conduces to maximum moral good (harmony of all interests involved) (*R. B. Perry*). **(3) LAW.** "Good", "right," defined in terms of obedience to law. **(a) Custom Theory:** Right = customary, sanctioned by tradition. **(b) Positive Law Theory** (Legalistic Ethics): Right = law of the land. **(c) Natural Law Theory:** Goodness or rightness = harmony with Law of Nature (*Stoics*), harmony with Law of God as Governor of Nature (*Aquinas*). **(d) Moral Law Theory:** Rightness = moral lawfulness; rightness of moral law consists in its being law (universal, necessary) (*Kant;* See *17*). **(4) SELF AS METAPHYSICAL PRINCIPLE. (a) Eudaemonism:** Good = what we ultimately aim at = happiness (life in accordance with human function [*Aristotle*]) or blessedness (life in vision of God [*Aquinas*]). **(b) Self-realization Theory:** Good = what we ultimately aim at = realizing our true self (*Hegel, Bradley*). **(5) PROCESS.** Good, right, viewed as deriving from dynamics of some significant process. **(a) Evolutionary Theory:** Good = fit for survival, in harmony with life-force, etc. Right = conducive to evolution of new values (*Julian Huxley*). **(b) Marxism:** Moral ideas are "ideological," i.e., they reflect economic status of class and represent class-interest as interest of whole society. That moral ideal is adequate which is adapted to the needs of a particular stage of economic-social development (*Marx, Engels*). **(c) Pragmatic Naturalism:** Moral judgments are hypotheses for resolving problematic situations of values, means-ends. Good = what promotes our course of activity, seen in the light of natural and social conditions and consequences for further activities. Right = what promotes mutual goods of community (*Dewey*).

## 16

**TELEOLOGICAL (GOAL-DIRECTED) SYSTEMS. A. CONTENT.** Emphasis on harmony, purpose, good, ends, consequences. An act's rightness is not in its motive but in its **purpose** or intended aim and in the act's **effects** on the self or on society. **B. HEDONISM. Pleasure** = sole intrinsic *good*. **Pain** = sole intrinsic *evil*. Good life, happiness, is life of maximum pleasure, minimum pain. **C. EGOISTIC HEDONISM.** Good life is life of maximum pleasure for *self* (*Aristippus, Epicurus*). **Chief virtue** = **prudence** (ability to forecast pleasure-pain effects of acts, avoid harm to self, seek inner harmony). **D. UNIVERSALISTIC (ALTRUISTIC) HEDONISM (Hedonistic Utilitarianism).** Ethical objective is greatest happiness (preponderance of pleasure over pain) for greatest number of people. **(1) QUANTITATIVE THEORY** (*Bentham*). Happiness is calculable in terms of the **quantity** of pleasure-pain (intensity, duration, probability, promptitude, fecundity, purity, social extent). Social morality is achieved by **sanctions** (physical, political, moral, religious). **(2) QUALITATIVE THEORY** (*J. S. Mill*): Pleasures differ qualitatively as well as quantitatively. Humans capable of higher pleasures than lower animals. Qualitative differences tested by preferences of those capable of experiencing both. (A discontented *Socrates* is better than a contented fool.) **E. IDEAL UTILITARIANISM.** Differs from **Hedonistic Utilitarianism** in that pleasure is not the sole good. Ethical objective is the maximization of good (*G. E. Moore, H. Rashdall*).

## 17

**DEONTOLOGICAL ("OUGHT") SYSTEMS. A. CONTENT.** Right, law, duty, conscience, obligation, virtue. Act's rightness lies in motive, virtuousness, or fulfillment of moral law regardless of consequences. **B. KANT'S FORMALISM. (1) ONLY GOOD WILL IS ABSOLUTELY GOOD.** Consequences do not determine the rightness of an act. An act is right if done in accordance with moral law because it is the law, not because of profit or pleasure. Man should act out of a sense of obligation, not inclination or desire. **(2) MORAL IMPERATIVE MUST BE CATEGORICAL, NOT HYPOTHETICAL. Hypothetical imperative** (means to an end): *If* you desire profit, *then* be honest (honesty is best policy). **Categorical imperative:** *Be* honest (because it is right, regardless of profit or loss). **(3) TEST OF LAWFULNESS.** Law is universal, necessary. Hence, rule of Categorical Imperative (which regards only the form not the matter of the act) is: *Obey moral law because it is law.* Kant proposes these maxims of behavior: **(a)** Act always in such a way that you can will your **maxim** (subjective principle of action as distinguished from objective principle which is practical law) to be the *universal law for all rational beings*. **(b)** Act so as to treat humanity, whether in your own person or in that of any other, always as an **end**, never as mere *means*. **(c)** Act as if you were a legislating member of the universal kingdom of ends. **C. METAPHYSICAL FOUNDATION OF MORALS.** Since **Nature** is a realm of *cause-effect*, and **morality** presupposes *freedom of will*, we postulate (though not knowing how to prove) human freedom, the immortality of our soul, and the existence of God.

## 18

**MIXED SYSTEMS. A. POWER THEORY. (1)** Nature is a scene of competition. Man's destiny is a struggle for power. He who **commands** lays down the law, thus determining the right. *Might makes right.* **(2)** Virtue is either **(a) strength,** might, or **(b) obedience** to commanding will and law. Hence the two-fold morality of strong and weak, master and subject. (*Thrasymachus* in **Plato's Republic**; also one aspect of *Nietzsche*). **(3)** *Hobbes'* variation of Power Theory: Reason, following the laws of Nature, leads men to social contract, **assigning** all power to a sovereign, who thereupon makes law, thus determining right. **B. STOICISM.** Nature is a harmonious rational whole; man a rational fragment of the whole. Man's happiness lies in following Nature's law. **(1) MAN'S ESSENTIAL FREEDOM:** his rational capacity to affirm or deny what Nature (Divine Reason) rules to be. **(2) MAN'S VIRTUE:** affirm Nature, conform to Nature. **(3) MAN'S HAPPINESS:** rational self-discipline, apathy (absolute control of desires, emotions) [*Epictetus, Seneca, Marcus Aurelius*].

## 19

**MIXED SYSTEMS—GREEK EUDAEMONISM. (Functionalism, Perfectionism).** Natural entities have essential functions or ends. Their good lies in the fulfillment of function: good life is life in accordance with **function. A. PLATONIC EUDAEMONISM** (*Plato's Republic*). **(1) MAN'S SOUL TRIPARTITE: reason, spirit, appetite. (2) FUNCTION OF SOUL:** guidance of human life in accordance with knowledge of good. **(3) SUBORDINATE FUNCTIONS:** reason rules; spirit is the principle of pride and ambition; a dynamic factor (*appetite*) acts through desire and needs to be controlled, organized. **(4) THE POLITICAL STATE IS TRIPARTITE** (by analogy to soul): **Guardians** (rulers), **Auxiliaries** (military), **Artisans** (workers). **(5) FUNCTION OF STATE:** to make possible realized human lives through justice, harmoniously organized human relations. **(6) SUBORDINATE FUNCTIONS: Guardians** to rule, legislate, organize. **Auxiliaries** to attack and defend. **Artisans** to work, reproduce, etc. **(7) VIRTUE** is excellence, the capacity to fulfill function. (*Good eye* is one which can see well; *good man* is one who can live well; *good state* is one which organizes human life well.) **Cardinal virtues**, both in man and state are *wisdom, courage, temperance, justice.* **(8) SEQUENCE OF STATES AND MEN** in order of decreasing degree of virtue: *Aristocratic, Timocratic, Oligarchic, Democratic, Tyrannical.* **In Aristocratic man**, reason governs, spirit and appetite obey, each performing its appointed task; similarly in Aristocratic or *best state.* **B. ARISTOTELIAN EUDAEMONISM** (*Aristotle's Nichomachean Ethics*). **(1) MAN'S FUNCTION** determined by his distinctive capacity (*differentia*) to reason. Hence man's function is the activity of his **soul** in accordance with reason, or in accordance with **human virtue** (not in a chance period of time, but in a complete life). **(2) VIRTUE TWOFOLD: intellectual** (learned), **moral** (practiced). Moral virtue is a habit of deliberate moral purpose, aiming at the *mean* relative to each of us, the mean being determined by reason or as a prudent man would determine it. **Virtue** lies between **excess** and **deficiency. EXAMPLES:** *courage* is a mean between foolhardiness and cowardice; *magnificence* is a mean between vulgarity and meanness. **(3)** right action is **(a)** voluntary, **(b)** based on deliberation, **(c)** aimed at good. **Good life** is an activity, not a state of mind. **(4) PLEASURE,** though not the goal of virtuous activity, nevertheless crowns that activity. Hence the virtuous life (life of reason) is also a happy life. **(5) HIGHEST HAPPINESS** is life in accordance with highest virtue. Highest virtue is the intellectual virtue of reason. Highest life is life of **contemplation,** akin to God's self-contemplation.

## 20

**MIXED SYSTEMS—COMMUNISM. A. CONTENT.** The moral ways of life vary historically with the **economic organization** of society, as a "superstructure" based on a class-system, e.g., *feudal ethics, capitalistic or bourgeois ethics.* The ruling class historically is constrained to disguise its self-interest in form of a pretended universal moral code (= the *"ideological"* character of a moral system). **B. IN A "TRUE" COMMUNIST SOCIETY: (1) NO CLASS-DIFFERENCE** so that the moral code reflects interest of entire society. **(2) NO DISGUISING** so that the ideological character disappears, falsehood is removed. *Moral code* harmonizes interests of total community.

## 21

**MIXED SYSTEMS—NATURALISTIC HUMANISM. A. CONTENT.** Man to be understood as a part of Nature, subject only to **natural conditions,** laws. **B. SPINOZA'S NATURALISM.** Man participates in Nature as a **body-mind.** Man innately impelled as are all things by a **striving** (*conatus*) to *preserve his being.* What aids him is good, what hinders is bad. **(1) PASSIONS,** leading to dependence on other things, hinder freedom, self-determination. **(2) HINDRANCES** are overcome by **knowledge.** Since knowledge gives freedom, its object is loved. Knowledge of God or Nature as the eternal substance is highest knowledge; the **intellectual love of God** is man's salvation and happiness. **(3) VIRTUES** are **emotions** controlled by rational activity (courage, self-assertion, generosity). **(4) VICES** are emotions correlated with irrational activity and hence harmful (hatred, envy, pity, humility). **(5) ALTRUISM AND SOCIAL COOPERATION** are good because of their usefulness toward the rational life of which they are a part. **C. PRAGMATIC NATURALISTIC HUMANISM. (DEWEY). (1) BIOLOGICAL AND SOCIAL EMPHASIS.** Man is an evolved, **socially conditioned animal.** Makes adjustive responses to changing situations so as to attain *self-development through social activity* and cultural life. His social life is a new, emergent level of reality, with new qualities, relations, etc. **Intelligence** is the supreme method and means of *human adaptation, adjustment.* **(2) FUNCTION OF INTELLIGENCE.** Solution of problems arising from changing conditions; analysis and removal of obstructions to human life-activity. **(3) MORAL EVALUATIONS.** Judgments concerning **effectiveness** of proposed conduct to resolve a given situation (without introducing further difficulties: *means-end continuum*). Hence, validity of moral ideas is judged by effectiveness as **means** or **instruments** for resolving human difficulties, opening way to enriched life. **(4) MORAL PROBLEMATIC SITUATIONS.** Always are specific, involving particular persons, places, issues; hence need for experimental temper of mind. **(5) EMPHASIS.** Democracy and social-intellectual freedoms, scientific study of man, inventiveness, growth of meanings and values in experience, socially shared values.

## 22

**OTHER MIXED SYSTEMS. A. IDEALISTIC SELF-REALIZATION THEORY.** (*Hegel, Bradley, Bosanquet, Royce*). **(1) TRUE SELF** is not the finite empirical self but the self as actualized in, or identified with, the **eternal Spirit** of which finite selves are only *manifestations.* **(2) MORALITY** arises with **conflict** between *desire* and *obligation,* wish and law. Moral progress is a resolution of this conflict, in which the willing, desiring self becomes more *rational* and the law of obligation becomes the *indwelling principle* of the self. The **finite individual life** matures into identification with the larger life of a *community of selves,* a whole such as the State. **(3) ULTIMATE TRUTH** of the self lies beyond the merely moral, in the identification with the Divine (as in religion or philosophy). **B. RELIGIOUS PERFECTIONISM (SELF-REALIZATION). (1) CHRISTIAN PREFECTIONISM OF AQUINAS** is a synthesis of *Platonic, Aristotelian, Hebrew-Christian* traditions. Purpose of life is achieved in fulfilling man's function. Man's function not only the natural one emphasized by the Greeks, but also and ultimately **supernatural. (2) MAN'S LAST END** is blessedness in knowledge of God, the *beatific vision of God.* **(3) THE FOUR CARDINAL VIRTUES OF THE GREEKS (prudence, justice, temperance, courage)** capped by the three **THEOLOGICAL VIRTUES (faith, hope, charity)** complete the moral structure of life.

## 23

**EPISTEMOLOGY: NATURE, REALITY, EXTENT OF KNOWLEDGE (KNOWER-KNOWN, SUBJECT-OBJECT). A. NAIVE REALISM. Commonsense view.** *Reid, G. E. Moore:* World of ordinary experience exists *independently* of perceiver and is just as it is experienced. **B. SUBJECTIVISM (Subjective Idealism).** *Berkeley:* Objects of knowledge do not exist independently of our consciousness of them. The world is a complex of ideas, sensations, etc., occurring within individual subjective minds. Reality consists of minds and their contents. Matter does not exist. Physical bodies are complexes of **sense-qualities. C. EPISTEMOLOGICAL DUALISM.** *Descartes, Locke, Critical Realists* like *Lovejoy, Santayana:* There is an **objective** world *outside the mind.* We know this world through representations of it in our minds (ideas, sensations, sense data, essences). **D. NEW REALISM.** English (*Moore, Russell, Alexander*), American (*Woodbridge, Holt, Montague, Perry*): Combines two theses. **(1) INDEPENDENCE OF KNOWN AND KNOWER.** An object of knowledge is not a construction of a mind, but is independent of the act of knowing. Sense data, physical objects, mathematical objects, other minds, all may exist whether or not we know them. **(2) PRESENTATIONAL NATURE OF KNOWLEDGE.** As opposed to *Epistemological Dualism,* knowledge is direct, immediate. The object known is or coincides with, the content already present for the knowing act. **E. PHENOMENALISM. (1) PURE PHENOMENALISM.** All things are **phenomena** or constructions out of phenomena. All we have a right to assume is **ideas,** impressions, or appearances. Both mind and matter are bundles or **sets of appearances.** *Hume* argued that merely rational analysis of knowledge leads to pure phenomenalism, whereas Nature, through habit and imagination, leads us to believe in substantial minds and things. Pure phenomenalism plays important role in **Radical Empiricism** of *James* and **Neutral Monism** of *Russell;* both were influenced by *J. S. Mill's* theory of material objects as *"permanent possibilities of sensation."* **(2) AGNOSTIC PHENOMENALISM** (*Kant, Spencer*). There is a reality (*thing-in-itself*) transcending consciousness, but we can know it only in its relations to our minds, as appearances, phenomena, not in itself as **noumenon. F. OBJECTIVE EPISTEMOLOGICAL IDEALISM.** Absolute idealism's view of knowledge, as in *T. H. Green, Hegel, Bradley, Bosanquet, Royce:* The **real** is manifested in, but *transcends,* our finite minds. What exists is mental, but deeper, more comprehensive than any finite mind and its contents. Knowing is the progress of the finite mind toward fuller conformity and identification with the *Absolute Idea, Mind, Spirit.* Truth is at once **mental** and **objective. G. PRAGMATISM (Instrumentalism)** *Dewey.* Knowledge is not the confrontation of subject with object, but the **resolving** of experiential problems, the transforming of an indeterminate, problematic situation into one which is *coherent* and *determinate* in the significances it contains. Knowledge is result of **experimental inquiry** (transformation of things of experience, from lacking to having significance).

## 24

**EPISTEMOLOGY: SOURCES, METHOD OF KNOWLEDGE. A. AUTHORITARIANISM.** In certain domains (e.g., *morals, politics, religion*) there are privileged individuals or institutions which function as sources of knowledge. As the witness can testify to the event, so the authority can transmit knowledge to us, handing on tradition, divine revelation, or divine command. **B. INTUITIONISM.** Intuition is a **direct apprehension of truth** which is not the result of reasoning or sense-perception: an *immediate,* nondiscursive, non-symbolic *penetration into the nature of the object* (*Bergson, the Mystics*). **C. RATIONALISM. (1) Reason** is capable of grasping basic truths *intuitively* and deriving all other truths from them by **a priori** rational procedures, logical demonstrations, etc. (*Aristotle, Aquinas, Descartes, Spinoza, Leibniz*). **(2) Rationalism** in *Absolute Idealism* opposes to this "linear" procedure the conception of **rational insight** as a growth of the *whole-vision* from indeterminate to determinate. In less extreme form, reason is necessary to distill out of sense experience or impart to sense experience, **universal necessary laws. D. EMPIRICISM.** Emphasizes indispensability of experience for knowledge. **(1) GENETIC EMPIRICISM** stresses experience as the original source of materials of knowledge. *Locke* said there were two sources of our ideas **—sensation** and **reflection** (introspection)—which provide the mind with its raw materials. **(2)** Most empiricism is also **evidential,** stressing experience as the test of *validity of beliefs,* as in *Pragmatism* (*James, Peirce, Dewey*) and *Logical Empiricism* (*Schlick, Carnap*). Empiricists admit *a priori* knowledge, if at all, only in logic and pure mathematics; otherwise all knowledge of fact is **a posteriori,** i.e., derived from and tested by experience. **(3)** Recent empiricists tend to give a larger role to reason in the **organization of knowledge,** stressing the *theoretical* aspects of science as contrasted with simple perception. They emphasize the tentative, hypothetico-experimental, and self-corrective character of science. **E. CONVENTIONALISM** (*P. Duhem, H. Poincaré*). A given body of data does not uniquely determine the theory which will interpret it. There is no "crucial experiment" but always a choice among **alternative hypotheses.** The choice cannot be made on grounds of evidence alone, but requires a decision or convention guided by practical, economic, aesthetic, or other motives.

## 25

**EPISTEMOLOGY: VALIDITY OF KNOWLEDGE (TRUTH/FALSEHOOD). A. CORRESPONDENCE THEORY** (*Aristotle, Locke*). Truth is **correspondence of thought** with *objective fact.* Test of truth is comparison between thought and fact: Is the fact as the thought thinks it or the statement states it? **B. COHERENCE THEORY** (*Hegel, Bradley, Bosanquet*). Truth is the **systematic coherence of Judgments.** Test of truth is *consistence of a judgment* with other judgments, its fitting into a coherent system of judgments, ultimately into a single absolute system or judgment. **C. PRAGMATISM (Instrumentalism).** Truth is the **effectiveness of an idea** used as a hypothesis. Test of truth is whether *idea works* when *tested* by experiment; guides us successfully in solving problems. Pragmatists vary in interpreting "success." *F. C. S. Schiller* viewed it in terms of **utility;** *James* as human **satisfaction;** *Peirce, Dewey* as **predictive power,** verifiability, utility in inquiry.

## 26

**AESTHETICS. A. DEFINITION.** Theory of art and beauty. **B. ART AS IMITATION (Mimesis, Mimetic Theory—** *Aristotle, Plato.*) Art imitates or depicts an object, whether individual, ideal, or universal. Thus for *Aristotle,* tragedy is an **imitation** (*representation*) of human action in which a good man comes to disaster because of some failing or defect. Its purpose is to effect a pleasurable **catharsis** (*purging*) of the emotions of pity and fear in the spectator. *Plato* also used the imitation theory to argue that art is inferior to philosophy and should be rigidly controlled by the state. **C. AESTHETIC HEDONISM.** Generally, the view that beauty or aesthetic value lies in the capacity of an object to *please* us in aesthetic contemplation. **(1) AQUINAS.** The beautiful is that which pleases in the mere apprehension of it. **(2) SANTAYANA.** Beauty is **"pleasure objectified,"** i.e., our pleasure projected spontaneously as a quality of the object, which then appears to us as beautiful. **D. ART AS REVELATION** (*Hegel, Schopenhauer*). Art embodies in concrete sensuous form a higher reality such as a **Platonic idea,** the **Absolute idea,** or (as in music acc. to *Schopenhauer*) the **Will. E. ART AS PLAY** (*F. Schiller, K. Lange, De Witt Parker*). Art enables human imagination to operate freely, through the order it imposes on the playful activity of the imagination. Allied theories make art an **escape** or **illusion,** as in *Freud* (art as the socially relevant expression of repressed wishes and desires) or *Lange* (art as a self-deception by which we escape into an ideal world). **F. ART AS EXPRESSION. (1) CROCE.** Art is an **expression of impressions,** i.e., the spiritual transformation of the raw materials of experience into *intuitions,* lyrical visions, of the concrete and individual. We achieve thereby a knowledge differing from the universalized, conceptualized knowledge of science. **(2) COLLINGWOOD.** Art is the **expressive objectification** of feeling, raising *feeling* to a level of *consciousness.* **G. ART AS COMMUNICATION. (1) TOLSTOY.** Views artistic expression as activity in which the artist transmits his emotions, by means of **consciously constructed external signs,** to other persons, who may then experience the same feelings. **(2)** Contrast the *Aesthetic Formalism* of *Clive Bell:* art is a **language of emotion,** where the emotion is a special aesthetic emotion experienced in apprehending a "significant form" rather than the represented content. **H. ART AS CONTEMPLATION AT A DISTANCE.** *Bullough* views the aesthetic attitude as one of **contemplation of an object,** achieved as a result of interposing *"psychical distance"* between it and our self. We *"distance"* an object by putting it out of gear with our practical needs and ends, and thereby allow ourselves to enjoy the contemplation of it. **I. ART AS EMPATHY** (*Lipps, Vernon Lee*). Empathy is *"Einfuhlung,"* literally *"feeling one's self into"* the object. In the aesthetic experience we project our own life-feelings, emotions, attitudes, activities into the object, and thus experience it as beautiful. **J. ART AS HEIGHTENED EXPERIENCE** (*Dewey*). Art purifies and heightens the features that make experience satisfying or consummatory, raising them **above the threshold of perception** so that we may enjoy them for their own sake.



DO NOT PLACE ON HOT SURFACES

KNOW THE FACTS ON THIS CHART AND YOU WILL GET

HIGHER MARKS IN CLASSWORK, HOMEWORK, TESTS AND EXAMS.

THERE ARE QUICK CHARTS IN MATH • SCIENCE • ENGLISH • ECO • HISTORY • LANGUAGES • PSYCH • SOC • PHILO • MUSIC

DATA-GUIDE EDITOR: JOSEPH L. LEON

PUBLISHED & MF'D BY DATA-GUIDE, INC. FLUSHING, N.Y. 11355

DATA-GUIDE QUICK CHART

# PRINCIPLES OF PSYCHOLOGY

A SIMPLIFIED SUMMARY & INSTANT REFERENCE

LEARN & REVIEW IN MINUTES

AUTHOR: Prof. B. Bugelski, Ph.D., U. of Buffalo.
CONSULTANT: Prof. R. Berry, Ph.D., Indiana U.

SOLID PLASTIC

SPEED STUDY TO SAVE YOU TIME AND WORK.

QUICK CHARTS PUT THE FACTS IN FULL VIEW:

PERMANENTLY FILL IN FOR MISSING NOTES & TEXTS.

QUICK CHARTS HELP YOU REMEMBER:

CONTAINS NO PAPER

## INTRODUCTION

**1 THE STIMULUS-RESPONSE (S → R) LAW.** **A. DETERMINISM AS APPLIED TO PSYCHOLOGY.** **Determinism** stresses that behavior is a function of specifiable variables. All responses are caused by or related to **stimuli** or **conditions.** If you know the **stimulus** and the **conditions**, you can predict **responses.** Thus, behavior is considered a function of stimulation: **B = f(S)** **B. DIFFICULTIES WITH THE S → R LAW. 1.** Difficult to define **stimuli** except with reference to the **responses. 2.** Most psychologists insist on considering the organism (**O**) as modifying the relationship between **S** and **R** and express the formula thus: **S → O → R. 3.** *B. F. Skinner* claims stimuli are too complex to describe and prefers to observe only responses. In Skinner's view, responses are thought of as **emitted** by the organism, not **elicited** by stimuli. Stimuli, when involved, merely set the stage for **responding.** They are then thought of as **discriminated** stimuli (**$S^D$**). **4.** *Clark L. Hull, C. E. Osgood et. al.*, think of additional **s → r** sequences as intervening between **S** and **R**; thus the *external world change* **S**, leads to some *internal event* **r**, which is followed by another *internal event* **s**, and these can continue initiating additional **s → r's** before the final observable **R.** These inner, covert, implicit **s → r** units are thought of as extra stages within the basic **S → R** framework. Thus, we speak of a **multiple-stage S → R Law** to account for delays between **S** and **R** (*as in thinking*).

**2 EXPERIMENTAL METHODS OF PSYCHOLOGY.** **A. BASIC PRINCIPLE. 1.** Involves a **controlled** situation where **observations** are made on some **dependent** variable (*a sample of behavior*) when one factor (*the* **independent** *variable*), assumed to be related to the behavior, is systematically **varied. 2.** Comparisons are made with a **control condition** where the independent variable is held **constant.** Independent variables normally considered are classified as **environmental, task,** and **subject** variables. **B. PSYCHOPHYSICAL METHODS.** Procedures for controlling the presentation of stimuli to *observers* (**O's**). **1. THE METHOD OF AVERAGE ERROR.** Here the **O** manipulates a stimulus and tries to adjust it to match a **standard.** His errors are **averaged.** These errors can be of two types, **constant** and **variable. Constant errors** indicate a general tendency to either over or underestimate. **Variable errors** measure the consistency with which **O** sets his stimulus. **2. METHOD OF LIMITS.** The *experimenter* (**E**), gradually changes a stimulus by **discrete** steps, until **O** reports equality or difference from some standard. **3. METHOD OF CONSTANT STIMULI.** **E** calls for repeated comparisons between some standard and a series of **fixed comparison stimuli** differing by known **small steps. C. CLINICAL METHODS. 1. TESTS.** These can be **objective,** with a limited range of possible answers; or **projective,** wherein the subject (**S**) is asked to react to vague, amorphous, or **unstructured** stimuli (*see 37:C*) in terms of his own background. **2. RATING SCALES.** A **trait** is described in terms of low to high in a number of steps. The **rater** indicates where he believes some person, item, or stimulus fits on the scale. Ratings suffer from **halo effects** (*see 37:F*). **3. INTERVIEWS AND QUESTIONNAIRES.** These suffer from possible deception, lack of rapport, overgeneralization. **4. OTHER METHODS: longitudinal** (*long term investigations, obtaining case histories, etc.*), or **cross-sectional** (*the more typical experimental operation*).

**3 BASIC METHODOLOGY REQUIREMENTS.** **A. RELIABILITY. 1.** To be acceptable as a fact or a method, it must be dependable and give about the same results every time it is used (**reproducible**). **2. Reliability** is commonly measured by **statistical** procedures involving **repeated** measuring. Comparing the differences between two sets of scores on the same people, by special techniques yields **coefficients of correlation. B. VALIDITY.** A finding must be **true:** *A test for mechanical ability should not measure intelligence.*

**4 BIOLOGICAL FOUNDATIONS OF PSYCHOLOGY.** **A. HEREDITY VS ENVIRONMENT. 1.** It is generally accepted that physical characteristics (*color of eyes, hair, length of nose, etc.*) are hereditary, but even these develop in certain specific environments which may **modify** them. **2.** Most **psychological** traits (*apprehensiveness, sociability, honesty, sense of humor, etc.*) are regarded as primarily **environmentally** determined. **3.** Man is regarded as a product of **heredity** and **environment,** with each affecting the other. He could not survive or develop without both. **B. MATURATION. 1.** Walking, talking, grasping, climbing, etc., are generally regarded as characteristics that develop in due time regardless of any special training or environmental influence (**maturational**). **2. Secondary sex characteristics** are also considered delayed hereditary products. **Sex behavior** on the other hand, is a result of both physiological development and environment. **3.** If an activity appears at about the same **time** *or in the same* **sequence**) in most people, and if training does not appear to be able to speed up this appearance, the activity is considered to be **maturational** in nature. **C. INSTINCT. 1.** In some animals (*insects, fish, birds*) some **behavior patterns** seem to be **universal,** appear without **training,** and seem to be **adjustive.** These are attributed to **inherited neural connections** (*built-in programs*). **2.** Due to the long infancy of man, there is a major problem in proving that something has *not* been **learned.** Similarly, wide variations in almost all human behavior patterns precludes establishing them as either **universal** or **adjustive. 3.** Psychologists agree generally that the instinct concept is of little or no help in accounting for human behavior. Even on the animal level, every effort is made to account for behavior in terms of **external stimuli** and **internal conditions. D. IMPRINTING AND OTHER EARLY EXPERIENCE. 1.** Some animals (*mostly birds*) show very **rapid** and **irreversible** learning or **imprinting** of a few responses shortly after birth. **2.** While there are no clear findings for **humans,** experiments on animals suggest marked influences on learning ability, emotionality, even sensory capacity, from **extreme** types of **environmental** manipulations in **infancy.**

**5 PHYSIOLOGICAL FACTORS—THE REFLEX ARC.** **A. TRADITIONAL VIEW.** Behavior, according to *Watson*, was assumed to follow the pattern of a **reflex.** A **sensory stimulus** would initiate a **neural impulse** in an **afferent** (*sensory*) nerve cell. This impulse would stimulate an **efferent** (*motor*) nerve cell by means of a **synapse** (*a functional, not physical, connection*). Frequent impulses across a synapse were supposed to make successive crossings easier (*learning, habit*). The motor nerve would carry the impulse to a muscle or gland (the **effector**). **B. MODERN VIEWS.** The **Central Nervous System** is now viewed as constantly active, as measured by electroencephalograms (**EEG**). (*The reflex arc is considered a "convenient fiction".*) An impulse that enters the **CNS** is **modified** or **influenced** (*facilitated or inhibited*) especially by the so-called **"reticular activating system"** (*a diffuse net-work of neurons in the brain stem or* **medulla**).

**6 CENTRAL NERVOUS SYSTEM (CNS).** Consists of the **brain** and **spinal chord. A. THE BRAIN.** Has some 12 billion **neurons** or nerve cells. Each cell has extensions or branches called **dendrites** *or* **proliferations** (*which pick up impulses*) and **axons** (*along which impulses travel toward other cells*). When neurons are stimulated they initiate a **train of impulses.** Even when not stimulated, neurons will discharge **spontaneously.** The function of neurons is a **connective** one, but no longer is this regarded as a static process. **B. FUNCTIONAL LOCALIZATION.** Experiments have shown that when certain **temporal lobe areas** are stimulated with weak electric shocks, emotional disturbance, specific memories, etc. can be induced. Yet, no single area of the brain is the *exclusive* seat of any *function, psychological process, aptitude or trait.* For example, though the principal vision connections are considered to be **localized** in the **occipital lobe** many other parts of the brain are involved in a visual response.

**7 AUTONOMIC NERVOUS SYSTEM (ANS).** An automatic **regulatory** system for maintaining general life processes (*breathing, digesting, eliminating, etc.*). It consists of two divisions: **A. THE PARASYMPATHETIC.** Divided into **cranial** and **sacral** sections. It functions under routine conditions. **B. THE SYMPATHETIC.** Considered more of an **emergency** system which functions in times of **stress** or strong emotion. It raises the level of general activity, increases heart rate, releases **adrenalin,** etc. **C. EFFECTORS.** Neural impulses terminate in actions of muscles and glands. **1. Muscles** of two types are involved: **striate** or skeletal muscles and **smooth** muscles (*visceral organs, blood vessels, intestines, etc.*). **2. Glands** are also of two types: **endocrine** or **ductless** (*pituitary, thyroid*) and **duct glands** (*salivary, tear*). The endocrine gland secretions affect other glands, body size, and growth rate.

## SENSATION

**8 SENSATION. A. DEFINITION.** Generally considered as an **internal process** inferred from a special class of observed **S → R** relations. **B. ATTRIBUTES OF SENSATION.** Sensations were described by *Wundtians* as possessing features of: **1. QUALITY.** Sub-sense of vision is *color*; of hearing, *pitch.* **2. INTENSITY.** loudness, brightness. **3. DURATION. 4. CLEARNESS. C. FUNCTIONAL CHARACTERISTICS OF SENSATION. 1. ADAPTATION.** Following prolonged stimulation, a sense may become **fatigued** or otherwise ineffective: *we do not hear clocks tick.* **2. CONTRAST.** A specific sensory experience can be enhanced or changed by background factors: *blue looks bluer on yellow background.* **3. SUMMATION EFFECTS.** Two sensations may sum to produce a third: *warm and cold stimulation may produce pain.* **4. MASKING.** Low tones make high tones hard to hear.

**9 THRESHOLDS. A. ABSOLUTE THRESHOLD.** For any sense, it is the **minimal** amount of stimulus energy that is necessary for a reaction to occur. **B. DIFFERENCE THRESHOLD OR DIFFERENCE LIMEN (DL).** The minimal amount of energy change required to detect a difference between one stimulus and another measured by **psychophysical** methods. **C. TERMINAL THRESHOLD.** The upper level of **intensity** of **stimulation** where further additions of energy are not **perceived.** At this point, one sense reaction may change to another type: *loud noise may become painful.*

**10 WEBER'S LAW (1834). Changes in sensation are relative to changes in stimulus strength. 1.** In an observer's report of a change in stimulus strength, the amount of change must be above some fixed fraction of the original or standard stimulus: **$\Delta s/s = k$**, where **S** is some *standard stimulus*, **$\Delta S$** is the *amount of change*, and **k** represents a *constant.* **2.** *Weber's Law* holds in the **middle range** of stimuli (*those intensities that are more or less normal in human experience*). **3.** Typical fractional constants are: for light, .01; for sound intensity, .25 to .30; for weight, .03.

**11 FECHNER'S LAW (1860).** *Fechner* attempted to measure sensation strengths themselves and not just stimulus differences. He restated *Weber's Law* to read **$S = k \log R$** where **S** is now the sensation itself, **k** is the *Weber* fraction and **R** is the stimulus strength. Lost adherents because of assumption that **Just Noticeable Differences** (*J.N.D.*) were equal throughout the range of stimuli.

**12 IMAGES.** Current positions regard images as **conditioned sensations,** with sensation itself regarded as some **neural aftermath** of sensory stimulation: *song produces image of loved-one.*

## PERCEPTION

**13 BASIC GESTALT PRINCIPLES.** Analysis of perceptions is unfruitful. Experiences must be taken as **pheonomenological givens.** Gestaltists hold that **"we see the world not as it is but as we are."** The observer, in effect, contributes to the organization of the sensory world through his own nature. **A. FIGURE-GROUND.** Any stimulus is perceived as a **figure** against a **background:** *A sound is heard against silence or some other "noise".* **B. GOOD FIGURE.** We always perceive the best figure possible from any stimulus, organizing it into a **meaningful whole. C. COMPLETION.** Broken or incomplete patterns are perceptually completed to form the most likely whole figures (**closure**). **D. PATTERNING.** Distinct and separate stimulus objects will be **organized** where possible (*because of similarity*) into groups, figures, etc.: *A series of successive musical notes becomes a melody with the individual notes of less significance than the pattern or arrangement.* **E. THE WHOLE IS GREATER THAN THE SUM OF ITS PARTS.** Our perceptions contain or consist of **wholes** not parts: *We see a wheel, not rim, hub, spokes; a triangle, not sides and angles.*

**14 OTHER FACTORS IN PERCEPTION.** **A. MOTIVATION AND SET.** We perceive what we **expect** to perceive in many situations. Instructions can determine our perceptions in **ambiguous** circumstances. **B. PERCEPTIONS CHANGE WITH AGE AND EXPERIENCE.** *A child and an adult will perceive the size of a room differently.* **C. PRINCIPLE OF CONSTANCY (GESTALT).** To some degree, we continue to see the same object even after physical change (*distance, color,* etc.): *a familiar object appears same size whether seen at 5' or 15'.*

**15 ABNORMALITIES OF PERCEPTION.** **A. ILLUSIONS.** Commonly shared **misinterpretations** of physical stimuli due to sense organ characteristics or experience: *railroad tracks appear to meet in the distance.* **B. HALLUCINATIONS.** Under influences of drugs, illness, or other physiological disturbance we may **perceive** stimulus objects that are *not* **physically** present: *hearing voices.*

## MOTIVATION

**16 DRIVES. A. PRIMITIVE BIOLOGICAL FACTORS. 1.** In an elementary sense any stimulus is a **motivator.** Some stimuli are generated from our own **body processes:** *changes that take place as we become hungry, thirsty, cold, tired, emotional, etc.* Such changes are presumed to generate activity in internal sense organs with these stimuli then initiating **restlessness** or **random activity. 2.** Body tissue changes are termed **drives.** The stimuli emanating from drives are called **drive stimuli ($S_d$). B. FUNCTIONS OF DRIVE STIMULI.** At first, only random activity can be expected (*as with an infant that is in pain from colic*). As drives are reduced or eliminated by some specific behavior, that behavior comes to be **learned** as a **response** to the **stimulation. C. LEARNED DRIVES. 1.** Some body conditions like those that prevail during emotion (*fear*), also can be thought of as **drives.** At first these only occur when some appropriate stimulation is present. Because other stimuli may also be present, the **emotional pattern** or **drive** may become attached to these stimuli through **conditioning** (*see 20*). Subsequently the drive may be initiated by some formerly **inadequate stimulus:** *Being bitten by a dog may result in fear of a bark.* **2.** Assuming that emotional patterns can be so conditioned, we could then have a great many stimuli for drives in addition to those not involving **conditioning. NOTE:** *O. H. Mowrer* suggests *hope* is an other **learned** drive.

**17 HUMAN MOTIVATION. A. SOCIAL MOTIVES. 1.** On the human level we do not usually observe behavior stemming immediately from biological drives. Instead we postulate such motives as a desire for **prestige,** a need for **companionship, attention, power,** etc. These social motives are presumed to be **drive states** that are learned, based on an assumed **anxiety** which, in turn, is learned as a **fear pattern** to various stimuli (*lack of money, loneliness, etc.*). **2.** On the positive side, parents who satisfy basic needs become **secondary reinforcers** (*see 23:B*) as do other social stimuli and these come to arouse *wants, wishes, etc.* (*See C below.*) **B. SETS AND WANTS. 1.** When a **primary** drive is reduced or satisfied, the **goal responses** (*consumption of food and attendent emotional responses of relaxation*) tend to come forward in time on future trials. They become **anticipatory** or **antedating. 2.** This can only be in terms of part or a **fraction** of the actual goal response because some of the total goal response depends on the presence of other stimuli: *food in the mouth.* Such **fractional anticipatory goal responses ($r_G$)** can be thought of as the **equivalents** of *attention, interest, sets, wishes, wants, desires, etc.*; and: **(a)** They occur in **advance** of the appearance of the **goal. (b)** They are responses and can become **conditioned** to other stimuli. **(c)** They depend on **prior** experience. **(d)** They prepare the organism for the **goal response. C. MOTIVES AND ATTITUDES.** Attitudes are *"all that we think and feel about some object, place, person, etc."* Such attitudes or **sets** amount to **anticipatory emotional reactions** that have been previously aroused in connection with the stimuli involved. **(1)** They prepare the organism for the action of responding to the **anticipated stimulus; (2)** restrict **field of attention (3)** when resistant to change are called **prejudices.**

**18 RELATION OF MOTIVATION TO OTHER PSYCHOLOGICAL AREAS. A. LEARNING. 1.** Drives and drive reduction are basic for **learning** (*Hull*). **2. Drives** determine **action** and there is no learning without action. **3.** Learning is dependent upon **attention,** which is often related to **motivation. 4. Retention** is dependent upon motivation. Unpleasant or **traumatic** experiences are often **repressed** or forgotten (*Freud*). **B. PERCEPTION.** Through the control of **sets** or **attitudes,** we may perceive only what we want to see. Various **needs** (*hunger, sex*) may affect dreams (*a kind of perception*) or our estimates of the relative desirability of stimuli: *estimated size of a coin might depend upon one's relative wealth.* **C. EMOTION.** Strong emotion may function as a drive both for **action** and **learning. D. CONFLICT AND PERSONALITY.** The basis for conflict is the **antagonism** between **motives.** According to *Freudian* doctrine, some motives are **unconscious** and express themselves in *dreams, accidents, slips of tongue, etc.*

## LEARNING

**19 NATURE OF LEARNING.** Our knowledge of learning is based on **inferences** from **performance. A. DEFINITION.** Learning is a change in **performance** resulting from **practice. 1.** The change is frequently assumed to relate to some hypothetical change in the **nervous system** (*new connections lowering of resistances at synapses*). Usually carries the implication of greater **efficiency** (smoother, faster, freer from errors). **2.** The nature of the **practice** is usually assumed to be not mere **repetition,** but **reinforced** repetition or correct occurrence. **B. PAVLOVIAN ("CLASSICAL") LEARNING.** Based on *Pavlov's* work with salivary responses of dogs. **C. INSTRUMENTAL ("OPERANT") LEARNING.** Originally based on *Thorndike's* work with cats which emphasized **trial and error.** So called because the learner operates on his environment and his activity is **instrumental** in changing it usually to point of bringing about some **reward.**

**20 CLASSICAL CONDITIONING. A. SUBSTITUTION OF CONTIGUOUS STIMULI. 1.** Originally an **unconditioned** (*natural, usual, or appropriate*) stimulus is required for some specific response called the **Unconditioned Response (UR). 2.** By presenting another (*neutral, inadequate, or inappropriate* stimulus called the **Conditioned Stimulus (CS)** just before the normal **Unconditioned Stimulus (US),** a new connection is presumed to be made between the **CS** and the **UR,** as in (a). Sometimes

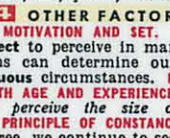



diagram is drawn to show some **Association Center (AC)** where the new connection is supposed to be formed (b). The *solid lines* show original connections; the *dotted lines,* acquired or learned connections. **B. HIGHER ORDER CONDITIONING.** After basic conditioning takes place a **CS** can now be used as a **US** for further conditioning by introducing a new **CS** and using the old **CS** to elicit the response. **C. GENERALIZATION.** Once conditioning has been established, other stimuli **similar** to the original in quality or quantity can also elicit the **CR,** although with **diminishing** efficiency as the **similarity** to the original **decreases. 1.** Generalization is greatest in the early stages of learning; with continued training it decreases. **2.** It is a *"something-for-nothing"* operation. We do not have to learn everything we know.

**21 EXTINCTION. A. EXPERIMENTAL EXTINCTION.** When the **CS** is followed by the **US,** conditioning is maintained and strengthened. If the **US** is omitted the **CR** gradually diminishes. **B. SPONTANEOUS RECOVERY.** After a period of time, an **extinguished response** may again be **elicited** by a **CS.** Succeeding extinctions occur more rapidly until there is no further spontaneous recovery. **C. GENERALIZATION OF EXTINCTION.** Extinguishing the original **CS** will also result in the reduction of the response to generalized stimuli. **D. THEORETICAL POSITIONS. 1.** It is widely held that extinction amounts to new learning (**counter conditioning**). **2.** *Pavlov* thought of it as the spread of **internal inhibition. 3.** *Hull* claimed any response generates some **reactive inhibition,** a state analogous to **fatigue.** This serves as a negative drive leading to a **resting response.** The **CS** comes to elicit this resting response if it is given frequently enough. **E. DISCRIMINATION. 1.** If a specific **CS** is reinforced while another on the **continuum** is *not*, the organism comes to respond to the first but not to the second. **2.** This procedure contrasts with generalization and allows the organism to respond only to **appropriate stimuli.**

**22 INSTRUMENTAL CONDITIONING. A. THE BASIC PRINCIPLE.** The substitution of responses **1.** Organisms are said to **emit** responses. **2.** Learning depends on the outcome of behavior. **3.** If a response is followed by a **reinforcer** (*reward*) its habit strength is said to be **increased. 4.** In strumental conditioning usually involves the **whole organism. B. DIFFERENCES FROM PAVLOVIAN CONDITIONING.** *Pavlov* emphasized stimuli and that stimuli **elicit** responses. The reinforcer come before the response. *Pavlovian* learning usually applies only to *one* organ or system. **C. SIMILARITIES. 1. Generalization, extinction spontaneous recovery, and discrimination** are common to both. **2. Secondary reinforcement** corresponds somewhat to higher order conditioning. **D. OTHER THEORETICAL POSITIONS. 1.** *Pavlovian* learning applies only to **ANS** phenomena. **2. Reinforcement** is not required for *Pavlovian* learning, mere **contiguity** of stimuli is enough. **3.** Only *Pavlovian* conditioning requires reinforcement. (*K. Spence* argues that food is an **US** and a reward). **4. Instrumental** learning is more like **voluntary behavior,** less **stimulus-bound.**

**23 PRINCIPLE OF REINFORCEMENT. A. PRIMARY REINFORCEMENT. 1. LAW OF EFFECT.** (*E. L. Thorndike*, 1898.) **S → R** bonds are **strengthened** when the response is followed by a **satisfying** state of affairs (*one the organism does nothing to avoid, frequently strives to attain*). **S → R** bonds are **weakened** when the response is followed by an **annoying** state of affairs (*one the organism does nothing to attain, frequently strives to avoid*). In 1932 *Thorndike* dropped the second part of the law, having concluded that **punishment** does not weaken bonds. **2. THE LAW OF PRIMARY REINFORCEMENT.** *Hull's* restatement of the **Law of Effect:** When a response is followed by a **decrease** in a drive the connections between any stimuli present and that response are **strengthened. 3. LAW OF REINFORCEMENT.** *B. F. Skinner:* When any response is followed by a **reinforcer** (*anything that has been found to increase the rate of the response*) the probability of recurrence of that response is strengthened. **B. SECONDARY REINFORCEMENT. 1.** A basic concept in explaining human learning. Any stimulus that accompanies a **primary** reinforcer (*food*) will acquire similar reinforcing properties. **2. ILLUSTRATION:** Animals will learn the path to a goal in a maze if the goal is of the same color as a food box, even if there is no food there. **C. THE ROLE OF PUNISHMENT. 1. SKINNER.** Punishment merely **suppresses** behavior. To get rid of behavior it must be allowed to occur without **reinforcement,** i.e., *it must extinguish.* **2. MOWRER** (1960). Revived punishment as an important factor. **(a)** When punishment follows some

response, it **inhibits** behavior through fear. (b) When an organism is punished (*hurt*) when not doing something, it learns to **escape** from such situations by performing whatever response is necessary. (c) Later it **avoids** the punishment itself by making the response **before the punishment** if some appropriate warning signal is presented. (d) In general, *Mowrer* holds that **emotional states** serve to guide **adjustive** behavior.

**24 HUMAN LEARNING.** **A. RELATION TO CONDITIONING.** 1. *Skinner* regards verbal learning as another form of **instrumental** behavior. 2. The acquisition of speech itself has not been satisfactorily explained. Once speech is learned, it is then a matter of reinforcing certain **patterns.**

**25 RESEARCH ON VERBAL LEARNING.** *Herman Ebbinghaus* (1880) introduced **nonsense** syllables (*two consonants and a vowel: "zom"*) as material for memory study. **A. SERIAL ORDER.** The subject learns a list, presented one at a time by some automatic device (**memory drum**). The usual pattern of learning shows: 1. **First** few and **last** few syllables are learned **earlier** than the rest, with most difficulty just beyond the **middle** of the list. 2. Grouping the results of a number of subjects will show a **negatively accelerated** learning curve: i.e., *learning is slower in the later trials.* 3. Distributed (**spaced**) practice is better than **massed** practice: i.e., *fewer trials are required if rests are allowed between trials.* 4. Learning is facilitated by **prior** familiarization with the syllables. 5. **Active** learning (*trying to anticipate and vocalizing*) is better than **passive.** 6. Nonsense syllables with **high association** values (*subject reminded of meaningful words*) are learned more easily than **low association** value syllables. 7. **Meaningful** words are learned better than nonsense materials. 8. The learning speed varies with the total **number of units**; becomes progressively poorer with additional units. 9. **Remote associations** are formed so that a syllable early in the list may invite a response of a syllable later in the list—such remote associations **interfere** with the learning. **B. PAIRED ASSOCIATES.** The syllables are presented as pairs with the first syllable serving as a **stimulus.** The learner has to **respond** with the appropriate **mate.** 1. The learning speed varies with the number of pairs. 2. Subjects tend to transform syllables into **meaningful** words which then serve as bridges or **mediators.** 3. Subjects tend to concentrate on one pair at a time or a few per trial and frequently learn a syllable in one trial. The additional trials are required to learn additional syllables. 4. Learning is subject to various influences of **familiarity** and **similarity,** the former **facilitating** and the latter **hampering** learning. **C. THE LEARNING OF SKILLS.** 1. **SENSORIMOTOR LEARNING.** Following a moving target with a pointer exemplifies a **sensorimotor task,** while learning meaningful words exemplifies a **verbal task.** In most tasks, there are both sensorimotor and verbal components. 2. **EARLY STUDIES.** Investigations with telegraphy have shown that: (a) There are substantial differences between individuals in the speed with which learning occurs. (b) Many subjects showed **plateaus** or lengthy periods of no improvement. The latter have been attributed to the presence of **habit hierarchies** or levels of habits.

**26 RETENTION — FORGETTING.** There are no good or bad memories, only good or bad learners. **A. FACTORS IN FORGETTING.** 1. **TIME.** A few psychologists postulate a factor of **disuse:** i.e., *that neural bonds weaken when not used*; however, the evidence is meager. 2. **INTERFERENCE.** Most theorists favor a competition or interference view. If original learning consists of $S_1 \rightarrow R_1$ and the subject now learns $S_1 \rightarrow R_2$ or if some generalized form of $S_1$ is involved, he will then be unable to recall $R_1$ because $R_2$ keeps recurring. This corresponds to an interference theory of extinction. **B. MEASURES OF RETENTION. Recall, recognition,** and **relearning.** Usually if one relearns forgotten material, it takes less time or effort than originally. The percent of effort saved is called a **savings score.** **C. RETROACTIVE INHIBITION.** The interference hypothesis applies to a standard experimental situation:

| 1. BASIC DESIGN: | LEARN A | LEARN B | TEST RETENTION FOR A |
|---|---|---|---|
| Experimental group | Yes | Yes | Yes |
| Control group | Yes | No | Yes |

The **interpolated** material B can be learned at any time after the **original** material A is learned and at anytime before the test for A. 2. **FINDINGS.** (a) The degree of interference is less if either A or B is learned well; (b) it is greatest with **incomplete** learning; (c) the degree of interference increases with the **similarity** between the materials learned in A and B. **D. GENERAL CONCLUSIONS.** If nothing **new** is learned after the learning of some response, that **response** will *not* be forgotten. Experiments with subjects who learn something just **prior** to sleep indicate that *less* is forgotten than if normal daytime routine is followed. **E. OTHER FACTORS IN FORGETTING.** 1. **INADEQUATE ORIGINAL LEARNING.** If original learning is **low in strength,** retention should not be expected. People sometimes learn without **conscious** intention of remembering. **Intention** to remember appears to be helpful but **incidental** learning (*learning without intent*) has been regularly shown to yield *poor* **retention.** 2. **INADEQUATE STIMULATION.** If the stimuli presented are **different** from the original, we can expect *poor* retention. A question asked in an unfamiliar way may not elicit desired answer. 3. **EMOTIONAL FACTORS.** Freud held that even mildly **traumatic** incidents might be **repressed** or driven into some **unconscious** level. Some psychologists argue that unpleasant events are more readily forgotten than pleasant.

**27 TRANSFER OF TRAINING.** The influence of past experience on future behavior. Transfer must be thought of as **positive** or **negative.** **A. PROACTIVE INHIBITION.** The terms implies *negative* transfer, however, it is evaluated in the same situation as is *positive* transfer.

| 1. EXPERIMENTAL DESIGN. | LEARN A | LEARN B | TEST RETENTION FOR B |
|---|---|---|---|
| Experimental group | Yes | Yes | Yes |
| Control group | No | Yes | Yes |

2. **EFFECTS.** If the experimental group retains B better, effects of learning A are assumed to have **transferred positively** to learning B. If B is retained less well, the effects are considered **negative.** Usually negative effects are due to **interference.** Positive effects are assumed to be due to the easier learning of **identical elements** (*Thorndike*) or to familiarity with the responses and/or stimuli. **B. TRANSFER AND RETROACTIVE INHIBITION—OSGOOD'S LAWS.** Attempted to describe the relationships between **transfer** and **retroactive inhibition** in the following laws (*from E. R. Hilgard*): 1. If new stimuli are **similar** to old and responses are **identical,** transfer is **positive** and high. 2. If new stimuli are less **similar** and responses still **identical,** positive **transfer is lower.** In both of these conditions, recall of original learning is high. 3. If stimuli are **identical** but responses only **similar,** there is little or no transfer and some **interference** in **recall** of the original. 4. If stimuli are **identical** but responses are **different,** the transfer effects will be **negative** and recall of original material greatly hampered. 5. If both stimuli and responses are **neutral,** there will be **no** systematic effects.

## THINKING

**28 THINKING AND PROBLEM SOLVING.** **A. CHARACTERISTICS OF THINKING.** 1. Occurs when there is no immediately available and appropriate **habit.** 2. Involves a **delay** between the stimulus and response. The thinking, if any, goes on during this delay. 3. Essentially a **covert, inner activity** which we know only by inferences from behavior. 4. It is a **symbolic** activity, assumed to be **mediational,** i.e., *it connects the external stimuli with a series of internal responses which in turn stimulate some overt response.* **B. THE TOOLS OF THINKING.** The mediational activity can be described as **pure stimulus acts** which might take various forms: 1. **Movement-produced** stimuli. 2. **Sub-vocal verbal behavior.** *Watson* described much thinking as amounting to *"talking to yourself."* 3. **Fractional antedating responses** ($r_g$) are partial aspects of previously learned goal responses. Such $r_g$'s can produce stimuli which may then initiate some overt activity. 4. **Neural processes.** *D. O. Hebb* postulated recurrent neural circuits or **cell assemblies** become organized in chains called phase sequences and can be self-completing when initiated. Such sequences amount to thoughts or thinking. **C. THINKING AS SUBSTITUTE ACTIVITY.** 1. All of the above **pure stimulus acts** are substitutes for directly observable behavior. 2. One substitute activity follows another until some successful or unsuccessful **overt** activity is initiated. Thinking is thus covert **"trial and error."** 3. Solutions of problems cannot be predicted. Even with the necessary background, **wrong set** (*improper perception of elements, etc.*), may interfere with successful associations.

**29 INSIGHT AND PROBLEM SOLVING.** **A. THEORY** The **Gestalt** psychologists argue that thinking is essentially a matter of **insight,** which in turn, amounts to a **"dynamic reorganization of the perceptual field,"** i.e., *the successful thinker comes to see the relationships between the parts of a problem in a new light.* Insight is related to the concept of **understanding** or **"getting the principle."** **B. CHARACTERISTICS OF INSIGHT.** Insight may not occur for a long time; when it does, it is sudden and is retained for a relatively long period. **C. OBJECTIONS. Behaviorists** claim that solutions come only from past experience. All successful solutions *must* come suddenly because they are the **last trials. Insight** is just a label, it explains nothing. Frequently the problem solver cannot explain how he solved the problem and may show little understanding. **D. THINKING AND REASONING. Reasoning** is concerned with the rules of **logic** and accuracy of **problem solving.**

## EMOTION

**30 EMOTIONAL BEHAVIOR.** **A. CHARACTERISTICS.** 1. Emotional reactions involve the **ANS.** When considering emotion we usually are concerned with strong **sympathetic** action. 2. Emotions are **disorganizing;** however, *R. Leeper* argues that unless **extreme** in nature, emotions can be organizing and may **serve as drives.** **B. KINDS AND NUMBER OF EMOTIONAL EXPRESSIONS.** 1. Some believe there is only one internal emotional state: **excitement.** Variations in expression come in terms of **stimuli.** 2. *Bridges*: We are born with a capacity for excitement, but by **maturation,** other emotional expressions can be differentiated in the first year or so of life. **Excitement** provides the basis for **distress** and **delight.** Out of delight grow *elation* and *affection* while out of distress *anger, fear,* and *disgust.* 3. *Watson*: We are born with three basic patterns: **fear, anger,** and **love.** By **conditioning,** these become attached to a variety of **stimuli** and make up our emotional reactions. **C. CURRENT VIEWS.** The number and kinds of emotional experience cannot be accurately determined because they cannot be readily identified. The most acceptable judgments are made when **stimuli** are known, thus emotion is a matter of **interpretation** of stimulus effects.

**31 THEORIES OF EMOTION.** **A. THE JAMES-LANGE THEORY.** Emotional experience is the **awareness** of bodily changes as they occur. Emotion comes after some **perception** and **response** to some stimulus: *We fear the bear because we run.* This view reverses the usual interpretation. Emotions become the **results** rather than **causes** of behavior. **B. THE CANNON EMERGENCY THEORY.** 1. Emotions are reactions to **stress** and **danger.** They are **homeostatic** adjustments that prepare the body for *"flight or fight"* 2. Emotions result when the **hypothalamus** (*in the mid-brain*) is subjected to intensive stimulation. Presumably the **ANS** in turn is stimulated with resulting increases in *heart rate, adrenalin secretion, etc.* 3. In opposition to *James, Cannon* regarded the **awareness** of bodily changes as **prior** to, or **simultaneous** with, **overt** action. **C. THE SCHLOSBERG THEORY.** Emotional facial expressions can be described in three dimensions: (1) a **pleasant-unpleasant** continuum; (2) an **attention-rejection** continuum (*corresponding to a negative or positive interest*). (3) an **intensity dimension** (*from sleep to excitement*) **D. THE DUFFY THEORY.** Emotions described in terms of **intensity** and **directional** dimension (*approach or avoidance*).

**32 RELATION OF EMOTION TO OTHER STATES OR CONDITIONS.** **A. LEARNING.** 1. The conditioning of emotions (*particularly fear*) is widely accepted. Some emotions—like fear—have indirect learning features in that we must acquire some **experience** before we can be frightened by such things as the dark, snakes, corpses. 2. *Hebb* says we must first learn how to behave in the **absence** of such stimuli; their appearance leaves us without a useful response. **B. MOTIVATION.** Fear is widely postulated as a **drive** and, with other similar drives, serves to control behavior through the **feedback of stimuli** generated by responses that arouse the drives. **C. PHYSIOLOGY.** Chronic activity of the **ANS** is alleged to be a factor in **psychosomatic illness.** According to *Selye*, **stress** may result in a three-phased pattern: an **alarm reaction** (*a shock state*), a **resistance phase,** and a **final exhaustion phase.** These phases follow extreme and prolonged **stress** and together they are called the **general adaptation syndrome.**

## ABNORMAL BEHAVIOR

**33 CONFLICT AND FRUSTRATION.** Frustrations are the results of **interference** with progress toward a **goal. Conflicts** are presumed to arise from opposing *forces, stimuli, motives,* or *emotions* which make progress difficult or impossible. **A. TYPES OF CONFLICTS.** (*Neal Miller*). 1. **APPROACH-APPROACH.** The organism is stimulated to approach **two different** objects. **Solution:** approach one, then the other. 2. **AVOIDANCE-AVOIDANCE.** The organism is caught between **two negative stimuli** (*"devil and the deep blue sea"*). **Solution:** get out of the field, if possible. 3. **APPROACH-AVOIDANCE.** The organism is stimulated to **approach** some goal; at the same time the approach involves **danger. Solution:** depends on strength of drives, degree of danger, etc. If positive and negative aspects are nicely **balanced,** there will be *vacillation, hesitation, resort to compromises, retreats into some defensive activity which quells the anxiety to some degree even if the conflict is not solved.* 4. **DOUBLE APPROACH-AVOIDANCE. Two attracting stimuli** affect the organism; approaching one means irretrievable loss of the other. **Solution:** similar to 3. above. **B. ASSUMPTIONS OF MILLER'S THEORY.** 1. The tendency to approach a goal depends on the **distance** to the goal just as the tendency to avoid depends on **nearness.** The strength of the tendencies to approach or avoid varies as a **gradient;** the **slope** of the **negative** gradient is steeper than that of the **positive.** This permits the gradients to cross. 2. The height of gradients can be changed by changes in **drive.**

**34 CONSEQUENCES OF CONFLICT.** **A. FRUSTRATION.** In general, conflict can be assumed from signs of **hesitation, vacillation,** and behavior patterns associated with frustration. Reactions will vary with past experience (*degree of learned frustration tolerance*) but in general we can expect: 1. **APATHY.** Substitution of goals, various **defensive tactics.** (*See B.*) 2. **AGGRESSION.** Usually directed at the source of the frustration. If it is impervious to assault, aggression may be **displaced** on innocent objects or persons. If the frustration is perceived as **self-caused,** aggression may be turned against the **self. B. DEFENSE MECHANISMS (FREUDIAN).** To escape conflict by resorting to forms of temporary **adjustment:** 1. **REPRESSION.** A form of forgetting of unpleasant or traumatic events (*they are pushed into the unconscious*). 2. **REGRESSION.** A falling back to old habits or methods of attaining goals (*crying*). 3. **REACTION FORMATION.** Over-reacting in the opposite direction (*showering affection on someone who is not loved*). 4. **PROJECTION.** Attributing one's *faults, weaknesses, wishes,* to others. 5. **SUBLIMATION.** A form of compensation (*engaging in a substitute activity*). 6. **RATIONALIZATION.** Finding excuses for already committed errors.

**35 BEHAVIOR DISORDERS.** **A. DEFINITIONS.** 1. **STATISTICAL. Abnormal** people are those at the **extremes** of the normal probability curve on any measurable trait. 2. **IDEAL.** Some ideal **normality** must be defined in terms of health, etc., deviations from the ideal would be abnormal. 3. **ADJUSTMENT.** People are not abnormal if they are **getting along:** *An anti-social person might make a satisfactory "space man."* **B. CAUSES OF ABNORMAL BEHAVIOR.** 1. **STRUCTURAL OR ORGANIC DISORDERS.** The result of *brain injury, disease,* or *faulty physical development,* etc. 2. **FUNCTIONAL DISORDERS.** Assumed to be learned because no physiological defect can be found. 3. **GENERAL.** Assumed as consequence of **faulty training, disturbing experiences;** thus deviations are **learned environmental** effects. **C. TYPES OF DISORDERS.** 1. **NEUROTICS.** People with relatively **mild** disorders, characterized by *inefficiency, complaints, ineffective defenses.* 2. **PSYCHOTICS.** (*Schizoid, Paranoid,* etc.) Severely disturbed, marked by inadequate **reality orientations** (*misinterpret own and others' roles*); often require hospitalization.

**36 THERAPIES.** Because of assumption that the disorder is learned, therapy basically amounts to **retraining** or **relearning.** **A. INTERPRETATIVE.** (*Directive, Freudian, analytic*). The patient reports on *memories, dreams, his attitudes,* etc., while the therapist tries to discover hidden (*from the patient*) meanings, uncover original and forgotten causes and have the patient acquire more **adjustive reactions** to stimuli previously arousing **anxiety. B. NON-DIRECTIVE.** (*Client-centered, Rogerian*). The patient is believed capable of effecting his own cure through assumed processes of **inner growth** if allowed to talk freely in a **"permissive"** atmosphere while the therapist echoes and clarifies the patient's feelings. **C. RECIPROCAL INHIBITION.** (*Wolpe*) The patient first **ranks** the the stimuli that cause him anxiety. Starting with the lowest of these, the therapist tries to **extinguish** the **anxiety responses** by *counter-conditioning* responses of **relaxation. D. SYMPTOM TREATMENT.** *Skinner* advocates extinction through **non-reinforcement** of undesirable behavior.

## PERSONALITY

**37 METHODS OF PERSONALITY MEASUREMENT OR ASSESSMENT.** **A. DEFINITION PROBLEM.** 1. G. Allport listed over 50 definitions emphasizing **individual differences, biological** and **social** factors, **social stimulus** values (*popularity*). 2. The **organization of traits** that distinguishes one person from another provides the closest agreement among the definitions. **B. ROLE PLAYING.** People behave differently in various situations (*father, judge, teacher*) complicating personality assessment. **C. THE TEST APPROACH.** Personality tests usually do not involve correct answers. They are often **"projective"** in nature. The subject is presumed to reveal his nature in his responses to **ambiguous stimuli** (*ink-blots, clouds*) or to pictures of people in various scenes (*Thematic Apperception Test,* **TAT.**) **D. INVENTORIES.** 1. Subjects indicate preferences, modes of action in common situations, symptoms. Patterns of answers are studied to determine **profiles.** 2. Difficult to **validate** tests since there is no accepted measurement **criterion.** 3. *Minnesota Multiphasic Personality Inventory* (MMPI): designed to get at numerous areas or sources of disturbance. **E. TYPOLOGY (THE APPROACH THROUGH THEORY).** 1. A preconceived view might hold that people can be divided into types: **introverts** and **extroverts** (*Jung*), or as by *Spranger,* into *theoretical, aesthetic, economic, political, social,* and *religious* types. 2. **Inventories** or **questionnaires** are then devised to determine to what degree a person may share these interests. **F. RATING SCALES (THE TRAIT APPROACH).** The most common tool. 1. Lists of adjectives are supplied, marked off on a graded line into varying numbers of steps labeled *very much, average, very little,* etc. The rater indicates how, in his **judgment,** a given subject fares with respect to some **trait** 2. **SHORTCOMINGS.** (a) Too few adjectives resulting in emphasis on **cardinal** or **general** traits. (b) Too many "yes-no" judgments; need **graded** (*continuous*) steps. (c) Raters frequently operate under a **halo** effect: *they form some general evaluation and then rate all traits fairly uniformly.* **G. CLASS-MEMBERSHIP APPROACH.** Inferences about personality are drawn from the **social** and **economic status** of a person or his family; *his political affiliations, clubs, job, racial or national origins,* etc. **H. GOALS AND ASPIRATIONS APPROACH.** 1. Assumes that individuals are motivated differently by **common goals:** *the need to be loved, to dominate, to attain prestige, etc.* 2. Needs are assessed by questionnaires, interviews, **projective** tests, etc. **I. PHYSICAL MEASUREMENT.** Some assume that body type is related to personality traits: *fat* (**pyknic, endomorph**); *lean* (**asthenic, ectomorph**); or *in-between* (**athletic, mesomorph**). **J. RESULTS OF INVESTIGATIVE TECHNIQUES.** There has been little or no success in **typing** people. They tend to spread out on all kinds of measurements into **normal distributions** with *a small percentage at the extremes, most in the middle.*

**38 THE INTERACTION OF THE EXTERNAL ENVIRONMENT, THE INHERITED BODY AND THE PERSONALITY.** **A. THE FREUDIAN VIEW.** (1) Men go through the same phases of **sexual development;** (2) pass, or fail to pass, through an **Oedipal** phase (*strong attachment to the mother, antagonism toward the father*); (3) are subject to the same **pressures or goals** (*satisfying unconscious* **id** *urges*) which may be in conflict with other goals (*conscience* or **Superego**). **B. THE JUNGIAN VIEW.** Man is unconsciously guided by the **experience of his race. C. THE ADLER VIEW.** Life is devoted to attainment of **power** as a **compensation** for felt **inferiority. D. THE BEHAVIORIST VIEW.** Personality amounts to learning specific **habits** to specific stimuli with **generalization** to unfamiliar situations. **E. CULTURAL INFLUENCES.** Anthropologists have forced psychologists to incorporate a **cultural view:** to take account of ways of *child rearing, customs, rules,* etc. Different societies create vast differences in **attitudes** and **behavior** with regard to the same stimuli (*money, women, death,* etc.).

## SOCIAL FACTORS

**39 THE STRUCTURE OF SOCIETY.** The **culture** (*arrangements for group living*) determines many aspects of daily life: *work choice, marriage partners, age of education,* etc. The culture especially **restricts** adolescents from activities for which they are **physically qualified** (*marriage, work,* etc.).

**40 THE GROUP.** Not merely an aggregate of people, members must have **common motives** and **interact** with each other. Some groups (*an audience*) may have a common motive but no interaction. **A. PRESSURES FOR CONFORMITY.** Often strong enough to make a person **deny the evidence** of his senses; the strength of the pressures depends upon the number of influencing people, their sex, status, etc. **B. THE LARGE GROUP.** An individual can lose himself in it (*anonymity*), reduce **personal** responsibility, and increase his personal safety and strength. Crowds may do things no individual would or could do by himself. **C. THE SMALL GROUP.** Here, there may be pressures for **conformity** which avoids argument. Committees, like crowds, can take *"group"* action without personal responsibility. This may lead to **corporate** or **institutional thinking** in which a **reality** is conferred upon non-existent things: *A university or a union looked on as existing independently of the people who make it up.*

**41 LEADERSHIP—THE INFLUENCE OF AN INDIVIDUAL ON A GROUP.** The people who are to be led determine the nature and qualifications of a leader. **A. THEORIES OF LEADERSHIP.** 1. **THE GREAT MAN.** A leader will not be denied. Because of **inherent** qualities he will rise to leadership, more or less regardless of circumstances. 2. **THE TIMES MAKE THE MAN.** At **crucial periods** men who express the **mood** and **goals** of a group will be forced into leadership. The same man in other times might prove unleaderlike.

**42 GROUP DYNAMICS.** Interaction of group members. **A. THE PROBLEM OF COHESION.** 1. Group morale. 2. Mutual **respect** and **submersion** of individual interest for the goals of the group. 3. Cohesiveness may be measured by special questionnaires or **"sociograms:"** *Group members secretly indicate which members of a group are liked and disliked by them; the likes and dislikes are then plotted to determine* **patterns of approval and rejection. B. THE FORMATION OF GROUPS.** Groups are formed when a **mutual problem** faces a number of individuals and where individual effort is **inadequate** to the task. **C. SURVIVAL OF GROUPS.** Groups often **create** reasons to continue after the original problems are solved. They adopt **new** goals or **merge** with others.

**43 SOCIAL CONTROL.** **A. EDUCATION.** A society controls the **curriculum** and **methods** of education. **B. PROPAGANDA.** Usually a **one-sided** presentation of issues with the purpose of supporting only one view using techniques like: (1) repetition; (2) the "band-wagon" approach (*"everybody's doing it"*); (3) "plain folks" (*the "common people" are with us*); (4) glittering generalities (*"this great and glorious cause"*); (5) "scape-goating" (*blaming defenseless groups*).



DO NOT PLACE ON HOT SURFACES

KNOW THE FACTS ON THIS CHART AND YOU WILL GET HIGHER MARKS IN CLASSWORK, HOMEWORK, TESTS AND EXAMS.

THERE ARE QUICK CHARTS IN MATH • SCIENCE • ENGLISH • ECO • HISTORY • LANGUAGES • PSYCH • SOC • PHILO • MUSIC

DATA-GUIDE EDITOR: JOSEPH L. LEON

PUBLISHED & MF'D BY DATA-GUIDE, INC. FLUSHING, N.Y. 11355

DATA-GUIDE QUICK CHART

# PRINCIPLES OF SOCIOLOGY

## A SIMPLIFIED SUMMARY & INSTANT REFERENCE

LEARN & REVIEW IN MINUTES

AUTHOR: Prof. Amos A. Hawley, Ph.D., Chairman, Department of Sociology, University of Michigan.
Consultant: Prof. Guenther Roth of Columbia University.

See also VOCABULARY FOR SOCIOLOGY

SOLID PLASTIC

SPEED STUDY TO SAVE YOU TIME AND WORK.

QUICK CHARTS PUT THE FACTS IN FULL VIEW:

PERMANENTLY FILL IN FOR MISSING NOTES & TEXTS.

QUICK CHARTS HELP YOU REMEMBER:

CONTAINS NO PAPER

**1 INTRODUCTION. A. DEF.** Sociology seeks knowledge about man's collective life through systematic, controlled observation of **human groups** (their development, interrelations, organization, etc.).

**2 THE SOCIAL NATURE OF MAN. A. SURVIVAL.** Unlike all other animals, man is born with no specific equipment for living in his environment. There are no known instances of **solitary existence** without early death and apparently there is no **social instinct.** Therefore to survive, man either lives in association with his fellows or he doesn't live. **B. INTERDEPENDENCE.** To a large extent man is a product of his group experience—he is a **social animal.** The word social denotes **interdependence,** (*mutual support and supplementation*). Interdependence is not only imperative for survival, but is also a condition for acquiring the ways of acting in a society: *knowledge, tastes, habits of perception, standards of judgment, moral precepts and even one's conception of himself.* It is the channel for securing **approbation, affection** and the satisfaction of other **affective needs.** Interdependence is a condition of life. The severest **sanction** imposed for a misdeed is **ostracism** (exclusion from one's group).

**3 SOCIALIZATION. A. DEFINITION.** The process by which one is transformed from an undisciplined organism to a disciplined member of society. Socialization includes the development of a **personality** as well as the acquisition of acceptable **behavior patterns.** While the process is most prolonged and significant in the period of youth, it recurs in attenuated form each time a person enters a new social situation. **B. CONCEPT OF SELF.** (1) An individual acquires conception of self from the **responses** of others to his acts. This concept begins in a **conversation of gestures:** *facial expressions, hand signs, oral (language), etc.* Of the various kinds of gestures, the **oral gesture** is most important for the individual can respond to it in the same way that others do and perceives the meaning of his behavior more quickly. As he grows **self-conscious** (*aware of himself as an object toward which others act in predictable ways*) he gains control over his own behavior and is able to assume a responsible position in his group. (2) Socialization is fulfilled however, when his conception of self is adapted to many **different others.** In a sense, all of the ways others will respond to his actions are **internalized** and the individual formulates a generalized conception of self. **C. THE PRIMARY GROUP.** (1) Socialization is most effectively carried on in a primary group—a small group in which relations are **unspecialized,** intimate, informal and lasting: *the family in certain of its aspects, the play group, association of friends.* Only in such groups is there sufficient attention devoted to the whole person over a long enough period of time, and with appropriate patience and tolerance, to foster the development of personality. (2) The primary group is primary not only in that it is the first to be experienced but in that it also performs a basic function in inculcating **elementary values** of the group and in providing the equally **elementary satisfaction** of needs. In the frequent renewals of socialization in adult life the primary group continues to occupy the same position of importance, though the process itself may be abbreviated.

**4 GROUP STRUCTURE.** As a system of **dependences,** most forms of groups [*For exceptions see 5: F.*] have a clearly marked structure. The structural parts are **roles** and **relationships. A. A HUMAN GROUP.** Two or more persons linked by ties of **mutual dependence** and by a set of **norms** prescribing expected ways of acting toward one another. A group has properties not possessed by the members taken separately, and must be studied as an entity. **B. ROLE.** (1) An established, routine way of acting involving one or more relationships: *husband, mother, minister, president, plumber, citizen, student.* Each individual can assume a number of different roles depending on the number of groups to which he belongs. He can also act in more than one role within a given group (*as husband and father, minister and church member*). (2) Role is a **group property** acted by different individuals at different times and places. **Role composition** is the stable feature of a group. **C. HIERARCHY.** (1) Roles tend to be arrayed in a *hierarchy.* In the key position is the role which performs a **critical function.** This is usually an activity which maintains the relations of group to the outside world. Other roles are ranked with reference to the proximity of their function to the key function. (2) The hierarchy tends also to be an **authority system** with the authority concentrated in the key function and graded through the succeeding ranks. **D. STATUS AND PRESTIGE.** (1) Hierarchy endows each type of role with corresponding **status,** an evaluation of the worth or importance of the role in the system. (2) A similar property, different in that it is attributed to individuals, is **prestige.** Prestige is accorded on the basis of the **quality** of role performance. Status and prestige, however, *do not* necessarily vary together. Role and status hierarchies not necessarily functional; in part may be historical legacies.

**5 TYPES OF GROUPS.** (*No standard taxonomy. The following is only illustrative of variety*). **A. FAMILY.** A union of parents and children. Can be highly ramified, extending to the limits of **kinship,** and can occur in a great number of forms. **B. ASSOCIATION.** An **extra-familial** group organized with reference to a special activity or interest: *club, church, business enterprise, political party, school, government.* **C. COMMUNITY.** A **territorially based** and limited group, relatively self-contained in that it has within it the arrangements and facilities for serving the day-to-day needs of its members. *Ranges from small village community to the great cities.* **D. SOCIETY.** Most inclusive of all groups. Embraces the entire system of relationships which unites a people and distinguishes it from other peoples. *Usually includes common territory, language, tradition and government.* **E. PRIMARY GROUP.** While it usually involves individuals who occupy clearly defined roles, its activation requires that they step out of their specialized roles and associate as equals, or as persons rather than as **functionaries. F. EXPRESSIVE GROUPS.** Relatively unstructured, transitory: *mobs, crowds, protest aggregates.* Often give rise to structured groups. **G. CATEGORIES. Unstructured aggregates** of individuals sharing common interests or common characteristics: *age grades, sex, social class, occupations, etc.* Often are transformed into **structured groups.**

**6 INTER-GROUP RELATIONS.** Society is a *system* of inter-group relations. **A. GROUP SPECIALIZATION.** Establishes linkages with other groups forming complex webs of direct and indirect relationships. **B. GROUP FEDERATION.** Groups of similar type often federate to pool their resources for mutual aid: *councils of churches, associations of manufacturers, federated garden clubs, national fraternities.* **C. OVERLAPPING MEMBERSHIPS.** Groups are informally connected through a sharing of members: *Any one individual may be a father of a family, a member of a work group, a club, a political party.*

**7 GROUP NORMS. A. DEFINITION.** A group involves a **normative system** as well as role structure. The norm is a standard way of acting. **Deviations** from the norm invite **sanctions** which vary from mild to harsh depending on seriousness of the deviation. The effect of norms is to insure that roles are performed correctly and consistently. **B. EXAMPLES:** Honesty, keeping one's word, respect for authority, promptness, awaiting one's turn, submission to majority decision in a democracy, humane treatment of one's fellows. **C. NORM CHARACTER, VALUES.** Many norms acquire a *sacred* character by becoming identified with religious or supernatural doctrine and criticism is not tolerated (*respect for the deceased*). All norms are included in the group's values and all involve judgments of rightness, goodness. Values also include ends and means.

**8 THE GROUP AND THE INDIVIDUAL.** The group places heavy emphasis on the **conformity** to norms, especially where change is infrequent. **A. RITUAL.** Conformity is reinforced through **ritual acts.** Generally ritual imposes discipline which subordinates individual to group (*the church service, rising to the national anthem, code of good manners*). **B. DEVIATION.** (1) If norms were observed literally and fully they would allow little individual freedom. Most norms include **latitudes of deviation** varying with the sacred element included. With the advancing complexity of society, group norms tend to become **principles** rather than detailed prescriptions. Thus literal observance in all situations may be a deviation. (*excessive honesty, rigid application of law*). (2) The **non-conformist** is often a person who insists on untempered observance of certain norms at expense of other related norms.

**9 CONVENTIONALIZED EVASIONS. A. ACCEPTED PRACTICES.** Virtually every society approves some conventional ways of evading norms. In some societies special days may be set aside for the violation of selected norms. In most, there are institutionalized evasion practices (*monogamy and serial marriages; free enterprise and government subsidy*). **B. FACTORS AFFECTING CONVENTIONALIZED EVASION.** (1) Necessity of providing occasional release from rigorous discipline (*Mardi-Gras, Fasching*). (2) Recognition of contingencies (*prostitution for the unattached or transient element*). (3) Imposition of norms of one group upon all related groups (*prohibition and the speakeasy*). (4) Incorporation of alien elements through migration or conquest and the lag in absorption (*Huguenots, Mormons*). (5) Uneven movement of change produces discontinuities—some groups change more rapidly than others (*subsistence farming vs. commercial farming*). (6) Disjunctiveness of groups in complex societies. (*See 27.*)

**10 INSTITUTION. A. DEF.** A group structure that is explicitly defined in a culture and commonly accepted as a right form of organization (*family, church, business enterprise, court of law*). Some are **universal** (*family*); others are **alternative** (*business enterprise*—corporation, partnership, proprietorship; *city government*—city-manager, mayor-council, commission). The origin of universal institutions usually antedate history, are often legendary and carry heavy **moral sanctions.** Alternative institutional forms have more recent origins and sanctions are **secular. B. INSTITUTIONAL ROLES.** (1) **ASCRIPTION:** Ascribed roles are assigned on basis of physical or other characteristics exclusive of merit (*age and sex roles, hereditary positions*). (2) **ACHIEVEMENT:** Achieved roles are acquired on merit, often through competition.

**11 STRATIFICATION. A. DIFFERENTIATION.** Every group, as it acquires a structure, becomes more or less clearly stratified. As roles become **differentiated** the people who occupy them tend also to be differentiated. Differentiation occurs with reference to (1) the worth of persons as inferred from **status;** (2) the life chances individuals are able to realize by virtue of their roles; and (3) the styles of life they are capable of enjoying. Stratification relates to the distribution of members of a group (*usually a society*) into graded categories on the basis of one or more of the above named factors or variables. When the gradation or **rank order** based on any one variable (*life chances*) corresponds closely to the gradation based on all other variables, a **class system** obtains: *Individuals who occupy similar status positions, have similar life chances, and similar life styles, constitute a social class.* **B. SOCIAL CLASS.** (1) **CLOSED SYSTEM:** Illustrated by extreme form known as the **caste system.** Castes are alleged to be genetically pure categories whose positions in society are caused by their **genetic properties.** hence inter-marriage is rigorously prohibited and there is very little movement of individuals from one caste to another. Other prescriptions in the culture may even enforce the maintenance of certain physical distances between members of different castes. **Examples of caste:** *Priest, warrior, merchantman, agriculturist, and laborer in Hindu society.* (2) **OPEN SYSTEM:** All other class systems are relatively open with some movement between classes possible. Movement is usually **inter**-generation rather than **intra**-generation. Openness of class system varies with the rate of change occurring in a society. Very stable societies have a higher degree of **closure** and more explicitly defined social classes than do unstable societies. Extreme openness means virtually no class system. (3) Change tends to make irrelevant the bases of pre-existing class system, thus breaking up the classes, and developing different bases for a new set of classes. Successive changes, however, can delay indefinitely the emergence of a new set of classes.

**12 CULTURE. A. SCOPE.** Refers to all forms of **learned behavior,** from the trivial to the important, the simple to the complex. Embraces *language, art, law, technology, etiquette, superstitions, religion, games, property, etc.* **B. COMPONENTS. (1) TRAITS:** Smallest unit part, such as the right to use property owned. **(2) COMPLEX:** A **cluster of traits** including the right of use of property owned, restrictions on use, responsibility for taxes and assessments, inheritance procedures and other arrangements for transferrals—all of which constitute the **property complex.**

**13 CULTURAL DIFFERENTIATION. (1)** While culture is universal and every society's culture has a similar outline to that of every other society, extensive differentiation prevails. **Languages** differ phonetically, grammatically and in vocabulary; **family** varies from the great and consanguineous to the small and conjugal; **religions** may be polytheistic or monotheistic, anthropomorphic or abstract; differences occur in sex behavior, myths, food preferences, judgments of right and wrong, habits of perception, etc. (2) Extent of differentiation reflects the diversity of environmental and other circumstances in which human groups are distributed. A group migrating to a new area brings a culture with it and promptly begins to change it to suit the new environment. Modifications occur, discoveries are made, contacts with alien people occur. (3) The spread of man over face of the earth has differentiated culture. An entirely unique culture occurs only when there is **complete isolation** from other groups. **Culture interchanges** are the rule, but occur unevenly from place to place.

**14 CULTURE AND GROUP. A. SUB-CULTURE.** Group is the **organization** of people; culture, the **behavior patterns** they (*usually a society*) possess. Every subgroup of a society (*family, social class, school, church*) acquires some behavior patterns peculiar to itself: **B. PARTICIPATION CLASSIFICATION OF CULTURE TRAITS.** (1) **Universals** are ways of acting common to all members of a society (*language, family forms, Moral values*). (2) **Specialities** are ways of acting peculiar to recognized categories (*medical ethics, female behavior*). [*See 4: B*] (3) **Alternatives** are different ways of pursuing a given end (*civil vs. church wedding, manner of dress*). **NOTE:** The more complicated and exposed to change the society, the fewer the universals relative to alternatives and specialties.

**15 ETHNOCENTRISM. A. DEF.** Tendency for members of a culture group to regard their culture as natural and therefore superior to that of other societies. The use of one's own culture as a standard for the appraisal of other cultures. **B. INTENSITY.** Varies with isolation of a society, but is never completely absent in any society. Group loyalty or **patriotism** may represent mild forms of ethnocentrism; intense nationalism constitutes an extreme form. Ethnocentrism has served as basis for justifying conquest, deprivation and enslavement of one people by another: *Anti-Semitism, Know-Nothingism, sectionalism,* etc.

**16 RACE AND CULTURE. A. DEF.** Race defined as an aggregate of people with similar biological characteristics: *color, stature, hair texture, cranial capacity, cephalic index.* Race is basically a social definition rationalized on a basis of physical characteristics. [*See 11:B.*] **B. MAIN RACIAL GROUPINGS. Caucasoid, Mongoloid, Negroid, Oceanian**—subdivided into twenty or more classes. Uncertainty results from difficulties of measuring characteristics and the fact that characteristics are continuous variables. **C. CAPABILITIES AND DIFFERENCES.** (1) No demonstrated relationship between racial characteristics and capability for learning and creativity. All so-called races have equal capacity. Characteristics are superficial. (2) Race and culture differences often coincide. They are not cause and effect but effects of other causes. Race results from prolonged **inbreeding** thru sustained **isolation;** a unique culture from limited interchange of ideas and experiences due to sustained isolation.

**17 POPULATION AND CULTURE.** Study of population provides crucial data for the study of social change. It is concerned mostly with the changing size and composition of social groups and societies, due to **fertility, mortality** and **migration. A. POPULATION SIZE.** Related to complexity of culture. Increase in size requires either more detailed elaboration of group structure or elimination of excess people. Societies or residence groups usually have mechanisms for controlling population size. **(1) TO ASSURE ENOUGH PEOPLE:** *multiple marriage, prohibition of celibacy, adoption, slavery, group consolidation.* **(2) TO PREVENT EXCESS:** *Marriage restriction, celibacy, infanticide, emigration.* **B. MALTHUSIAN THEORY** (*Classic formulation of growth*). In any given state of the arts and where resources are brought to full use, *food* can only be increased **arithmetically** while *population* can increase **geometrically.** Hence population always tends to outrun food supply. Growth is checked at **subsistence level** by *vice, moral restraint, famine, disease and war.* Theory limited by the assumption of a fixed state of the arts; where technology changes, the theory does not apply. **C. GROWTH PATTERNS. (1)** In an isolated, subsistence society, fertility is more or less constant and the death rate fluctuates widely due to an inadequate control over the environment. Alternating periods of growth and decline result and growth is not cumulative. (2) In societies involved in exchange relations, growth tends to be cumulative. **D. DEMOGRAPHIC TRANSITION.** Birth and death rates move as follows: (1) Birth rates remain at high level while death rates fall. (2) Increasing excess of births over deaths causes an accelerating growth rate. (3) Later, the birth rate enters a decline, moving faster than the death rate, hence population growth decelerates.

**18 SOCIAL CHANGE—THEORIES OF HISTORY A. DIALECTIC: THESIS-ANTITHESIS-SYNTHESIS** (*Hegel, Marx*). **(1)** A prevailing principle of organization is opposed eventually by a **counter-principle.** From the resulting clash and interaction a **new principle** emerges. **(2)** The theory is useful for organizing historical materials retrospectively; not useful for forecasting. Tends to ignore the instrumentalities of change. **B. NATURAL HISTORY. (1)** The rise and fall of civilization through youthful, mature, and senile stages (*Spengler*). **(2)** Subject to same criticisms as above. Analogy of society with individual organism is exaggerated. **C. EVOLUTION. (1)** Holds that society passes from lower to higher forms through a series of well-marked stages (*savagery, barbarism, civilization*). **(2)** Analogy with organic evolution questionable. Identification of stages a major problem. No evidence that history of Western peoples will be repeated elsewhere in same way. Provides no clues on instrumentalities of change.

**19 RESTRICTED THEORIES OF SOCIAL CHANGE A. INVENTION AND DIFFUSION. (1) Invention** refers to the discoveries and development of new ideas. **(2) Diffusion** is spread of culture elements from place to place through contacts between peoples. **(3)** Absorption of culture of one people by a different people is described as **acculturation. (4)** Evidence indicates that **invention** is dependent on diffusion and is most probable and most frequent where there is the greatest intermingling of diverse culture elements. (*There are numerous recorded instances of simultaneous invention.*) **B. CULTURE LAG.** The unevenness of change in society and culture. **Mechanical** and **technical traits** are subject to most rapid change. **Ideological** and **moral elements** slowest to change. The consequence of lag or unevenness is inconsistency and friction among the components of society and culture.

**20 EXPANSION. A. DEVELOPMENT OF A CENTER (1)** Expansion is a **developmental process** in which a society grows larger and more complex. It involves the growth of a center and the territorial extension of the center's influence. **(2)** The center originates from locations which are the intersection points of routes of travel: *crossing of paths, confluences of streams, convergence of land and water routes.* At such points people of diverse experiences and cultures meet and interact. Exchanges of products occur, with **diffusion** and **invention** most rapid. **B. ACCESS. (1)** Size and organization of a society, other things being equal, is a function of the **extent of access** it has to the surrounding world. The scope of a location's access defines the resources at its disposal, the influences to which it is exposed and the problems with which it must cope. **(2) Access is greatest** at route intersections and is largely effected by the efficiency of the facilities for transportation and communication. **(3) Access is measured** by the frequency of contact with people and events lying beyond the immediate habitat. **NOTE: CONTRACTION** (*alternative to expansion.*) May result from dissipation of resources through misuse or from infringement by expansion of a neighboring society.

**21 URBANIZATION. A. DESCRIPTION.** The basic community form of Western industrialized society. **(1)** Urban organization is centered on a city, usually situated at an intersection of regional and interregional **access routes.** Smaller cities, towns and villages scattered over the hinterland are closely linked to the **principal city** which provides a variety of services for the region: *transportation, marketing, central office, shopping, entertainment, etc.* **(2)** The principal city is a gateway to the world at large for the smaller community. There is no self-sufficiency; all the settlements are closely inter-linked. **B. CONDITIONS FOR THE APPEARANCE OF URBAN ORGANIZATION: (1) STRATEGIC SITE** relative to transport routes. **(2) IMPROVEMENT OF TECHNOLOGY. Transportation** and **communication; food production** to support a larger non-agricultural population and to provide buying power for agriculturists' **manufacturing** implements and other consumer goods for intra- and inter-regional exchange. **(3) POPULATION.** Increase sufficient to staff a more complex organization; **(4) TRADE GROWTH** between and within regions. **(5) CENTRALIZATION OF POLITICAL POWER.** Assures order and observance of norms.

**22 POPULATION. A. MIGRATION. (1)** Technological improvements result in a decline of the death rate and a rapid population growth. [*See 17:D.*] Surplus population transferred from agricultural to urban areas through **rural** to **urban migration. (2)** Growth of cities through 19th century depended on migration, since the natural population increase in the cities was not sufficient to provide growth. Migration to cities mostly by young adults with females more numerous than males. Selectivity with reference to intelligence extensively investigated but results are inconclusive. **B. BALANCE OF BIRTH AND DEATH. (1) Vital rates** low in cities—*death rates* 10 to 12 per 1,000, *birth rates* 15 to 18 per 1,000. Similar rates diffuse to rural areas especially where contact with cities is frequent. **(2)** In urban organization *death rates* fairly constant from season to season and year to year. Effectively controlled through adequate food supply and sanitary and medical technology. *Birth rates* fluctuate between fairly wide extremes (*from 15 to 25 per 1,000*) and vary with marriage rate and economic conditions. **(3)** Population size in any locality is determined mainly by the volume of **economic opportunity** rather than by resource supply. Hence **migration** is a more important adjustment factor than the balance of births and deaths.

**23 ECONOMY. A. MECHANIZED INDUSTRY.** Mass production using large capital investments, market oriented. Trend toward mechanization of extractive and manufacturing industries, releasing workers for growing service industry. **B. CAPITAL OWNERSHIP. Capital** basis of wealth rather than land. Land enters the market as a **commodity** subject to buying and selling. Scale of activities very large. **C. REGIONAL INTERDEPENDENCE.** Local self-sufficiency eliminated and local and regional specialization and market exchanges prevail. The system is highly vulnerable to external shifts and changes such as *resource exhaustion, new competitor technological changes, political movements.*

**24 DIVISION OF LABOR. A. SIZE.** Household too small and unselective of membership to serve as an effective unit of economy. Units are **extra-familial** associations (*factories, corporations, stores, agencies*) all highly **specialized. B. OPERATION.** Individual specialization extensive, increasingly on **task** rather than industry basis. Labor force in relation to population small by comparison with that of village and town organization. Age and sex differentiation loses significance as a dependable basis for **ascription** of roles. *[See 31.]* **C. TECHNOLOGY.** Mechanized, highly technical, changeable. Increasing number of human skills built into machines. Uses small quantities of labor power per unit of product. Relies primarily on mechanical sources of power. Transportation and communication are rapid and inexpensive.

**25 STANDARDIZATION. A. BEHAVIOR-PROCESSES-EXCHANGE. (1)** A large number of people living together within a single organization require **standardization of behavior.** This is necessary for close cooperation, efficient communication and exchange, and the supplying of daily needs. **(2)** Rise of national states furthered the development of urban organization by establishing and enforcing **standard norms** governing processes and exchanges: *language, coinage, weights and measures, rules of conduct, roles, modes of dress,* techniques of all kinds are driven toward uniformity. **(3)** Tendency is pervasive—uniformity may extend to *habits of thought, judgment of values and personality traits.* **B. SPECIALIZATION. (1)** Standardization is not offset by extensive *specialization.* Unlike specialization by industry, as in town organization, **specialization by task** does not differentiate ways of life of individuals and families. **Task specialists** use standardized tools and receive a uniform wage for a standard work week in a uniform type of work organization; each of which is governed by a standard accounting system, standard governmental regulation and the requisites of a single, inclusive economy. **(2)** Standardization tendency is less coercive in the highly skilled and professional specialities. Such activities are less subject to administrative controls and more open to individual judgment and innovation. **C. MOBILITY.** Standardization facilitates mobility. Movement within and between cities and regions are not hampered by **culture differences.** Workers can shift from job to job with little or no retraining. Uniformity of consumer habits and of access to facilities simplifies **inter-class mobility. NOTE:** Barriers to movement are reduced by standardization, not eliminated.

**26 THE HOUSEHOLD UNIT. A. CHANGING FUNCTION OF HOUSEHOLD. (1)** The enlarged scale of collective activities and the specialization fostered by urban organization cause **functions** to leave the household. The separation of work from the home has relieved the household of its basic economic function and reduced its economic role to that of a **consuming unit.** It has removed the **income earner** from the household for a large part of each day with a resultant change in the character of husband-wife relations from husband dominance toward **equality of mates. (2)** The loss of functions has brought about new spatial arrangements of work places and residences, altering the accessibility requirements of both. Furthermore, industry has absorbed production and preservation of food, the making of cloth and clothing, utensils and furnishings, leisure time equipment, the distribution of news, etc. Medical care, care of the aged, supervision of delinquent children, marital and family counseling, much recreation and even income savings have been taken over by formally organized, **extra-familial groups.** Numerous other optional services are available: *day-care of children, launderies, domestic and financial consultation, etc.* **B. CHANGING SIZE OF HOUSEHOLD.** With the increase in scale and number of specialized extra-familial units, and the loss of household functions to such units, the size of the household declines. Loss of the **producing function** removed the economic imperative for having children; hence birth rate declined. In the U.S., average size of household declined from almost 6 in 1890, to 3.3 persons per household in 1960. This reduction was brought about by the elimination of non-family members (*servants*) from households and by the growing tendency of the **parental generation** to live apart from a **conjugal unit** of son or daughter as well as by a decline in the birth rate. **C. STATUS OF WOMEN. (1)** Transfer of functions from household to other units, mechanization of household tasks, decline of birth rate have all contributed to the rising status of women. **(2)** Greater freedom permits more education, greater participation in community activities and gainful employment outside the household. Sex division of labor becomes less distinct. **(3)** Change is spread over time, thus there tends to be a lag in the adjustment of women to their enlarged range of opportunity. **D. HOUSEHOLD'S BASIC FUNCTION. (1)** Despite losses and changes, the household unit is still important. It remains the principal basis of **primary relations** and carries the main responsibility for the care and nurture of children. **(2)** The household performs numerous **nonfunctions,** such as final states of food preparation, maintenance of clothing and accessories, supplementation of formal education, provision of leisure time pursuits, and performs the **last stage** in distribution of income. **(3)** It is the principal **consuming unit** of the economy and is instrumental in maintaining the individual's prestige.

**27 DISJUNCTIVENESS OF GROUPS. A. MEMBERSHIP PATTERN.** Urban organization comprises a multiplicity of more or less specialized groups which relate to one another only with reference to the particular function performed and exchanged. The **unit of membership** in such groups is the individual rather than the family or household (*as in town organization*). In many cases membership in one group is not a criterion for membership or participation in another, though there are frequent instances of groups within groups (*industrial work group and labor union, student body and fraternity*). **B. DEVIATION.** Disjunctiveness poses a problem of maintaining the **coherence of norms** in society. Concentration on special functions tends to lead groups to adapt to circumstances of the function and thus to depart from **accepted norms.** Deviation partially restrained by overlapping memberships which provide channels of **inter-group communication** through which norms are reasserted.

**28 FORMAL GROUP MEMBERSHIP. A. PURPOSE.** The complexity, impersonality and individualization characteristic of urban organization make exceedingly difficult the pursuit of individual interests or the expression of individual opinion. A solution is found in the establishment of a formally organized group among persons sharing a **common interest** to foster, protect and exercise that interest: *fraternal orders, hobby clubs, professional associations, garden clubs, athletic clubs.* **B. MEMBERSHIP PATTERN.** These groups occur most frequently among persons who have time and are articulate. Frequency of group membership varies with social-economic level: is greater in suburbs than in central cities, and also among young adults. Whether formal group membership is an alternative to informal associations is undetermined.

**29 INDIVIDUALISM. A. DEPENDENCE.** Under the village and town organization an individual is dependent on a small number of **kinsmen** and friends. In urban organization the individual is dependent on a large number of strangers and groups of strangers. Though he is no less dependent, the transferral of dependence emancipated the individual from the **family unit** which formerly encompassed all of his daily life. Individualism thus means that he has options as to *stores to shop in, firms to work for, clubs to join, avocations to pursue.* **B. ROLE CONFLICT. (1)** The individual is left with the main responsibility for reconciling the demands of the roles he occupies in the different groups. Occasionally, different **role requirements** are contrary and even contradictory: *prowess in the gang and obedience to parents; business success and observance of religious tenets; professional achievement and adequate discharge of family responsibility.* **(2)** Inability to resolve conflicting demands can lead to profound disturbances in the person or to withdrawal from more inflexible group situation.

**30 THE DIFFUSION OF URBAN ORGANIZATION.** The urban mode of life tends to spread over the total area for which the city is the economic and cultural center. **A. EFFECTS. (1)** As rural areas are brought into the market economy of the city they necessarily adopt urban forms of organization: *patterns of behavior, fashions and fads, and interests of all kinds assume urban form.* **(2)** The process is accelerated by modern forms of communication — *radio, television, daily newspapers.* The fact that **urbanity** is no longer confined to cities is a characteristic of contemporary civilization.

**31 LARGE-SCALE ORGANIZATION.** Development of the urban organization increases the size of groups in all areas of collective life, *excepting the family.* **A. SMALL GROUP DEFICIENCIES.** Small groups are unable to cope with the scale of events encountered in urban society. They lack the resources and organization needed for maintaining extensive communications, for the exercise of effective influence, for mass production and mass marketing, or for other forms of participation in a complicated and territorially extensive system. **B. RATIONALITY. (Secularization).** Growth generates a corresponding **degree of rationality** in group organization. The task of coordinating and administering the large unit demands **systematization** wherever possible. This is particularly characteristic of **specialized groups,** where specialization renders the group vulnerable to external shifts and changes: *market changes, technological innovations, population movements, political upheavals, fashion and fads.* **C. RATIONALITY FOSTERED BY: (1)** Massing in cities of strangers who share no common tradition; **(2)** culture accumulation, especially technical culture; **(3)** prevalence of monetary system for communicating values; **(4)** general movement toward standardization. **D. FORMAL STRUCTURE.** The effect of increased rationality is to magnify and sharpen the normal contours of group structure **(1) Specialization** of task is developed intensively, with processes and jurisdiction made explicit. **(2) Hierarchy** is elaborated as coordinating task is subdivided and graded in responsibility. **(3) Rules and sanctions** become highly developed and enunciated. **(4) Technical qualifications** govern the recruiting of the individual who is subordinated impersonally to the specification of the highly formalized organization. **E. PROBLEMS OF FORMALITY.** While the **elaboration of structure** enhances efficiency of group activity, it also threatens a group with inertia and loss of flexibility. **(1) The rule of formal procedure** and prescribed standard tends to put a premium on **conformity** and to stifle the exercise of imagination and judgment. **(2) The rigid hierarchial structure** tends to remove leaders or administrators from operating problems, making them dependent on extended channels of communication for needed information. **(3) Fear of sanctions** for errors in lower levels of hierarchy leads to feeding misinformation into the communication channel. **(4) Formality and impersonality** of the system tends to ignore individual differences and needs. **(5) The** conflict between the efficient, formal system and the need for flexibility has placed heavy responsibility on the administrative role. This has given rise to the field of professional administration. **(6) Conformity on rules** and procedures tends to obscure the objectives of organization. **F. INFORMAL DEVELOPMENT. Formality** and **impersonality** in a system foster the development of numerous informal arrangements and associations. **(1)** Informal groups serve as a means of controlling and even exploiting the mechanical discipline of the organization. Lodged in **friendship groups** and **cliques** of various kinds, they are usually formed within the **strata** of the hierarchy, and comprise persons who occupy roles having a similar status. **(2)** Each informal group acquires a **set of norms** regarding *rate of production, relations with superiors, protection of members from organizational sanctions, etc.* They are often supportive of the objectives of the organization as well as restrictive against excessive or arbitrary demands. **(3)** Informal groups are powerful factors in regulating the rate and character of **internal change** in an organization and tend to shape change to accord with **group norms. G. EXTERNAL INFLUENCES.** Internal affairs of the large-scale organization are also influenced by **formally organized** groups external to it: *labor unions, professional associations, government.* Some of these external groups originated from the informal groups within the organization or from federations of many local associations. The function of such external groups (*excluding government*) is to protect and enlarge the prerogatives of **classes of roles.** Though often a stimulus for change, they can constitute another source of rigidity (*A.M.A. vs. Fed. medical insurance*). **H. HOMOGENEITY OF FORM.** All group organizations are **subsystems** in a more inclusive system. All must be able to *communicate, render accounts* and *share common facilities.* Hence they gravitate toward a standard form, subject to size variations and some functional peculiarities.

**32 STRATIFICATION IN URBAN ORGANIZATION.** The rise of urban organization dissolves traditional classes. **A. NEW BASE. (1)** The relation to land and family membership lose their relevance as the basis of stratification. Substituted for them are **capital ownership and income, occupational achievement and consumption power. (2)** Urban organization also redistributes functional positions and opportunities and reconstitutes life styles. The process is a result of change and continues as long as change continues to occur. There is no close correlation between **functional position** (*occupation*), **opportunity** (*income*) and **life style** (*education*). Persons on a high level in one or two areas may be low in others. **(3)** Since there are no clear-cut classes, such designations as *upper, middle* and *lower classes* are gross approximations resting mainly on statistical generalization rather than on indications of **class consciousness** or **homogeneity** within strata. **B. STRATIFICATION OF INTERESTS. (1)** Individuals who occupy similar positions in an organization acquire similar characteristics and develop similar interest. Associations for leisure time purposes are formed among persons with *similar occupations, similar education* or *similar consumption habits.* The associations formed within occupational categories are often **heterogeneous** with respect to education and/or consumption habits. **(2)** Persons in lower occupational and income levels belong to few organizations, do little travelling and read few books and magazines. The converse is true in the upper levels. **C. OPPORTUNITY VARIABLES.** Affected by position in **occupational hierarchy: (1)** In U.S. about 23 per cent of factory workers' children apply for college admission; 45 per cent of white collar workers' children; 75 per cent of children of professional and executive workers. **(2)** Life-time earnings of workers with *elementary education* average $75,000, of *high school graduates* $170,000 and of *college graduates* $270,000. **NOTE:** The finding that some 85 per cent of Americans report themselves as being middle class probably means that that proportion of the population have one or more characteristics that fall in mid range of occupation, income, or education.

**33 SOCIAL MOBILITY. A. DEF.** The change of position on occupational, income or educational scales. **B. DIRECTION.** Though movement can be downward or upward, it is usually to **adjacent stratum,** seldom skipping a stratum. **(1)** Movement is mainly **inter-generational:** *son enters a stratum different from that of father.* **(2)** An individual once established in a stratum rarely moves out of it, though he may move from job to job within a given stratum **(3) Social mobility** is usually accompanied by **residential mobility. NOTE:** New stratum positions are usually accompanied by a different **distribution of interactions.**

**34 FACTORS CONTRIBUTING TO MOBILITY. A. EXPANSION OF ORGANIZATION. (1)** Creates many new administrative, professional, skilled and other specialized categories of occupation. **(2)** Natives of area tend to move up to fill them; they are then replaced in lower positions by newcomers to the organization. **B. TECHNOLOGICAL CHANGE. (1)** Develops new positions and obsolescence of old ones. **(2)** Decline of agriculture and mining, forces sons into new and different positions. **(3) Mechanization** of factory production and other formerly manual tasks reduces unskilled labor opportunities and expands semi-skilled area. **(4)** In general, technological history has been—first, the growth of **primary activities** (*extractive*), then the growth of **secondary activities** (*manufacturing*) at the expense of the primary, and finally the growth of **tertiary activities** (*service and administration*) at the expense of both primary and secondary activities. **C. RISING LEVEL OF LIVING. (1)** Improvements in the **productivity** of an economy frees increasing numbers of people from the labor force to attend schools, which in turn are multiplied and enlarged to absorb the growing enrollment. This increased level of **mass education** reduces effective **income differentials.** The wide differences are reduced further by the growth of *contractual savings programs, social security, pension plans, etc.* **(2)** Style of life differences are also reduced by: *mass production of standardized items, the development of installment buying and the growth of credit facilities.* **(3)** Shortening of the work-week and increase of leisure time gives greater opportunities for consumption and cultivation of avocations. **D. EXCESS REPRODUCTION.** When the number of progeny exceed opportunities in the parents' stratum, the new generation must move into other strata (*often forcing* **rural to urban movement**). The reduction of social-economic differentials decreases this type of movement. **E. VALUE SYSTEM.** Although **values** are but a reflection of organizational and material circumstances, they may have **differential incidence** on individuals: Emphasis on *personal advancement* or "*getting ahead*" and the *censure of the lazy* contribute to **mobility aspirations.** *Education* is highly valued as a means for upward mobility. The goal of **security** is pursued through striving for **mobility.**

**35 STRESSES AND STRAINS INDUCED BY SOCIAL MOBILITY. A. FAMILY RELATIONS.** May be strained by a difference in **strata positions** between parents and children because of **contrary norms** and difficulties in communication. Status difference leads to alienation and **residential separation** which interfere with mutual aid. **B. CONFLICT IN THE NEW STRATA. (1)** Lack of correlation among **rank positions** induces strains especially for the upwardly mobile person: *One might move up the* **occupation scale** *without a corresponding movement on the* **education scale,** *or upward on the education scale without similar movement on other scales.* **(2)** Inconsistency in rank positions affects one's acceptability to individuals and groups who are in the higher stratum (*the nouveau riche*). These inconsistencies often provoke internal disturbances in the person. A similar reverse effect occurs with downward mobility when individuals do not accept the implications of the lower status positions. **NOTE:** The completion of a vertical movement usually involves **three generations. C. COMMUNITY CONFLICT.** The upwardly mobile compete for leadership positions (*business, social, political*) and for appropriate residential space. The resultant frictions can be productive or wasteful.

**36 NORMATIVE ORDER IN URBAN ORGANIZATION.** Urban organization tends to displace **sacred** with **secular norms** (which are *rational, formal* and *impersonal*). Norms tend to be seen as a means to ends rather than as ends in themselves. **A. NORMATIVE INTEGRATION. (1)** The prevalence of specialization causes a differentiation of norms among groups and categories. Normative integration is less a matter of unanimous agreement and universal sharing of norms than of **logical consistency** among divergent though related norms. Reconciliation of discrepancies is achieved partly through **inter-group adjustments** (*as between organized labor and management groups*) and partly through effect of overlapping group memberships. **(2)** Failure of such mechanism leads to legislation and an enlargement of the body of rational and formal norms.

**37 STRESSES AND FRICTION IN THE NORMATIVE ORDER.** Due largely to change which may create new circumstances to which no existing norms pertain. **A. AVAILABILITY OF NEW LABOR FORCES—PROBLEMS: (1)** The necessity to provide full and productive use of the energies of women released from household duties by mechanization. **(2)** Increased longevity and the lack of a role for the aged. **(3)** The gap between age at leaving school for many adolescents and age at which they can secure employment. **B. UNRELATED AND CONTRARY NORMS.** Observable in cities containing colonies of foreign-born migrants who tend to preserve their foreign cultures. **C. OBSOLESCENSE OF NORMS.** Persistence of rules governing an occupation after **technological shifts** have significantly altered character of the occupation: *railroad engineers and firemen.* **D. PEER GROUP ASSOCIATIONS.** Decline of the family as a producing and mutual aid unit has also impaired its effectiveness as a **primary group.** Primary associations are sought in peer groups, that is *associations within age categories.* This problem occurs at all ages, but is most serious with adolescents. The **adolescent peer group** tends to develop norms peculiar to itself and often contrary to **societal norms.** *The delinquent gang is a peer group which demands loyalty of its members and bestows prestige on basis of successful resistance to societal norms.* **E. ANONYMITY.** Impersonality in an urban setting makes the **evasion of norms** easy. The city provides a haven for habitual evaders. **F. EXPERIENCE.** Discrepancies exist between societal norms and actual experience: *norm of equality of opportunity vs. strata differences in opportunities; norm of justice for all vs. the greater susceptibility to arrest of minority group and lower social economic levels.* **G. ROLE CONFLICT.** Generated by memberships in two or more **disjunctive groups.**

**38 DEVIATIONS FROM SOCIETAL NORMS. A. GEOGRAPHIC.** Most prevalent in cities lowest in rural areas. Statistical difference due in part to a greater frequency of arrest in cities. Rural area deviations are more frequently handled informally. Deviations in cities take the form of offenses against persons. **B. STRATA.** Deviation resulting in arrests are concentrated in lower social-economic levels and are due partly to the greater susceptibility to arrest and partly to inadequate socialization. The latter can be related to the frequency of broken homes and employment of mother as well as inadequate resources and opportunities: *congested living quarters, low income.* Deviations in upper strata come to attention of police less frequently and those individuals have greater resources for defending themselves.

**39 MASS SOCIETY AND SOCIAL MOVEMENTS. A. DEF.** Society viewed in its aspect as a large number of more or less anonymous individuals living in fairly standardized ways. **B. COMMUNICATION.** Communication with society in the mass requires mass communications, media that reach all parts of the whole simultaneously (*newspapers, radio, television*). Communication tends to exploit the least common denominator, the commonplace which the least able person can grasp. **C. STANDARDS.** The mass is held to be responsive only to the garish and tawdry, not to the refinements of ideas, art or moral issues. It is impressed by the vulgarly dramatic (*pomp and circumstances, a vicious crime*) and it is susceptible to rumor (*the invasion from Mars, 1938*) to contagion of ideas (*fads, propaganda*). A mass is a fertile basis for a **social movement.**

**40 PATTERN OF SOCIAL MOVEMENT. A. BACKGROUND.** A concerted effort by many people can be directed at modifying an existing element or at the creation of a new element of organization. This generally occurs in periods of change; hence the effort is a vehicle of change. It occurs most often when relatively large numbers of people are released from old associations and are confronted with new and unpredictable circumstances. **B. INITIATION.** Movement is initiated when the stirrings of a number of like-minded people are mobilized by the appearance of a spokesman and leader. Most effective in the early phase is the **charismatic leader:** *the forceful visionary who has the appeal of a prophet endowed with mystical sanction.* (*e.g., Gandhi*). This leader's appeal is usually enhanced by his being an **amateur and of the people.** Such a leader can enlist fervent support from his followers and can convert the doubtful to his cause. **C. IDEOLOGY AND PROGRAM.** Leadership evolves an ideology and a program (*Roosevelt, the New Deal*). Dedicated amateurs take positions as lieutenants and proceed to build an organization (*Anti-Saloon League*) through formation of chapters or cells and the establishment of communication links between them. **D. LEGITIMACY.** While the movement thrives on opposition, it is likely to end as merely another protest effort unless it gains a token of **legitimacy** (*election of its leader to a major office*), or an invitation from some constituted authority to negotiate (*union movement*). **E. ORGANIZATION AND EXPANSION.** With the establishment of legitimacy a replacement of leadership begins. Professional or administrative type leaders replace the dedicated amateurs (*partly through purges*), though the charismatic leader may remain because of his propaganda or symbolic value. Then membership campaigns are initiated, resources are accumulated and a many-faceted assault on the "enemy" is launched. Failure to recruit administrative leadership usually leads to dissolution of movement (*Townsend movement*). **F. MODIFICATION and CONSERVATISM.** Success of the movement affects the change sought and also begins a modification of program objectives and radical proposals. Remnants of the movement grow more and more conservative as they are faced with the problems of dealing with the organizational complexities of societal government (*Social Credit movement of Alberta, Canada*).





DATA-GUIDE EDITOR: JOSEPH L. LEON

PUBLISHED & M'F'D BY DATAGUIDE, INC. FLUSHING, N.Y. 11355

DATA-GUIDE QUICK CHART

# ESSENTIALS OF STATISTICS

SIMPLIFIED SUMMARY ◆ INSTANT REFERENCE ◆ PERMANENT MEMORY-AID

LEARN & REVIEW IN MINUTES

AUTHOR: Prof. R. Perlman, Ph.D., Adelphi University

SOLID PLASTIC

SPEED STUDY TO SAVE YOU TIME AND WORK.

QUICK CHARTS PUT THE FACTS IN FULL VIEW:

PERMANENTLY FILL IN FOR MISSING NOTES & TEXTS.

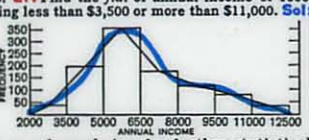
QUICK CHARTS HELP YOU REMEMBER:

CONTAINS NO PAPER

## 1 FREQUENCY DISTRIBUTIONS (f.d.).

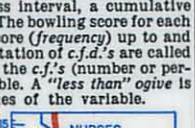
**A. CONSTRUCTION.** Divide the range, i.e. the difference between the largest and the smallest value, into **classes** of equal width or **interval** (*c.i.*). Too few classes will obscure the pattern of the distribution; too many will prevent concise presentation of the data. A few items at the ends of the distribution may be grouped together in an **open-end** (i.e., "under" or "over") **class.** The *f.d.* is completed by recording the number of items, or frequency (*f*) of each class. **E1:** Find the *f.d.* of annual income of 1000 plumbers, fewer than 5% earning less than $3,500 or more than $11,000. **Sol:**

| c. i. | f |
|---|---|
| under $3,500 | 38 |
| 3,500 – 4,999 | 197 |
| 5,000 – 6,499 | 359 |
| 6,500 – 7,999 | 176 |
| 8,000 – 9,499 | 114 |
| 9,500 – 10,999 | 82 |
| $11,000 and over | 34 |




**Note:** Since class intervals are of equal size, for further statistical treatment the lower limit of the *f.d.* is simply assumed to be $2,000 and the upper limit $12,500. **B. GRAPHIC REPRESENTATION. (1) A histogram** or bar graph is a set of rectangles each having the width of the class interval and altitude of the class frequency. **(2) A frequency polygon** connects the frequencies (altitudes) at the mid-value of each class with straight lines. **(3) A frequency curve** is the smooth continuous curve through the plotted points of the polygon. (See Fig. 1.) **C. TYPES OF FREQUENCY CURVES. (1) Normal Curve:** symmetrical bell-shaped curve important in statistical inference. The normal distribution (*n.d.*) occurs when many factors, no one of which is dominant, determine the value of the variable (heights of men or women, I.Q.'s, grades). **(2) Platykurtic:** symmetrical distribution with too few items at the center, and too many in the tails, of the curve (flat-topped). **(3) Skewed curves** (assymetrical): **(a)** *positive* (right skewed) when values stretch far out to the right of the peak of the distribution (*wage-income data*); **(b)** *negative* (left-skewed) with frequencies trailing far off to the left (below) the major point of concentration (*ages of corp. executives*). **(4) J-shaped curves:** frequencies increasing as the variable grows larger (*death rate per thousand for different age groups*). **(5) Reverse J-shaped curve:** frequencies declining as the variable increases (*age of cars on the road*). **D. CUMULATIVE FREQUENCY DISTRIBUTION AND OGIVES. (1)** When the frequency assigned to each class interval represents the total frequencies up to and including that class interval, a cumulative frequency distribution (*c.f.d.*) is formed. **E:** The bowling score for each frame (*class interval*) measures the total score (*frequency*) up to and including that frame. **(2)** Graphic representation of *c.f.d.'s* are called **ogives.** *A "more than" ogive* is the curve of the *c.f.'s* (number or percentage) > particular values of the variable. A *"less than" ogive* is the curve of the *c.f.'s* < particular values of the variable.

**E2:** Plot "more than" ogives for monthly incomes of doctors and nurses, given: **(1)** 15% of the doctors are interns working at a minimum $200 rate. **(2)** High-income doctors earn more than high-income nurses. **(3)** 61% of both doctors and nurses earn more than $400 per month.




**Summary:** A greater percentage of nurses earn more than any wage less than $400 per mo.; the reverse is true for wages above $400 per mo.

## 2 AVERAGES (MEASURES OF CENTRAL TENDENCY).

**A. CALCULATION OF ARITHMETIC MEAN ($\bar{X}$). (1) Basic Method:** Add the values of the variable $X$ and divide this sum ($\Sigma$ = sum of) by the number of values $N$; $\bar{X} = \Sigma X/N$. For grouped date, the assumption is made that all the items in each *c.i.* are equal to the **mid-value** (*M.V.*) of the interval; $\bar{X} = \frac{\Sigma f(M.V.)}{\Sigma f}$ **(2) Short method:** Add to a purely arbitrary value ($a$) the mean of the deviations ($d = X - a$); $\bar{X} = a + (\Sigma d/N)$. For grouped data, the arbitrary point is taken as a *M.V.* of one of the classes of the distribution; $\bar{X} = (M.V._a) + (\Sigma fd/\Sigma f)$, where $d$ now is difference between the *M.V.* of each class and the arbitrary mid-value. Since the class intervals are of equal size, *c.i.* may be factored out of the second term; $\bar{X} = (M.V._a) + \frac{\Sigma fd'}{\Sigma f}(c.i.)$, (where $d'$ is diff. between class *M.V.* and arbitrary *M.V.*, measured in class interval units).

| c. i. | f | d' | fd' |
|---|---|---|---|
| under $3,500 | 38 | −2 | −76 |
| 3,500 – 4,999 | 197 | −1 | −197 |
| 5,000 – 6,400 | 359 | 0 | 0 |
| 6,500 – 7,900 | 176 | 1 | 176 |
| 8,000 – 9,400 | 114 | 2 | 228 |
| 9,500 – 10,900 | 82 | 3 | 246 |
| $11,000 and over | 34 | 4 | 136 |
| | 1,000 | | 513 |

**E3:** Find the arithmetic mean of a distribution of plumbers' incomes. **Sol:** $M.V._a = \$5750$

$$\bar{X} = M.V._a + \frac{\Sigma fd}{\Sigma f}(c.i.)$$

$$\bar{X} = \$5750 + \frac{513}{1000}(\$1000) = \$6520$$

**B. MEDIAN (1) When the data is arranged in order of magnitude,** the middle item (half above, half below) is the median (*Md.*). If the number of items is even, the *Md.* is the average of the two middle items. **(2) For grouped data,** the median is the value that divides the area under the frequency curve in half. Assuming that the values are spread evenly through each class, the *Md.* by interpolation is $Md = L + \frac{(N/2 - F)}{f}(c.i.)$, where $L$ is the lower limit of the class in which the median lies, $F$ is the cumulative frequency up to but not including that class, and $f$ is the frequency of the *Md.* class.

**E4:** Find the median income of the plumbers. **Sol:** $N/2 = 1000/2 = 500$; median lies in 5000–6400 class; $L = \$5000$, $F = 235$, $f = 359$, $c.i. = \$1500$.

$$\text{Substituting: } Md = \$5000 + \frac{(500 - 235)}{359}(\$1500) = \$6107.$$

**C. RELATIONSHIP OF MEDIAN AND MEAN. (1) In positively skewed distributions,** the mean is greater than the median since only the mean is greatly influenced by extreme values. (Compare the mean of $6520 in E3 with the median of $6107 in E4.) **(2) In negatively skewed distributions** the median is greater than the mean. **D. THE MODE. (1) In positively skewed distributions,** both median and mean are greater than the mode (*MO.*), the value of the variable having the greatest frequency. **(2) In negatively skewed distributions,** *MO.* is greater than both. **(3) For distributions that are moderately skewed,** the mean is three times further from the mode than from the median; $\bar{X} - MO = 3(\bar{X} - Md.)$. **(4) In symmetrical distributions,** including the normal, the three measures have the same value. **E. GEOMETRIC MEAN (G.M.).** Used when rates of change are involved. It is the $n$th-root of the product of $n$ values; $G.M. = \sqrt[n]{x_1 \cdot x_2 \ldots \cdot x_n}$.

**E5:** Find the geometric mean of 3, 9, and 27. **Sol:** $G.M. = \sqrt[3]{3 \cdot 9 \cdot 27} = \sqrt[3]{729} = 9$. Compare $\bar{X}$ with $G.M.$: $\bar{X} = 13$ is that value which, if substituted for each of the individual values, yields the same *sum* (39); $GM = 9$ is the value which, if substituted for the individual values, yields the same *product* (729).

**F. HARMONIC MEAN (H).** The reciprocal of the mean of the reciprocals of the values; $H = N \div \left(\sum \frac{1}{x}\right)$. Used to average time rates, or price data measured in number of units per dollar.

**E6:** A car travels 30 mph. the first mile, 40 mph. the second mile, and 60 mph. the third mile. What is its average rate of speed?

**Sol:** $H = \frac{3}{1/30 + 1/40 + 1/60} = \frac{3}{0/120} = 40$ mph. Note that the arithmetic mean rate (43 1/3 mph.) *is incorrect because it fails to consider that the car takes longer at the slower rates to travel one mile.*

## 3 DISPERSION.

**A. PARTITION MEASURES.** The range (largest minus smallest value) is heavily influenced by extreme values. **(1) The quartile deviation** is half the distance between the first and third quartile values of the distribution; it reflects the degree of central concentration. **(2) The mean deviation** (*M.D.*) is the average of the deviations about the mean ($X - \bar{X} = x$). Since the algebraic sum of deviations about the mean is always zero, the mean deviation is found by ignoring signs (shown by the absolute value sign ||); $M.D. = (\Sigma|x|) \div N$. The quartile deviation and mean deviation are not capable of further mathematical treatment. **B. STANDARD DEVIATION (σ or *s.d.*).** **(1)** It is the square-root of the mean of the squared deviations about the mean. Alternatively, it is the square-root of the mean of the squared values minus the square of $\bar{X}$: $\sigma = \sqrt{\frac{\Sigma x^2}{N}} = \sqrt{\frac{\Sigma X^2}{N} - \left(\frac{\Sigma X}{N}\right)^2}$

For grouped data, using the short method of calculation: $\sigma = (c.i.)\sqrt{\frac{\Sigma f(d')^2}{\Sigma f} - \left(\frac{\Sigma fd'}{\Sigma f}\right)^2}$

**E7:** Find the standard deviation of the wages of the plumbers. **Sol:**

| c. i. | f | d | fd' | f(d')² |
|---|---|---|---|---|
| under $3,500 | 38 | −2 | −76 | 152 |
| 3,500 – 4,900 | 197 | −1 | −197 | 197 |
| 5,000 – 6,400 | 359 | 0 | 0 | 0 |
| 6,500 – 7,900 | 176 | 1 | 176 | 176 |
| 8,000 – 9,400 | 114 | 2 | 228 | 456 |
| 9,500 – 10,900 | 82 | 3 | 246 | 738 |
| $11,000 and over | 34 | 4 | 136 | 544 |
| | 1,000 | | 513 | 2,263 |

$$\sigma = \$1500\sqrt{\frac{2263}{1000} - \left(\frac{513}{1000}\right)^2}$$

$$\sigma = \$1500\sqrt{2.263 - .263}$$

$$\sigma = \$2121$$

**C. RELATIVE VARIATION—COEFFICIENT OF VARIATION (V).** The *s.d.* is often inadequate for comparing the dispersions of two sets of data, either because the values for one set tend to be larger, or because they are expressed in different units. The coefficient of variation ($V$) avoids these pitfalls by expressing the *s.d.* as a % of the mean: $V = (\sigma/\bar{X})(100)$.

**E8:** Compare the dispersion of income of the plumbers with that of their years of experience: mean length of time working is 12 years; standard deviation 3 years. **Sol:** $V$ (income) = ($2121 ÷ $6520) x 100 = 32.6%, $V$ (experience) = (3 yrs. ÷ 12 yrs.) x 100 = 25.0%. There is greater relative dispersion in their income than in their experience.

Another application is in determination of whether a value from one set of data is more outstanding than a value in another set.

**E9:** Who was a better hitter, a player who batted .330 in 1940 when the mean average of all players was .280 with a *s.d.* of .040, or a player who batted .300 in 1960 when the mean for all players was .260 with a *s.d.* of .030? **Sol:** It is impossible to determine who was the better hitter since conditions were different in the two years. To determine who was more exceptional in his group, it can be calculated that the 1940 player was $\frac{.330 - .280}{.040} = 1.25$ *s.d.'s* above his group's mean, while the 1960 player was $\frac{.300 - .260}{.030} = 1.33$ *s.d.'s* above his group's mean. Therefore, the 1960 player was more outstanding.

**D. DISPERSION OF NORMAL DISTRIBUTION. (1) Binomial Expansion.** The *n.d.* is the limit of the binomial distribution as the number of possibilities for an event to occur ($n$) increases. Thus, the expansion of $(p+q)^n$ yields a *n.d.* as $n$ approaches infinity where $p$ is the probability an event will occur and ($q$), that it will not occur.

**E10:** If ten coins are tossed, what are the probabilities that 10 or 9 or 8 or 7 . . . or 0 heads will fall. **Sol:** $p = q = \frac{1}{2}$, $n = 10$. Substitute: $(p+q)^n =$

$$p^n + np^{n-1}q + \frac{n(n-1)}{1 \cdot 2}p^{n-2}q^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}p^{n-3}q^3 + \cdots + q^n$$

$$(\tfrac{1}{2} + \tfrac{1}{2})^{10} = (\tfrac{1}{2})^{10} + 10(\tfrac{1}{2})^9(\tfrac{1}{2}) + \frac{10(9)}{1 \cdot 2}(\tfrac{1}{2})^8(\tfrac{1}{2})^2 + \frac{10(9)(8)}{1 \cdot 2 \cdot 3}(\tfrac{1}{2})^7(\tfrac{1}{2})^3 + \cdots + (\tfrac{1}{2})^{10}.$$

The probabilities for 10, 9, 8, 7, . . . 0 heads, respectively, are $\frac{1}{1024}, \frac{10}{1024}, \frac{45}{1024}, \frac{120}{1024}, \frac{210}{1024}, \frac{252}{1024}, \frac{210}{1024}, \frac{120}{1024}, \frac{45}{1024}, \frac{10}{1024}$, and $\frac{1}{1024}$.

The graph of this distribution with relative frequencies on the $Y$-axis is a perfectly symmetrical 10 sided polygon. If $n$ had been larger, the number of sides would have been greater until the smooth normal curve had resulted from an infinite $n$.

**(2) Normal Deviate (T).** In *n.d.'s*, the deviation of a value from the mean expressed in standard deviation units is $T = (X - \bar{X}) \div \sigma$. Most $T$-Tables give the cumulative relative frequencies (*c.r.f.*) for values of $T$ from 0 up to at least +3. Since the normal curve is symmetrical, the frequencies for the corresponding ± $T$ values are the same.

**E11:** If the reading time of a magazine article for a large group of individuals is normally distributed, and $\bar{X}$ is 25 minutes and the *s.d.* 5 minutes, what percentage of the group would take from 23 to 29 minutes to read the article? **Sol:** For those in the specified range who read faster than the mean, the lower limit $T = \frac{23 \text{ min.} - 25 \text{ min.}}{5 \text{ min.}} = -.4$. For those who read more slowly than the mean, the upper limit $T = \frac{29 \text{ min.} - 25 \text{ min.}}{5 \text{ min.}} = +.8$. Referring to the $T$-Table, a $T$ of .4 includes 15.54% of the cases and a $T$ of .8 includes 28.81% of the cases. Therefore, 15.54%+28.81% or 44.35% of the readers would take from 23 to 29 minutes.

**Relative frequencies (*r.f.*)** often used in making statistical inferences can be read from the $T$-table. Thus, 50% of the distribution, or the probable error, falls within the range $\bar{X} \pm .67\sigma$; 95% lies within $\bar{X} \pm 1.96\sigma$; and 99% within $\bar{X} \pm 2.58\sigma$. The 3-sigma limit ($\bar{X} \pm 3\sigma$) includes all but .27% of the *n.d.*, i.e. virtually all cases. **E. FITTING A NORMAL CURVE.** Involves the calculation of theoretical frequencies ($f$) for each class interval of a given distribution that would result were the distribution normal: **(1)** Find the lower limit (*LL*) of each class. **(2)** Using the $T$-Table, find the *c.r.f.* from the mean to this point. Subtract that value from the similar value found for the class upper limit (*UL*), to find the *r.f.* in the *c.i.* **(3)** Multiply the *r.f.* by the total $N$ in the distribution to find theoretical $f$. **(4)** Repeat for all classes.

**E12:** Fit a normal curve to the income distribution of the plumbers. **Sol:** The following tabular form shows the derivation of $f$ for each class. (For clarity, the *UL* of each class is written to equal the *LL* of the next larger class.) Having previously found the mean ($6520) and *s.d.* ($2121) for the distribution, $T$ for each class limit can be found. For example,

$T$-value for $3500 is ($3500—$6520) ÷ $2121 = −1.42.

| c.i. | T of LL | c.r.f. of LL | T of UL | c.r.f. of UL | r.f. of c.i. | f | actual f |
|---|---|---|---|---|---|---|---|
| under $3,500 | −∞ | .5000 | −1.42 | .4222 | .0778 | 78 | 38 |
| 3,500 – 5,000 | −1.42 | .4222 | −.72 | .2642 | .1580 | 158 | 197 |
| 5,000 – 6,500 | −.72 | .2642 | −.09 | .0359 | .2283 | 228 | 359 |
| 6,500 – 8,000 | −.09 | .0359 | .70 | .2580 | .2939 | 294 | 176 |
| 8,000 – 9,500 | .70 | .2580 | 1.40 | .4192 | .1612 | 161 | 114 |
| 9,500 – 11,000 | 1.40 | .4192 | 2.11 | .4826 | .0634 | 63 | 82 |
| $11,000 and over | 2.11 | .4826 | ∞ | .5000 | .0174 | 17 | 34 |

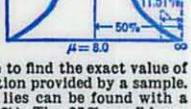
**F. DIFFERENCES BETWEEN THEORETICAL AND ACTUAL FREQUENCIES. (1)** They may merely reflect chance; with a larger amount of data, the distribution might have come closer to the normal. **(2)** The data might have come from a population group which does not follow the normal pattern. The application of **statistical inference** determines which explanation is more valid. **G. STATISTICAL INFERENCE.** Drawing conclusions about a large population from the analysis of a (small) sample. Items must be chosen on **random basis,** i.e., every member must have an equal and independent chance of being selected.

## 4 SAMPLING DISTRIBUTION.

**A. SAMPLE MEANS. (1)** The theoretical *f.d.* of means of an infinite number of samples, each of size $N$ drawn from a large population, is called the **distribution of sample means.** When the population distribution does not diverge extremely far from the normal, the sample means tend to be normally distributed as $N$ increases. (Arbitrarily, when $N > 30$ we assume a *n.d.*) **(2)** The mean of the sample means equals the mean of the population, $\mu$. The *s.d.* of the sample means, called the **standard error** ($\sigma_{\bar{x}}$), depends on the *s.d.* of the population ($\sigma$) and on the size of the sample, according to the following relationship: $\sigma_{\bar{x}} = (\sigma/\sqrt{N})$. When the *s.d.* of the population is unknown, the *s.d.* of the sample ($s$) is used in its place. (For $N > 30$, $s$ closely approximates $\sigma$). $T$ for a sample mean is the same formula as for any value of a *n.d.* Thus, $T = (\bar{X}_i - \mu) \div (\sigma)$ where $\bar{X}_i$ is the mean of a particular sample and $\mu$ the mean of the sample means.

**B. APPLICATIONS OF THE SAMPLE DISTRIBUTION.**

**E13:** If the mean diameter of a large group of rings is (or is assumed to be) 8 inches with a *s.d.* of 3 inches, what is the probability that a random sample of 36 such rings would average more than 8.6 inches in diameter? **Sol:** $\sigma_{\bar{X}} = 3/\sqrt{36} = .5$, then $T = (8.6 - 8.0) \div .50 = 1.2$. Referring to the $T$-Table, the probability that the mean of the sample would fall between the mean of the sample means, and a value 1.20$\sigma_{\bar{X}}$ greater, is 38.49%. Thus the probability that the mean will be 8.6 or more is 50%—38.49% = 11.51%.




**C. FINDING THE TRUE MEAN.** It is impossible to find the exact value of the population or true mean from information provided by a sample. But a range within which the true mean lies can be found with a definite degree of probability (95% or 99%). The **95% confidence level** indicates that, while the true mean ($\mu$) may or may not lie within the specified range, the odds are 19 to 1 that it does. Assuming a *n.d.*, the 95% limits are $\bar{X} \pm 1.96\sigma_{\bar{x}}$, since the probability that a value is more (or less) than true mean by at least $\bar{X} \pm 1.96\sigma_{\bar{x}}$ is 2½%.

**E14:** A random sample of 49 students runs 100 yds. in the average time of 12 seconds with a *s.d.* of 1 sec. Using a 95% confidence level, within what range does the true mean running time of all students lie? **Sol:** If the true mean is greater than the upper 95% limit, it can be as little as 12 sec. +1.96 (1 sec./$\sqrt{49}$) = 12.28 sec. Similarly, if the true mean is less than the lower limit, it can be as much as 12 sec. −1.96 (1 sec./$\sqrt{49}$) = 11.72 sec. Thus, the 95% range of the true mean is 11.72 ↔ 12.28 sec. There is an equal probability of 2.5% that the true mean lies above or below this range.

Widening confidence limits increases the probability that the specified range includes the true mean, but weakens the importance of the conclusion by widening the range. **D. SMALL SAMPLES.** For $N \leq 30$, the *s.d.* of the sample can no longer be used as an approximation of the population *s.d.* to find $T$. For problems involving small samples the **t-table** is used instead of the $T$-Table. As $N$ grows larger, the $t$-distribution approaches the *n.d.* Since there is, in effect, a different table for every value of $N \leq 30$, most texts provide $t$-tables which list only critical values for degrees of freedom in sample ($\mathbf{n = N - 1}$). One degree of freedom is lost when $s$ is used as an estimate of $\sigma$.

**E15:** A new process costs a firm an additional $100 per hour. In 10 tests, output per hour is increased an average of 50 units with a *s.d.* of 18 units per hour. What must be the selling price of the product for a 95% probability of making some additional profit with the new process? **Sol:** To find the lowest value of the true mean gain in production ($\mu$), put the given values in the $t$-equation:

$$t = \frac{\bar{X}_i - \mu}{\frac{s}{\sqrt{N-1}}} = \frac{50 - \mu}{\frac{18}{\sqrt{10-1}}} = \frac{50 - \mu}{6}$$

The 90% level for $t$ when $n = 9$ is 1.83. The 95% level is not used because, if there is to be a 5% chance of loss, only one side (the lower) of the symmetrical $t$-distribution relates to the problem. Thus, $1.83 = (50 - \mu)/6$ and $\mu = 39.0$ units per hour; a value which has only a 5% chance of being below the true mean gain. Since the additional cost of the new process is $100 per hour and there is a 95% probability that the gain in production is at least 39 units per hour, if the product is priced at $2.56 per unit ($100/hr. ÷ 39.0 units/hr.), the probability that the firm will make additional profits from new process is 95%.

## 5 DIFFERENCE BETWEEN TWO MEANS AND NULL HYPOTHESIS.

**A. THE NORMAL DEVIATE.** Used in testing the hypothesis that two samples were chosen from the same population. If two samples are drawn from the same population, ($\bar{X}_1 - \bar{X}_2$) is the difference between the two means. If a large number of pairs of samples are examined, the differences would be normally distributed with a mean of zero. The *s.d.* of the distribution of the difference between means ($\sigma_D$) is measured by: $\sigma_D = \sqrt{\sigma_{\bar{x}_1}^2 + \sigma_{\bar{x}_2}^2} = \sqrt{\frac{\sigma_1^2}{N_1} + \frac{\sigma_2^2}{N_2}}$.

**B. THE NULL HYPOTHESIS.** The true mean difference between the means of paired samples from a given population, is zero. A pair of samples may have different means as a result of: **(1)** the chance factors of sampling, or **(2)** the samples' coming from different populations. To test the null hypothesis, that the difference between the sample means is not significant (i.e., is due only to chance), requires setting up confidence limits in accordance with the $T$ (or $t$)-table.

**E16:** In a mechanical aptitude test, a group of 32 men score a mean of 103 with a *s.d.* of 10 and on the same test, 64 women average 95 with a *s.d.* of 12. Is there a significant sex difference in the test results?

**Sol:** $T = \frac{\bar{X}_1 - \bar{X}_2}{\sigma_D} = \frac{103 - 95}{\sqrt{(10^2/32) + (12^2/64)}} = \frac{8}{\sqrt{5.38}} = 3.45$. Value exceeds the critical value at the 5% level of 1.96. The possibility that chance alone could have accounted for the 8 point difference between the two means is too remote to be accepted; the null hypothesis is rejected. There is a significant sex difference in the test scores.

**NOTE:** Two types of error are possible in testing the null hypothesis. **Type I error:** the hypothesis may be true, yet be rejected by the test. $T$ may exceed the confidence limits, yet the true mean may be zero and the samples come from the same population. **Type II error:** the hypothesis may be false, yet be accepted by the test. $T$ may fall within the confidence limits, but the true mean may really not be zero, and the samples come from different populations.

## 6 VARIANCE ANALYSIS.

When more than two means are to be tested for significant difference, variance analysis is used. The dispersion among the means, measured by variance ($\sigma^2$), is compared with the dispersion within the classifications. If the former variance is significantly greater than the latter, the classifications represent different populations. The **F-table** provides critical values in testing whether the ratio of the two variances could be attributed to chance.

**E17:** A time-and-motion expert wishes to test whether there is any difference in speed in performing a mechanical operation using three different methods. Six workers, chosen at random, perform the operation with the results shown. **Sol:** Since the grand mean of the test (12 sec.) is used to estimate the true grand mean, a degree of freedom is lost by the three means, $N_{\bar{X}} - 1 = 2$. Since the mean of each sample is used to estimate true mean of method, a degree of freedom is lost in each sample. Therefore, degrees of freedom within the classification $= N - 3 = 15$. Thus, the variance among the means equals the sum of the squared deviations of the values, assuming each value equals the mean of its classification, divided by 2. The variance within is the sum of the squared deviations of each value about the mean of its classification divided by 15. A variance table puts the results in concise form.

| | Method | | |
|---|---|---|---|
| Worker | A | B | C |
| 1 | 10 sec. | 11 | 10 |
| 2 | 8 | 12 | 13 |
| 3 | 8 | 11 | 15 |
| 4 | 12 | 10 | 16 |
| 5 | 10 | 12 | 15 |
| 6 | 12 | 16 | 15 |

| | Sum of Squared Deviation | Degrees of Freedom | Variance |
|---|---|---|---|
| Among | 48 | 2 | 24 |
| Within | 62 | 15 | 4.13 |

$F = \frac{24}{4.13} = 5.81$

The $F$-table reveals that for 2 and 15 degrees of freedom, an $F$ as high as 5.81 would occur because of chance alone with less than 5% and with more than 1% probability. So, if the 5% level is used, we conclude that the three methods yield significantly different results; but not at the 1% level.

## 7 CHI-SQUARE.

$\chi^2$ is a measure of difference between observed and theoretical frequences. $\chi^2 = \sum\left(\frac{(f_o - f)^2}{f}\right)$. When observed ($f_o$) and theoretical ($f$) frequencies are equal, $\chi^2 = 0$. $\chi^2$ is used in testing whether actual differences are greater than could be expected on the basis of chance variations. Chi-square table gives critical values of $\chi^2$ for specified probability levels for degrees of freedom ($n$) from 1 to 30.

**E18:** Test whether there was a different reaction in two cities to an administration foreign policy measure from the following table of responses: **Sol:** $f$ is found for each classification in each city by taking the ratio which the total of each classification is to total responses, as its theoretical ratio of total responses for each city. For "disapproved" responses in City 1, $f = (280/1000)300 = 84$.

| RESPONSE | City 1 | City 2 | Total |
|---|---|---|---|
| Approved | 160 | 440 | 600 |
| Disapproved | 100 | 180 | 280 |
| No Opinion | 40 | 80 | 120 |
| | 300 | 700 | 1000 |

| City 1 | $f_o$ | $f$ | $(f_o-f)^2/f$ |
|---|---|---|---|
| Approved | 160 | 180 | 2.22 |
| Disapproved | 100 | 84 | 3.05 |
| No Opinion | 40 | 36 | .44 |
| | 300 | 300 | 5.71 |

| City 2 | $f_o$ | $f$ | $(f_o-f)^2/f$ |
|---|---|---|---|
| Approved | 440 | 420 | .95 |
| Disapproved | 180 | 196 | 1.30 |
| No Opinion | 80 | 84 | .19 |
| | 700 | 700 | 2.44 |

$\chi^2 = 5.71 + 2.44 = 8.15$

**Calculations for finding $\chi^2$.** In the six cells, four degrees of freedom were lost . . . one because the grand total sums of $f_o$ and $f$ must agree (1000), one because the sums of $f_o$ and $f$ for City 1 and City 2 must also agree (300 and 700), and two because $f_o$ and $f$ must agree for any two of the classifications (600 for approved, 280 for disapproved). Therefore, $n = 2$. For $n = 2$, the Chi-square table gives limits of $\chi^2$ of 5.991 at the 95% level and 9.210 at the 99% level. Thus, since $\chi^2 = 8.15$, the chance that there is no significant difference in reaction between the two cities is less than 5% and greater than 1%.

## 8 TIME SERIES, SECULAR TREND.

The long-term movement of a series over time reflects: **(1)** growth pattern of the variable, **(2)** variations due to cyclical and random forces, and **(3)** variations resulting from seasonal factors (when series is of monthly instead of annual data). The long-term trend of the data can be depicted by a free-hand line drawn through the data plotted on ordinary graph paper. (When the trend appears to change at a steady rate, the line should be drawn through the data plotted on semi-log paper.) **The two purposes of a trend line are: (a)** To indicate whether the value of the variable at a particular time exceeds or falls short of the value to be expected from the long-term growth experience of the variable. **(b)** To extrapolate the trend, not for the purpose of predicting the exact size of a future value, but for a standard against which the value can be compared. **B. MOVING AVERAGES.** The average value of the variable for a number of years is taken to represent the trend value for the mid-year of the span. The moving average trend value for the following year is found by omitting the first year of the previous span and adding the next after its last year. And so on.

**E19:** Data on juvenile delinquency in a community for the years 1954-1960 are given in the second column of this table. Find the 3-year moving average. **Summary:** The moving average smooths out the original data. A longer span for the average yields smoother results, but loses more years of the original data. For an even span of years, the average can be centered later by taking a two-year average of each two consecutive trend values.

| Year | Cases (per 1,000) age 14-18 yrs. | 3-year moving total | 3-year moving average |
|---|---|---|---|
| 1954 | 52 | — | — |
| 1955 | 58 | 160 | 53.3 |
| 1956 | 50 | 165 | 55.0 |
| 1957 | 57 | 172 | 57.3 |
| 1958 | 65 | 198 | 66.0 |
| 1959 | 76 | 232 | 77.3 |
| 1960 | 91 | — | — |

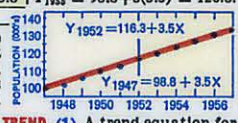
**C. THE LEAST-SQUARES LINE.** A more refined measure of linear trend. The line of least squares is that line drawn through the data such that the algebraic sum of the deviations about it is equal to zero. The sum of the squared deviations about it are less than about any other line. The line, of the form $Y_c = a+bX$, where $Y_c$ is the calculated value of the variable and $X$ the year for which $Y_c$ is calculated, is found by solving the two normal equations: $\Sigma Y = Na+b\Sigma X$ and $\Sigma XY = a\Sigma X+b\Sigma X^2$. (For non-linear trends there would be additional equations containing the $X$ terms raised to higher powers.)

**E20:** Find the least squares trend line of population changes in a city for 1947-1957 (data in fourth column of the following table). **Sol:**

| Year | X | $X^2$ | Y in 000's | X • Y | $Y_c$ |
|---|---|---|---|---|---|
| 1947 | 0 | 0 | 100 | 0 | 98.8 |
| 1948 | 1 | 1 | 102 | 102 | 102.3 |
| 1949 | 2 | 4 | 108 | 216 | 105.8 |
| 1950 | 3 | 9 | 110 | 330 | 109.3 |
| 1951 | 4 | 16 | 111 | 444 | 112.8 |
| 1952 | 5 | 25 | 113 | 565 | 116.3 |
| 1953 | 6 | 36 | 119 | 714 | 119.8 |
| 1954 | 7 | 49 | 122 | 854 | 123.3 |
| 1955 | 8 | 64 | 126 | 1008 | 126.8 |
| 1956 | 9 | 91 | 131 | 1179 | 130.3 |
| 1957 | 10 | 100 | 137 | 1370 | 133.8 |
| | 55 | 385 | 1279 | 6782 | |

$1279 = 11a+55b$
$6782 = 55a+385b$
Solving simultaneously, $a = 98.8$, $b = 3.5$
Trend equation: $Y_c = 98.8+3.5X$
Find $Y_c$ for each year by substituting value for $X$ in trend equation and solve.
For example: $Y_{1955} = 98.8+8(3.5) = 126.8$.

A shorter method of calculating the trend involves placing the 0 point for $X$ at the mean of the years. Then $\Sigma X = 0$ and both normal equations contain only one unknown each.




**D. SHIFTING FROM ANNUAL TO MONTHLY TREND. (1)** A trend equation for U.S. population, $Y_c$ for 1950 = 150+3$X$, signifies that $Y_c$ for July 1, 1950 = 150 million. To change from annual to monthly trend, it is necessary to divide the slope ($b$) of the line by 12 and advance the series ½ month to center in the middle of the month. (Add ½ of $b$/12 to $a$.)

**E21:** If U.S. population follows the trend $Y_c$ for 1950 = 150+3$X$, find the trend value for March, 1956. **Sol:** $Y_c$ for 1950 = 150 + 3$X$. Changing to a monthly basis: $Y_c = 150 + .25X_m$ (for July 1, 1950). Centering for July 15, 1960: $Y_c = 150.125 + .25X_m$. March 1956 is 68 months after July, 1950. Thus, $Y_c = 150.125 + .25\,(68) = 167.125$ million for March, 1956.

**(2)** For some series, the change from annual to monthly basis requires treatment of the fact that an annual figure represents the accumulated value of the variable for the year, as in production or sales data. In these cases, $a$ **monthly** = 1/12 $a$ **annually** and $b$ **monthly** = 1/144 $b$ **annually.**

**E22:** If steel production in the U.S. follows the trend $Y_c$ for 1950 = 72 million tons + 2.88 million tons $X$, find the trend value for February, 1962. **Sol:** $Y_c$ = 72+2.88 $X$, 2.88 is the annual change in the trend value. The monthly change in the annual value is .24. The monthly change in the monthly value is .02. On a monthly basis, $Y_c$ for July, 1950 = 6+.02$X_m$. For March 1962, 140 months later, $Y_c = 6 + .02\,(140) = 8.8$ million tons. The annual rate for the March 1962 trend value is 8.8 × 12 = 105.6 million tons.

## 9 SEASONALS.

Apart from the long-term growth factor, monthly data of time series are subject to seasonal influences: e.g., ice cream sales are greater in July than December, vice-versa for department store sales. **The seasonal component** measures the expected ratio of actual monthly values to the average monthly value over a year. An index of 60 for a month means that month's value, based on past experience, averages 60% of 1/12 the annual value. A value ($A$) is **deseasonalized** when it is divided by the seasonal component ($S$) for each month. When seasonals of a series change over time, as with industrial production, new seasonal components must be calculated.

## 10 CYCLICAL-RANDOM COMPONENT.

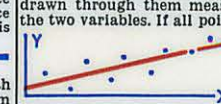
The ratio of deseasonalized actual monthly value to its trend (T) yields its **cyclical-random component.** In symbols $C = A/(T \cdot S)$. Thus if a monthly value is greater than its trend multiplied by its seasonal component, the cyclical-random component is "positive" i.e. $C > 1$, reflecting either a cyclical upswing in the series or merely a random unassignable force. Randomness is more likely present when $C$ passes unity from one month to the next: For example, a very hot July followed by a cool Aug. may result in a $C > 1$ for July beer sales and $C < 1$ for Aug.

**E23:** The trend equation for the index of department store sales is $Y_c$ for 1955 = 100 + 3.6$X$ and the seasonal component for August is 60 and for December, 180. Which was a relatively better month for sales, August 1956 when the index was 68 or December 1960, when the index was 220? **Sol:** $Y_c$ for 1955 = 100+3.6$X$. On a monthly basis, $Y_c = 100+.3X_m$ (July 1, 1955) and $Y_c = 100.15+.3X$ on July 15, 1955. For August 1956, 13 months after the base month, the trend value $Y_c = 100.15+.3\,(13) = 104.05$. The December 1960 trend value is then $Y_c = 100.15+.3\,(65) = 119.65$. Multiply the value for each month by its seasonal component: (104.05) (.60) = 62.43 [August 1956]; (119.65) (180) = 215.37 [Dec. 1960]. Divide the actual value of each month by its trend x seasonal: $\frac{68}{62.43} = 1.09$ [Aug. 1956], $\frac{220}{215.37} = 1.02$ [Dec. 1960]. Thus August 1956 was the relatively better month for sales even though its actual index was less than one-third that of Dec. 1960. It had a $C$ of 1.09, or—in the usual presentation of $C$ as a percentage above normal—+9 compared to +2 for Dec. 1960. In tabular summary:

| | A | T | S | T×S | $\frac{A}{T\times S}$ | $\left(\frac{A}{T\times S}-1\right)100$ |
|---|---|---|---|---|---|---|
| Aug. 1956 | 68 | 104.05 | 60 | 62.43 | 1.09 | +9 |
| Dec. 1960 | 220 | 119.65 | 180 | 215 37 | 1.02 | +2 |

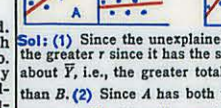
## 11 LINEAR CORRELATION. A. GENERAL CONSIDERATIONS.

If values of variable $Y$ are chosen at random and plotted on a graph in order of their selection, no discernible pattern of distribution would tend to emerge. If on the other hand, each $Y$ were associated with a variable $X$, then a pattern of $Y$'s relative to their $X$'s might appear. For example, if income ($Y$) of a group of men were plotted against their years of schooling ($X$), then $Y$ would tend to increase with $X$. That $Y$ increases as $X$ does, need not necessarily mean that $X$ causes $Y$, but merely indicates that there is "correlation" between the variables. If a scatter diagram of the pairs of $Y$'s and $X$'s is plotted (see below), the closeness of fit of the $Y$'s about the line of least squares drawn through them measures the extent of relationship between the two variables. If all points had fallen on the line, $Y$ would have been entirely "explained" by $X$, and the coefficient of correlation would have been $r = +1$. Values of $r$ are negative when $Y$ tends to decrease as $X$ increases. However, if the line of least squares had been horizontal, $Y$ would have been independent of $X$, and $r$ would have been zero. In this case, since the sum of the squared deviation of the values about the mean is less than about any other value, the equation of the line would have been $Y = \bar{Y}$. The mean of the squared deviations about the mean of $Y$ is the variance of $Y$, which equals $\sigma_Y^2$. This is the total variation of the variable $Y$. The mean of the squared deviations about the line of least squares—**the line of regression**—is the square of the standard error of estimate ($S_Y^2$). This variation remaining after the least squares line is drawn is the **"unexplained" variation of Y.** The mean of the squared deviations of the calculated values of $Y_c$ about the mean of $Y$, ($\sigma_{Y_c}^2$) is **"explained" variation of Y.** The **coefficient of determination** ($d$) is ratio of "explained" variation to total variation: $d = (\sigma_{Y_c}^2) \div (\sigma_Y^2)$.




Since $\sigma_{Y_c}^2$ is usually difficult to calculate, and since total variation is the sum of its "explained" and "unexplained" components, the alternative $d = (1-S_Y^2) \div (\sigma_Y^2)$ is used in calculation. The **coefficient of correlation** $r$, the most widely used measure of relationship, is the square root of $d$. Thus $r = \sqrt{1-(S_Y^2/\sigma_Y^2)}$.

**E24:** From the following three pairs of scatter diagrams find which of each pair reveals the greater value for $r$.




**Sol: (1)** Since the unexplained variation is the same in each case, $A$ shows the greater $r$ since it has the steeper regression line, or the greater variation about $\bar{Y}$, i.e., the greater total variation. Thus $1-(S_Y^2/\sigma_Y^2)$ is greater for $A$ than $B$. **(2)** Since $A$ has both the smaller $S_Y^2$ and the greater $\sigma_Y^2$, $A$ has the greater $r$. **(3)** $A$ has the greater $S_Y^2$, but it also has the greater $\sigma_Y^2$, so that the answer cannot be determined. While $A$ has more "unexplained" variation, its regression line explained more than $B$'s.

**B. CALCULATION. (1) Basic method.** To find $r$ from its basic formula it is necessary to find the values of its components. This entails finding $\sigma_Y^2$, the **regression equation**, and $S_Y^2$.

**E25:** Find the coefficient of correlation between scores in tests of reading ($X$) and spelling ($Y$) for a group of children (data in solution).

| X | Y | XY | $X^2$ | $Y_c$ | $(Y-\bar{Y})$ | $(Y-\bar{Y})^2$ | $(Y-Y_c)$ | $(Y-Y_c)^2$ |
|---|---|---|---|---|---|---|---|---|
| 52 | 65 | 3380 | 2704 | 66.4 | −19 | 361 | −1.4 | 1.96 |
| 58 | 72 | 4176 | 3364 | 71.2 | −12 | 144 | .8 | .64 |
| 60 | 68 | 4080 | 3600 | 72.8 | −16 | 256 | −4.8 | 23.04 |
| 64 | 83 | 5312 | 4096 | 76.0 | −1 | 1 | 7.0 | 49.00 |
| 72 | 88 | 6336 | 5184 | 82.4 | +4 | 16 | 5.6 | 31.36 |
| 76 | 78 | 5928 | 5776 | 85.6 | −6 | 36 | −7.6 | 57.76 |
| 78 | 82 | 6396 | 6084 | 87.2 | −2 | 4 | −4.2 | 27.04 |
| 82 | 93 | 7626 | 6724 | 90.4 | +9 | 81 | 2.6 | 6.76 |
| 83 | 98 | 8134 | 6889 | 91.2 | +14 | 196 | 6.8 | 46.24 |
| 85 | 94 | 7990 | 7225 | 92.8 | +10 | 100 | 1.2 | 1.44 |
| 88 | 92 | 8096 | 7744 | 95.2 | +8 | 64 | −3.2 | 10.24 |
| 90 | 95 | 8550 | 8100 | 96.8 | +11 | 121 | −1.8 | 3.24 |
| 888 | 1,008 | 76,004 | 67,490 | | | 1380 | | 258.72 |

Solving for the equation of the regression line 1008 = 12$a$+888$b$ and 76,004 = 888$a$+67,490$b$; hence $a = 24.8$, $b = .8$; $Y_c = 24.8+.8X$; the values of $X$ are substituted in equation to fill out the $Y_c$ column.

$$r = \sqrt{1-\frac{S_Y^2}{\sigma_Y^2}} = \sqrt{1-\frac{\Sigma(Y-Y_c)^2}{\Sigma(Y-\bar{Y})^2}} = \sqrt{1-\frac{259}{1380}} = \sqrt{.81} = .90$$

In the above example, good reading ability correlates with good spelling results. **C. PRODUCT MOMENT METHOD.** A shorter method for finding $r$ can be derived from the basic formula $r = \frac{P}{(\sigma_X)(\sigma_Y)}$ where $P$, **the product moment**, equals $\frac{\Sigma XY}{N} - \bar{X} \cdot \bar{Y}$. Here there is no need to find the regression equation and calculated $Y$ values.

**E26:** From the data of the preceding example, find $r$ by the product moment method.

**Sol:** $\bar{X} = 74$, $\bar{Y} = 84$ $\quad r = \frac{P}{(\sigma_X)(\sigma_Y)} = \frac{(\Sigma XY/N)-(\bar{X}\cdot\bar{Y})}{\sqrt{\left[\frac{\Sigma X^2}{N}-(\bar{X})^2\right]\cdot\left[\frac{\Sigma Y^2}{N}-(\bar{Y})^2\right]}}$

The table of the preceding example provides all of the data needed for finding $r$ except $\Sigma Y^2$. Squaring 65, 72, 68, etc., $\Sigma Y^2 = 86{,}052$. Thus,

$$r = \frac{(76{,}004/12)-(74\times 84)}{\sqrt{\left[\frac{67{,}490}{12}-(74)^2\right]\left[\frac{86{,}052}{12}-(84)^2\right]}} = \frac{118}{17{,}020} = .90$$

## 12 TESTS OF SIGNIFICANCE OF r. A. NULL HYPOTHESIS.

To conclude from a sample whether the population $r$ is significant, the null hypothesis that the population $r = 0$, is tested. For large samples, if the population $r$ were zero, $r$'s of successive samples would be normally distributed about zero; while $\sigma_r$, dependent only on the size of the sample, would be found from $\sigma_r = 1/\sqrt{N-1}$. $N$ is the number of pairs of observations in the sample.

**E27:** If from 257 observations, $r = .30$, is the null hypothesis accepted? **Sol:** $\sigma_r = \frac{1}{\sqrt{N-1}} = \frac{1}{\sqrt{257-1}} = \frac{1}{16} = .0625$. Thus, $T = \frac{.30-0}{.0625} = 4.8$. Using the 95% level, $T$ of 4.8 is greater than the critical value of 1.96. The null hypothesis is rejected. The sample $r$ of .30 indicates significant correlation For small samples, ($N \leq 30$) as in examples of statistical inference above, the $t$-table is used. Furthermore, the distribution of sample $r$'s is no longer symmetrical, and consequently $\sigma_r$ is no longer dependent on $N$ alone. $\sigma_r$ is found from the formula $\sigma_r = (\sqrt{1-r^2}) \div (\sqrt{N-2})$.

**E28:** If in a sample of 18 observations, $r$ is .4, is the $r$ significant? **Sol:** $\sigma_r = \frac{\sqrt{1-(.4)^2}}{\sqrt{18-2}} = .23$; $t = \frac{.4-0}{.23} = 1.74$ which is less than the critical value of 2.12 for $n = 16$. (Note: degrees of freedom = $N-2$.) Thus, the null hypothesis is accepted; the data does not refute the hypothesis that the true population $r$ is zero.

**B. THE TRUE VALUE OF r AND THE Z-TRANSFORMATION.** Since the range of $r$ extends only from −1 to +1, the distribution of sample $r$'s for any population value of $r$, other than zero, is not symmetrical. Therefore, use is made of the **Z-transformation** to find values of $Z$ corresponding to all possible values of $r$. The two important attributes of $Z$ that make it useful in finding the range of the true value of $r$ from a sample are: **(a)** The $Z$-distribution is approximately normal regardless of the true value of $r$. **(b)** The standard deviation of $Z$ is dependent only on $N$; $\sigma_Z = 1/\sqrt{N-3}$.

**E29:** Find the range within which the population $r$ lies with 95% confidence from a sample of 67 observations having an $r$ of .56. **Sol:** From the $Z$-table, $r$ of .56 corresponds to $Z$ of .63; $\sigma_Z = \frac{1}{\sqrt{67-3}} = .125$. Using the 95% confidence level, the range for $Z$ is .63 ±.125(1.96) = .38 ↔ .88. Transforming range for $Z$ to a range for $r$, $Z$-table yields a range for population $r$ of .36 ↔ .71.

**C. RELIABILITY AND VALIDITY.** The conclusions of a sample are reliable if similar conclusions can be expected to be reached on the basis of another sample taken from the population. A sample coefficient of correlation may be considered reliable if the range of the true $r$ based on it is $0 < \text{true } r < 1$ or $-1 < \text{true } r < 0$. Clearly, under this criterion, a sample is unreliable if it satisfies the null hypothesis. A test is valid if it yields accurate measurements of the quality it attempts to measure. Applying this concept to correlation, if one wishes to measure musical ability in a large number of children, and the most reliable test would be too long, he could design two (or more) shorter tests. He could correlate their results to the long standard test, for a small number of children, and use whichever short test showed the greatest correlation with the standard test.

## 13 RANK CORRELATION.

It is impossible to find $r$ for variables which are not numerical, such as behavior, attitude, etc. But these variables may still be ranked according to their standing. The **coefficient of rank correlation** $\rho$ is found according to the formula $\rho = 1-\frac{6\Sigma D^2}{N(N^2-1)}$, where $D$ is the difference in ranking between a pair of variables.

**E30:** Find $\rho$ for the rankings of ambition and ability for the six workers in this table. **Summary:** $\rho$ may also be used as a quick, but rough approximation to $r$ for measurable data. The major shortcoming of $\rho$ as a measure of relationship is that a high value for $\rho$ may obscure substantial differences within rankings.

| Ambition | Ability | D | $D^2$ |
|---|---|---|---|
| 1 | 2 | −1 | 1 |
| 2 | 3 | −1 | 1 |
| 3 | 1 | 2 | 4 |
| 4 | 6 | −2 | 4 |
| 5 | 4 | 1 | 1 |
| 6 | 5 | 1 | 1 |
| | | | 12 |

$\rho = 1-\frac{6(12)}{6(36-1)} = .66$

**E31:** In correlating wages with experience, if worker $A$ with 1 year's experience, earns $1.30 per hour, $B$ has 1.1 year's experience and $3.00 per hour, and $C$ 12 years' experience and $3.10 per hour, the ranks are in perfect agreement, but no strong relationship between experience and wages is indicated.

## 14 INDEX NUMBERS. A. SIMPLE PRICE INDEXES. (1) Calculation.

A price index is a measure of a relative change in a group of prices from the **base period** to another period. **Simple or unweighted indexes** do not take into account the relative importance of each of the items that comprise the index. The unweighted arithmetic mean of price relatives. **U.A.M.R.** = $\Sigma(P_1/P_0) \div N$, where $P_1/P_0$ is a price relative or ratio of a price in a given period to its value in the base period multiplied by 100, is really weighted by the amount of each item that can be bought for $100.

**E32:** Find the unweighted arithmetic mean of price relatives in 1960 for the foods included in this table, 1950 as base.

| FOOD | P1950 | P1960 | $\frac{P1960}{P1950}$ |
|---|---|---|---|
| eggs(doz.) | .50 | .60 | 120 |
| butter(lb.) | .75 | .75 | 100 |
| milk (qt.) | .20 | .25 | 125 |
| | 1.45 | 1.60 | 345 |

**The unweighted aggregative index** relates the sum of the prices of the components of the index in a given time period to their sum during the base period **U.A.I.** = $\Sigma P_1/\Sigma P_0$. The items in this "unweighted" index are in effect weighted by the magnitude of their price.

**U.A.M.R.** = $\frac{\Sigma \frac{P_1}{P_0}}{N} = \frac{345}{3} = 115.0$

**E33:** Find the *U.A.I.* for food prices from the data of the preceding problem. **Sol:** **U.A.I.** = $\Sigma P_1/\Sigma P_0 = 160/145 = 110.3$.

The implicit weighting of the two simple indexes explains the different results of the examples. The most expensive item, butter, is weighted more heavily in the *U.A.I.* than in the *U.A.M.R.* Since butter did not change in price over the period and so showed the least relative increase of the three items, the index that weighted it more heavily gave the smaller increase for the price index —110.3 for the *U.A.I.* compared to 115.0 for the *U.A.M.R.* **(2) The time reversal test.** Individual price relatives meet the time reversal test in that the relative of $P_1$ with $P_0$ as base multiplied by the relative of $P_0$ with $P_1$ as base always equals 100. The *U.A.M.R.* meets the time reversal test but the *U.A.M.R.* does not.

**E34:** For eggs in E32; $(P_1/P_0)(P_0/P_1) = 120 \times 833 = 100$. The *U.A.I.* meets this test, but the *U.A.M.R.* does not. With 1960 as base, the *U.A.M.R.* index for 1950 is 87.7. The time reversal value of 115.0 × 87.7 = 108.7, demonstrates the upward type bias of this index.

**B. WEIGHTED PRICE INDEXES.** Weighting an index gives each item an importance in determining the value of the index equivalent to its relative importance among the items. When each price is weighted by quantity, the resultant value represents the item's share of the total value of all the items that comprise the index. **C. THE LASPEYRES AND PAASCHE INDEXES. The Laspeyres Index** weights the price of each item by its quantity sold in the base period; $L = (\Sigma P_1 \cdot Q_0) \div (\Sigma P_0 \cdot Q_0)$. **The Paasche formula** weights each item by its quantity sold in the given year; $P = (\Sigma P_1 \cdot Q_1) \div (\Sigma P_0 \cdot Q_1)$. **E35:** Find the Laspeyres and Paasche index for the items in E32. (Quantity data in the table below.) **Sol:**

| FOOD | P1950 | Q1950 | P1960 | Q1960 | P1950 • Q1950 | P1960 • Q1950 |
|---|---|---|---|---|---|---|
| eggs(doz) | .50 | 20 | .60 | 30 | 100 | 120 |
| butter(lb) | .75 | 10 | .75 | 12 | 75 | 75 |
| milk(qt) | .20 | 40 | .25 | 60 | 80 | 100 |

| P1960 • Q1960 | P1950 • Q1960 |
|---|---|
| 180 | 150 |
| 90 | 90 |
| 150 | 120 |

$L = \frac{\Sigma P1960 \cdot Q1950}{\Sigma P1950 \cdot Q1950} = \frac{295}{255} = 115.7$

$P = \frac{\Sigma P1960 \cdot Q1960}{\Sigma P1950 \cdot Q1960} = \frac{420}{360} = 116.6$

The use of **given year weights** (*Paasche Index*) gives the index an **upward bias**; while **base year weights** (*Laspeyres Index*) give a **downward bais.** But, since the Laspeyres is equivalent to an arithmetic average with an upward type bias, and the Paasche is equivalent to a harmonic average with a downward type bias, according to the time reversal test each of the indexes contains both an upward and downward bias. Thus, the two forms tend to yield similar index values.

**D. THE FACTOR REVERSAL TEST AND THE IDEAL INDEX.** If the factor reversal test is met, the change in value for the items in the index from the base period to the given year period should equal the product of the price and quantity index over the period.

Thus, $\frac{\Sigma P_1 \cdot Q_1}{\Sigma P_0 \cdot Q_0} = \frac{\Sigma P_1 \cdot Q_0}{\Sigma P_0 \cdot Q_0} \times \frac{\Sigma Q_1 \cdot P_0}{\Sigma Q_0 \cdot P_0}$ for Laspeyres index to meet the test. For Paasche index, $\frac{\Sigma P_1 \cdot Q_1}{\Sigma P_0 \cdot Q_0} = \frac{\Sigma P_1 \cdot Q_1}{\Sigma P_0 \cdot Q_1} \times \frac{\Sigma Q_1 \cdot P_1}{\Sigma Q_0 \cdot P_1}$.

**E36:** Make the factor reversal test for both indexes from data in the above example. **Sol:** $\frac{\Sigma P_1 \cdot Q_1}{\Sigma P_0 \cdot Q_0} = \frac{\Sigma P1960 \cdot Q1960}{\Sigma P1950 \cdot Q1950} = \frac{420}{255} = 164.7$

| Testing the Laspeyres Index | Testing the Paasche Index |
|---|---|
| $\frac{\Sigma P1960 \cdot Q1950}{\Sigma P1950 \cdot Q1950} \times \frac{\Sigma Q1960 \cdot P1950}{\Sigma Q1950 \cdot P1950}$ | $\frac{\Sigma P1960 \cdot Q1960}{\Sigma P1950 \cdot Q1960} \times \frac{\Sigma Q1960 \cdot P1960}{\Sigma Q1950 \cdot P1960}$ |
| = 115.7 × (360/255) = 163.4 | = 116.6 × (420/295) = 166.0 |

Neither index meets the test. The Laspeyres form falls short by 1.3 points, the same magnitude by which the Paasche form exceeds the test. The ideal index, the square root of the product of the two forms of the weighted index, meets the factor reversal test:

$(\Sigma P_1 Q_1) \div (\Sigma P_0 Q_0) = \sqrt{L} \cdot P$ for price × $\sqrt{L} \cdot P$ for quantities.

**E37:** Perform the factor reversal tests for the ideal index from the above data. **Sol:** $(\Sigma P_1 Q_1) \div (\Sigma P_0 Q_0) = 164.7$; $\sqrt{L.P.}$ for price = $\sqrt{115.7 \times 116.6} = \sqrt{134.9}$; $\sqrt{L.P.}$ for quantities = $\sqrt{(366/255) \times (420/295)} = \sqrt{201.1}$; $\sqrt{L.P.}$ for prices multiplied by $\sqrt{L.P.}$ for quantities = $\sqrt{134.9}$ $\sqrt{201.1} = \sqrt{271.3} = 164.7$. The ideal index meets the factor reversal test.

**E. LINK RELATIVES.** A price index relates prices at a given time to prices in the base period. **Link relatives** relate price changes to the price level of the preceding year. Thus, they give year-to-year percentage changes in price.

**E38:** From the following table of consumers price index, find the link relatives for each year. **Summary:** The link relatives for each year were found from the ratio (multiplied by 100) of a year's index to that of the preceding year. For example, the link relative for 1957 was (120.2/116.2) × (100) = 103.4.

| Year | C.P.I. (1947-1949 = 100) | Link Relatives |
|---|---|---|
| 1953 | 114.4 | .... |
| 1954 | 114.8 | 100.3 |
| 1955 | 114.5 | 99.7 |
| 1956 | 116.2 | 101.5 |
| 1957 | 120.2 | 103.4 |
| 1958 | 123.5 | 102.7 |
| 1959 | 124.6 | 100.9 |

**Data-Guide's®**
**PLASTIC REFERENCE CHARTS**
*To Learn Faster, Remember More!*

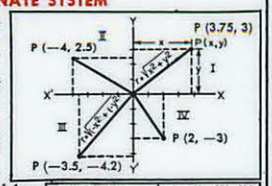
# Trigonometry

**AUTHOR**
*Prof. Arnold Fass of Queens College, N. Y.*
**Data-Guide Editor**
*Joseph L. Leon*

Published and MF'd by Data-Guide, Inc. Flushing, N. Y.


## I. RECTANGULAR COORDINATE SYSTEM

**A. POSITION OF A POINT.** **1. The ABSCISSA** or $x$-coordinate of any point is its horizontal distance from the $y$-axis; "+" if to the right of the $y$-axis, "−" if to the left. **2. The ordinate** or $y$-coordinate of any point is its vertical distance from the $x$-axis; "+" if above the $x$-axis, "−" if below. A point is then identified by the pair of numbers $(x, y)$ called its *coordinates*. **3. The radius vector** of any point is the line segment from the *origin* to the point; its length "$r$" is always "+". If the point has coordinates $(x, y)$, $\mathbf{r = \sqrt{x^2 + y^2}}$.




| QUADRANT | I | II | III | IV |
|---|---|---|---|---|
| ABSCISSA ($x$) | + | − | − | + |
| ORDINATE ($y$) | + | + | − | − |
| RADIUS VECTOR ($r$) | + | + | + | + |

**II. ANGLES. A. DEFINITION. 1.** Angle is formed when two rays are drawn from a common point, the *vertex*. An angle is measured by the amount of rotation from one of the lines (the initial line) to the other (the terminal line). Rotation in the *ccw* direction produces a *positive angle*; in the *cw* direction, a *negative angle*. **2. The standard position of an angle:** vertex at the origin, initial line the positive part of the $x$-axis.

**B. DEGREE.** The degree is divided into sixty minutes (60′); each minute into sixty seconds (60″). A circle contains 360°.

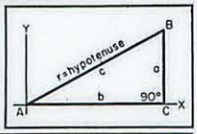
**C. RADIAN.** An angle that has its vertex at the center of a circle and its sides intercepting an arc equal in length to the radius measures one radian. *A circle contains $2\pi$ radians (360°).*

**D. MEASUREMENT OF ANGLES.** Either in degrees or radians.
**1 radian** = $180°/\pi = 57.296° = 57° 17′ 45″$
**1 degree** = $\pi/180° = .0175$ radians; $1′ = .0002909$ radians
**number of degrees ÷ 180 = number of radians ÷ $\pi$**
**1. Degrees ➔ radians,** multiply degrees by $\pi/180°$ (= .0175) or divide by 57.3. **E:** $315° = 315 \times \pi/180 = 7\pi/4$ radians. **E:** $12° = 12 \times .01745 = .2094$ radians.
**2. Radians ➔ degrees,** multiply radians by $180/\pi$ or 57.3. **E:** $3\pi/2$ rad $= 3\pi/2 \cdot 180/\pi = 270°$ **E:** .8 rad = (.8) (57.3) = 45.84°

**E. ARC LENGTH.** In a circle of radius $r$, a central angle of $\theta$ radians intercepts an arc of length $\mathbf{s = r\theta}$ and $\theta = \mathbf{s/r}$. **E:** Find the length of the arc intercepted by a central angle of 18° in a circle of radius 5 in. **Sol:** $18° = \pi/10$ rad; $s = 5 \times \pi/10 = 1.57$ in.

## III. TRIGONOMETRIC FUNCTIONS

**A. DEFINITION.** The ratios of sides of right triangles or of distances in a rectangular coordinate system. **NOTE:** In all triangles, a capital letter will denote an angle; the same lower case letter will denote the side opposite that angle.




| FUNCTION | ANGLE IN STANDARD POSITION | ANGLE IN A RIGHT TRIANGLE |
|---|---|---|
| sine $\theta$ (sin) | ordinate / radius vector $= \frac{y}{r}$ | side opposite / hypotenuse $= \frac{a}{c}$ |
| cosine $\theta$ (cos) | abscissa / radius vector $= \frac{x}{r}$ | side adjacent / hypotenuse $= \frac{b}{c}$ |
| tangent $\theta$ (tan) | ordinate / abscissa $= \frac{y}{x}$ | side opposite / side adjacent $= \frac{a}{b}$ |
| cotangent $\theta$ (cot) | abscissa / ordinate $= \frac{x}{y}$ | side adjacent / side opposite $= \frac{b}{a}$ |
| secant $\theta$ (sec) | radius vector / abscissa $= \frac{r}{x}$ | hypotenuse / side adjacent $= \frac{c}{b}$ |
| cosecant $\theta$ (csc) | radius vector / ordinate $= \frac{r}{y}$ | hypotenuse / side opposite $= \frac{c}{a}$ |

## IV. ACUTE ANGLES. A. DEFINITION.

Angles between 0° and 90° are *acute*. Two acute angles whose sum is 90° are *complementary*: the two acute angles in a right triangle ($\mathbf{A + B = 90°}$).

$\sin A = \frac{a}{c} = \cos B = \cos(90° - A)$ | $\cos A = \frac{b}{c} = \sin B = \sin(90° - A)$

$\tan A = \frac{a}{b} = \cot B = \cot(90° - A)$ | $\cot A = \frac{b}{a} = \tan B = \tan(90° - A)$

$\sec A = \frac{c}{b} = \csc B = \csc(90° - A)$ | $\csc A = \frac{c}{a} = \sec B = \sec(90° - A)$

**B. VARIATION OF TRIGONOMETRIC FUNCTIONS.** Between 0° and 90°, the sin, tan, and sec of an angle *increase* as the angle *increases*; the cos, cot, and csc *decrease* as the angle *increases*.

**C. USING TRIG. TABLES. 1. Given angle,** find function. **E:** Find the cosine of 56°43′.

| ANGLE | COSINE |
|---|---|
| 56°50′ | 0.5471 |
| 56°43′ | cosine |
| 56°40′ | 0.5495 |

10 : 3 :: $d$ : −.0024

Write: $\frac{3}{10} = \frac{d}{-.0024}$; $d = -0.00072$
Cos 56°43′ = 0.5495 − 0.0007 = 0.5488

**2. Given function,** find angle. **E:** Find $A$ if $\sin A = 0.4454$.

| ANGLE | SINE |
|---|---|
| 26°30′ | 0.4462 |
| angle | 0.4454 |
| 26°20′ | 0.4436 |

10 : $d$ :: .0026 : .0018

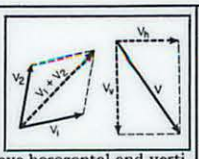
Write: $\frac{d}{10} = \frac{.0018}{.0026}$; $d = 7$
Angle $A$ = 26°20′ + 7′ = 26°27′

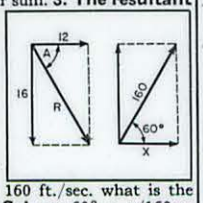
**D. SOLVING RIGHT TRIANGLES.** If two sides, or one side and one acute angle of a right triangle are known, the remaining sides and angles can be determined. **PROCEDURE:** Draw the triangle; label the parts known and the parts wanted. Choose the applicable formulas listed in IV:A then substitute known values and solve. You may also use the Pythagorean theorem: $\mathbf{c^2 = a^2 + b^2}$.

**E. ACCURACY OF COMPUTED RESULTS FOR ALL TRIANGLES.** (The letters S.F. stand for significant figures.)

| S.F. IN SIDE LENGTHS | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| ANGLES TO NEAREST | deg. | ten min. | min. | .1′ (5 place tables) |
| S.F. IN RESULTS | 2 | 3 | 4 | 5 |

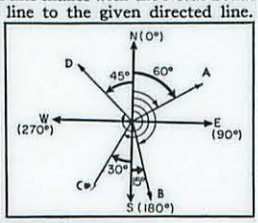
**F. VECTORS.** Quantities that have both magnitude and direction, e.g. force, velocity, distance of a point from the origin (radius vector), etc. **1. The sum** of two vectors can be represented as the diagonal of a parallelogram in which the two vectors form the adjacent sides. **2. Components.** Any vector may be shown to have horozontal and vertical vector components, and equals their sum. **3. The resultant** of two forces is the single force which can produce the same effect on a body that the two original forces do and is treated as a vector sum. **E1:** What single force would produce the same effect as a horozontal pull of 12 lbs. and a downward pull of 16 lbs. **Sol:** $R = \sqrt{(12)^2 + (16)^2} = 20$ lbs; $\tan A = 16/12 = 53° 10′$. **E2:** A gun muzzle is elevated 60° from the horozontal. If the muzzle velocity of a projectile is 160 ft./sec. what is the horozontal component of the velocity. **Sol:** $\cos 60° = x/160$ so that $160 \cos 60° = 160\ (.5000) = 80$.




**G. NAVIGATION.** *Bearing* and *course* indicates direction. **1. Bearing.** The acute angle a directed line makes with the *North-South line*; measured from the $N$-$S$ line to the given directed line. **2. Course.** The angle a directed line makes with the *North direction*; measured clockwise from North to the directed line.

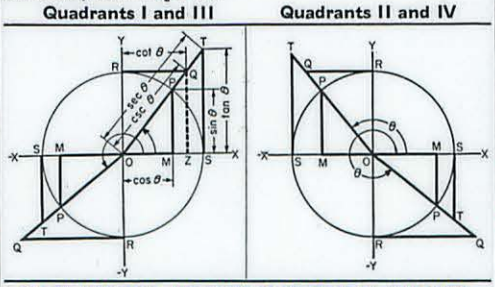
| | BEARING | COURSE |
|---|---|---|
| A | N60°E | 60° |
| B | S15°E | 165° |
| C | S30°W | 210° |
| D | N45°W | 315° |




## V. FUNCTION VARIATION

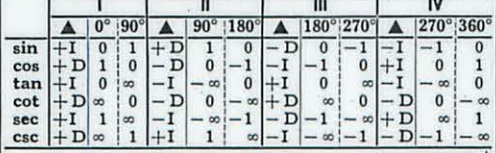
**A. CIRCULAR REPRESENTATION.** In a circle of unit radius (1) $OP = OR = OS = 1$. In each diagram, if $OP$ is the terminal side of the angle: $\sin\theta = MP$, $\csc\theta = OQ$, $\cos\theta = OM$, $\sec\theta = OT$, $\tan\theta = ST$, $\cot\theta = RQ = OZ$.

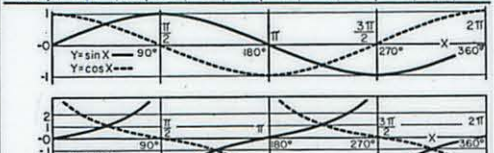
Quadrants I and III | Quadrants II and IV

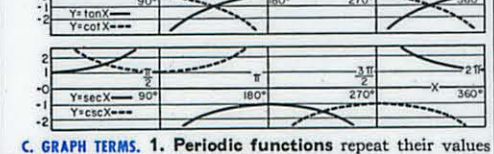



**B. SIGN, VARIATION, AND RANGE OF FUNCTIONS BY QUADRANTS.** NOTE: ▲-column shows sign of function ("+" or "−") and whether it increases (I) or decreases (D).

| | I | | | II | | | III | | | IV | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | ▲ | 0° | 90° | ▲ | 90° | 180° | ▲ | 180° | 270° | ▲ | 270° | 360° |
| sin | +I | 0 | 1 | +D | 1 | 0 | −D | 0 | −1 | −I | −1 | 0 |
| cos | +D | 1 | 0 | −D | 0 | −1 | −I | −1 | 0 | +I | 0 | 1 |
| tan | +I | 0 | ∞ | −I | −∞ | 0 | +I | 0 | ∞ | −I | −∞ | 0 |
| cot | +D | ∞ | 0 | −D | 0 | −∞ | +D | ∞ | 0 | −D | 0 | −∞ |
| sec | +I | 1 | ∞ | −I | −∞ | −1 | −D | −1 | −∞ | +D | ∞ | 1 |
| csc | +D | ∞ | 1 | +I | 1 | ∞ | −I | −∞ | −1 | −D | −1 | −∞ |

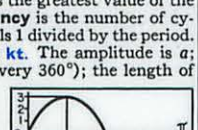


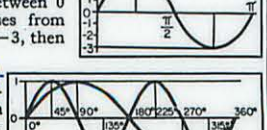




**C. GRAPH TERMS. 1. Periodic functions** repeat their values at regular intervals called *periods*: $\sin x$ and $\cos x$ are periodic at 360° ($2\pi$ rad.); $\tan x$ at 180° ($\pi$ rad.). The period is the same as the cycle. **2. Continuous functions** have unbroken curves ($\sin x$, $\cos x$) giving specific values for every angle. **Discontinuous functions** have broken curves ($\tan x$) and do not have values for each angle. **3. Amplitude** is the greatest value of the ordinate of a function. **4. The frequency** is the number of cycles of the function within 360°; it equals 1 divided by the period.

**D. FUNCTIONS $s = a \sin kt$, $s = a \cos kt$.** The amplitude is $a$; the frequency equals $k/2\pi$ ($k$ cycles every 360°); the length of the period is therefore $2\pi/k$. **E:** Graph $y = 3 \sin 2x$. **Sol:** Period is $2\pi/2 = \pi$; the amplitude is 3. Thus between 0 and $\pi$ the function increases from 0 to 3 and then decreases to −3, then returns to 0.




**E. GRAPH SOL. OF TRIG. EQUATIONS. E:** Find values of $\theta$ between 0° and 360° for which $\sin x = \sin 2x$. **Sol:** $\sin x = \sin 2x$ at $x$ = 0°, 60°, 180°, 300°, 360°.




| θ(deg.) | θ(rad.) | sin θ | cos θ | tan θ | cot θ | sec θ | csc θ |
|---|---|---|---|---|---|---|---|
| 0° | 0 | 0 | 1 | 0 | — | 1 | — |
| 30° | $\frac{\pi}{6}$ | 1/2 .500 | $\sqrt{3}/2$ .866 | $\sqrt{3}/3$ .577 | $\sqrt{3}$ 1.732 | $2\sqrt{3}/3$ 1.155 | 2 2.000 |
| 45° | $\frac{\pi}{4}$ | $\sqrt{2}/2$ .707 | $\sqrt{2}/2$ .707 | 1 1.000 | 1 1.000 | $\sqrt{2}$ 1.414 | $\sqrt{2}$ 1.414 |
| 60° | $\frac{\pi}{3}$ | $\sqrt{3}/2$ .866 | 1/2 .500 | $\sqrt{3}$ 1.732 | $\sqrt{3}/3$ .577 | 2 2.000 | $2\sqrt{3}/3$ 1.155 |
| 90° | $\frac{\pi}{2}$ | 1 | 0 | — | 0 | — | 1 |
| 120° | $\frac{2\pi}{3}$ | $\sqrt{3}/2$ .866 | −1/2 −.500 | $-\sqrt{3}$ −1.732 | $-\sqrt{3}/3$ − .577 | −2 −2.000 | $2\sqrt{3}/3$ 1.155 |
| 150° | $\frac{5\pi}{6}$ | 1/2 .500 | $-\sqrt{3}/2$ −.866 | $-\sqrt{3}/3$ − .577 | $-\sqrt{3}$ −1.732 | $-2\sqrt{3}/3$ −1.155 | 2 2.000 |
| 180° | $\pi$ | 0 | −1 | 0 | — | −1 | — |
| 210° | $\frac{7\pi}{6}$ | −1/2 −.500 | $-\sqrt{3}/2$ −.866 | $\sqrt{3}/3$ .577 | $\sqrt{3}$ 1.732 | $-2\sqrt{3}/3$ −1.155 | −2 −2.000 |
| 240° | $\frac{4\pi}{3}$ | $-\sqrt{3}/2$ −.866 | −1/2 −.500 | $\sqrt{3}$ 1.732 | $\sqrt{3}/3$ .577 | −2 −2.000 | $-2\sqrt{3}/3$ −1.155 |
| 270° | $\frac{3\pi}{2}$ | −1 | 0 | — | 0 | — | −1 |
| 300° | $\frac{5\pi}{3}$ | $-\sqrt{3}/2$ −.866 | 1/2 .500 | $-\sqrt{3}$ −1.732 | $-\sqrt{3}/3$ − .577 | 2 2.000 | $-2\sqrt{3}/3$ −1.155 |
| 330° | $\frac{11\pi}{6}$ | −1/2 −.500 | $\sqrt{3}/2$ .866 | $-\sqrt{3}/3$ − .577 | $-\sqrt{3}$ −1.732 | $2\sqrt{3}/3$ 1.155 | −2 −2.000 |
| 360° | $2\pi$ | 0 | 1 | 0 | — | 1 | — |

## VI. TRIG. FUNCTIONS (TF) OF ANY ANGLE

**A. COMPLEMENTS.** Any function of a positive acute angle is equal to the cofunction of the complementary angle (see IV:A).

**B. THE RELATED ANGLE ($\theta_r$).** An angle may be expressed as a multiple of 180° plus or minus an acute angle. This acute angle is called the related angle: $\theta = \mathbf{(n \cdot 180°) \pm (\theta_r)}$ **E:** $91° = (180°) - (89°)$; $205° = (180°) + (25°)$; $680° = (4 \cdot 180°) - (40°)$. The angle $\theta_r$ is the positive acute angle that must be added to or subtracted from $\theta$ to obtain a multiple of 180°.

**THE RELATED ANGLE BY QUADRANTS**

| | |
|---|---|
| QUADRANTS I AND III | $\theta = (n \cdot 180°) + (\theta_r)$ $\quad \theta_r = \theta - (n \cdot 180°)$ |
| QUADRANTS II AND IV | $\theta = (n \cdot 180°) - (\theta_r)$ $\quad \theta_r = (n \cdot 180°) - \theta$ $= (n \cdot 179°60′) - \theta$ $= (n \cdot 179°59′60″) - \theta$ |

**C. TF OF ANY POSITIVE ANGLE, RELATED ANGLE METHOD. 1.** Determine the quadrant containing the terminal side of the angle $\theta$ and obtain the proper sign for TF($\theta$). **2.** Compute the related angle $\theta_r$. **3.** Affix the sign of TF($\theta$) obtained in Step 1 to TF($\theta_r$). **E:** $\sin 335°$ **Sol:** $\theta$ in quadrant IV, sin negative; $\theta_r = (2 \cdot 180°) - (335°) = 25°$; thus $\sin 335° = -\sin 25° = -.4226$ **E:** $\tan 600°$ **Sol:** $\theta$ in quadrant III, tan positive; $\theta_r = 600° - (3 \cdot 180°) = 60°$; thus $\tan 600° = \tan 60° = 1.7321$ **E:** $\cos 123°16′53″$ **Sol:** $\theta$ in quadrant II, cos negative; $\theta_r = (179°59′60″ - 123° 16′53″) = 56°43′7″$; thus $-\cos 56°43′7″$. Interpolate:

60 : 7 :: $x$ : 24

| ANGLE | COSINE |
|---|---|
| 56°43′ | 0.54878 |
| 56°43′7″ | cosine |
| 56°44′ | 0.54854 |

Write: $\frac{7}{60} = \frac{x}{24}$; $x = 2.8$ or 3

$\cos 123°16′53″ = -0.54875$

**D. NEGATIVE ANGLES.** Change to functions of positive angles:

| FUNCTION SIGN CHANGES | FUNCTION SIGN UNCHANGED |
|---|---|
| $\sin(-\theta) = -\sin\theta$ $\quad \tan(-\theta) = -\tan\theta$ | $\cos(-\theta) = \cos\theta$ |
| $\csc(-\theta) = -\csc\theta$ $\quad \cot(-\theta) = -\cot\theta$ | $\sec(-\theta) = \sec\theta$ |

**E:** $\sin(-215°)$ **Sol:** $\sin(-215°) = -\sin 215°$; $\theta$ in quadrant III, sin negative; $\theta_r = 215° - 180° = 35°$; thus $-\sin 215° = -(-\sin 35°) = -(-.5736) = +.5736$ **E:** $\cos(-183°) = +\cos 183° = -\cos(183° - 180°) = -\cos 3°$.

**E. INVERSE TRIG. FUNCTIONS. 1. Definition.** If $y = \sin\theta$, then $\theta$ is an angle whose sin is "$y$"; written $\theta = \arcsin y$ or $\theta = \sin^{-1} y$ (arc cos $\theta$, arc tan $\theta$, arc cot $\theta$, arc sec $\theta$, arc csc $\theta$). Arc cos 1/2 is an angle whose cos is 1/2; $\tan^{-1}\sqrt{3}$ is an angle whose tangent is $\sqrt{3}$. Since trig. functions are periodic there are an infinite number of angles for any given function value. **E:** arc cos 1/2 **Sol:** $\cos 60° = \cos 300° = \cos(-60°) = \cos 420°$. . etc. $= 1/2$. In fact $\cos(n \cdot 360° \pm 60°) = 1/2$. **2. Principal value** of an inverse trig. function is the angle with the smallest numerical value. Where a positive and a negative angle are numerically equal but smaller than any other, positive angle is principal one.

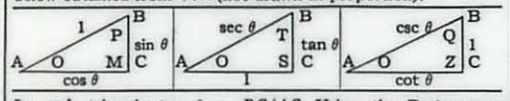
| FUNCTION | arc sin θ / arc csc θ | arc tan θ / arc cot θ | arc cos θ / arc sec θ |
|---|---|---|---|
| PRINCIPAL VALUE | between −90° and +90° | | between 0° and 180° |

**E:** Principal value of arc cos 1/2 is 60°. **E:** Find the principal value of arc tan $(-\sqrt{3}/3)$ **Sol:** $\tan 30° = \sqrt{3}/3$; $\tan(-30°) = -\sqrt{3}/3$; arc tan $-\sqrt{3}/3 = -30°$ **E:** $\sin^{-1}(2)$ No solution; $\sin x$ between −1 and +1. **E:** Arc sin (.4147) **Sol:** 24° 30′

## VII. FUNDAMENTAL RELATIONSHIPS

| FUNCTION | RECIPROCAL | RATIO OR QUOTIENT | PRODUCT | PYTHAGOREAN |
|---|---|---|---|---|
| $\sin\theta = \frac{y}{r}$ | $\frac{1}{\csc\theta}$ | $\frac{\cos\theta}{\cot\theta}$; $\frac{\tan\theta}{\sec\theta}$ | $\cos\theta \tan\theta$ | $\sin^2\theta = 1 - \cos^2\theta$ $\quad \sin\theta = \pm\sqrt{1 - \cos^2\theta}$ |
| $\cos\theta = \frac{x}{r}$ | $\frac{1}{\sec\theta}$ | $\frac{\cot\theta}{\csc\theta}$; $\frac{\sin\theta}{\tan\theta}$ | $\sin\theta \cot\theta$ | $\cos^2\theta = 1 - \sin^2\theta$ $\quad \cos\theta = \pm\sqrt{1 - \sin^2\theta}$ |
| $\tan\theta = \frac{y}{x}$ | $\frac{1}{\cot\theta}$ | $\frac{\sin\theta}{\cos\theta}$; $\frac{\sec\theta}{\csc\theta}$ | $\sin\theta \sec\theta$ | $\tan^2\theta = \sec^2\theta - 1$ $\quad \tan\theta = \pm\sqrt{\sec\theta^2 - 1}$ |
| $\cot\theta = \frac{x}{y}$ | $\frac{1}{\tan\theta}$ | $\frac{\cos\theta}{\sin\theta}$; $\frac{\csc\theta}{\sec\theta}$ | $\cos\theta \csc\theta$ | $\cot^2\theta = \csc\theta^2 - 1$ $\quad \cot\theta = \pm\sqrt{\csc^2\theta - 1}$ |
| $\sec\theta = \frac{r}{x}$ | $\frac{1}{\cos\theta}$ | $\frac{\tan\theta}{\sin\theta}$; $\frac{\csc\theta}{\cot\theta}$ | $\csc\theta \tan\theta$ | $\sec^2\theta = \tan^2\theta + 1$ $\quad \sec\theta = \pm\sqrt{\tan^2\theta + 1}$ |
| $\csc\theta = \frac{r}{y}$ | $\frac{1}{\sin\theta}$ | $\frac{\sec\theta}{\tan\theta}$; $\frac{\cot\theta}{\cos\theta}$ | $\cot\theta \sec\theta$ | $\csc^2\theta = \cot^2\theta + 1$ $\quad \csc\theta = \pm\sqrt{\cot^2\theta + 1}$ |

**A. EXPRESSING ONE FUNCTION IN TERMS OF ANOTHER.** Triangles below obtained from V: A (not drawn in proportion).




In each triangle $\tan\theta = BC/AC$. Using the Pythagorean relationship and the sides of the function triangles, $\tan\theta = \dfrac{\sin\theta}{\pm\sqrt{1-\sin^2\theta}} = \dfrac{\pm\sqrt{1-\cos^2\theta}}{\cos\theta} = \pm\sqrt{\sec^2\theta - 1}$; etc.

**B. SUMMARY.** Sign is determined by terminal quadrant of function. Relationships true for all quadrants.

| | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ | $\cot\theta$ | $\sec\theta$ | $\csc\theta$ |
|---|---|---|---|---|---|---|
| $\sin\theta$ | | $\pm\sqrt{1-\cos^2\theta}$ | $\pm\dfrac{\tan\theta}{\sqrt{1+\tan^2\theta}}$ | $\pm\dfrac{1}{\sqrt{1+\cot^2\theta}}$ | $\pm\dfrac{\sqrt{\sec^2\theta-1}}{\sec\theta}$ | $\dfrac{1}{\csc\theta}$ |
| $\cos\theta$ | $\pm\sqrt{1-\sin^2\theta}$ | | $\pm\dfrac{1}{\sqrt{1+\tan^2\theta}}$ | $\pm\dfrac{\cot\theta}{\sqrt{1+\cot^2\theta}}$ | $\dfrac{1}{\sec\theta}$ | $\pm\dfrac{\sqrt{\csc^2\theta-1}}{\csc\theta}$ |
| $\tan\theta$ | $\pm\dfrac{\sin\theta}{\sqrt{1-\sin^2\theta}}$ | $\pm\dfrac{\sqrt{1-\cos^2\theta}}{\cos\theta}$ | | $\dfrac{1}{\cot\theta}$ | $\pm\sqrt{\sec^2\theta-1}$ | $\pm\dfrac{1}{\sqrt{\csc^2\theta-1}}$ |
| $\cot\theta$ | $\pm\dfrac{\sqrt{1-\sin^2\theta}}{\sin\theta}$ | $\pm\dfrac{\cos\theta}{\sqrt{1-\cos^2\theta}}$ | $\dfrac{1}{\tan\theta}$ | | $\pm\dfrac{1}{\sqrt{\sec^2\theta-1}}$ | $\pm\sqrt{\csc^2\theta-1}$ |
| $\sec\theta$ | $\pm\dfrac{1}{\sqrt{1-\sin^2\theta}}$ | $\dfrac{1}{\cos\theta}$ | $\pm\sqrt{1+\tan^2\theta}$ | $\pm\dfrac{\sqrt{1+\cot^2\theta}}{\cot\theta}$ | | $\pm\dfrac{\csc\theta}{\sqrt{\csc^2\theta-1}}$ |
| $\csc\theta$ | $\dfrac{1}{\sin\theta}$ | $\pm\dfrac{1}{\sqrt{1-\cos^2\theta}}$ | $\pm\dfrac{\sqrt{1+\tan^2\theta}}{\tan\theta}$ | $\pm\sqrt{1+\cot^2\theta}$ | $\pm\dfrac{\sec\theta}{\sqrt{\sec^2\theta-1}}$ | |

## VIII. MULTIPLE ANGLE FORMULAS

**A. SUMS OR DIFFERENCE OF TWO ANGLES. 1. The sine** of the sum (*difference*) of two angles equals the product of the sine of the first angle and the cosine of the second plus (*minus*) the product of the cosine of the first and the sine of the second: $\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y$ **E:** $\sin 75° = \sin(45° + 30°) = \sin 45°\cos 30° + \cos 45°\sin 30° = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4} = .966$

**2. The cosine** of the sum (*difference*) of two angles is equal to the product of the cosines of the two angles minus (*plus*) the product of their sines: $\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$

**3. The tangent** of the sum (*difference*) of two angles is equal to the sum (*difference*) of their tangents divided by 1 minus (*plus*) the product of their tangents:

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x\tan y} \qquad \cot(x \pm y) = \frac{\cot x\cot y \mp 1}{\cot y \pm \cot x}$$

**B. TWICE AN ANGLE 1. The sine** of twice an angle equals twice the product of the sine and cosine of the angle: $\sin 2x = 2\sin x\cos x$ **E:** $\sin 40° = 2\sin 20°\cos 20° = 2(.3420)(.9397) = .6428$ **2. The cosine** of twice an angle is equal to the square of the cosine of the angle minus the square of the sine of the angle; or twice the square of the cosine of the angle minus 1; or 1 minus twice the square of the sine of the angle: $\cos 2x = \cos^2x - \sin^2x = 2\cos^2x - 1 = 1 - 2\sin^2x$ **E:** $\cos 60° = \cos^2 30° - \sin^2 30° = \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$; $\cos 60° = 1 - 2\sin^2(30°) = 1 - 2\cdot(1/2)^2 = 1 - (1/2) = 1/2$ **3. The tangent** of twice an angle equals twice the tangent of the angle divided by 1 minus the square of the tangent of the angle:

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x} \qquad \cot 2x = \frac{\cot^2 x - 1}{2\cot x}$$

**C. HALF AN ANGLE. 1. The sine** (*cosine*) of half an angle equals $\pm$ the square root of 1 minus (*plus*) the cosine of the angle all divided by two. **NOTE:** The sign of the radical is determined by the quadrant of the angle $x/2$:

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}} \qquad \cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$

**E:** $\sin 15° = \sin\frac{30°}{2} = \sqrt{\frac{1-\cos 30°}{2}} = \sqrt{\frac{1-.866}{2}} = \sqrt{.067} = .259$ **2. The tangent** of half an angle equals $\pm$ the square root of 1 minus the cosine of the angle divided by 1 plus the cosine of the angle:

$$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} \qquad \cot\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{1-\cos x}}$$

**E:** $\tan 30° = \sqrt{\frac{1-\cos 60°}{1+\cos 60°}} = \sqrt{\frac{1-1/2}{1+1/2}} = \sqrt{\frac{1}{3}} = .577$

**D. SUMS AND DIFFERENCES OF FUNCTIONS. 1. The sum** (*difference*) **of the sines** of two angles is equal to twice the sine (*cosine*) of half the sum of the angles times the cosine (*sine*) of half the difference of the angles: $\sin x + \sin y = 2\sin 1/2(x+y)\cos 1/2(x-y)$; $\sin x - \sin y = 2\cos 1/2(x+y)\sin 1/2(x-y)$ **E:** $\sin 75° + \sin 15° = 2\sin 1/2(90°)\cos 1/2(60°) = 2\sin 45°\cos 30° = 2(\sqrt{2}/2)(\sqrt{3}/2) = \sqrt{6}/2 = 1.125$ **2. The sum of the cosines** of two angles equals twice the cosine of half the sum of the angles times the cosine of half their difference: $\cos x + \cos y = 2\cos 1/2(x+y)\cos 1/2(x-y)$ **3. The difference of the cosines** of two angles equals minus two times the sine of half the sum of the angles times the sine of half their difference $\cos x - \cos y = -2\sin 1/2(x+y)\sin 1/2(x-y)$

**E. PRODUCT OF TWO FUNCTIONS.**

$$\sin x\sin y = 1/2[\cos(x-y) - \cos(x+y)]$$
$$\cos x\cos y = 1/2[\cos(x+y) + \cos(x-y)]$$
$$\sin x\cos y = 1/2[\sin(x+y) + \sin(x-y)]$$

## IX. TRIG. IDENTITIES

**A. DEFINITION.** Statement that two expressions are equal for all values of the angle or angles involved, when the functions appearing are defined. **E:** $\csc\theta = 1/\sin\theta$ is an identity but $\tan\theta = \sin\theta$ is true only for some values of $\theta$.

**B. TYPES OF SOLUTION. 1. Transform one member into the other. E:** Verify $\cot^2x - \cos^2x = \cot^2x\cos^2x$ **Sol:** $\cot^2x - \cos^2x = \dfrac{\cos^2x}{\sin^2x} - \cos^2x = \dfrac{\cos^2x - \sin^2x\cos^2x}{\sin^2x} = \dfrac{\cos^2x(1-\sin^2)}{\sin^2x} = \dfrac{\cos^2x}{\sin^2x}\cdot\cos^2x = \cot^2x\cos^2x$ **2. Transform both members. E:** Show that $\dfrac{1-\sin A}{1+\sin A} = (\sec A - \tan A)^2$ **Sol:** $\dfrac{1-\sin A}{1+\sin A} = \dfrac{(1-\sin A)^2}{(1-\sin A)(1+\sin A)} = \dfrac{(1-\sin A)^2}{1-\sin^2 A} = \dfrac{(1-\sin A)^2}{\cos^2 A}$;

$$(\sec A - \tan A)^2 = \left(\frac{1}{\cos A} - \frac{\sin A}{\cos A}\right)^2 = \left(\frac{1-\sin A}{\cos A}\right)^2 = \frac{(1-\sin A)^2}{\cos^2 A}$$

## X. TRIG. EQUATIONS

**A. DEFINITION.** Statement that two expressions are equal for certain values of the angle or angles involved. These values are called the *roots* of the equation and are usually determined in the interval between 0° and 360°. The other roots can be obtained by adding multiples of 360°. **E:** $\sin^2x = 1$ when $\sin x = \pm1$, hence when $x = 90°$ or $270°$ but also when $x = 450°, 630°, -90°$, etc.

**B. SOLUTION. 1.** Reduce all expressions to functions of a single angle using transformation methods, elimination and algebraic simplification. **2.** Replace different functions with one function; solve algebraically for this function. **3.** Check all solutions.

**C. LINEAR TRIG. EQUATIONS. E:** $3\tan x + \sqrt{3} = 0$ **Sol:** $3\tan x = -\sqrt{3}$, $\tan x = -\sqrt{3}/3$; since $\tan x$ is negative, $x$ is in quadrant II or IV and $\arctan\sqrt{3}/3 = 30°$. Thus $x = 180° - 30° = 150°$ or $360° - 30° = 330°$.

**D. QUADRATIC EQUATIONS IN THE FORM $ax^2 + c = 0$. E:** $4\cos^2\theta - 3 = 0$ **Sol:** $4\cos^2\theta = 3$, $\cos^2\theta = 3/4$, $\cos\theta = \pm\sqrt{3}/2$; thus $\theta = 30°, 150°, 210°, 330°$.

**E. QUADRATIC EQUATIONS IN THE FORM $ax^2 + bx = 0$. E:** $2\sin^2x = \sin x$ **Sol:** $2\sin^2x - \sin x = 0$, $\sin x(2\sin x - 1) = 0$, $\sin x = 0$ or $1/2$; thus $x = 0°, 30°, 150°, 180°, 360°$.

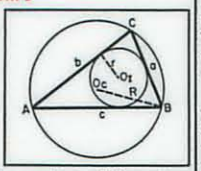
**F. COMPLETE QUADRATIC. E:** $3\sin^2x - 2\sin x - 1 = 0$ **Sol:** $(3\sin x + 1)(\sin x - 1) = 0$; $\sin x = -1/3$ or 1. For $\sin x = 1$, $x = 90°$, for $\sin x = -.333$, $x = 19°30'$. **E:** $2\sin^2x + 3\sin x - 2 = 0$. **Sol:** $(2\sin x - 1)(\sin x + 2) = 0$, $\sin x = 1/2$ or $-2$. Thus arc $\sin 1/2 = 30°, 150°$; for arc $\sin -2$, no solution.

## XI. OBLIQUE TRIANGLE RELATIONSHIPS

**A. SUM OF ANGLES.** $A+B+C=180°$

**B. LAW OF SINES.** Lengths of the sides of a triangle are proportional to the sines of the angles opposite them:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$




**C. LAW OF COSINES.** The square of the length of a side of a triangle equals the sum of the squares of the lengths of the other two sides minus twice the product of these two sides times the cosine of the angle between them:

$$a^2 = b^2 + c^2 - 2bc\cos A \qquad b^2 = a^2 + c^2 - 2ac\cos B \qquad c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc} \qquad \cos B = \frac{a^2+c^2-b^2}{2ac} \qquad \cos C = \frac{a^2+b^2-c^2}{2ab}$$

**D. LAW OF TANGENTS.** The sum of two sides of a triangle divided by their difference equals the tangent of half of the sum of the angles opposite these sides divided by the tangent of half of the difference of these angles:

$$\frac{a+b}{a-b} = \frac{\tan 1/2(A+B)}{\tan 1/2(A-B)} \qquad \frac{b+c}{b-c} = \frac{\tan 1/2(B+C)}{\tan 1/2(B-C)} \qquad \frac{c+a}{c-a} = \frac{\tan 1/2(C+A)}{\tan 1/2(C-A)}$$

If $a < b$, $b < c$, or $c < a$, reverse order of sides, angles.

**E. OTHER LENGTHS.** Let $s$ be the semiperimeter (half the sum of the lengths of the sides), $r$ the radius of the inscribed circle, $R$ the radius of the circumscribed circle, then:

$$s = \frac{a+b+c}{2} \qquad r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \qquad R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$$

**F. ANGLES IN TERMS OF THE SIDES. 1. The sine** of half an angle equals the square root of [$s$ minus one adjacent side times $s$ minus the other adjacent side divided by the product of the adjacent sides]:

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \qquad \sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}} \qquad \sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

**2. The cosine** of half an angle equals the square root of [$s$ times $s$ minus opposite side divided by product of the adjacent sides]:

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} \qquad \cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}} \qquad \cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

**3. The tangent** of half an angle equals the square root of [$s$ minus one adjacent times $s$ minus the other adjacent side divided by the product of $s$ and $s$ minus the opposite side]. It also equals the radius of the inscribed circle divided by $s$ minus the side opposite the angle:

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{r}{s-a} \qquad \tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} = \frac{r}{s-b} \qquad \tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \frac{r}{s-c}$$

**G. MOLLWEIDE'S EQUATIONS.** To check solution of a triangle.

$$\frac{a-b}{c} = \frac{\sin 1/2(A-B)}{\cos 1/2C} \quad \text{or} \quad \frac{b-a}{c} = \frac{\sin 1/2(B-A)}{\cos 1/2C}$$

**H. AREA OF A TRIANGLE. 1.** In terms of two sides and the included angle, the area is half the product of the two sides times the sine of the included angle:

$$K = \frac{1}{2}ab\sin C \qquad K = \frac{1}{2}ac\sin B \qquad K = \frac{1}{2}bc\sin A$$

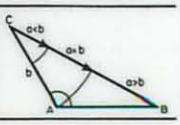
**2.** In terms of three sides, the area is the square root of the product of $s$ and $s$ minus each of the sides. $K = \sqrt{s(s-a)(s-b)(s-c)}$ **3.** In terms of the sides and the radius of the circumscribed circle (R) or inscribed circle (r): $K = abc/4R$ $K = rs$

## XII. SOLUTION OF OBLIQUE TRIANGLES

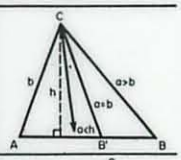
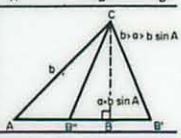
**A. GIVEN ONE SIDE AND TWO ANGLES (S.A.A. OR A.S.A.). Solution:** Find the third angle from $A + B + C = 180°$; the other two sides by the Law of Sines. **Check:** Mollweide's equations. **AREA:** Find another adjacent side by the Law of Sines then substitute in $K = 1/2$ product of the sides times the sine of the included angle.

**B. GIVEN TWO SIDES AND AN ANGLE OPPOSITE ONE SIDE (S.S.A.).** Called the ambiguous case, the number of angles and sides to be determined depends upon number of triangles formed.

**ANGLE GREATER THAN OR EQUAL TO 90°**

| | | |
|---|---|---|
| 1. The opposite side less than the adjacent side: $a < b$ | 0 |  |
| 2. The opposite side equals the adjacent side: $a = b$ | 0 | |
| 3. The opposite side greater than the adjacent side: $a > b$ | 1 | |

**ANGLE LESS THAN 90°**

| | | |
|---|---|---|
| 4. The opposite side greater than the adjacent side: $a > b$ | 1 ABC |  |
| 5. The opposite side equals the adjacent side (isocles): $a = b$ | 1 A′BC | |
| 6. The opposite side less than *altitude to third side: $a < b$ sin A | 0 | |
| 7. The opposite side less than the adjacent side but greater than the altitude to the third side: $a < b$ but $a > b$ sin A | 2 AB′C AB″C |  |
| 8. The opposite side equals the altitude to the third side (right triangle): $a = b$ sin A | 1 ABC | |

***NOTE:** The altitude to the third side equals the adjacent side times the sine of the angle: $h = b\sin A$.

**Solution:** Determine the number of solutions. Find the angle opposite the adjacent side (angle $B$) using Law of Sines ($\sin B = b\sin A/a$). Find the third angle from $A + B + C = 180°$ and the third side by the Law of Sines. (In situation 7, each triangle is solved separately.) **Check:** Mollweides equations. **AREA:** If three sides have been determined use $K = \sqrt{s(s-a)(s-b)(s-c)}$. If two sides and an included angle known solve as in A.

**C. GIVEN TWO SIDES AND THE INCLUDED ANGLE (S.A.S.). Solution 1:** Find the remaining side using the Law of Cosines; the second angle from the Law of Cosines or Law of Sines; the third angle from $A + B + C = 180°$. **Solution 2:** Find the two angles with the Law of Tangents; the third side from the Law of Sines. **Check:** Law of Sines. **AREA:** Substitute directly in $K = 1/2$ product of sides times sine of included angle.

**D. GIVEN THREE SIDES (S.S.S.). Solution 1:** Find one angle using Law of Cosines; the second angle by Law of Cosines or Law of Sines; the third angle from $A + B + C = 180°$. **Solution 2:** Find any two angles using the Half Angle formulas (XI:F); the third angle from $A + B + C = 180°$. **Check:** Mollweide's Equations. **AREA:** $K = \sqrt{s(s-a)(s-b)(s-c)}$

**E. TYPICAL EXAMPLES. NOTE:** Subtract 10 from the logs of all trig. functions except when log tan $\theta$ and log cot $\theta > 1$.

**E1:** $C = 68°22'$, $A = 13°15'$, $a = 1432$ **Sol: S.A.A.**

$A+B+C = 180°$
$13°15'+B+68°22' = 180°$
$B = 98°23'$

$\dfrac{b}{\sin B} = \dfrac{a}{\sin A}$ $\quad \dfrac{b}{\sin 98°23'} = \dfrac{a}{\sin 13°15'}$

$\log 1432 = 3.15594$
$(+)\log\sin 98°23' = 9.99533 - 10$
$= 13.15127 - 10$
$(-)\log\sin 13°15' = 9.36022 - 10$
$\log b = 3.79105$
$b = 6180.9$

$\dfrac{c}{\sin C} = \dfrac{a}{\sin A}$ $\quad \dfrac{c}{\sin 68°22'} = \dfrac{a}{\sin 13°15'}$

$\log 1432 = 3.15594$
$(+)\log\sin 68°22' = 9.96828 - 10$
$= 13.12422 - 10$
$(-)\log\sin 13°15' = 9.36022 - 10$
$\log c = 3.76400$
$c = 5807.6$

**E2:** Does a triangle exist if $a = 12$, $b = 20$, $A = 43°$? **Sol: S.S.A.,** ambiguous case. Evaluating $b\sin A = (20)(.6820) = 13.64$ shows that $a$ is less than $b\sin A$; no triangle exists.

**E3:** Find $b$ if $a = 6$, $c = 8$, $B = 22°$ **Sol: S.A.S.**; substitute $b^2 = c^2 + a^2 - 2ac\cos B = 8^2 + 6^2 - 2\cdot8\cdot6\cdot\cos 22° = 100 - 96(0.9272) = 100 - 89.01 = 10.99$; $b = 3.3$

**E4:** $b = 2657$, $c = 1826$, $A = 21°54'$ **Sol: S.A.S., method 2.**

$b = 2657$
$c = 1826$
$b + c = 4483$
$b - c = 831$

$A + B + C = 180°$
$21°54' + B + C = 180°$
$B + C = 158°6'$
$\frac{1}{2}(B + C) = 79°3'$

$\dfrac{b+c}{b-c} = \dfrac{\tan\frac{1}{2}(B+C)}{\tan\frac{1}{2}(B-C)}$ $\quad \dfrac{4483}{831} = \dfrac{\tan 79°3'}{\tan\frac{1}{2}(B-C)}$

$\log\tan 79°3' = 0.71338$
$(+)\log 831 = 2.91960$
$= 13.63298 - 10$
$(-)\log 4483 = 3.65157$
$\log\tan\frac{1}{2}(B-C) = 9.98141 - 10$
$\frac{1}{2}(B-C) = 43°46'26''$
$\frac{1}{2}(B+C) = 79°3'$
$B = 122°49'26''$
$C = 35°16'34''$

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$ $\quad \dfrac{a}{\sin 21°54'} = \dfrac{2657}{\sin 122°49'26''}$

$\log 2657 = 3.42439$
$(+)\log\sin 21°54' = 9.57169 - 10$
$= 12.99608 - 10$
$(-)\log\sin 122°49'26'' = 9.92445 - 10$
$\log a = 3.07163$
$a = 1176.6$

**E5:** If $a = 4$, $b = 5$, $c = 6$, find $A$. **Sol: S.S.S.**; substitute $\cos A = \dfrac{b^2+c^2-a^2}{2bc} = \dfrac{25+36-16}{2\times5\times6} = \dfrac{45}{60} = .7500$; $A = 48°35'$

**E6:** $a = 545.7$, $b = 697.3$, $c = 813.4$. **Sol: S.S.S., method 2.**

$2s = 2056.4$

$s = 1028.2$
$s - a = 482.5$
$s - b = 330.9$
$s - c = 214.8$

$\cos\dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$ $\quad \cos\dfrac{A}{2} = \sqrt{\dfrac{(1028.2)(482.5)}{(697.3)(813.4)}}$

$\log 1028.2 = 3.01208$
$(+)\log 4825 = 2.68350$
$\log\text{Num} = 15.69558 - 10$
$\log 697.3 = 2.84342$
$(+)\log 813.4 = 2.91030$
$\log\text{Dem} = 5.75372$

$\cos\dfrac{B}{2} = \sqrt{\dfrac{s(s-b)}{ac}}$ $\quad \cos\dfrac{B}{2} = \sqrt{\dfrac{(1028.2)(330.9)}{(545.7)(813.4)}}$

$\log 1028.2 = 3.01208$
$(+)\log 330.9 = 2.51970$
$\log\text{Num} = 15.53178 - 10$
$\log 545.7 = 2.73695$
$(+)\log 813.4 = 2.91030$
$\log\text{Dem} = 5.64725$

$\frac{1}{2}[\log\text{Num} - \log\text{Dem}] = \frac{1}{2}[9.94186 - 10]$ $\quad \frac{1}{2}[\log\text{Num} - \log\text{Dem}] = \frac{1}{2}[9.88453 - 10]$

$\log\cos\frac{A}{2} = 4.97093 - 5$ $\quad \log\cos\frac{B}{2} = 4.94227 - 5$

$\frac{A}{2} = 20°43.8'$; $A = 41°27.6'$ $\quad \frac{B}{2} = 28°53.6'$; $B = 57°47.2'$

$A+B+C = 180°$ $\quad 41°27.6'+57°47.2'+C = 180°$ $\quad C = 80°45.2'$