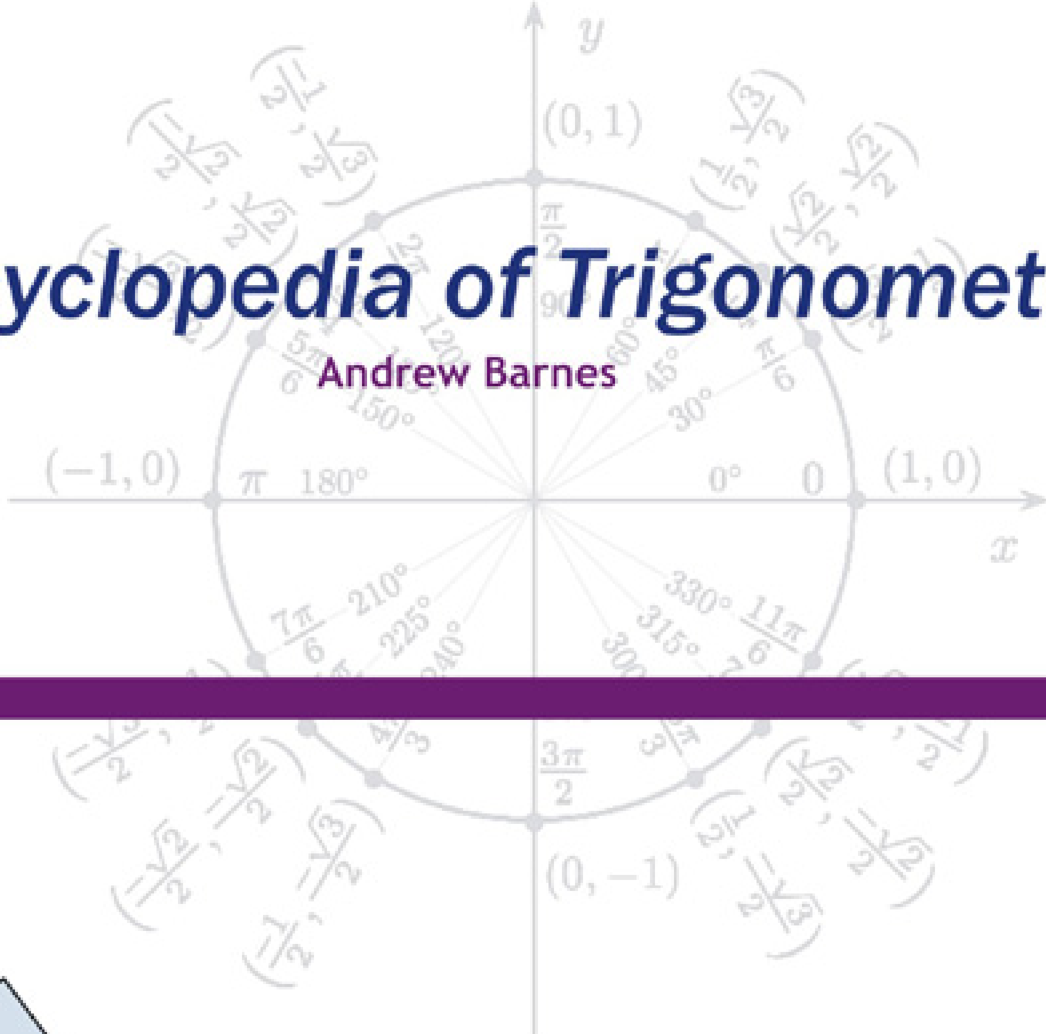
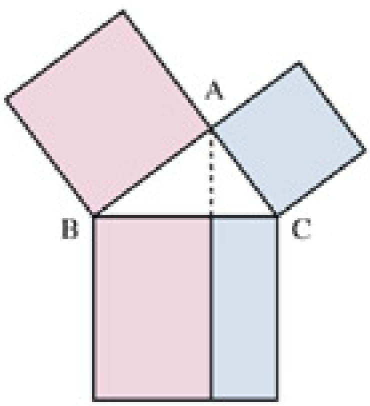
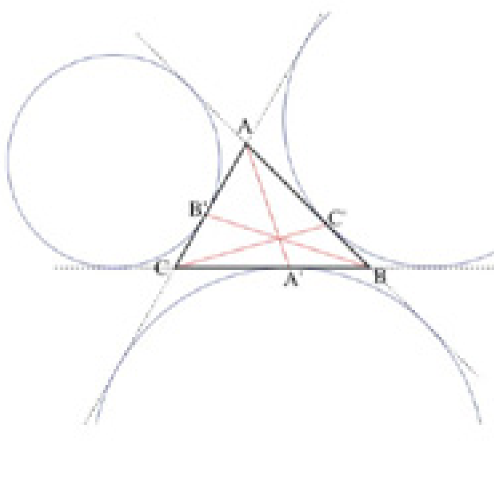
Encyclopedia of Trigonometry

Andrew Barnes







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Trigonometry

Introduction

Trigonometery is the study of triangles. “Tri” is Ancient Greek word for three, “gon” means side, “metry” measurementtogether they make “measuring three sides”. If you know some facts about a triangle, such as the lengths of it sides, then using trignometry you can find out other facts about that triangleits area, its angles, its center, the size of the largest circle that can be drawn inside it. As a consequencethe Ancient Greeks were able to use trigonometery to calculate the distance from the Earth to the Moon.

Trigonometry starts by examining a particularly simplified trianglethe right-angle triangle. More complex triangles can be built by joining right-angle triangles together. More complex shapes, such as squares, hexagons, circles and ellipses can be constructed from two or more triangles. Ultimately, the universe we live in, can be mapped through the use of triangles.

Trigonometry is an important, fundamental step in your mathematical education. From the seemingly simple shape, the right triangle, we gain tools and insight that help us in further practical as well as theoretical endeavors. The subtle mathematical relationships between the right triangle, the circle, the sine wave, and the exponential curve can only be fully understood with a firm basis in trigonometry.

Trigonometry is a system of mathematics, based generally on circles and triangles, that is used to solve complex problems (again, mainly involving circles and triangles). Extensions of various algebraic formulas, namely the Pythagorean theorem, are utilized.

In Review

Here are some useful formulas that should be learned before delving into trigonometry:

Pythagorean Theorem$a^2 + b^2 = c^2$, in a right triangle where a and b are the two sides, and c is the hypotenuse.

Pythagorean Triples3-4-5 (the two smaller values being the sides, with the larger the hypotenuse), 5-12-13, 7-24-25, 8-15-17, and any multiples of these (including 6-8-10, 10-24-26, etc.)

Properties of Special Right Triangles

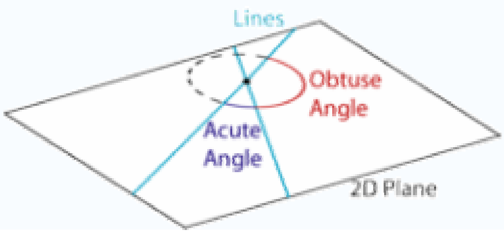
In a 45-45-90 right triangle, i.e., an isosceles right triangle, let one leg be x. The hypotenuse is then x$\sqrt{2}$ In a 30-60-90 right triangle, let the short leg, i.e. the leg opposite the 30° angle, be x. The hypotenuse is then equal to 2x and the longer leg, i.e. the leg opposite the 60° angle, is equal to x*$\sqrt{3}$.

In simple terms

Simple introduction

Introduction to Angles

If you are unfamiliar with angles, where they come from, and why they are actually required, this section will help you develop your understanding.



An angle between two lines in a 2-dimesional plane

In two dimensional flat space, two lines that are not parallel lines meet at an **angle**. Suppose you wish to measure the angle between two lines exactly so that you can tell a remote friend about itdraw a circle with its center located at the meeting of the two lines, making sure that the circle is small enough to cross both lines, but large enough for you to measure the distance along the circle's edge, the circumference, between the two cross points. Obviously this distance depends on the size of the circle, but as long as you tell your friend both the radius of the circle used, and the length along the circumference, then your friend will be able to reconstruct the angle exactly.

Triangle Definition

We know that triangles have three sideseach side is a lineif we choose any two sides of the triangle, then we have chosen two lines, which must therefore meet at an angle. There are three ways that we can pick the two sides, so a triangle must have three angleshence tri-angles, shortened to triangle.

This argument does not always work in reverse though. If you give me three angles, for example three right angles, I cannot make a triangle from them. This is also a problem with sidesI can give you three lengths that do not make the sides of a triangleyour height, the height of the nearest tree, the distance from the top of the tree to the center of the sun.

Triangle Ratios

Angles are not affected by the length of linesan angle is invariant under transformations of scale. Given the three angles of a known triangle, you can draw a new triangle that is similar to the old triangle but not necessarily congruent. Two triangles are called similar if they have the same angles as each other, they are congruent if they have the same size angles and matching sides have the same length. Given a pair of similar triangles, if the lengths of two matching sides are equal, then the lengths of the other sides are also equal, and so the triangles are congruent. More generally, the ratios of the lengths of matching pairs of sides between two similar triangles are equal. In particular, a triangle can be divided into 4 congruent triangles by connecting the mid-points on the sides of the original triangle together, each congruent triangle is similar on half scale to the original triangle.

Consequently, angles are useful for making comparisons between similar triangles.

Right Angles

An angle of particular significance is the right-anglethe angle at each corner of a square or a rectangle. Indeed, a rectangle can always be divided into two triangles by drawing a line through opposing corners of the rectangle. This pair of triangles has interesting properties. First, the triangles are identical. Second, they each have one angle which is a right-angle.

Imagine sitting at a table drawing a rectangle on a sheet of paper, and then dividing the rectangle into a pair of triangles by drawing a line from one corner to another. Perhaps the two triangles are different? A friend could sit on the other side of the table and draw the exact same thing by re-using your lines. The triangle nearest you has been produced by the exact same process as the triangle nearest your friend, therefore the two triangles are congruent, that is, identical. Try cutting a rectangle into two triangles to check.

Each triangle of this pair of triangles has one right anglethe rectangle has four right-angles, we split two of them by drawing from corner to corner, the remaining two were distributed equally to the identical triangles, so each triangle got one right-angle.

Thus every right angled triangle is half a rectangle.

A rectangle has four sides of two different lengthstwo long sides and two short sides. When we split the rectangle into two identical right angled triangles, each triangles got a long side and a short side from the rectangle, each triangle also got a copy of the split line. The split line has a Greek name"hypotenuse", Trigonometry has been defined as 'Many cheerful facts about the square of the hypotenuse'.

So the area of a right angled triangle is half the area of the rectangle from which it was split. Looking at a right angled triangle we can tell what the long and short sides of that rectangle were, they are the sides, the lines, that meet at a right angle. The area of the rectangle is the long side times the short side. The area of a right angled triangle is therefore half as much.

Right Triangles and Measurement

Given a right angled triangle, we could split the right angle into two equal, binary, halves using the following procedureusing a compass, draw a circle whose center is where the two right angled sides meet (rememberthe long and the short side from the rectangle meet at the right angle), make the radius of the circle the same as the length of the short side. Mark where the circle crosses the long sidedraw a circle around this mark whose radius is the length of the hypotenuse. Draw another circle with radius the length of the hypotenuse centered on the point at which the hypotenuse meets the short side. These last two circles will meet at two points. Draw a line between these two points and observe that the line splits the right angle into two equal angles.

Either of these new equal angles could again be split into two more equal, binary, halves using the same procedure. This process can be continued indefinately to get angles as small as we like. We can add and subtract these new angles together bydrawing a line and marking a point on it, then after choosing any two fractions of the right-angle, juxtapositioning each one so that its point split from the original right angle point touches the point marked on the line and one of its sides lies along the line. The new angle formed is a new binary fraction of the right angle.

So by splitting the right angle and recombining the angles so created, we can get a whole new range of angles. In principle we could use these binary fractions of the right angle to measure any unknown angle by finding the best fitting binary fraction of the right angle. This measure would be"Measurement of angles in binary fractions of a right angle" a system for measuring the size of angles in which the size of the right-angle is considered to be one, just as in measuring length, the size of the Olde King's foot is considered to be one.

Introduction to Radian Measure

Of course, it is equally possible to start with a different sized angle split into binary fractions. We might consider the angle made by half a circle to be one, or the angle made by a full circle to be one.

Trigonometry is simplified if we choose the following strange angle as oneDraw a circle, draw a radius of the circle, and then from the point where the radius meets the circumference of the circle, measure one radius length along the circumference of the circle moving counterclockwise and make a mark. Draw a line from this mark back to the center of the circle. The angle so formed is considered to be of size one in trigonometry; in order to tell your friends that this is the method you were using, you would say ‘measuring in radians’.

Does it matter what size circle is used to measure in radians? Perhaps in large circles using one radius length along the circumference will produce a different angle than that produced by a small circle? When measuring out one radius length along the circumference of the circle we might proceed as followsuse a bit of string of length one radius of the circle, and stick one end to the starting point. Stick a small loop on the other end of the string and thread it onto the circumference. Move the loop along the circumference until the string is pulled tight. The end of the string must reach beyond one radian, because the string is now in a straight line with the circumference taking a longer path outside. To fix this, find the half way point of the string, stick a loop there, thread it onto the circumference of the circle and pull the string until it is tight. The end point of the string is now closer to the one radian mark because the string follows the circumference more closely, although still cutting across, this time in two sections. We can keep on improving the fit of the string against the circumference by dividing each new section of the string in half, sticking on a loop, threading the loop onto the circumference and pulling the string tight.

If we were to draw lines between neighboring loops where they touch the circumference and from each loop to the center of the circle, we would get a lot of isoceles triangles whose longest sides were one circle radius in length.

Now draw a small circle inside the first circle, with the same center, but with half the radius. It too has lots of isoceles triangles extending from its center to its circumference whose longest sides are one new circle radius. The longest sides of the new isoceles triangles are half the size of the matching sides in the old isoceles triangles. The old and new isoceles triangles share a common apex angle, and because they are isoceles, their other angles must also match. Therefore the old and new isoceles triangles are similar with half scale. Therefore the short edges of the new isoceles triangles are half the size of the old ones. Therefore the total length of the short sides of the new triangles is close to half the length of the old radius, that is close to the length of the radius of the new circle. Hence the new triangles also delimit an angle as close to one radian as we like. Hence the definition of an angle of one radian is unaffected by the size of the circle used to define it.

Using Radians to Measure Angles

Once we have an angle of one radian, we can chop it up into binary fractions as we did with the right angle to get a vast range of known angles with which to measure unknown angles. A protractor is a device which uses this technique to measure angles approximately. To measure an angle with a protractorplace the marked center of the protractor on the corner of the angle to be measure, align the right hand zero radian line with one line of the angle, and read off where the other line of the angle crosses the edge of the protractor. A protractor is often transparent with angle lines drawn on it to help you measure angles made with short linesthis is allowed because angles do not depend on the length of the lines from which they are made.

If we agree to measure angles in radians, it would be useful to know the size of some easily defined angles. We could of course simply draw the angles and then measure them very accurately, though still approximately, with a protractorhowever, then we would then be doing physics, not mathematics.

The ratio of the length of the circumference of a circle to its radius is defined as 2π, where **π** is an invariant independent of the size of the circle by the argument above. Hence if we were to move 2π radii around the circumference of a circle from a given point on the circumference of that circle, we would arrive back at the starting point. We have to conclude that the size of the angle made by one circuit around the circumference of a circle is 2π radians. Likewise a half circuit around a circle would be π radians. Imagine folding a circle in half along an axis of symmetrythe resulting crease will be a diameter, a straight line through the center of the circle. Hence a straight line has an angle of size π radians.

The angles of a triangle add up to a half circle angle, that is, an angle of size π radians. To see this, draw any triangle, mark the midpoints of each side and connect them with lines to subdivide the original triangles into 4 similar copies at half scale. A copy of each of the original triangles three interior angles are juxtapositioned at each mid point, demonstrating that for any triangle the interior angles sum to a straight line, that is an angle of size π radians. This is true in particular for right angled triangles, which always have one angle of size $\pi/2$ radians, therefore the other two angles must also sum to a total size of $\pi - \pi/2 = \pi/2$ radians. If two sides of a right angled triangle have equal lengths, i.e. it is an isoceles right angled triangle, then each of the two other angles must be of size $½*π/2 = π/4$ radians.

Folding a half circle in half again produces a quarter circle which must therefore have an angle of size $π/2$ radians. Is a quarter circle a right angle? To see that it isdraw a square whose corner points lie on the circumference of a circle. Draw the diagonal lines that connect opposing corners of the square, by symmetry they will pass through the center of the circle, to produce 4 similar triangles. Each such triangle is isoceles, and has an angle of size $2π/4 = π/2$ radians where the two equal length sides meet at the center of the circle. Thus the other two angles of the triangles must be equal and sum to $π/2$ radians, that is each angle must be of size $π/4$ radians. We know that such a triangle is right angled, we must conclude that an angle of size $π/2$ radians is indeed a right angle.

Interior Angles of Regular Polygons

To demonstrate that a square can be drawn so that each of its four corners lies on the circumference of a single circleDraw a square and then draw its diagonals, calling the point at which they cross the center of the square. The center of the square is (by symmetry) the same distance from each corner. Consequently a circle whose center is co-incident with the center of the square can be drawn through the corners of the square.

A similar argument can be used to find the interior angles of any regular polygon. Consider a polygon of **n** sides. It will have n corners, through which a circle can be drawn. Draw a line from each corner to the center of the circle so that n equal apex angled triangles meet at the center, each such triangle must have an apex angle of $2π/n$ radians. Each such triangle is isoceles, so its other angles are equal and sum to $π - 2π/n$ radians, that is each other angle is $(π - 2π/n)/2$ radians. Each corner angle of the polygon is split in two to form one of these other angles, so each corner of the polygon has $2*(π - 2π/n)/2$ radians, that is $π - 2π/n$ radians.

This formula predicts that a square, where the number of sides n is 4, will have interior angles of $π - 2π/4 = π - π/2 = π/2$ radians, which agrees with the calculation above.

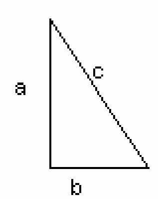
Likewise, an equilateral triangle with 3 equal sides will have interior angles of $π - 2π/3 = π/3$ radians.

A hexagon will have interior angles of $π - 2π/6 = 4π/6 = 2π/3$ radians which is twice that of an equilateral trianglethus the hexagon is divided into equilateral triangles by the splitting process described above.

Summary and Extra Notes

In summaryit is possible to make deductions about the sizes of angles in certain special conditions using geometrical arguments. However, in general, geometry alone is not powerful enough to determine the size of unknown angles for any arbitrary triangle. To solve such problems we will need the help of trigonmetric funcions.

In principle, all angles and trigonometric functions are defined on the unit circle. The term *unit* in mathematics applies to a single measure of any length. We will later apply the principles gleaned from unit measures to a larger (or smaller) scaled problems. All the functions we need can be derived from a triangle inscribed in the unit circleit happens to be a right-angled triangle.



A Right Triangle

The center point of the unit circle will be set on a Cartesian plane, with the circle's centre at the origin of the plane — the point (0,0). Thus our circle will be divided into four sections, or quadrants.

Quadrants are always counted counter-clockwise, as is the default rotation of angular velocity ω (omega). Now we inscribe a triangle in the first quadrant (that is, where the *x*- and *y*-axes are assigned positive values) and let one leg of the angle be on the *x*-axis and the other be parallel to the *y*-axis. (Just look at the illustration for clarification). Now we let the hypotenuse (which is always 1, the radius of our *unit* circle) rotate counter-clockwise. You will notice that a new triangle is formed as we move into a new quadrant, not only because the sum of a triangle's angles cannot be beyond 180°, but also because there is no way on a two-dimensional plane to imagine otherwise.

Angle-values simplified

Imagine the angle to be nothing more than exactly the size of the triangle leg that resides on the x-axis (the cosine). So for any given triangle inscribed in the unit circle we would have an angle whose value is the distance of the triangle-leg on the x-axis. Although this would be possible in principle, it is much nicer to have a independent variable, let's call it phi, which does not change sign during the change from one quadrant into another and is easier to handle (that means it is not necessarily always a decimal number).

!!Notice that all sizes and therefore angles in the triangle are mutually directly proportional. So for instance if the x-leg of the triangle is short the y-leg gets long.

That is all nice and well, but how do we get the actual length then of the various legs of the triangle? By using translation tables, represented by a function (therefore arbitrary interpolation is possible) that can be composed by algorithms such as taylor. Those translation-table-functions (sometimes referred to as LUT, Look up tables) are well known to everyone and are known as sine, cosine and so on. (Whereas of course all the abovementioned latter ones can easily be calculated by using the sine and cosine).

In fact in history when there weren't such nifty calculators available, printed sine and cosine tables had to be used, and for those who needed interpolated data of arbitrary accuracy - taylor was the choice of word.

So how can I apply my knowledge now to a circle of any scale. Just multiply the scaling coefficient with the result of the trigonometric function (which is referring to the unit circle).

And this is also why $\cos(\varphi)^2 + \sin(\varphi)^2 = 1$, which is really nothing more than a veiled version of the pythagorean theorem$\cos(\varphi) = a;\sin(\varphi) = b;a^2 + b^2 = c^2$, whereas the $c = 1^2 = 1$, a peculiarity of most unit constructs. Now you also see why it is so comfortable to use all those mathematical unit-circles.

Another way to interprete an angle-value would beA angle is nothing more than a translated 'directed'-length into which the information of the actual quadrant is packed and the applied type of trigonometric function along with its sign determines the axis ('direction'). Thus something like the translation of a (x,y)-tuple into polar coordinates is a piece of cake. However due to the fact that information such as the actual quadrant is 'translated' from the sign of x and y into the angular value (a multitude of 90) calculations such as for instance the division in polar-form isn't equal to the steps taken in the non-polar form.

Oh and watch out to set the right signs in regard to the number of quadrant in which your triangle is located. (But you'll figure that out easily by yourself).

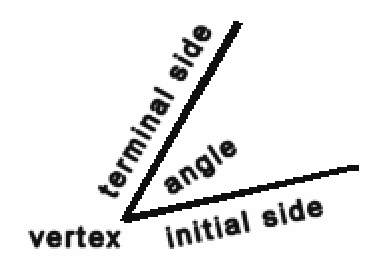
I hope the magic behind angles and trigonometric functions has disappeared entirely by now, and will let you enjoy a more in-depth study with the text underneath as your personal tutor.

Radian and Degree Measure

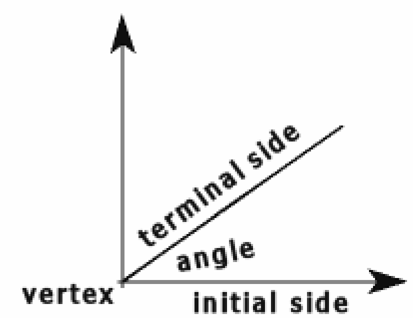
A Definition and Terminology of Angles

An angle is determined by rotating a ray about its endpoint. The starting position of the ray is called the *initial side* of the angle. The ending position of the ray is called the *terminal side*. The endpoint of the ray is called its *vertex*. *Positive angles* are generated by counter-clockwise rotation. *Negative angles* are generated by clockwise rotation. Consequently an angle has four partsits vertex, its initial side, its terminal side, and its rotation.

An angle is said to be in *standard position* when it is drawn in a cartesian coordinate system in such a way that its vertex is at the origin and its initial side is the positive x-axis.



Definition of an Angle



An angle in standard position

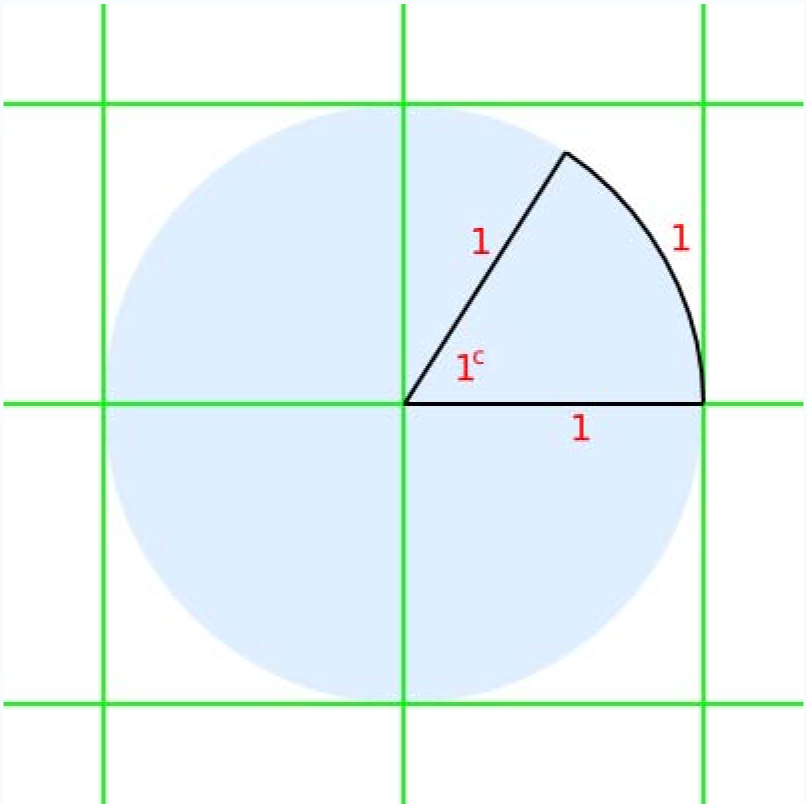
The radian measure

One way to measure angles is in radians. To signify that a given angle is in radians, a superscript c, or the abbreviation rad might be used. **If no unit is given on an angle measure, the angle is assumed to be in radians.**

$$\frac{3\pi}{2}^c \equiv \frac{3\pi}{2} \text{ rad.} \equiv \frac{3\pi}{2}$$

Defining a radian

A single radian is defined as the angle formed in the minor sector of a circle, where the minor arc length is the same as the radius of the circle.



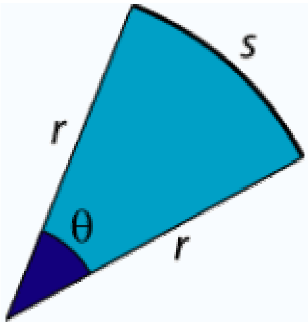
Defining a radian with respect to the unit circle.

$$1 \approx 57.296°$$

Measuring an angle in radians

The size of an angle, in radians, is the length of the circle arc *s* divided by the circle radius *r*.

$$\text{angle in radians} = \frac{s}{r}$$



Measuring an Angle in Radians

We know the circumference of a circle to be equal to $2\pi r$, and it follows that a central angle of one full counterclockwise revolution gives an arc length (or circumference) of $s = 2\pi r$. Thus 2 π radians corresponds to 360°, that is, there are 2π radians in a circle.

Converting from Radians to Degrees

Because there are 2π radians in a circle:

To convert degrees to radians:

$$\theta^c = \theta^\circ \times \frac{\pi}{180}$$

To convert radians to degrees:

$$\phi^\circ = \phi^c \times \frac{180}{\pi}$$

Exercises

Exercise 1

Convert the following angle measurements from degrees to radians. Express your answer exactly (in terms of π).

a) 180 degrees

b) 90 degrees

c) 45 degrees

d) 137 degrees

Exercise 2

Convert the following angle measurements from radians to degrees.

5. $\frac{\pi}{3}$

6. $\frac{\pi}{6}$

7. $\frac{7\pi}{3}$

8. $\frac{3\pi}{4}$

Answers

Exercise 1

a) π

b) $\frac{\pi}{2}$

c) $\frac{\pi}{4}$

d) $\frac{137\pi}{180}$

Exercise 2

a) 60°

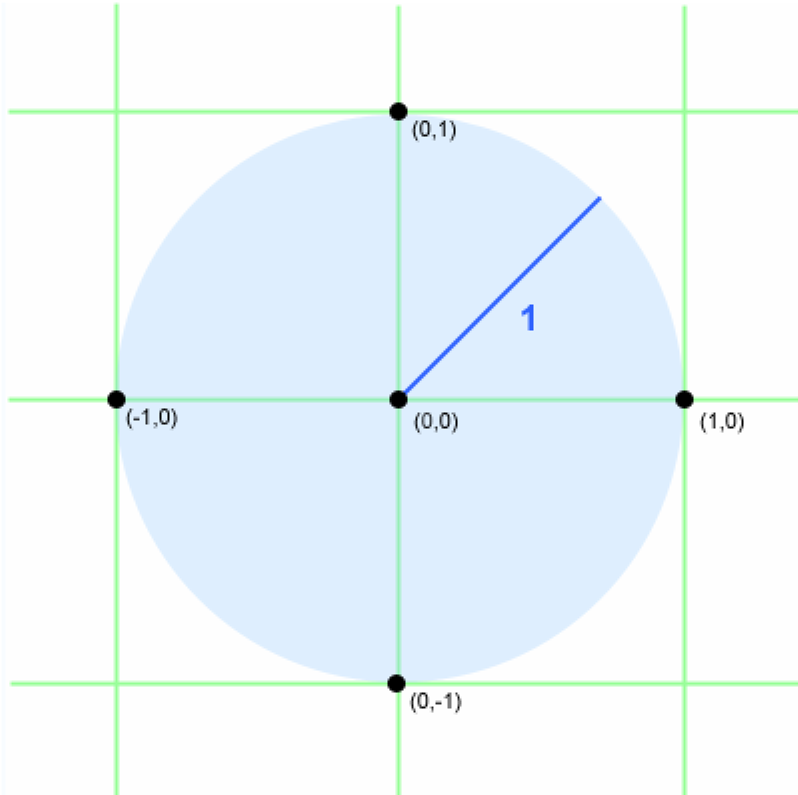
b) 30°

c) 420°

d) 135°

The Unit Circle

The Unit Circle is a circle with its center at the origin (0,0) and a radius of one unit.



The Unit Circle

Angles are always measured from the positive x-axis (also called the “right horizon”). Angles measured counterclockwise have *positive* values; angles measured clockwise have *negative* values.

A unit circle with certain exact values marked on it is available at Wikipedia. It is well worth the effort to memorize the values of sine and cosine on the unit circle (cosine is equal to x while sine is equal to y) included in this link.

Trigonometric-Angular Functions

Geometrically defining sine and cosine

In the unit circle shown here, a unit-length radius has been drawn from the origin to a point (x,y) on the circle.

Image:Defining sine and cosine.png
Defining sine and cosine

A line perpendicular to the x-axis, drawn through the point (x,y), intersects the x-axis at the point with the abscissa *x*. Similarly, a line perpendicular to the y-axis intersects the y-axis at the point with the ordinate *y*. The angle between the x-axis and the radius is α.

We define the basic trigonometric functions of any angle α as follows:

$$\begin{array}{lcl} \text{Sine :} & \sin(\alpha) & = & y \\ \text{Cosine :} & \cos(\alpha) & = & x \end{array}$$

$\tan\theta$ can be algebraically defined.

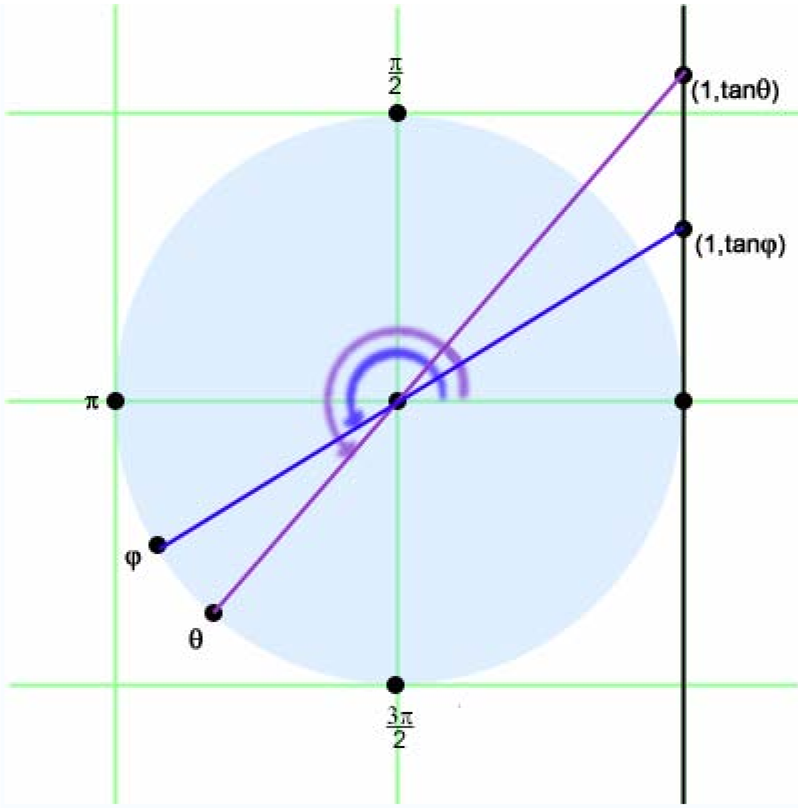
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cos\theta \neq 0$$

$$\tan\alpha = \frac{y}{x} \qquad x \neq 0$$

These three trigonometric functions can be used whether the angle is measured in degrees or radians as long as it specified which, when calculating trigonometric functions from angles or vice versa.

Geometrically defining tangent

In the previous section, we algebraically defined tangent, and this is the definition that we will use most in the future. It can, however, be helpful to understand the tangent function from a geometric perspective.



Geometrically defining tangent

A line is drawn at a tangent to the circle $x = 1$. Another line is drawn from the point on the radius of the circle where the given angle falls, through the origin, to a point on the drawn tangent. The ordinate of this point is called the tangent of the angle.

Domain and range of circular functions

Any size angle can be the input to sine or cosine — the result will be as if the largest multiple of 2π (or 360°) were subtracted from the angle. The output of the two functions is limited by the absolute value of the radius of the unit circle, $|1|$.

	domain	range
sine	R	$[-1, 1]$
cosine	R	$[-1, 1]$

R represents the set of all real numbers.

No such restrictions apply to the tangent, however, as can be seen in the diagram in the preceding section. The only restriction on the domain of tangent is that odd integer multiples of $\frac{\pi}{2}$ are undefined, as a line parallel to the tangent will never intersect it.

	domain	range
tangent	$R \setminus \{\cdots, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots\}$	R

Applying the trigonometric functions to a right-angled triangle

If you redefine the variables as follows to correspond to the sides of a right triangle:
- x = a (adjacent)
- y = o (opposite)
- a = h (hypotenuse)

Right Angle Trigonometry

Construction of Right Triangles

Right triangles are easily constructed:

1. Construct a diameter of a circle. Call the points where the diameter cuts the circle *a* and *b*.
2. Choose a distinct point *c* on the circle.
3. Construct line segments connecting these three points.

Then Δabc is a right triangle. This right triangle can be further divided into two isosceles triangles by adding a line segment from *c* to the center of the circle.

To simplify the following discussion, we specify that the circle has diameter 1 and is oriented such that the diameter drawn above runs left to right. We shall denote the angle of the triangle at the right θ and that at the left φ. The sides of the right triangle will be labeled, starting on the right, *a, b, c,* so that *a* is the rightmost side, *b* the leftmost, and *c* the diameter. We know from earlier that side *c* is opposite the right angle and is called the **hypotenuse.** Side *a* is opposite angle φ, while side *b* is opposite angle θ. We reiterate that the diameter, now called *c*, shall be assumed to have length 1 except where otherwise noted.

By constructing a right angle to the diameter at one of the points where it crosses the circle and then using the method outlined earlier for producing binary fractions of the right angle, we can construct one of the angles, say θ, as an angle of known measure between 0 and π / 2. The measure of the other angle φ is then $\pi - \pi/2 - \theta = \pi/2 - \theta$, the **complement** of angle θ. Likewise, we can bisect the diameter of the circle to produce lengths which are binary fractions of the length of the diameter. Using a compass, a binary fractional length of the diameter can be used to construct side *a* (or *b*) having a known size (with regard to the diameter *c*) from which side *b* can be constructed.

Using Right Triangles to find Unknown Sides

Finding unknown sides from two other sides

From the Theorem of Pythagoras, we know that:

$$a^2 + b^2 = c^2.$$

, c has been defined as of length one, so that:

$$b^2 = 1 - a^2.$$

and we can calculate the length of b squared. It may happen that b squared is a fraction such as 1/4 for which a square root can be explicitly found, in this case as 1/2, alternatively, we could use Newton's Method to find an approximate value for b. It often happens that b squared is what we want to find; it is not always necessary to find the exact value of the square root.

Finding unknown sides from a side and a (non-right) angle

As we can construct an uncountably large number of known angles for θ, and likewise an uncountably large number of sides a of known length, it would seem plausible that some of these known angles and known lengths should coincide in the same right angled triangle so that we could construct a convenient table relating the size of θ to the length of sides a and c for known values of θ and a with c assumed to be of length one. Even better would be a function, we will call it $\cos$, which for any value of θ, gave the corresponding value of a, that is, a function $\frac{a}{c} = \cos\theta$ which would save us the work of constructing angles and lengths and making difficult deductions from them.

Of course, given an angle θ, we could construct a right angled triangle using ruler and compass that had θ as one of its angles, we could then measure the length of the side that corresponds to a to evaluate the $\cos$ function. Such measurements would necessarily be in-exact; it would be a problem in physics to see how accurately such measurements can be made; using trigonometry we can make precise predictions with which the results of these physical measurements can be compared.

Some explicit values for the $\cos$ function are known. For $\theta = 0$, sides a and c are coincidental $a = c = \frac{a}{c} = 1$, so $\cos 0 = 1$. For $\theta = \frac{\pi}{2}$, sides b and c are coincidental and of length 1, and side a is of zero length, consequently $a = 0$, $c = 1$. $\frac{a}{c} = 0$ and $\cos\frac{\pi}{2} = 0$.

The simplest right angle triangle we can draw is the isoceles right angled triangle, it has a pair of angles of size $\frac{\pi}{4}$ radians, and if its hypotenuse is considered to be of length one, then the sides a and b are of length $\sqrt{\frac{1}{2}}$ as can be verified by the theorem of Pythagoras. If the side a is chosen to be the same length as the radius of the circle containing the right angled triangle, then the right hand isoceles triangle obtained by splitting the right angled triangle from the circumference of the circle to its center is an equilateral triangle, so θ must be $\frac{\pi}{3}$, and φ must be $\frac{\pi}{6}$ and b^2 must be $1 - \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$.

Properties of the cosine and sine functions

Period

Adding an integer multiple of 2π to θ moves the point at which the right angle touches the circumference of the circle adds an integral number of full circles to the point arrving back at the same position. So cos(θ+2πn) = cos(θ).

Half Angle and Double Angle Formulas

We can derive a formula for cos(θ/2) in terms of cos(θ) which allows us to find the value of the cos() function for many more angles. To derive the formula, draw an isoceles triangle, draw a circle through its corners, connect the center of the circle with radii to each corner of the isoceles triangle, extend the radius through the apex of the isoceles triangle into a diameter of the circle and connect the point where the diameter crosses the other side of the circle with lines to the other corners of the isoceles triangle.

Therefore:

$$\cos(\theta/2) = \sqrt{\frac{1/2 + \cos\theta}{2}}$$

which gives a method of calculating the cos() of half an angle in terms of the cos() of the original angle. For this reason * is called the "Cosine Half Angle Formula".

The half angle formula can be applied to split the newly discovered angle which in turn can be split again ad infinitum. Of course, each new split involves finding the square root of a term with a square root, so this cannot be recommended as an effective procedure for computing values of the cos() function.

Equation * can be inverted to find cos(θ) in terms of cos(θ)/2:

$$cos(\theta/2) = \sqrt{\frac{1 + \cos\theta}{2}}, \;\; \cos^2(\theta/2) = \frac{1 + \cos(\theta)}{2}$$

```
=>     cos(θ/2)**2 - 1/2 = cos(θ)/2)

=> 2 * cos(θ/2)**2 - 1   = cos(θ)
```

substituting δ = θ/2 gives:

```
  2 * cos(δ)**2 - 1     = cos(2δ)
```

that is a formula for the cos() of double an angle in terms of the cos() of the original angle, and is called the "cosine double angle sum formula".

Angle Addition and Subtraction formulas

To find a formula for $\cos(\theta_1 + \theta_2)$ in terms of $\cos\theta_1$ and $\cos\theta_2$construct two different right angle triangles each drawn with side c having the same length of one, but with $\theta_1 \neq \theta_2$, and therefore angle $\psi_1 \neq \psi_2$. Scale up triangle two so that side a_2 is the same length as side c_1. Place the triangles so that side c_1 is coincidental with side a_2, and the angles θ_1 and θ_2 are juxtaposed to form angle $\theta_3 = \theta_1 + \theta_2$ at the origin. The circumference of the circle within which triangle two is imbedded (circle 2) crosses side a_1 at point g, allowing a third right angle to be drawn from angle $\varphi_2$to point g . Now reset the scale of the entire figure so that side c_2 is considered to be of length 1. Side a_2 coincidental with side c_1 will then be of length $\cos\theta_1$, and so side a_1 will be of length $\cos\theta_1 \cdot \cos\theta_2$in which length lies point g. Draw a line parallel to line a_1 through the right angle of triangle two to produce a fourth right angle triangle, this one imbedded in triangle two. Triangle 4 is a scaled copy of triangle 1, because:

(1) it is right angled, and

(2) $$\theta_4 + (\pi - \frac{\pi}{2} - \theta_1 - \theta_2) = \varphi_2 = \pi - \frac{\pi}{2} - \theta_2 \Rightarrow \theta_4 = \theta_1$$.

The length of side b4 is cos(φ2)cos(φ4) = cos(φ2)cos(φ1) as θ4 = θ1 Thus point g is located at length:

```
cos(θ1+θ2) = cos(θ1)cos(θ2) - cos(φ1)cos(φ2), where θ1+φ1 = θ2+φ2 = π/2
```

giving us the "Cosine Angle Sum Formula".

Proof that angle sum formula and double angle formula are consistent

We can apply this formula immediately to sum two equal angles:

```
cos(2θ) = cos(θ+θ) = cos(θ)cos(θ) - cos(φ)cos(φ) where θ+φ = π/2
        = cos(θ)**2 - cos(φ)**2           (I)
```

From the theorem of Pythagoras we know that:

```
a**2 + b**2 = c**2
```

in this case:

```
  cos(θ)**2 + cos(φ)**2  = 1**2                  where θ+φ = π/2
=>            cos(φ)**2  = 1 - cos(θ)**2
```

Substituting into (I) gives:

```
cos(2θ)   = cos(θ)**2 -      cos(φ)**2         where θ+φ = π/2
          = cos(θ)**2 - (1 - cos(θ)**2)
          = 2 * cos(θ)**2 - 1
```

which is identical to the "Cosine Double Angle Sum Formula":

```
2 * cos(δ)**2 - 1 = cos(2δ)
```

Derivative of the Cosine Function

We are now in a position to evaluate the expression:

```
cos(θ+δθ)/ δθ = (cos(θ)cos(δθ) - cos(δθ)cos(δφ)) /δθ   where θ+φ = δθ+δφ = π/2.
```

If we let δθ tend to zero we get an increasingly accurate expression for the rate at which cos(θ) varies with θ. If we let δθ tend to zero, cos(δθ) will tend to one, δφ will tend to π/2, cos(δφ) will tend to zero, and cos(δφ)/δθ will tend to oneallowing us to write in the limit:

```
 (cos(θ+δθ) - cos(θ))/δθ = (cos(θ)cos(δθ)-cos(φ)cos(δφ)-cos(θ)) /δθ   where θ+φ = δθ+δφ =
π/2.
                         = (cos(θ) * 1   -cos(φ) * δθ  -cos(θ)) /δθ
                         = -cos(φ) * δθ /δθ
                         = -cos(φ)                                   where θ+φ = π/2.

                         = -cos(π/2 - θ)
```

The function cos(π/2 - θ) occurs with such prevalence that it has been given a special name:

```
 sin(θ) = cos(π/2 - θ)
        = cos(φ) where θ+φ = π/2.
```

"sin" is pronounced "sine".

The above result can now be stated more succinctly:

```
 (cos(θ+δθ) - cos(θ))/δθ = -sin(θ)
```

often further abbreviated to:

```
 d cos(θ)/δθ = -sin(θ)
```

or in wordsthe rate of chage of cos(θ) with θ is -sin(θ).

Pythagorean identity

Armed with this definition of the sin() function, we can restate the Theorem of Pythagoras for a right angled triangle with side c of length one, from:

```
   cos(θ)**2 + cos(φ)**2 = 1**2  where θ+φ = π/2
```

to:

```
   cos(θ)**2 + sin(θ)**2 = 1.
```

We can also restate the "Cosine Angle Sum Formula" from:

```
cos(θ1+θ2) = cos(θ1)cos(θ2) - cos(φ1)cos(φ2), where θ1+φ1 = θ2+φ2 = π/2
```

to:

```
cos(θ1+θ2) = cos(θ1)cos(θ2) - sin(θ1)sin(θ2)
```

Sine Formulas

The price we have to pay for the notational convenince of this new function sin() is that we now have to answer questions likeIs there a "Sine Angle Sum Formula". Such questions can always be answered by taking the cos() form and selectively replacing cos(θ)**2 by 1 - sin(θ)**2 and then using algebra to simplify the resulting equation. Applying this technique to the "Cosine Angle Sum Formula" produces:

```
   cos(θ1+θ2)       =      cos(θ1)cos(θ2) - sin(θ1)sin(θ2)
=> cos(θ1+θ2)**2     =     (cos(θ1)cos(θ2) - sin(θ1)sin(θ2))**2
=> 1 - cos(θ1+θ2)**2 = 1 - (cos(θ1)cos(θ2) - sin(θ1)sin(θ2))**2
=>  sin(θ1+θ2)**2          =  1  -  (cos(θ1)**2*cos(θ2)**2  +  sin(θ1)**2*sin(θ2)**2  -
2*cos(θ1)cos(θ2)sin(θ1)sin(θ2))   -- Pythagoras on left, multiply out right hand side
                     =  1  -  (cos(θ1)**2*(1-sin(θ2)**2)  +  sin(θ1)**2*(1-cos(θ2)**2)  -
2*cos(θ1)cos(θ2)sin(θ1)sin(θ2)) -- Carefully selected Pythagoras again on the left hand
side
                     =    1    -    (cos(θ1)**2-cos(θ1)**2*sin(θ2)**2    +    sin(θ1)**2-
sin(θ1)**2*cos(θ2)**2 - 2*cos(θ1)cos(θ2)sin(θ1)sin(θ2)) -- Multiplied out
                     =  1  -  (1            -cos(θ1)**2*sin(θ2)**2                   -
sin(θ1)**2*cos(θ2)**2 - 2*cos(θ1)cos(θ2)sin(θ1)sin(θ2)) -- Carefully selected Pythagoras
                     =                                                cos(θ1)**2*sin(θ2)**2
sin(θ1)**2*cos(θ2)**2 + 2*cos(θ1)cos(θ2)sin(θ1)sin(θ2)) -- Algebraic simplification
                     =                                                      (cos(θ1)sin(θ2)
+sin(θ1)cos(θ2))**2
```

taking the square root of both sides produces the the "Sine Angle Sum Formula"

```
=> sin(θ1+θ2) = cos(θ1)sin(θ2) + sin(θ1)*cos(θ2)
```

We can use a similar technique to find the "Sine Half Angle Formula" from the "Cosine Half Angle Formula":

cos(θ/2) = squareRoot(1/2 + cos(θ)/2).

We know that 1 - cos(θ/2)**2 = sin(θ/2) **2, so squaring both sides of the "Cosine Half Angle Formula" and subtracting from one:

```
=> (1 - cos(θ/2)**2) =       1 - (1/2 + cos(θ)/2)
=>      sin(θ/2)**2  =           1/2 - cos(θ)/2
=>      sin(θ/2)     = squareRoot(1/2 - cos(θ)/2)
```

So far so good, but we still have a cos(θ/2) to get rid of. Use Pythagoras again to get the "Sine Half Angle Formula":

```
        sin(θ/2)     = squareRoot(1/2 - squareRoot(1 - sin(θ)**2)/2)
```

or perhaps a little more legibly as:

```
        sin(θ/2)     = squareRoot(1/2)*squareRoot(1 - squareRoot(1 - sin(θ)**2))
```

On the Use of Computers for Algebraic Manipuation

The algebra to perform such manipulations tends to be horrendousthe horrors can be reduced by using automated symbolic algebraic manipulation software to confirm that each line of the derivation has been carried out correctly. However, such software is unable, in of itself, to perform the set of deductions above without human interventionyou will note that at various points in the above sequence, the Theorem of Pythagoras was cunningly applied for reasons that only became obvious towards the end of the derivation.

Alternatively, one can draw a plausible geometric diagram, make careful deductions about unknown sides and angles from known sides and angles. Again, software can check that each inference is consistent with the known facts, however, such software cannot decide which diagram to draw in the first place, nor which are the significant inferences to be made. You will no doubt have experienced the alarming feeling when manipulating such equations or diagrams that the numbers of terms in the equations, or the numbers of lines, sides and circles on the diagram, are doubling after each possible inference, multiplying exponentially beyond your control, with no solution in sight. Undirected software rapidly looses itself in this exponential maze despite the protestations of the salaried scientists held captive by Microsoft at Cambridge University, who protest, I think too much, that Alan Turing and Kurt Godell were wrong, and that "Machines can do Mathematics" if only they are given enouigh time to program (that is direct) them, to do so. Fortunately, the most that machines can do, isassist humans with the tedious but necessary task of verifying each logical deduction. Machines cannot see the beauty of geometry, nor find directions to solve equations.

As is possible to define many other ratios of sides in an right angled triangle, the ratio of side b to side a for example. There will surely be relationships between these newly defined auxillary trigonometric functions - it would be very odd if there were not. However, all such subsidary functions can be dealt with by the properties of the cos() function alone; any newly defined functions must earn their keep by reducing complexity, not increasing ittheir existence often creates more definitions to learn without generating any significantly new knowledge. Mathematics should be derivable from as few ideas as possible, that is from quality without quantity. Those of you, other than the physics students liberated by Feynmann in Brazil, who are facing exams produced at the behest of tyrannical governments will be reduced to the tedium of learning the many intricate relationships between these auxillary functions, so that a bureaucrat with a fetish for metrics can kowtow to public opinionplease keep in mind that quantity and quality are not necessarily the same in mathematics; that increasing the number of auxillary functions and their interrelationships merely aggrevates the number of possibilities in each step of the solution space, without offering you any useful guidance as to which would be the most to use.

Sine derivative

We discovered earlier that the rate of change of cos(θ) with θ is -cos(φ) where θ+φ = π/2, or in the new fangled notation introduced above-sin(θ). If we increase θ by δθ, we reduce φ by δθ, that is the rate of change of φ with θ is -1,

```
=> dφ / dθ = -1, => -dθ / dφ = 1.
```

We have also proved above that

```
d cos(θ)/dθ = -sin(θ)
```

And we know from the definition that:

```
cos(θ) = sin(φ)
```

Putting these knowns together we can find the rate of change of sin(θ) with θ:

```
 d sin(θ)/dθ = d cos(φ)/dθ = d cos(φ)/dθ * -dθ/dφ = -1* d cos(φ)/dφ = -1* -cos(θ) =
cos(θ)
```

that is the rate of change of sin(θ) with is cos(θ). To summarize:

```
d  sin(θ) / dθ =  cos(θ)
d  cos(θ) / dθ = -sin(θ)
d -sin(θ) / dθ = -cos(θ)
d -cos(θ) / dθ =  sin(θ)
```

We are going round in circles! Notice that we have to apply the rate of change operation 4 times to get back to where we started.

Relationships-between exponential function and trigonometric functions

A similar function is the exponential function e(θ) which is defined by the statementthe rate of change of e(θ)is e(θ) and the value of e(0) is 1. Here we only have to apply the rate of change operator once to get back where we started. Explicitly:

```
 (e(θ+δθ)-e(θ)) / δθ tends to e(θ) as δθ tends to zero.
```

which can be rewritten replacing θ by iθ where i is any constant number, to get:

```
 (e(iθ+δiθ)-e(iθ)) / δiθ tends to e(iθ) as δθ tends to zero.
```

because i is a constant, δiθ = iδθ, performing this subsitution, we get:

```
    (e(iθ+iδθ)-e(iθ))  / iδθ tends to e(iθ) as δθ tends to zero.
=>  (e(i(θ+δθ))-e(iθ)) / iδθ tends to e(iθ) as δθ tends to zero.
=>  d e(iθ) / dθ  = i*e(iθ)
```

That is, the rate of change of e(iθ) with θ is i*e(iθ). We can continue this process to find the rate of change of i*e(iθ) with θ:

```
d i*e(iθ) / dθ                                    is the limit of
(i* e(i(θ+δθ))-i*e(iθ)) / δθ as δθ tends to zero, which is the same as:
 i*(e(i(θ+δθ))-  e(iθ)) / δθ as δθ tends to zero, which is:
 i* d e(iθ) / dθ = i*i*e(iθ).
```

Performing this 4 time sucessively yields and comparing with the same action on the cos() function:

```
 d       e(iθ) / dθ  =       i*e(iθ)               d   cos(θ) / dθ  = - sin(θ)
 d     i*e(iθ) / dθ  =     i*i*e(iθ)               d - sin(θ) / dθ  = - cos(θ)
 d   i*i*e(iθ) / dθ  =   i*i*i*e(iθ)               d - cos(θ) / dθ  =   sin(θ)
 d i*i*i*e(iθ) / dθ  = i*i*i*i*e(iθ)               d   sin(θ) / dθ  =   cos(θ)
```

If only there was a number i, such that i*i = -1, and hence i*i * i*i = -1 * -1 = 1, then we could relate the function cos() to the function e(). Fortunately, there is a number that will workthe square roots of -1. From here on i will denote a square root of -1. -i*-i = (-1)*i*(-1)*i = (-1)*(-1)*i*i = (1)*(-1)=-1 is also a solution. We can expect then that cos(θ) is some linear combination of e(iθ) and e(-iθ), perhaps:

```
cos(θ) = A*e(iθ)+B*e(-iθ)
```

We know that cos(0) = 1, so:

```
cos(0) = 1 = A*e(i*0)+B*e(-i*0) = A*e(0)+B*e(0) = A + B
```

Finding the rate of change with θ:

```
=> d cos(θ) / dθ = d (A*e(iθ)) / dθ +d  (B*e(-iθ))/dθ
=> - sin(θ)      =  i*A*e(iθ)       + -i*B*e(-iθ) = - cos(π/2-θ)
```

setting θ = 0, remembering that cos(π/2) = 0

```
=> 0 = iA - iB = i(A-B) => A = B
```

So now we know that A+B = 1 and A = B, so A+A = 1, so A = 1/2 and B = 1/2.

To summarize what we know so far:

```
cos(θ) = e(iθ)/2 + e(-iθ)/2   where θ is in radians and i is a square root of -1.
```

Replacing θ by -θ gives conversly:

```
cos(-θ) = e(-iθ)/2 + e(iθ)/2, the same formula, so we must have that cos(θ) =  cos(-θ).
```

Given:

```
cos(θ) = e(iθ)/2 + e(-iθ)/2
```

We can find the rate of change with θ of both sides to fined the sin() in terms of e()

```
   -sin(θ)  =  i+e( iθ)/2 - i*e(-iθ)/2
 =>  sin(θ)  =  i*e(-iθ)/2 - i*e( iθ)/2
```

Substituting -θ for θ gives:

```
     sin(-θ)  =  i*e(iθ)/2 - i*e(-iθ)/2 = -sin(θ)
```

thus sin() is an odd function, compare this to cos() which is an even function because cos(θ) = cos(-θ).

We can find the function e() in terms of cos() and sin():

```
 cos(θ) + i * sin(θ)  =  e(iθ)/2 + e(-iθ)/2 + i*i*e(-iθ)/2 - i*i*e(iθ)/2
                      =  e(iθ)/2 + e(-iθ)/2 -     e(-iθ)/2 +     e(iθ)/2
                      =  e(iθ)
```

This is called "Euler's Formula".

From the "Cosine Double Angle Formula", we know that:

```
cos(2θ) = 2 * cos(θ)**2 - 1
```

Let θ = π/2, so that cos(π/2) = 0, then:

```
   cos(2π/2) = 2 * cos(π/2)**2 - 1

=> cos (π)   = 2 * 0**2        - 1

=> cos (π)   = - 1
```

By Pythagoras:

```
   sin(π)**2 + cos(π)**2 = 1
=> sin(π)**2 + -1**2     = 1
=> sin(π)**2             = 0
=> sin(π)                = 0
```

Consequently, we can evaluate e(iπ) as:

```
 e(iπ) = cos(π) + i * sin(π) = -1 + i * 0 = -1;
```

Similarily, we can evaluate e(-iθ) from e(iθ):

```
   e( iθ) = cos( θ) + i * sin( θ)
=> e(-iθ) = cos(-θ) + i * sin(-θ)
=> e(-iθ) = cos(θ)  - i * sin( θ)  as cos() is even, sin() is odd
```

This result allows us to evaluate e(iθ)e(-iθ):

```
   e(iθ)e(-iθ) = (cos(θ) + i * sin(θ))(cos(θ) - i * sin(θ))
               = (cos(θ)**2 - i*i*sin(θ)**2)
               =  cos(θ)**2 +     sin(θ)**2        as i*i = -1
               =  1                                Pythagoras
```

Starting again from the "Cosine Double Angle Formula", we know that:

```
cos(2θ) = 2 * cos(θ)**2 - 1
```

Replace cos() by its formulation in e():

```
cos(θ) = e(iθ)/2 + e(-iθ)/2
```

to get:

```
e(2iθ)/2 + e(-2iθ)/2 = 2 * (e(iθ)/2 + e(-iθ)/2)**2                      - 1
                     = 2 * (e(iθ)/2)**2+(e(-iθ)/2)**2 +2e(iθ)e(-iθ)/4) - 1

                     =     (e(iθ)**2)/2+(e(-iθ)**2)/2 + e(iθ)e(-iθ)    - 1
```

But

```
e(iθ)e(-iθ) = 1
```

so we continue the algrebraic simplification to get:

```
 e(2iθ)/2 + e(-2iθ)/2 =     (e(iθ)**2)/2+(e(-iθ)**2)/2 + e(iθ)e(-iθ)    - 1
                      =     (e(iθ)**2)/2+(e(-iθ)**2)/2 + 1              - 1
                      =     (e(iθ)**2)/2+(e(-iθ)**2)/2
```

Again

```
e(iθ)e(-iθ) = 1
```

so we are forced to conclude that

```
 e( 2iθ) = e( iθ)**2  and
 e(-2iθ) = e(-iθ)**2   for any angle θ.
```

The e() function is behaving like exponeniation, that is we can write:

```
 e(iθ) = e**iθ
```

where e is some number whose value is as yet unknown, which is the solution to the equation:

```
 e**iπ = -1
```

As the e() function behaves like an exponential:

```
e(i*θ1) * e(i*θ2) = e**(i*θ1) * e** (i*θ2) = e**(i*(θ1+θ2)) = e(i(θ1+θ2))
```

In particular:

```
(cos(θ) + i * sin(θ))**n =  e(iθ)**n
                         = (e**iθ)**n
                         =  e**inθ
                         =  e(inθ)
                         = (cos(nθ) + i * sin(nθ))
```

Lets try this formula out with n = 2:

```
(cos(θ) + i * sin(θ))**2                       = cos(2θ) + i * sin(2θ)
```

=> (cos(θ)**2 - sin(θ))**2 + 2 * i * cos(θ)sin(θ) = cos(2θ) + i * sin(2θ)

Now, the number i is manifestly not a real number, as no real number is a solution to the equation i*i = -1, yet both the cos() and sin() functions produce real numbered results, they are, after all, just the ratios of the lengths of the sides of triangles. Consequently in the above we can equate the parts of the equations which are separated by being multiplied by i to get two equations:

```
cos(2θ) = cos(θ)**2 - sin(θ))**2
sin(2θ) = 2*cos(θ)*sin(θ)
```

Recall the "Cosine Angle Sum Formula" of:

```
cos(θ1+θ2) = cos(θ1)cos(θ2) - sin(θ1)sin(θ2)
```

Set θ = θ1 = θ2 to get the identical result:

```
cos(θ+θ) = cos(2θ) = cos(θ)cos(θ) - sin(θ)sin(θ)
```

Likewise the Recall the "Sine Angle Sum Formula" of:

```
sin(θ1+θ2) = cos(θ1)sin(θ2) + sin(θ1)cos(θ2)
```

Set θ = θ1 = θ2 to get the identical result:

```
sin(θ+θ) = sin(2θ) = cos(θ)sin(θ) + sin(θ)cos(θ) = 2*sin(θ)cos(θ)
```

Using Cosine and Sine Angle Sum Formulae and equating parts, we can deduce that:

```
e(i*θ1) * e(i*θ2))
 = (cos(θ1) + i * sin(θ1)) * (cos(θ2) + i * sin(θ2))
 = (cos(θ1)*cos(θ2) - sin(θ1)*sin(θ2) + i(cos(θ1)sin(θ2)) + cos(θ2)sin(θ1))
 =  cos(θ1 + θ2) + isin(θ1 + θ2)
 =  e(i(θ1+θ2))
```

wherein the e() function demonstrates its exponential nature to perfection.

The ability of i to partition single equations into two orthogonal simultaneous equations makes expressions of the form e(iθ), and hence trigonometery, invaluable in such diverse applications as electronicssimultaneously representing current and voltage in the same equation; and quantum mechanics, where it is necessary to represent position and momentum, or time and energy as pairs of variables partitioned by the uncertainty principle.

Taylor series approximations for the trig. functions

Having learnt a good deal about the properties of the cos() function, it might be helpful to know how to calculate cos(θ) for a given θ. We know that cos() can be defined in terms of the e() function which has the remarkable property that d e(θ) / dθ = e(θ), and e(0) = 1. Perhaps it is possible to represent e() by an infinite polynomial:

```
e(θ) = a0 + a1*θ + a2*θ**2 + a3*θ**3 + ...  an*θ**n +  ....
```

in which case evaluating at θ = 0 yields a0 = 1. To find the rate of change with θ of the n'th term:

```
d an*θ**n / dθ = (an*(θ+δθ)**n - an*(θ)**n)           / δθ as δθ tends to zero
               = an *((θ+δθ)**n - θ**n)               / δθ as δθ tends to zero
               = an *((θ**n+n*δθ*θ**(n-1)+... - θ**n) / δθ as δθ tends to zero
               = an *(n*δθ*θ**(n-1)+...)              / δθ as δθ tends to zero
               = an * n*θ*(n-1)+...
```

where the ... represent terms multiplied by at least δθ**2, and which therefore tend to zero as δθ tends to zero.

Finding the rate of change with respect to θ of the polynomial representing e(θ) produces:

```
e(θ) = a1*θ + 2 * a2*θ**1 + 3 * a3*θ**2 + ...  n * an*θ**(n-1) +  ....
```

Again evaluating at θ = 0, we get a1 = 1

Repeating the whole process, we get:

```
e(θ) = 2 * a2 + 3 * 2* a3*θ + ...  n * (n-1) * an*θ**(n-2) +  ....
```

and evaluating at θ = 0, we get a2 = 1/2, and in general, an = 1/n!, wher n! means n*(n-1)*(n-2)*...3*2*1.

Putting these results together, we get:

```
e(θ) = 1 + θ + θ**2/(2*1) + θ**3/(3*2*1)2 + ... θ**n / n! +  ....
```

Recalling that e(θ) = e**θ, and thus that e(1) = e**1 = e, we find e to be:

e = 1+1+1/2+1/6+1/24+.... 1/n!+ ...

which is approximately equal to 2.71828183

To calculate cos(θ), we would apply the formula:

```
   2*cos(θ) = e(iθ)+e(-iθ)
            = 1 + iθ - θ**2/2 - iθ**3/6 + θ**4/24 + ...
            + 1 +-iθ - θ**2/2 --iθ**3/6 + θ**4/24 + ...
            = 2      -2θ**2/2          +2θ**4/24 + ...
 => cos(θ)  = 1 - θ**2/2 + θ**4/24 + ... (-1)**n*θ**(2n)/((2n)!)+ ...
```

in which the odd terms cancel out and the even terms double and alternate in sign. Substituting -θ for θ does not change the result as θ occurs with even powers, hence the earlier phrase that cos() is an even function.

Using the Taylor series to calculate π

Having calculated, e and cos(), lets see if we can calculate π. Draw a circle with radius of length one, and draw diameters to divide the circle up into n segments of equal area, and hence equal angles θ; connect the ends of each diameter to its neighbors to get a regular polygon of n equal length sides. Each segment of the polygon is an isoceles triangle imbedded within a segment of the circle. The length of the sides og the polygon is less than the distance along the circumference of the circle between each corner of the polygon, the circumference length is exactly θ because we are using radians to measure angles within a circle of radius of length one. We can bisect each side of the polygon to construct a regular polygon with 2*n equal sides,the angle between each diameter is now θ/2 - lets us call the angle between diameters the interior angle. As the interior angle gets smaller and smaller, the lengths measured between two adjacent corners of the polygon, measured first along the side of the polygon, the choord length, and secondly along the circumference of the circle, the arc length, agree more and more closely. We can get this agreement as close as we might like by dividing the interior angle often enough, although to get eact agreement we would have to divide the interior angle an infinte number of times, clearly impossible in the real world of physics, but within the realm of mathematics it is possible to contemplate such an infinite process.

We have already defined the length of the circumference of a circle of diameter of length one as having the value π, so now all we need to do is find the total length of the sides of the polygons we are using to approximate π and then extrapolate as the number of sides approaches infinity to get an estimate for the value of π.

Let us start the process off using a polygon of 4 sides, a square, which has an interior angle of a right angle, that is, π/2. Draw a square, then draw its diagonals, and a circle through the corners of the square, to get 4 chords and 4 arcs. Scale the drawing so that the radius of the circle drawn is exactly 1/2. Each chord is the base of isoceles triangle which can be bisected into two identical right angled triangles. The hypotenuse or side c of of each of these right angle triangles is a radius, we have scaled the drawing above so that each such radius will be of length 1/2. The angle θ of each such right angle triangle is 1/2*π/2 = π/4. The length of side b in each such right angle triangle has length 1/2*sin(π/4) = 1/2*sin(2π/8). If we bisect each interior angle of the square we get a total of 8 of these right angled triangles, hence our first approximation to the length of the circumference of a circle of diameter of length one which is defined to be π, is 8 * the length of side b:

```
    π = 8 * 1/2 * sin(2*π/8)
=> 2π = n * sin(2*π/n) where n = 8.
```

If we extend each side a of each of the each such right angle triangles to meet the circumference of the circle at points, and then draw lines from each of these points to the points neighboring points where the diagonals of the square cross the circumference we will construct the octogon - a polygon with 8 equal sides - twice the number of the square, with interior angles of π/4, half that of the square. The argument used above to calculate the lengths of the sides of the square can be applied to the octogon. When this argument was first applied to the square it seems unnecessarily complicatedit would have been easier to notice that diagonals of the square conveniently meet at right angles allowing us to apply Pythagoras immediately to calculate the length of the side of a square drawn in a circle of radius of length 1/2 as squareRoot(1/2**2+1/2**2) = squareRoot(1/4+1/4) = squareRoot(1/2). While this answer is perfectly correct for a square, it does not work for any other polygon as their diagonals do not meet at such a convenient angle, while the formula:

```
=> 2π = n * sin(2*π/n)
```

works for any value of n.

All we need now is the value of the sin() function for polygions with many sides, that is with very small interior angles, or more specifically, with n large. We do know the sin() function for one angle:

```
sin(π/2) = sin(2π/4) = 1
```

The "Sine Half Angle Formula" lets us calculate the value of the sin() fiction for half angle of known angles:

```
    sin(θ/2)      = squareRoot(1/2)*squareRoot(1 - squareRoot(1 - sin(θ)**2))

 => sin(2π/8)     = squareRoot(1/2)*squareRoot(1 - squareRoot(1 - 1**2))
                  = squareRoot(1/2)
 => sin(2π/16)    = squareRoot(1/2)*squareRoot(1 - squareRoot(1 - squareRoot(1/2)**2))
                  = squareRoot(1/2)*squareRoot(1 - squareRoot(1 - 1/2))
                  = squareRoot(1/2 - 1/2*squareRoot(1/2))
 =>  sin(2π/32)      =  squareRoot(1/2)*squareRoot(1  -  squareRoot(1  -  (squareRoot(1/2  -
1/2*squareRoot(1/2)))**2))
                  =    squareRoot(1/2)*squareRoot(1    -    squareRoot(1    -    1/2    +
1/2*squareRoot(1/2)))
                  = squareRoot(1/2)*squareRoot(1 - squareRoot(1/2 + 1/2*squareRoot(1/2)))
                  = squareRoot(1/2 - 1/2*squareRoot(1/2 + 1/2*squareRoot(1/2)))
```

Its easy to guess from this pattern that:

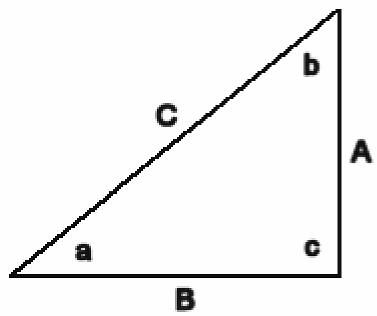
```
 =>  sin(2π/64)       =  squareRoot(1/2  -  1/2*squareRoot(1/2  +  1/2*squareRoot(1/2  +
1/2*squareRoot(1/2))))
```

and so on. This formula does give an approximate value for π, an 8 digit calculator produced a value of 3.13 which is within 1% of the true value. However, this formula cannot be ergarded as an efficient way of calculating π accuartely to many digitsthe constant need to take square roots insures a continual loss of precision.

We have now built up some basic trigonometric results.

Trigonometric defintitions

We have defined the *sine, cosine,* and *tangent* functions using the unit circle. Now we can apply them to a right triangle.



A right triangle

This triangle has *sides A* and *B*. The angle between them, *c*, is a right angle. The third side, *C*, is the *hypotenuse.* Side *A* is *opposite* angle *a*, and side *B* is *adjacent* to angle *a*.

Applying the definitions of the functions, we arrive at these useful formulas:

```
sin(a) = A / C  or opposite side over hypotenuse
cos(a) = B / C  or adjacent side over hypotenuse
tan(a) = A / B  or opposite side over adjacent side
```

This can be memorized using the mnenomic 'SOHCAHTOA' (sin = opposite over hypothenuse, cosine = adjacent over hypotenuse, tangent = opposite over adjacent).

Exercises(Draw a diagram!)

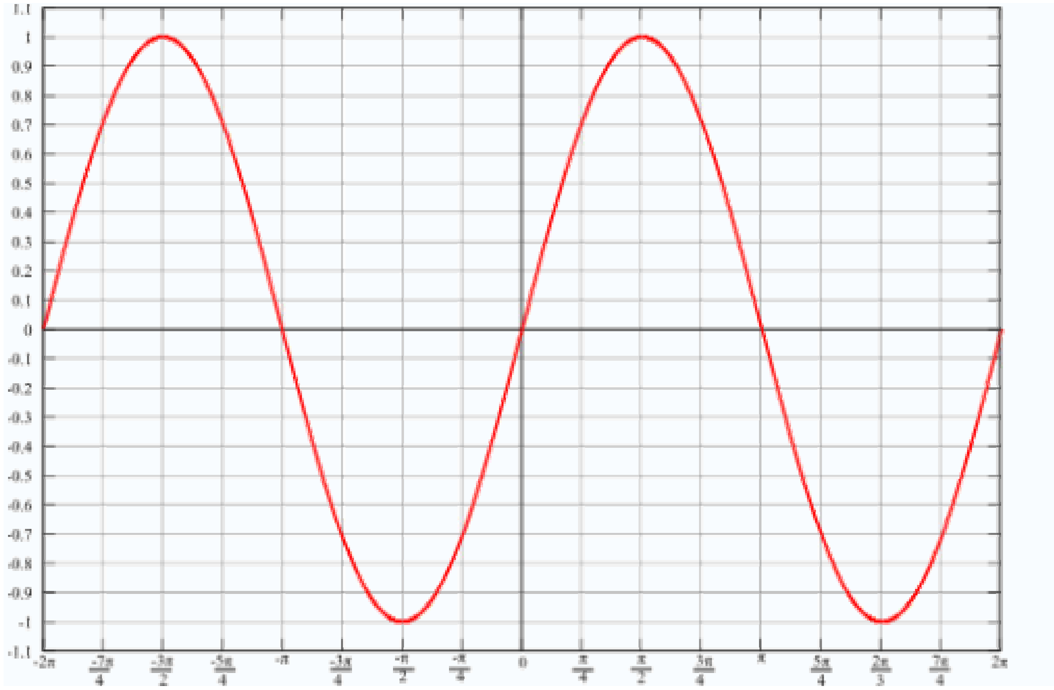
1. A right triangle has side A = 3, B = 4, and C = 5. Calculate the following:

```
sin(a), cos(a), tan(a)
```

2. A different right triangle has side C = 6 and sin(a) = 0.5 . Calculate side A.

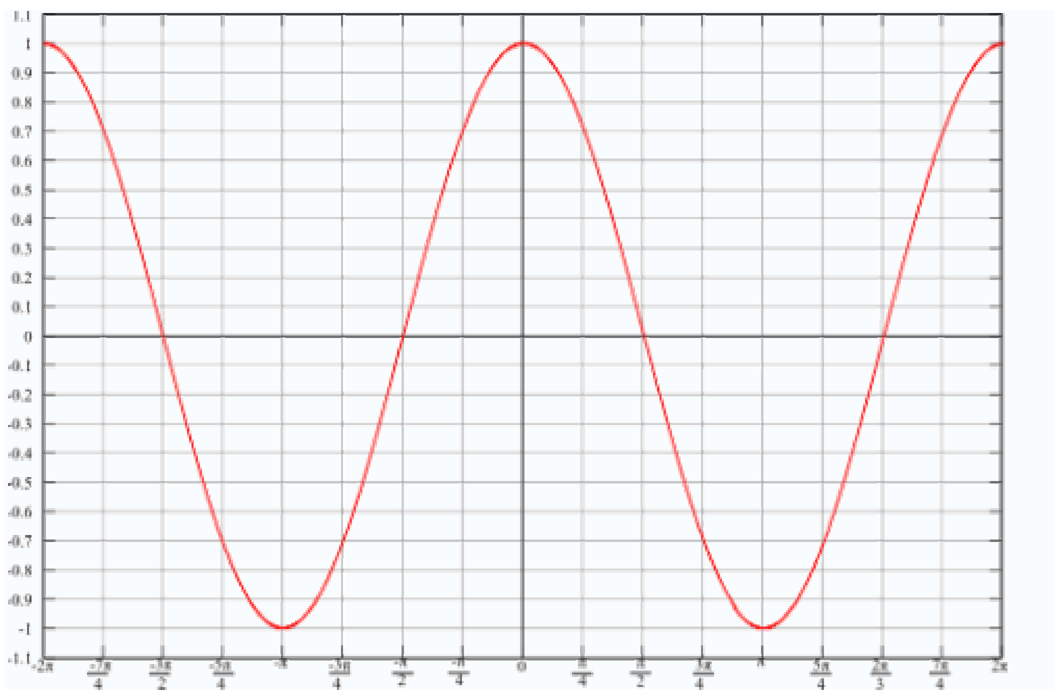
Graphs of Sine and Cosine Functions

The graph of the sine function looks like this:



Careful analysis of this graph will show that the graph corresponds to the unit circle. X is essentially the degree measure(in radians), while Y is the value of the sine function.

The graph of the cosine function looks like this:

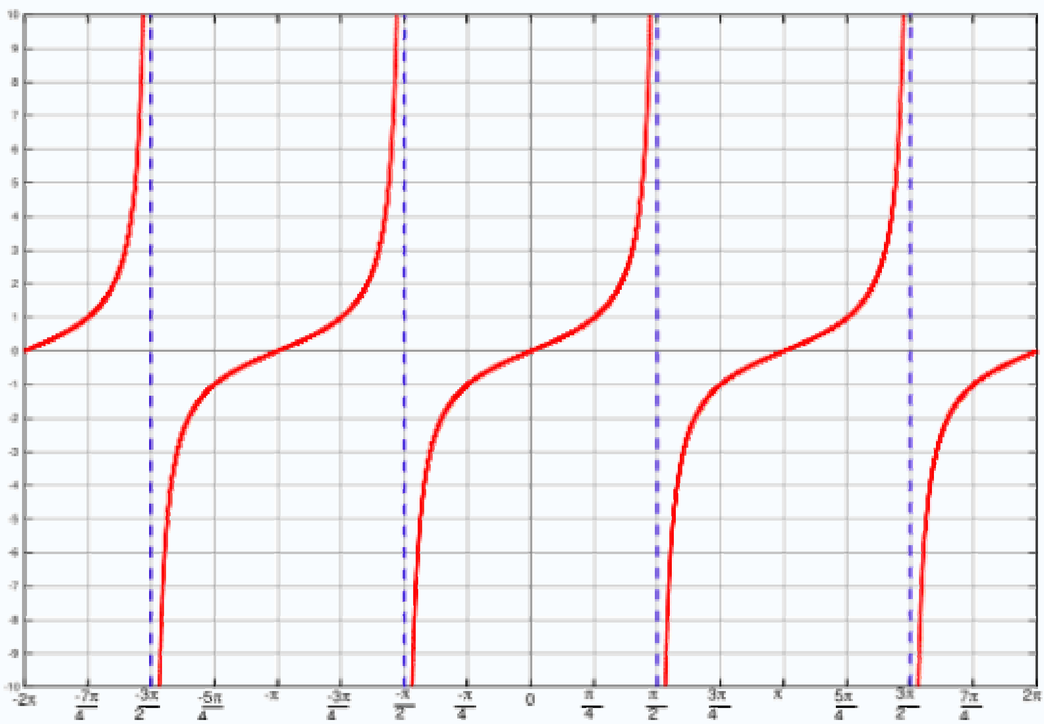


As with the sine function, analysis of the cosine function will show that the graph corresponds to the unit circle. One of the most important differences between the sine and cosine functions is that sine is an odd function while cosine is an even function.

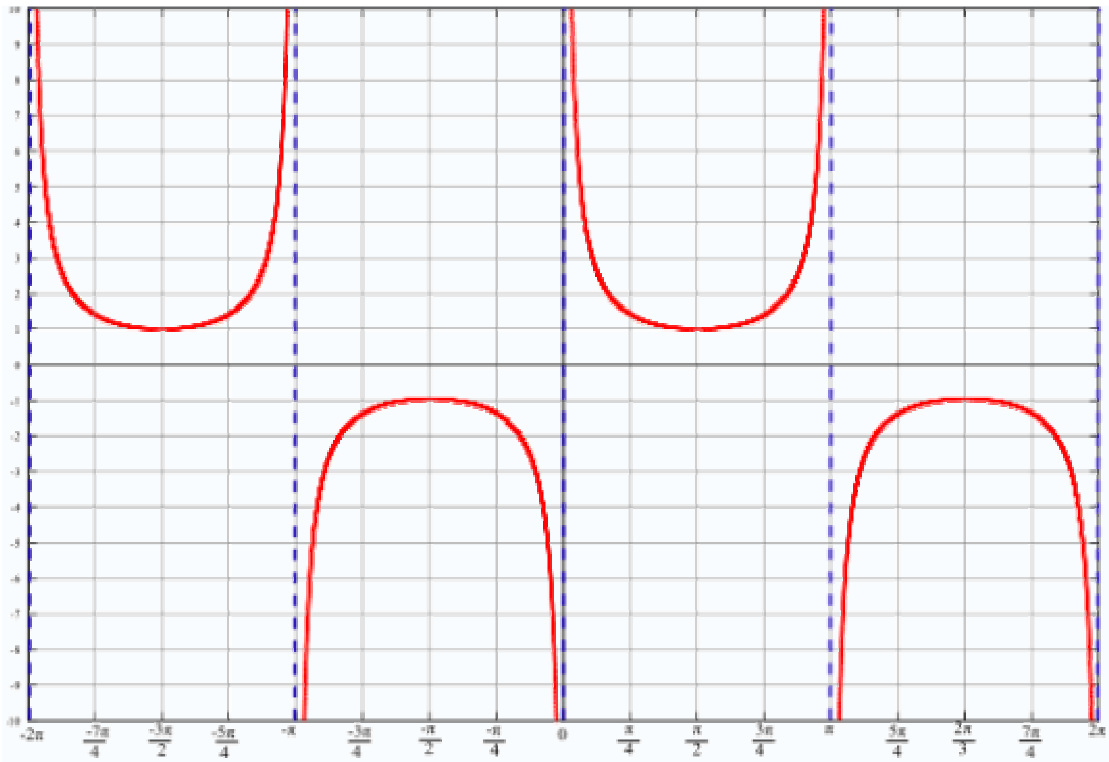
Sine and cosine are periodic functions; that is, the above is repeated for preceding and following intervals with length 2π.

Graphs of Other Trigonometric Functions

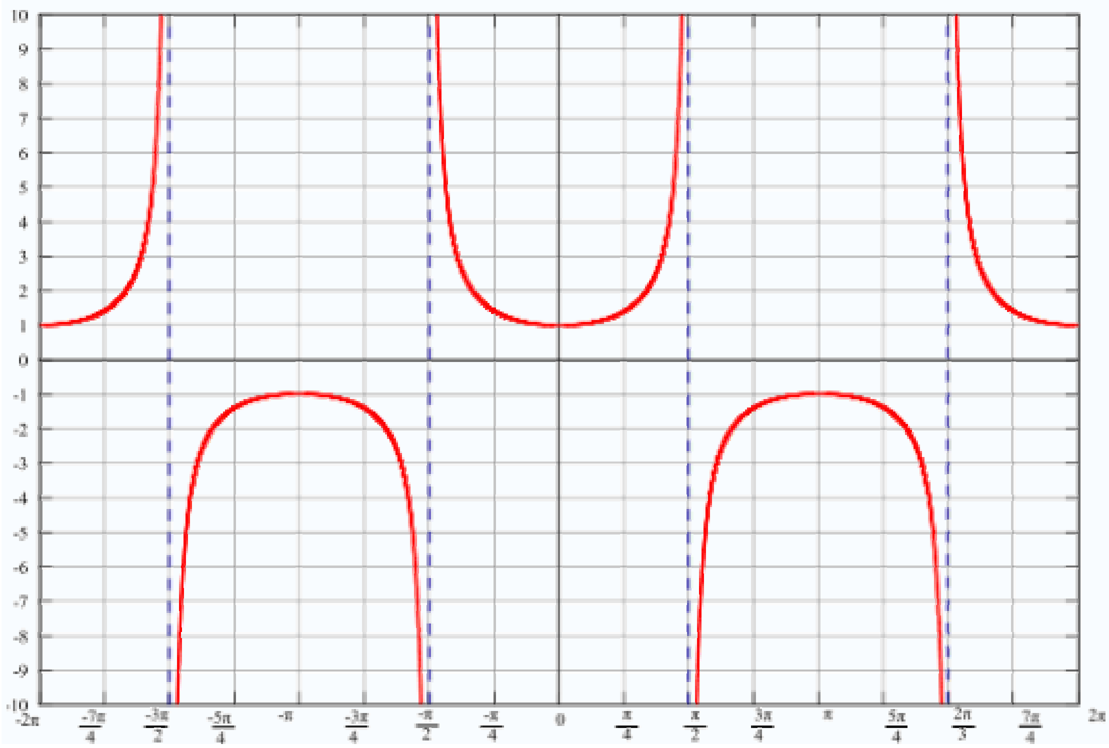
A graph of $\tan(x)$. $\tan(x)$ is defined as $\frac{\sin(x)}{\cos(x)}$.



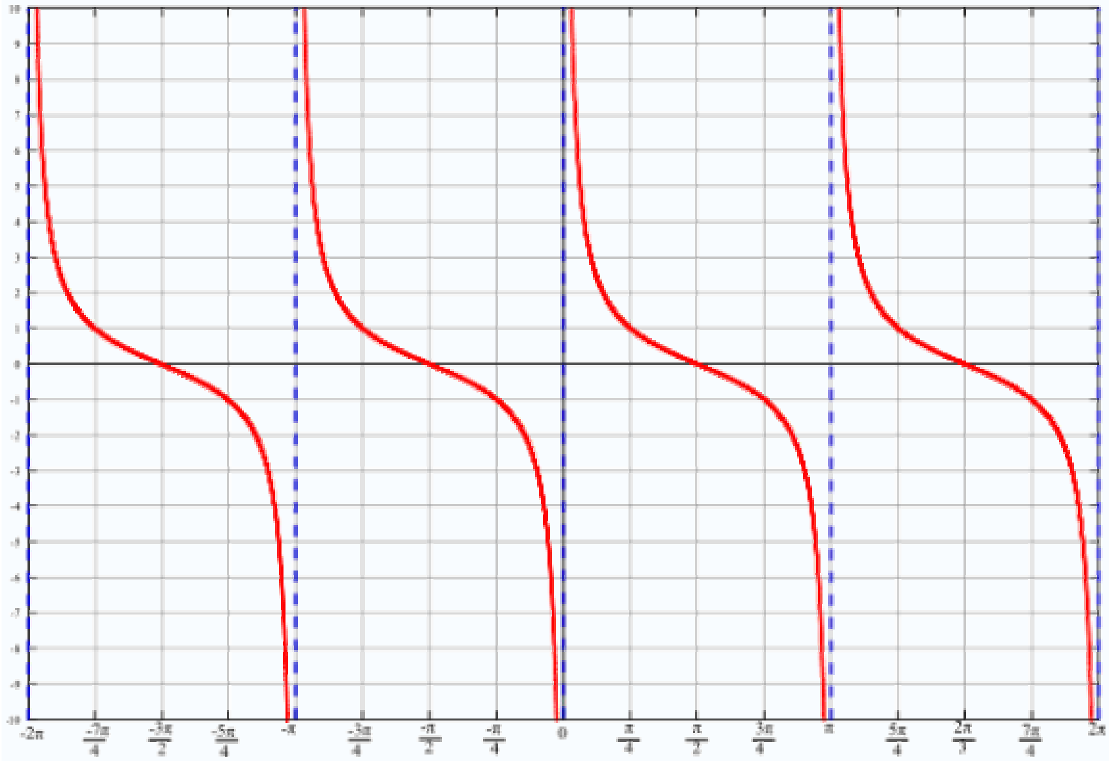
A graph of $\csc(x)$. $\csc(x)$ is defined as $\frac{1}{\sin(x)}$.



A graph of $\sec(x)$. $\sec(x)$ is defined as $\frac{1}{\cos(x)}$.



A graph of $\cot(x)$. $\cot(x)$ is defined as $\frac{1}{\tan(x)}$ or $\frac{\cos(x)}{\sin(x)}$.



Note that $\tan(x)$, $\sec(x)$, and $\csc(x)$ are unbounded.

Inverse-Trigonometric Functions

The Inverse Functions, Restrictions, and Notation

While it might seem that **inverse trigonometric functions** should be relatively self defining, some caution is necessary to get an inverse *function* since the trigonometric functions are not one-to-one. To deal with this issue, some texts have adopted the convention of allowing $\sin^{-1}x$, $\cos^{-1}x$, and $\tan^{-1}x$ (all with lower-case initial letters) to indicate the inverse *relations* for the trigonometric functions and defining new functions $\mathrm{Sin}x$, $\mathrm{Cos}x$, and $\mathrm{Tan}x$ (all with initial capitals) to equal the original functions but with restricted domain, thus creating one-to-one functions with the inverses $\mathrm{Sin}^{-1}x$, $\mathrm{Cos}^{-1}x$, and $\mathrm{Tan}^{-1}x$. For clarity, we will use this convention. Another common notation used for the inverse functions is the "arcfunction" notation $\mathrm{Sin}^{-1}x = \arcsin x$, $\mathrm{Cos}^{-1}x = \arccos x$, and $\mathrm{Tan}^{-1}x = \arctan x$ (the arcfunctions are sometimes also capitalized to distinguish the inverse *functions* from the inverse *relations*). The arcfunctions may be so named because of the relationship between radian measure of angles and arclength--the arcfunctions yeild arc lengths on a unit circle.

The restrictions necessary to allow the inverses to be functions are standard $\mathrm{Sin}^{-1}x$ has range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; $\mathrm{Cos}^{-1}x$ has range $[0, \pi]$; and $\mathrm{Tan}^{-1}x$ has range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (these restricted ranges for the inverses are the restricted domains of the capital-letter trigonometric functions). For each inverse function, the restricted range includes first-quadrant angles as well as an adjacent quadrant that completes the domain of the inverse function and maintains the range as a single interval.

It is important to note that because of the restricted ranges, the inverse trigonometric functions do not necessarily behave as one might expect an inverse function to behave. While $\mathrm{Sin}^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right) = \mathrm{Sin}^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ (following the expected $\mathrm{Sin}^{-1}(\sin x) = x$), $\mathrm{Sin}^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right) = \mathrm{Sin}^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$! For the inverse trigonometric functions, $f^{-1} \circ f(x) = x$ only when x is in the range of the inverse function. The other direction, however, is less tricky $f \circ f^{-1}(x) = x$ for all x to which we can apply the inverse function.

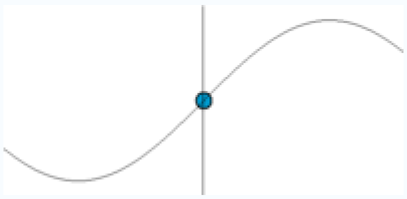
The Inverse Relations

For the sake of completeness, here are definitions of the inverse trigonometric *relations* based on the inverse trigonometric *functions*:

- $\sin^{-1} x = \left\{\mathrm{Sin}^{-1} x + 2\pi n, n \in \mathbb{Z}\right\} \cup \left\{\pi - \mathrm{Sin}^{-1} x + 2\pi n, n \in \mathbb{Z}\right\}$ (the sine function has period 2π, but within any given period may have two solutions and $\sin x = \sin(\pi - x)$)
- $\cos^{-1} x = \left\{\pm \mathrm{Cos}^{-1} x + 2\pi n, n \in \mathbb{Z}\right\}$ (the cosine function has period 2π, but within any given period may have two solutions and cosine is even--$\cos x = \cos(-x)$)
- $\tan^{-1} x = \left\{\mathrm{Tan}^{-1} x + \pi n, n \in \mathbb{Z}\right\}$ (the tangent function has period π and is one-to-one within any given period)

Applications and Models

Simple harmonic motion



Simple harmonic motion. Notice that the position of the dot matches that of the sine wave.

Simple harmonic motion (SHM) is the motion of an object which can be modeled by the following function:

$$x = A\sin\left(\omega t + \phi\right)$$

or

$$x = c_1\cos\left(\omega t\right) + c_2\sin\left(\omega t\right)$$

where $c_1 = A \sin \varphi$ and $c_2 = A \cos \varphi$.

In the above functions, A is the amplitude of the motion, ω is the angular velocity, and φ is the phase.

The velocity of an object in SHM is

$$v = A\omega\cos\left(\omega t + \phi\right)$$

The acceleration is

$$a = -A\omega^2\sin\left(\omega t + \phi\right)$$

Springs and Hooke's Law

An application of this is the motion of a weight hanging on a spring. The motion of a spring can be modeled approximately by Hooke's Law:

$F = -kx$

where F is the force the spring exerts, x is the position of the end of the spring, and k is a constant characterizing the spring (the stronger the spring, the higher the constant).

Calculus-based derivation

From Newton's laws we know that *F* = *ma* where *m* is the mass of the weight, and *a* is its acceleration. Substituting this into Hooke's Law, we get

$$ma = -kx$$

Dividing through by *m*:

$$a = -\frac{k}{m}x$$

The calculus definition of acceleration gives us

$$x'' = -\frac{k}{m}x$$
$$x'' + \frac{k}{m}x = 0$$

Thus we have a second-order differential equation. Solving it gives us

$$x = c_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}t\right) \quad \textbf{(2)}$$

with an independent variable *t* for time.

We can change this equation into a simpler form. By lettting c_1 and c_2 be the legs of a right triangle, with angle φ adjacent to c_2, we get

$$\sin\phi = \frac{c_1}{\sqrt{c_1^2 + c_2^2}}$$
$$\cos\phi = \frac{c_2}{\sqrt{c_1^2 + c_2^2}}$$

and

$$c_1 = \sqrt{c_1^2 + c_2^2}\sin\phi$$
$$c_2 = \sqrt{c_1^2 + c_2^2}\cos\phi$$

Substituting into (2), we get

$$x = \sqrt{c_1^2 + c_2^2} \sin\phi \cos\left(\sqrt{\frac{k}{m}} t\right) + \sqrt{c_1^2 + c_2^2} \cos\phi \sin\left(\sqrt{\frac{k}{m}} t\right)$$

Using a trigonometric identity, we get:

$$x = \sqrt{c_1^2 + c_2^2} \left[\sin\left(\phi + \sqrt{\frac{k}{m}} t\right) + \sin\left(\phi - \sqrt{\frac{k}{m}} t\right)\right] + \sqrt{c_1^2 + c_2^2} \left[\sin\left(\sqrt{\frac{k}{m}} t + \right.\right.$$

$$x = \sqrt{c_1^2 + c_2^2} \sin\left(\sqrt{\frac{k}{m}} t + \phi\right)$$ **(3)**

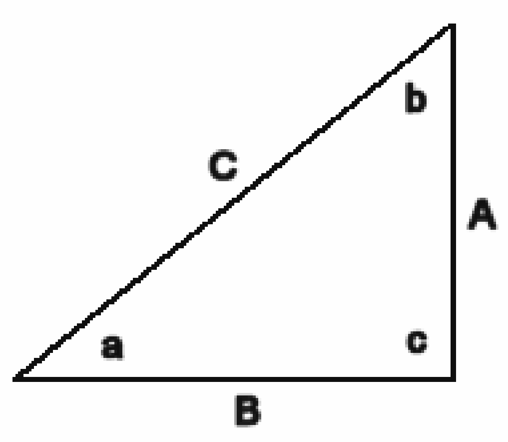
Let $A = \sqrt{c_1^2 + c_2^2}$ and $\omega^2 = \frac{k}{m}$. Substituting this into (3) gives

$$x = A \sin(\omega t + \phi)$$

Analytic Trigonometry

Using Fundamental Identities

Some of the fundamental trigometric identities are those derived from the Pythagorean Theorem. These are defined using a right triangle:



A right triangle

By the Pythagorean Theorem,

$A^2 + B^2 = C^2$ {1}

Dividing through by C^2 gives

$$\left(\frac{A}{C}\right)^2 + \left(\frac{B}{C}\right)^2 = \left(\frac{C}{C}\right)^2 = 1$$ {2}

We have already defined the sine of *a* in this case as *A*/*C* and the cosine of *a* as *B*/*C*. Thus we can substitute these into {2} to get

$\sin^2 a + \cos^2 a = 1$

Related identities include:

$\sin^2 a = 1 - \cos^2 a$ or $\cos^2 a = 1 - \sin^2 a$
$\tan^2 a + 1 = \sec^2 a$ or $\tan^2 a = \sec^2 a - 1$
$1 + \cot^2 a = \csc^2 a$ or $\cot^2 a = \csc^2 a - 1$

Other Fundamental Identities include the **Reciprocal**, **Ratio**, and **Co-function** identities

Reciprocal identities

$$\csc a = \frac{1}{\sin a} \quad \sec a = \frac{1}{\cos a} \quad \cot a = \frac{1}{\tan a}$$

Ratio identities

$$\tan a = \frac{\sin a}{\cos a} \quad \cot a = \frac{\cos a}{\sin a}$$

Co-function identities (in radians)

$$\cos a = \sin\left(\frac{\pi}{2} - a\right) \quad \csc a = \sec\left(\frac{\pi}{2} - a\right) \quad \cot a = \tan\left(\frac{\pi}{2} - a\right)$$

Verifying-Trigonometric Identities

To verify an identity means to make sure that the equation is true by setting both sides equal to one another.

There is no set method that can be applied to verifying identities; there are, however, a few different ways to start based on the identity which is to be verified.

Introduction

Trigonometric identities are used in both course texts and in real life applications to abbreviate trigonometric expressions. It is important to remember that merely verifying an identity or altering an expression is not an end in itself, but rather that identities are used to simplify expressions according to the task at hand. Trigonometric expressions can always be reduced to sines and cosines, which are more managable than their inverse counterparts.

To verify an identity:

1) Always try to reduce the larger side first.

2) Sometimes getting all trigonometric functions on one side can help.

3) Remember to use and manipulate already existing identities. The Pythagorean identities are usually the most useful in simplifying.

4) Remember to factor if needed.

5) Whenever you have a squared trigonometric function such as Sin^2(t), always go to your pythagorean identities, which deal with squared functions.

Easy example $1 / cot(t) = sin(t) / cos(t)$

```
1/cot(t)= tan(t) ,so
```
$tan(t) = sin(t) / cos(t)$

tan(t) is the same as sint(t)/cos(t) and therefore can be rewritten as $sin(t) / cos(t) = sin(t) / cos(t)$

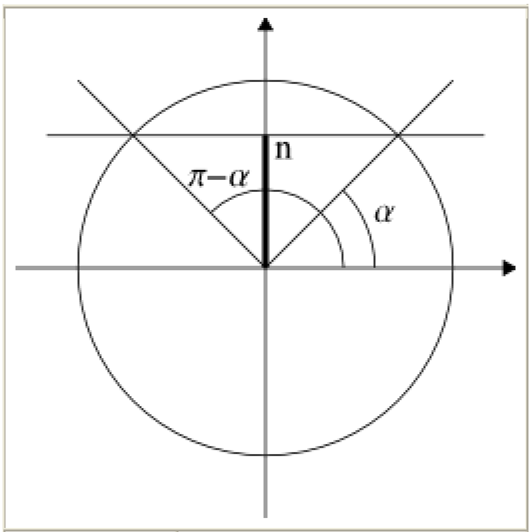
Identity verified

Solving-Trigonometric Equations

Trigonometric equations involve finding an unknown which is an argument to a trigonometric function.

Basic trigonometric equations

sin *x* = n



n	$\sin x = n$
$\|n\| < 1$	$x = \alpha + 2k\pi$ $x = \pi - \alpha + 2k\pi$ $\alpha \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$
$n = -1$	$x = -\frac{\pi}{2} + 2k\pi$
$n = 0$	$x = k\pi$
$n = 1$	$x = \frac{\pi}{2} + 2k\pi$
$\|n\| > 1$	$x \in \varnothing$

The equation $\sin x = n$ has solutions only when n is within the interval [-1; 1]. If n is within this interval, then we need to find an α such that:

$$\alpha = \sin^{-1} n$$

The solutions are then:

$$x = \alpha + 2k\pi$$
$$x = \pi - \alpha + 2k\pi$$

Where k is an integer.

In the cases when n equals 1, 0 or -1 these solutions have simpler forms which are summarizied in the table on the right.

For example, to solve:

$$\sin\frac{x}{2} = \frac{\sqrt{3}}{2}$$

First find α:

$$\alpha = \sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$$

Then substitute in the formulae above:

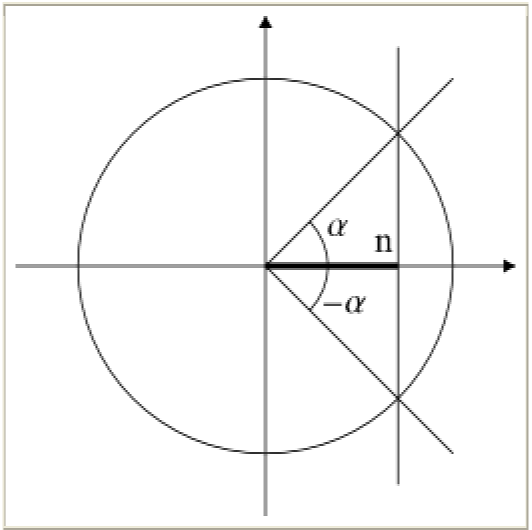
$$\frac{x}{2} = \frac{\pi}{3} + 2k\pi$$
$$\frac{x}{2} = \pi - \frac{\pi}{3} + 2k\pi$$

Solving these linear equations for x gives the final answer:

$$x = \frac{2\pi}{3}(1 + 6k)$$
$$x = \frac{4\pi}{3}(1 + 3k)$$

Where k is an integer.

cos *x* = n



n	$\cos x = n$
$\|n\| < 1$	$x = \pm\alpha + 2k\pi$ $\alpha \in [0;\pi]$
$n = -1$	$x = \pi + 2k\pi$
$n = 0$	$x = \frac{\pi}{2} + k\pi$
$n = 1$	$x = 2k\pi$
$\|n\| > 1$	$x \in \varnothing$

Like the sine equation, an equation of the form $\cos x = n$ only has solutions when n is in the interval [-1; 1]. To solve such an equation we first find the angle α such that:

$$\alpha = \cos^{-1} n$$

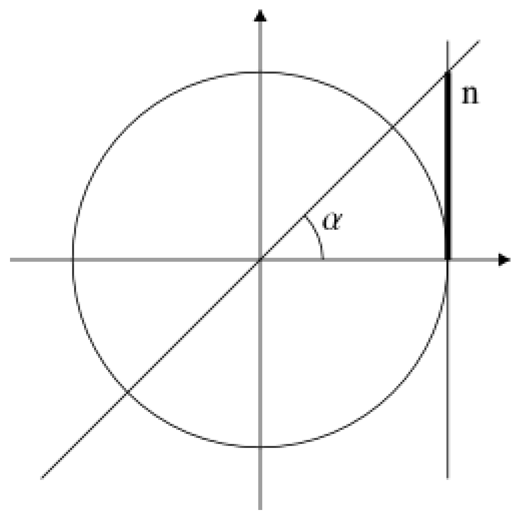
Then the solutions for x are:

$$x = \pm\alpha + 2k\pi$$

Where k is an integer.

Simpler cases with n equal to 1, 0 or -1 are summarized in the table on the right.

tan *x* = n



n	$\tan x = n$
General case	$x = \alpha + k\pi$ $\alpha \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$
$n = -1$	$x = -\frac{\pi}{4} + k\pi$
$n = 0$	$x = k\pi$
$n = 1$	$x = \frac{\pi}{4} + k\pi$

An equation of the form $\tan x = n$ has solutions for any real n. To find them we must first find an angle α such that:

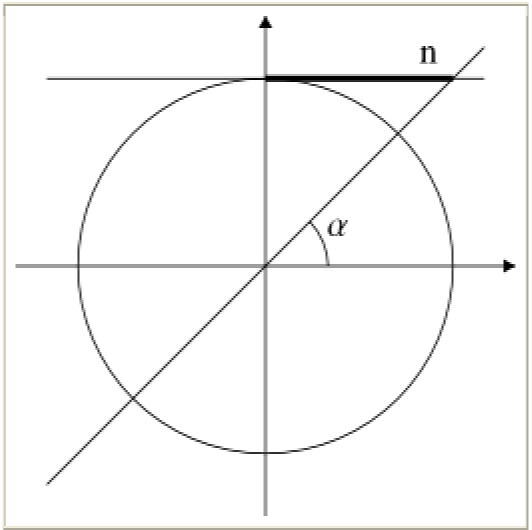
$$\alpha = \tan^{-1} n$$

After finding α, the solutions for x are:

$$x = \alpha + k\pi$$

When n equals 1, 0 or -1 the solutions have simpler forms which are shown in the table on the right.

cot *x* = n



n	$\cot x = n$
General case	$x = \alpha + k\pi$ $\alpha \in [0; \pi]$
$n = -1$	$x = -\frac{3\pi}{4} + k\pi$
$n = 0$	$x = \frac{\pi}{2} + k\pi$
$n = 1$	$x = \frac{\pi}{4} + k\pi$

The equation $\cot x = n$ has solutions for any real n. To find them we must first find an angle α such that:

$$\alpha = \cot^{-1} n$$

After finding α, the solutions for x are:

$$x = \alpha + k\pi$$

When n equals 1, 0 or -1 the solutions have simpler forms which are shown in the table on the right.

csc *x* = n and sec *x* = n

The trigonometric equations csc *x* = n and sec *x* = n can be solved by transforming them to other basic equations:

$$\csc x = n \Leftrightarrow \frac{1}{\sin x} = n \Leftrightarrow \sin x = \frac{1}{n}$$

$$\sec x = n \Leftrightarrow \frac{1}{\cos x} = n \Leftrightarrow \cos x = \frac{1}{n}$$

Further examples

Generally, to solve trigonometric equations we must first transform them to a basic trigonometric equation using the trigonometric identities. This sections lists some common examples.

a sin *x* + *b* cos *x* = *c*

To solve this equation we will use the identity:

$$a\sin x + b\cos x = \sqrt{a^2+b^2}\sin(x+\alpha)$$
$$\alpha = \begin{cases} \tan^{-1}(b/a), & \text{if } a>0 \\ \pi + \tan^{-1}(b/a), & \text{if } a<0 \end{cases}$$

The equation becomes:

$$\sqrt{a^2+b^2}\sin(x+\alpha) = c$$
$$\sin(x+\alpha) = \frac{c}{\sqrt{a^2+b^2}}$$

This equation is of the form $\sin x = n$ and can be solved with the formulae given above.

For example we will solve:

$$\sin 3x - \sqrt{3}\cos 3x = -\sqrt{3}$$

In this case we have:

$$a = 1, b = -\sqrt{3}$$
$$\sqrt{a^2+b^2} = \sqrt{1^2 + \left(-\sqrt{3}\right)^2} = 2$$
$$\alpha = \tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$$

Apply the identity:

$$2\sin\left(3x - \frac{\pi}{3}\right) = -\sqrt{3}$$

$$\sin\left(3x-\frac{\pi}{3}\right)=-\frac{\sqrt{3}}{2}$$

So using the formulae for $\sin x = n$ the solutions to the equation are:

$$3x-\frac{\pi}{3}=-\frac{\pi}{3}+2k\pi \Leftrightarrow x=\frac{2k\pi}{3}$$
$$3x-\frac{\pi}{3}=\pi+\frac{\pi}{3}+2k\pi \Leftrightarrow x=\frac{\pi}{9}(6k+5)$$

Where k is an integer.

Sum and Difference Formulas

Cosine Formulas

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$$

Sine Formulas

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

$$\sin(a-b) = \sin a \cos b - \cos a \sin b$$

$$\sin 2a = 2\sin a \cos a$$

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$$

Tangent Formulas

$$\tan(a+b) = \frac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)}$$
$$\tan(a-b) = \frac{\tan(a)-\tan(b)}{1+\tan(a)\tan(b)}$$
$$\tan 2A = \frac{2\tan A}{1-\tan^2 A} = \frac{2\cot A}{\cot^2 A - 1} = \frac{2}{\cot A - \tan A}$$
$$\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}} = \frac{\sin A}{1+\cos A} = \frac{1-\cos A}{\sin A}$$

Derivations

- cos(a + b) = cos a cos b - sin a sin b
- cos(a - b) = cos a cos b + sin a sin b

Using cos(a + b) and the fact that cosine is even and sine is odd, we have

```
cos(a + (-b)) = cos a cos (-b) - sin a sin (-b)
              = cos a cos b - sin a (-sin b)
              = cos a cos b + sin a sin b
```

- sin(a + b) = sin a cos b + cos a sin b

Using cofunctions we know that sin a = cos (90 - a). Use the formula for cos(a - b) and cofunctions we can write

```
sin(a + b) = cos(90 - (a + b))
           = cos((90 - a) - b)
           = cos(90 -a)cos b + sin(90 - a)sin b
           = sin a cos b + cos a sin b
```

- sin(a - b) = sin a cos b - cos a sin b

Having derived sin(a + b) we replace b with "-b" and use the fact that cosine is even and sine is odd.

```
sin(a + (-b)) = sin a cos (-b) + cos a sin (-b)
              = sin a cos b + cos a (-sin b)
              = sin a cos b - cos a sin b
```

Multiple-Angle and Product-to-sum Formulas

Multiple-Angle Formulas

- sin(2a) = 2 sin a cos a
- cos(2a) = $\cos^2$ a - $\sin^2$ a = 1 - 2 $\sin^2$ a = 2 $\cos^2$ a - 1
- tan(2a) = (2 tan a)/(1 - $\tan^2$ a)

Proofs for Double Angle Formulas

Recall that: $cos(a + b) = cos(a)cos(b) - sin(a)sin(b)$

Using a = b in the above formula yields:

$$cos(2a) = cos(a + a) = cos(a)cos(a) - sin(a)sin(a) = cos^2(a) - sin^2(a)$$

From the last (rightmost) term, two more identities may be derived. One containing only a sine:

$$cos(2a) = cos^2(a) - sin^2(a) = cos^2(a) - sin^2(a) + 0 = cos^2(a) - sin^2(a) + [sin^2(a) - sin^2(a)] = [cos^2(a) + sin^2(a)] - 2sin^2(a) = 1 - 2sin^2(a)$$

And one containing only a cosine:

$$cos(2a) = cos^2(a) - sin^2(a) = cos^2(a) - sin^2(a) + 0 = cos^2(a) - sin^2(a) + [cos^2(a) - cos^2(a)] = 2cos^2(a) - [cos^2(a) + sin^2(a)] = 2cos^2(a) - 1$$

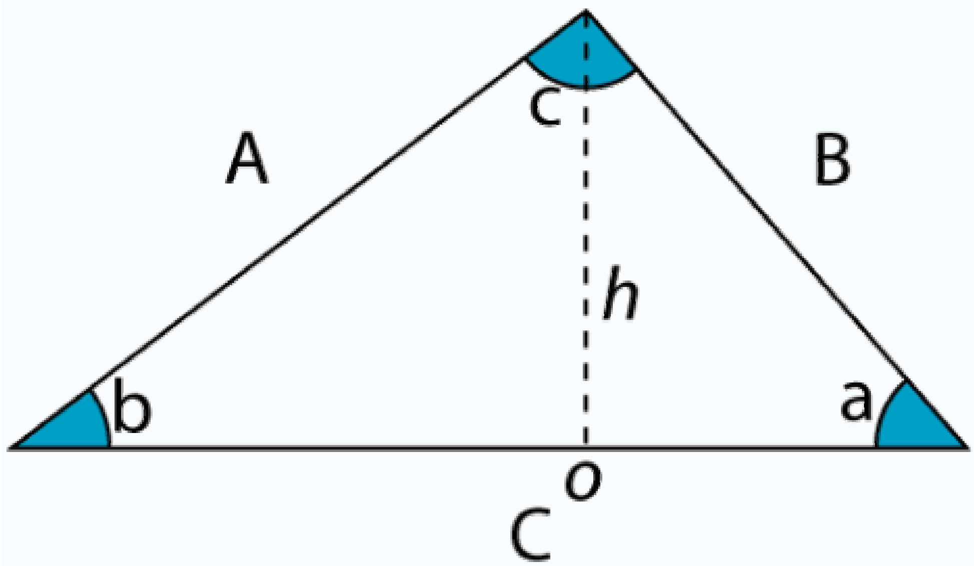
$sin(a + b) = sin(a)cos(b) + sin(b)cos(a)$; a = b for sin(2a)

$$sin(a)cos(a) + sin(a)cos(a) = 2cos(a)sin(a) = sin(2a)$$

Additional-Topics in Trigonometry

Law of Sines

Consider this triangle:



It has three sides

- A, length *A*, opposite angle *a* at vertex a
- B, length *B*, opposite angle *b* at vertex b
- C, length *C*, opposite angle *c* at vertex c

The perpendicular, *oc*, from line *ab* to vertex *c* has length *h*

The Law of Sines states that:

$$\frac{A}{\sin a} = \frac{B}{\sin b} = \frac{C}{\sin c}$$

The law can also be written as the reciprocal:

$$\frac{\sin a}{A} = \frac{\sin b}{B} = \frac{\sin c}{C}$$

Proof

The perpendicular, *oc*, splits this triangle into two right-angled triangles. This lets us calculate *h* in two different ways

- Using the triangle cao gives

  $h = B \sin a$

- Using the triangle cbo gives

  $h = A \sin b$

- Eliminate h from these two equations

  $A \sin b = B \sin a$

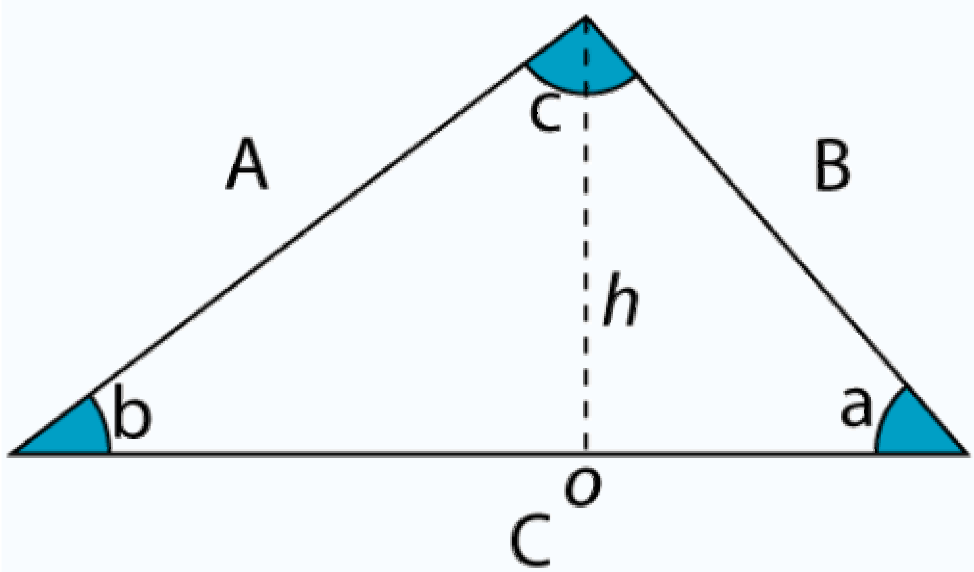
- Rearrange

  $$\frac{A}{\sin a} = \frac{B}{\sin b}$$

By using the other two perpendiculars the full law of sines can be proved.

Law of Cosines

Consider this triangle:



It has three sides

- a, length *a*, opposite angle *A* at vertex A

- b, length *b*, opposite angle *B* at vertex B
- c, length *c*, opposite angle *C* at vertex C

The perpendicular, oc, from line ab to vertex c has length *h*

The Law of Cosines states that:

$$a^2 = b^2 + c^2 - 2bc \cdot \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cdot \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cdot \cos C$$

Proof

The perpendicular, *oc*, divides this triangle into two right angled triangles, *aco* and *bco*.

First we will find the length of the other two sides of triangle *aco* in terms of known quantities, using triangle *bco*.

$h = a \sin B$

Side c is split into two segments, total length *c*.

ob, length $c \cos B$
ao, length $c - a \cos B$

Now we can use Pythagoras to find *b*, since $b^2 = ao^2 + h^2$

$$\begin{aligned} b^2 &= (c - a\cos B)^2 + a^2 \sin^2 B \\ &= c^2 - 2ac\cos B + a^2 \cos^2 B + a^2 \sin^2 B \\ &= a^2 + c^2 - 2ac\cos B \end{aligned}$$

The corresponding expressions for *a* and *c* can be proved similarly.

Solving Triangles

One of the most common applications of trigonometry is solving triangles—finding missing sides and/or angles given some information about a triangle. The process of solving triangles can be broken down into a number of cases.

Given a Right Triangle

If we know the triangle is a right triangle, one side and one of the non-right angles or two sides uniquely determine the triangle. (It is also worth noting that if we are given a triangle not specified to be a right triangle, but two given angles are complementary, then the third angle must be a right angle, so we have a right triangle.) Use right triangle trigonometry to find any missing information.

Given Three Sides (SSS)

Given three sides of a triangle, we can use the Law of Cosines to find any of the missing angles (the alternate form $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$ may be helpful here).

Given Two Sides and the Included Angle (SAS)

Given two sides and the angle included by the two given sides, we can apply the Law of Cosines to find the missing side, then procede as above to find the missing angles.

Given Two Angles and a Side (AAS or ASA)

Given two angles, we can find the third angle (since the sum of the measures of the three angles in a triangle is a straight angle). Knowing all three angles and one side, we can use the Law of Sines to find the missing sides.

Given Two Sides and an Angle Not Included by the Two Sides (SSA or the Ambiguous Case)

Here, we run into trouble—the given information may not uniquely determine a triangle. One of the two given sides is opposite the given angle, so we can apply the Law of Sines to try to find the angle opposite the other given side. When we do this (given sides a and b and angle A), we'll have $\sin B = \frac{b \sin A}{a}$ and there are three possibilities $\sin B > 1$, $\sin B = 1$, or $\sin B < 1$.

If $\sin B > 1$, then there is no angle B that meets the given information, so no triangle can be formed with the given sides and angle.

If $\sin B = 1$, then we have a right triangle with right angle at B and we can proceed as above for right triangles.

If $\sin B < 1$, then there are two possible measures for angle B, one accute (the inverse sine of the value of $\sin B$) and one obtuse (the supplement of the accute one). The accute angle will give a triangle and we can find the missing information as above now that we have two angles. The obtuse angle may or may not give a triangle--when we attempt to compute the third angle, it may be that $A + B$ is more than a straight angle, leaving no room for angle C. If there is room for an angle C, then there is a second possible triangle determined by the given information and we can find the last missing length as above.

Vectors in the Plane

In practice, one of the most useful applications of trigonometry is in calculations related to vectors, which are frequently used in Physics. A vector is a quantity which has both magnitude (such as three or eight) and direction (such as north or 30 degrees south of east). It is represented in diagrams by an arrow, often pointing from the origin to a specific point.

A plane vector $\vec{A}$ can be expressed in two ways -- as the sum of a horizontal vector of magnitude A_x and a vertical vector of magnitude A_y, or in terms of its angle θ and magnitude $|\vec{A}|$ (or simply A). These two methods are called "rectangular" and "polar" respectively.

Rectangular to Polar conversion

For simplicity, assume $\vec{A}$ is in the first quadrant and has x-component A_x and y-component A_y. Given these components, we want to find the angle θ and the magnitude A.

If we draw all three of these vectors, they form a right triangle. It is easy to see that $\tan\theta = \frac{A_y}{A_x}$, or $\theta = \arctan\frac{A_y}{A_x}$ (A vector with an angle of zero is defined to be pointing directly to the right.) Furthermore, by the Pythagorean Theorem, ${A_x}^2 + {A_y}^2 = A^2$, or $A = \sqrt{{A_x}^2 + {A_y}^2}$.

Polar to Rectangular conversion

This is essentially the same problem as above, but in reverse. Here, θ and A are known and we want to calculate the values of A_x and A_y.

Using the same triangle as above, we can see that $\cos\theta = \frac{A_x}{A}$, or $A_x = A\cos\theta$. Also, $\sin\theta = \frac{A_y}{A}$, or $A_y = A\sin\theta$.

Review of conversions

- $\theta = \arctan\frac{A_y}{A_x}$

- $\left|\vec{A}\right| = A = \sqrt{A_x{}^2 + A_y{}^2}$
- $A_x = A\cos\theta$
- $A_y = A\sin\theta$

Vectors and Dot Products

< Trigonometry

Consider the vectors **U** and **V** (with respective magnitudes **|U|** and **|V|**). If those vectors enclose an angle θ then the dot product of those vectors can be written as:

$$\mathbf{U}\cdot\mathbf{V} = |\mathbf{U}||\mathbf{V}|\cos(\theta)$$

If the vectors can be written as:

$$\mathbf{U} = (U_x, U_y, U_z)$$
$$\mathbf{V} = (V_x, V_y, V_z)$$

then the *dot product* is given by:

$$\mathbf{U}\cdot\mathbf{V} = U_xV_x + U_yV_y + U_zV_z$$

For example,

$$(1,2,3)\cdot(2,2,2) = 1(2) + 2(2) + 3(2) = 12.$$

and

$$(0,5,0)\cdot(4,0,0) = 0.$$

We can interpret the last case by noting that the product is zero because the angle between the two vectors is 90 degrees.

Trigonometric Form of the Complex Number

$$z = a + bi = r\left(\cos\phi\ + i\sin\phi\right)$$

where

- *i* is the Imaginary Number $\left(i = \sqrt{-1}\right)$
- the modulus $r = \mathrm{mod}(z) = |z| = \sqrt{a^2 + b^2}$
- the argument $\phi = \arg(z)$ is the angle formed by the complex number on a polar graph with one real axis and one imaginary axis. This can be found using the right angle trigonometry for the trigonometric functions.

This is sometimes abbreviated as $r\left(\cos\phi + i\sin\phi\right) = r\mathrm{cis}\phi$ and it is also the case that $r\mathrm{cis}\phi = re^{i\phi}$ (provided that φ is in radians).

Trigonometry References

Trigonometric Unit Circle and Graph Reference

The Unit Circle

The unit circle is a commonly used tool in trigonometry because it helps the user to remember the special angles and their trigonometric functions. The unit circle is a circle drawn with its center at the origin of a graph(0,0), and with a radius of 1. All angles are measured starting from the x-axis in quadrant one and may go around the unit circle any number of degrees. points on the outside of the circle that are in line with the terminal(ending) sides of the angles are very useful to know, as they give the trigonometric function of the angle through their coordinants. The format is (cos, sin).

Trigonometric Formula Reference

The principal identities in trigonometry are:

$\sin^2\theta + \cos^2\theta = 1$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Four trigonometric functions are 2π periodic:

$\sin\theta = \sin(\theta + 2\pi)$

$\cos\theta = \cos(\theta + 2\pi)$

$\csc\theta = \csc(\theta + 2\pi)$

$\sec\theta = \sec(\theta + 2\pi)$

Two trigonometric functions are π periodic:

$\tan\theta = \tan(\theta + \pi)$

$\cot\theta = \cot(\theta + \pi)$

Formulas involving sums of angles are as follows:

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Substituting $\beta = \alpha$ gives the **double angle formulae**

$\sin(2\alpha) = 2\sin\alpha\cos\alpha$

$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$

Substituting $\sin^2\alpha + \cos^2\alpha = 1$ gives

$\cos(2\alpha) = 2\cos^2\alpha - 1$

$\cos(2\alpha) = 1 - 2\sin^2\alpha$

Trigonometric Identities Reference

Pythagoras

1. $\sin^2(x) + \cos^2(x) = 1$
2. $1 + \tan^2(x) = \sec^2(x)$
3. $1 + \cot^2(x) = \csc^2(x)$

These are all direct consequences of Pythagoras's theorem.

Sum / Difference of angles

1. $\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$
2. $\sin(x \pm y) = \sin(x)\cos(y) \pm \sin(y)\cos(x)$
3. $\tan(x \pm y) = \dfrac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)}$

Product to Sum

1. $2\sin(x)\sin(y) = \cos(x - y) - \cos(x + y)$
2. $2\cos(x)\cos(y) = \cos(x - y) + \cos(x + y)$
3. $2\sin(x)\cos(y) = \sin(x - y) + \sin(x + y)$

Sum and difference to product

1. $A\sin(x) + B\cos(x) = C\sin(x + y)$

where $C = \sqrt{A^2 + B^2}$ and $y = \tan^{-1}(B/A)$

1. $\sin\alpha + \sin\beta = 2\sin\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}$
2. $\sin\alpha - \sin\beta = 2\cos\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}$
3. $\cos\alpha + \cos\beta = 2\cos\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}$
4. $\cos\alpha - \cos\beta = -2\sin\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}$

Double angle

1. $\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$
2. $\sin(2x) = 2\sin(x)\cos(x)$
3. $\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$

These are all direct consequences of the sum/difference formulae

Half angle

1. $\cos(\frac{x}{2}) = \pm\sqrt{\frac{1+\cos(x)}{2}}$
2. $\sin(\frac{x}{2}) = \pm\sqrt{\frac{1-\cos(x)}{2}}$
3. $\tan(\frac{x}{2}) = \frac{1-\cos(x)}{\sin(x)} = \frac{\sin(x)}{1+\cos(x)} = \pm\sqrt{\frac{1-\cos(x)}{1+\cos(x)}}$

In cases with $\pm$, the sign of the result must be determined from the value of $\frac{x}{2}$. These derive from the $\cos(2x)$ formulae.

Power Reduction

1. $\sin^2\theta = \frac{1-\cos 2\theta}{2}$
2. $\cos^2\theta = \frac{1+\cos 2\theta}{2}$
3. $\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$

Even / Odd

1. $\sin(-\theta) = -\sin(\theta)$
2. $\cos(-\theta) = \cos(\theta)$
3. $\tan(-\theta) = -\tan(\theta)$
4. $\csc(-\theta) = -\csc(\theta)$
5. $\sec(-\theta) = \sec(\theta)$
6. $\cot(-\theta) = -\cot(\theta)$

Calculus

1. $\frac{d}{dx}[\sin x] = \cos x$
2. $\frac{d}{dx}[\cos x] = -\sin x$
3. $\frac{d}{dx}[\tan x] = \sec^2 x$
4. $\frac{d}{dx}[\sec x] = \sec x \tan x$
5. $\frac{d}{dx}[\csc x] = -\csc x \cot x$
6. $\frac{d}{dx}[\cot x] = -\csc^2 x$

Natural Trigonometric Functions of Primary Angles

NoteSome values in the table are given in forms that include a radical in the denominator — this is done both to simplify recognition of reciprocal pairs and because the form given in the table is simpler in some sense.

θ**(radians)**	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$	θ**(degrees)**
0	0	1	0	*undefined*	1	*undefined*	0°
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2	30°
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1	$\sqrt{2}$	$\sqrt{2}$	45°

$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$	60°
$\frac{\pi}{2}$	1	0	*undefined*	0	*undefined*	1	90°
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	-2	$\frac{2}{\sqrt{3}}$	120°
$\frac{3\pi}{4}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$	135°
$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$-\frac{2}{\sqrt{3}}$	2	150°
π	0	-1	0	*undefined*	-1	*undefined*	180°
$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$-\frac{2}{\sqrt{3}}$	-2	210°
$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1	-1	$-\sqrt{2}$	$-\sqrt{2}$	225°
$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	-2	$-\frac{2}{\sqrt{3}}$	240°
$\frac{3\pi}{2}$	-1	0	*undefined*	0	*undefined*	-1	270°

$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	2	$-\frac{2}{\sqrt{3}}$	300°
$\frac{7\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$	315°
$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$\frac{2}{\sqrt{3}}$	-2	330°
2π	0	1	0	*undefined*	1	*undefined*	360°

Notice that for certain values of X, the tangent, secant, cosecant, and cotangent functions are undefined. This is because these functions are defined as $\frac{\sin(x)}{\cos(x)}$, $\frac{1}{\cos(x)}$, $\frac{1}{\sin(x)}$, and $\frac{\cos(x)}{\sin(x)}$, respectively. Since an expression is undefined if it is divided by zero, the functions are therefore undefined at angle measures where the denominator (the sine or cosine of X, depending on the trigonometric function) is equal to zero. Take, for example, the tangent function. If the tangent function is analyzed at 90 degrees ($\frac{\pi}{2}$ radians), the function is then equivalent to $\frac{\sin(0)}{\cos(0)}$, or $\frac{1}{0}$, which is an undefined value.

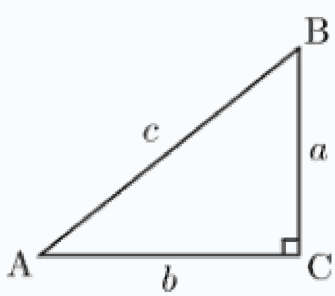
Trigonometry (from the Greek *trigonon* = three angles and *metron* = measure) is a branch of mathematics which deals with triangles, particularly triangles in a plane where one angle of the triangle is 90 degrees (right triangles). Triangles on a sphere are also studied, in spherical trigonometry. Trigonometry specifically deals with the relationships between the sides and the angles of triangles, that is, the trigonometric functions, and with calculations based on these functions.

This robotic arm on the International Space Station is operated by controlling the angles of its joints. Calculating the final position of the astronaut at the end of the arm requires repeated use of the trigonometric functions of those angles.

Trigonometry has important applications in many branches of pure mathematics as well as of applied mathematics and, consequently, much of science.

Overview

Basic definitions



In this right trianglesin(A) = a/c; cos(A) = b/c; tan(A) = a/b.

The shape of a right triangle is completely determined, up to similarity, by the value of either of the other two angles. This means that once one of the other angles is known, the ratios of the various sides are always the same regardless of the size of the triangle. These ratios are traditionally described by the following trigonometric functions of the known angle:

- The **sine** function (sin), defined as the ratio of the leg opposite the angle to the hypotenuse.
- The **cosine** function (cos), defined as the ratio of the adjacent leg to the hypotenuse.
- The **tangent** function (tan), defined as the ratio of the opposite leg to the adjacent leg.

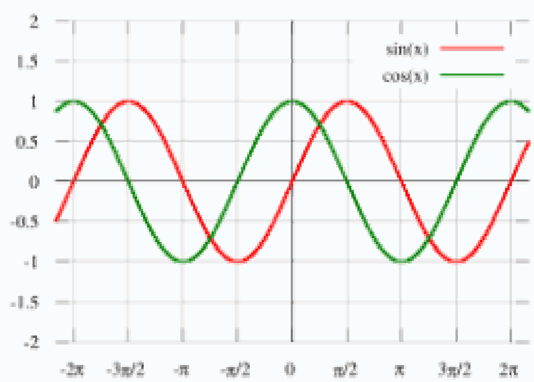
The **adjacent leg** is the side of the angle that is not the hypotenuse. The **hypotenuse** is the side opposite to the 90 degree angle in a right triangle; it is the longest side of the triangle.

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos A = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin A}{\cos A}$$

The reciprocals of these functions are named the **cosecant** (csc), **secant** (sec) and **cotangent** (cot), respectively. The inverse functions are called the **arcsine**, **arccosine**, and **arctangent**, respectively. There are arithmetic relations between these functions, which are known as trigonometric identities.

With these functions one can answer virtually all questions about arbitrary triangles, by using the law of sines and the law of cosines. These laws can be used to compute the remaining angles and sides of any triangle as soon as two sides and an angle or two angles and a side or three sides are known. These laws are useful in all branches of geometry since every polygon may be described as a finite combination of triangles.

Extending the definitions



Graphs of the functions sin(*x*) and cos(*x*), where the angle *x* is measured in radians.

The above definitions apply to angles between 0 and 90 degrees (0 and π/2 radians) only. Using the unit circle, one may extend them to all positive and negative arguments (see trigonometric function). The trigonometric functions are periodic, with a period of 360 degrees or 2π radians. That means their values repeat at those intervals.

The trigonometric functions can be defined in other ways besides the geometrical definitions above, using tools from calculus or infinite series. With these definitions the trigonometric functions can be defined for complex numbers. The complex function **cis** is particularly useful

$$\operatorname{cis}(x) = \cos x + i \sin x = e^{ix}$$

See Euler's formula.

Mnemonics

The *sine*, *cosine* and *tangent* ratios in right triangles can be remembered by SOH CAH TOA (sine-opposite-hypotenuse cosine-adjacent-hypotenuse tangent-opposite-adjacent). It is commonly referred to as "Sohcahtoa" by some American mathematics teachers, who liken it to a (nonexistent) Native American girl's name. See trigonometry mnemonics for other memory aids.

Calculating trigonometric functions

Generating trigonometric tables

Trigonometric functions were among the earliest uses for mathematical tables. Such tables were incorporated into mathematics textbooks and students were taught to look up values and how to interpolate between the values listed to get higher accuracy. Slide rules had special scales for trigonometric functions.

Today scientific calculators have buttons for calculating the main trigonometric functions (sin, cos, tan and sometimes cis) and their inverses. Most allow a choice of angle measurement methods, degrees, radians and, sometimes, grad. Most computer programming languages provide function libraries that include the trigonometric functions. The floating point unit hardware incorporated into the microprocessor chips used in most personal computers have built in instructions for calculating trigonometric functions.

Early history of trigonometry

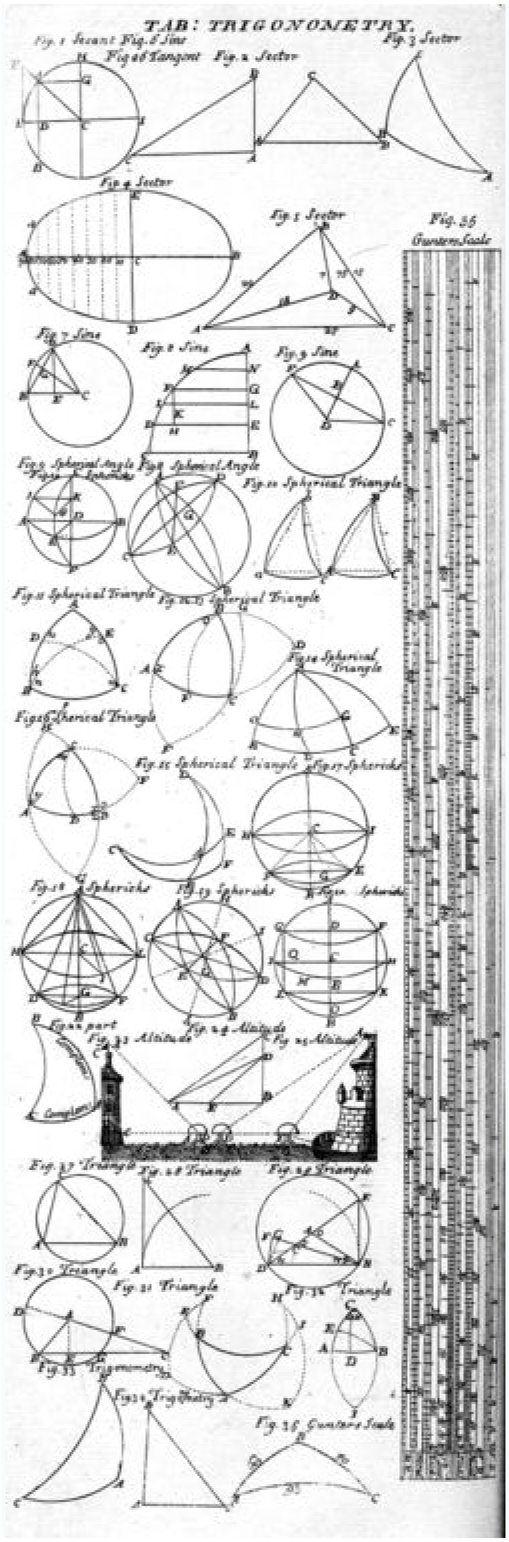


Table of Trigonometry, 1728 *Cyclopaedia*

The origins of trigonometry can be traced to the civilizations of ancient Egypt, Mesopotamia and the Indus Valley, more than 4000 years agoThe common practice of measuring angles in degrees, minutes and seconds comes from the Babylonian's base sixty system of numeration.

Some experts believe that trigonometry was originally invented to calculate sundials, a traditional exercise in the oldest books.

The first recorded use of trigonometry came from the Hellenistic mathematician Hipparchus circa 150 BC, who compiled a trigonometric table using the sine for solving triangles. Ptolemy further developed trigonometric calculations circa 100 AD.

The Sulba Sutras written in India, between 800 BC and 500 BC, correctly compute the sine of π/4 (45°) as 1/√2 in a procedure for circling the square (the opposite of squaring the circle).

The ancient Sinhalese, when constructing reservoirs in the Anuradhapura kingdom, used trigonometry to calculate the gradient of the water flow.

The Indian mathematician Aryabhata in 499, gave tables of half chords which are now known as sine tables, along with cosine tables. He used *zya* for sine, *kotizya* for cosine, and *otkram zya* for inverse sine, and also introduced the versine.

Another Indian mathematician, Brahmagupta in 628, used an interpolation formula to compute values of sines, up to the second order of the Newton-Stirling interpolation formula.

In the 10th century, the Persian mathematician and astronomer Abul Wáfa introduced the tangent function and improved methods of calculating trigonometry tables. He established the angle addition identities, e.g. sin ($a + b$), and discovered the sine formula for spherical geometry:

$$\frac{\sin(A)}{\sin(a)} = \frac{\sin(B)}{\sin(b)} = \frac{\sin(C)}{\sin(c)}$$

Also in the late 10th and early 11th centuries, the Egyptian astronomer Ibn Yunus performed many careful trigonometric calculations and demonstrated the formula $\cos(a)\cos(b) = 1 / 2[\cos(a + b) + \cos(a - b)]$.

Indian mathematicians were the pioneers of variable computations algebra for use in astronomical calculations along with trigonometry. Lagadha (circa 1350-1200 BC) is the first person thought to have used geometry and trigonometry for astronomy, in his *Vedanga Jyotisha*.

Persian mathematician Omar Khayyám (1048-1131) combined trigonometry and approximation theory to provide methods of solving algebraic equations by geometrical means. Khayyam solved the cubic equation $x^3 + 200x = 20x^2 + 2000$ and found a positive root of this cubic by considering the intersection of a rectangular hyperbola and a circle. An approximate numerical solution was then found by interpolation in trigonometric tables.

Detailed methods for constructing a table of sines for any angle were given by the Indian mathematician Bhaskara in 1150, along with some sine and cosine formulae. Bhaskara also developed spherical trigonometry.

The 13th century Persian mathematician Nasir al-Din Tusi, along with Bhaskara, was probably the first to treat trigonometry as a distinct mathematical discipline. Nasir al-Din Tusi in his *Treatise on the Quadrilateral* was the first to list the six distinct cases of a right angled triangle in spherical trigonometry.

In the 14th century, Persian mathematician al-Kashi and Timurid mathematician Ulugh Beg (grandson of Timur) produced tables of trigonometric functions as part of their studies of astronomy.

The mathematician Bartholemaeus Pitiscus published an influential work on trigonometry in 1595 which may have coined the word “trigonometry”.

Applications of trigonometry

Marine sextants like this are used to measure the angle of the sun or stars with respect to the horizon. Using trigonometry and an accurate clock, the position of the ship can then be determined from several such measurements.

There are an enormous number of applications of trigonometry and trigonometric functions. For instance, the technique of triangulation is used in astronomy to measure the distance to nearby stars, in geography to measure distances between landmarks, and in satellite navigation systems. The sine and cosine functions are fundamental to the theory of periodic functions such as those that describe sound and light waves.

Fields which make use of trigonometry or trigonometric functions include astronomy (especially, for locating the apparent positions of celestial objects, in which spherical trigonometry is essential) and hence navigation (on the oceans, in aircraft, and in space), music theory, acoustics, optics, analysis of financial markets, electronics, probability theory, statistics, biology, medical imaging (CAT scans and ultrasound), pharmacy, chemistry, number theory (and hence cryptology), seismology, meteorology, oceanography, many physical sciences, land surveying and geodesy, architecture, phonetics, economics, electrical engineering, mechanical engineering, civil engineering, computer graphics, cartography, crystallography and game development.

Common formulae

Trigonometric identity
Trigonometric function

Certain equations involving trigonometric functions are true for all angles and are known as *trigonometric identities*. Many express important geometric relationships. For example, the Pythagorean identities are an expression of the Pythagorean Theorem. Here are some of the more commonly used identities, as well as the most important formulae connecting angles and sides of an arbitrary triangle. For more identities see trigonometric identity.

Trigonometric identities

Pythagorean identities

$$\sin^2 A + \cos^2 A = 1, \qquad 1 + \tan^2 A = \sec^2 A, \qquad 1 + \cot^2 A = \csc^2 A$$

Sum and difference identities

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$
$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Double-angle identities

$$\sin 2A = 2 \sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A} = \frac{2\cot A}{\cot^2 A - 1} = \frac{2}{\cot A - \tan A}$$

Half-angle identities

Note that $\pm$in these formulae does not mean both are correct, it means it may be either one, depending on the value of *A/2*.

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$$
$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$$
$$\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}} = \frac{\sin A}{1+\cos A} = \frac{1-\cos A}{\sin A}$$

Triangle identities

Law of sines

The **law of sines** for an arbitrary triangle states:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

or equivalently:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

Law of cosines

The **law of cosines** (also known as the cosine formula) is an extension of the Pythagorean theorem to arbitrary triangles:

$$c^2 = a^2 + b^2 - 2abcosC,$$

or equivalently:

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

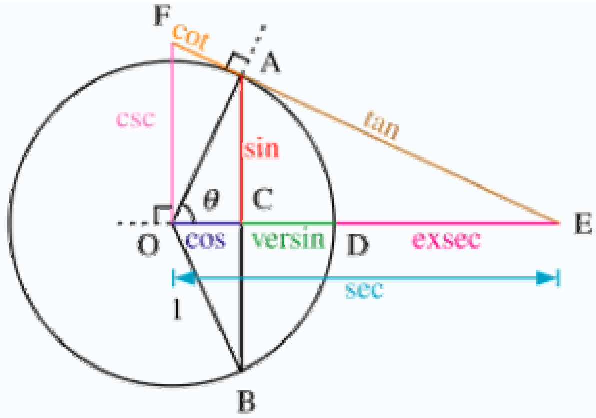
Law of tangents

The **law of tangents**:

$$\frac{a+b}{a-b} = \frac{\tan[\frac{1}{2}(A+B)]}{\tan[\frac{1}{2}(A-B)]}$$

Trigonometric function

“Sine” redirects here. For other uses, see Sine (disambiguation).



All of the trigonometric functions of an angle θ can be constructed geometrically in terms of a unit circle centered at *O*.

In mathematics, the **trigonometric functions** are functions of an angle; they are important when studying triangles and modeling periodic phenomena, among many other applications. They are commonly defined as ratios of two sides of a right triangle containing the angle, and can equivalently be defined as the lengths of various line segments from a unit circle. More modern definitions express them as infinite series or as solutions of certain differential equations, allowing their extension to positive and negative values and even to complex numbers. All of these approaches will be presented below.

The study of trigonometric functions dates back to Babylonian times, and a considerable amount of fundamental work was done by Persian and Greek mathematicians.

In modern usage, there are six basic trigonometric functions, which are tabulated below along with equations relating them to one another. Especially in the case of the last four, these relations are often taken as the *definitions* of those functions, but one can define them equally well geometrically or by other means and then derive these relations. A few other functions were common historically (and appeared in the earliest tables), but are now seldom used, such as the versine (1 − cos θ) and the exsecant (sec θ − 1). Many more relations between these functions are listed in the article about trigonometric identities.

Function	**Abbreviation**	**Relation**
Sine	sin	$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$
Cosine	cos	$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$
Tangent	tan	$\tan\theta = \frac{1}{\cot\theta} = \frac{\sin\theta}{\cos\theta} = \cot\left(\frac{\pi}{2} - \theta\right)$

Cotangent	cot	$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = \tan\left(\frac{\pi}{2} - \theta\right)$
Secant	sec	$\sec\theta = \frac{1}{\cos\theta} = \csc\left(\frac{\pi}{2} - \theta\right)$
Cosecant	csc (or cosec)	$\csc\theta = \frac{1}{\sin\theta} = \sec\left(\frac{\pi}{2} - \theta\right)$

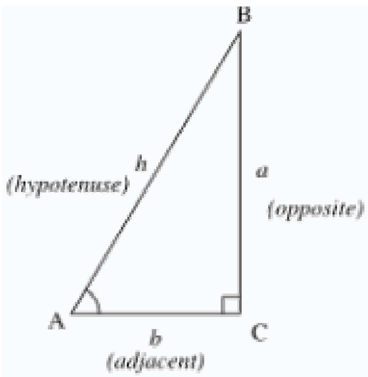
History

History of trigonometric functions

The earliest use of sine appears in the *Sulba Sutras* written in ancient India from the 8^{th} century BC to the 6^{th} century BC. Trigonometric functions were later studied by Hipparchus of Nicaea (180-125 BC), Ptolemy, Aryabhata (476–550), Varahamihira, Brahmagupta, Muḥammad ibn Mūsā al-Ḵwārizmī, Abu’l-Wafa, Omar Khayyam, Bhaskara II, Nasir al-Din Tusi, Ghiyath al-Kashi (14^{th} century), Ulugh Beg (14^{th} century), Regiomontanus (1464), Rheticus, and Rheticus’ student Valentin Otho. Madhava (c. 1400) made early strides in the analysis of trigonometric functions in terms of infinite series. Leonhard Euler’s *Introductio in analysin infinitorum* (1748) was mostly responsible for establishing the analytic treatment of trigonometric functions in Europe, also defining them as infinite series and presenting “Euler’s formula”, as well as the near-modern abbreviations *sin., cos., tang., cot., sec.,* and *cosec.*

The notion that there should be some standard correspondence between the length of the sides of a triangle and the angles of the triangle comes as soon as one recognises that similar triangles maintain the same ratios between their sides. That is, for any similar triangle the ratio of the hypotenuse (for example) and another of the sides remains the same. If the hypotenuse is twice as long, so are the sides. It is just these ratios that the trig. functions express.

Right triangle definitions



A right triangle always includes a 90° (π/2 radians) angle, here labeled C. Angles A and B may vary. Trigonometric functions specify the relationships between side lengths and interior angles of a right triangle.

In order to define the trigonometric functions for the angle *A*, start with an arbitrary right triangle that contains the angle *A*:

We use the following names for the sides of the triangle:

- The *hypotenuse* is the side opposite the right angle, or defined as the longest side of a right-angled triangle, in this case **h**.
- The *opposite side* is the side opposite to the angle we are interested in, in this case **a**.
- The *adjacent side* is the side that is in contact with the angle we are interested in and the right angle, hence its name. In this case the adjacent side is **b**.

All triangles are taken to exist in the Euclidean plane so that the inside angles of each triangle sum to π radians (or 180°); therefore, for a right triangle the two non-right angles are between zero and π/2 radians. The reader should note that the following definitions, strictly speaking, only define the trigonometric functions for angles in this range. We extend them to the full set of real arguments by using the unit circle, or by requiring certain symmetries and that they be periodic functions.

1) The **sine** of an angle is the ratio of the length of the opposite side to the length of the hypotenuse. In our case

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{h}.$$

Note that this ratio does not depend on the particular right triangle chosen, as long as it contains the angle *A*, since all those triangles are similar.

The set of zeroes of sine is

$$\left\{ n\pi \middle| n \in \mathbb{Z} \right\}.$$

2) The **cosine** of an angle is the ratio of the length of the adjacent side to the length of the hypotenuse. In our case

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{h}.$$

The set of zeroes of cosine is

$$\left\{\frac{\pi}{2} + n\pi \middle| n \in \mathbb{Z}\right\}.$$

3) The **tangent** of an angle is the ratio of the length of the opposite side to the length of the adjacent side. In our case

$$\tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b}.$$

The set of zeroes of tangent is

$$\left\{n\pi \middle| n \in \mathbb{Z}\right\}.$$

The same set of the sine function since

$$\tan A = \frac{\sin A}{\cos A}.$$

The remaining three functions are best defined using the above three functions.

4) The **cosecant** csc(A) is the multiplicative inverse of sin(A), i.e. the ratio of the length of the hypotenuse to the length of the opposite side:

$$\csc A = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{h}{a}.$$

5) The **secant** sec(A) is the multiplicative inverse of cos(A), i.e. the ratio of the length of the hypotenuse to the length of the adjacent side:

$$\sec A = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{h}{b}.$$

6) The **cotangent** cot(A) is the multiplicative inverse of tan(A), i.e. the ratio of the length of the adjacent side to the length of the opposite side:

$$\cot A = \frac{\text{adjacent}}{\text{opposite}} = \frac{b}{a}.$$

Mnemonics

There are a number of mnemonics for the above definitions, for example *SOHCAHTOA* (sounds like “soak a toe-a” or “sock-a toe-a” depending upon the use of American English or British English, or even “soccer tour”). It means:

- SOH ... **s**in = **o**pposite/**h**ypotenuse
- CAH ... **c**os = **a**djacent/**h**ypotenuse
- TOA ... **t**an = **o**pposite/**a**djacent.

Many other such words and phrases have been contrived. For more see mnemonic.

Slope definitions

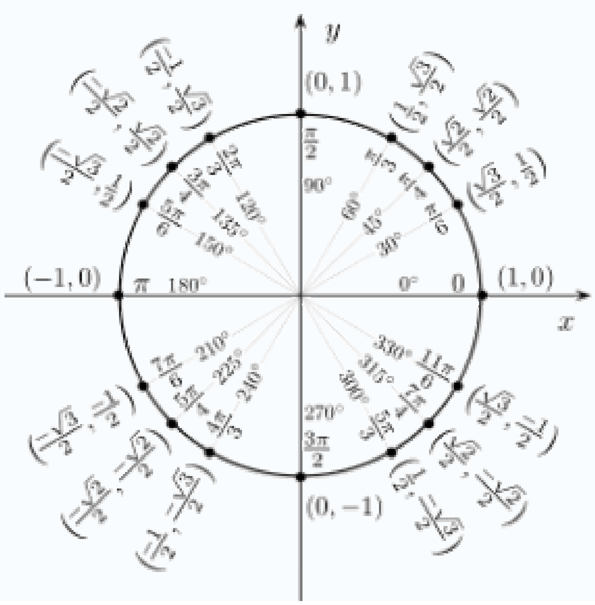
Equivalent to the right-triangle definitions, the trigonometric functions can be defined in terms of the *rise*, *run*, and *slope* of a line segment relative to some horizontal line. The slope is commonly taught as “rise over run” or rise/run. The three main trigonometric functions are commonly taught in the order sine, cosine, tangent. With a unit circle, this gives rise to the following matchings:

1. Sine is first, rise is first. Sine takes an angle and tells the rise.
2. Cosine is second, run is second. Cosine takes an angle and tells the run.
3. Tangent is the slope formula that combines the rise and run. Tangent takes an angle and tells the slope.

This shows the main use of tangent and arctangentconverting between the two ways of telling the slant of a line, *i.e.*, angles and slopes. (Note that the arctangent or “inverse tangent” is not to be confused with the *cotangent*, which is 1 divided by the tangent.)

While the radius of the circle makes no difference for the slope (the slope doesn’t depend on the length of the slanted line), it does affect rise and run. To adjust and find the actual rise and run, just multiply the sine and cosine by the radius. For instance, if the circle has radius 5, the run at an angle of 1 is 5 cos(1).

Unit-circle definitions

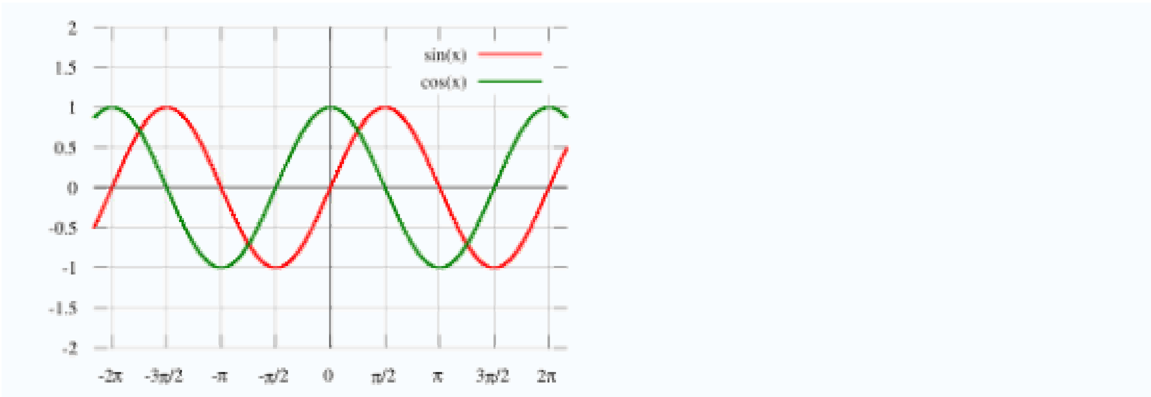


The unit circle

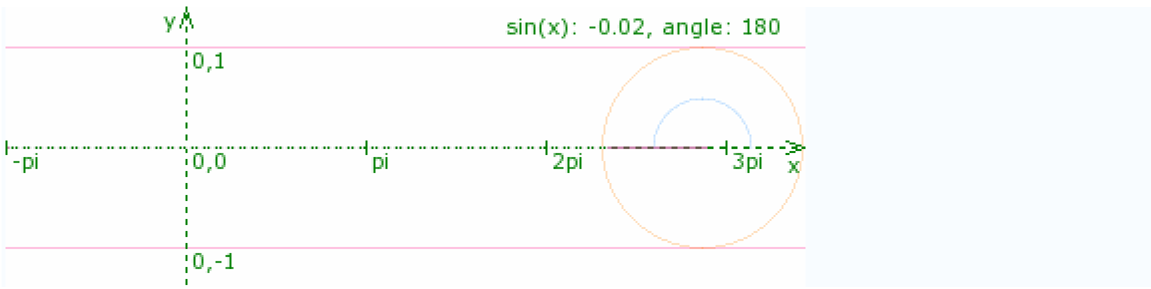
The six trigonometric functions can also be defined in terms of the unit circle, the circle of radius one centered at the origin. The unit circle definition provides little in the way of practical calculation; indeed it relies on right triangles for most angles. The unit circle definition does, however, permit the definition of the trigonometric functions for all positive and negative arguments, not just for angles between 0 and π/2 radians. It also provides a single visual picture that encapsulates at once all the important triangles used so far. The equation for the unit circle is:

$$x^2 + y^2 = 1$$

In the picture, some common angles, measured in radians, are given. Measurements in the counter clockwise direction are positive angles and measurements in the clockwise direction are negative angles. Let a line making an angle of θ with the positive half of the *x*-axis intersect the unit circle. The *x*- and *y*-coordinates of this point of intersection are equal to cos θ and sin θ, respectively. The triangle in the graphic enforces the formula; the radius is equal to the hypotenuse and has length 1, so we have sin θ = *y*/1 and cos θ = *x*/1. The unit circle can be thought of as a way of looking at an infinite number of triangles by varying the lengths of their legs but keeping the lengths of their hypotenuses equal to 1.



The *f*(*x*) = sin(*x*) and *f*(*x*) = cos(*x*) functions graphed on the cartesian plane.



A sin(x) animation which shows the connection between circle and sine.

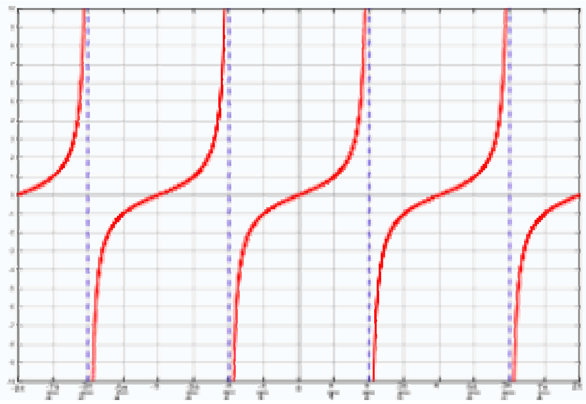
For angles greater than 2π or less than −2π, simply continue to rotate around the circle. In this way, sine and cosine become periodic functions with period 2π:

$$\sin\theta = \sin\left(\theta + 2\pi k\right)$$
$$\cos\theta = \cos\left(\theta + 2\pi k\right)$$

for any angle θ and any integer *k*.

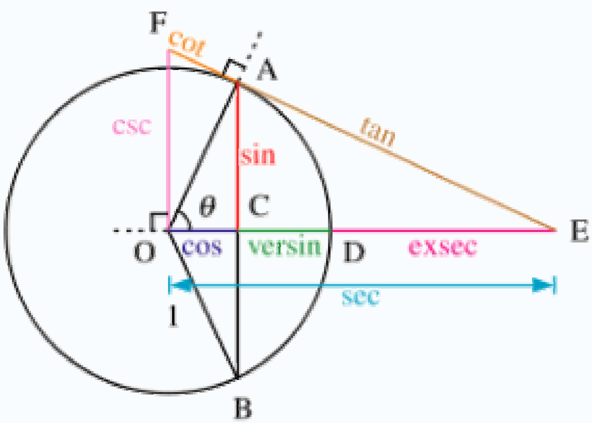
The *smallest* positive period of a periodic function is called the *primitive period* of the function. The primitive period of the sine, cosine, secant, or cosecant is a full circle, i.e. 2π radians or 360 degrees; the primitive period of the tangent or cotangent is only a half-circle, i.e. π radians or 180 degrees. Above, only sine and cosine were defined directly by the unit circle, but the other four trigonometric functions can be defined by:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$
$$\csc\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$



The *f*(x) = tan(x) function graphed on the cartesian plane.

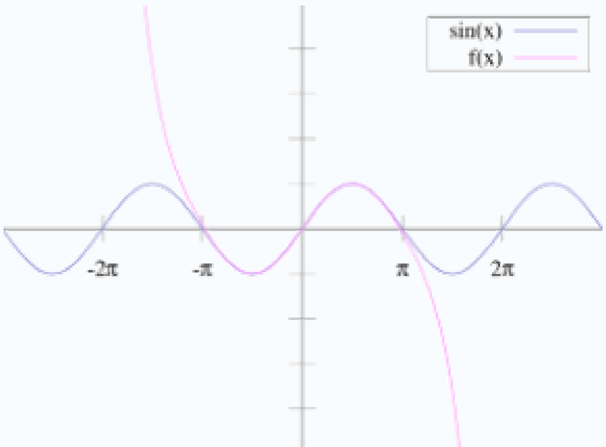
To the right is an image that displays a noticeably different graph of the trigonometric function f(θ)= tan(θ) graphed on the cartesian plane. Note that its x-intercepts correspond to that of sin(θ) while its undefined values correspond to the x-intercepts of the cos(θ). Observe that the function's results change slowly around angles of *k*π, but change rapidly at angles close to (*k*/2)π. The graph of the tangent function also has a vertical asymptote at θ = *k*π/2. This is the case because the function approaches infinity as θ approaches *k*/π from the left and minus infinity as it approaches *k*/π from the right.



All of the trigonometric functions can be constructed geometrically in terms of a unit circle centered at *O*.

Alternatively, *all* of the basic trigonometric functions can be defined in terms of a unit circle centered at *O* (shown at right), and similar such geometric definitions were used historically. In particular, for a chord *AB* of the circle, where θ is half of the subtended angle, sin(θ) is *AC* (half of the chord), a definition introduced in India (see above). cos(θ) is the horizontal distance *OC*, and versin(θ) = 1 − cos(θ) is *CD*. tan(θ) is the length of the segment *AE* of the tangent line through *A*, hence the word *tangent* for this function. cot(θ) is another tangent segment, *AF*. sec(θ) = *OE* and csc(θ) = *OF* are segments of secant lines (intersecting the circle at two points), and can also be viewed as projections of *OA* along the tangent at *A* to the horizontal and vertical axes, respectively. *DE* is exsec(θ) = sec(θ) − 1 (the portion of the secant outside, or *ex*, the circle). From these constructions, it is easy to see that the secant and tangent functions diverge as θ approaches π/2 (90 degrees) and that the cosecant and cotangent diverge as θ approaches zero. (Many similar constructions are possible, and the basic trigonometric identities can also be proven graphically.)

Series definitions



The sine function (blue) is closely approximated by its Taylor polynomial of degree 7 (pink) for a full cycle centered on the origin.

Using only geometry and properties of limits, it can be shown that the derivative of sine is cosine and the derivative of cosine is the negative of sine. (Here, and generally in calculus, all angles are measured in radians; see also the significance of radians below.) One can then use the theory of Taylor series to show that the following identities hold for all real numbers *x*:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

These identities are often taken as the *definitions* of the sine and cosine function. They are often used as the starting point in a rigorous treatment of trigonometric functions and their applications (*e.g.*, in Fourier series), since the theory of infinite series can be developed from the foundations of the real number system, independent of any geometric considerations. The differentiability and continuity of these functions are then established from the series definitions alone.

Other series can be found:

$$\tan x = \sum_{n=0}^{\infty} \frac{U_{2n+1} x^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n} (2^{2n} - 1) B_{2n} x^{2n-1}}{(2n)!}$$

$$= x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots, \qquad \text{for } |x| < \frac{\pi}{2}$$

where

U_n is the *n*th up/down number,
B_n is the *n*th Bernoulli number, and
E_n (below) is the *n*th Euler number.

When this is expressed in a form in which the denominators are the corresponding factorials, and the numerators, called the “tangent numbers”, have a combinatorial interpretationthey enumerate alternating permutations of finite sets of odd cardinality.

$$\csc x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2 (2^{2n-1} - 1) B_{2n} x^{2n-1}}{(2n)!}$$

$$= \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \frac{31x^5}{15120} + \cdots, \qquad \text{for } 0 < |x| < \pi$$

$$\sec x = \sum_{n=0}^{\infty} \frac{U_{2n} x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n E_n x^{2n}}{(2n)!}$$

$$= 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots, \qquad \text{for } |x| < \frac{\pi}{2}$$

When this is expressed in a form in which the denominators are the corresponding factorials, the numerators, called the “secant numbers”, have a combinatorial interpretationthey enumerate alternating permutations of finite sets of even cardinality.

$$\cot x = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} B_{2n} x^{2n-1}}{(2n)!}$$

$$= \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \cdots, \qquad \text{for } 0 < |x| < \pi$$

From a theorem in complex analysis, there is a unique analytic extension of this real function to the complex numbers. They have the same Taylor series, and so the trigonometric functions are defined on the complex numbers using the Taylor series above.

Relationship to exponential function and complex numbers

It can be shown from the series definitions that the sine and cosine functions are the imaginary and real parts, respectively, of the complex exponential function when its argument is purely imaginary:

$$e^{i\theta} = \cos\theta + i\sin\theta\,.$$

This relationship was first noted by Euler and the identity is called Euler’s formula. In this way, trigonometric functions become essential in the geometric interpretation of complex analysis. For example, with the above identity, if one considers the unit circle in the complex plane, defined by e^{ix}, and as above, we can parametrize this circle in terms of cosines and sines, the relationship between the complex exponential and the trigonometric functions becomes more apparent.

Furthermore, this allows for the definition of the trigonometric functions for complex arguments *z*:

$$\begin{aligned} \sin z &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \frac{e^{\imath z} - e^{-\imath z}}{2\imath} = -\imath \sinh(\imath z) \\ \cos z &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = \frac{e^{\imath z} + e^{-\imath z}}{2} = \cosh(\imath z) \end{aligned}$$

where $i^2 = -1$. Also, for purely real *x*,

$$\begin{aligned} \cos x &= \operatorname{Re}(e^{\imath x}) \\ \sin x &= \operatorname{Im}(e^{\imath x}) \end{aligned}$$

It is also known that exponential processes are intimately linked to periodic behavior.

Definitions via differential equations

Both the sine and cosine functions satisfy the differential equation

$$y'' = -y.$$

That is to say, each is the additive inverse of its own second derivative. Within the 2-dimensional vector space *V* consisting of all solutions of this equation, the sine function is the unique solution satisfying the initial conditions $y(0) = 0$ and $y'(0) = 1$, and the cosine function is the unique solution satisfying the initial conditions $y(0) = 1$ and $y'(0) = 0$. Since the sine and cosine functions are linearly independent, together they form a basis of *V*. This method of defining the sine and cosine functions is essentially equivalent to using Euler’s formula. (See linear differential equation.) It turns out that this differential equation can be used not only to define the sine and cosine functions but also to prove the trigonometric identities for the sine and cosine functions.

The tangent function is the unique solution of the nonlinear differential equation

$$y' = 1 + y^2$$

satisfying the initial condition *y*(0) = 0. There is a very interesting visual proof that the tangent function satisfies this differential equation; see Needham's *Visual Complex Analysis.*

The significance of radians

Radians specify an angle by measuring the length around the path of the circle and constitute a special argument to the sine and cosine functions. In particular, only those sines and cosines which map radians to ratios satisfy the differential equations which classically describe them. If an argument to sine or cosine in radians is scaled by frequency,

$$f(x) = \sin(kx); k \neq 0, k \neq 1$$

then the derivatives will scale by *amplitude*.

$$f'(x) = k\cos(kx).$$

Here, *k* is a constant that represents a mapping between units. If *x* is in degrees, then

$$k = \frac{\pi}{180^\circ}.$$

This means that the second derivative of a sine in degrees satisfies not the differential equation

$$y'' = -y,$$

but

$$y'' = -k^2 y;$$

similarly for cosine.

This means that these sines and cosines are different functions, and that the fourth derivative of sine will be sine again only if the argument is in radians.

Identities

List of trigonometric identities.

Many identities exist which interrelate the trigonometric functions. Among the most frequently used is the **Pythagorean identity,** which states that for any angle, the square of the sine plus the square of the cosine is always 1. This is easy to see by studying a right triangle of hypotenuse 1 and applying the Pythagorean theorem. In symbolic form, the Pythagorean identity reads,

$$(\sin x)^2 + (\cos x)^2 = 1,$$

which is more commonly written in the following way:

$\sin^2 x + \cos^2 x = 1.$

Other key relationships are the **sum and difference formulas,** which give the sine and cosine of the sum and difference of two angles in terms of sines and cosines of the angles themselves. These can be derived geometrically, using arguments which go back to Ptolemy; one can also produce them algebraically using Euler's formula.

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \qquad \sin(x-y) = \sin x \cos y - \cos x \sin$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y \qquad \cos(x-y) = \cos x \cos y + \sin x \sin$$

When the two angles are equal, the sum formulas reduce to simpler equations known as the **double-angle formulas.**

For integrals and derivatives of trigonometric functions, see the relevant sections of table of derivatives, table of integrals, and list of integrals of trigonometric functions.

Definitions using functional equations

In mathematical analysis, one can define the trigonometric functions using functional equations based on properties like the sum and difference formulas. Taking as given these formulas and the Pythagorean identity, for example, one can prove that only two real functions satisfy those conditions. Symbolically, we say that there exists exactly one pair of real functions *s* and *c* such that for all real numbers *x* and *y*, the following equations hold:

$$s(x)^2 + c(x)^2 = 1,$$
$$s(x+y) = s(x)c(y) + c(x)s(y),$$
$$c(x+y) = c(x)c(y) - s(x)s(y),$$

with the added condition that

$$0 < xc(x) < s(x) < x \text{ for } 0 < x < 1.$$

Other derivations, starting from other functional equations, are also possible, and such derivations can be extended to the complex numbers. As an example, this derivation can be used to define trigonometry in Galois fields.

Computation

The computation of trigonometric functions is a complicated subject, which can today be avoided by most people because of the widespread availability of computers and scientific calculators that provide built-in trigonometric functions for any angle. In this section, however, we describe more details of their computation in three important contextsthe historical use of trigonometric tables, the modern techniques used by computers, and a few “important” angles where simple exact values are easily found. (Below, it suffices to consider a small range of angles, say 0 to π/2, since all other angles can be reduced to this range by the periodicity and symmetries of the trigonometric functions.)

Generating trigonometric tables

Prior to computers, people typically evaluated trigonometric functions by interpolating from a detailed table of their values, calculated to many significant figures. Such tables have been available for as long as trigonometric functions have been described (see History above), and were typically generated by repeated application of the half-angle and angle-addition identities starting from a known value (such as sin(π/2)=1).

Modern computers use a variety of techniques.One common method, especially on higher-end processors with floating point units, is to combine a polynomial approximation (such as a Taylor series or a rational function) with a table lookup — they first look up the closest angle in a small table, and then use the polynomial to compute the correction. On simpler devices that lack hardware multipliers, there is an algorithm called CORDIC (as well as related techniques) that is more efficient, since it uses only shifts and additions. All of these methods are commonly implemented in hardware for performance reasons.

Exact trigonometric constants

Finally, for some simple angles, the values can be easily computed by hand using the Pythagorean theorem, as in the following examples. In fact, the sine, cosine and tangent of any integer multiple of π/60 radians (three degrees) can be found exactly by hand.

Consider a right triangle where the two other angles are equal, and therefore are both π/4 radians (45 degrees). Then the length of side *b* and the length of side *a* are equal; we can choose *a* = *b* = 1. The values of sine, cosine and tangent of an angle of π/4 radians (45 degrees) can then be found using the Pythagorean theorem:

$$c = \sqrt{a^2 + b^2} = \sqrt{2}.$$

Therefore:

$$\sin\left(\pi/4\right) = \sin\left(45^\circ\right) = \cos\left(\pi/4\right) = \cos\left(45^\circ\right) = {1 \over \sqrt{2}},$$
$$\tan\left(\pi/4\right) = \tan\left(45^\circ\right) = {\sqrt{2} \over \sqrt{2}} = 1.$$

To determine the trigonometric functions for angles of π/3 radians (60 degrees) and π/6 radians (30 degrees), we start with an equilateral triangle of side length 1. All its angles are π/3 radians (60 degrees). By dividing it into two, we obtain a right triangle with π/6 radians (30 degrees) and π/3 radians (60 degrees) angles. For this triangle, the shortest side = ½, the next largest side =(√3)/2 and the hypotenuse = 1. This yields:

$$\sin\left(\pi/6\right) = \sin\left(30^\circ\right) = \cos\left(\pi/3\right) = \cos\left(60^\circ\right) = {1 \over 2},$$
$$\cos\left(\pi/6\right) = \cos\left(30^\circ\right) = \sin\left(\pi/3\right) = \sin\left(60^\circ\right) = {\sqrt{3} \over 2},$$
$$\tan\left(\pi/6\right) = \tan\left(30^\circ\right) = \cot\left(\pi/3\right) = \cot\left(60^\circ\right) = {1 \over \sqrt{3}}.$$

Inverse functions

inverse trigonometric function

The trigonometric functions are periodic, so we must restrict their domains before we are able to define a unique inverse. In the following, the functions on the left are *defined* by the equation on the right; these are not proved identities. The principal inverses are usually defined as:

$$
\begin{array}{llll}
\text{for} & -\frac{\pi}{2} \le y \le \frac{\pi}{2}, & y = \arcsin(x) & \text{if and only if} \quad x = \sin(y) \\
\text{for} & 0 \le y \le \pi, & y = \arccos(x) & \text{if and only if} \quad x = \cos(y) \\
\text{for} & -\frac{\pi}{2} < y < \frac{\pi}{2}, & y = \arctan(x) & \text{if and only if} \quad x = \tan(y) \\
\text{for} & -\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \ne 0, & y = \operatorname{arccsc}(x) & \text{if and only if} \quad x = \csc(y) \\
\text{for} & 0 \le y \le \pi, y \ne \frac{\pi}{2}, & y = \operatorname{arcsec}(x) & \text{if and only if} \quad x = \sec(y) \\
\text{for} & -\frac{\pi}{2} < y < \frac{\pi}{2}, y \ne 0, & y = \operatorname{arccot}(x) & \text{if and only if} \quad x = \cot(y)
\end{array}
$$

For inverse trigonometric functions, the notations $\sin^{-1}$ and $\cos^{-1}$ are often used for arcsin and arccos, etc. When this notation is used, the inverse functions are sometimes confused with the multiplicative inverses of the functions. The notation using the “arc-“ prefix avoids such confusion, though “arcsec” may occasionally be confused with “arcsecond”.

Just like the sine and cosine, the inverse trigonometric functions can also be defined in terms of infinite series. For example,

$$\arcsin z = z + \left(\frac{1}{2}\right)\frac{z^3}{3} + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)\frac{z^5}{5} + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)\frac{z^7}{7} + \cdots$$

These functions may also be defined by proving that they are antiderivatives of other functions. The arcsine, for example, can be written as the following integral:

$$\arcsin(x) = \int_0^x \frac{1}{\sqrt{1 - z^2}}\,\mathrm{d}z, \quad |x| < 1$$

Analogous formulas for the other functions can be found at Inverse trigonometric function. Using the complex logarithm, one can generalize all these functions to complex arguments:

$$
\begin{aligned}
\arcsin(z) &= -i \log\left(iz + \sqrt{1 - z^2}\right) \\
\arccos(z) &= -i \log\left(z + \sqrt{z^2 - 1}\right) \\
\arctan(z) &= \frac{i}{2} \log\left(\frac{1 - iz}{1 + iz}\right)
\end{aligned}
$$

Properties and applications

Uses of trigonometry

The trigonometric functions, as the name suggests, are of crucial importance in trigonometry, mainly because of the following two results.

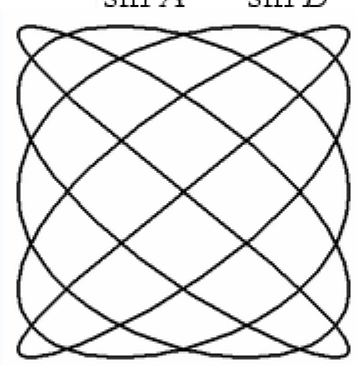
Law of sines

The **law of sines** for an arbitrary triangle states:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

also known as:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$



A Lissajous curve, a figure formed with a trigonometry-based function.

It can be proven by dividing the triangle into two right ones and using the above definition of sine. The common number (sin*A*)/*a* occurring in the theorem is the reciprocal of the diameter of the circle through the three points *A*, *B* and *C*. The law of sines is useful for computing the lengths of the unknown sides in a triangle if two angles and one side are known. This is a common situation occurring in *triangulation*, a technique to determine unknown distances by measuring two angles and an accessible enclosed distance.

Law of cosines

The **law of cosines** (also known as the cosine formula) is an extension of the Pythagorean theorem:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

also known as:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Again, this theorem can be proven by dividing the triangle into two right ones. The law of cosines is useful to determine the unknown data of a triangle if two sides and an angle are known.

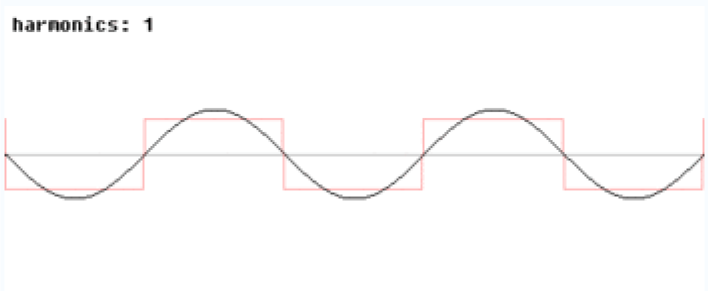
If the angle is not contained between the two sides, the triangle may not be unique. Be aware of this ambiguous case of the Cosine law.

Other useful properties

There is also a **law of tangents**:

$$\frac{a+b}{a-b} = \frac{\tan[\frac{1}{2}(A+B)]}{\tan[\frac{1}{2}(A-B)]}$$

Periodic functions



Animation of the additive synthesis of a square wave with an increasing number of harmonics

The sine and the cosine functions also appear when describing a simple harmonic motion, another important concept in physics. In this context the sine and cosine functions are used to describe one dimension projections of the uniform circular motion, the mass in a string movement, and a small angle approximation of the mass on a pendulum movement.

The trigonometric functions are also important outside of the study of triangles. They are periodic functions with characteristic wave patterns as graphs, useful for modelling recurring phenomena such as sound or light waves. Every signal can be written as a (typically infinite) sum of sine and cosine functions of different frequencies; this is the basic idea of Fourier analysis, where trigonometric series are used to solve a variety of boundary-value problems in partial differential equations. For example the square wave, can be written as the Fourier series

$$x_{\text{square}}(t) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin\left((2k-1)t\right)}{(2k-1)}.$$

Generating trigonometric tables

In mathematics, tables of trigonometric functions are useful in a number of areas. Before the existence of pocket calculators, **trigonometric tables** were essential for navigation, science and engineering. The calculation of mathematical tables was an important area of study, which led to the development of the first mechanical computing devices.

Modern computers and pocket calculators now generate trigonometric function values on demand, using special libraries of mathematical code. Often, these libraries use pre-calculated tables internally, and compute the required value by using an appropriate interpolation method.

Interpolation of simple look-up tables of trigonometric functions are still used in computer graphics, where accurate calculations are either not needed, or cannot be made fast enough.

Another important application of trigonometric tables and generation schemes is for fast Fourier transform (FFT) algorithms, where the same trigonometric function values (called *twiddle factors*) must be evaluated many times in a given transform, especially in the common case where many transforms of the same size are computed. In this case, calling generic library routines every time is unacceptably slow. One option is to call the library routines once, to build up a table of those trigonometric values that will be needed, but this requires significant memory to store the table. The other possibility, since a regular sequence of values is required, is to use a recurrence formula to compute the trigonometric values on the fly. Significant research has been devoted to finding accurate, stable recurrence schemes in order to preserve the accuracy of the FFT (which is very sensitive to trigonometric errors).

Half-angle and angle-addition formulas

Historically, the earliest method by which trigonometric tables were computed, and probably the most common until the advent of computers, was to repeatedly apply the half-angle and angle-addition trigonometric identities starting from a known value (such as sin(π/2)=1, cos(π/2)=0). The relevant identities, the first recorded derivation of which is by Ptolemy, are:

$$\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos(x)}{2}}$$

$$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos(x)}{2}}$$
$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$$
$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$

Various other permutations on these identities are possible (for example, the earliest trigonometric tables used not sine and cosine, but sine and versine).

A quick, but inaccurate, approximation

A quick, but inaccurate, algorithm for calculating a table of N approximations s_n for $\sin(2\pi n/N)$ and c_n for $\cos(2\pi n/N)$ is:

$s_0 = 0$
$c_0 = 1$
$s_{n+1} = s_n + d \times c_n$
$c_{n+1} = c_n - d \times s_n$

for $n = 0,...,N-1$, where $d = 2\pi/N$.

This is simply the Euler method for integrating the differential equation:

$ds / dt = c$
$dc / dt = -s$

with initial conditions $s(0) = 0$ and $c(0) = 1$, whose analytical solution is $s = \sin(t)$ and $c = \cos(t)$.

Unfortunately, this is not a useful algorithm for generating sine tables because it has a significant error, proportional to $1/N$.

For example, for $N = 256$ the maximum error in the sine values is ~0.061 ($s_{202} = -1.0368$ instead of −0.9757). For $N = 1024$, the maximum error in the sine values is ~0.015 ($s_{803} = -0.99321$ instead of −0.97832), about 4 times smaller. If the sine and cosine values obtained were to be plotted, this algorithm would draw a logarithmic spiral rather than a circle.

A better, but still imperfect, recurrence formula

A simple recurrence formula to generate trigonometric tables is based on Euler’s formula and the relation:

$$e^{i(\theta+\Delta\theta)} = e^{i\theta} \times e^{i\Delta\theta}$$

This leads to the following recurrence to compute trigonometric values s_n and c_n as above:

$$c_0 = 1$$
$$s_0 = 0$$
$$c_{n+1} = w_r\, c_n - w_i\, s_n$$
$$s_{n+1} = w_i\, c_n + w_r\, s_n$$

for $n = 0, ..., N - 1$, where $w_r = \cos(2\pi/N)$ and $w_i = \sin(2\pi/N)$. These two starting trigonometric values are usually computed using existing library functions (but could also be found e.g. by employing Newton's method in the complex plane to solve for the primitive root of $z^N - 1$).

This method would produce an *exact* table in exact arithmetic, but has errors in finite-precision floating-point arithmetic. In fact, the errors grow as $O(\varepsilon N)$ (in both the worst and average cases), where ε is the floating-point precision.

A significant improvement is to use the following modification to the above, a trick (due to Singleton) often used to generate trigonometric values for FFT implementations:

$$c_0 = 1$$
$$s_0 = 0$$
$$c_{n+1} = c_n - (\alpha c_n + \beta\, s_n)$$
$$s_{n+1} = s_n + (\beta\, c_n - \alpha\, s_n)$$

where $\alpha = 2\sin^2(\pi/N)$ and $\beta = \sin(2\pi/N)$. The errors of this method are much smaller, $O(\varepsilon \sqrt{N})$ on average and $O(\varepsilon N)$ in the worst case, but this is still large enough to substantially degrade the accuracy of FFTs of large sizes.

Numerical analysis

Numerical analysis is the study of approximate methods for the problems of *continuous mathematics* (as distinguished from discrete mathematics).

For thousands of years, man has used mathematics for construction, warfare, engineering, accounting and many other puposes. The earliest mathematical writing is perhaps the famous Babylonian tablet Plimpton 322, dating from approximately 1800 BC. On it one can read a list of pythagorean triplestriples of numbers, like (3,4,5), which are the lengths of the sides of a right-angle triangle. The Babylonian tablet YBC 7289 gives an approximation of $\sqrt{2}$, which is the length of the diagonal of a square whose side measures one unit of length. Being able to compute the sides of a triangle (and hence, being able to compute square roots) is extremely important, for instance, in carpentry and construction. If the roof of a house makes a right angle isosceles triangle whose side is 3 meters long, then the central support beam must be $\sqrt{18} \approx 4.2426$meters longer than the side beams.

Numerical analysis continues this long tradition of practical mathematical calculations. Much like the Babylonian approximation to $\sqrt{2}$, modern numerical analysis does not seek exact answers, because exact answers are impossible to obtain in practice. Instead, much of numerical analysis is concerned with obtaining approximate solutions while maintaining reasonable bounds on errors.

Numerical analysis naturally finds applications in all fields of engineering and the physical sciences, but in the 21st century, the life sciences and even the arts have adopted elements of scientific computations. Ordinary differential equations appear in the movement of heavenly bodies (planets, stars and galaxies); optimization occurs in portfolio management; numerical linear algebra is essential to quantitative psychology; stochastic differential equations and Markov chains are essential in simulating living cells for medicine and biology.

General introduction

We will now outline several important themes of numerical analysis. The overall goal is the design and analysis of techniques to give approximate solutions to hard problems. To fix ideas, the reader might consider the following problems and methods:

- If a company wants to put a toothpaste commercial on television, it might produce five commercials and then choose the best one by testing each one on a focus group. This would be an example of a Monte Carlo optimization method.
- To send a rocket to the moon, rocket scientists will need a rocket simulator. This simulator will essentially be an integrator for an ordinary differential equation.
- Car companies can improve the crash safety of their vehicles by using computer simulations of car crashes. These simulations are essentially solving partial differential equations numerically.
- Hedge funds (secretive financial companies) use tools from all fields of numerical analysis to calculate the value of stocks and derivatives more precisely than other market participants.
- Airlines use sophisticated optimization algorithms to decide ticket prices, airplane and crew assignments and fuel needs. This field is also called operations research.

Direct and iterative methods

Direct vs iterative methods
Consider the problem of solving $3x^3 + 4 = 28$

Some problems can be solved exactly by an algorithm. These algorithms are called *direct methods*. Examples are Gaussian elimination for solving systems of linear equations and the simplex method in linear programming.

However, no direct methods exist for most problems. In such cases it is sometimes possible to use an iterative method. Such a method starts from a guess and finds successive approximations that hopefully converge to the solution. Even when a direct method does exist, an iterative method may be preferable if it is more efficient or more stable.

Discretization

A discretization example

A race car drives along a track for two hours. What distance has it covered?

If we know that the race car was going at 150Km/h after one hour, we might guess that the total distance travelled is 300Km. If we know in addition that the car was travelling at 140Km/h at the 20 minutes mark, and at 180Km/h at the 1:40 mark, then we can divide the two hours into three blocks of 40 minutes. In the first block of 40 minutes, the car travelled roughly 93.3Km, in the interval between 0:40 and 1:20 the car travelled roughly 100Km, and in the interval from 1:20 to 2:00, the car travelled roughly 120Km, for a total of 313.3Km.

Essentially, we have taken the continuously varying speed

for the unknown quantity x.

Direct Method

	$3x^3 + 4 = 28$.
Subtract 4	$3x^3 = 24$.
Divide by 3	$x^3 = 8$.
Take cube roots	$x = 2$.

For the iterative method, apply the bisection method to $f(x) = 3x^3 + 24$. The initial values are $a = 0, b = 3, f(a) = 4, f(b) = 85$.

Iterative Method

a	b	mid	f(mid)
0	3	1.5	14.125
1.5	3	2.25	38.17...
1.5	2.25	1.875	23.77...
1.875	2.25	2.0625	30.32...

We conclude from this table that the solution is between 1.875 and 2.0625. The algorithm might return any number in that range with an error less than 0.2.

v(*t*) and approximated it using a speed $\hat{v}(t)$ which is constant on each of the three intervals of 40 minutes.

How far is Schumacher driving?

Furthermore, continuous problems must sometimes be replaced by a discrete problem whose solution is known to approximate that of the continuous problem; this process is called *discretization*. For example, the solution of a differential equation is a function. This function must be represented by a finite amount of data, for instance by its value at a finite number of points at its domain, even though this domain is a continuum.

The generation and propagation of errors

The study of errors forms an important part of numerical analysis. There are several ways in which error can be introduced in the solution of the problem.

Round-off

Round-off errors arise because it is impossible to represent all real numbers exactly on a finite-state machine (which is what all practical digital computers are).

On a pocket calculator, if one enters 0.0000000000001 (or the maximum number of zeros possible), then a +, and then 100000000000000 (again, the maximum number of zeros possible), one will obtain the number 100000000000000 again, and not 100000000000000.0000000000001. The calculator's answer is incorrect because of *roundoff* in the calculation.

Truncation and discretization error

Truncation errors are committed when an iterative method is terminated and the approximate solution differs from the exact solution. Similarly, discretization induces a discretization error because the solution of the discrete problem does not coincide with the solution of the continuous problem. For instance, in the iteration above to compute the solution of $3x^3 + 4 = 28$, after 10 or so iterations, we conclude that the root is roughly 1.99 (for example). We therefore have a truncation error of 0.01.

Once an error is generated, it will generally propagate through the calculation. For instance, we have already noted that the operation + on a calculator (or a computer) is inexact. It follows that a calculation of the type a+b+c+d+e is even more inexact.

Numerical stability and well posedness

Numerical stability and well posedness

Ill posed problem Take the function $f(x) = 1 / (x - 1)$. The data is *x* and the output is $f(x)$. Note that $f(1.1) = 10$ and $f(1.001) = 1000$. So a change in x of less than 0.1 turns into a change in $f(x)$ of nearly 1000. Evaluating $f(x)$ near $x = 1$ is an ill-posed (or ill-conditioned) problem.

Well-posed problem By contrast, the function $f(x) = \sqrt{x}$ is continuous so the problem of computing $\sqrt{x}$is well-posed.

Numerically unstable method Still, some algorithms which are meant to compute $\sqrt{x}$are fallible. Consider the following iterationStart with x_1 an approximation of $\sqrt{2}$(for instance, take $x_1 = 1.4$) and then use the iteration $x_{k+1} = (x_k^2 - 2)^2 + x_k.$Then the iterates are

$x_1 = 1.4$
$x_2 = 1.4016$
$x_3 = 1.4028614885...$
$x_4 = 1.403884186...$
$x_{1000000} = 1.414213437...$

On the other hand, if we start from $x_1 = 1.42$, we obtain the iterates

$x_2 = 1.42026896$
$x_3 = 1.42056...$
$x_{20} = 1.445069...$
$x_{27} = 9.34181...$
$x_{28} = 7280.2284...$

and the iteration diverges. Because this algorithm fails for certain initial guesses near $\sqrt{2}$, we say that it is numerically unstable.

Numerically stable method The

This leads to the notion of numerical stabilityan algorithm is numerically stable if an error, once it is generated, does not grow too much during the calculation. This is only possible if the problem is well-conditioned, meaning that the solution changes by only a small amount if the problem data are changed by a small amount. Indeed, if a problem is ill-conditioned, then any error in the data will grow a lot.

However, an algorithm that solves a well-conditioned problem may or may not be numerically stable. An art of numerical analysis is to find a stable algorithm for solving a well-posed mathematical problem.

Newton method for $\sqrt{a}$ is $x_1 = 1$ and $x_{k+1} = \frac{x_k + a/x_k}{2}$ and is extremely stable. The first few iterations for $a = 2$ are

$x_1 = 1$
$x_2 = 1.5$
$x_3 = 1.416...$
$x_4 = 1.414215...$
$x_5 = 1.41421356237469...$
$x_6 = 1.41421356237309...,$

which is essentially exact to the last displayed digit. The method will work well even if we change a, x_1 and/or introduce small errors at every step of the computation.

Areas of study

The field of numerical analysis is divided in different disciplines according to the problem that is to be solved.

Computing values of functions

One of the simplest problems is the evaluation of a function at a given point. The most straightforward approach, of just plugging in the number in the formula is sometimes not very efficient. For polynomials, a better approach is using the Horner scheme, since it reduces the necessary number of multiplications and additions. Generally, it is important to estimate and control round-off errors arising from the use of floating point arithmetic.

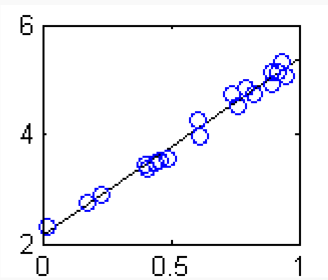
Interpolation, extrapolation and regression

Examples of Interpolation, extrapolation and regression

InterpolationIf the temperature at noon was 20 degrees centigrade and at 2pm we observe that the temperature is 14 degrees centigrade, we might guess that at 1pm the temparature was in fact the average of 14 and 20, which is 17 degrees centigrade. This would correspond to a linear interpolation of the temperature between the times of noon and 2pm.

ExtrapolationIf the gross domestic product of a country has been growing an average of 5% per year and was 100 billion dollars last year, we might extrapolate that it will be 105 billion dollars this year.

RegressionGiven two points on a piece of paper, there is a line that goes through both points. Given twenty points on a piece of paper, what is the line that goes through them? In most cases, there is no straight line going through the twenty points. In linear regression, given *n* points, we compute a line that passes as close as possible to those *n* points.



A line through 20 points?

Interpolation solves the following problemgiven the value of some unknown function at a number of points, what value does that function have at some other point between the given points? A very simple method is to use linear interpolation, which assumes that the unknown function is linear between every pair of successive points. This can be generalized to polynomial interpolation, which is sometimes more accurate but suffers from Runge's phenomenon. Other interpolation methods use localized functions like splines or wavelets.

Extrapolation is very similar to interpolation, except that now we want to find the value of the unknown function at a point which is outside the given points.

Regression is also similar, but it takes into account that the data is imprecise. Given some points, and a measurement of the value of some function at these points (with an error), we want to determine the unknown function. The least squares-method is one popular way to achieve this.

Solving equations and systems of equations

Another fundamental problem is computing the solution of some given equation. Two cases are commonly distinguished, depending on whether the equation is linear or not. For instance, the equation $2x + 5 = 3$ is linear while $2x^2 + 5 = 3$ is not.

Much effort has been put in the development of methods for solving systems of linear equations. Standard direct methods i.e. methods that use some matrix decomposition are Gaussian elimination, LU decomposition, Cholesky decomposition for symmetric (or hermitian) and positive-definite matrix, and QR decomposition for non-square matrices. Iterative methods such as the Jacobi method, Gauss-Seidel method, successive over-relaxation and conjugate gradient method are usually preferred for large systems.

Root-finding algorithms are used to solve nonlinear equations (they are so named since a root of a function is an argument for which the function yields zero). If the function is differentiable and the derivative is known, then Newton's method is a popular choice. Linearization is another technique for solving nonlinear equations.

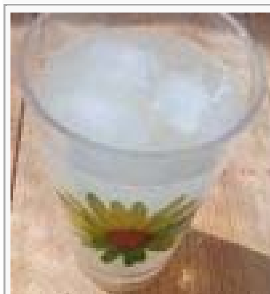
Solving eigenvalue or singular value problems

Several important problems can be phrased in terms of eigenvalue decompositions or singular value decompositions. For instance, the spectral image compression algorithm is based on the singular value decomposition. The corresponding tool in statistics is called principal component analysis. One application is to automatically find the 100 top subjects of discussion on the web, and to then classify each web page according to which subject it belongs to.

Optimization

Optimization (mathematics)

Optimization, differential equations



OptimizationSay you sell lemonade at a lemonade stand, and notice that at 1$, you can sell 197 glasses of lemonade per day, and that for each increase of 0.01$, you will sell one less lemonade per day. The optimization problem is to find the price of lemonade at which your profit per day is largest possible, ignoring production costs. If you could charge 1.485$, you would maximize your profit, but due to the constraint of having to charge a whole cent amount, charging 1.49$ per glass will yield the maximum profit of 220.52$ per day.

Optimization problems ask for the point at which a given function is maximized (or minimized). Often, the point also has to satisfy some constraints.

The field of optimization is further split in several subfields, depending on the form of the objective function and the constraint. For instance, linear programming deals with the case that both the objective function and the constraints are linear. A famous method in linear programming is the simplex method.

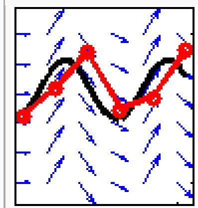
The method of Lagrange multipliers can be used to reduce optimization problems with constraints to unconstrained optimization problems.

Evaluating integrals

Numerical integration

How much paint would you need to give the Statue of Liberty a fresh coat? She is 151 feet tall and her waist is 35' across,so a first approximation is that it would require the same amount of paint you would need to paint a 151x35x35 room. Counting the four walls and the ceiling, that would make a surface area of 22,365 square feet. One gallon of paint covers about 350 square feet, so by this estimate, we might require 64 gallons of paint.

Numerical integration



Differential equation If you set up 100 fans to blow air from one end of the room to the other and then you drop a feather into the wind, what happens? The feather will follow the air currents, which may be very complex. One approximation is to measure the speed at which the air is blowing near the feather every second, and advance the simulated feather as if it were moving in a straight line at that same speed for one second, before measuring the wind speed again. This is called the Euler method for solving an ordinary differential equation.

However, it may be more precise to estimate each piece on its own. We can approximate the parts below the neck with a 95 feet tall cylinder whose radius is 17 feet. The head is roughly a sphere of radius 15 feet. The arm is another 42' long cylinder with a radius of 6', the tablet is a 24'x14'x2' box. Adding all the surface areas gives roughly 15385 square feet, which requires an approximate 44 gallons of paint.

To further improve our estimate, we would measure the folds in her cloth, how non-spherical her head really is and so on. This process is called *numerical integration*.

Numerical integration, in some instances also known as numerical quadrature, asks for the value of a definite integral. Popular methods use one of the Newton-Cotes formulas (like the midpoint rule or Simpson's rule) or Gaussian quadrature. These methods rely on a "divide and conquer" strategy, whereby an integral on a relatively large set is broken down into integrals on smaller sets. In higher dimensions, where these methods become prohibitively expensive in terms of computational effort, one may use Monte Carlo or quasi-Monte Carlo methods, or, in modestly large dimensions, the method of sparse grids.

Differential equations

sNumerical ordinary differential equations, Numerical partial differential equations.

Numerical analysis is also concerned with computing (in an approximate way) the solution of differential equations, both ordinary differential equations and partial differential equations.

Partial differential equations are solved by first discretizing the equation, bringing it into a finite-dimensional subspace. This can be done by a finite element method, a finite difference method, or (particularly in engineering) a finite volume method. The theoretical justification of these methods often involves theorems from functional analysis. This reduces the problem to the solution of an algebraic equation.

Software

List of numerical analysis software

Since the late twentieth century, most algorithms are implemented and run on a computer. The Netlib repository contains various collections of software routines for numerical problems, mostly in Fortran and C. Commercial products implementing many different numerical algorithms include the IMSL and NAG libraries; a free alternative is the GNU Scientific Library.

MATLAB is a popular commercial programming language for numerical scientific calculations, but there are commercial alternatives such as S-PLUS and IDL, as well as free and open source alternatives such as FreeMat, GNU Octave (similar to Matlab), R (similar to S-PLUS) and certain variants of Python. Performance varies widelywhile vector and matrix operations are usually fast, scalar loops vary in speed by more than an order of magnitude.

Many computer algebra systems such as Mathematica or Maple (free software systems include SAGE, Maxima, Axiom, calc and Yacas), can also be used for numerical computations. However, their strength typically lies in symbolic computations. Also, any spreadsheet software can be used to solve simple problems relating to numerical analysis.

Exact trigonometric constants

Exact constant expressions for trigonometric expressions are sometimes useful, mainly for simplifying solutions into radical forms which allow further simplification.

All values of sine, cosine, and tangent of angles with 3° increments are derivable using identitiesHalf-angle, Double-angle, Addition/subtraction and values for 0°, 30°, 36° and 45°. Note that 1° = π/180 radians.

This article is incomplete in at least two senses. First, it is always possible to apply a half-angle formula and find an exact expression for the cosine of one-half the smallest angle on the list. Second, this article exploits only the first two of five known Fermat primes3 and 5. One could in principle write down formulae involving the angles 2π/17, 2π/257, or 2π/65537, but they would be too unwieldy for most applications. In practice, all values of sine, cosine, and tangent not found in this article are approximated using the techniques described at *Generating trigonometric tables*.

Table of constants

Values outside [0°,45°] angle range are trivially extracted from circle axis reflection symmetry from these values. (See Trigonometric identity)

0° Fundamental

$$\sin 0^\circ = 0$$
$$\cos 0^\circ = 1$$
$$\tan 0^\circ = 0$$

3° - 60-sided polygon

$$\sin\frac{\pi}{60} = \sin 3^\circ = \frac{2(1-\sqrt{3})\sqrt{5+\sqrt{5}} + \sqrt{2}(\sqrt{5}-1)(\sqrt{3}+1)}{16}$$
$$\cos\frac{\pi}{60} = \cos 3^\circ = \frac{2(1+\sqrt{3})\sqrt{5+\sqrt{5}} + \sqrt{2}(\sqrt{5}-1)(\sqrt{3}-1)}{16}$$
$$\tan\frac{\pi}{60} = \tan 3^\circ = \frac{\left((2-\sqrt{3})(3+\sqrt{5})-2\right)\left(2-\sqrt{2(5-\sqrt{5})}\right)}{4}$$
$$\cot\frac{\pi}{60} = \cot 3^\circ = \frac{\left((2+\sqrt{3})(3+\sqrt{5})-2\right)\left(2+\sqrt{2(5-\sqrt{5})}\right)}{4}$$

6° - 30-sided polygon

$$\sin\frac{\pi}{30} = \sin 6^\circ = \frac{(\sqrt{6})\sqrt{5-\sqrt{5}} - (\sqrt{5}+1)}{8}$$
$$\cos\frac{\pi}{30} = \cos 6^\circ = \frac{(\sqrt{2})\sqrt{5-\sqrt{5}} + \sqrt{3}(\sqrt{5}+1)}{8}$$
$$\tan\frac{\pi}{30} = \tan 6^\circ = \frac{(\sqrt{2})\sqrt{5-\sqrt{5}} - \sqrt{3}(\sqrt{5}-1)}{2}$$
$$\cot\frac{\pi}{30} = \cot 6^\circ = \frac{\sqrt{3}(3+\sqrt{5}) + \sqrt{50+22\sqrt{5}}}{2}$$

9° - 20-sided polygon

$$\sin\frac{\pi}{20} = \sin 9^\circ = \frac{\sqrt{2}(\sqrt{5}+1) - 2\sqrt{5-\sqrt{5}}}{8}$$
$$\cos\frac{\pi}{20} = \cos 9^\circ = \frac{\sqrt{2}(\sqrt{5}+1) + 2\sqrt{5-\sqrt{5}}}{8}$$
$$\tan\frac{\pi}{20} = \tan 9^\circ = \sqrt{5}+1-\sqrt{5+2\sqrt{5}}$$
$$\cot\frac{\pi}{20} = \cot 9^\circ = \sqrt{5}+1+\sqrt{5+2\sqrt{5}}$$

12° - 15-sided polygon

$$\sin\frac{\pi}{15} = \sin 12^\circ = \frac{(\sqrt{2})\sqrt{5+\sqrt{5}} - \sqrt{3}(\sqrt{5}-1)}{8}$$
$$\cos\frac{\pi}{15} = \cos 12^\circ = \frac{(\sqrt{6})\sqrt{5+\sqrt{5}} + (\sqrt{5}-1)}{8}$$
$$\tan\frac{\pi}{15} = \tan 12^\circ = \frac{(\sqrt{3})(3-\sqrt{5}) - \sqrt{50-22\sqrt{5}}}{2}$$
$$\cot\frac{\pi}{15} = \cot 12^\circ = \frac{\sqrt{3}(\sqrt{5}+1) + \sqrt{2}\sqrt{5+\sqrt{5}}}{2}$$

15° - 12-sided polygon

$$\sin\frac{\pi}{12} = \sin 15^\circ = \frac{\sqrt{2}\left(\sqrt{3}-1\right)}{4}$$

$$\cos\frac{\pi}{12} = \cos 15^\circ = \frac{\sqrt{2}\left(\sqrt{3}+1\right)}{4}$$

$$\tan\frac{\pi}{12} = \tan 15^\circ = 2-\sqrt{3}$$

$$\cot\frac{\pi}{12} = \cot 15^\circ = 2+\sqrt{3}$$

18° - 10-sided polygon

$$\sin\frac{\pi}{10} = \sin 18^\circ = \frac{\sqrt{5}-1}{4}$$

$$\cos\frac{\pi}{10} = \cos 18^\circ = \frac{\sqrt{2(5+\sqrt{5})}}{4}$$

$$\tan\frac{\pi}{10} = \tan 18^\circ = \frac{\sqrt{5(5-2\sqrt{5})}}{5}$$

$$\cot\frac{\pi}{10} = \cot 18^\circ = \sqrt{5+2\sqrt{5}}$$

21° - Sum 9° + 12°

$$\sin\frac{7\pi}{60} = \sin 21^\circ = \frac{2(\sqrt{3}+1)\sqrt{5-\sqrt{5}}-\sqrt{2}(\sqrt{3}-1)(1+\sqrt{5})}{16}$$

$$\cos\frac{7\pi}{60} = \cos 21^\circ = \frac{2(\sqrt{3}-1)\sqrt{5-\sqrt{5}}+\sqrt{2}(\sqrt{3}+1)(1+\sqrt{5})}{16}$$

$$\tan\frac{7\pi}{60} = \tan 21^\circ = \frac{\left(2-(2+\sqrt{3})(3-\sqrt{5})\right)\left(2-\sqrt{2(5+\sqrt{5})}\right)}{4}$$

$$\cot\frac{7\pi}{60} = \cot 21^\circ = \frac{\left(2-(2-\sqrt{3})(3-\sqrt{5})\right)\left(2+\sqrt{2(5+\sqrt{5})}\right)}{4}$$

22.5° - Octagon

$$\sin\frac{\pi}{8} = \sin 22.5^\circ = \frac{\sqrt{2-\sqrt{2}}}{2}$$

$$\cos\frac{\pi}{8}=\cos 22.5^\circ=\frac{\sqrt{2+\sqrt{2}}}{2}$$
$$\tan\frac{\pi}{8}=\tan 22.5^\circ=\sqrt{2}-1$$
$$\cot\frac{\pi}{8}=\cot 22.5^\circ=\sqrt{2}+1$$

24° - Sum 12° + 12°

$$\sin\frac{2\pi}{15}=\sin 24^\circ=\frac{\sqrt{3}(\sqrt{5}+1)-\sqrt{2}\sqrt{5-\sqrt{5}}}{8}$$
$$\cos\frac{2\pi}{15}=\cos 24^\circ=\frac{\sqrt{6}\sqrt{5-\sqrt{5}}+\sqrt{5}+1}{8}$$
$$\tan\frac{2\pi}{15}=\tan 24^\circ=\frac{\sqrt{50+22\sqrt{5}}-\sqrt{3}(3+\sqrt{5})}{2}$$
$$\cot\frac{2\pi}{15}=\cot 24^\circ=\frac{\sqrt{2}\sqrt{5-\sqrt{5}}+\sqrt{3}(\sqrt{5}-1)}{2}$$

27° - Sum 12° + 15°

$$\sin\frac{3\pi}{20}=\sin 27^\circ=\frac{2\sqrt{5+\sqrt{5}}-\sqrt{2}(\sqrt{5}-1)}{8}$$
$$\cos\frac{3\pi}{20}=\cos 27^\circ=\frac{2\sqrt{5+\sqrt{5}}+\sqrt{2}(\sqrt{5}-1)}{8}$$
$$\tan\frac{3\pi}{20}=\tan 27^\circ=\sqrt{5}-1-\sqrt{5-2\sqrt{5}}$$
$$\cot\frac{3\pi}{20}=\cot 27^\circ=\sqrt{5}-1+\sqrt{5-2\sqrt{5}}$$

30° - Hexagon

$$\sin\frac{\pi}{6}=\sin 30^\circ=\frac{1}{2}$$
$$\cos\frac{\pi}{6}=\cos 30^\circ=\frac{\sqrt{3}}{2}$$
$$\tan\frac{\pi}{6}=\tan 30^\circ=\frac{\sqrt{3}}{3}$$

$$\cot\frac{\pi}{6} = \cot 30^\circ = \frac{3}{\sqrt{3}} = \sqrt{3}$$

33° - Sum 15° + 18°

$$\sin\frac{11\pi}{60} = \sin 33^\circ = \frac{2(\sqrt{3}-1)\sqrt{5+\sqrt{5}} + \sqrt{2}(1+\sqrt{3})(\sqrt{5}-1)}{16}$$
$$\cos\frac{11\pi}{60} = \cos 33^\circ = \frac{2(\sqrt{3}+1)\sqrt{5+\sqrt{5}} + \sqrt{2}(1-\sqrt{3})(\sqrt{5}-1)}{16}$$
$$\tan\frac{11\pi}{60} = \tan 33^\circ = \frac{\left(2-(2-\sqrt{3})(3+\sqrt{5})\right)\left(2+\sqrt{2(5-\sqrt{5})}\right)}{4}$$
$$\cot\frac{11\pi}{60} = \cot 33^\circ = \frac{\left(2-(2+\sqrt{3})(3+\sqrt{5})\right)\left(2-\sqrt{2(5-\sqrt{5})}\right)}{4}$$

36° - Pentagon

$$\sin\frac{\pi}{5} = \sin 36^\circ = \frac{\sqrt{2(5-\sqrt{5})}}{4}$$
$$\cos\frac{\pi}{5} = \cos 36^\circ = \frac{\sqrt{5}+1}{4}$$
$$\tan\frac{\pi}{5} = \tan 36^\circ = \sqrt{5-2\sqrt{5}}$$
$$\cot\frac{\pi}{5} = \cot 36^\circ = \frac{\sqrt{5(5+2\sqrt{5})}}{5}$$

39° - Sum 18°+ 21°

$$\sin\frac{13\pi}{60} = \sin 39^\circ = \frac{2(1-\sqrt{3})\sqrt{5-\sqrt{5}} + \sqrt{2}(\sqrt{3}+1)(\sqrt{5}+1)}{16}$$
$$\cos\frac{13\pi}{60} = \cos 39^\circ = \frac{2(1+\sqrt{3})\sqrt{5-\sqrt{5}} + \sqrt{2}(\sqrt{3}-1)(\sqrt{5}+1)}{16}$$
$$\tan\frac{13\pi}{60} = \tan 39^\circ = \frac{\left((2-\sqrt{3})(3-\sqrt{5})-2\right)\left(2-\sqrt{2(5+\sqrt{5})}\right)}{4}$$

$$\cot\frac{13\pi}{60}=\cot 39^\circ=\frac{\left((2+\sqrt{3})(3-\sqrt{5})-2\right)\left(2+\sqrt{2(5+\sqrt{5})}\right)}{4}$$

42° - Sum 21° + 21°

$$\sin\frac{7\pi}{30}=\sin 42^\circ=\frac{\sqrt{6}\sqrt{5+\sqrt{5}}-(\sqrt{5}+1)}{8}$$
$$\cos\frac{7\pi}{30}=\cos 42^\circ=\frac{\sqrt{2}\sqrt{5+\sqrt{5}}+\sqrt{3}(\sqrt{5}-1)}{8}$$
$$\tan\frac{7\pi}{30}=\tan 42^\circ=\frac{\sqrt{3}(\sqrt{5}+1)-\sqrt{2}\sqrt{5+\sqrt{5}}}{2}$$
$$\cot\frac{7\pi}{30}=\cot 42^\circ=\frac{\sqrt{50-22\sqrt{5}}+\sqrt{3}(3-\sqrt{5})}{2}$$

45° - Square

$$\sin\frac{\pi}{4}=\sin 45^\circ=\frac{\sqrt{2}}{2}$$
$$\cos\frac{\pi}{4}=\cos 45^\circ=\frac{\sqrt{2}}{2}$$
$$\tan\frac{\pi}{4}=\tan 45^\circ=1$$
$$\cot\frac{\pi}{4}=\cot 45^\circ=1$$

Notes

Uses for constants

As an example of the use of these constants, consider a dodecahedron with the following volume, where *e* is the length of an edge:

$$V=\frac{5e^3\cos 36^\circ}{\tan^2 36^\circ}$$

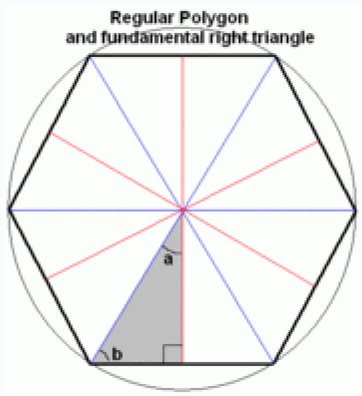
Using

$$\cos 36^\circ = \frac{\sqrt{5}+1}{4}$$
$$\tan 36^\circ = \sqrt{5-2\sqrt{5}}$$

this can be simplified to:

$$V = \frac{e^3\left(15+7\sqrt{5}\right)}{4}$$

Derivation triangles



Regular polygon (*N*-sided) and its fundamental right triangle. Angle*a* = 180/*N* °

The derivation of sine, cosine, and tangent constants into radial forms is based upon the constructability of right triangles.

Here are right triangles made from symmetry sections of regular polygons are used to calculate fundamental trigonometric ratios. Each right triangle represents three points in a regular polygona vertex, an edge center containing that vertex, and the polygon center. A *N*-gon can be divided into 2*N* right triangle with angles of {180/*N*, 90−180/*N*, 90} degrees, for *N* = 3, 4, 5, ...

Constructibility of 3, 4, 5, and 15 sided polygons are the basis, and angle bisectors allow multiples of two to also be derived.

- Constructible
  - 3×2^X-sided regular polygons, *X* = 0, 1, 2, 3, ...
    - 30°-60°-90° triangle - triangle (3 sided)
    - 60°-30°-90° triangle - hexagon (6-sided)
    - 75°-15°-90° triangle - dodecagon (12-sided)
    - 82.5°-7.5°-90° triangle - icosikaitetragon (24-sided)
    - 86.25°-3.75°-90° triangle - tetracontakaioctagon (48-sided)
    - ...
  - 4×2^X-sided

        - 45°-45°-90° triangle - square (4-sided)
        - 67.5°-22.5°-90° triangle - octagon (8-sided)
        - 88.75°-11.25°-90° triangle - hexakaidecagon (16-sided)
        - ...
    - 5×2^X-sided
        - 54°-36°-90° triangle - pentagon (5-sided)
        - 72°-18°-90° triangle - decagon (10-sided)
        - 81°-9°-90° triangle - icosagon (20-sided)
        - 85.5°-4.5°-90° triangle - tetracontagon (40-sided)
        - 87.75°-2.25°-90° triangle - octacontagon (80-sided)
        - ...
    - 15×2^X-sided
        - 78°-12°-90° triangle - pentakaidecagon (15-sided)
        - 84°-6°-90° triangle - tricontagon (30-sided)
        - 87°-3°-90° triangle - hexacontagon (60-sided)
        - 88.5°-1.5°-90° triangle - hectoicosagon (120-sided)
        - 89.25°-0.75°-90° triangle - dihectotetracontagon (240-sided)
    - ... (Higher constructible regular polygons don’t make whole degree angles17, 51, 85, 255, 257...)
- Nonconstructable (with whole or half degree angles) - No finite radical expressions involving real numbers for these triangle edge ratios are possible because of Casus Irreducibilis.
    - 9×2^X-sided
        - 70°-20°-90° triangle - enneagon (9-sided)
        - 80°-10°-90° triangle - octakaidecagon (18-sided)
        - 85°-5°-90° triangle - triacontakaihexagon (36-sided)
        - 87.5°-2.5°-90° triangle - heptacontakaidigon (72-sided)
        - ...
    - 45×2^X-sided
        - 86°-4°-90° triangle - tetracontakaipentagon (45-sided)
        - 88°-2°-90° triangle - enneacontagon (90-sided)
        - 89°-1°-90° triangle - hectaoctacontagon (180-sided)
        - 89.5°-0.5°-90° triangle - trihectohexacontagon (360-sided)
        - ...

How can the trig values for sin and cos be calculated?

The trivial ones

In degree format0, 90, 45, 30 and 60 can be calculated from their triangles, using the pythagorean theorem.

n π over 10

The multiple angle formulas for functions of 5x, where x = {18, 36, 54, 72, 90} and 5x = {90, 180, 270, 360, 540}, can be solved for the functions of x, since we know the function values of 5x. The multiple angle formulas are
$\sin 5x = 16\sin^5 x - 20\sin^3 x + 5\sin x$
$\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x$

- When sin 5x = 0 or cos 5x = 0, we let y = sin x or y = cos x and solve for y

$16y^5 - 20y^3 + 5y = 0$
One solution is zero, and the resulting 4th degree equation can be solved as a quadratic in y-squared.

- When sin 5x = 1 or cos 5x = 1, we again let y = sin x or y = cos x and solve for y

$16y^5 - 20y^3 + 5y - 1 = 0$
which factors into
$(y - 1)(4y^2 + 2y - 1)^2 = 0$

n π over 20

9° is 45 - 36, and 27° is 45 - 18; so we use the subtraction formulas for sin and cos.

n π over 30

6° is 36 - 30, 12° is 30 - 18, 24° is 54 - 30, and 42° is 60 - 18; so we use the subtraction formulas for sin and cos.

n π over 60

3° is 18 - 15, 21° is 36 - 15, 33° is 18 + 15, and 39° is 54 - 15, so we use the subtraction (or addition) formulas for sin and cos.

How can the trig values for tan and cot be calculated?

Tangent is sine divided by cosine, and cotangent is cosine divided by sine. Set up each fraction and simplify.

Plans for simplifying

Rationalize the denominator

If the denominator is a square root, multiply the numerator and denominator by that radical.
If the denominator is the sum or difference of two terms, multiply the numerator and denominator by the conjugate of the denominator. The conjugate is the identical, except the sign between the terms is changed.
Sometimes you need to rationalize the denominator more than once.

Split a fraction in two

Sometimes it helps to split the fraction into the sum of two fractions and then simplify both separately.

Squaring and square rooting

If there is a complicated term, with only one kind of radical in a term, this plan may help. Square the term, combine like terms, and take the square root. This may leave a big radical with a smaller radical inside, but it is often better than the original.

Simplification of nested radical expressions

Nested radicals

In general nested radicals cannot be reduced.

But if for $\sqrt{a+b\sqrt{c}}$,

$R=\sqrt{a^2-b^2c}$ is rational,

and both $d=\pm\sqrt{\frac{a\pm R}{2}}$ and $e=\pm\sqrt{\frac{a\pm R}{2c}}$ are rational,

with the appropriate choice of the four $\pm$ signs,

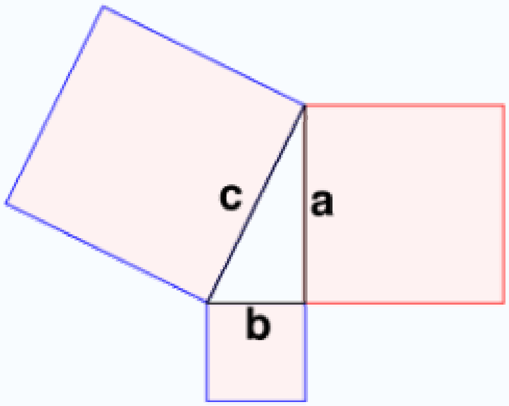
then $\sqrt{a+b\sqrt{c}}=d+e\sqrt{c}$

Example:

$$4\sin 18^\circ=\sqrt{6-2\sqrt{5}}=\sqrt{5}-1$$

Pythagorean theorem

In mathematics, the **Pythagorean theorem** or **Pythagoras' theorem** is a relation in Euclidean geometry among the three sides of a right triangle. The theorem is named after the Greek mathematician Pythagoras, who by tradition is credited with its discovery,although knowledge of the theorem almost certainly predates him.



The theorem is as follows:

> *In any right triangle, the area of the square whose side is the hypotenuse (the side of a right triangle opposite the right angle) is equal to the sum of areas of the squares whose sides are the two legs (i.e. the two sides other than the hypotenuse).*

If we let *c* be the length of the hypotenuse and *a* and *b* be the lengths of the other two sides, the theorem can be expressed as the equation

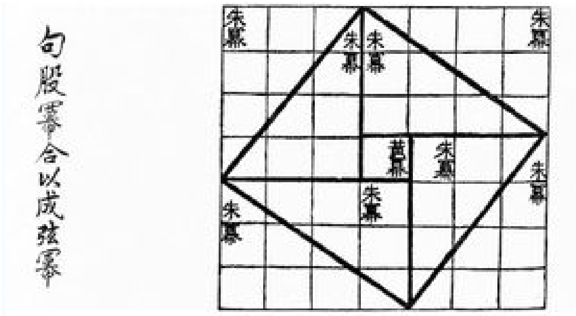
$$a^2 + b^2 = c^2.$$

This equation provides a simple relation among the three sides of a right triangle so that if the lengths of any two sides are known, the length of the third side can be found. A generalization of this theorem is the law of cosines, which allows the computation of the length of the third side of any triangle, given the lengths of two sides and the size of the angle between them.

This theorem may have more known proofs than any other. *The Pythagorean Proposition*, a book published in 1940, contains 370 proofs of Pythagoras' theorem, including one by American President James Garfield.

The converse of the theorem is also true:-

> *For any three positive numbers a, b, and c such that $a^2 + b^2 = c^2$, there exists a triangle with sides a, b and c, and every such triangle has a right angle between the sides of lengths a and b.*



Visual proof for the (3, 4, 5) triangle as in the Chou Pei Suan Ching 500–200 BC

History

The history of the theorem can be divided into three partsknowledge of Pythagorean triples, knowledge of the relationship between the sides of a right triangle, and proofs of the theorem.

Megalithic monuments from 4000 BC in Egypt, and in the British Isles from circa 2500 BC, incorporate right triangles with integer sidesbut the builders did not necessarily understand the theorem. Bartel Leendert van der Waerden conjectures that these Pythagorean triples were discovered algebraically

Written between 2000–1786 BC, the Middle Kingdom Egyptian papyrus *Berlin 6619* includes a problem, whose solution is a Pythagorean triple.

During the reign of Hammurabi, the Mesopotamian tablet *Plimpton 322*, written between 1790 and 1750 BC, contains many entries closely related to Pythagorean triples.

The *Baudhayana Sulba Sutra*, written in the 8th century BC in India, contains a list of Pythagorean triples discovered algebraically, a statement of the Pythagorean theorem, and a geometrical proof of the Pythagorean theorem for an isosceles right triangle.

The *Apastamba Sulba Sutra* (circa 600 BC) contains a numerical proof of the general Pythagorean theorem, using an area computation. Van der Waerden believes that “it was certainly based on earlier traditions”. According to Albert Bŭrk, this is the original proof of the theorem, and Pythagoras copied it on his visit to India. Many scholars find van der Waerden and Bŭrk’s claims unsubstantiated.

Pythagoras, whose dates are commonly given as 569–475 BC, used algebraic methods to construct Pythagorean triples, according to Proklos’s commentary on Euclid. Proklos, however, wrote between 410 and 485 AD. According to Sir Thomas L. Heath, there is no attribution of the theorem to Pythagoras for five centuries after Pythagoras lived. However, when authors such as Plutarch and Cicero attributed the theorem to Pythagoras, they did so in a way which suggests that the attribution was widely known and undoubted.

Around 400 BC, according to Proklos, Plato gave a method for finding Pythagorean triples that combined algebra and geometry. Circa 300 BC, in Euclid's *Elements*, the oldest extant axiomatic proof of the theorem is presented.

Written sometime between 500 BC and 200 AD, the Chinese text *Chou Pei Suan Ching* ( ), (*The Arithmetical Classic of the Gnomon and the Circular Paths of Heaven*) gives a visual proof of the Pythagorean theorem — in China it is called the "Gougu Theorem" ( ) — for the (3, 4, 5) triangle. During the Han Dynasty, from 200 BC to 200 AD, Pythagorean triples appear in *The Nine Chapters on the Mathematical Art*, together with a mention of right triangles.

There is much debate on whether the Pythagorean theorem was discovered once or many times. B.L. van der Waerden asserts a single discovery, by someone in Neolithic Britain, knowledge of which then spread to Mesopotamia circa 2000 BC, and from there to India, China, and Greece by 600 BC. Most scholars disagree however, and favor independent discovery.

More recently, Shri Bharati Krishna Tirthaji in his book Vedic Mathematics claimed ancient Indian Hindu Vedic proofs for the Pythagoras Theorem.

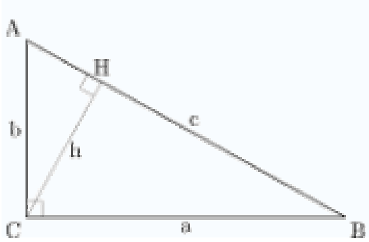
Proofs

This theorem may have more known proofs than any other (the law of quadratic reciprocity being also a contender for that distinction); the book *Pythagorean Proposition*, by Elisha Scott Loomis, contains over 350 proofs. James Garfield, who later became President of the United States, devised an original proof of the Pythagorean theorem in 1876. The external links below provide a sampling of the many proofs of the Pythagorean theorem.

Some arguments based on trigonometric identities (such as Taylor series for sine and cosine) have been proposed as proofs for the theorem. However, since all the fundamental trigonometric identities are proved using the Pythagorean theorem, there cannot be any trigonometric proof. (See also begging the question.)

For similar reasons, no proof can be based on analytic geometry or calculus.

Geometrical proof



Proof using similar triangles

Like many of the proofs of the Pythagorean theorem, this one is based on the proportionality of the sides of two similar triangles.

Let *ABC* represent a right triangle, with the right angle located at *C*, as shown on the figure. We draw the altitude from point *C*, and call *H* its intersection with the side *AB*. The new triangle *ACH* is similar to our triangle *ABC*, because they both have a right angle (by definition of the altitude), and they share the angle at *A*, meaning that the third angle will be the same in both triangles as well. By a similar reasoning, the triangle CBH is also similar to *ABC*. The similarities lead to the two ratios:

$$\frac{AC}{AB} = \frac{AH}{AC} \text{ and } \frac{CB}{AB} = \frac{HB}{CB}.$$

These can be written as:

$AC^2 = AB \times AH$ and $CB^2 = AB \times HB.$

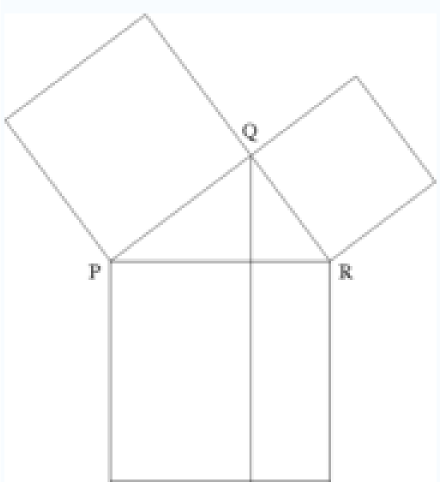
Summing these two equalities, we obtain:

$$AC^2 + CB^2 = AB \times AH + AB \times HB = AB \times (AH + HB) = AB^2.$$

In other words, the Pythagorean theorem:

$$AC^2 + BC^2 = AB^2.$$

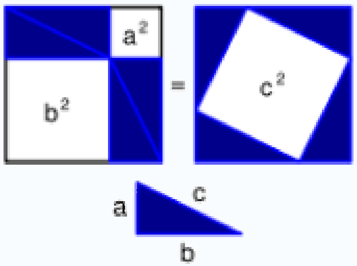
Euclid's proof



Proof in Euclid's *Elements*

In Euclid's *Elements*, the Pythagorean theorem is proved by an argument along the following lines. Let *P*, *Q*, *R* be the vertices of a right triangle, with a right angle at *Q*. Drop a perpendicular from *Q* to the side opposite the hypotenuse in the square on the hypotenuse. That line divides the square on the hypotenuse into two rectangles, each having the same area as one of the two squares on the legs.

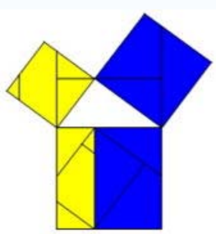
Visual proof



Proof using area subtraction

A visual proof is given by this illustration. The area of each large square is $(a + b)^2$. In both, the area of four identical triangles is removed. The remaining areas, $a^2 + b^2$ and c^2, are equal. Q.E.D.

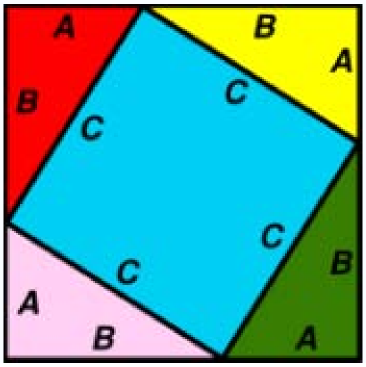
This proof is indeed very simple, but it is not *elementary*, in the sense that it does not depend solely upon the most basic axioms and theorems of Euclidean geometry. In particular, while it is quite easy to give a formula for area of triangles and squares, it is not as easy to prove that the area of a square is the sum of areas of its pieces. In fact, proving the necessary properties is harder than proving the Pythagorean theorem itself. For this reason, axiomatic introductions to geometry usually employ another proof based on the similarity of triangles (see above).



Proof using rearrangement

A second graphic illustration of the Pythagorean theorem fits parts of the sides' squares into the hypotenuse's square. A related proof would show that the repositioned parts are identical with the originals and, since the sum of equals are equal, that the corresponding areas are equal. To show that a square is the result one must show that the length of the new sides equals *c*.

Algebraic proof



A square created by aligning four right angle triangles and a large square.

A more algebraic variant of this proof is provided by the following reasoning. Looking at the illustration, the area of each of the four red, yellow, green and pink right-angled triangles is given by:

$$\frac{1}{2}ab$$

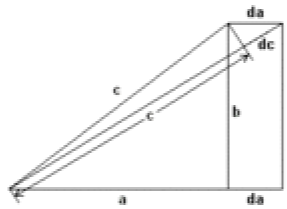
The blue square in the middle has side length c, so its area is c^2. Thus the area of everything together is given by:

$$4\left(\frac{1}{2}ab\right)+c^2.$$

However, as the large square has sides of length $a+b$, we can also calculate its area as $(a+b)^2$, which expands to $a^2+2ab+b^2$. This can be shown by considering the angles.

Proof by differential equations

One can arrive at the Pythagorean theorem by studying how changes in a side produce a change in the hypotenuse in the following diagram and employing a little calculus.



Proof using differential equations

As a result of a change in side *a*,

$$\frac{da}{dc} = \frac{c}{a}$$

by similar triangles and for differential changes. So

$$c\,dc = a\,da$$

upon separation of variables. A more general result is

$$c\ dc = a\,da + b\,db$$

which results from adding a second term for changes in side *b*.

Integrating gives

$$c^2 = a^2 + b^2 + \text{constant}.$$
$$a = b = c = 0 \Rightarrow \text{constant} = 0$$

So

$$c^2 = a^2 + b^2.$$

As can be seen, the squares are due to the particular proportion between the changes and the sides while the sum is a result of the independent contributions of the changes in the sides which is not evident from the geometric proofs. From the proportion given it can be shown that the changes in the sides are inversely proportional to the sides. The differential equation suggests that the theorem is due to relative changes and its derivation is nearly equivalent to computing a line integral. A simpler derivation would leave b fixed and then observe that

$$a = 0 \Rightarrow c^2 = b^2 = \text{constant}.$$

It is doubtful that the Pythagoreans would have been able to do the above proof but they knew how to compute the area of a triangle and were familiar with figurate numbers and the gnomon, a segment added onto a geometrical figure. All of these ideas predate calculus and are an alternative for the integral.

The proportional relation between the changes and their sides is at best an approximation, so how can one justify its use? The answer is the approximation gets better for smaller changes since the arc of the circle which cuts off *c* more closely approaches the tangent to the circle. As for the sides and triangles, no matter how many segments they are divided into the sum of these segments is always the same. The Pythagoreans were trying to understand change and motion and this led them to realize that the number line was infinitely divisible. Could they have discovered the approximation for the changes in the sides? One only has to observe that the motion of the shadow of a sundial produces the hypotenuses of the triangles to derive the figure shown.

Rational trigonometry

For a proof by the methods of rational trigonometry, see Pythagoras' theorem proof (rational trigonometry).

Proof of the converse

> For any three positive numbers a, b, and c such that $a^2 + b^2 = c^2$, there exists a triangle with sides a, b and c, and every such triangle has a right angle between the sides of lengths a and b.

This converse also appears in Euclid's *Elements*. It can be proven using the law of cosines (see below under Generalizations), or by the following proof:

Let ABC be a triangle with side lengths a, b, and c, with $a^2 + b^2 = c^2$. We need to prove that the angle between the a and b sides is a right angle. We construct another triangle with a right angle between sides of lengths a and b. By the Pythagorean theorem, it follows that the hypotenuse of this triangle also has length c. Since both triangles have the same side lengths a, b and c, they are congruent, and so they must have the same angles. Therefore, the angle between the side of lengths a and b in our original triangle is a right angle.

Consequences and uses of the theorem

Pythagorean triples

Pythagorean triple

A Pythagorean triple consists of three positive integers a, b, and c, such that $a^2 + b^2 = c^2$. In other words, a Pythagorean triple represents the lengths of the sides of a right triangle where all three sides have integer lengths. Evidence from megalithic monuments on the

British Isles shows that such triples were known before the discovery of writing. Such a triple is commonly written (*a*, *b*, *c*), and a well-known example is (3, 4, 5).

A Pythagorean triple is primitive if *a*, *b*, and *c* have no common divisor other than 1. There are infinitely many primitive triples, and all Pythagorean triples can be explicitly generated using the following formulachoose two integers *m* and *n* with $m > n$, and let $a = m^2 - n^2$, $b = 2mn$, $c = m^2 + n^2$. Then we have $a^2 + b^2 = c^2$. Also, any multiple of a Pythagorean triple is again a Pythagorean triple.

Pythagorean triples allow the construction of right angles. The fact that the lengths of the sides are integers means that, for example, tying knots at equal intervals along a string allows the string to be stretched into a triangle with sides of length three, four, and five, in which case the largest angle will be a right angle. This methods was used to step masts at sea and by the Egyptians in construction work.

A generalization of the concept of Pythagorean triples is a triple of positive integers *a*, *b*, and *c*, such that $a^n + b^n = c^n$, for some *n* strictly greater than 2. Pierre de Fermat in 1637 claimed that no such triple exists, a claim that came to be known as Fermat’s last theorem. The first proof was given by Andrew Wiles in 1994.

The existence of irrational numbers

One of the consequences of the Pythagorean theorem is that irrational numbers, such as the square root of two, can be constructed. A right triangle with legs both equal to one unit has hypotenuse length square root of two. The Pythagoreans proved that the square root of two is irrational, and this proof has come down to us even though it flew in the face of their cherished belief that everything was rational. According to the legend, Hippasus, who first proved the irrationality of the square root of two, was drowned at sea as a consequence.

Distance in Cartesian coordinates

The distance formula in Cartesian coordinates is derived from the Pythagorean theorem. If (x_0, y_0) and (x_1, y_1) are points in the plane, then the distance between them, also called the Euclidean distance, is given by

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}.$$

More generally, in Euclidean *n*-space, the Euclidean distance between two points $A = (a_1, a_2, \dots, a_n)$ and $B = (b_1, b_2, \dots, b_n)$, is defined, using the Pythagorean theorem, as:

$$\sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots + (a_n - b_n)^2} = \sqrt{\sum_{i=1}^{n}(a_i - b_i)^2}$$

Generalizations

The Pythagorean theorem was generalised by Euclid in his *Elements:*

> If one erects similar figures (see Euclidean geometry) on the sides of a right triangle, then the sum of the areas of the two smaller ones equals the area of the larger one.

The Pythagorean theorem is a special case of the more general theorem relating the lengths of sides in any triangle, the law of cosines:

$$a^2 + b^2 - 2ab\cos\theta = c^2,$$

where θ is the angle between sides *a* and *b*.

When θ is 90 degrees, then cos(θ) = 0, so the formula reduces to the usual Pythagorean theorem.

Given two vectors **v** and **w** in a complex inner product space, the Pythagorean theorem takes the following form:

$$\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 + 2\,\mathrm{Re}\,\langle\mathbf{v}, \mathbf{w}\rangle$$

In particular, $\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$ if and only if **v** and **w** are orthogonal.

Using mathematical induction, the previous result can be extended to any finite number of pairwise orthogonal vectors. Let $\mathbf{v}_1$, $\mathbf{v}_2$,..., $\mathbf{v}_n$ be vectors in an inner product space such that $<\mathbf{v}_i, \mathbf{v}_j> = 0$ for $1 \le i < j \le n$. Then

$$\left\|\sum_{k=1}^{n}\mathbf{v}_k\right\|^2 = \sum_{k=1}^{n}\|\mathbf{v}_k\|^2$$

The generalisation of this result to *infinite-dimensional* real inner product spaces is known as Parseval’s identity.

When the theorem above about vectors is rewritten in terms of solid geometry, it becomes the following theorem. If lines AB and BC form a right angle at B, and lines BC and CD form a right angle at C, and if CD is perpendicular to the plane containing lines AB and BC, then the sum of the squares of the lengths of AB, BC, and CD is equal to the square of AD. The proof is trivial.

Another generalization of the Pythagorean theorem to three dimensions is **de Gua's theorem**, named for Jean Paul de Gua de MalvesIf a tetrahedron has a right angle corner (a corner like a cube), then the square of the area of the face opposite the right angle corner is the sum of the squares of the areas of the other three faces.

There are also analogs of these theorems in dimensions four and higher.

In a triangle with three acute angles, $\alpha + \beta > \gamma$ holds. Therefore, $a^2 + b^2 > c^2$ holds.

In a triangle with an obtuse angle, $\alpha + \beta < \gamma$ holds. Therefore, $a^2 + b^2 < c^2$ holds.

Edsger Dijkstra has stated this proposition about acute, right, and obstuse triangles in this language:

$$\operatorname{sgn}(\alpha + \beta - \gamma) = \operatorname{sgn}(a^2 + b^2 - c^2)$$

where α is the angle opposite to side a, β is the angle opposite to side b and γ is the angle opposite to side c.

The Pythagorean theorem in non-Euclidean geometry

The Pythagorean theorem is derived from the axioms of Euclidean geometry, and in fact, the Euclidean form of the Pythagorean theorem given above does not hold in non-Euclidean geometry. (It has been shown in fact to be equivalent to Euclid's Parallel (Fifth) Postulate.) For example, in spherical geometry, all three sides of the right triangle bounding an octant of the unit sphere have length equal to $\pi / 2$; this violates the Euclidean Pythagorean theorem because $(\pi/2)^2 + (\pi/2)^2 \neq (\pi/2)^2$. However, in hyperbolic geometry, the Pythagorean theorem does hold in the limit of small distances.

This means that in non-Euclidean geometry, the Pythagorean theorem must necessarily take a different form from the Euclidean theorem. There are two cases to consider—spherical geometry and hyperbolic plane geometry; in each case, as in the Euclidean case, the result follows from the appropriate law of cosines:

For any right triangle on a sphere of radius R, the Pythagorean theorem takes the form

$$\cos\left(\frac{c}{R}\right) = \cos\left(\frac{a}{R}\right)\,\cos\left(\frac{b}{R}\right).$$

By using the Maclaurin series for the cosine function, it can be shown that as the radius R approaches infinity, the spherical form of the Pythagorean theorem approaches the Euclidean form.

For any triangle in the hyperbolic plane (with Gaussian curvature −1), the Pythagorean theorem takes the form

$$\cosh c = \cosh a \, \cosh b$$

where cosh is the hyperbolic cosine.

By using the Maclaurin series for this function, it can be shown that as a hyperbolic triangle becomes very small (i.e., as *a*, *b*, and *c* all approach zero), the hyperbolic form of the Pythagorean theorem approaches the Euclidean form.

Pythagoras's theorem and complex numbers

This proof is only valid if *a* and *b* are real. If *a* and/or *b* have imaginary parts, Pythagoras's theorem breaks down because the concept of areas loses its meaning because in the complex plane loci of the type $y = f(x)$ (which includes straight lines) cannot separate an inside from an outside because there they are 2-dimensional (y, iy) in a 4-dimensional space (x, ix, y, iy).

Cultural references to the Pythagorean theorem

- In *The Wizard of Oz*, when the Scarecrow receives his diploma from the Wizard, he immediately exhibits his "knowledge" by reciting a mangled and incorrect version of the theorem"The sum of the square roots of any two sides of an isosceles triangle is equal to the square root of the remaining side. Oh, joy, oh, rapture. I've got a brain!"
- In an episode of *The Simpsons*, homage is paid to the Oz Scarecrow's quote, thus turning the theorem into a cultural reference to a cultural reference. After finding a pair of Henry Kissinger's glasses at the Springfield Nuclear Power Plant, Homer puts them on and quotes the scarecrow verbatim. A man in a nearby toilet stall then yells out "That's a *right* triangle, you idiot!"
- In 2000, Uganda released a coin with the shape of a right triangle. The tail has an image of Pythagoras and the Pythagorean theorem, accompanied with the mention "Pythagoras Millennium".