Topology

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1 Set and Map

1.1 Set and Element

Sets and elements are the most basic concepts of mathematics. Given any element x and any set X, either x belongs to X (denoted $x \in X$), or x does not belong to X (denoted $x \notin X$).

A set can be presented by listing all its elements.

Example 1.1

- $X_1 = \{1, 2, 3, \cdots, n\}$ is the set of all integers between 1 and n;
- $X_2 = \{2, -5\}$ is the set of all numbers satisfying the equation $x^2 + 3x - 10 = 0$.

Example 1.2

- $Y_1 = \{a, b, c, \cdots, x, y, z\}$ is the set of all latin alphabets;
- $Y_2 = \{\text{red}, \text{green}, \text{blue}\}$ is the set of basic colours (whose combinations form all the other colours);
- $Y_3 = \{\text{red}, \text{yellow}\}$ is the set of colours in the Chinese national flag;
- The set Y_4 of all registered students in this class is the list of names the registration office gave to me.

A set can also be presented by describing the properties its elements satisfy.

Example 1.3

- Natural numbers $\mathbf{N} = \{n : n \text{ is obtained by repeatedly adding 1 to itself}\}$;
- Prime numbers $\mathbf{P} = \{p : p \text{ is an integer, and is not a product of two integers not equal to 0, 1, } -1\}$;
- Rational numbers $\mathbf{Q} = \{\frac{m}{n} : m \text{ and } n \text{ are integers}\}$;
- Open interval $(a, b) = \{x : x \in \mathbf{R}, a < x < b\}$;
- Closed interval $[a, b] = \{x : x \in \mathbf{R}, a \le x \le b\}$;
- Real polynomials $\mathbf{R}[t] = \{p(t) = a_0 + a_1 t + a_1 t^2 + \cdots + a_n t^n : a_i \text{ are real numbers}\}$;
- Continuous functions $C[0, 1] = \{f : \lim_{x \to a} f(x) = f(a) \text{ for any } 0 \le a \le 1\}$ on $[0, 1]$;
- Unit sphere $S^2 = \{(x_1, x_2, x_3) : x_i \in \mathbf{R}, x_1^2 + x_2^2 + x_3^2 = 1\}$ is contained in $\mathbf{R}^3$;
- $X_1 = \{x : x \in \mathbf{N}, x \le n\}$;
- $X_2 = \{x : x^2 + 3x - 10 = 0\}$.

Exercise 1.1 Present the following sets: the set $\mathbf{Z}$ of integers; the unit sphere S^n in $\mathbf{R}^{n+1}$; the set $\mathbf{GL}(n)$ of invertible $n \times n$ matrices; the set of latin alphabets in your name.

Lte X and Y be sets. We say X is a *subset* of Y if $x \in X$ implies $x \in Y$. In this case, we denote $X \subset Y$ (X is *contained in* Y) or $Y \supset X$ (Y *contains* X). We clearly have

$$X \subset Y, \quad Y \subset Z \quad \Longrightarrow \quad X \subset Z.$$

The *power set* $\mathcal{P}(X)$ (somtimes also denoted as 2^X) of a set X is the collection of all subsets of X.

The special notation $\emptyset$ is reserved for the *empty set*, the set with no elements. The empty set is a subset of any set.

Example 1.4 The following are all the subsets of the set $\{1, 2, 3\}$:

$$\emptyset,\ \{1\},\ \{2\},\ \{3\},\ \{1, 2\},\ \{1, 3\},\ \{2, 3\},\ \{1, 2, 3\}.$$

Therefore the power set of $\{1, 2, 3\}$ is

$$\mathcal{P}\{1, 2, 3\} = \Big\{\emptyset,\ \{1\},\ \{2\},\ \{3\},\ \{1, 2\},\ \{1, 3\},\ \{2, 3\},\ \{1, 2, 3\}\Big\}$$

of all subsets of $\{1, 2, 3\}$ is called the power set of $\{1, 2, 3\}$.

The construction of power sets demonstrates that sets themselves can become elements of some other sets (which we usually call *collection of sets*). For example, the set

$$\Big\{\emptyset,\ \{1, 2\},\ \{1, 3\},\ \{2, 3\}\Big\}$$

is the collection of subsets of $\{1, 2, 3\}$ with even number of elements.

Exercise 1.2 How many elements are in the power set of $\{1, 2, \cdots, n\}$ (The answer suggests the reason for denoting the power set by 2^X)? How many of these contain even number of elements? How many contain odd number?

Exercise 1.3 List all elements in $\mathcal{P}(\mathcal{P}\{1, 2\})$, the power set of the power set of $\{1, 2\}$.

Exercise 1.4 Given any real number $\epsilon > 0$, find a real number $\delta > 0$, such that

$$\{x : |x - 1| < \delta\} \subset \{x : |x^2 - 1| < \epsilon\}.$$

Finally, we say X and Y are *equal* if $X \subset Y$ and $X \supset Y$. In other words, in order to show $X = Y$, we need to verify

- $x \in X$ implies $x \in Y$;
- $y \in Y$ implies $y \in X$.

Exercise 1.5 Use the definition to show that the set X_2 in Example 1.1 is equal to the set X_2 in Example 1.3.

Exercise 1.6 We say X is a *proper subset* of Y if $X \subset Y$ and $X \neq Y$. Show that if X is a proper subset of Y and Y is a proper subset of Z, then X is a proper subset of Z.

Exercise 1.7 Can you find two sets X and Y such that X is a subset of Y and Y is a subset of X?

1.2 Set Operations

The *union* of two sets X and Y is

$$X \cup Y = \{x : x \in X \text{ or } x \in Y\}.$$

Note that in the union, we allow the possibility that x is in both X and Y. The *intersection* of two sets is

$$X \cap Y = \{x : x \in X \text{ and } x \in Y\}.$$

If $X \cap Y = \emptyset$, then there is no common elements shared by X and Y, and we say X and Y are *disjoint*. Note the logical difference between the use of *or* in the union and the use of *and* in the intersection.

The union and the intersection have the following properties:

- $X \cap Y \subset X \subset X \cup Y$;
- Commutativity: $Y \cup X = X \cup Y$, $Y \cap X = X \cap Y$;
- Associativity: $(X \cup Y) \cup Z = X \cup (Y \cup Z)$, $(X \cap Y) \cap Z = X \cap (Y \cap Z)$;
- Distibutivity: $(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$, $(X \cap Y) \cup Z = (X \cup Z) \cap (Y \cup Z)$;
- $X \cup \emptyset = X$, $X \cap \emptyset = \emptyset$.

The properties are not hard to prove. For example, the proof of the associativity of the union involves the verification of $(X \cup Y) \cup Z \subset X \cup (Y \cup Z)$ and $(X \cup Y) \cup Z \supset X \cup (Y \cup Z)$. The first inclusion is verified as follows

$$\begin{aligned} & x \in (X \cup Y) \cup Z \\ \Rightarrow\ & x \in X \cup Y \text{ or } x \in Z \\ \Rightarrow\ & x \in X, \text{ or } x \in Y, \text{ or } x \in Z \\ \Rightarrow\ & x \in X, \text{ or } x \in Y \cup Z \\ \Rightarrow\ & x \in X \cup (Y \cup Z). \end{aligned}$$

The second inclusion can be verified similarly.

The *difference* of two sets X and Y is

$$X - Y = \{x : x \in X \text{ and } x \notin Y\}.$$

We also call $X - Y$ the *complement* of Y in X. The following are some useful properties about the difference:

- $X - Y = \emptyset \Leftrightarrow X \subset Y$, $X - Y = X \Leftrightarrow X \cap Y = \emptyset$;
- deMorgan's Law: $X - (Y \cup Z) = (X - Y) \cap (X - Z)$; $X - (Y \cap Z) = (X - Y) \cup (X - Z)$.

We note that deMorgan's law says that the complement operation converts the union to the intersection and vice versa.

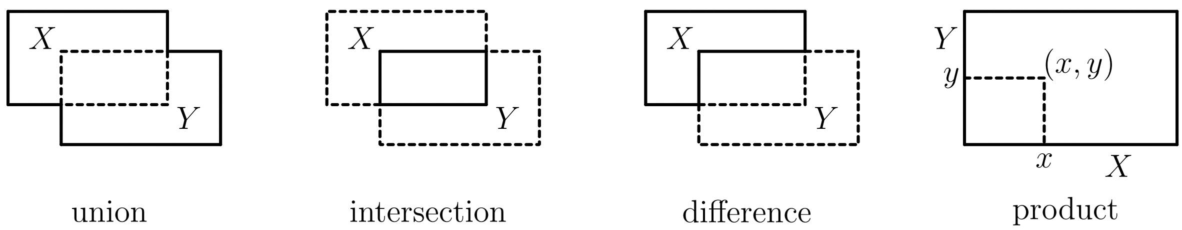


Figure 1: operations on sets

The (cartesian) *product* of two sets X and Y is

$$X \times Y = \{(x, y) : x \in X, y \in Y\}.$$

We also use X^n to denote the product of n copies of X.

When we mix several operations, we usually first take product $\times$, then take union $\cup$ or intersection $\cap$, and finally take difference $-$. For example, $x \in (X - Y \times Z) \cap W - U$ means $x \in X$, $x \notin Y \times Z$, $x \in W$, and $x \notin U$. The convention is similar to first taking multiplication and division, and then taking summation and subtraction in numerical computations.

Exercise 1.8 Prove the properties of the union, the intersection, and the difference.

Exercise 1.9 Find all the unions and intersections among $A = \{\emptyset\}$, $B = \{\emptyset, A\}$, $C = \{\emptyset, A, B\}$.

Exercise 1.10 Express the following using sets X, Y, Z and operations $\cup$, $\cap$, $-$:

1. $A = \{x : x \in X \text{ and } (x \in Y \text{ or } x \in Z)\}$;
2. $B = \{x : (x \in X \text{ and } x \in Y) \text{ or } x \in Z\}$;
3. $C = \{x : x \in X,\ x \notin Y, \text{ and } x \in Z\}$.

Exercise 1.11 Determine which of the following statements are true:

1. $X \subset Z$ and $Y \subset Z \Rightarrow X \cup Y \subset Z$;
2. $X \subset Z$ and $Y \subset Z \Rightarrow X \cap Y \subset Z$;
3. $X \subset Z$ or $Y \subset Z \Rightarrow X \cup Y \subset Z$;
4. $Z \subset X$ and $Z \subset Y \Rightarrow Z \subset X \cap Y$;
5. $Z \subset X$ and $Z \subset Y \Rightarrow Z \subset X \cup Y$;
6. $Z \subset X \cap Y \Rightarrow Z \subset X$ and $Z \subset Y$;
7. $Z \subset X \cup Y \Rightarrow Z \subset X$ and $Z \subset Y$;
8. $X - (Y - Z) = (X - Y) \cup Z$;
9. $(X - Y) - Z = X - Y \cup Z$;
10. $X - (X - Z) = Z$;
11. $X \cap (Y - Z) = X \cap Y - X \cap Z$;
12. $X \cup (Y - Z) = X \cup Y - X \cup Z$;
13. $(X \cap Y) \cup (X - Y) = X$;
14. $X \subset U$ and $Y \subset V \Leftrightarrow X \times Y \subset U \times V$;
15. $X \times (U \cup V) = X \times U \cup X \times V$;
16. $X \times (U - V) = X \times U - X \times V$;
17. $(X - Y) \times (U - V) = X \times U - Y \times V$.

Exercise 1.12 Let A and B be subsets of X, prove that

$$A \subset B \Leftrightarrow X - A \supset B \Leftrightarrow A \cap (X - B) = \emptyset.$$

1.3 Maps

A *map* from a set X to a set Y is a process f that assigns, for each $x \in X$, a unique $y = f(x) \in Y$. Sometimes, especially when $X = Y$, we also call f a *transformation.* In case Y is a set of numbers, we also call f a *function.*

We call X and Y the *domain* and *range* of the map. We also call y the *image* (or the *value*) of x. We may denote a map by the notation

$$f : X \to Y, \quad x \mapsto f(x),$$

or the notation

$$f(x) = y : X \to Y,$$

in which all the ingredients are indicated. We may omit some parts of the notation if the part is clear from the context.

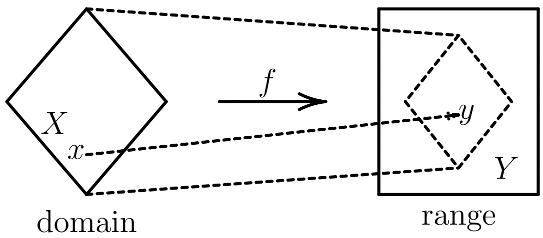


Figure 2: domain, range, and image

We emphasis that in order for a process f to be a map, two important properties have to be established:

1. For any $x \in X$, the process always goes through to the end and produces some $y \in Y$;
2. For any $x \in X$, the end result $y \in Y$ of the process must be unique. In other words, the process is *single-valued.*

It is especially important that whenever we define a new map f by certain process, we have to subtantiate the process by showing that it always works (as in item 1) and the outcome is unambiguous (as in item 2). Once substantiated, we say f is *well-defined.*

Example 1.5 By the map $f(x) = 2x^2 - 1 : \mathbf{R} \to \mathbf{R}$ (equivalently, $f : \mathbf{R} \to \mathbf{R}, x \mapsto 2x^2 - 1$), we mean the following process: For any $x \in \mathbf{R}$, we first multiply x to itself to get x^2, then we multiply 2 and subtract 1 to get $2x^2 - 1$. Since each step always works and gives unique outcome, the process is a map.

Example 1.6 The square root function $f(x) = \sqrt{x} : [0, \infty) \to \mathbf{R}$ is the following process: For any $x \geq 0$, we find a *non-negative* number y, such that multiplying y to itself yields x. Then $f(x) = y$. Again, since y always exists and is unique, the process is a map.

Note that if we change $[0, \infty)$ to $\mathbf{R}$, then y does not exist for negative $x \in \mathbf{R}$, we cannot find a number y so that multiplying y to itself is -1. In other words, the first condition for the process to be a map is violated, and $\sqrt{x} : \mathbf{R} \to \mathbf{R}$ is not a map.

On the other hand, suppose we modify the process by no longer requiring y to be non-negative. Then for any $x \in [0, \infty)$, the process always go through, except we will have two outcomes (one positive, one negative) in general. Thus the second condition is violated and the process is not a map.

Example 1.7 The map $R_\theta(x_1, x_2) = (x_1 \cos\theta - x_2 \sin\theta, x_2 \sin\theta + x_1 \cos\theta) : \mathbf{R}^2 \to \mathbf{R}^2$ is the following process: For any point (x_1, x_2) on the plane $\mathbf{R}^2$, we rotate (x_1, x_2) around the origin by angle θ in counterclockwise direction.

Example 1.8 The map $\text{Age} : Y_4 \to \mathbf{N}$ is the process of subtracting the birth year from the current year.

Example 1.9 The map $FA : Y_2 \to Y_1$, $FA(\text{red}) = r$, $FA(\text{green}) = g$, $FA(\text{blue}) = b$ is the process of assigning the first alphabet of the name of the colour. The map $A : Y_2 \to \mathcal{P}(Y_1)$, $A(\text{red}) = \{d, e, r\}$, $A(\text{green}) = \{e, g, n, r\}$, $A(\text{blue}) = \{b, e, l, u\}$ is the process of assigning all the alphabets in the name of the colour.

Exercise 1.13 Suppose we want to combine two maps $f : X \to Z$ and $g : Y \to Z$ to get a new map $h : X \cup Y \to Z$ as follows

$$h(x) = \begin{cases} f(x) & \text{if } x \in X \\ g(x) & \text{if } x \in Y \end{cases}.$$

What is the condition for h to be a map?

Exercise 1.14 For a subset $A \subset X$, define the *characteristic function*

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \quad : \quad X \to \mathbf{R}$$

Prove the following equalities

$$\chi_{A\cap B} = \chi_A \chi_B, \quad \chi_{X-A} + \chi_A = 1.$$

Moreover, express $\chi_{A\cup B}$ in terms of χ_A and χ_B.

For a map $f : X \to Y$, the *image* of a subset $A \subset X$ is

$$f(A) = \{f(a) : a \in A\} = \{y : y = f(a) \quad \text{for some } a \in A\}.$$

In the other direction, the *preimage* of a subset $B \subset Y$ is

$$f^{-1}(B) = \{x : f(x) \in B\}.$$

In case B is a single point, we have the preimage

$$f^{-1}(y) = \{x : f(x) = y\}$$

of a point $y \in Y$.

Example 1.10 For the map $f(x) = 2x^2 - 1 : \mathbf{R} \to \mathbf{R}$, both the domain and the range are $\mathbf{R}$. The image of the whole domain $\mathbf{R}$ is $[-1, \infty)$. Moreover, the image of $[0, \infty)$ is also $[-1, \infty)$. The preimage of $[0, \infty)$ is $(-\infty, -\frac{1}{\sqrt{2}}] \cup [\frac{1}{\sqrt{2}}, \infty)$. The preimage of 1 is $\{1, -1\}$.

For the rotation map $R_\theta(x_1, x_2) = (x_1 \cos\theta - x_2 \sin\theta, x_1 \sin\theta + x_2 \cos\theta) : \mathbf{R}^2 \to \mathbf{R}^2$. Both the domain and the range are $\mathbf{R}^2$. The image of the whole domain is also $\mathbf{R}^2$. Moreover, the image (and the preimage) of any circle centered at the origin is the circle itself. If the circle is not centered at the origin, then the image (and the preimage) is still a circle, but at a different location.

For the first alphabet map $FA : Y_2 \to Y_1$, the domain is the three colours, the range is all the alphabets, and the image is $\{b, g, r\}$. The preimage of $\{a, b, c\}$ is $\{\text{blue}\}$. The preimage of $\{a, b, c, d, e\}$ is still $\{\text{blue}\}$. The preimage of $\{u, v, w, x, y, z\}$ is $\emptyset$.

Exercise 1.15 Determine which of the following are true. If not, whether at least one direction is true:

1. $A \subset B \Leftrightarrow f(A) \subset f(B)$;
2. $A \subset B \Leftrightarrow f^{-1}(A) \subset f^{-1}(B)$;
3. $A \cap B = \emptyset \Leftrightarrow f(A) \cap f(B) = \emptyset$;
4. $A \cap B = \emptyset \Leftrightarrow f^{-1}(A) \cap f^{-1}(B) = \emptyset$.

Exercise 1.16 Determine which of the following equalities are true. If not, determine whether some inclusion still holds:

1. $f(A \cup B) = f(A) \cup f(B)$;
2. $f(A \cap B) = f(A) \cap f(B)$;
3. $f(A - B) = f(A) - f(B)$;
4. $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$;
5. $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$;
6. $f^{-1}(A - B) = f^{-1}(A) - f^{-1}(B)$;
7. $f(f^{-1}(A)) = A$;
8. $f^{-1}(f(A)) = A$.

Exercise 1.17 For the rotation R_θ, what is the image (or the preimage) of a straight line in $\mathbf{R}^2$? When is the image (or the preimage) the same as the original line?

Exercise 1.18 Let A and B be subsets of X. What are $\chi_A(B)$ and $\chi_A^{-1}(1)$? Moreover, let C be a subset of $\mathbf{R}$, what is $\chi_A^{-1}(C)$?

Exercise 1.19 Let $f : X \to Y$ be a map. Show that the "image map" $F : \mathcal{P}(X) \to \mathcal{P}(Y), A \mapsto f(A)$ is indeed a map. Is the similar "preimage map" also a map?

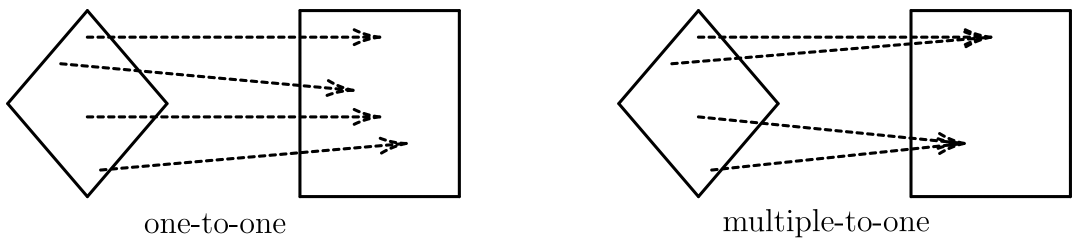


Figure 3: one-to-one and multiple-to-one

We say a map $f : X \to Y$ is *one-to-one* (or *injective*) if

$$f(x_1) = f(x_2) \quad \Longrightarrow \quad x_1 = x_2.$$

In other words, different elements of X are mapped by f to different elements of Y. We say f is *onto* (or *surjective*) if

$$y \in Y \quad \Longrightarrow \quad y = f(x) \quad \text{for some } x \in X.$$

In other words, the image of the whole domain $f(X) = Y$. We say f is a *one-to-one correspondence* (or *bijective*) if it is one-to-one and onto.

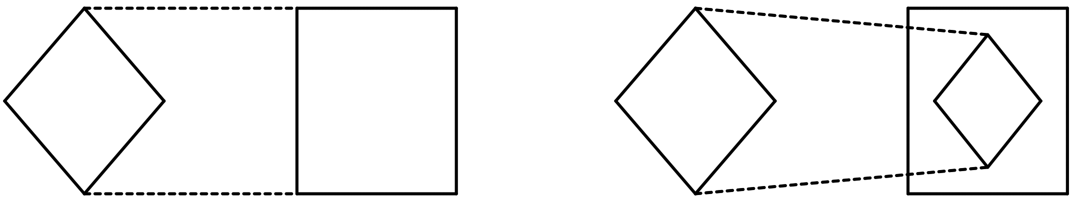


Figure 4: onto and not onto

Example 1.11 The map $f(x) = 2x^2 - 1 : \mathbf{R} \to \mathbf{R}$ is neither one-to-one nor onto. However, if we change the range to $[-1, \infty)$, then the map $f(x) = 2x^2 - 1 : \mathbf{R} \to [-1, \infty)$ is onto (but still not one-to-one).

The square root function $f(x) = \sqrt{x} : [0, \infty) \to \mathbf{R}$ is one-to-one but not onto. Again by changing the range to $[0, \infty)$, the function $f(x) = \sqrt{x} : [0, \infty) \to [0, \infty)$ becomes one-to-one and onto.

Example 1.12 The rotation map R_θ is a one-to-one correspondence.

The Age map is (almost certainly) not one-to-one and (definitely) not onto.

The first alphabet map FA is ono-to-one but not onto. If we increase the number of colours in Y_2, then FA is getting more likely to be onto but still not one-to-one.

Exercise 1.20 Add one colour into Y_2 so that FA is no longer one-to-one. Also add enough colours into Y_2 so that FA becomes a one-to-one correspondence.

Exercise 1.21 Show that $f : X \to Y$ is one-to-one if and only if for any $y \in Y$, the preimage $f^{-1}(y)$ contains at most one point.

Exercise 1.22 Is the "image map" in Exercise 1.19 onto or one-to-one? What about the "preimage map"? Will your conclusion be different if we assume f to be one-to-one or onto?

Exercise 1.23 Given a map $f : X \to Y$, we may modify the range from Y to the image $f(X)$ and think of f as a map $\hat{f} : X \to f(X)$. Show that

1. $\hat{f}$ is onto;
2. If f is one-to-one, then $\hat{f}$ is a one-to-one correspondence.

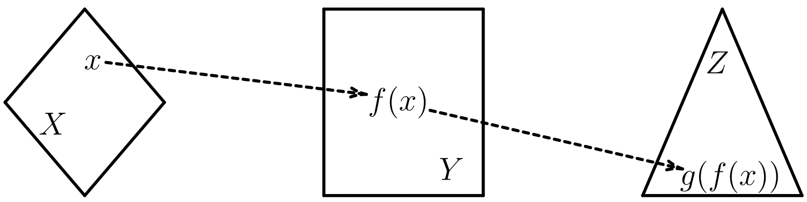


Figure 5: composition $g \circ f$

Given two maps $f : X \to Y$ and $g : Y \to Z$ (note that the range of f is the domain of g), the *composition* $g \circ f$ is defined as follows

$$(g \circ f)(x) = g(f(x)) : X \to Z.$$

A map $f : X \to Y$ is *invertible* if there is another map $g : Y \to X$ in the opposite direction, such that

$$g \circ f = id_X, \qquad f \circ g = id_Y.$$

In other words,

$$g(f(x)) = x, \qquad f(g(y)) = y.$$

We also call g the *inverse* of f.

The most important fact about the invertible map is the following.

Proposition 1.1 *A map is invertible if and only if it is a one-to-one correspondence.*

Proof: We only prove one-to-one correspondence implies invertible. The converse is left as exercises (see Exercises 1.24 and 1.25).

Let $f : X \to Y$ be a one-to-one correspondence. Then we define $g : Y \to X$ to be the following process: For any $y \in Y$, we find $x \in X$, such that $f(x) = y$. Then we take $g(y) = x$.

We need to verify that such a process always goes through. The key is the existence of x. Such existence is guranteed by the fact that f is onto.

We need to verify that the result of such a process is unique. The key is the uniqueness of x. Such uniqueness is guranteed by the fact that f is one-to-one.

Thus we have established that g is a well-defined map. It is then easy to verify that g is indeed the inverse of f.

□

Example 1.13 Let $A \subset X$ be a subset. Then we have the natural *inclusion map* $i(a) = a : A \to X$. For any map $f : X \to Y$, the composition $f \circ i : A \to Y$ is the *restriction* of f on A. The map is often denoted by $f|_A$.

Example 1.14 The map $f(x) = 2x^2 - 1 : \mathbf{R} \to \mathbf{R}$ is not invertible. However, if we change the domain and the range to get $f(x) = 2x^2 - 1 : [0, \infty) \to [-1, \infty)$, then the map is invertible, with $g(y) = \sqrt{\frac{y+1}{2}} : [-1, \infty) \to [0, \infty)$ as the inverse map.

Example 1.15 The composition $R_{\theta_2} \circ R_{\theta_1}$ means a rotation by angle θ_1 followed by another rotation by angle θ_2. Clearly the effect is the same as a rotation by angle $\theta_1 + \theta_2$. Therefore we have $R_{\theta_2} \circ R_{\theta_1} = R_{\theta_1+\theta_2}$.

Exercise 1.24 A *left inverse* of $f : X \to Y$ is a map $g : Y \to X$ such that $g \circ f = id_X$. Prove that if f has a left inverse, then f is one-to-one.

Exercise 1.25 A *right inverse* of $f : X \to Y$ is a map $g : Y \to X$ such that $f \circ g = id_Y$. Prove that if f has a right inverse, then f is onto.

Exercise 1.26 Prove that the preimage of the composition is computed by $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$.

Exercise 1.27 Note that the characteristic function in Exercise 1.14 is actually a map from X to $\{0, 1\}$. Since $\{0, 1\}$ is a two-point set, the set of all maps from X to $\{0, 1\}$ is also denoted by 2^X. Thus we have the "meta-characteristic map"

$$\chi : \mathcal{P}(X) \to 2^X, \quad A \mapsto \chi_A.$$

Prove that the map

$$2^X \to \mathcal{P}(X), \quad f \mapsto f^{-1}(1)$$

is the inverse of χ. In particular, the number of subsets of X is the same as the number of maps from X to $\{0, 1\}$ (see Exercise 1.2).

Exercise 1.28 Is the integration map

$$\mathcal{I} : C[0,1] \to C[0,1], \quad f(t) \mapsto \int_0^t f(x)dx$$

ono-to-one? Is the map onto?

1.4 Equivalence Relation, Partition, and Quotient

In this section, we discuss the following three mutually equivalent concepts: equivalence relation, partition, and quotient.

We first introduce the concepts on a set X. An *equivalence relation* on X is a collection of pairs, denoted $x \sim y$ for $x, y \in X$, such that the following properties are satisfied:

- **symmetry** If $x \sim y$, then $y \sim x$;
- **reflexivity** For any $x \in X$, we have $x \sim x$;
- **transitivity** If $x \sim y$ and $y \sim z$, then $x \sim z$.

A *partition* of X is a decomposition of X into a disjoint union of nonempty subsets.

$$X = \bigcup_{i \in I} X_i, \quad X_i \neq \emptyset, \quad X_i \cap X_j = \emptyset \text{ for } i \neq j. \tag{1}$$

A *quotient* of X is a surjective map $q : X \to I$. Before explaining how the three concepts are equivalent, let us first look at some examples.

Example 1.16 For numbers $x, y \in \mathbf{R}$, define $x \sim y$ if x and y have the same sign. This is an equivalence relation. The corresponding partition is $\mathbf{R} = \mathbf{R}_- \cup \mathbf{R}_0 \cup \mathbf{R}_+$, with $\mathbf{R}_- = (-\infty, 0)$, $\mathbf{R}_0 = \{0\}$, and $\mathbf{R}_+ = (0, \infty)$. The corresponding quotient map is

$$q(x) = \begin{cases} + & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ - & \text{if } x < 0 \end{cases} \quad : \quad \mathbf{R} \to \{+, 0, -\}.$$

On the other hand, define $x \sim y$ if $x \leq y$. The relation is reflexive and transitive, but not symmetric. Therefore this is not an equivalence relation.

Example 1.17 For integers $x, y \in \mathbf{Z}$, define $x \sim y$ if $x - y$ is even. This is an equivalence relation. The corresponding partition is $\mathbf{Z} = \mathbf{Z}_{\text{even}} \cup \mathbf{Z}_{\text{odd}}$, where $\mathbf{Z}_{\text{even}}$ ($\mathbf{Z}_{\text{odd}}$) denotes the subset of all even (odd) numbers. The corresponding quotient map is

$$q(x) = \begin{cases} \text{even} & \text{if } x \text{ is even} \\ \text{odd} & \text{if } x \text{ is odd} \end{cases} \quad : \quad \mathbf{Z} \to \{\text{even}, \text{odd}\}.$$

Similarly, if we define $x \sim y$ when $x - y$ is a multiple of 3, then we also have an equivalence relation. The corresponding partition is $\mathbf{Z} = \mathbf{Z}_0 \cup \mathbf{Z}_1 \cup \mathbf{Z}_2$, where $\mathbf{Z}_i$ is all the integers that have remainder i after devided by 3. The corresponding quotient is a surjective map onto $I = \{0, 1, 2\}$.

On othe other hand, if we define $x \sim y$ when $x - y$ is odd, then the relation is not an equivalence relation.

Example 1.18 On the product $X \times Y$ of two nonempty sets, define $(x_1, y_1) \sim (x_2, y_2)$ if $x_1 = x_2$. This is an equivalence relation. The corresponding partition is

$$X \times Y = \bigcup_{x \in X} x \times Y,$$

and the corresponding quotient is

$$q(x, y) = x : X \times Y \to Y.$$

Example 1.19 Let X be the all the people in the world. Consider the following relations:

1. *descendant relation*: $x \sim y$ if x is a descendant of y;
2. *blood relation*: $x \sim y$ if x and y have the same ancesters;
3. *sibling relation*: $x \sim y$ if x and y have the same parents.

The descendant relation is neither reflexive nor symmetric, although it is transitive. The blood relation is reflexive and symmetric but not transitive. The sibling relation has all three properties and is an equivalence relation.

Exercise 1.29 In Example 1.16, we have a relation on $\mathbf{R}$ that is reflexive and transitive but not symmetric. Which properties fail and which hold for the following relations on $\mathbf{R}$:

1. $x \sim y$ if $|x| < 1$ and $|y| < 1$;
2. $x \sim y$ if $|x - y| < 1$.

Exercise 1.30 Determine which of the following are equivalence relations on $\mathbf{R}^2$:

1. $(x_1, x_2) \sim (y_1, y_2)$ if $x_1^2 + y_1 = x_2^2 + y_2$;
2. $(x_1, x_2) \sim (y_1, y_2)$ if $x_1 \le x_2$ or $y_1 \le y_2$;
3. $(x_1, x_2) \sim (y_1, y_2)$ if (x_1, x_2) is obtained from (y_1, y_2) by some rotation around the origin;
4. $(x_1, x_2) \sim (y_1, y_2)$ if (x_1, x_2) is a scalar multiple of (y_1, y_2).

Exercise 1.31 Determine which of the following are equivalence relations:

1. $X = \mathbf{R}$, $x \sim y$ if $x - y$ is an integer;
2. $X = \mathbf{C}$, $x \sim y$ if $x - y$ is a real number;
3. $X = \mathbf{Z} - \{0\}$, $x \sim y$ if $x = ky$ for some $k \in X$;
4. $X = \mathbf{Q} - \{0\}$, $x \sim y$ if $x = ky$ for some $k \in X$;
5. $X = \mathcal{P}(\{1, 2, \cdots, n\})$, $A \sim B$ if $A \cap B \neq \emptyset$.

Exercise 1.32 Suppose $\sim$ is an equivalence relation on X. Suppose $A \subset X$ is a subset. We may restrict the equivalence relation to A by defining $a \sim_A b$ for $a, b \in A$ if $a \sim b$ by considering a and b as elements of X. Show that $\sim_A$ is an equivalence relation on A.

Exercise 1.33 Suppose a relation $\sim$ on X is reflexive and transitive. Prove that if we force the symmetry by adding $x \sim y$ (new relation) whenever $y \sim x$ (existing relation), then we have an equivalence relation.

Now let us find out how the concepts of equivalence relations and partitions are related. Given an equivalence relation $\sim$ on a set X and an element $x \in X$, the subset

$$[x] = \{y : y \sim x\}$$

of all elements related to x is the *equivalence class* determined by x. We also call x a *representative* of the equivalence class.

Proposition 1.2 *Any two equivalence classes are either identical or disjoint. In fact, we have* $[x] = [y]$ *when* $x \sim y$ *and* $[x] \cap [y] = \emptyset$ *when* $x \not\sim y$. *Moreover, the whole set* X *is the union of all equivalence classes.*

Proof: Suppose $x \sim y$. Then for any $z \in [x]$, we have $z \sim x$ by the definition of $[x]$. Then from $z \sim x$, $x \sim y$, and the transitivity of $\sim$, we have $z \sim y$, i.e., $z \in [y]$. This proves $[x] \subset [y]$. On the other hand, from $x \sim y$ and the symmetry of $\sim$, we have $y \sim x$. Thus it can be similarly proved that $[y] \subset [x]$, and we conclude that $[x] = [y]$.

Now we want to show $[x] \cap [y] = \emptyset$ when $x \not\sim y$. This is the same as proving $[x] \cap [y] \neq \emptyset$ implies $x \sim y$. Thus we assume $z \in [x] \cap [y]$. Then we have $z \sim x$ and $z \sim y$ by the definition of equivalence classes. By $z \sim x$ and symmetry, we have $x \sim z$. By $x \sim z$, $z \sim y$, and transitivity, we have $x \sim y$.

Finally, by the reflexivity, we have $x \in [x]$. Therefore any element of X is in some equivalence class. This implies that X is equal to the union of equivalence classes.

□

Proposition 1.2 tells us that any equivalence relation on X induces a partition into disjoint union of equivalence classes. Conversely, given a partition (1), we may define a relation on X as follows:

$$x \sim y \quad \text{if } x \text{ and } y \text{ are in the same subset } X_i.$$

It is easy to see that this is an equivalence relation, and the equivalence classes are given by

$$[x] = X_i \quad \text{if } x \in X_i.$$

This establishes the one-to-one correspondence between equivalence relations on X and partitions of X.

Example 1.20 Consider the same sign equivalence relation in Example 1.16. We have

$$[x] = \begin{cases} (0, \infty) = \mathbf{R}_+ & \text{if } x > 0 \\ \{0\} = \mathbf{R}_0 & \text{if } x = 0 \\ (-\infty, 0) = \mathbf{R}_- & \text{if } x < 0 \end{cases}.$$

For the even difference equivalence relation in Example 1.17, we have

$$[x] = \begin{cases} 2\mathbf{Z} = \mathbf{Z}_{\text{even}} & \text{if } x \text{ is even} \\ 2\mathbf{Z} + 1 = \mathbf{Z}_{\text{odd}} & \text{if } x \text{ is odd} \end{cases}.$$

For the first coordinate equivalence relation in Example 1.18, we have

$$[(x, y)] = x \times Y.$$

Example 1.21 Let V be a vector space. A straightline L in V passing through the origin $\mathbf{0}$ is a one-dimensional subspace. For any two such straight lines L_1 and L_2, we have either $L_1 = L_2$ or $L_1 \cap L_2 = \mathbf{0}$. Therefore the collection of all $L - \mathbf{0}$ form a partition of $V - \mathbf{0}$.

To see the equivalence relation coreesponding to the partition, we consider vectors $u, v \in V - \mathbf{0}$, they belong to the same one-dimensional subspace if and only if one is a (nonzero) scalar multiple of the other: $u = cv$ for some $c \in \mathbf{R} - 0$ ($c \in \mathbf{C} - 0$ if V is a complex vector space).

The corresponding quotient is called the *projective space*.

Next we find out the relation between the concepts of partitions and quotients. Given by a partition (1), we may construct the set of subsets in the partition, which may be identified as the index set for the subsets ($\cong$ means "identify"):

$$X/\!\sim\ = \{[x] : x \in X\} = \{X_i : i \in I\} \cong I,$$

We call $X/\!\sim$ the *quotient set* of the partition (and of the corresponding equivalence relation). Moreover, we have the canonical map

$$q : X \to I, \quad q(x) = i \quad \text{if } x \in X_i,$$

called the *quotient map*. Since all X_i are non-empty, the quotient map is surjective.

Concersely, given a surjective map $q : X \to I$, we define $X_i = q^{-1}(i)$, the preimage of $i \in I$. Then it is not hard to see that this construction is the converse of the construction above. This establishes the one-to-one correspondence between partitions of X and quotients of X.

We note that $x, y \in X$ are in the same X_i (i.e., $x \sim y$) if and only if $q(x) = q(y)$. Besides Examples 1.16 through 1.18, here are more examples on the relation between the three concepts.

Exercise 1.34 Consider the surjective map $q(x) = |x| : \mathbf{C} \to [0, \infty)$ as a quotient map. Find the equivalence relation and the partition associated with the map.

Exercise 1.35 Consider the equivalence relation on $\mathbf{C}$ defined by $x \sim y$ if $x = ry$ for some real number $r > 0$. Find the partition and the quotient associated with the map.

Exercise 1.36 On any set X, define $x \sim y$ if $x = y$. Show that this is an equivalence relation. What are the partition and the quotient associated to the relation?

Exercise 1.37 For those equivalence relations in Exercise 1.31, find the associated partition and the quotient.

Exercise 1.38 Let $\mathcal{F}$ be the collection of all finite sets. For $A, B \in \mathcal{F}$, define $A \sim B$ if there is a bijective map $f : A \to B$. Prove that this is an equivalence relation. Moreover, identify the quotient set as the set of non-negative integers and the quotient map as the number of elements in a set. The exercise leads to a general theory of counting.

Exercise 1.39 Let V be a vector space. Let $H \subset V$ be a vector subspace. For $x, y \in V$, define $x \sim y$ if $x - y \in H$.

1. Prove that $\sim$ is an equivalence relation on V;
2. Construct additions and scalar multiplications on the quotient set $V/\!\sim$ so that it is also a vector space;
3. Prove that the quotient map $V \to V/\!\sim$ is a linear transformation.

The quotient vector space $V/\!\sim$ is usally denoted by V/H.

The equivalence relation and the quotient are related by the following universal property.

Proposition 1.3 *Let $\sim$ be an equivalence relation on a set X, with the associated quotient $q : X \to X/\!\sim$. Let $f : X \to Y$ be a map such that $x_1 \sim x_2$ implies $f(x_1) = f(x_2)$. Then there is a unique map $\bar{f} : X/\!\sim \to Y$, such that the following diagram is commutative*

$$\begin{array}{ccc} X & \xrightarrow{q} & X/\!\sim \\ {\scriptstyle f}\searrow & & \swarrow{\scriptstyle \bar{f}} \\ & Y & \end{array}$$

i.e., $\bar{f} \circ q = f$.

Proof: The requirement $\bar{f} \circ q = f$ means $\bar{f}([x]) = f(x)$. We may take this as the definition of $\bar{f}$. All we need to show is that $\bar{f}$ is well-defined.

Suppose $[x] = [y]$. Then we have $x \sim y$. By the condition on f, we have $f(x) = f(y)$. Therefore we indeed have $\bar{f}([x]) = \bar{f}([y])$. This verifies that the definition of $\bar{f}$ is not ambiguous. □

Example 1.22 Consider the first coordinate relation on $X \times Y$. If a map $f : X \times Y \to Z$ satisfies the condition of Proposition 1.3, then we have $f(x, y_1) = f(x, y_2)$ for any $x \in X$ and $y_1, y_2 \in Y$. In other words, $f(x, y) = \bar{f}(x)$ is essentially a function of the first variable. Here we recall from Example 1.18 that the quotient set of $X \times Y$ is X, and the quotient map $X \times Y \to X$ is the projection to the first coordinate.

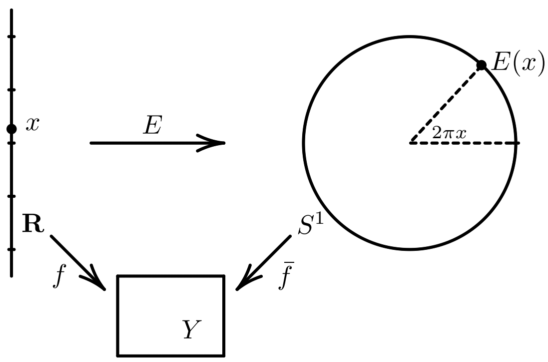


Figure 6: quotient of **R** with interger difference and periodic maps

Example 1.23 On **R**, define $x \sim y$ if $x - y$ is an integer. We verify that $\sim$ is an equivalence relation. First, since $x - x = 0$ is an integer, we have symmetry. Second, suppose $x - y$ is an integer, then $y - x = -(x - y)$ is also an integer. This verifies the reflexivity. Finally, if $x - y$ and $y - z$ are integers, then $x - z = (x - y) + (y - z)$ is also an integer. Therefore $\sim$ is transitive.

Next, we identify the quotient set with the unit circle $S^1 = \{e^{i\theta} : \theta \in \mathbf{R}\}$ in the complex plane, and identify the quotient map with the exponential map $E(x) = e^{i2\pi x} : \mathbf{R} \to S^1$. First, if $x = y + n$ for some integer n, then we clearly have $E(x) = e^{i2\pi y} e^{i2\pi n} = e^{i2\pi y} = E(y)$. Therefore by Proposition 1.3, we have a unique map $\bar{E} : \mathbf{R}/\!\sim \to S^1$, such that $\bar{E}([x]) = e^{i2\pi x}$.

We claim that $\bar{E}$ is invertible. By Proposition 1.1, it suffices to show $\bar{E}$ is one-to-one and onto. Since every point on S^1 is of the form $e^{i\theta} = \bar{E}([\frac{\theta}{2\pi}])$, we see $\bar{E}$ is onto. On the other hand, if $\bar{E}([x]) = \bar{E}([y])$, then from $e^{i2\pi x} = e^{i2\pi y}$ we conclude that $x - y$ is an integer. Therefore we have $[x] = [y]$. This proves that $\bar{E}$ is one-to-one.

Since $\bar{E}$ is invertible, we may use the map to identify the quotient set $\mathbf{R}/\!\sim$ with the circle S^1. Then from $[x] = [y] \Leftrightarrow E(x) = E(y)$, we may conclude that E is identified with the quotient map.

Finally, we note that a map $f : \mathbf{R} \to Y$ satisfies the condition of Proposition 1.3 if and only if $f(x) = f(x+1)$, i.e., f is a periodic map with period 1. By Proposition 1.3 and the discussion above, we conclude that periodic maps with period 1 are in one-to-one correspondence with the maps $\bar{f} : S^1 \to Y$ from the sphere.

Exercise 1.40 Let $\sim_1$ and $\sim_2$ be two equivalence relations on X, such that $x \sim_1 y$ implies $x \sim_2 y$. How are the quotient sets $X/\!\sim_1$ and $X/\!\sim_2$ related?

Exercise 1.41 Let $\sim_X$ and $\sim_Y$ be equivalence relations on X and Y. A map $f : X \to Y$ is said to *preserve the relation* if $x_1 \sim_X x_2$ implies $f(x_1) \sim_Y f(x_2)$. Prove that such a map induces a unique map $\bar{f} : X/\!\sim_X \to Y/\!\sim_Y$ such that the following diagram is commutative

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ q_X \downarrow & & \downarrow q_Y \\ X/\!\sim_X & \xrightarrow{\bar{f}} & Y/\!\sim_Y \end{array}$$

i.e., $q_Y \circ f = \bar{f} \circ q_X$.

2 Metric Space

2.1 Metric

Recall the definition of the continuity of a function $f : \mathbf{R} \to \mathbf{R}$ at a: For any $\epsilon > 0$, there is $\delta > 0$, such that

$$|x - a| < \delta \quad \Longrightarrow \quad |f(x) - f(a)| < \epsilon. \tag{2}$$

Loosely speaking, this means

$$x \text{ is close to } a \quad \Longrightarrow \quad f(x) \text{ is close to } f(a)$$

where the closeness is measured by the distance function $d(u, v) = |u - v|$ between numbers.

Observing that the absolute values in (2) are distances, we can easily extend the notion of continuity by considering appropriate distance functions. For example, by introducing the distance

$$d((x_1, \cdots, x_n), (a_1, \cdots, a_n)) = \sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2} \tag{3}$$

between two points in $\mathbf{R}^n$, we may define the continuity of multivariable functions $f(x_1, \cdots, x_n) : \mathbf{R}^n \to \mathbf{R}$. For another example, we may define the distance

$$d(f(t), g(t)) = \max_{0 \le t \le 1} |f(t) - g(t)|$$

between two continuous functions on $[0, 1]$. Then for the map $\mathcal{I} : f(t) \mapsto \int_0^1 f(t)dt$, we have

$$d(f, g) < \delta \quad \Longrightarrow \quad |\mathcal{I}(f) - \mathcal{I}(g)| = \left| \int_0^1 f(t)dt - \int_0^1 g(t)dt \right| < \epsilon = \delta.$$

Therefore we say $\mathcal{I}$ is a continuous map.

Definition 2.1 A *metric space* is a set X, together with a *metric* (also called *distance*)

$$d : X \times X \to \mathbf{R},$$

satisfying the following properties

- **positivity** $d(x, y) \ge 0$, and $d(x, y) = 0 \Leftrightarrow x = y$;
- **symmetry** $d(x, y) = d(y, x)$;
- **triangle inequality** $d(x, y) + d(y, z) \ge d(x, z)$.

Example 2.1 Over any set X, the function

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \ne y \end{cases}$$

is a metric, called *discrete metric*. Such a metric space X is a *discrete space*.

Example 2.2 Let X be all the subway stations in Hong Kong. For subway stations $x, y \in X$, let $d(x, y)$ be the subway fare from station x to station y. Then d is a metric on X.

However, if X is changed to all the bus stops in Hong Kong and $d(x, y)$ is the lowest bus fare (adding fares for several bus rides if necessary) from a stop x to another stop y. Then d does not satisfy the symmetry condition, although it clearly satisfies the other two conditions. Thus the lowest bus fare in Hong Kong is not a metric.

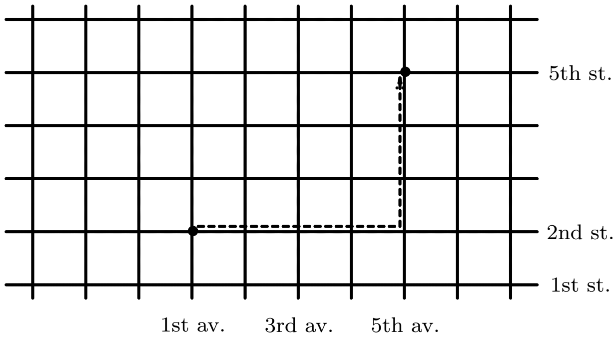
On the other hand, the bus fares are calculated in different ways in different cities. In some other cities (such as the cities where the bus fare is independent of the length of the ride), the lowest bus fare is indeed a metric.

Example 2.3 The following are some commonly used metrics on $\mathbf{R}^n$:

$$\begin{array}{ll} \textit{Euclidean metric}: & d_2((x_1,\cdots,x_n),(y_1,\cdots,y_n)) = \sqrt{(x_1-y_1)^2+\cdots+(x_n-y_n)^2}, \\ \textit{taxicab metric}: & d_1((x_1,\cdots,x_n),(y_1,\cdots,y_n)) = |x_1-y_1|+\cdots+|x_n-y_n|, \\ L_\infty\textit{-metric}: & d_\infty((x_1,\cdots,x_n),(y_1,\cdots,y_n)) = \max\{|x_1-y_1|,\cdots,|x_n-y_n|\}. \end{array} \tag{4}$$

More generally, for $p \geq 1$, we have

$$L_p\textit{-metric}: \quad d_p((x_1,\cdots,x_n),(y_1,\cdots,y_n)) = \sqrt[p]{|x_1-y_1|^p+\cdots+|x_n-y_n|^p}. \tag{5}$$



To go from 1st av. and 2nd st. to 5th av. and 5th st., one needs to drive 7 blocks.

Figure 7: taxicab metric

To show d_p satisfies the triangle inequality, we note that if $p \geq 1$, then $\phi(t) = t^p$ satisfies $\phi''(t) \geq 0$ for $t \geq 0$. This implies that t^p is a *convex function* for non-negative t. In particular, we have

$$(\lambda t + (1-\lambda)s)^p \leq \lambda t^p + (1-\lambda)s^p, \qquad \text{for} \quad 0 \leq \lambda \leq 1, \quad t, s \geq 0. \tag{6}$$

By substituting

$$t = \frac{|x_i - y_i|}{d_p(x,y)}, \quad s = \frac{|y_i - z_i|}{d_p(y,z)}, \quad \lambda = \frac{d_p(x,y)}{d_p(x,y)+d_p(y,z)}$$

into (6), we have

$$\frac{|x_i-z_i|^p}{(d_p(x,y)+d_p(y,z))^p} \leq \frac{(|x_i-y_i|+|y_i-z_i|)^p}{(d_p(x,y)+d_p(y,z))^p} \leq \frac{d_p(x,y)\dfrac{|x_i-y_i|^p}{d_p(x,y)^p}+d_p(y,z)\dfrac{|y_i-z_i|^p}{d_p(y,z)^p}}{d_p(x,y)+d_p(y,z)}.$$

By summing over $i = 1, \cdots, n$, we have

$$\frac{d_p(x,z)^p}{(d_p(x,y)+d_p(y,z))^p} \leq \frac{d_p(x,y)\dfrac{d_p(x,y)^p}{d_p(x,y)^p}+d_p(y,z)\dfrac{d_p(y,z)^p}{d_p(y,z)^p}}{d_p(x,y)+d_p(y,z)} = 1.$$

Then taking the p-th root gives us the triangle inequality.

Example 2.4 Let $C[0,1]$ be the set of all continuous functions on $[0,1]$. The L_∞-*metric* on $C[0,1]$ is

$$d(f,g) = \max_{0\le t\le 1}|f(t)-g(t)|.$$

The L_1-*metric* on $C[0,1]$ is

$$d(f,g) = \int_0^1 |f(t)-g(t)|dt.$$

Example 2.5 For a fixed prime number p, the p-*adic metric* on the set $\mathbf{Q}$ of rational numbers is defined as follows: Let x, y be different rational numbers. Then we have the unique expression

$$x - y = \frac{m}{n}p^a,$$

where m, n, a are integers, and neither m nor n is divisible by p. Define

$$d_{p\text{-adic}}(x,y) = \frac{1}{p\text{-factor of } (x-y)} = p^{-a}.$$

In addition, we define $d_{p\text{-adic}}(x,x) = 0$.

For example, for $x = \frac{7}{6}$ and $y = \frac{3}{4}$, we have $x - y = \frac{5}{12} = 2^{-2}\cdot 3^{-1}\cdot 5$. Therefore

$$\begin{aligned}
\text{2-factor of } (x-y) \text{ is } 2^{-2} &\implies d_{2\text{-adic}}(x,y) = 2^2 = 4,\\
\text{3-factor of } (x-y) \text{ is } 3^{-1} &\implies d_{3\text{-adic}}(x,y) = 3^1 = 3,\\
\text{5-factor of } (x-y) \text{ is } 5^{1} &\implies d_{5\text{-adic}}(x,y) = 5^{-1} = 0.2,\\
\text{7-factor of } (x-y) \text{ is } 7^{0} &\implies d_{7\text{-adic}}(x,y) = 7^{-0} = 1.
\end{aligned}$$

We need to verify that the p-adic metric satisfies the three conditions. The only nontrivial verification is the triangle inequality. In fact, we will prove a stronger inequality

$$d_{p\text{-adic}}(x,z) \le \max\{d_{p\text{-adic}}(x,y), d_{p\text{-adic}}(y,z)\}. \tag{7}$$

The triangle inequality follows from this (prove this!).

The inequality is clearly true when at least two of x, y, z are the same. Thus we assume x, y, z are distinct. Let $d_{p\text{-adic}}(x,y) = p^{-a}$, $d_{p\text{-adic}}(y,z) = p^{-b}$. Then

$$x - y = \frac{m}{n}p^a, \qquad y - z = \frac{k}{l}p^b, \tag{8}$$

and p does not divide any of k, l, m, n. Without loss of generality, we may assume $a \le b$. Then

$$x - z = (x-y) + (y-z) = \frac{m}{n}p^a + \frac{k}{l}p^b = \frac{ml + nkp^{b-a}}{nl}p^a.$$

Since $a \le b$, the numerator of the fraction is an integer, which can contribute p^c only for some non-negative c. On the other hand, the denominator of the fraction cannot contribute any powers of p because neither n nor l contains powers of p. As a result, the p-factor of $x - z$ is $p^c p^a = p^{c+a}$ for some $c \ge 0$. Then we conclude

$$d_{p\text{-adic}}(x,z) = p^{-(c+a)} \le p^{-a} = \max\{p^{-a}, p^{-b}\},$$

where the last equality comes from the assumption $a \le b$.

Exercise 2.1 We would like to construct a metric on the set $X = \{1,2,3\}$ of three points. Suppose we have chosen $d(1,2) = 2$ and $d(1,3) = 3$. What numbers can we choose as $d(2,3)$?

Exercise 2.2 Explain why d_p given by (5) is not a metric on $\mathbf{R}^n$ in case $p < 1$.

Exercise 2.3 Prove that the Euclidean metric and the taxicab metric satisfy the inequalities $d_2(x,y) \leq d_1(x,y) \leq \sqrt{n} d_2(x,y)$. Find similar inequalities between other pairs (such as Euclidean and L_∞) of metrics on $\mathbf{R}^n$.

Exercise 2.4 Prove that for any two points x and y in $\mathbf{R}^n$, we have $d_\infty(x,y) = \lim_{p\to\infty} d_p(x,y)$. The equality is the reason for the terminology L_∞-metric.

Exercise 2.5 For $p \geq 1$, define the L_p-metric on $C[0,1]$. Prove that your definition satisfies the triangle inequality.

Exercise 2.6 Prove that metrics satisfy $|d(x,y) - d(y,z)| \leq d(x,z)$ and $|d(x,y) - d(z,w)| \leq d(x,z) + d(y,w)$.

Exercise 2.7 Prove that if $d(x,y)$ satisfies the positivity and the triangle inequality conditions, then $\bar{d}(x,y) = d(x,y) + d(y,x)$ is a metric.

Exercise 2.8 Prove that if a function $d(x,y)$ satisfies

- $d(x,y) = 0 \Leftrightarrow x = y$,
- $d(y,x) + d(y,z) \geq d(x,z)$,

then $d(x,y)$ is a metric.

Exercise 2.9 Which of the following are metrics on $\mathbf{R}$?

1. $d(x,y) = (x-y)^2$;
2. $d(x,y) = \sqrt{|x-y|}$;
3. $d(x,y) = |x^2 - y^2|$.

Exercise 2.10 Suppose d is a metric on X. Which among $\min\{d,1\}$, $\dfrac{d}{1+d}$, $\sqrt{d}$, d^2 are also metrics?

Exercise 2.11 For what numbers k is it true that d is a metric implies d^k is also a metric.

Exercise 2.12 Suppose d_1 and d_2 are metrics on X. Which among $\max\{d_1,d_2\}$, $\min\{d_1,d_2\}$, d_1+d_2, $|d_1-d_2|$ are also metrics?

Exercise 2.13 Let d_x and d_Y be metrics on X and Y. Use d_X and d_Y to construct at least twometrics on the the product set $X \times Y$.

Exercise 2.14 Let the power set $\mathcal{P}(\mathbf{N})$ be the collection of all subsets of the set $\mathbf{N}$ of natural numbers. For example, the following are considered as three points in $\mathcal{P}(\mathbf{N})$:

$$\begin{aligned} E &= \{2,4,6,8,\cdots\} = \text{all even numbers}, \\ P &= \{2,3,5,7,\cdots\} = \text{all prime numbers}, \\ S &= \{1,4,9,16,\cdots\} = \text{all square numbers}. \end{aligned}$$

Define the distance between $A, B \in \mathcal{P}(\mathbf{N})$ to be

$$d(A,B) = \begin{cases} 0 & \text{if } A = B \\ \dfrac{1}{\min((A-B)\cup(B-A))} & \text{if } A \neq B \end{cases}$$

For example, from $(E-P)\cup(P-E) = \{3,4,5,\cdots\}$, we have $d(E,P) = \dfrac{1}{3}$. Show

1. $d(A,C) \leq \max\{d(A,B), d(B,C)\}$ (*Hint:* $(A-C) \subset (A-B)\cup(B-C)$);
2. d is a metric on X;
3. $d(\mathbf{N}-A, \mathbf{N}-B) = d(A,B)$.

The last property means that the *complement map* $A \mapsto \mathbf{N}-A$ is an *isometry*.

2.2 Ball

Let (X, d) be a metric space. An (open) *ball* of radius $\epsilon > 0$ centered at a is

$$B_d(a, \epsilon) = \{x : d(x, a) < \epsilon\}.$$

The closeness between two points has the following useful set-theoretical interpretation

$$d(x, y) < \epsilon \quad \Longleftrightarrow \quad x \in B_d(y, \epsilon) \quad \Longleftrightarrow \quad y \in B_d(x, \epsilon).$$

If X has discrtete metric, then

$$B(a, \epsilon) = \begin{cases} \{a\} & \text{if } \epsilon \leq 1 \\ X & \text{if } \epsilon > 1 \end{cases}$$

Figure 8 shows the balls with respect to different metrics on $\mathbf{R}^2$. In $C[0, 1]$ with L_∞-metric, $g \in B(f, \epsilon)$ means (see See Figure 9).

$$|f(t) - g(t)| < \epsilon \quad \text{for all } 0 \leq t \leq 1.$$

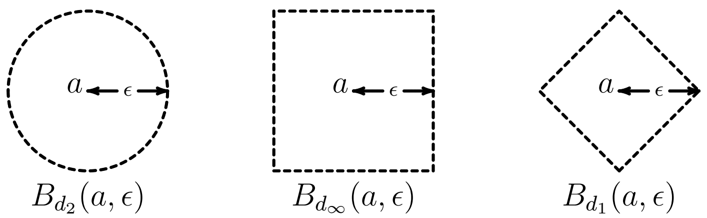


Figure 8: balls in different metrics of $\mathbf{R}^2$

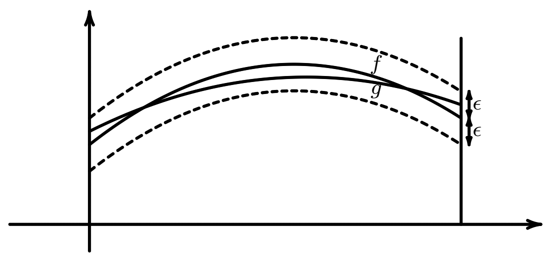


Figure 9: g is in the ϵ-ball around f in L_∞-metric

For the 2-adic metric on rational numbers $\mathbf{Q}$, the ball $B_{\text{2-adic}}(0, 1)$ consists of all rational numbers of the form $\dfrac{m}{n}$, where m is any even integer and n is any odd integer.

Exercise 2.15 Let $\mathbf{R}^2$ have the Euclidean metric. Find the biggest ϵ, such that $B((1, 1), \epsilon)$ is contained in $B((0, 0), 2) \cap B((2, 1), 2)$. Moreover, do the same problem with L_∞-metric in place of the Euclidean metric.

Exercise 2.16 For any $x \in \mathbf{R}^n$ and $\epsilon > 0$, find $\delta > 0$, such that $B_{\text{Euclidean}}(x, \delta) \subset B_{\text{taxicab}}(x, \epsilon)$. Moreover, do the similar problem for other pairs (such as Euclidean and L_∞) of metrics on $\mathbf{R}^n$.

Hint: Use Exercise 2.3.

Exercise 2.17 Which of the following functions in $C[0,1]$ are in $B_{L_1}(1,1)$? Which are in $B_{L_\infty}(1,1)$?

1. $f_1(t) = 0$;
2. $f_2(t) = t$;
3. $f_2(t) = t + \frac{1}{2}$.

Moreover, can you find a function in $B_{L_\infty}(1,1)$ but not in $B_{L_1}(1,1)$.

Exercise 2.18 Describe numbers in $B_{\text{2-adic}}(0,2)$. Describe numbers in $B_{\text{2-adic}}(0,2) - B_{\text{2-adic}}(0,1)$. Moreover, prove $B_{\text{2-adic}}(0,3) = B_{\text{2-adic}}(0,4)$.

Exercise 2.19 A metric satisfying the inequality (7) is called an *ultrametric*. Thus the p-adic metric is an ultrametric. Prove that if d is an ultrametric and $y \in B_d(x,\epsilon)$, then $B_d(x,\epsilon) = B_d(y,\epsilon)$.

Exercise 2.20 What is the ball $B(E, 0.1)$ in the metric space in Exercise 2.14?

Exercise 2.21 In the Euclidean space with Euclidean metric, we have $B(x_1,\epsilon_1) = B(x_2,\epsilon_2) \Rightarrow x_1 = x_2$ and $\epsilon_1 = \epsilon_2$. Is this always true in any metric space?

Exercise 2.22 Prove that metric spaces are *Hausdorff*: If $x \neq y$, then there is $\epsilon > 0$, such that $B(x,\epsilon) \cap B(y,\epsilon) = \emptyset$.

2.3 Open Subset

The system of balls in a metric space gives rise to the concept of open subsets.

Definition 2.2 A subset U of a metric space X is *open* if any point of U is contained in a ball inside U.

In other words, for any $a \in U$, there is an $\epsilon > 0$, such that the ball $B(a,\epsilon) \subset U$.

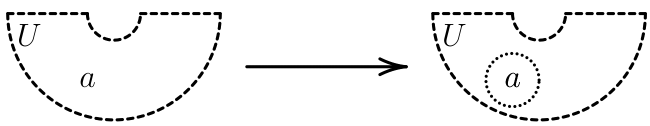


Figure 10: definition of open subset

To get some feeling toward the concept, let us consider the two subsets of the plane $\mathbf{R}^2$ in Figure 11. The subset A does not contain the boundary. Any point inside A has a ball around it and is contained in A. The closer the point is to the boundary, the smaller the ball must be. The important fact is that such a ball, as long as it is small enough, can always be found. Therefore A is open. In contrast, B contains the boundary. At a boundary point, no matter how small the ball is, it always contains points outside of B. So we cannot find a ball around the point and contained in B. Therefore B is not open.

Example 2.6 Let X be a discrete metric space. Then for any subset $A \subset X$ and $a \in A$, we have $B(a,1) = \{a\} \subset A$. Therefore any subset is open in the discrete space.

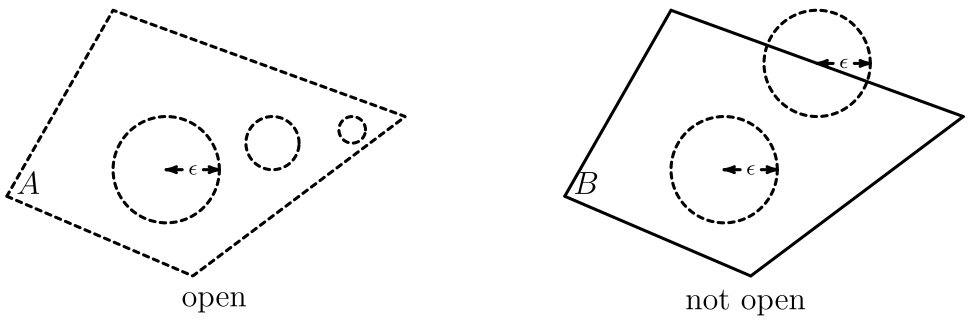


Figure 11: open and not open subsets

Example 2.7 We call (a, b) an *open* interval because it is indeed open in $\mathbf{R}_{\text{usual}}$[1]. In fact, for any $x \in (a, b)$, we have $\epsilon = \min\{x - a, b - x\} > 0$ and $B(x, \epsilon) = (x - \epsilon, x + \epsilon) \subset (a, b)$ (draw a picture to see the meaning of this sentence).

The single point subset $\{a\} \subset \mathbf{R}_{\text{usual}}$ is not open. However, according to the previous example, if we change the metric from the usual one to the discrete one, then $\{a\}$ is indeed open in $\mathbf{R}_{\text{discrete}}$. This example indicates that *openness depends on the choice of the metrics.*

The closed interval $[0, 1]$ is not open in $\mathbf{R}_{\text{usual}}$ because no matter how small ϵ is, the ball $B(0, \epsilon) = (-\epsilon, \epsilon)$ around $0 \in [0, 1]$ is not contained in $[0, 1]$. Intuitively, the situation is similar to the subset B in Figure 11: 0 is a "boundary point" of $[0, 1]$.

Example 2.8 We may think of the "closed interval" $[0, 1]$ as a subset of $X = [0, 1] \cup [2, 3]$ (again equipped with the usual metric). Note that the balls in X are

$$B_X(a, \epsilon) = \{x \in X : |x - a| < \epsilon\} = ([0, 1] \cup [2, 3]) \cap (a - \epsilon, a + \epsilon).$$

In particular, for any $a \in [0, 1]$ (including $a = 0$ and $a = 1$), we have $B_X(a, 1) \subset [0, 1]$. This shows that $[0, 1]$ is open in $[0, 1] \cup [2, 3]$. This example indicates that *openness also depends on the choice of the ambient space.*

Example 2.9 We claim that the subset $U = \{f \in C[0, 1] : f(t) > 0 \text{ for all } 0 \leq t \leq 1\}$ of all positive functions is open in $C[0, 1]_{L_\infty}$. To prove this, for any positive function f, we need to find some $\epsilon > 0$, such that any function in the ball $B_{L_\infty}(f, \epsilon)$ is positive.

Since $[0, 1]$ is a bounded closed interval, f has positive lower bound $\epsilon > 0$. Suppose $h \in B_{L_\infty}(f, \epsilon)$. Then we have $|h(t) - f(t)| < \epsilon$ for all $0 \leq t \leq 1$. This implies $h(t) - f(t) > -\epsilon$, so that

$$h(t) > f(t) - \epsilon \geq 0.$$

Thus we have shown that any function in $B_{L_\infty}(f, \epsilon)$ is positive.

The following result provides lots of examples of open subsets.

Lemma 2.3 *Any ball $B(a, \epsilon)$ is open.*

Proof: We need to show that if $x \in B(a, \epsilon)$, then we can find small enough radius $\delta > 0$, such that $B(x, \delta) \subset B(a, \epsilon)$. From Figure 12, we see that any δ satisfying $0 < \delta < \epsilon - d(x, a)$ should be sufficient for our purpose.

[1]To simplify presentation, we will often use subscripts to indicate the metric (and later on the topology) on the set. Thus $\mathbf{R}_{\text{usual}}$ means $\mathbf{R}$ with the usual metric $d(x, y) = |x - y|$, and $\mathbf{R}_{\text{discrete}}$ means $\mathbf{R}$ with the discrete metric.

Of course we need to rigorously prove our intuition. First of all, the assumption $x \in B(a,\epsilon)$ implies $d(x,a) < \epsilon$. Therefore we can always find δ satisfying $0 < \delta < \epsilon - d(x,a)$. Fix one such δ, we then need to verify $B(x,\delta) \subset B(a,\epsilon)$, which means

$$d(y,x) < \delta \quad \overset{?}{\Longrightarrow} \quad d(y,a) < \epsilon.$$

This can be proved by using the triangle inequality,

$$d(y,a) \le d(y,x) + d(x,a) < \delta + d(x,a) < \epsilon - d(x,a) + d(x,a) = \epsilon.$$

□

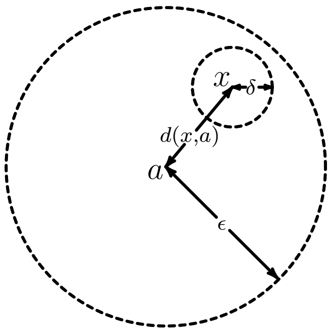


Figure 12: $B(a,\epsilon)$ is open

As an application of the lemma, the interval (a,b) is open because $(a,b) = B\left(\frac{a+b}{2}, \frac{b-a}{2}\right)$.

Exercise 2.23 Prove that in a finite metric space, the single point subsets $\{x\}$ are open. What about the two point subsets?

Exercise 2.24 Which of the following are open subsets of $\mathbf{R}^2_{\text{Euclidean}}$?

1. $U_1 = \{(x,y) : x < 0\}$;
2. $U_2 = \{(x,y) : 2 < x + y < 6\}$;
3. $U_3 = \{(x,y) : x^2 + y^2 < 1 \text{ or } (x,y) = (1,0)\}$;
4. $U_4 = \{(x,y) : x > 2 \text{ and } y \le 3\}$.

Exercise 2.25 Determine whether the interval $(0,1]$ is open as a subset of each of the following spaces equipped with the usual metric $d(x,y) = |x - y|$:

1. $A = \mathbf{R} = (-\infty,\infty)$;
2. $B = (0,\infty)$;
3. $C = (-\infty,1]$;
4. $D = (0,1]$;
5. $E = [0,1]$;
6. $F = \{-1\} \cup (0,1]$.

Exercise 2.26 Which of the following are open subsets of $\mathbf{Q}_{\text{2-adic}}$?

1. $U_1 =$ all integers;
2. $U_2 =$ all even integers;
3. $U_3 =$ all numbers of the form $\dfrac{\text{odd integer}}{\text{even integer}}$;
4. $U_4 =$ all numbers of the form $\dfrac{\text{even integer}}{\text{odd integer}}$;
5. $U_5 =$ all rational numbers with absolute value < 1.

Exercise 2.27 Which of the following are open subsets of the metric space in Exercise 2.14?

1. $U_1 = \{A \subset \mathbf{N} : \text{all numbers in } A \text{ are } \geq 10\}$;
2. $U_2 = \{A \subset \mathbf{N} : \text{all numbers in } A \text{ are } \leq 10\}$;
3. $U_3 = \{A \subset \mathbf{N} : \text{all numbers in } A \text{ are even}\}$;
4. $U_4 = \{A \subset \mathbf{N} : 100 \text{ is not in } A\}$.

Exercise 2.28 Is U in Example 2.9 an open subset of $C[0,1]_{L_1}$?

Exercise 2.29 Let d_X and d_Y be metrics on X and Y. Prove the following:

1. $d_{X\times Y}((x_1,y_1),(x_2,y_2)) = \max\{d_X(x_1,x_2), d_Y(y_1,y_2)\}$ is a metric on $X \times Y$;
2. $U \subset X$ and $V \subset Y$ are open subsets $\Rightarrow U \times V \subset X \times Y$ is an open subset with respect to $d_{X\times Y}$.

Exercise 2.30 Suppose d_1 and d_2 are two metrics on X. Prove the following:

1. $d_1(x,y) \leq c\, d_2(x,y)$ for some constant $c > 0 \Rightarrow$ Open in (X, d_1) implies open in (X, d_2);
2. $c_1\, d_2(x,y) \leq d_1(x,y) \leq c_2\, d_2(x,y)$ for some constants $c_1, c_2 > 0 \Rightarrow d_1$ and d_2 give the same topology (i.e., open in (X, d_1) is equivalent to open in (X, d_2));
3. The different metrics on $\mathbf{R}^n$ in Example 2.3 give the same topology;
4. Open in $C[0,1]_{L_1} \Rightarrow$ Open in $C[0,1]_{L_\infty}$. But the converse is not true;
5. Any two metrics on a finite set give the same topology.

The following theorem contains the most important properties of open subsets. The properties will become the axioms for the concept of topology.

Theorem 2.4 *The open subsets of X satisfy the following properties:*

1. *$\emptyset$ and X are open;*
2. *Any unions of open subsets are open;*
3. *Finite intersections of open subsets are open.*

Note that infinite intersections of open subsets may not be open. A counterexample is given by $\cap(-\frac{1}{n}, \frac{1}{n}) = \{0\}$ in $\mathbf{R}_{\text{usual}}$.

Proof of Theorem 2.4: The first property is trivial.

Before proving the second property, we try to understand what we need to do. Let U_i be open subsets of X (this is our *assumption*), i being some indices. The second property says that the subset $U = \cup U_i$ is also open. By the definition, this means that

$$x \in U \quad \Longrightarrow \quad \text{a ball } B(x, \epsilon) \subset U,$$

(this should be our *conclusion*). The left picture in Figure 13 describes the situation.

Keeping in mind the assumption and the conclusion, we prove the second property.

$$\begin{array}{lll} & x \in U = \cup U_i & \text{(assumption)} \\ \Rightarrow & x \in U_i \text{ some index } i & \text{(definition of union)} \\ \Rightarrow & B(x,\epsilon) \subset U_i \text{ for some } \epsilon > 0 & \text{(definition of open)} \\ \Rightarrow & B(x,\epsilon) \subset U. & (U \text{ contains } U_i) \end{array}$$

For the third property, we only need to consider the union of two open subsets. In other words, we need to show that if U_1 and U_2 are open (*assumption*), then

$$x \in U = U_1 \cap U_2 \quad \Longrightarrow \quad \text{a ball } B(x, \epsilon) \subset U$$

(*conclusion*). The right picture in Figure 13 describes the situation.

$$\begin{array}{lll} & x \in U = U_1 \cap U_2 & \text{(assumption)} \\ \Rightarrow & x \in U_1, x \in U_2 & \text{(definition of intersection)} \\ \Rightarrow & B(x,\epsilon_1) \subset U_1, B(x,\epsilon_2) \subset U_2 & \text{(definition of open)} \\ \Rightarrow & B(x,\epsilon) \subset B(x,\epsilon_1) \cap B(x,\epsilon_2) \subset U_1 \cap U_2 = U. & \end{array}$$

The second $\Rightarrow$ is true for some choice of $\epsilon_1, \epsilon_2 > 0$. In the third $\Rightarrow$, we take any ϵ satisfying $0 < \epsilon \leq \min\{\epsilon_1, \epsilon_2\}$.

□

Figure 13: union and intersection of open subsets

As an application of the theorem, we know that in $\mathbf{R}_{\text{usual}}$, any (possibly infinite) union of open intervals is open. It turns out that the converse is also true.

Theorem 2.5 A subset of $\mathbf{R}_{\text{usual}}$ is open $\Leftrightarrow$ It is a disjoint union of open intervals.

Proof: Let $U \subset \mathbf{R}$ be an open subset. We will show that U is a disjoint union of open intervals.

The first step is to construct the open intervals. For any $x \in U$, we consider the collection of open intervals (a, b) between x and U (i.e., satisfying $x \in (a, b) \subset U$). Since U is open, the collection is not empty. Thus we may define

$$a_x = \inf_{x \in (a,b) \subset U} a, \quad b_x = \sup_{x \in (a,b) \subset U} b.$$

Intuitively, (a_x, b_x) is the largest open interval between x and U, and should be one of the intervals described in the theorem. We expect U to be the union of these intervals (for various $x \in U$).

Next we prove $U = \cup_{x \in U}(a_x, b_x)$. By the construction of (a_x, b_x), we clearly have $x \in (a_x, b_x) \subset \cup_{x \in U}(a_x, b_x)$ for any $x \in U$. This implies $U \subset \cup_{x \in U}(a_x, b_x)$. Conversely, if $y \in (a_x, b_x)$, then either $a_x < y \leq x$ or $x \leq y < b_x$. In case $a_x < y \leq x$, by the definition of a_x, we have an interval (a, b), such that $a < y \leq x$ and $x \in (a, b) \subset U$. This implies $y \in (a, b) \subset U$. By the similar argument, we may prove that $x \leq y < b_x$ implies $y \in U$. Combining the two cases, we find that $(a_x, b_x) \subset U$. Therefore we also have $\cup_{x \in U}(a_x, b_x) \subset U$.

Having proved $U = \cup_{x \in U}(a_x, b_x)$, we turn to the disjointness. It is indeed possible for (a_x, b_x) and (a_y, b_y) to intersect for distinct x and y in U. However, we claim that if the intersection of (a_x, b_x) and (a_y, b_y) is not empty, then the two intervals must be identical.

Suppose $z \in (a_x, b_x) \cap (a_y, b_y)$. Then $z \in U$. From $z \in (a_x, b_x) \subset U$ and the fact that (a_z, b_z) is the largest open interval between z and U, we have $(a_x, b_x) \subset (a_z, b_z) \subset U$. Then since (a_x, b_x) is the largest open interval between x and U, we conclude that $(a_x, b_x) = (a_z, b_z)$. Similarly, $(a_y, b_y) = (a_z, b_z)$. This proves that (a_x, b_x) and (a_y, b_y) are idential.

Thus we see that in the union $U = \cup_{x \in U}(a_x, b_x)$, the open intervals on the right side are either disjoint or identical. By eliminating the duplications on the right side, we may express U as a disjoint union of open intervals.

□

Another application of Theorem 2.4 is the following useful technical result.

Lemma 2.6 *A subset of a metric space is open if and only if it is a union of (open) balls.*

Proof: It follows from Lemma 2.3 and the second part of Theorem 2.4 that a union of balls is open. Conversely, suppose U is open, then for any $x \in U$, we can find $\epsilon_x > 0$, such that $B(x, \epsilon_x) \subset U$ (the subscript x in ϵ_x is used to indicate the possible dependency of ϵ on x). After constructing one such ball for each $x \in U$, we have $\cup_{x \in U} B(x, \epsilon_x) \subset U$. On the other hand, $U = \cup_{x \in U}\{x\} \subset \cup_{x \in U} B(x, \epsilon_x)$. Therefore we have

$$U = \cup_{x \in U} B(x, \epsilon_x),$$

which expresses U as a union of balls.

□

Exercise 2.31 What are the open subsets of the space $X = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}$ with the usual metric?

Exercise 2.32 Fix two continuous functions $a(t)$ and $b(t)$. Prove that the subset

$$U = \{f \in C[0,1] : a(t) > f(t) > b(t) \text{ for all } 0 \le t \le 1\}$$

is an open subset in the L_∞-metric. What about L_1-metric?

Exercise 2.33 Is $U = \{f \in C[0,1] : f(0) > 1\}$ an open subset of $C[0,1]_{L_1}$? Open subset of $C[0,1]_{L_\infty}$? What about $V = \{f \in C[0,1] : f(0) > 1, f(1) < 0\}$?

Exercise 2.34 Prove that in a finite metric space, any subsets are open.
Hint: From Exercise 2.23, we already know single point subsets are open.

Exercise 2.35 Prove that in any metric space X, the complement $X - \{x\}$ of any point x is an open subset of X. What about the complement of two points?

Exercise 2.36 For a subset A of a metric space X and $\epsilon > 0$, define the *ϵ-neighborhood* of A to be

$$A^\epsilon = \{x \in X : \quad d(x,a) < \epsilon \text{ for some } a \in A\}$$

Prove that A^ϵ is an open subset of X. (Note that if $A = \{a\}$ is a single point, the $A^\epsilon = a^\epsilon = B(a,\epsilon)$ is the ball of radius ϵ around a.)

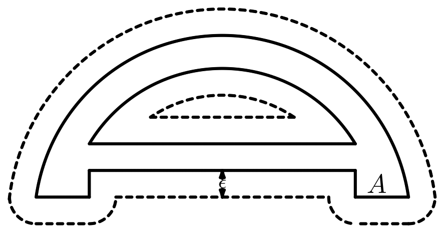


Figure 14: ϵ-neighborhood of A

2.4 Continuity

As promised before, the notion of continuity can be defined for maps between metric spaces.

Definition 2.7 A map $f : (X, d_X) \to (Y, d_Y)$ between metric spaces is *continuous* if for any $a \in X$ and $\epsilon > 0$, there is $\delta > 0$, such that

$$d_X(x,a) < \delta \quad \Longrightarrow \quad d_Y(f(x), f(a)) < \epsilon. \tag{9}$$

Example 2.10 Let X be a discrete metric space and Y be any metric space. Then any map $f : X \to Y$ is continuous. To see this, for any given a and ϵ, we simply take $\delta = 1$. Because X is discrete, $d(x,a) < 1$ implies $x = a$ and $d(f(x), f(a)) = 0 < \epsilon$.

Example 2.11 The evaluation map $E(f) = f(0) : C[0,1]_{L_\infty} \to \mathbf{R}_{\text{usual}}$ is continuous. In fact, we have

$$d(E(f), E(g)) = |f(0) - g(0)| \le \max_{0 \le t \le 0} |f(t) - g(t)| = d_{L_\infty}(f,g).$$

This implies that (9) holds for E if we take $\delta = \epsilon$.

If we change L_∞-metric to L_1-metric, then E is no longer continuous. In order to see this, we take a in (9) to be the constant zero function 0 and $\epsilon = 1$. For any $\delta > 0$, we will find a function f, such that the left of (9) holds:

$$d_{L_\infty}(f, 0) = \int_0^1 |f(t)|dt < \delta,$$

but the right side fails:

$$d(E(f), E(0)) = |f(0)| \geq 1.$$

The function can be easily constructed by considering its graph (draw the graph by yourself): We have a straight line connecting $f(0) = 1$ to $f(\delta) = 0$ and then $f = 0$ on $[\delta, 1]$. Specifically, f is given by the following formula:

$$f(t) = \begin{cases} 1 - \delta^{-1}t & \text{for } 0 \leq t \leq \delta \\ 0 & \text{for } \delta \leq t \leq 1 \end{cases}.$$

Example 2.12 Let $f : X \to Y$ and $g : X \to W$ be continuous maps. Then we consider $h(x) = (f(x), g(x)) : X \to Y \times W$, where $Y \times W$ has the metric given in the first part of Exercise 2.29. We prove that h is also continuous.

For any given $a \in X$ and $\epsilon > 0$, we can find $\delta_1 > 0$ and $\delta_2 > 0$, such that $d_X(x, a) < \delta_1$ implies $d_Y(f(x), f(a)) < \epsilon$ and $d_X(x, a) < \delta_2$ implies $d_W(g(x), g(a)) < \epsilon$. Then $d_X(x, a) < \min\{\delta_1, \delta_2\}$ implies $d_{Y\times W}(h(x), h(a)) = \max\{d_Y(f(x), f(a)), d_W(g(x), g(a))\} < \epsilon$. Thus by taking $\delta = \min\{\delta_1, \delta_2\}$, we established (9) for h.

Example 2.13 We know from calculus that the summation, as a map from $\mathbf{R}^2_{\text{usual}}$ to $\mathbf{R}_{\text{usual}}$, is continuous. A generalization of this fact is left as Exercise 2.38.

Now let us consider the summation $\mathbf{Q}^2_{p\text{-adic}} \to \mathbf{Q}_{p\text{-adic}}, (x, y) \mapsto x + y$, where the p-adic metric on $\mathbf{Q}^2$ is given in the first part of Exercise 2.29. We claim that

$$d_{p\text{-adic}}(x_1 + x_2, y_1 + y_2) \leq d_{p\text{-adic}}((x_1, x_2), (y_1, y_2)) = \max\{d_{p\text{-adic}}(x_1, y_1), d_{p\text{-adic}}(x_2, y_2)\}. \tag{10}$$

The inequality is similar to (7) and the proof is also similar. In fact, we let

$$x_1 - x_2 = \frac{m}{n}p^a,\ d_{p\text{-adic}}(x_1, x_2) = p^{-a}; \quad y_1 - y_2 = \frac{k}{l}p^b,\ d_{p\text{-adic}}(y_1, y_2) = p^{-b},$$

which is similar to (8). Then by the similar argument as in Example 2.5, we have $c \geq 0$, such that

$$d_{p\text{-adic}}(x_1 + x_2, y_1 + y_2) = p^{-(c+\min\{a,b\})} \leq \max\{p^{-a}, p^{-b}\}.$$

The ineqiality (10) implies that (9) is true for our summation map if we take $\delta = \epsilon$.

Exercise 2.37 Suppose we change $<$ in (9) to $\leq$. Will the concept of continuity change?

Exercise 2.38 Use ϵ-δ definition of the continuity to prove that if maps $f, g : X \to \mathbf{R}_{\text{usual}}$ are continuous, then the sum $f + g$ and the product $f \cdot g$ are also continuous.

Exercise 2.39 Determine the continuity of the integration map

$$\mathcal{I} : C[0, 1] \to \mathbf{R}_{\text{usual}}, \quad f(t) \mapsto \int_0^1 f(t)dt$$

for the L_1- as well as the L_∞-metrics on $C[0, 1]$.

Exercise 2.40 Prove that the summation map $C[0, 1]^2_{L_\infty} \to C[0, 1]_{L_\infty}, (f, g) \mapsto f + g$ is continuous. What about the multiplication map? What about L_1-metric in place of L_∞-metric.

Exercise 2.41 Fix a point a in a metric space X. Prove that $f(x) = d(x, a) : X \to \mathbf{R}_{\text{usual}}$ is continuous.

Exercise 2.42 Let (X, d) be a metric space and let $X \times X$ have the metric given in the first part of Exercise 2.29. Prove that the map $d : X \times X \to \mathbf{R}_{\text{usual}}$ is continuous.

Exercise 2.43 Let $f : X \to Y$ and $g : Z \to W$ be continuous maps. Prove that the map $h(x, z) = (f(x), g(z)) : X \times Z \to Y \times W$, where the metric on $X \times Z$ and $Y \times W$ are given in the first part of Exercise 2.29, is continuous.

Exercise 2.44 Let $f : X \to Y$ be a map between metric spaces. Prove that if there is a constant $c > 0$ such that $d_Y(f(x), f(y)) \le c\, d_X(x, y)$, then f is continuous.

Exercise 2.45 Prove that the identity map $id : \mathbf{R}_{\text{discrete}} \to \mathbf{R}_{\text{usual}}$ is continuous, while the other identity map $id : \mathbf{R}_{\text{usual}} \to \mathbf{R}_{\text{discrete}}$ is not continuous.

Exercise 2.46 Prove that the identity map $id : C[0,1]_{L_\infty} \to C[0,1]_{L_1}$ is continuous.

Exercise 2.47 Let $f : (X, d) \to Y$ be a continuous map. Suppose d is another metric on X such that $d'(x_1, x_2) \le c\, d(x_1, x_2)$ for a constant $c > 0$. Is $f : (X, d') \to Y$ still continuous? What if d and d' satisfy the inequality $d(x_1, x_2) \le c\, d'(x_1, x_2)$ instead? What if we modify the metric on Y in similar way?

Note: This is a continuation of Exercise 2.30.

Exercise 2.48 Prove that the multiplication map is continuous under the p-adic metric.

Exercise 2.49 Consider the metric space in Exercise 2.14. We fix a subset E of $\mathbf{N}$.

1. Prove that $d(A \cup E, B \cup E) \le d(A, B)$ and then use this to prove that the map $A \mapsto A \cup E$ is continuous;
2. Is the map $A \mapsto A \cap E$ continuous? What about the map $A \mapsto A - E$?

The implication (9) may be rephrased in terms of balls:

$$f(B(a, \delta)) \subset B(f(a), \epsilon). \tag{11}$$

Equivalently, this means

$$B(a, \delta) \subset f^{-1}(B(f(a), \epsilon)). \tag{12}$$

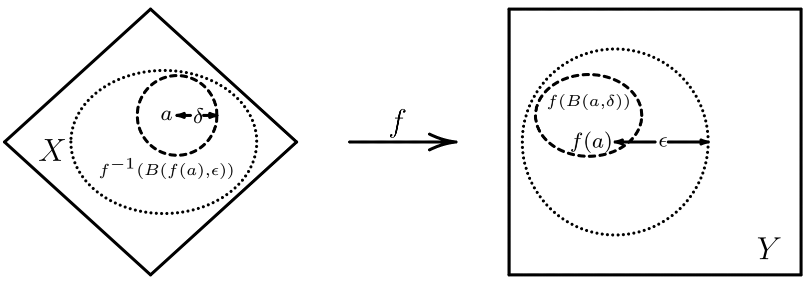


Figure 15: continuity in terms of balls

Theorem 2.8 *The following are equivalent for a map $f : X \to Y$ between metric spaces:*

1. *The map f is continuous;*

2. *The preimage $f^{-1}(U)$ of any open subset $U \subset Y$ is in X;*

3. *The preimage $f^{-1}(B(y,\epsilon))$ of any ball in Y is open in X.*

Proof: Suppose $f : X \to Y$ is continuous. Let $U \subset Y$ be an open subset. The following shows that $f^{-1}(U)$ is open,

$$\begin{array}{lll} & a \in f^{-1}(U) & \text{(assumption)} \\ \Rightarrow & f(a) \in U & \text{(definition)} \\ \Rightarrow & B(f(a),\epsilon) \subset U \quad \text{for some } \epsilon > 0 & (U \text{ is open}) \\ \Rightarrow & B(a,\delta) \subset f^{-1}(B(f(a),\epsilon)) \subset f^{-1}(U) \quad \text{for some } \delta > 0. & \text{(continuity (12))} \end{array}$$

This completes the proof that the first property implies the second.

As a consequence of Lemma 2.3, the second property implies the third.

Finally, assume f has the third property. We need to show that f is continuous. For any $a \in X$ and $\epsilon > 0$, we take $y = f(a)$ in the third property and find that $f^{-1}(B(f(a),\epsilon))$ is open. Since $a \in f^{-1}(B(f(a),\epsilon))$, we have some $\delta > 0$ such that (12) holds. This proves the continuity of f.

□

Corollary 2.9 *The compositions of continuous maps are continuous.*

Proof: Suppose $f : X \to Y$ and $g : Y \to Z$ are continuous. Let $U \subset Z$ be open. We need to show $(gf)^{-1}(U) = f^{-1}(g^{-1}(U))$ is open. The continuity of g implies that $g^{-1}(U)$ is open. The continuity of f then further implies that $f^{-1}(g^{-1}(U))$ is open.

□

Example 2.14 Consider the summation map $\sigma(x,y) = x + y : \mathbf{R}^2_{\text{Euclidean}} \to \mathbf{R}_{\text{usual}}$. To check the continuity of σ, we only need to consider the preimage of balls in $\mathbf{R}$, which are simply open intervals. Note that $\sigma^{-1}(a,b)$ is an open strip in $\mathbf{R}^2$ at 135 degrees. It is intuitively clear (and not difficult to show) that $\sigma^{-1}(a,b)$ is indeed open. Therefore σ is continuous.

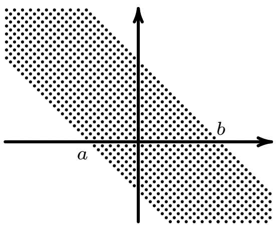


Figure 16: $\sigma^{-1}(a,b)$

Note that by the second property in Theorem 2.8, the continuity property depends only on open subsets. Since according to the third part of Exercise 2.30, the open subsets with regard to the Euclidean metric on $\mathbf{R}^2$ is the same as the open subsets with regard to the taxicab metric (and many other metrics, including all the L_p-metrics), σ is also continuous if we equip $\mathbf{R}^2$ with the taxicab metric.

Example 2.15 The subset of triples (x,y,z) satisfying the equation $1 < x^2 + y^2 + z^2 + xy + yz + zx < 2$ is an open subset of $\mathbf{R}^3_{\text{Euclidean}}$. To see this, we consider the continuous map $f(x,y,z) = x^2 + y^2 + z^2 + xy + yz + zx : \mathbf{R}^3 \to \mathbf{R}_{\text{usual}}$. Then the subset we are concerned with is $f^{-1}(1,2)$, which by Theorem 2.8 is open.

Example 2.16 The set $M(n,n)$ of all $n \times n$ matrices may be identified with $\mathbf{R}^{n^2}$ in an obvious way. With the Euclidean metrics, the map $\det : M(n,n) \to \mathbf{R}$ is continuous, because det is a big multivariable polynomial. Therefore $\det^{-1}(\mathbf{R} - \{0\})$ is an open subset of $M(n,n)$. This open subset is simply all the invertible matrices.

Example 2.17 In Example 2.11, we have seen that the evaluation map $E(f) = f(0) : C[0,1]_{L_\infty} \to \mathbf{R}_{\text{usual}}$ is continuous. As a consequence of this, the subset $U = \{f \in C[0,1] : f(0) > 1\} = E^{-1}(1,\infty)$ is L_∞-open (see Exercise 2.33). More generally, by the same argument as in Example 2.11, the evaluation $E_t(f) = f(0) : C[0,1]_{L_\infty} \to \mathbf{R}_{\text{usual}}$ at (fixed) t is also continuous. In particular, the subset (for fixed a, t, ϵ)

$$B(a,t,\epsilon) = \{f \in C[0,1] : \quad |f(t) - a| < \epsilon\} = E_t^{-1}(a-\epsilon, a+\epsilon)$$

is L_∞-open.

In Exercise 2.33, we also see that U is not L_1-open. This implies that E is not continuous as a map from the L_1-metric to the usual metric.

Exercise 2.50 As in Example 2.14, draw the pictures of the preimages of an open interval under the substraction map, the multiplication map, and the division map. Then explain why these maps are continuous.

Exercise 2.51 Let X be a finite metric space. Prove that any map $f : X \to Y$ is continuous.

Exercise 2.52 Using the conclusions of Examples 2.12 and 2.14, Exercises 2.43 and 2.50, and Corollary 2.9, explain why the map f in Example 2.15 is continuous.

Exercise 2.53 Let A be a subset of a metric space. Consider the distance function from A:

$$d(x,A) = \inf\{d(x,a) : a \in A\} : \quad X \to \mathbf{R}.$$

1. Prove $|d(x,A) - d(y,A)| \le d(x,y)$ and the continuity of $d(?,A)$;
2. Use the continuity of $d(?,A)$ to give another proof of Exercise 2.36.

Exercise 2.54 In Example 2.10, we see that any map *from* a discrete metric space is continuous. How about the maps *to* a discrete metric space?

The answer gives us examples of discontinuous *maps such that the* images *of open subsets are open.*

2.5 Limit Point

Definition 2.10 $x \in X$ is a *limit point* of $A \subset X$ if for any $\epsilon > 0$, there is a point a, such that

1. $a \in A$,
2. $a \neq x$,
3. $d(a,x) < \epsilon$.

The collection of limit points of A is denoted A'.

In other words, there are points of A that are arbitrarily close but not equal to x. Equivalently, $(A - x) \cap B(x,\epsilon) \neq \emptyset$ for any $\epsilon > 0$. For the relation between this definition and the one in terms of convergent sequences that you probably learned in calculus, see Exercise 2.56.

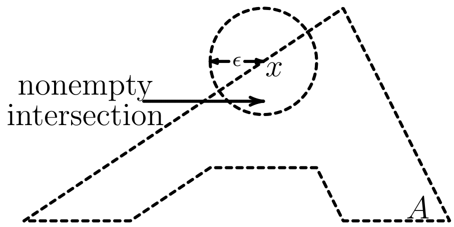


Figure 17: x is a limit point of A

Example 2.18 Consider the following subsets of $\mathbf{R}_{\text{usual}}$: $A_1 = [0,1] \cup \{2\}$, $A_2 = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots\right\}$, $A_3 = \mathbf{Q}$, $A_4 = \mathbf{Z}$. The set of limit points are respectively $A_1' = [0,1]$, $A_2' = \{0\}$, $A_3' = \mathbf{R}$, $A_4' = \emptyset$.

2 is not a limit point of A_1 because $a \in A_1$ and $d(a,2) < 0.5 \Rightarrow a = 2$. Therefore the three conditions cannot be satisfied at the same time. Intuitively, the reason for 2 not to be a limit point is because it is an *isolated point of* A, i.e., there are no other points of A nearby it. The same intuition applies to A_4, in which all the points are isolated, so that no points are limit points.

Example 2.19 In $\mathbf{R}^n_{\text{Euclidean}}$, the closed ball $\bar{B}(a,\epsilon) = \{x \in \mathbf{R}^n : d(x,a) \le \epsilon\}$ is the set of limit points of the ball $B(a,\epsilon)$.

Example 2.20 A foundamental theorem in approximation theory is that any continuous function on $[0,1]$ may be uniformly approximated by polynomials. Specifically, this says that for any continuous function $f(t)$ and $\epsilon > 0$, there is a polynomial $p(t)$, such that $|f(t) - p(t)| < \epsilon$ for all $0 \le t \le 1$. It follows from this that the limit points of the subset $\mathbf{R}[t] \subset C[0,1]_{L_\infty}$ of polynomials is the whole space $C[0,1]$.

Example 2.21 Consider the subset $A = \{f : f(1) = 1\}$ of $C[0,1]$. We would like to find out whether the constant zero function 0 is a limit point of A.

Suppose we equip $C[0,1]$ with the L_∞-metric. Then the three conditions in the definition means the following for a function f (taken as a in the definition): $f(1) = 1$, $f \ne 0$, $|f(t)| < \epsilon$ for all $0 \le t \le 1$. The conditions are contradictory for $\epsilon = 1$. Therefore 0 is not a limit point of A in the L_∞-metric.

Now we take L_1-metric instead. The conditions become $f(1) = 1$, $f \ne 0$, $\int_0^1 |f(t)|dt < \epsilon$. No matter how small ϵ is, we can always find a big N ($N > \epsilon^{-1}$ is good enough), such that $f(t) = t^N$ satisfies the three conditions. Therefore 0 is a limit point of A in the L_1-metric.

Example 2.22 By taking $\epsilon = 1$, we see that any subset of a discrete metric space has no limit points. In particular, the set of limit points of $B(a,1) = \{a\}$ is $\emptyset$, which is different from the "closed ball" $\bar{B}(a,1)$ =whole space. This is in contrast with the balls in a Euclidean space (see Example 2.19).

Exercise 2.55 Can the set A' of limit points of A be bigger than, smaller than, or equal to A. Provide examples to illustrate all the possibilities.

Exercise 2.56 Prove that x is a limit point of A if and only if there is a sequence a_n of points in A, such that $a_i \ne a_j$ for $i \ne j$ and $d(a_n, x) \to 0$ as $n \to \infty$. In other words, there is a *non-repetitive* sequence in A converging to x.

Exercise 2.57 Prove that a finite subset of a metric spaces has no limit points.

Exercise 2.58 Suppose x is a limit point of a subset A, and U is an open subset containing x. Prove that x is a limit point of $A \cap U$, and $A \cap U$ contains infinitely many points.

Exercise 2.59 Suppose $f : X \to Y$ is continuous. If x is a limit point of A, is it necessarily true that $f(x)$ is a limit point of $f(A)$?

Exercise 2.60 Suppose x is a limit point of a subset A of a metric space X. Is x still a limit point of A if we modify the metric as in Exercise 2.47 (also see Exercise 2.30)?

Exercise 2.61 Prove true statements and provide counterexamples to wrong ones:

1. $A \subset B \Rightarrow A' \subset B'$;
2. $(A \cup B)' = A' \cup B'$;
3. $(A \cap B)' = A' \cap B'$;
4. $(X - A)' = X - A'$;
5. $A'' \subset A'$.

Exercise 2.62 Let A be a subset of a metric space. Let $d(x, A)$ be the distance function defined in Exercise 2.53. Prove that $d(x, A) = 0$ if and only if $x \in A$ or $x \in A'$.

2.6 Closed Subset

The notion of closed subsets is motivated by the intuition that nothing "escapes" from them.

Definition 2.11 $C \subset X$ is *closed* if all limit points of C are inside C.

Our examples on limit points gives us many examples (and counterexamples) of closed subsets.

In Example 2.18, only A'_1 and A'_4 are contained respectively in A_1 and A_4. Therefore A_1 and A_4 are closed, while A_2 and A_3 are not closed. From Example 2.19, we see that points of distance ϵ from a are limit points of the Euclidean ball $B(a, \epsilon)$. Since such points are not in the Euclidean ball, $B(a, \epsilon)$ is not closed. From Example 2.20, the subset $\mathbf{R}[t]$ of polynomials is not closed in $C[0, 1]_{L_\infty}$. In fact, it is also not closed in $C[0, 1]_{L_1}$. By 2.22, any subset of a discrete metric space is closed. Finally, by Exercise 2.57, any finite subset of a metric space has no limit points and is therefore closed.

Exercise 2.63 The following are the four possibilities for a subset A of a metric space X.

1. A is open and closed;
2. A is open but not closed;
3. A is closed but not open;
4. A is neither open nor closed.

For each possibility, construct an example of X and A, with $A \neq \emptyset, X$. Moreover, prove that the only open and closed subsets of $X = \mathbf{R}_{\text{usual}}$ are $\emptyset$ and $\mathbf{R}$.

Exercise 2.64 Continuing Exercise 2.60, suppose A is a closed subset of a metric space X. Is A still closed if we modify the metric as in Exercise 2.47?

Exercise 2.65 Prove that in an ultrametric space (see Exercise 2.19), any ball is also closed.

Now we discuss major properties of closed subsets.

Theorem 2.12 *$C \subset X$ is closed if and only if $X - C$ is open.*

Proof:

$$\begin{aligned} & X - C \text{ is open} \\ \Leftrightarrow\ & x \in X - C \text{ implies } B(x,\epsilon) \subset X - C \text{ for some } \epsilon > 0 \\ \Leftrightarrow\ & x \notin C \text{ implies } C \cap B(x,\epsilon) = \emptyset \text{ for some } \epsilon > 0 \\ \Leftrightarrow\ & x \notin C \text{ implies } (C - x) \cap B(x,\epsilon) = \emptyset \text{ for some } \epsilon > 0 \\ \Leftrightarrow\ & x \notin C \text{ implies } x \text{ is not a limit point of } C \\ \Leftrightarrow\ & x \text{ is a limit point of } C \text{ implies } x \text{ is in } C \\ \Leftrightarrow\ & C \text{ is closed.} \end{aligned}$$

The third equivalence comes from $C = C - x$ for $x \notin C$.

□

As an application, we know from Example 2.6 that any subset of a discrtete space is open. By the theorem, any subset of a discrete space is also closed.

By making use of $f^{-1}(Y - C) = X - f^{-1}(C)$, Theorem 2.12 implies the following.

Corollary 2.13 *A map between metric spaces is continuous if and only if the preimage of any closed subset is closed.*

By making use of deMorgan's law $X - \cup A_i = \cap(X - A_i)$ and $X - \cap A_i = \cup(X - A_i)$, Lemma 2.4 and Theorem 2.12 imply the following.

Corollary 2.14 *The closed subsets of a metric space X satisfy the following properties:*

1. *$\emptyset$ and X are closed;*
2. *Any intersections of closed subsets are closed;*
3. *Finite unions of closed subsets are closed.*

Example 2.23 The solutions of the equation $x^2 - y^2 = 1$ is a pair of hyperbola in $\mathbf{R}^2$. From the picture it is easy to see that the solution set H is a closed subset of $\mathbf{R}^2$. Here is a rigorous way of proving the closedness of H: Consider the map $f(x, y) = x^2 - y^2 : \mathbf{R}^2 \to \mathbf{R}$, both sides with the usual metric. Since the map is continuous and $\{1\} \subset \mathbf{R}$ is a closed subset, we see that $f^{-1}(1)$ is a closed subset. But $f^{-1}(1)$ is nothing but H.

The argument generalizes to higher dimensional cases, when it is hard to argue directly from the picture that some solution set is closed. For example, the set of 2×2 singular matrices is a closed subset of $\mathbf{R}^4$ because $d(x, y, z, w) = \det \begin{pmatrix} x & y \\ z & w \end{pmatrix} = xw - yz : \mathbf{R}^4 \to \mathbf{R}$ is continuous and $\{0\} \subset \mathbf{R}$ is closed.

Exercise 2.66 Find suitable maps and the prove the following subsets of the Euclidean space are closed.

1. Solution set of $-1 \le x^3 - y^3 + 2z^3 - xy - 2yz + 3zx \le 3$;
2. Solution set of the system of equations: $x^4 + y^4 = z^4 + w^4$, $x^3 + z^3 = y^3 + z^3$;

3. The set of $n \times n$ orthogonal matrices (as a subset of $\mathbf{R}^{n^2}$).

Exercise 2.67 The quadratic forms $q(x_1, \cdots, x_n) = \sum_{1 \le i,j \le n} a_{ij} x_i x_j$, $a_{ij} = a_{ji}$, in n-variables can be identified with the Euclidean space $\mathbf{R}^{\frac{n(n+1)}{2}}$. A quadratic form q is called *semi-positive definite* if $q(x_1, \cdots, x_n) \ge 0$ for any vector $(x_1, \cdots, x_n)$. Prove that with the usual metric on $\mathbf{R}^{\frac{n(n+1)}{2}}$, the set of semi-positive definite quadratic forms is a closed subset.

Exercise 2.68 Continuing Exercise 2.29, prove that if $A \subset X$ and $B \subset Y$ are closed subsets, then $A \times B \subset X \times Y$ is a closed subset.

Exercise 2.69 Find an example showing that an infinite union of closed subsets may not be closed.

Exercise 2.70 Which of the following are closed subsets of $\mathbf{R}^2_{\text{usual}}$?

1. $C_1 = \{(x, y) : x = 0, y \le 5\}$;
2. $C_2 = \{(x, y) : x \text{ is a positive integer, } y \text{ is an integer}\}$;
3. $C_3 = \{(x, y) : x^2 + y^2 < 1 \ \text{ or } \ (x, y) = (1, 0)\}$;
4. $C_4 = \{(x, y) : y = x^2\}$.

Exercise 2.71 Determine whether the interval $(0, 1]$ is closed as a subset of each of the following spaces equipped with the usual metric $d(x, y) = |x - y|$:

1. $A = \mathbf{R} = (-\infty, \infty)$;
2. $B = (0, \infty)$;
3. $C = (-\infty, 1]$;
4. $D = (0, 1]$;
5. $E = [0, 1]$;
6. $F = \{-1\} \cup (0, 1]$.

Exercise 2.72 Is $C = \{f \in C[0, 1] : f(0) \ge 0\}$ a closed subset of $C[0, 1]_{L_1}$? Closed subset of $C[0, 1]_{L_\infty}$? What about $D = \{f \in C[0, 1] : f(0) \ge 0, f(1) \le 0\}$? What about $E = \{f \in C[0, 1] : f(0) \ge f(1)\} \subset C[0, 1]$?

Exercise 2.73 Use the conclusion of Exercise 2.62 to characterize closed subsets in terms of the distance function.

3 Graph and Network

3.1 Seven Bridges in Königsberg

Perhaps the first work which deserves to be considered as the beginnings of topology is due to Leonhard Euler[2]. In 1736 Euler published a paper entitled “Solutio problematis ad geometriam situs pertinentis” which translates into English as “The solution of a problem relating to the geometry of position”. The title itself indicates that Euler was aware that he was dealing with a different type of geometry where distance was not relevant.

The problem studied in that paper was concerned with the geography of the city of Königsberg. The city was divided by a river. In the river there were two islands. Seven bridges connected the two parts of the city and the two islands together.

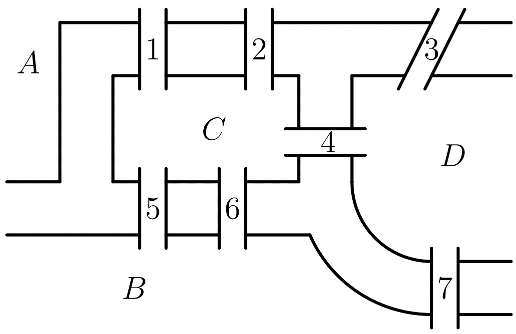


Figure 18: the city of Königsberg

Euler’s questions is: Is it possible to cross the seven bridges in a single journey (i.e., cross each bridge exactly once)? The problem may be abstracted into Figure 19, in which each region of the city becomes a point and each bridge becomes a line connecting two points.

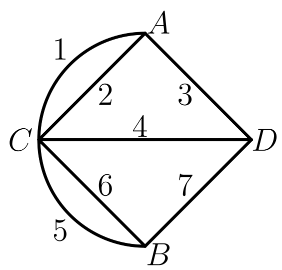


Figure 19: the graph of the city of Königsberg

Definition 3.1 A *graph* consists of finitely many points (called *vertices*), and finitely many lines (called *edges*) connecting these points. We have the following concepts about a graph:

1. A *path* is a sequence of vertices $V_1, V_2, \cdots, V_k$, and a sequence of edges $E_1, E_2, \cdots, E_{k-1}$, such that E_i connects V_i and V_{i+1}. We say the path connects the vertices V_1 and V_k;

[2]Born April 15, 1707 in Basel, Switzerland; Died September 18, 1783 in St. Petersburg, Russia

2. A *cycle* is a path such that $V_1 = V_k$;

3. A graph is *connected* if any two vertices can be connected by a path;

4. The *degree* of a vertex is the number of edges emanating from it.

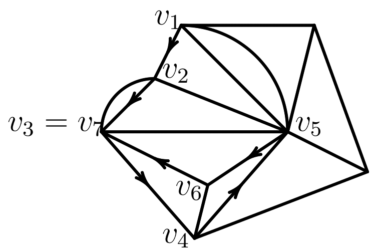


Figure 20: a path in a graph

The Königsberg graph is connected. The degrees of A, B, D are 3. The degree of C is 5.

Euler's problem can be generalized to the following: Given a graph, is it possible to find a path, so that each edge appears exactly once in the path?

Theorem 3.2 *Let G be a connected graph. Then the following are equivalent:*

1. *There is a path in which each edge appears exactly once;*

2. *There are at most two vertices with odd degree.*

We will need the following preparatory result (Theorem 3.3) to prove the theorem above. In the remaining part of this section, we will discuss this result and its applications. The proof of Theorem 3.2 will be deferred to the next section.

Theorem 3.3 *In any graph we have* $\sum_{\text{all vertices } V} \deg(V) = 2(\text{number of edges})$.

In case of the Königsberg graph, we have

$$\deg(A) + \deg(B) + \deg(C) + \deg(D) = 3 + 3 + 3 + 5 = 2 \times 7.$$

We analyse more closely why we have this equality for the Königsberg graph. In $\deg(A)$, we are counting the edges labeled 1, 2, and 3. In $\deg(C)$, we are counting the edges labeled 1, 2, 4, 5, and 6. Thus the edge labeled 3 appears once in $\deg(A)$ and once in $\deg(C)$. Moreover, it does not appear in the other two degrees. In fact, because each edge has exactly two end vertices, each of the seven edges appear in exactly two degrees in the summation. This is why we have twice of seven on the right side of the equation.

The discussion on the Königsberg graph can be generalized to any graph. Again, the key property of the graph is that each edge has exactly two end vertices. Therefore each graph appears exactly twice in the summation $\sum_{\text{all vertices } V} \deg(V)$. Consequently, the summation is twice of the number of vertices.

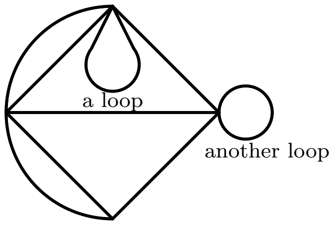


Figure 21: loops

If we are little bit more careful, we need to worry about a special case. A *loop* is an edge such that the two ends are the same vertex. Suppose a loop is attached to a vertex V. Then the loop is emanating from V in two ways. Therefore in $\deg(V)$, this loop is counted twice. This is balanced by the right side of the formula.

Exercise 3.1 Can you find a graph with the following numbers as the degrees of the vertices:

1. 1, 2, 3;
2. 2, 2, 3;
3. 1, 2, 2, 3, 3, 4, 4;
4. 2, 2, 2, 3, 3, 3, 5.

Can you conceive a theorem regarding possible list of numbers as the degrees of all vertices in a graph? What about connected graph?

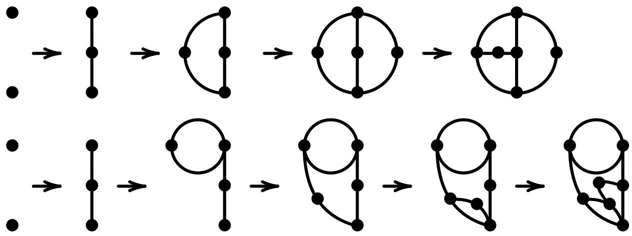


Figure 22: possible developments of "sprouts" starting from two points

Next is an interesting application of the degree formula in Theorem 3.3. The game "sprouts" starts with a number of dots on a sheet of paper. Several players take turns to draw a curve and add a new dot anywhere along the new curve. The curve must satisfy the following conditions

1. starts and ends at dots (possibly the same);
2. cannot cross itself;
3. cannot cross any previous curves;
4. cannot pass any other dot.

Another rule of the game is

5. there cannot be more than three curves emanating from any dot.

The first player who cannot add more curve loses.

Theorem 3.4 *Starting with n dots, it takes at most $3n$ steps to finish the game.*

Proof: The game creates a graph, with dots as the vertices, and (half) curves as edges. Each step creates one more vertex, and two more edges. Therefore after N steps, the number of vertices is $v = n + N$, and the number of edges is $e = 2N$.

By the fifth rule, the degree of each vertex is ≤ 3. Therefore by Theorem 3.3, we have inequality $2e \leq 3v$. Putting $v = n+N$ and $e = 2N$ into the inequality, we obtain $4N \leq 3(n+N)$. This is the same as $N \leq 3n$.

□

As a matter of fact, the game has at most $3n - 1$ steps.

Exercise 3.2 What would be the outcome of the game if we make any one of the following changes to the rule:

1. The fifth rule is changed from three curves to four curves;
2. Two dots are added in each step instead of just one dot;
3. The game is carried out over a torus or a Möbius band instead of the plane.

3.2 Proof of One-Trip Criteria

In this section, we prove Theorem 3.2.

Proof of $1 \Rightarrow 2$: Let α be a path satisfying 1. Suppose α connects a vertex V to a vertex V'. We consider the vertices passed when we traverse along α.

Whenever α passes a vertex W different from V and V', α must arrive W along an edge and leave W along another edge. This pair of edges will not be passed again, by the assumption on α. Moreover, each edge emanating from W is passed sooner or later. Therefore

$$\deg(W) = 2(\text{number of times } \alpha \text{ passes } W),$$

so that the degree of any vertex other than V and V' is even.

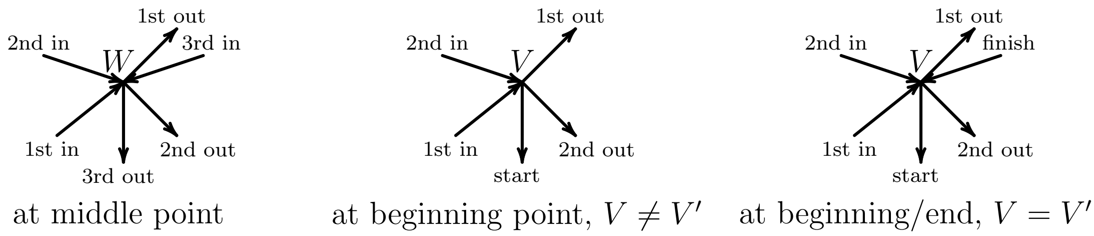


Figure 23: what happens for the path at a vertex

The counting is the same at V, except the starting edge along which α leaves V. If α does not end at V (i.e., $V' \neq V$), then this extra edge gives us

$$\deg(V) = 1 + 2(\text{number of times } \alpha \text{ comes back to } V \text{ after starting from } V).$$

Similarly, in case $V' \neq V$ we have

$$\deg(V') = 2(\text{number of times } \alpha \text{ reaches } V' \text{ before finally finishing at } V') + 1.$$

Therefore V and V' are the only two vertices with odd degrees.

If $V = V'$, we have

$$\deg(V) = 2(\text{number of times } \alpha \text{ passes } V) + 2,$$

where 2 counts the edges along which α starts from and ends at V. Therefore all vertices have even degree.

□

We note that in case we do have a pair of vertices with odd degrees, the proof above shows that the pair must be the two end vertices of the path α. If we do not have vertices of odd degrees, then the path must be a cycle.

Proof of $2 \Rightarrow 1$: Inspired by the discussion above, we will actually try to prove the following:

1. If all the vertices have even degrees, then starting from any vertex, we are able to find a cycle in which each edge appears exactly once;

2. If two vertices V and V' have odd degrees, and all the other vertices have even degrees, then we are able to find a path beginning from V and ending at V', and in which each edge appears exactly once.

We prove these statements by induction on the number of edges. If the graph has only one edge, then the statements are obviously true. Now suppose the statements are true for all (connected) graphs with number of edges $< n$, $n \geq 2$. Then we consider a connected graph G with the number of edges $= n$.

Suppose all vertices of G have even degrees. We need to prove the first statement by starting from an arbitrary vertex V. We pick an edge E emanating from V. We denote by V' the other vertex of E. Then we remove E and get a graph G' with $n-1$ edges. Now we need to consider two cases.

Case 1: If E is loop, i.e., $V = V'$, then $\deg_{G'}(V) = \deg_G(V) - 2$ is even. Moreover, for any other vertex W, $\deg_{G'}(W) = \deg_G(W)$ remains even. The graph G' is clearly still connected. By applying the first statement in the inductive hypothesis to G', we can find a cycle α' of G' beginning and ending at V, such that each edge of G' appears exactly once in the path. Let α be the path that begins at V, follows α' to arrive back at V, and then follows E back to V again. Then α is a cycle in which each edge of G appears exactly once.

Case 2: If E is not a loop, i.e., $V \neq V'$, then $\deg_{G'}(V) = \deg_G(V) - 1$, $\deg_{G'}(V') = \deg_G(V') - 1$ are odd. Moreover, for any other vertex W, $\deg_{G'}(W) = \deg_G(W)$ remains even.

We claim that G' has to be connected. If not, then G' is split into two disjoint parts G'_1 and G'_2, such that V is a vertex of G'_1 and V' is a vertex of G'_2 (since G is connected, the break off must happen at E). In particular, G'_1 and G'_2 are two graphs with exactly one odd degree vertex each. But it is a consequence of Theorem 3.3 that this cannot happen (see Exercise 3.1). Thus we conclude that G' has to be connected.

Now we apply the second statement in the inductive hypothesis to the connected graph G', which has V ans V' as the only odd degree vertices. we can find a path α' of G' connecting V to V', such that each edge of G' appears exactly once in α'. Now let α be the path that begins at V, follows α' to arrive at V', and then follows E to V. Then α is a cycle in which each edge appears exactly once.

In case all except two vertices have even degrees, we need to take E to be the edge emanating from an odd degree point. Then we have to consider the possibility that removing E may break G into two parts. The details is a little bit more complicated. But the key idea is similar. We omit the proof here.

□

Exercise 3.3 In which of the graphs in Figure 24, can you find a path, so that each edge appears exactly once? If possible, find such a path.

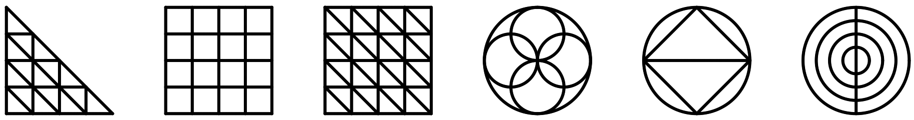


Figure 24: can you make one trip?

Exercise 3.4 Construct one more bridge in Königsberg, so that Euler's problem has a solution.

Exercise 3.5 Given a connected graph, how many edges do you have to add in order for Euler's problem to have solution?

Exercise 3.6 What if you are allowed to take two journeys instead of just one?

3.3 Euler Formula

The next step in freeing mathematics from being a subject about measurement was also due to Euler. In 1750 he wrote a letter to Christian Goldbach[3] which gives Euler's famous formula for a connected graph on a 2-dimensional sphere

$$v - e + f = 2$$

[3]Born March 18, 1690 in Königsberg, Prussia (now Kaliningrad, Russia); Died November 20, 1764 in Moscow, Russia. Goldbach is best remembered for his conjecture, made in 1742 in a letter to Euler, that every even integer greater than 2 is the sum of two primes. The conjecture is still open.

where v is the number of vertices, e is the number of edges and f is the number of faces (regions of the sphere divided by the graph).

It is interesting to realize that this, really rather simple, formula seems to have been missed by Archimedes and Descartes although both wrote extensively on polyhedra. Again the reason must be that to everyone before Euler, it had been impossible to think of geometrical properties without measurement being involved.

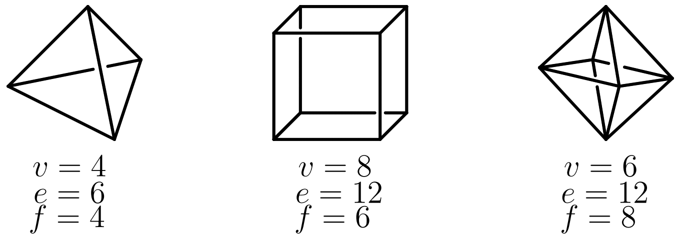


Figure 25: numbers of vertices, edges, and faces

If we punch a hole on the sphere, then we get a plane. Correspondingly, any graph on the sphere and not touching the hole becomes a graph on the plane. This graph enclose several finite regions and one infinite region in the plane. For a graph in the plane, usually we only count the number of finite regions. This number f is one less than the corresponding number for the graph in the sphere. Therefore for planar graphs the Euler formula is the following.

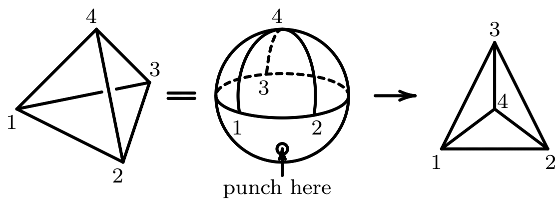


Figure 26: converting spherical Euler formula to planar Euler formula

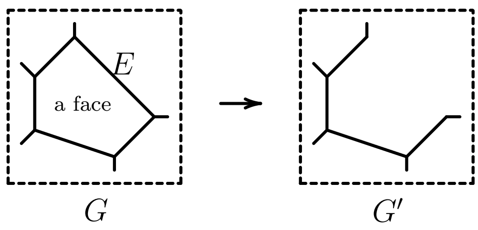
Theorem 3.5 *For any connected planar graph, we have*

$$v - e + f = 1.$$

Proof: We prove by induction on the number of edges of a graph. If there is only one edge, then we have two possibilities: (i) The edge has two distinct end vertices. Then $v = 2$, $e = 1$, $f = 0$. (ii) The edge is a loop. Then $v = 1$, $e = 1$, $f = 1$. In either case, we have $v - e + f = 1$.

Assume the theorem is true for graphs with $e < n$. Let G be a connected graph in plane with $e = n$.

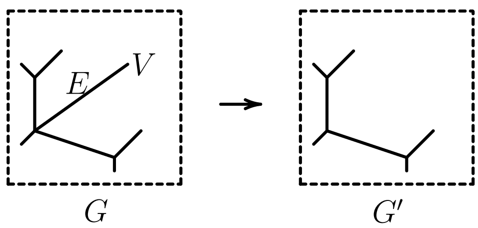
If $f \neq 0$, then there is at least one finite region. Let E be an edge on the boundary of one finite region. If we remove E, then the new graph G' is still connected. Since E is on the boundary of one finite region, we get one less region after removing E. Of course we also get one less edge and the same number of vertices after removing E. Therefore $f(G') = f(G) - 1$, $e(G') = e(G) - 1$, $v(G') = v(G)$. Since $e(G') = n - 1 < n$, by inductive assumption we have $v(G') - e(G') + f(G') = 1$. Therefore $v(G) - e(G) + f(G) = 1$.



$$v' = v$$
$$e' = e - 1$$
$$f' = f - 1$$

Figure 27: proof of Euler formula in case $f \neq 0$

If $f = 0$, then there is no finite region. We claim that there must be a vertex with degree 1. We prove the claim by contradiction. Assume all the vertices have degrees ≥ 2. Then we start from some vertex and going around the graph along the edges. Because any vertex has at least two edges emanating from it, we can choose a path without "backtracking": going from one end of an edge to the other end and then coming back. Because there are finitely many vertices, sooner or later the path must cross itself at some vertex. In this way, we are able to find a cycle, which must enclose a finite region. This finite region may be further divided by the graph. In any case, we should have $f > 0$, a contradiction.



$$v' = v - 1$$
$$e' = e - 1$$
$$f' = f$$

Figure 28: proof of Euler formula in case $f = 0$

Thus we have established the existence of vertices of degree 1. Let V be one such vertex, with only one edge E emanating from it. If we remove V and E, then the new graph G' is still connected. Moreover, we have $f(G') = f(G)$, $e(G') = e(G) - 1$, $v(G') = v(G) - 1$. Since $e(G') = n - 1 < n$, by inductive assumption we have $v(G') - e(G') + f(G') = 1$. Therefore $v(G) - e(G) + f(G) = 1$.

□

3.4 Embedding Graphs

In this section, we discuss an interesting application of Euler formula.

Consider three houses and three wells in Figure 29. Can you construct paths from each house to each well without having any two paths crossing each other? Think of the houses and wells as vertices and the paths connecting them as edges. Then the situation may be abstracted to the left graph $K_{3,3}$ in Figure 30.

The answer to the question is no, according to the following famous theorem due to Kazimierz Kuratowski[4].

[4]Born February 2, 1896 in Warsaw, Poland; Died June 18, 1980 in Warsaw, Poland.

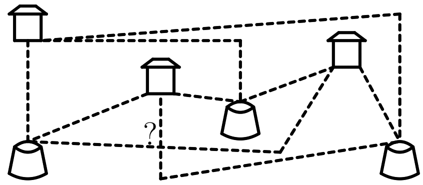


Figure 29: three houses and three wells

Theorem 3.6 *A graph is planar if and only if it does not contain any of the graphs $K_{3,3}$ and K_5 in Figure 30.*

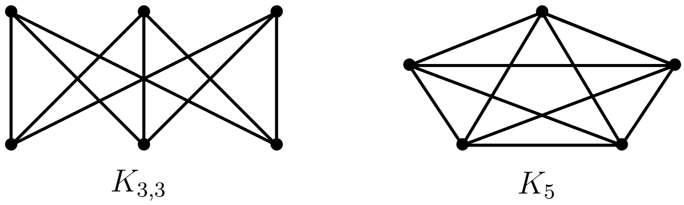


Figure 30: basic non-planar graphs

The proof of the theorem is rather complicated. Here we only show that the two graphs are not planar.

(Partial) Proof: Consider the graph K_5 first. Suppose it is planar. Then by the Euler formula, $v - e + f = 1$ and $e = 10$, $v = 5$, we have $f = 6$. Now the boundary of each face consists of at least three edges (because the boundary is a cycle). Moreover, any "internal edge" is on the boundary of exactly two faces, and any "boundary edge" is on the boundary of only one face. Therefore if we let e_i and e_b be the numbers of internal and boundary edges, then

$$e_i + e_b = e = 10, \quad 2e_i + e_b \geq 3f = 18.$$

From $2(10 - e_b) + e_b \geq 18$ we see that $e_b \leq 2$. However, this would imply that the graph looks like either one in Figure 31. In case $e_b = 2$, there are two edges joining the same pair of vertices. In case $e_b = 1$, there is a loop. Neither situation can be found inside the two graphs. The contradiction proves that K_5 cannot be planar.

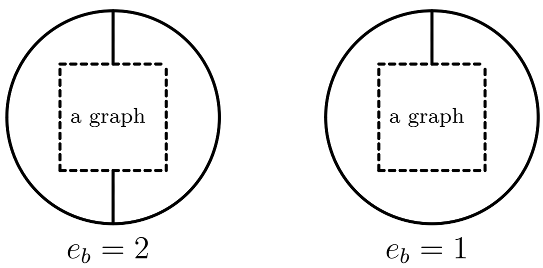


Figure 31: what K_5 would be like if it is embedded in plane

For the graph $K_{3,3}$, we have $e = 9$, $v = 6$, and from $v - e + f = 1$, $f = 4$. The similar argument would lead to $e_i + e_b = 9, 2e_i + e_b \geq 3f = 12$, so that $e_b \leq 6$, which would not lead to contradiction as before. Thus one more input is needed. Observe that any cycle in the graph should consist of at least 4 edges, and the boundary of any face must be a cycle. Therefore any face has at least 4 edges on its boundary. This shows that the inequality may be improved to $2e_i + e_b \geq 4f = 16$. Then we are able to conclude $e_b \leq 2$, and the same contradiction arises.

□

Exercise 3.7 Construct embeddings of $K_{3,3}$ and K_5 in the torus.

3.5 Platonic Solids

Plato[5] founded, on land which had belonged to Academos, a school of learning called the Academy. The institution was devoted to research and instruction in philosophy and sciences. Over the door of the Academy was written: *Let no one unversed in geometry enter here.*

The five Platonic solids are given in Figure 32. They are characterized by the extreme symmetry in numbers: All the faces have the same number of edges on the boundary; All the vertices have the same number of edges emanating from them (i.e., same degree).

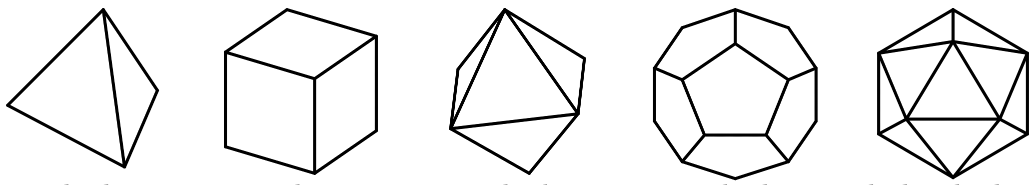


Figure 32: platonic solids

Theorem 3.7 *The five platonic solids are the only ones with such symmetry.*

Proof: We need to show the following are the only possibilities for v, e, f:

	v	e	f
tetrahedron	4	6	4
cube	8	12	6
octahedron	6	12	8
icosahedron	12	30	20
dodecahedron	20	30	12

Suppose each face has m edges on the boundary, and each vertex has n edges emanating from it. The geometric consideration tells us that we should assume m, n and e to be ≥ 3. Then by Theorem 3.3,

$$nv = 2e.$$

[5]Born 427 BC in Athens, Greece; Died 347 BC in Athens, Greece

Moreover, if we count the number of edges from the viewpoint of faces, we get mf. However, this counts each edge twice, because each edge is on the boundary of exactly two faces. Therefore we get

$$mf = 2e$$

Substituting these into Euler's formula, we get

$$v - e + f = \frac{2e}{n} - e + \frac{2e}{m} = 2,$$

or

$$\frac{1}{n} - \frac{1}{2} + \frac{1}{m} = \frac{1}{e}.$$

We need to solve the equation, under the assumption that m, n, e are integers ≥ 3.

From the equation, we get

$$\frac{2}{\min\{m,n\}} \geq \frac{1}{n} + \frac{1}{m} = \frac{1}{2} + \frac{1}{e} > \frac{1}{2}.$$

Therefore we have $\min\{m,n\} < 4$. On the other hand, we have assumed $m, n \geq 3$. Therefore we must have $\min\{m,n\} = 3$.

Suppose $m = 3$. Then we have

$$\frac{1}{n} - \frac{1}{6} = \frac{1}{e} > 0.$$

Thus $n < 6$ and we have the following possibilities

1. $m = 3, n = 3$. Then $e = 6$ and $v = 4, f = 4$;
2. $m = 3, n = 4$. Then $e = 12$ and $v = 6, f = 8$;
3. $m = 3, n = 5$. Then $e = 30$ and $v = 12, f = 20$.

On the other hand, we may assume $n = 3$. This means that we may exchange m and n in the discussion above. As a result, $v = \frac{2e}{n}$ and $f = \frac{2e}{m}$ are also exchanged, and we have the following possibilities

1. $m = 3, n = 3$. Then $e = 6$ and $v = 4, f = 4$;
2. $m = 4, n = 3$. Then $e = 12$ and $v = 8, f = 6$;
3. $m = 5, n = 3$. Then $e = 30$ and $v = 20, f = 12$.

Combining the six cases together, we get the five cases in the table.

□

Exercise 3.8 What is the condition on positive integers v, e, f that can be realized by a connected graph embeded in the sphere?

Exercise 3.9 Suppose we have a connected graph embeded in a sphere such that any face has exactly three edges. Prove that $e = 3v - 6$, $f = 2v - 4$.

Hint: Prove $3f = 2e$ and then combine this with the Euler formula.

4 Topology

4.1 Topological Basis and Subbasis

The key topological concepts and theories for metric spaces can be introduced from balls. In fact, if we carefully examine the definitions and theorems about open subsets, closed subsets, continuity, etc., then we see that we have used exactly two key properties about the balls. Whether or not we have a metric, as long as we have a system of balls satisfying these two properties, we should be able to develop similar topological theory. This observation leads to the concept of topological basis.

Definition 4.1 A *topological basis* on a set X is a collection $\mathcal{B}$ of subsets of X, such that

1. $x \in X \Rightarrow x \in B$ for some $B \in \mathcal{B}$;
2. $x \in B_1 \cap B_2$, with $B_1, B_2 \in \mathcal{B} \Rightarrow x \in B \subset B_1 \cap B_2$ for some $B \in \mathcal{B}$.

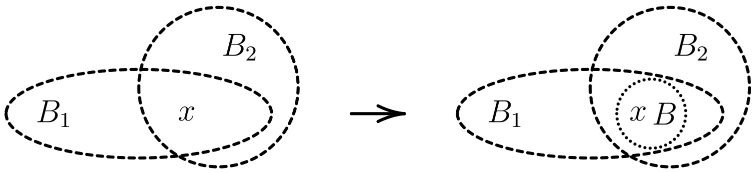


Figure 33: definition of topological basis

We may think of subsets in $\mathcal{B}$ as "generalized balls". These subsets are not necessarily produced from a distance function. The following lemma is a useful and simple way of recognizing topological basis (the converse is however not true).

Lemma 4.2 *Suppose a collection $\mathcal{B}$ satisfies the first condition and*

$$B_1, B_2 \in \mathcal{B} \quad \Longrightarrow \quad B_1 \cap B_2 \in \mathcal{B} \quad \text{or} \quad B_1 \cap B_2 = \emptyset.$$

Then $\mathcal{B}$ is a topological basis.

Given the property in the lemma, the second condition for topological basis is met by taking $B = B_1 \cap B_2$.

Example 4.1 The following collections are all topological bases on $\mathbf{R}$ because they satisfy the condition of Lemma 4.2.

$$\begin{aligned}
\mathcal{B}_1 &= \{(a, b) : a < b\} \\
\mathcal{B}_2 &= \{[a, b) : a < b\} \\
\mathcal{B}_3 &= \{(a, b] : a < b\} \\
\mathcal{B}_4 &= \mathcal{B}_1 \cup \mathcal{B}_2 \\
\mathcal{B}_5 &= \{(a, +\infty) : \text{all } a\} \\
\mathcal{B}_6 &= \{(-\infty, a) : \text{all } a\} \\
\mathcal{B}_7 &= \{(a, b) : a < b, \text{ and } a,\ b \text{ are rational}\} \\
\mathcal{B}_8 &= \{[a, b) : a < b, \text{ and } a,\ b \text{ are rational}\} \\
\mathcal{B}_9 &= \{\mathbf{R} - F : F \text{ is finite}\} \\
\mathcal{B}_{10} &= \{[a, b] : a \leq b\}
\end{aligned}$$

On the other hand, $\mathcal{B}_2 \cup \mathcal{B}_3$, $\mathcal{B} = \{\text{open intervals of length } 1\}$, and $\mathcal{B}' = \{[a,b] : a < b\}$ are not topological bases. To see $\mathcal{B}_2 \cup \mathcal{B}_3$ is not a topological basis, let us consider $0 \in (-1,0] \cap [0,1)$. If B satisfies $0 \in B \subset (-1,0] \cap [0,1)$, then we necessarily have $B = \{0\}$, which is not in $\mathcal{B}_2 \cup \mathcal{B}_3$. Therefore the second condition for the topological basis is not satisfied.

Example 4.2 The following collections are all topological bases on $\mathbf{R}^2$.

$$
\begin{aligned}
\mathcal{B}_1 &= \{(a,b) \times (c,d) : a < b, c < d\} \\
\mathcal{B}_2 &= \{\text{open disks}\} \\
\mathcal{B}_3 &= \{\text{open triangles}\} \\
\mathcal{B}_4 &= \{[a,b) \times [c,d) : a < b, c < d\} \\
\mathcal{B}_5 &= \{[a,b) \times (c,d) : a < b, c < d\} \\
\mathcal{B}_6 &= \{(a,b) \times (c,d) : a < b,\ c < d, \text{ and } a,b,c,d \text{ are rational}\}
\end{aligned}
$$

We may apply Lemma 4.2 to show that $\mathcal{B}_1$, $\mathcal{B}_4$, $\mathcal{B}_5$, $\mathcal{B}_6$ are topological bases. Although $\mathcal{B}_2$ and $\mathcal{B}_3$ do not satisfy the condition of Lemma 4.2, they are still topological bases (see Figure 34).

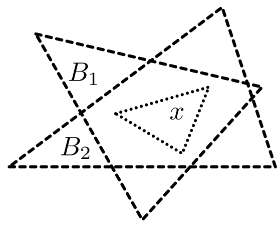


Figure 34: $\mathcal{B}_3$ is a topological basis

Example 4.3 The collection $\mathcal{B} = \Big\{\{1\}, \{4\}, \{1,2\}, \{1,3\}\Big\}$ of subsets of $X = \{1,2,3,4\}$ is a topological basis by Lemma 4.2. However, the collection $\mathcal{S} = \Big\{\{4\}, \{1,2\}, \{1,3\}\Big\}$ is not a topological basis. It violates the second condition with $x = 1$, $B_1 = \{1,2\}$, and $B_2 = \{1,3\}$.

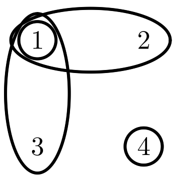


Figure 35: a topological basis on four points

Example 4.4 For any integer $n > 0$, numbers $a_1, \cdots, a_n$, points $0 \le t_1, \cdots, t_n \le 1$, and $\epsilon > 0$, define

$$B(a_1, \cdots, a_n, t_1, \cdots, t_n, \epsilon) = \{f \in C[0,1] : \quad |f(t_1) - a_1| < \epsilon, \cdots, |f(t_n) - a_n| < \epsilon\}.$$

Then the collection $\mathcal{B} = \{B(a_1, \cdots, a_n, t_1, \cdots, t_n, \epsilon) : \text{all } n,\ a_1, \cdots, a_n,\ t_1, \cdots, t_n,\ \epsilon\}$ is a topological basis of $C[0,1]$.

Exercise 4.1 For any set X, prove that the collections $\mathcal{B}_1 = \{X - F : F \subset X \text{ is finite}\}$ and $\mathcal{B}_2 = \{X - C : C \subset X \text{ is countable}\}$ are topological bases.

Exercise 4.2 Prove the balls in a metric space form a topological basis. More precisely, prove the following: Suppose $d(x, x_1) < \epsilon_1$ and $d(x, x_2) < \epsilon_2$. Then there is $\epsilon > 0$, such that $d(x, y) < \epsilon$ implies $d(y, x_1) < \epsilon_1$ and $d(y, x_2) < \epsilon_2$.

Exercise 4.3 Let $\mathcal{B}$ be a topological basis on X, and $f : Y \to X$ be a map. Is $f^{-1}(\mathcal{B}) = \{f^{-1}(B) : B \in \mathcal{B}\}$ a topological basis on Y?

Exercise 4.4 Let $\mathcal{B}$ be a topological basis on X, and $f : X \to Y$ be a map. Is $f(\mathcal{B}) = \{f(B) : B \in \mathcal{B}\}$ a topological basis on Y?

Exercise 4.5 Let $\mathcal{B}_X$ and $\mathcal{B}_Y$ be a topological basis on X. Is $\mathcal{B}_X \times \mathcal{B}_Y = \{B \times C : B \in \mathcal{B}_X, C \in \mathcal{B}_Y\}$ a topological basis on $X \times Y$?

Exercise 4.6 Suppose $\mathcal{B}$ and $\mathcal{B}'$ are topological bases on X. Which of the following are also topological bases?

1. $\mathcal{B}_1 = \{B \in \mathcal{B} \text{ or } \mathcal{B}'\}$;
2. $\mathcal{B}_2 = \{B \in \mathcal{B} \text{ and } \mathcal{B}'\}$;
3. $\mathcal{B}_3 = \{B \in \mathcal{B} \text{ and } \notin \mathcal{B}'\}$;
4. $\mathcal{B}_4 = \{B \cap B' : B \in \mathcal{B} \text{ and } B' \in \mathcal{B}'\}$;
5. $\mathcal{B}_5 = \{B \cup B' : B \in \mathcal{B} \text{ and } B' \in \mathcal{B}'\}$.

For any collection $\mathcal{S}$ of subsets of X, the collection

$$\mathcal{B} = \{S_1 \cap \cdots \cap S_n : \quad S_i \in \mathcal{S}, n \geq 1\} \cup \{X\}$$

is, by Lemma 4.2, a topological basis. We call $\mathcal{S}$ the *topological subbasis* that induces (or generates) $\mathcal{B}$. Note that we add X to the collection for the sole purpose of making sure the first condition for topological bases is satisfied. If $\mathcal{S}$ satisfies $\cup_{S \in \mathcal{S}} S = X$, then we do not need to include X.

Example 4.5 Let $\mathcal{S} = \{\text{open intervals of length } 1\}$ be as in Example 4.1. Then $\mathcal{S}$ is a topological subbasis on $\mathbf{R}$. The topological basis induced by $\mathcal{S}$ is $\mathcal{B} = \{\text{open intervals of length} \leq 1\}$. Note that we do not need to include the whole space $\mathbf{R}$ here.

Another example of topological subbasis on $\mathbf{R}$ is given by $\mathcal{S} = \{(a, \infty) : \text{all } a\} \cup \{(-\infty, a) : \text{all } a\}$. Since $(a, b) = (-\infty, b) \cap (a, \infty)$, it induces the subbasis $\mathcal{B}_1$ in Example 4.1.

Example 4.6 On any set X, the topological subbasis $\mathcal{S} = \{X - \{x\} : \text{all } x \in X\}$ induces the topological basis $\mathcal{B} = \{X - F : F \subset X \text{ is finite}\}$.

Example 4.7 In Example 4.4, we constructed a topological basis on $C[0, 1]$ by considering arbitrary number of points in $[0, 1]$. If we take only one point in $[0, 1]$, then

$$\mathcal{S} = \{B(a, t, \epsilon) : \quad \text{all } a,\ t,\ \epsilon\}$$

is no longer a topological basis. The topological basis generated by $\mathcal{S}$ consists of subsets of the form

$$B(a_1, t_1, \epsilon_1) \cap \cdots \cap B(a_n, t_n, \epsilon_n)$$

where n is arbitrary. The topological basis in Example 4.4 is different from this. However, we will see that the two topological bases will produce the same topological theory on $C[0, 1]$.

Exercise 4.7 Find topological subbases that induce the topological bases $\mathcal{B}_2$ and $\mathcal{B}_3$ in Example 4.1.

Exercise 4.8 Show that the subbasis $\mathcal{S} = \{(a, b) \times \mathbf{R} : a < b\} \cup \{\mathbf{R} \times (c, d) : c < d\}$ induces the topological basis $\mathcal{B}_1$ in Example 4.2.

Exercise 4.9 For any integer $n > 0$, let $S_n = \{kn : k \in \mathbf{Z}\}$ be the set of all multiples of n. For example, S_2 is the set of all even numbers. Describe the topological basis on $\mathbf{Z}$ induced by the subbasis $\mathcal{S} = \{S_p : p \text{ is a prime number}\}$.

4.2 Open Subset

As in metric spaces, we may introduce the concept of open subsets from a topological basis.

Definition 4.3 Given a topological basis $\mathcal{B}$ on X, a subset U of X is *open* if it has the following property

$$x \in U \implies x \in B \subset U \quad \text{for some } B \in \mathcal{B}.$$

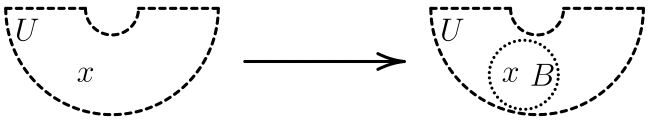


Figure 36: definition of open subset

Example 4.8 We would like to determine whether $(0,1)$ is an open subset with respect to some of the topological bases in Example 4.1.

$\mathcal{B}_1$: For any $x \in (0,1)$, we have $x \in B \subset (0,1)$ by choosing $B = (0,1) \in \mathcal{B}_1$. Therefore $(0,1)$ is open.

$\mathcal{B}_2$: For any $x \in (0,1)$, we have $x \in B \subset (0,1)$ by choosing $B = [x,1) \in \mathcal{B}_2$. Therefore $(0,1)$ is open.

$\mathcal{B}_5$: We cannot never find $B \in \mathcal{B}_5$, such that $x \in B \subset (0,1)$, because any such B would be an infinite interval. Consequently, $(0,1)$ is not open.

Example 4.9 On the left of Figure 37, we have an open triangle plus one open (meaning: excluding end points) edge. This is open with respect to the topological basis $\mathcal{B}_4$ in Example 4.2. In the middle, we change the location of the edge and the subset is no longer open. On the right, we change the edge from open to closed (meaning: including end points) and find that the subset is also not open with respect to $\mathcal{B}_4$.

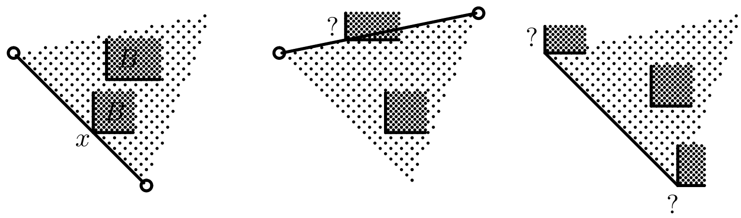


Figure 37: open subset in $\mathcal{B}_4$

Example 4.10 Consider the topological basis in Example 4.3. By

$$1 \in \{1\} \subset \{1,2,3\}, \quad 2 \in \{1,2\} \subset \{1,2,3\}, \quad 3 \in \{1,3\} \subset \{1,2,3\},$$

we see that the subset $\{1,2,3\}$ is open. Moreover, since there is no subset in the topological basis lying between 2 and $\{2,3\}$, $\{2,3\}$ is not open.

Example 4.11 In Example 2.9, we have shown that the subset $U = \{f \in C[0,1] : f(t) > 0 \text{ for all } 0 \le t \le 1\}$ of all positive functions is open with respect to the L_∞-metric. Here we will argue that U is not open with respect to the topological basis $\mathcal{B}$ in Example 4.4.

Let $f \in U$ and $f \in B = B(a_1, \cdots, a_n, t_1, \cdots, t_n, \epsilon)$, we need to find a non-positive function in B. This would imply that $B \not\subset U$ and therefore there is no B between f and U. We may choose a point t_0 distinct from any of $t_1, \cdots, t_n$. By modifying f around t_0, it is not difficult to construct a continuous function g such that $g(t_0) = -1$, $|g(t_1) - a_1| < \epsilon$, $\cdots$, $|g(t_n) - a_n| < \epsilon$. Clearly, $g \in B$ and $g \notin U$, and g is what we are looking for.

Exercise 4.10 Prove that any $B \in \mathcal{B}$ is open with respect to $\mathcal{B}$.

Exercise 4.11 Determine whether $[0,1]$, $(0,1]$, and $(0,+\infty)$ are open with respect to each of the topological bases in Example 4.1.

Exercise 4.12 Which of $\{2,3\}$, $\{1,2,4\}$, $\{2,3,4\}$ are open with respect to the topological basis $\mathcal{B}$ in Example 4.3.

Exercise 4.13 Determine which disk in Figure 38 plus part of open (meaning: excluding end points) arc boundary are open with respect to each of the topological bases in Example 4.2.

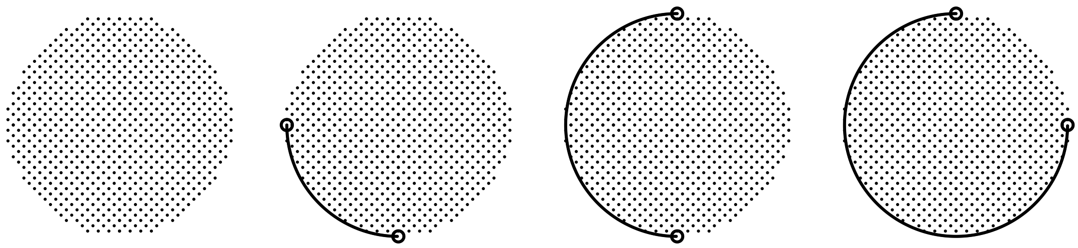


Figure 38: which are open subsets

Exercise 4.14 Show that the subset $U = \{f \in C[0,1] : f(0) > 1\}$ is an open subset with respect to the topological basis in Example 4.4.

Exercise 4.15 Let $\mathcal{B}_X$ and $\mathcal{B}_Y$ be topological bases on X and Y. Show that $\mathcal{B}_X \cup \mathcal{B}_Y$ is a topological basis on the disjoint union $X \coprod Y$. What are the open subsets of $X \coprod Y$ with respect to this topological basis?

The open subsets in the metric spaces have three properties given by Theorem 2.4. We still have the three properties by starting from topological basis.

Theorem 4.4 *The open subsets of X satisfy the following properties:*

1. *$\emptyset$ and X are open;*
2. *Any unions of open subsets are open;*
3. *Finite intersections of open subsets are open.*

Proof: By default, $\emptyset$ is open because there is no points in $\emptyset$ for us to verify the condition. On the other hand, the first condition for the topological basis implies X is open.

The proof of the second property of open subsets in Theorem 2.4 can be used here without much change.

The proof of the third property is also similar. Let U_1, U_2 be open, and let $x \in U_1 \cap U_2$. Then by definition we have $x \in B_1 \subset U_1$, $x \in B_2 \subset U_2$ for some $B_1, B_2 \in \mathcal{B}$. Then by the second property of topological basis, we have $x \in B \subset B_1 \cap B_2 \subset U_1 \cap U_2$ for some $B \in \mathcal{B}$. Thus we have verified the condition for $U_1 \cap U_2$ to be open.

□

Theorem 4.4 has the following useful consequence, comparable to Lemma 2.6.

Lemma 4.5 *A subset is open* $\Leftrightarrow$ *it is a union of some subsets in the topological basis.*

Proof: If U is open, then for each $x \in U$, we can find $B_x \in \mathcal{B}$, such that $x \in B_x$. We have

$$U = \cup_{x\in U}\{x\} \subset \cup_{x\in U} B_x \subset U.$$

This implies $U = \cup_{x\in U} B_x$, a union of subsets in the topological basis.

Conversely, any subset in $\mathcal{B}$ is open (see Exercise 4.10). Therefore by Theorem 4.4, any union of subsets in $\mathcal{B}$ is open.

□

Theorem 4.4 also leads to the concept of topology.

Definition 4.6 A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X, satisfying the three properties in Theorem 4.4:

1. $\emptyset, X \in \mathcal{T}$;
2. $U_i \in \mathcal{T}$ implies $\cup U_i \in \mathcal{T}$;
3. $U_1, U_2 \in \mathcal{T}$ implies $U_1 \cap U_2 \in \mathcal{T}$.

Example 4.12 Any metric space is a topological space. Note that different metrics may produce the same topology. For example, if two metrics are related as in the second part of Exercise 2.30, then openness with respect to either topology is the same. In particular, the various L_p-metrics on $\mathbf{R}^n$ in Example 2.3 are topologically the same, and we will denote the topology by $\mathbf{R}^n_{\text{usual}}$.

Example 4.13 On any set X we have the *discrete topology*

$$\mathcal{T}_{\text{discrete}} = \{\text{all subsets of } X\},$$

and the *trivial topology*

$$\mathcal{T}_{\text{trivial}} = \{\emptyset, X\}.$$

It is easy to see both $\mathcal{T}_{\text{discrete}}$ and $\mathcal{T}_{\text{trivial}}$ have the three properties. Moreover, the discrete topology is the topology induced from the discrete metric.

Example 4.14 There is only one topology on a single point space $X = \{1\}$:

$$\mathcal{T} = \{\emptyset, \{1\}\}.$$

There are four topologies on a two point space $X = \{1, 2\}$:

$$\begin{array}{ll} \mathcal{T}_1 = \{\emptyset, \{1,2\}\}, & \mathcal{T}_2 = \{\emptyset, \{1\}, \{1,2\}\}, \\ \mathcal{T}_3 = \{\emptyset, \{2\}, \{1,2\}\}, & \mathcal{T}_4 = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}. \end{array}$$

Note that if we exchange the two points, then $\mathcal{T}_3$ and $\mathcal{T}_4$ are interchanged. Exchanging the two points is a *homeomorphism* between the two topologies.

Figure 40 contains some examples of topologies on a three point space $X = \{1, 2, 3\}$. Again we see that some topologies may be homeomorphic. Moreover, we also see that some topologies are "larger" and some are "smaller". The total number of topologies on a three point space is 29. If we count homeomorphic topologies as the same one, then the total number is 10.

It is still an unsolved problem to find a nice formula on the number of topologies on a set of n elements.

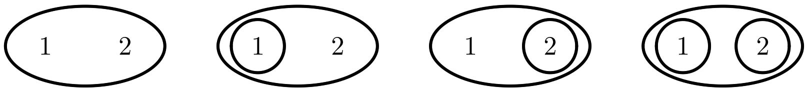


Figure 39: topologies on two points

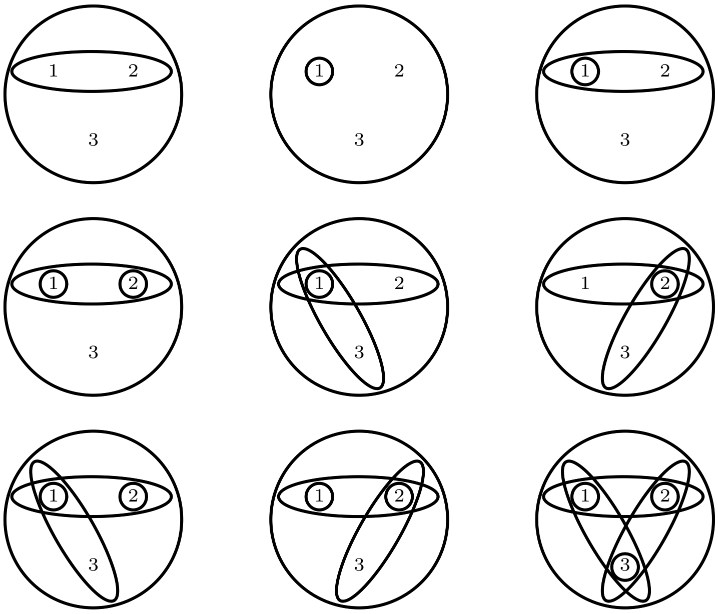


Figure 40: some topologies on three points

Example 4.15 The collection

$$\mathcal{S} = \Big\{\{4\}, \{1,2\}, \{1,3\}\Big\}$$

in Example 4.3 is not a topology on $X = \{1,2,3,4\}$. To get a topology, we need to include all the finite intersections. The new intersections not appearing in $\mathcal{S}$ are $\{4\} \cap \{1,2\} = \emptyset$ and $\{1,2\} \cap \{1,3\} = \{1\}$, and $\mathcal{S}$ is enlarged to

$$\mathcal{B} = \Big\{\emptyset, \{1\}, \{4\}, \{1,2\}, \{1,3\}\Big\}.$$

This is the topological basis in Example 4.3, and is the topological basis induced by $\mathcal{S}$.

To get a topology, we still need to take unions. In fact, according to Lemma 4.5. Taking all possible unions of subsets in $\mathcal{B}$ will give us the topology. The following are the new unions not appearing in $\mathcal{B}$

$$\{1\} \cup \{4\} = \{1,4\}, \quad \{1,2\} \cup \{1,3\} = \{1,2,3\}, \quad \{1,2\} \cup \{4\} = \{1,2,4\},$$

$$\{1,3\} \cup \{4\} = \{1,3,4\}, \quad \{1,2\} \cup \{1,3\} \cup \{4\} = \{1,2,3,4\}.$$

By adding all of these to $\mathcal{B}$, we get the following topology

$$\mathcal{T} = \Big\{\emptyset, \{1\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{1,2,3,4\}\Big\}.$$

Example 4.16 On any set X we have the *finite complement topology*

$$\mathcal{T} = \{X - F : F \text{ is a finite subset}\} \cup \{\emptyset\},$$

The second and the third properties mostly follow from

$$\cup(X - F_i) = X - \cap F_i, \quad \cap_{\text{finite}}(X - F_i) = X - \cup_{\text{finite}} F_i,$$

and the fact that both $\cap F_i$ and $\cup_{\text{finite}} F_i$ are finite. I said "mostly follow" because strictly speaking, we also have to consider the case some $X - F_i$ are replaced by $\emptyset$.

The finite complement topology is induced by the topological (sub)basis in Example 4.6.

We make some remarks on the concept of topology.

1. Why use open subsets to define topology?

As we have seen in the theory of metric spaces (especially Theorems 2.8 and 2.12), all the key topological concepts and theories depends only on the open subsets. Therefore open subsets is the most fundamental concept in topology. This is the reason why we use the key properties of open subsets as the definition of the concept of topology. The situation is similar to linear algebra, where the most fundamental concepts are summation and scalar multiplication, and their properties are used as the definition of the concept of vector spaces.

2. How to present a topology in practice?

Although open subsets is the most fundamental concept, for practical reason we almost never present specific topologies by listing all the open subsets. The Examples 4.12 through 4.16 are very rare ones in terms of the presentation. Even for $\mathbf{R}^2_{\text{Euclidean}}$, for example, it is really impossible to make a complete list of all the open subsets. For all practical purposes, the topologies are almost always presented as generated from some topological basis or subbasis.

Because of the fundamental nature of open subsets, the definitions of topological concepts in subsequent sections will be based on open subsets. On the other hand, we will always try to interprete the concepts in terms of topological basis, which will be useful for practical applications.

3. How to construct the suitable topological (sub)basis for your specific purpose?

In metric spaces, the balls are used to describe the closeness of points. As generalization of balls, the topological basis should play the similar role. In other words, if you intuitively feel the need to describe certain closeness, then you need to construct some subsets so that being inside one such subset matches your idea of closeness.

The choice of balls in metric spaces clearly reflects the closeness we try to describe. For a non-metric example, we illustrate the thinking by looking at Example 4.7. The example is motivated by the *pointwise convergence* of a sequence of functions $\{f_n\}$: For any $0 \le t \le 1$, the sequence $\{f_n(t)\}$ is convergent. In other words, for any fixed t, there is a number a, such that for any $\epsilon > 0$, we have $|f_n(t) - a| < \epsilon$ for sufficiently large n. The condition can be rephrased as f_n being in the subset $B(a, t, \epsilon)$ for sufficiently large n. As a result, we should use $B(a, t, \epsilon)$ (for all choices of a, t, ϵ) as the building blocks for the topological basis. Since these subsets do not form a topological basis, we need to generate a topological basis from these subsets, as we have done in Example 4.7.

Because of the motivation behind Example 4.7, the topology induced by the topological subbasis in the example is called the *pointwise convergence topology* on $C[0, 1]$.

Exercise 4.16 Show that on any set X, the following is a topology

$$\mathcal{T} = \{X - C : C \text{ is a countable subset}\} \cup \{\emptyset\}.$$

Exercise 4.17 Which of following are topologies?

1. $X = \mathbf{R}$, $\mathcal{T} = \{(a, \infty) : a \in \mathbf{R}\} \cup \{\emptyset, \mathbf{R}\}$;
2. $X = \mathbf{R}$, $\mathcal{T} = \{(a, \infty) : a \in \mathbf{Z}\} \cup \{\emptyset, \mathbf{R}\}$;
3. $X = \mathbf{Z}$, $\mathcal{T} = \{\text{subsets with at least two elements}\} \cup \{\emptyset, \mathbf{Z}\}$;
4. $X = \mathbf{Z}$, $\mathcal{T} = \{\text{subsets with even number of elements}\}$;
5. $X = \mathbf{Z}$, $\mathcal{T} = \{\text{subsets containing no odd numbers}\} \cup \{\mathbf{Z}\}$;
6. $X = \{1, 2, 3, 4\}$, $\mathcal{T} = \{\{1, 2\}, \{1, 2, 3, 4\}\}$;
7. $X = \{1, 2, 3, 4\}$, $\mathcal{T} = \{\emptyset, \{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3, 4\}\}$;
8. $X = \mathbf{R}^2$, $\mathcal{T} = \{\text{all open polygons}\} \cup \{\emptyset, \mathbf{R}^2\}$.

Exercise 4.18 Fix a point x_0 in a set X. Verify that all subsets containing x_0, plus the empty set, is a topology on X (called *included point topology*). Also verify that all subsets not containing x_0, plus the whole set X, is a topology on X (called *excluded point topology*).

The topologies may be generalized by replacing x_0 with a fixed subset $A \subset X$.

Exercise 4.19 Show that a subset A is open if and only if for any $x \in A$, there is an open subset U, such that $x \in U \subset A$.

Exercise 4.20 Prove that if any single point is open, then the topology is discrete.

Exercise 4.21 Prove true statements and provide counterexample to false statements.

1. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on X, then $\mathcal{T}_1 \cap \mathcal{T}_2$ is a topology on X;
2. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on X, then $\mathcal{T}_1 \cup \mathcal{T}_2$ is a topology on X;
3. If $\mathcal{T}$ is a topology on X, then $\mathcal{T}' = \{V \subset X : V \text{ is not in } \mathcal{T}\}$ is a topology on X;
4. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on X_1 and X_2, then $\mathcal{T} = \{U \times V : U \in \mathcal{T}_1, V \in \mathcal{T}_2\}$ is a topology on $X \times Y$;
5. If $\mathcal{T}$ is a topology on X and $Y \subset X$, then $\mathcal{T}' = \{U \cap Y : U \in \mathcal{T}\}$ is a topology on Y;
6. If $\mathcal{T}$ is a topology on X and $f : X \to Y$ is a map, then $f(\mathcal{T}) = \{f(U) : U \in \mathcal{T}\}$ is a topology on Y;
7. If $\mathcal{T}$ is a topology on X and $f : X \to Y$ is a map, then $\mathcal{T}' = \{V \subset Y : f^{-1}(V) \in \mathcal{T}\}$ is a topology on Y;
8. If $\mathcal{T}$ is a topology on X and $f : Y \to X$ is a map, then $f^{-1}(\mathcal{T}) = \{f^{-1}(U) : U \in \mathcal{T}\}$ is a topology on Y;
9. If $\mathcal{T}$ is a topology on X and $f : Y \to X$ is a map, then $\mathcal{T}' = \{V \subset Y : f(V) \in \mathcal{T}\}$ is a topology on Y.

4.3 Comparing Topologies

In Example 4.14, we see that sometimes different topologies on the same set can be compared.

Definition 4.7 Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the same set. If

$$U \text{ is open in } \mathcal{T} \quad \Longrightarrow \quad U \text{ is open in } \mathcal{T}',$$

then we say $\mathcal{T}$ is *coarser* than $\mathcal{T}'$ and we also say $\mathcal{T}'$ is *finer* than $\mathcal{T}$.

In other words, "larger" topologies are finer, and "smaller" topologies are coarser.

Example 4.17 The discrete topology is finer than any topology. It is the *finest* topology.
The trivial topology is coarser than any topology. It is the *coarsest* topology.

Example 4.18 According to Exercise 2.30, if two metrics on X satisfy $d_1(x,y) \le c\, d_2(x,y)$ for some constant $c > 0$, then open subsets with respect to d_1 are open subsets with respect to d_2. In other words, the d_1 induce coarser topology than d_2.

Example 4.19 In Example 4.15, we started with a collection $\mathcal{S}$ of subsets that does not satisfy the conditions for topology and constructed a topology $\mathcal{T}$. Since we have only added those *necessary* subsets in order to satisfy the conditions for topology, $\mathcal{T}$ is the coarsest topology such that any subset in $\mathcal{S}$ is open.

Example 4.20 In Example 2.9, we have shown that the subset $U = \{f \in C[0,1] : f(t) > 0 \text{ for all } 0 \le t \le 1\}$ is open with respect to the L_∞-metric and In Example 4.11, we have also shown that U ia not open with respect to the topological basis in Example 4.4. Therefore the L_∞-topology is not coarser than the topology induced by the topological basis in Example 4.4.

For practical purpose, it is useful to know how to compare topologies in terms of the topological bases.

Lemma 4.8 *Suppose topologies $\mathcal{T}$ and $\mathcal{T}'$ are constructed from the topological bases $\mathcal{B}$ and $\mathcal{B}'$. Then $\mathcal{T}$ is coarser than $\mathcal{T}'$ if and only if the following is true:*

$$x \in B \text{ for some } B \in \mathcal{B} \quad \Longrightarrow \quad x \in B' \subset B \text{ for some } B' \in \mathcal{B}'.$$

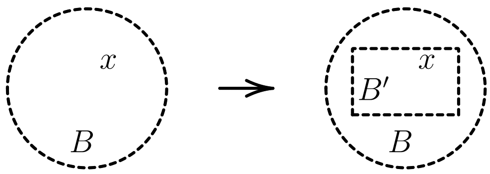


Figure 41: comparing topological basis

Proof: Let $U \in \mathcal{T}$. Then under the condition of the lemma, we have

$$\begin{array}{lll} & x \in U & \\ \Rightarrow & x \in B \subset U \text{ for some } B \in \mathcal{B} & (\mathcal{B} \text{ is a basis for } \mathcal{T}) \\ \Rightarrow & x \in B' \subset B \text{ for some } B' \in \mathcal{B}' & (\text{condition of lemma}) \\ \Rightarrow & x \in B' \subset U \text{ for some } B' \in \mathcal{B}' & (B' \subset B \subset U) \end{array}$$

Since $\mathcal{B}'$ is a basis for $\mathcal{T}'$, we conclude that U is open in $\mathcal{T}'$.

Conversely, suppose we know $\mathcal{T}$ is coarser than $\mathcal{T}'$. Then

$$\begin{array}{lll} & x \in B \text{ for some } B \in \mathcal{B} & \\ \Rightarrow & x \in B,\ B \text{ open in } \mathcal{T} & (\text{Exercise 4.10}) \\ \Rightarrow & x \in B,\ B \text{ open in } \mathcal{T}' & (\mathcal{T} \text{ is coarser than } \mathcal{T}') \\ \Rightarrow & x \in B' \subset B \text{ for some } B' \in \mathcal{B}' & (\mathcal{B}' \text{ is a basis for } \mathcal{T}') \end{array}$$

This verifies the condition of the lemma.

□

The following is a useful consequence of the Lemma.

Corollary 4.9 *If $\mathcal{B} \subset \mathcal{B}'$, then $\mathcal{B}$ induces coarser topology than $\mathcal{B}'$.*

The argument in Example 4.19 can also be generalized to the following useful result.

Lemma 4.10 *If all subsets in a topological basis $\mathcal{B}$ (or subbasis $\mathcal{S}$) are open in a topology $\mathcal{T}$, then $\mathcal{B}$ (or $\mathcal{S}$) induces coarser topology than $\mathcal{T}$.*

Proof: By Lemma 4.5, any open subset U with respect to $\mathcal{B}$ is of the form $U = \cup_i B_i$, with $B_i \in \mathcal{B}$. Since B_i are assumed to be open in $\mathcal{T}$, we see that U is also open with respect to $\mathcal{T}$. Thus we have proved that open with respect to $\mathcal{B}$ impies with respect to $\mathcal{T}$.

Now let $\mathcal{B}$ be the topological basis induced by a topological subbasis $\mathcal{S}$. Then each $B \in \mathcal{B}$ is either the whole space or a finite intersection of subsets in $\mathcal{S}$. In th first case, B is open in $\mathcal{T}$ by the first condition for topology. In the second case, B is open in $\mathcal{T}$ by the third condition for topology and the assumption that subsets in $\mathcal{S}$ are open in $\mathcal{T}$. Thus we have verified that $\mathcal{B}$ satisfies the conditions of this lemma, and the conclusion about $\mathcal{S}$ follows from the conclusion about $\mathcal{B}$.

□

Example 4.21 In Example 4.1 we considered the following two topological bases:

$$\mathcal{B}_1 = \{(a, b) : a < b\}, \quad \mathcal{B}_2 = \{[a, b) : a < b\}.$$

$\mathcal{B}_1$ gives $\mathbf{R}_{\text{usual}}$, or simply $\mathbf{R}_u$. $\mathcal{B}_2$ gives the *lower limit topology* which we will denote by $\mathbf{R}_{\text{lower limit}}$, or simply $\mathbf{R}_l$.

The following argument shows that $\mathbf{R}_u$ is coarser than $\mathbf{R}_l$:

$$x \in (a, b) \implies x \in [x, b) \subset (a, b).$$

On the other hand, if $\mathbf{R}_l$ were coarser than $\mathbf{R}_u$, then we would have

$$x \in [x, b) \implies x \in (a, c) \subset [x, b) \quad \text{for some } a < c.$$

However, $x \in (a, c)$ implies $a < x$ and $(a, c) \subset [x, b)$ implies $a \geq x$. This contradiction shows that $\mathbf{R}_l$ is not coarser than $\mathbf{R}_u$. Thus we conclude that $\mathbf{R}_u$ is *strictly coarser* than $\mathbf{R}_l$.

Example 4.22 In Example 4.2 we considered topological bases $\mathcal{B}_1$ and $\mathcal{B}_3$, consisting of rectangles and triangles on the plane $\mathbf{R}^2$. Figure 42 shows that the two topological bases give us the same topology on $\mathbf{R}^2$. In fact, this topology is also the same as $\mathbf{R}^2_{\text{usual}}$.

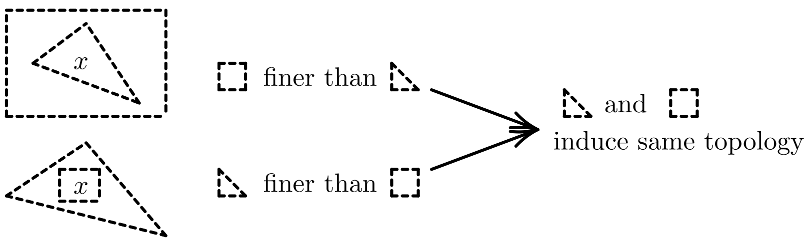


Figure 42: $\mathcal{B}_1$ and $\mathcal{B}_3$ induce same topology

Example 4.23 Let $\mathcal{B}$ and $\mathcal{B}'$ be the topological bases in Examples 4.4 and 4.7. We will show that they induce the same topology (the pointwise convergent topology).

Since $\mathcal{B} \subset \mathcal{B}'$ (by taking $\epsilon_1 = \cdots = \epsilon_n = \epsilon$), by Corollary 4.9 we see that $\mathcal{B}$ induces coarser topology than $\mathcal{B}'$. Conversely, the topological basis $\mathcal{B}'$ is indeced by the topological subbasis $\mathcal{S}$ in Example 4.7. Since each subset $B(a, t, \epsilon) \in \mathcal{S}$ is open in the topology induced by $\mathcal{B}$ (because $B(a, t, \epsilon)$ is also a subset in $\mathcal{B}$), by Lemma 4.10, $\mathcal{B}'$ induces (i.e., $\mathcal{S}$ induces) coarser topology than $\mathcal{B}$.

Example 4.24 We compare the L_∞-topology with the pointwise convergence topology. In Example 2.17, we have argued that the subsets $B(a, t, \epsilon)$ in the topological subbasis $\mathcal{S}$ of Example 4.7 are L_∞-open. Thus by Lemma 4.10, the pointwise convergence topology is coarser than the L_∞-topology. Combined with Example 4.20, we see that the pointwise convergence topology is strictly coarser than the L_∞-topology.

Exercise 4.22 Compare the topologies induced from the topological bases in Example 4.1.

Exercise 4.23 Compare the topologies induced from the topological bases in Example 4.2.

Exercise 4.24 Find the coarsest topology on $X = \{1, 2, 3, 4\}$, such that the subsets $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$ are open.

Exercise 4.25 Find the coarsest topology on $\mathbf{R}$, such that for any number a, both $(-\infty, a)$ and (a, ∞) are open.

Exercise 4.26 Use the fourth part of Exercise 2.30 to compare the L_1-topology with the L_∞-topology on $C[0, 1]$. How do you compare the L_1-topology and the pointwise convergent topology?

Exercise 4.27 Prove that for any subbasis $\mathcal{S}$, the topology induced by $\mathcal{S}$ is the coarsest topology such that all subsets in $\mathcal{S}$ are open.

Exercise 4.28 Show that Lemma 4.10 still holds if we replace the topological subbasis by a topological basis. Is the converse of the lemma true?

Exercise 4.29 Let $\mathcal{B}$ be a topological basis on X. We may think of $\mathcal{B}$ as a topological *subbasis* and then consider the topological basis $\mathcal{B}'$ induced from $\mathcal{B}$. Show that $\mathcal{B}$ and $\mathcal{B}'$ induce the same topology.

Exercise 4.30 Let $\mathcal{T}$ be a topology on X. We may think of $\mathcal{T}$ as a topological *basis*. Prove that the topology induced by (the topological basis) $\mathcal{T}$ is still (the topology) $\mathcal{T}$.

4.4 Closed Subset and Limit Point

The notion of limit points introduced in Section 2.6 can be adapted to the general topological spaces. Thus we define x is a limit point of $A \subset X$ if for any open U containing x, there is a, such that

1. $a \in A$;
2. $a \neq x$;
3. $a \in U$.

In other words, we have

$$(A - x) \cap U \neq \emptyset.$$

It is easy to see that if the topology is induced by a topological basis $\mathcal{B}$, then in the definition of limit points, we may replace U by $B \in \mathcal{B}$.

Example 4.25 We would like to find the limit points of $(0, 1)$ with respect to some of the topological bases in Example 4.1.

In $\mathbf{R}_{\text{usual}}$, the limit points of $(0, 1)$ form the closed interval $[0, 1]$.

In $\mathbf{R}_{\text{lower limit}}$, any number $0 \leq x < 1$ is a limit point. To prove this, we consider $x \in B = [y, x + \epsilon)$ for any $y \leq x$ and $\epsilon > 0$. We can always find a number a satisfying $x < a < \min\{1, x + \epsilon\}$. This number a satisfies the three conditions in the definition of limit points, so that x is a limit point. If $x < 0$, then we have $x \in [x, 0)$ and $[x, 0)$ contains no point in $(0, 1)$. Similarly, if $x \geq 1$, then we have $x \in [x, x + 1)$ and $[x, x + 1)$ contains no point in $(0, 1)$. In either case, we cannot find a number a satisfying the first condition in the definition of limit points. Therefore we conclude that the limit points of $(0, 1)$ in the lower limit topology form the interval $[0, 1)$.

Finally, we try to find the limit points of $(0, 1)$ with respect to $\mathcal{B}_5 = \{(a, \infty) : \text{all } a\}$. If $x > 1$, then for $x \in B = (1, \infty)$ we have $(0, 1) \cap B = \emptyset$. In particular, we can never have $((0, 1) - x) \cap B \neq \emptyset$, and x is not a limit point. If $x \leq 1$, then for any $x \in B = (a, \infty)$, we have $a < x \leq 1$. This implies that $((0, 1) - x) \cap B \supset (\max\{a, 0\}, 1) - x$, and $(\max\{a, 0\}, 1)$ is never empty. Thus x is a limit point. In conclusion, the limit points of $(0, 1)$ with respect to $\mathcal{B}_5$ is $(-\infty, 1]$.

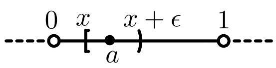


Figure 43: $0 \leq x < 1$ is a limit point of $(0, 1)$ in $\mathbf{R}_{\text{lower limit}}$

Figure 44: $x < 0$ and $x \geq 1$ are not limit points of $(0, 1)$ in $\mathbf{R}_{\text{lower limit}}$

Example 4.26 We study the limit points in the topology constructed in Example 4.15. In order to cut the number of open subsets we need to consider, we will make use of the topological basis $\mathcal{B}$ in Example 4.3.

We try to find the limit point of $A = \{1\}$. $x = 1$ is not a limit point because $a \in A$ and $a \neq x$ are always contradictory. $x = 2$ is a limit point because we only need to consider $x \in B = \{1, 2\}$, for which we can find $a = 1$ satisfying the three conditions. Similarly, $x = 3$ is also a limit point. Finally, $x = 4$ is not a limit point because by choosing $x \in B = \{4\}$, the conditions $a \in A$ and $a \in B$ become contradictory. Thus we conclude that the limit points of $\{1\}$ is $\{1\}' = \{2, 3\}$.

The limit points of $\{2\}$, $\{3\}$, $\{4\}$ are all empty.

Example 4.27 Let us consider the pointwise convergent topology discussed in Examples 4.4, 4.7, and 4.23. We claim that any function in $C[0, 1]$ is a limit point of the unit ball in L_1-metric:

$$A = \left\{ f \in C[0, 1] : \quad d_{L_1}(f, 0) = \int_0^1 |f(t)| dt < 1 \right\}.$$

To show any $g \in C[0, 1]$ is a limit point of A, we assume $g \in B$ (you may use B in either topological basis you like) and need to find f in $(A - g) \cap B$. Pick any $t_0 \in [0, 1]$ distinct from $t_1, \cdots, t_n$. Then it is easy to construct (see Figure 45) a continuous function f satisfying

$$f(t_0) \neq g(t_0),\ f(t_1) = g(t_1),\ \cdots,\ f(t_n) = g(t_n),\ \int_0^1 |f(t)| dt < 1.$$

The first inequality implies $f \neq g$. The equalities in the middle imply $f \in B$. The last inequality implies $f \in A$.

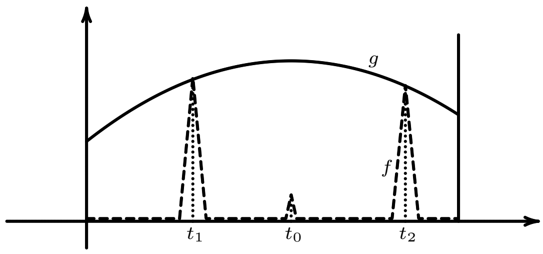


Figure 45: f is equal to g at two points and has small integration

Exercise 4.31 Find the limit points of $(0,1) \cup \{2\}$, $\{n^{-1}\}$, $[0,+\infty)$ with respect to each of the topological bases in Example 4.1.

Exercise 4.32 Let $A \subset \mathbf{R}$. Prove that x is a limit point of A in the lower limit topology if and only if there is a strictly decreasing sequence in A converging to x.

Exercise 4.33 Find the limit points of the disk $\{(x,y) : x^2+y^2<1\}$ with respect to each of the topological bases in Example 4.2.

Exercise 4.34 What are the limit points of any subset in the discrete topology? What about the trivial topology? What about the finite complement topology?

Exercise 4.35 Find the limit points of each subset $\{1,2\}$, $\{1,2,3\}$, $\{1,2,4\}$ in the topology $\mathcal{T}$ in Example 4.15.

Exercise 4.36 For which of the following subsets of $C[0,1]$ is it true that any function is a limit point in the pointwise convergent topology?

1. A_1 = all positive functions;
2. A_2 = all functions satisfying $\displaystyle\int_0^1 |f(t)|dt > 1$;
3. A_3 = all functions satisfying $f(0) = 0$;
4. A_4 = all functions such that $f(r)$ is rational for any rational $r \in [0,1]$.

Exercise 4.37 Prove that if x is a limit point of A, then x is also a limit point of $A - x$.

Exercise 4.38 Suppose x is a limit point of A. Is x still a limit point of A in a coarser topology? What about in a finer topology?

Exercise 4.39 Do the conclusions of Exercise 2.58 still hold for general topological spaces?

Exercise 4.40 Do Exercise 2.61 again for general topological spaces.

Theorem 2.12 suggests the following definition of closed subsets in a topological space.

Definition 4.11 A subset C of a topological space X is *closed* if $X - C$ is open.

By replacing $B(x, \epsilon)$ with open U in the proof, we see that Theorem 2.12 is still valid. Therefore a subset is closed if and only if it contains all its limit points.

Since $X - (X - U) = U$, it is easy to see that $U \subset X$ is open if and only if the complement $X - U$ is closed. Moreover, by deMorgan's law, we may convert Theorem 4.4 to the following three properties of closed subsets (compare Corollary 2.14).

Theorem 4.12 *The closed subsets of a topological space X satisfy the following properties:*

1. *$\emptyset$ and X are closed;*
2. *Any intersections of closed sets are closed;*
3. *Finite unions of closed sets are closed.*

Example 4.28 In the discrete topology, all subsets are open and closed. In the trivial topology, the only closed subsets are $\emptyset$ and the whole space.

Example 4.29 The closed subsets in the finite complement topology (see Example 4.16) are the finite subsets and the whole space.

Example 4.30 From Example 4.25, we see that $(0, 1)$ does not contain its limit points with respect to any of the topological bases $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_5$. Thus $(0, 1)$ is not closed with respect to any of the three topology.

Example 4.31 We would like to find the coarsest topology on $X = \{1, 2, 3, 4\}$, such that the subsets $\{1, 2, 3\}$, $\{3, 4\}$, $\{2, 4\}$ are closed. The requirement is the same as that the subsets $\{4\}$, $\{1, 2\}$, $\{1, 3\}$ are open. Since $\emptyset$ and X should always be closed, the problem becomes finding the coarsest topology on $X = \{1, 2, 3, 4\}$, such that all subsets in the collection $\mathcal{S}$ of Example 4.15 are open. The answer is the topology at the end of Example 4.15.

Exercise 4.41 Determine whether $(0, 1) \cup \{2\}$, $\{n^{-1}\}$, $[0, +\infty)$ are closed with respect to each of the topological bases in Example 4.1.

Exercise 4.42 When we say $[a, b]$ is a closed interval, we really mean it is closed with respect to the usual topology. Study whether $[a, b]$ is still closed with respect to the topological bases in Example 4.1.

Exercise 4.43 Determine whether the disk $\{(x, y) : x^2 + y^2 < 1\}$ is closed with respect to each of the topological bases in Example 4.2.

Exercise 4.44 Is $C = \{f \in C[0, 1] : f(0) \geq 0\}$ a closed subset in the pointwise convergent topology? What about $D = \{f \in C[0, 1] : f(0) \geq 0, f(1) \leq 0\}$? What about $E = \{f \in C[0, 1] : f(0) \geq f(1)\} \subset C[0, 1]$?

Exercise 4.45 Use the conclusion of Exercise 2.62 to characterize closed subsets in terms of the distance function.

Exercise 4.46 Suppose U is open and C is closed. Show that $U - C$ is open and $C - U$ is closed.

Exercise 4.47 Suppose a topological space X has the property that any single point subset is closed. Prove that for any subset A of X, we have $A' \subset A''$.

Exercise 4.48 Find the coarsest topology on $X = \{1, 2, 3, 4\}$ such that $\{1, 2\}$, $\{3, 4\}$ are open and $\{1, 2\}$, $\{2, 3\}$ are closed.

Exercise 4.49 Prove that the finite complement topology is the coarsest topology such that $\{x\}$ is closed for any $x \in X$.

Note: Compare with Exercise 4.20.

4.5 Closure

Since being a closed subset is equivalent to containing all the limit points, it is natural to expect that by adding limit points to any subset A, one should get a closed subset. The result is the *closure* $\bar{A}$ of A, given by the following equivalent descriptions.

Lemma 4.13 *For any subset A, the following characterize the same subset $\bar{A}$:*

1. $\bar{A} = A \cup \{\textit{limit points of } A\}$;
2. $\bar{A} = \cap\{C : C \textit{ is closed, } A \subset C\}$;
3. *$\bar{A}$ is the smallest closed subset containing A.*

Proof: Denote by A_1, A_2, A_3 the three subsets given by the three descriptions.

By Theorem 4.12, A_2 ia closed. Then

$$\left\{\begin{array}{l}\text{(i) } A \subset A_2 \\ \text{(ii) } A_2 \text{ is closed} \\ \text{(iii) } A_3 \text{ is smallest satisfying (i) and (ii)}\end{array}\right. \quad \Longrightarrow \quad A_3 \subset A_2.$$

On the other hand, the fact that A_3 is closed and contains A implies that A_3 is one of the C's in the description of A_2. Since any such C contains A_2, we conclude that $A_2 \subset A_3$. Thus we have shown $A_2 = A_3$.

Next we prove $A_1 \subset A_2$. By definition, $A \subset A_2$. Therefore we need to show

$$\left\{\begin{array}{l}\text{(i) } x \text{ is a limit point of } A \\ \text{(ii) } C \text{ is closed} \\ \text{(iii) } A \subset C\end{array}\right. \quad \Longrightarrow \quad x \in C.$$

If x were not in C, then $x \in U = X - C$, which is open because of (ii). Now by (iii), $(A - x) \cap U \subset C \cap U = \emptyset$. Therefore x is not a limit point of A, in contradiction with (i).

Finally we prove $A_3 \subset A_1$. By the description of A_3, we only need to show A_1 is closed. Equivalently, we need to show $X - A_1$ is open.

Suppose $x \in X - A_1$, i.e., $x \notin A$ and x is not a limit point of A. Since x is not a limit point of A, we have some some open U, such that $x \in U$ and $(A - x) \cap U = \emptyset$. Since $x \notin A$, we have $(A - x) \cap U = A \cap U$. Therefore we conclude that

$$x \in U \subset X - A, \quad \text{for some open } U.$$

Now U is disjoint from A. We also claim that U does not contain any limit points of A. In fact, for any point $y \in U$, we have $(A - y) \cap U \subset A \cap U = \emptyset$, which implies y is not a limit point of A.

Thus we conclude that U contains neither points of A nor limit points of A. Therefore we have

$$x \in U \subset X - A_1, \quad \text{for some open } U.$$

Since this holds for any point $x \in X - A_1$, we conclude that $X - A_1$ is open.

□

The following is a useful way of characterizing points in the closure.

Lemma 4.14 *A point x is in the closure of A $\Leftrightarrow$ for open U containing x, the intersection $A \cap U \neq \emptyset$. Moreover, if the topology is given by a topological basis, then U can be replaced by subsets in the basis.*

Proof: We prove the opposite statement:

$$\begin{aligned}
& x \notin \bar{A} \\
\Leftrightarrow \quad & x \notin A, x \text{ is not a limit point of } A \\
\Leftrightarrow \quad & x \notin A, (A - x) \cap U = \emptyset \text{ for some open } U \text{ containing } x \quad \text{(definition of limit point)} \\
\Leftrightarrow \quad & x \notin A, A \cap U = \emptyset \text{ for some open } U \text{ containing } x \\
\Leftrightarrow \quad & A \cap U = \emptyset \text{ for some open } U \text{ containing } x \quad \text{(second statement implies } x \notin A)
\end{aligned}$$

□

Example 4.32 From Example 4.25, we see that the closure of $(0, 1)$ in the usual topology is $[0, 1]$. The closure in the lower limit topology is $[0, 1)$. And the closure with respect to $\mathcal{B}_5$ is $(-\infty, 1]$.

Example 4.33 In the topological space considered in Example 4.26, the closure of $\{1\}$ is $\{1\} \cup \{2, 3\} = \{1, 2, 3\}$. Alternatively, from the topology $\mathcal{T}$ in Example 4.15 we find all the closed subsets (by taking complements)

$$\{1, 2, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3\}, \{3, 4\}, \{2, 4\}, \{2, 3\}, \{4\}, \{3\}, \{2\}, \emptyset.$$

Among these, $\{1, 2, 3\}$ is the smallest that contains $\{1\}$.

Example 4.34 In a metric space, recall the distance $d(x, A)$ from a point x to a subset A in Exercise 2.53. By Exercise 2.62 and the first description in Lemma 4.13, we have

$$\bar{A} = \{x \in X : \quad d(x, A) = 0\}.$$

We can also prove the equivalence between $x \in \bar{A}$ and $d(x, A) = 0$ by making use of Lemma 4.14. Note that the lemma still holds if we replace U by a ball $B(x, \epsilon)$ with arbitrarily small $\epsilon > 0$. Then $x \in \bar{A}$ means $A \cap B(x, \epsilon) \neq \emptyset$ for any $\epsilon > 0$. However, a point a in $A \cap B(x, \epsilon)$ means $a \in A$ and $d(x, a) < \epsilon$. Thus the nonemptyness simply means the distance between x and points of A can e arbitrarily small. This in turn means exactly $d(x, A) = 0$.

Example 4.35 From Example 2.20, we see that $C[0, 1]$ is the closure of all the polynomials in L_∞-topology. In fact, $C[0, 1]$ is also the closure of all the polynomials in L_1-topology as well as the pointwise convergent topology (you will see the reason after doing Exercise 4.57).

On the other hand, Example 4.27 tells us that $C[0, 1]$ is the closure of the L_1-unit ball in pointwise convergent topology. However, it follows from the previous example that the closure of the L_1-unit ball in L_1- as well as the L_∞-topology is the closed L_1-unit ball

$$\left\{ f \in C[0, 1] : \quad d_{L_1}(f, 0) = \int_0^1 |f(t)| dt \leq 1 \right\}.$$

instead of all the functions.

Exercise 4.50 Prove that A is closed $\Leftrightarrow$ $A = \bar{A}$.

Exercise 4.51 Find the closure of each subset $\{1, 2\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$ in the topology $\mathcal{T}$ in Example 4.15.

Exercise 4.52 Find the closure of $(-\sqrt{2}, \sqrt{2})$ and $\{n^{-1} : n \in \mathbf{N}\}$ with respect to each of the topological bases in Example 4.1.

Exercise 4.53 Find the closure of the disk $\{(x, y) : x^2 + y^2 < 1\}$ with respect to each of the topological bases in Example 4.2.

Exercise 4.54 In a metric space, is it always true that the closed ball $\bar{B}(a, \epsilon) = \{x : d(x, a) \leq \epsilon\}$ is the closure of the open ball $B(a, \epsilon)$?

Exercise 4.55 Consider the topology on $\mathbf{N}$ induced by the topological subbasis in Exercise 4.9. Find the limit points and the closures of $\{10\}$, $\{2, 5\}$, and $S_{10} = \{\text{all multiples of } 10\}$.

Exercise 4.56 Find the closure of the *topologist's sine curve*

$$\Sigma = \left\{(x, \sin\frac{1}{x}) : \quad x > 0\right\}$$

in the usual topology of $\mathbf{R}^2$.

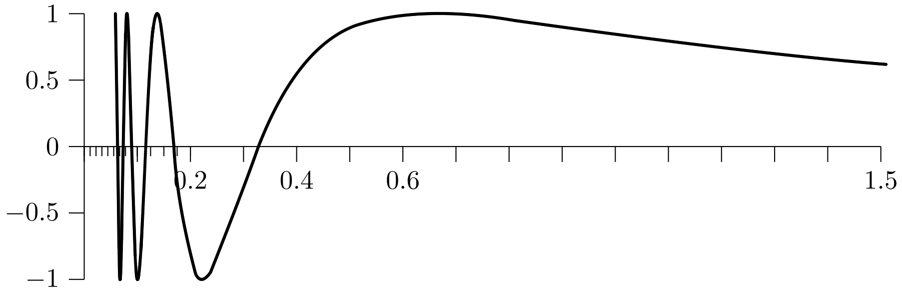


Figure 46: topologist's sine curve

Exercise 4.57 Suppose we have two topologies on X, one finer than the other. For a subset $A \subset X$, can you say anything about its limit points and closures with respect to the two topologies?

Exercise 4.58 Prove true statements and provide counterexamples to wrong ones:

1. $A \subset B \Rightarrow \bar{A} \subset \bar{B}$;
2. $\overline{A \cup B} = \bar{A} \cup \bar{B}$;
3. $\overline{A \cap B} = \bar{A} \cap \bar{B}$;
4. $\overline{X - A} = X - \bar{A}$;
5. $\bar{\bar{A}} = \bar{A}$.

Exercise 4.59 Prove that U is open $\Leftrightarrow$ The inclusion $\bar{A} \cap U \subset \overline{A \cap U}$ holds for any subset A.

Exercise 4.60 Let A be a subset of a topological space X.

1. Use the complement and closure operations to construct the biggest open subset $\dot{A}$ contained in A;
2. Prove that $x \in \dot{A}$ if and only if there is an open subset U, such that $x \in U \subset A$;
3. Study the properties of $\dot{A}$. For example, consider problems similar to Exercises 4.50 and 4.58.

Note: The subset $\dot{A}$ is the interior *of A. The concept is complementary to the concept of closure. Moreover, the open subset U in the second item can be replaced by topological basis.*

Exercise 4.61 (Kuratowski's 14-Set Theorem) Starting from a subset $A \subset X$, we may apply the closure operation and the complement operation alternatively to get subsets A, $X - A$, $\bar{A}$, $\overline{X - A}$, $X - \bar{A}$, $X - \overline{X - A}$, etc.

1. Prove that for fixed A, we can produce at most 14 distinct subsets by using the two operations;
2. Find a subset A of $\mathbf{R}_{\text{usual}}$ for which the maximal number 14 is obtained.

5 Basic Topological Concepts

5.1 Continuity

Theorem 2.8 suggests us how to generalize the notion of continuity to maps between topological spaces.

Definition 5.1 A map $f : X \to Y$ between topological spaces is *continuous* if

$$U \subset Y \text{ open} \quad \Longrightarrow \quad \text{preimage } f^{-1}(U) \subset X \text{ is open.}$$

Example 5.1 Any map *from* a discrete topology is continuous. Any map *to* a trivial topology is continuous.

Example 5.2 Consider the constant map $f(x) = b : X \to Y$, where b is a fixed element in Y. For any open subset U of Y, we have

$$f^{-1}(U) = \begin{cases} X & \text{if } b \in U \\ \emptyset & \text{if } b \in U \end{cases}$$

Thus $f^{-1}(U)$ is always an open subset of X, and f is continuous.

Example 5.3 Consider the map

$$\rho(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \quad : \quad \mathbf{R}_{\text{lower limit}} \to \mathbf{R}_{\text{usual}}$$

For any open subset U of $\mathbf{R}_{\text{usual}}$, the preimage

$$\rho^{-1}(U) = \begin{cases} \emptyset & \text{if } 0, 1 \in U \\ [0, \infty) & \text{if } 0 \notin U, 1 \in U \\ (-\infty, 0) & \text{if } 0 \in U, 1 \notin U \\ \mathbf{R} & \text{if } 0, 1 \notin U \end{cases}$$

which are all open in $\mathbf{R}_{\text{lower limit}}$. Thus ρ is continuous.

On the other hand, the function $\rho(-t)$ is not continuous because the preimge $(-\infty, 0]$ of $(-1, \infty)$ is not open in $\mathbf{R}_{\text{lower limit}}$.

Since most topologies are given by bases or subbases, the following is very useful.

Lemma 5.2 *Suppose $\mathcal{B}$ is a topological basis, or $\mathcal{S}$ is a topological subbasis, for a topological space Y. Then for a map $f : X \to Y$, the following are equivalent:*

1. *f is continuous;*
2. *$f^{-1}(B)$ is open for any $B \in \mathcal{B}$;*
3. *$f^{-1}(S)$ is open for any $S \in \mathcal{S}$.*

The first and the second are equivalent because any open subset is a union of subsets in the topological basis (Lemma 4.5), and $f^{-1}(\cup B_i) = \cup f^{-1}(B_i)$. The second and the third are equivalent because the induced topological basis consists of finite intersections of subsets in the subbasis, and $f^{-1}(S_1 \cap \cdots \cap S_n) = f^{-1}(S_1) \cap \cdots \cap f^{-1}(S_n)$.

Moreover, we note that the properties for continuous maps between metric spaces are still valid (and the same proofs apply). These include: A map is continuous $\Leftrightarrow$ the preimage of any closed subset is closed. The composition of continuous maps is continuous.

Example 5.4 In Example 2.14, we have seen that the summation map $\sigma(x,y) = x+y : \mathbf{R}^2 \to \mathbf{R}$ is continuous, if both sides have the usual topology. Now let us give $\mathbf{R}$ the lower limit topology and give $\mathbf{R}^2$ the topology induced by $\mathcal{B}_4$ or $\mathcal{B}_5$ in Example 4.2.

By Lemma 5.2, the key point is whether $\sigma^{-1}[a,b)$ is open. The subset is shown on the left of Figure 47. It is easy to see that $\sigma^{-1}[a,b)$ is open in $\mathcal{B}_4$ but not open in $\mathcal{B}_5$. Therefore the summation, as a map into the lower limit topology, is continuous if $\mathbf{R}^2$ has the topology from $\mathcal{B}_4$, and is not continuous if $\mathbf{R}^2$ has the topology from $\mathcal{B}_5$.

We may also consider the difference map $\delta(x,y) = x - y : \mathbf{R}^2 \to \mathbf{R}$, which is continuous if both sides have the usual topology. If we again give $\mathbf{R}$ the lower limit topology, then we need to consider $\delta^{-1}[a,b)$, which is depicted on the right of Figure 47. The conclusion is that, with topology on $\mathbf{R}^2$ coming from either $\mathcal{B}_4$ or $\mathcal{B}_5$, δ is not continuous.

Figure 47: $\sigma^{-1}[a,b)$ and $\delta^{-1}[a,b)$

Example 5.5 In Example 2.11, we proved that the evaluation map $E(f) = f(0) : C[0,1] \to \mathbf{R}_{\text{usual}}$ is continuous if we give $C[0,1]$ the L_∞-topology. Now we claim that E is also continuous if we give $C[0,1]$ the pointwise convergent topology.

Since $\mathbf{R}_{\text{usual}}$ has a subbasis consisting of (a,∞) and $(-\infty,a)$, by Lemma 5.2, it is sufficient to show $E^{-1}(a,\infty)$ and $E^{-1}(-\infty,a)$ are open. Specifically, we have

$$E^{-1}(a,\infty) = \{f \in C[0,1] : E(f) \in (a,\infty)\} = \{f \in C[0,1] : f(0) > a\}.$$

By the argument similar to Exercise 4.14, we conclude $E^{-1}(a,\infty)$ is open in the pointwise convergent topology. By similar argument, we also conclude $E^{-1}(-\infty,a)$ is open.

Exercise 5.1 Describe all the continuous maps from $\mathbf{R}_{\text{usual}}$ to the space $\{1,2\}$ equipped with the topology $\{\emptyset, \{1\}, \{1,2\}\}$.

Exercise 5.2 Prove that the only continuous maps from $\mathbf{R}_{\text{finite complement}}$ to $\mathbf{R}_{\text{usual}}$ are the constants.

Exercise 5.3 Study the continuity of the multiplication map $\mu(x,y) = xy : \mathbf{R}^2 \to \mathbf{R}$, where $\mathbf{R}$ has the lower limit topology and $\mathbf{R}^2$ has the topology induced by $\mathcal{B}_4$ or $\mathcal{B}_5$ in Example 4.2.

Exercise 5.4 Describe (by providing a topological basis, for example) the coarsest topology on $\mathbf{R}$ such that functions (to $\mathbf{R}$ with the usual topology) of the form

$$\rho_a(t) = \begin{cases} 1 & t \geq a \\ 0 & t < a \end{cases}$$

are continuous for all a.

Exercise 5.5 Suppose $[0,\infty)$ has the usual topology. Find the coarsest topology on $\mathbf{R}$, such that $f(t) = t^2 : \mathbf{R} \to [0,\infty)$ is continuous.

Exercise 5.6 Let $X = \{1, 2, 3, 4\}$. Let $\mathbf{R}$ hve the topology induced by the topological basis $\mathcal{B}_5$ in Example 4.1. Let $f : X \to \mathbf{R}$ be the map that takes the point $x \in X$ to the number $x \in \mathbf{R}$.

1. Find the coarsest topology $\mathcal{T}$ on X such that $\{1\}$ is open, $\{2, 3\}$ is closed, and f is continuous;
2. Find the limit points and the closure of $\{4\}$ in $\mathcal{T}$.

Exercise 5.7 Prove that all the continuous maps $f : \mathbf{R}_{\text{usual}} \to \mathbf{R}_{\text{lower limit}}$ are constant maps.

Hint: Use the last conclusion of Exercise 2.63 and the fact that $[a, \infty)$ is open and closed in the lower limit topology.

Exercise 5.8 Let X be a topological space such that any map $f : X \to Y$ is continuous. Is it true that X must be a discrete space?

Exercise 5.9 Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on X. Prove that the identity map $id : (X, \mathcal{T}) \to (X, \mathcal{T}')$ is continuous $\Leftrightarrow$ $\mathcal{T}'$ is coarser than $\mathcal{T}$.

Exercise 5.10 Consider a continuous map $f : X \to Y$ between topological spaces. If we make the topology on X coarser, is f still continuous? What if we make the topology finer? What if we modify the topology on Y instead of X.

Exercise 5.11 Let Y be a topological space. Let $f : X \to Y$ be a map. Find the coarsest topology on the set X, such that f is continuous.

Given topological bases $\mathcal{B}_X$ and $\mathcal{B}_Y$ for X and Y, we look at the second description in Lemma 5.2 in more detail. The openness of $f^{-1}(B)$ for $B \in \mathcal{B}_Y$ means

$$x \in f^{-1}(B) \text{ for some } B \in \mathcal{B}_Y \quad \Longrightarrow \quad x \in B' \subset f^{-1}(B) \text{ for some } B' \in \mathcal{B}_X.$$

Thus we get the following criterion of continuity in terms of topological bases.

Lemma 5.3 *Suppose $\mathcal{B}_X$ and $\mathcal{B}_Y$ are topological bases for X and Y. Then a map $f : X \to Y$ is continuous if and only if the following is true:*

$$f(x) \in B \text{ for some } B \in \mathcal{B}_Y \quad \Longrightarrow \quad x \in B' \text{ and } f(B') \subset B \text{ for some } B' \in \mathcal{B}_X.$$

The criterion is quite similar to the ϵ-δ criterion for continuous maps between metric spaces. We also note that the third description in Lemma 5.2 leads to the following criterion for contiuity

$$f(x) \in S \text{ for some } S \in \mathcal{S}_Y \quad \Longrightarrow \quad x \in B' \text{ and } f(B') \subset S \text{ for some } B' \in \mathcal{B}_X.$$

Example 5.6 By Lemma 5.3, a map $f : \mathbf{R}_{\text{lower limit}} \to \mathbf{R}_{\text{usual}}$? is continuous if and only if the following holds

$$f(x) \in (a, b) \quad \Longrightarrow \quad \text{there is } [c, d) \text{ such that } x \in [c, d), f[c, d) \subset (a, b). \tag{13}$$

We claim that it is sufficient to take $(a, b) = (x - \epsilon, x + \epsilon)$ and $[c, d) = [x, x + \delta)$. In other words, we will show that the condition (13) is equivalent to the following condition

$$f(x) \in (x-\epsilon, x+\epsilon) \text{ for some } \epsilon > 0 \quad \Longrightarrow \quad \text{there is } \delta > 0 \text{ such that } x \in [x, x+\delta), f[x, x+\delta) \subset (x-\epsilon, x+\epsilon). \tag{14}$$

To see why (13) implies (14), we start with $f(x) \in (x - \epsilon, x + \epsilon)$. By taking $(a, b) = (x - \epsilon, x + \epsilon)$ in (13), we find $[c, d)$ satisfying $x \in [c, d), f[c, d) \subset (x - \epsilon, x + \epsilon)$. Let $\delta = d - x$. Then $\delta > 0$, $[x, x + \delta) = [x, d) \subset [c, d)$, and $f[x, x + \delta) \subset f[c, d) \subset (x - \epsilon, x + \epsilon)$. Thus we have verified (14) under the assumption that (13) holds.

To see why (14) implies (13), we start with $f(x) \in (a,b)$. By taking ϵ to be any postive number no bigger than $x-a$ and $b-x$, we have $x \in (x-\epsilon, x+\epsilon) \subset (a,b)$. Then by (13), we find $\delta > 0$ satisfying $f[x, x+\delta) \subset (x-\epsilon, x+\epsilon)$. In other words, the right side of (13) holds for $[c,d) = [x, x+\delta)$.

Finally, we may simplify (14) to the following statement: For any x and $\epsilon > 0$, there is $\delta > 0$ such that

$$x \le y < x+\delta \quad \Longrightarrow \quad |f(y) - f(x)| < \epsilon.$$

This means f is *right continuous* in calculus: $\lim_{y \to x, y \ge x} f(y) = f(x)$.

Exercise 5.12 Given toplogical bases on X and Y, what do you need to do in order to show that a map $f : X \to Y$ is *not* continuous?

Exercise 5.13 What topology would you put on $\mathbf{R}$ on the left, so that the continuity of $f : \mathbf{R} \to \mathbf{R}_{\text{usual}}$ means left continuous in calculus?

Exercise 5.14 What topology would you put on $\mathbf{R}^2$, so that the continuity of $f(x,y) : \mathbf{R}^2 \to \mathbf{R}_{\text{usual}}$ means f is continuous in x variable (but not necessarily in y variable)[6]?

Exercise 5.15 Describe continuous maps $f : \mathbf{R}_{\text{lower limit}} \to \mathbf{R}_{\text{lower limit}}$. Then prove that $f, g : \mathbf{R}_{\text{lower limit}} \to \mathbf{R}_{\text{lower limit}}$ are continuous $\Rightarrow f+g$ is continuous. What about $f-g$?

Exercise 5.16 Prove that the integration map

$$\mathcal{I} : C[0,1]_{\text{pt conv}} \to \mathbf{R}_{\text{usual}}, \quad f(t) \mapsto \int_0^1 f(t)dt$$

is not continuous.

Exercise 5.17 Suppose f is a continuous map. If x is a limit point of A, is it necessarily true that $f(x)$ is a limit point of $f(A)$?

Exercise 5.18 Prove that f is continuous $\Leftrightarrow f(\bar{A}) \subset \overline{f(A)}$.

Exercise 5.19 Prove that f is continuous $\Leftrightarrow \overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$.

Exercise 5.20 Let $f, g : X \to Y$ be two continuous maps. Suppose $A \subset X$ is a subset satisfying $\bar{A} = X$. If $f(a) = g(a)$ for any $a \in A$, is it necessarily true that $f(x) = g(x)$ for all $x \in X$?

Exercise 5.21 A function $f : X \to \mathbf{R}$ is *upper semicontinuous* if $f^{-1}(a, \infty)$ is open (in the usual topology) for all a. It is *lower semicontinuous* if $f^{-1}(-\infty, a)$ is open for all a. Prove the following

1. f is upper semicontinuous $\Leftrightarrow$ for any $a < f(x)$ there is an open $U \subset X$, such that $y \in U$ implies $a < f(y)$. Moreover, U can be replaced by subsets in a topological basis;
2. f is continuous $\Leftrightarrow$ f is both upper and lower semicontinuous;
3. f is upper semicontinuous $\Leftrightarrow$ $-f$ is lower semicontinuous;
4. f and g are upper semicontinuous $\Leftrightarrow$ $f+g$ is upper semicontinuous;
5. The characteristic function (see Exercise 1.14) $\chi_A(x)$ is upper semi-continuous $\Leftrightarrow$ A is closed;
6. The characteristic function $\chi_A(x)$ is lower semi-continuous $\Leftrightarrow$ A is open;

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[6]If you want f to be continuous in x variable, in y variable, but not necessarily in both variables together, then you need to put "plus topology" on $\mathbf{R}^2$. See D.J. Velleman: *Multivariable Calculus and the Plus Topology*, AMS Monthly, vol106, 733-740 (1999)

5.2 Homeomorphism

In Example 4.14, we noticed that among the four topologies on the two point space $X = \{1, 2\}$, $\mathcal{T}_2$ and $\mathcal{T}_3$ are interchanged via the map $f : X \to X, f(1) = 2, f(2) = 1$. We have also seen the equivalences between some topologies on $X = \{1, 2, 3\}$.

Definition 5.4 A map $f : X \to Y$ between topological spaces is a *homeomorphism* if

1. f is invertible;
2. Both f and f^{-1} are continuous.

Two topological spaces are *homeomorphic* if there is a homeomorphism between them.

To show X and Y are homeomorphic, we need to construct an invertible map $f : X \to Y$, such that

$$U \subset X \text{ is open} \iff f(U) \subset Y \text{ is open}.$$

In other words, f induces a one-to-one correspondence between the open subsets.

Homeomorphic spaces are considered as the "same" from the topological viewpoint. In this regard, the concept of homeomorphism (called *isomorphism* in general) is a very universal and powerful one. For example, as far as making one trip across the seven bridges in Königsberg is concerned, the simple graph is isomorphic to the network of actual bridges. For another example, we have learned in linear algebra that all finite dimensional vector spaces are isomorphic to the standard Euclidean space. The fact enables us to translate special properties about Euclidean spaces into general properties about linear algebra.

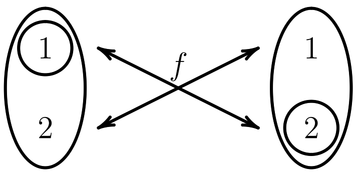


Figure 48: homeomorphism between $\mathcal{T}_2$ and $\mathcal{T}_3$

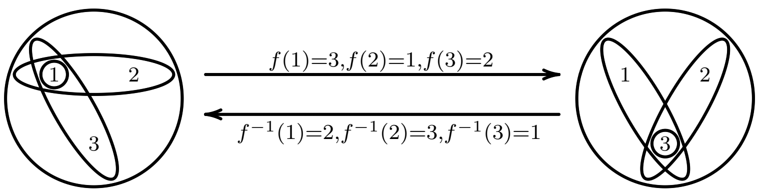


Figure 49: homeomorphic topologies on three points

Example 5.7 The map $f(x) = 3x - 2 : \mathbf{R}_{\text{usual}} \to \mathbf{R}_{\text{usual}}$ is a homeomorphism because f is invertible, and both f and $f^{-1}(y) = \frac{1}{3}y + \frac{2}{3}$ are continuous.

The map $f(x) = \frac{x}{1 - x^2} : (-1, 1) \to \mathbf{R}_{\text{usual}}$ tells us that $(-1, 1)$ (with the usual topology) and $\mathbf{R}_{\text{usual}}$ are homeomorphic. Another homeomorphism is given by $g(x) = \tan \frac{\pi x}{2}$.

Example 5.8 The topological bases $\mathcal{B}_2$ in Example 4.1 induces the lower limit topology $\mathbf{R}_{\text{lower limit}}$. Similarly, the topological basis $\mathcal{B}_3$ induces the *upper* limit topology $\mathbf{R}_{\text{upper limit}}$. The map

$$f(x) = -x : \quad \mathbf{R}_{\text{lower limit}} \to \mathbf{R}_{\text{upper limit}}$$

is a homeomorphism. This follows from the fact that f induces a one-to-one correspondence between $\mathcal{B}_2$ and $\mathcal{B}_3$.

Example 5.9 Consider the identity map

$$id : \quad (X, \mathcal{T}) \to (X, \mathcal{T}')$$

between two topologies on X. The map id is continuous if and only if $\mathcal{T}'$ is coarser than $\mathcal{T}$. The inverse map $id^{-1} = id$ is continuous if and only if $\mathcal{T}$ is coarser than $\mathcal{T}'$. Therefore id is a homeomorphism if and only if $\mathcal{T} = \mathcal{T}'$.

Example 5.10 Consider the interval $[0, 1)$ and the circle S^1, both with the usual topology. The restriction of the map $E(x) = e^{i2\pi x}$ (see Example 1.23) to $[0, 1)$ is invertible and continuous. However, E is not a homeomorphism since E^{-1} is not continuous (U is open in $[0, 1)$, but $E(U) = (E^{-1})^{-1}(U)$ is not open in S^1).

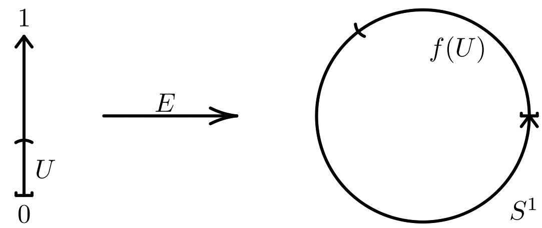


Figure 50: one-to-one correspondence and continuity do not imply homeomorphism

Exercise 5.22 Consider $X = \{1, 2, 3, 4\}$ with the topology given by Example 4.15. Find all homeomorphisms from X to itself.

Exercise 5.23 Prove that all the open intervals, with the usual topology, are homeomorphic. What about closed intervals?

Exercise 5.24 Describe how the unit disk $\{(x, y) : x^2 + y^2 < 1\}$ and the unit square $(0, 1) \times (0, 1)$, with the usual topology, are homeomorphic.

Exercise 5.25 Let $X = \mathbf{R} \cup \{-\infty, +\infty\}$ be the straight line with two infinities. We give X a topology induced by the following topological basis

$$\mathcal{B} = \Big\{(a, b)\Big\} \bigcup \Big\{\{-\infty\} \cup (-\infty, b)\Big\} \bigcup \Big\{(a, +\infty) \cup \{+\infty\}\Big\}$$

Prove that X is homeomorphic to $[0, 1]$.

Exercise 5.26 Describe all homeomorphisms from $\mathbf{R}_{\text{usual}}$ to itself.

Exercise 5.27 Prove that $[0, 1]$ and $(0, 1)$ are not homeomorphic by showing that any continuous map $f : [0, 1] \to (0, 1)$ is not onto. Similarly, prove that $[0, 1)$ is not homeomorphic to either $[0, 1]$ or $(0, 1)$.

Exercise 5.28 Prove that if X and Y are homeomorphic, and Y and Z are homeomorphic, then X and Z are homeomorphic.

Exercise 5.29 What is the condition for two discrete topological spaces to be homeomorphic? What about two spaces with trivial topology?

Exercise 5.30 A map $f : X \to Y$ betwwen metric spaces is an *isometry*[7] if f is invertible and $d_X(x, y) = d_Y(f(x), f(y))$. Show that any isometry is a homeomorphism. Then find all isometries from $\mathbf{R}$ (with the usual metric) to itself.

Since homeomorphisms induce one-to-one correspondence between open subsets, for any property (concept, theory) derived from open subsets, such a property (concept, theory) in one space exactly corresponds to the same property in the other one via f. For example, we have

$$x \text{ is a limit point of } A \quad \Longleftrightarrow \quad f(x) \text{ is a limit point of } f(A).$$

Definition 5.5 *A property P about topological spaces is called a* topological property *if for any homeomorphic spaces X and Y, X has $P \Leftrightarrow Y$ has P. A quantity I associated to topological spaces is called a* topological invariant *if X and Y are homeomorphic $\Rightarrow I(X) = I(Y)$.*

We have seen that any property (quantity) described by open subsets is a topological property. For example, being a limit point is a topological property. The other topological properties include closedness for subsets, being closure, being discrete topology. We will also study important topological properties such as Hausdorff, connected, and compact. Among the simplest topological invariants is the Euler number, which we will study later on. More advanced topological invariants include (covering) dimension, foundamental group, homology group, and homotopy group.

Example 5.11 A metric space is *bounded* if there is an upper bound for the distance between its points. With the usual metric, $\mathbf{R}$ is unbounded, while $(0, 1)$ is bounded. Since $\mathbf{R}$ and $(0, 1)$ are homeomorphic and have different boundedness property, we see that boundedness is not a topological property. The boundedness is a *metrical property* (a property preserved by isometries), though.

Example 5.12 A topological space X is *metrizable* if the topology can be induced by a metric on X. Metrizability is a topological property, because if d induces the topology on X, and $f : Y \to X$ is a homeomorphism, then

$$d_Y(y_1, y_2) = d(f(y_1), f(y_2))$$

is a metric on Y that induces the topology on Y.

Example 5.13 A topological space is *second countable* if it is generated by a countable topological basis. The second countability is a topological property: If $f : X \to Y$ is a homeomorphism, and a countable collection $\mathcal{B}$ of subsets is a topological basis for X, then $f(\mathcal{B}) = \{f(B) : B \in \mathcal{B}\}$ is a countable topological basis for Y.

The collection $\mathcal{B}_7$ in Example 4.1 is a countable basis for the usual topology on $\mathbf{R}$. By taking product of $\mathcal{B}_7$ with itself, we also see that the usual topology on $\mathbf{R}^n$ is second countable.

On the other hand, we claim that the lower limit topology on $\mathbf{R}$ is not second countable. Assume $\mathcal{B}$ is a topological basis for the lower limit topology. Since $[x, \infty)$ is open in the lower limit topology, we should have

[7] Isometry is isomorphism between metric spaces, just like homeomorphism is isomorphism between topological spaces.

some $B_x \in \mathcal{B}$, such that $x \in B_x \subset [x, \infty)$. In particular, we see that $x = \min B_x$. Since $x \neq y$ implies B_x and B_y have different minimums, we conclude that B_x and B_y are different. Therefore the number of subsets in $\mathbf{R}$ is at least as many as $\mathbf{R}$, which is not countable.

As a consequence of the above discussion, the lower limit topology is not homeomorphism to the usual topology.

Example 5.14 A *local topological basis* at a point $x \in X$ is a collection $\mathcal{B}_x$ of open subsets containing x, such that the following holds:

$$x \in U, U \text{ open} \quad \Longrightarrow \quad x \in B \subset U \quad \text{for some } B \in \mathcal{B}_x.$$

We note the similarily between the property above and the definition of open subset in terms of a topological basis. The only difference here is that x is fixed.

A topological space is *first countable* if any point has a countable local topological basis. By reason similar to the second countability, the first countability is also a topological property.

The most important example of first countable spaces is metric spaces. For any x in a metric space, the collection $\mathcal{B}_x = \{B(x, \frac{1}{n}) : n \in \mathbf{N}\}$ of balls around x is a countable local topological basis. In particular, if a topological space is metrizable, it has to be first countable.

We claim that the pointwise convergence topology is not first countable. Assuming the countrary, we have a sequence of open subsets U_1, U_2, $\cdots$, forming a local topological basis of the constant function 0. For each U_i, we can find some $B_i = B(0, \cdots, 0, t_{i1}, \cdots, t_{in_i}, \epsilon_i)$ between 0 and U_i. Since the set of points t_{ij}, $i = 1, 2, \cdots$, $1 \leq j \leq n_i$, is countable, we can find some $0 \leq t_0 \leq 1$ not equal to any of these t_{ij}. Now 0 is a point in the open subset $B(0, t_0, 1)$. Therefore we should have some i so that U_i is between 0 and $B(0, t_0, 1)$. Correspondingly, we should have $B_i \subset B(0, t_0, 1)$ for this particular i. But since t_0 is different from any of $t_{i1}, \cdots, t_{in_i}$, we can easily construct a function in B_i but not in $B(0, t_0, 1)$ (see Example 5.21 for detailed construction). The contradiction shows that the pointwise convergence topology cannot be first countable. Consequently, the topology is not metrizable and therefore is not homeomorphic to the L_1- or L_∞-topology.

Exercise 5.31 Study the homeomorphisms between the topologies in Example 4.1.

1. Show that the following is a topological property: The complement of any open subset, except $\emptyset$, is finite. Then use the property to prove that the finite complement topology is not homeomorphic to any of the other topologies in the example;
2. Show that the following is a topological property (called T_1- or *Fréchet space*): For any two distict points x, y, there is an open subset U, such that $x \in U$ and $y \notin U$. Then use the property to prove that the topology induced by $\mathcal{B}_5$ is not homeomorphic to those induced by $\mathcal{B}_1$, $\mathcal{B}_2$, etc.
3. What topological property can be used to show that the lower limit topology is not homeomorphic to the topology induced by $\mathcal{B}_8$?

Exercise 5.32 Prove that second countability implies the first countability. Moreover, find a topolgical space that is first countable but not second countable.

5.3 Subspace

Definition 5.6 Let Y be a subset of a topoogical space $(X, \mathcal{T})$. The subspace topology on Y is

$$\mathcal{T}_Y = \{Y \cap U : \quad U \text{ is an open subset of } X\}$$

Strictly speaking, one needs to verify $\mathcal{T}_Y$ satisfy the three axioms for a topology. This was given as the fifth part of Exercise 4.21.

As we have seen in the case of metric spaces, sometimes we have to be specific in saying whether a subset $V \subset Y$ is open in Y or in X. The two statements might mean different things.

It is also convenient to see subspace topology from the viewpoint of topological (sub)bases.

Lemma 5.7 *If $\mathcal{B}$ is a topological basis for X, then*

$$\mathcal{B}_Y = \{Y \cap B : B \in \mathcal{B}\}$$

is a topological basis for the subspace topology on Y. Similar statement is also true for topological subbases.

Proof: Let $V \subset Y$ be open with respect to the subspace topology. Then $V = Y \cap U$ for some open subset $U \subset X$. By Lemma 4.5, since $\mathcal{B}$ is a topological basis for X, we have $U = \cup B_i$ for some $B_i \in \mathcal{B}$. Therefore $V = Y \cap (\cup B_i) = \cup(Y \cap B_i)$ is, as a union of subsets in $\mathcal{B}_Y$ (and by Lemma 4.5 again), open with repect to $\mathcal{B}_Y$.

On the other hand, since any $B \in \mathcal{B}$ is open in X, any subset in $\mathcal{B}_Y$ is open in the subspace topology. Then by Lemma 4.10, any open subset with respect to $\mathcal{B}_Y$ is also open in th subspace topology.

The claim about subbases follows from the fact about the topological bases and $\cap(Y \cap S_i) = Y \cap (\cap S_i)$.

□

The subspace topology also has the following useful interpretation.

Lemma 5.8 *Let X be a topological space and $Y \subset X$ is a subset. The subspace topology on Y is the coarsest topology such that the inclusion map $i(y) = y : Y \to X$ is continuous.*

Proof: Note that for any subset $U \subset X$, the preimage $i^{-1}(U) = U \cap Y$. Therefore for i to be continuous is equivalent to $U \cap Y$ is an open subset of Y for any open subset U of X. The subspace topology on Y is clealy the coarsest topology having this property.

□

Example 5.15 The usual topology on $\mathbf{R}$ is given by open intervals. By taking intersections with $[0, 1]$, we get the following topological basis for the subspace topology on $[0, 1]$:

$$\{(a, b) : 0 \le a < b \le 1\} \cup \{[0, b) : 0 < b \le 1\} \cup \{(a, 1] : 0 \le a < 1\} \cup \{[0, 1]\}.$$

Example 5.16 Let have the topology induced by the topological basis $\mathcal{B}_5$ in Example 4.2. We would like to find out the subspace topology on a straight line L in $\mathbf{R}^2$.

We describe the topology by naturally identifying L with the real line. We need to split our discussion into two cases. If L is not vertical, then the topological basis on L is given by $\mathcal{B}_4$ in Example 4.1, which induces the lower limit topology. If L is vertical, then the topological basis on L is given by $\mathcal{B}_1$ in Example 4.1, which induces the usual topology.

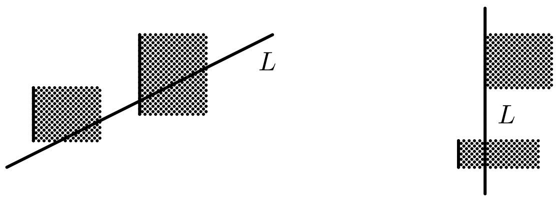


Figure 51: subspace topology on $L \subset \mathbf{R}_l \times \mathbf{R}$

Example 5.17 Let $Y \subset C[0,1]$ be polynomials of degree ≤ 2. Then the pointwise convergent topology induces a subspace topology on Y. We claim that the natural identification $p(a,b,c) = a + bt + ct^2 : \mathbf{R}^3_{\text{usual}} \to Y$ is a homeomorphism.

First we show that p is continuous. By Exercise 5.38, this is the same as $P(a,b,c) = a + bt + ct^2 : \mathbf{R}^3_{\text{usual}} \to C[0,1]_{\text{pt conv}}$ being continuous (we change p to P to indicate the change of range). Then we need to look at the preimage $P^{-1}(B(\alpha, t_0, \epsilon))$ of the topological subbasis of pointwise convergence. Note that

$$P^{-1}(B(\alpha, t_0, \epsilon)) = \{(a,b,c) : \quad |\pi(a,b,c) - \alpha| < \epsilon\} = \pi^{-1}(\alpha - \epsilon, \alpha + \epsilon),$$

where (for fixed t_0)

$$\pi(a,b,c) = a + bt_0 + ct_0^2 : \quad \mathbf{R}^3_{\text{usual}} \to \mathbf{R}_{\text{usual}}.$$

Since π is continuous, we see that $P^{-1}(B(\alpha, t_0, \epsilon))$ is an open subset of $\mathbf{R}^3_{\text{usual}}$. This completes the proof that p is continuous.

Now we construct a continuous inverse of p. The evaluations at three points give us a map $E(f) = \left(f(0), f(\frac{1}{2}), f(1)\right) : C[0,1]_{\text{pt conv}} \to \mathbf{R}^3_{\text{usual}}$. By the proof of Example 5.5, each evaluation is continuous. By Example 2.12, their combination is also continuous. Then by Exercise 5.39, the restriction $E|_Y : Y \to \mathbf{R}^3_{\text{usual}}$ is also continuous.

The composition $T = E|_Y \circ p(a,b,c) = \left(a, a + \frac{1}{2}b + \frac{1}{4}c, a + b + c\right)$ is an invertible linear tranformation. It is easy to see that both T and T^{-1} are continuous maps from $\mathbf{R}^3_{\text{usual}}$ to itself. Thus $T^{-1} \circ E|_Y$ is the continuous inverse of p.

Exercise 5.33 Consider the topological space in Example 4.15. Which subspace is homeomorphic to $\{1,2,3\}$ with the topology $\left\{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,3\}, \{1,2,3\}\right\}$. Moreover, find the closure of $\{4\}$ in the subspace $\{1,2,4\}$.

Exercise 5.34 Find the subspace topology of straight lines in the plane equipped with various topologies induced by the topological bases in Example 4.1.

Exercise 5.35 Find the subspace topology on $\mathbf{N}$ relative to each of the topological bases in Example 4.1.

Exercise 5.36 Let $Y \subset X$ have the subspace topology. Prove that $C \subset Y$ is closed if and only if $C = Y \cap D$ for some closed $D \subset X$.

Exercise 5.37 Let $Y \subset X$ be an open subset. Prove that a subset of Y is open in Y if and only if it is open in X.

Exercise 5.38 Let $Y \subset X$ have the subspace topology. Prove that a map $f : Z \to Y$ is continuous if and only if it is continuous when viewed as a map into the topological space X (i.e., if and only if $i \circ f$ is continuous).

Exercise 5.39 Let $Y \subset X$ have the subspace topology. Prove that $f : X \to Z$ is continuous $\Rightarrow$ The restriction $f|_Y : Y \to Z$ is continuous.

Exercise 5.40 Let $X = \cup_i X_i$ be a union of open subsets. Prove that $f : X \to Y$ is continuous $\Leftrightarrow$ the restrictions $f|_{X_i} : X_i \to Y$ are continuous. Is the similar statement true for closed subsets?

Exercise 5.41 Suppose $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$ be unions of open subsets.

1. Suppose $f_1 : X_1 \to Y_1$, $f_2 : X_2 \to Y_2$, $f_0 : X_1 \cap X_2 \to Y_1 \cap Y_2$ are homeomorphisms, such that $f_1(x) = f_0(x) = f_2(x)$ for $x \in X_1 \cap X_2$, prove that X and Y are homeomorphic;
2. If X_1 is homeomorphic to Y_1, X_2 is homeomorphic to Y_2, and $X_1 \cap X_2$ is homeomorphic to $Y_1 \cap Y_2$, can you conclude that X is homeomoprhic to Y?
3. Is the openness condition necessary?

Exercise 5.42 Let $Y \subset X$ have the subspace topology and let $A \subset Y$ be a subset. Can you say something between the closures of A in Y and in X?

Exercise 5.43 Prove that if we give $C[0,1]$ the L_1- or L_∞-topology, then the map p in Example 5.17 is still a homeomorphism. In fact, the homeomorphism still holds if we consider polynomials of other degrees.

We end the section with a discussion about subspaces in metric spaces.

Let (X, d) be a metric space. Let $Y \subset X$ be a subset. Then we may construct two topologies on Y.

1. The restriction d_Y of d on Y is also a metric. The metric space (Y, d_Y) can be considered as a *subspace from the metric viewpoint*. This metric induces a topology $\mathcal{T}_d$ on Y, with topological basis $\mathcal{B}_d$ consisting of balls of the form $B(y, \epsilon) \cap Y$, where $B(y, \epsilon)$ denotes a ball in X, and $y \in Y$, $\epsilon > 0$.

2. d induces a topology on X, and this further induces a subspace topology $\mathcal{T}_s$ on Y, with topological basis $\mathcal{B}_s$ consisting of balls of the form $B(x, \epsilon) \cap Y$, where $x \in X$, $\epsilon > 0$.

We claim that $\mathcal{T}_d = \mathcal{T}_s$. First from the descriptions of two topological bases, we see that $\mathcal{B}_d \subset \mathcal{B}_s$. This implies $id : (Y, \mathcal{T}_s) \to (Y, \mathcal{T}_d)$ is continuous. On the other hand, to prove $id : (Y, \mathcal{T}_d) \to (Y, \mathcal{T}_s)$ is continuous, it is sufficient to show the inclusion $i : (Y, d_Y) \to (X, d)$ is continuous (see Exercise 5.38). Then it follows from Exercise 2.44 and $d(i(y_1), i(y_2)) = d_Y(y_1, y_2)$ that i is indeed continuous.

The equivalence of the two topologies illustrates the compatibility between metric spaces and subspace topology.

5.4 Product

Given topological spaces X and Y, how can we make the product $X \times Y$ into a topological space? Naturally, we would like to have

$$U \subset X \text{ and } V \subset Y \text{ are open} \quad \Longrightarrow \quad U \times V \subset X \times Y \text{ is open.}$$

However, the collection

$$\mathcal{B}_{X \times Y} = \{U \times V : \quad U \subset X \text{ and } V \subset Y \text{ are open}\}$$

does not satisfy the second condition in the definition of topology. On the other hand, the collection is indeed a topological basis (by Lemma 4.2, for example).

Definition 5.9 The *product topology* on $X \times Y$ is the topology induced from the topological basis $\mathcal{B}_{X\times Y}$.

Lemma 5.10 *If $\mathcal{B}_X$ and $\mathcal{B}_Y$ are topological bases for X and Y, then*

$$\mathcal{B}_X \times \mathcal{B}_Y = \{B_1 \times B_2 : \quad B_1 \in \mathcal{B}_X, B_2 \in \mathcal{B}_Y\}$$

is a topological basis for the product topology.

Proof: We appy Lemma 4.8 to compare $\mathcal{B}_{X\times Y}$ and $\mathcal{B}_X \times \mathcal{B}_Y$. First for any $B_1 \times B_2 \in \mathcal{B}_X \times \mathcal{B}_Y$, B_1 and B_2 are open in X and Y (see Exercise 4.10). Thus $\mathcal{B}_X \times \mathcal{B}_Y \subset \mathcal{B}_{X\times Y}$, and $\mathcal{B}_X \times \mathcal{B}_Y$ induces coarser topology than $\mathcal{B}_{X\times Y}$. Conversely, for any $(x, y) \in U \times V \in \mathcal{B}_{X\times Y}$, we have $x \in U$ and U is open in X. Therefore we can find $B_1 \in \mathcal{B}_X$, such that $x \in B_1 \subset U$. Similarly, we can find $B_2 \in \mathcal{B}_Y$, such that $y \in B_2 \subset V$. Then we have

$$(x, y) \in B_1 \times B_2 \subset U \times V \quad \text{for some } B_1 \times B_2 \in \mathcal{B}_X \times \mathcal{B}_Y.$$

This shows that $\mathcal{B}_{X\times Y}$ induces coarser topology than $\mathcal{B}_X \times \mathcal{B}_Y$.

□

Example 5.18 Let $X = \{1, 2\}$ have the topology $\mathcal{T}_2$ in Example 4.14. The product topology on $X \times X$ is illustrated in Figure 52.

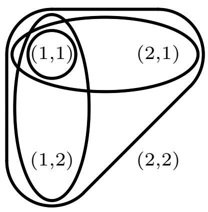


Figure 52: product topology

Example 5.19 The product topology on $\mathbf{R}^2$ of two copies of $\mathbf{R}_{\text{usual}}$ has a topological basis given by open rectangles ($\mathcal{B}_1$ in Example 4.2). This topology is simply $\mathbf{R}^2_{\text{usual}}$.

The topologies on $\mathbf{R}^2$ induced by the topological bases $\mathcal{B}_4$ and $\mathcal{B}_5$ in Example 4.2 are the product topologies $\mathbf{R}_{\text{lower limit}} \times \mathbf{R}_{\text{lower limit}}$ and $\mathbf{R}_{\text{lower limit}} \times \mathbf{R}_{\text{usual}}$.

Example 5.20 For any topological space X, we claim the *diagonal map* $\Delta(x) = (x, x) : X \to X \times X$ is continuous. According to Lemma 5.2, we only need to verify that $\Delta^{-1}(U \times V)$ is open for any open U and V in X. Since $\Delta^{-1}(U \times V) = U \cap V$, this is indeed the case.

As a matter of fact, the diagonal map is an *embedding*[8]. In other words, the map $\delta(x) = (x, x) : X \to \Delta(X)$ is a homeomorphism, where the image $\Delta(X)$ has the subspace topology.

By Exercise 5.38 and the continuity of Δ, we see that δ is continuous. To see $\delta^{-1}(x, x) = x$ is also continuous, we need to check that for any open $U \subset X$, $\delta(U)$ is an open subset of $\Delta(X)$. Since we clearly have $\delta(U) = \Delta(X) \cap (U \times U)$, by the openness of $U \times U$ in $X \times X$, we see that $\delta(U)$ is indeed open.

[8]The map P in Example 5.17 is also an embedding.

Example 5.21 We show that the total evaluation map (as opposed to the evaluation at 0 only, in Example 5.5)

$$E(f,t) = f(t) : \quad C[0,1]_{\text{pt conv}} \times [0,1]_{\text{usual}} \to \mathbf{R}_{\text{usual}}$$

is not continuous. According to Lemmas 5.3 and 5.10, the continuity would mean the following: Given f, t, and $\epsilon > 0$, there is $B(a_1,\cdots,a_n,t_1,\cdots,t_n,\delta_1)$ and $\delta_2 > 0$, such that $f \in B(a_1,\cdots,a_n,t_1,\cdots,t_n,\delta_1)$ and

$$g \in B(a_1,\cdots,a_n,t_1,\cdots,t_n,\delta_1), \quad |t'-t| < \delta_2 \quad \Longrightarrow \quad |g(t') - f(t')| < \epsilon.$$

To see why such a statement cannot be true, for any choice of $B(a_1,\cdots,a_n,t_1,\cdots,t_n,\delta_1)$ and δ_2, we can always first find some t' such that

$$t' \neq t_i, \quad \text{for} \quad i = 1,\cdots,n; \quad \text{and} \quad |t'-t| < \delta_2.$$

Then we can find a continuous function g such that

$$g(t_i) = a_i \quad \text{for} \quad i = 1,\cdots,n; \quad \text{and} \quad |g(t') - f(t')| > \epsilon.$$

This g violates the statement we want to have. As a result, E is not continuous.

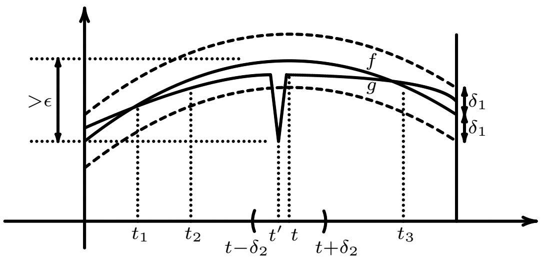


Figure 53: evaluation is not continuous

Exercise 5.44 Prove that $\mathcal{S} = \{U \times Y : U \subset X \text{ is open}\} \cup \{X \times V : V \subset Y \text{ is open}\}$ is a topological subbasis for the product topology on $X \times Y$.

Exercise 5.45 Prove that the product topology is the coarsest topology such that the projections $X \times Y \to X$ and $X \times Y \to Y$ are continuous.

Exercise 5.46 Prove that the projections $X \times Y \to X$ and $X \times Y \to Y$ map open subsets to open subsets.

Exercise 5.47 Let $f : X \to Y$ and $g : X \to Z$ be maps. Let $h(x) = (f(x), g(x)) : X \to Y \times Z$. Prove that h is continuous $\Leftrightarrow$ f and g are continuous.

Exercise 5.48 Prove that if $f, g : X \to \mathbf{R}_{\text{usual}}$ are continuous, then for any constants α, β, the linear combination $\alpha f + \beta g$ and the product fg are also continuous. Then use this to show that if $f, g, h : X \to \mathbf{R}_{\text{usual}}$ are continuous, then $\{x : f(x) < g(x) < h(x)\}$ is an open subset of X, and $\{x : f(x) \leq g(x) \leq h(x)\}$ is a closed subset.

Exercise 5.49 Let X and Y topological spaces. Fix $b \in Y$ and consider the subset $X \times b \subset X \times Y$. With subspace topology, $X \times b$ is a topological space. Prove that the natural map $i(x) = (x, b) : X \to X \times b$ is a homeomorphism (i.e., $x \mapsto (a, b)$ is am embedding of X into $X \times Y$).

Exercise 5.50 The *graph map* of $f : X \to Y$ is $\gamma(x) = (x, f(x)) : X \to X \times Y$.

1. Prove that f is continuous $\Leftrightarrow$ γ is continuous;

2. Prove that if f is continuous, then the graph map is an embedding.

Exercise 5.51 Let $\mathbf{R}^2$ have the product topology of the topological bases $\mathcal{B}_2$ and $\mathcal{B}_5$ in Example 4.1. Find the closure of a straight line in $\mathbf{R}^2$.

Exercise 5.52 Let $\mathbf{R}^2$ have the product topology of the usual topology and the finite complement topology. Find the closure of a solid triangle in $\mathbf{R}^2$.

Exercise 5.53 Describe a homeomorphism between $[0,1) \times [0,1)$ and $[0,1] \times [0,1)$, both having the usual topology.

Note: This shows that if $X \times Z$ is homeomorphic to $Y \times Z$, it does not necessarily follow that X is homeomorphic to Y.

Exercise 5.54 Prove that if the topology on $C[0,1]$ is replaced by the L_∞-topology, then the evaluation map in Example 5.21 is continuous. What about L_1-topology?

Exercise 5.55 Prove true statements and provide counterexample to false statements:

1. $A \subset X$ and $B \subset Y$ are closed $\Rightarrow A \times B \subset X \times Y$ is closed;
2. $A \times B \subset X \times Y$ is closed $\Rightarrow A \subset X$ and $B \subset Y$ are closed;
3. (x,y) is a limit point of $A \times B \subset X \times Y \Rightarrow x$ is a limit point of A and y is a limit point of B;
4. x is a limit point of A and y is a limit point of $B \Rightarrow (x,y)$ is a limit point of $A \times B \subset X \times Y$;
5. $\overline{A \times B} = \bar{A} \times \bar{B}$;
6. $(f,g) : X \times Y \to Z \times W$ is continuous $\Leftrightarrow f : X \to Z$ and $g : Y \to W$ are continuous.

We end the section with a discussion about products in metric spaces.

Let (X, d_X) and (Y, d_Y) be metric spaces. Then

$$d_{X\times Y}((x_1,y_1),(x_2,y_2)) = \max\{d_X(x_1,x_2), d_Y(y_1,y_2)\}$$

is a metric on $X \times Y$. This is one natural way of introducing product metric. We claim that this metric induces the product topology.

Denote by $\mathcal{B}_X$, $\mathcal{B}_Y$, $\mathcal{B}_{X\times Y}$ the topological bases of balls in respective metrics. Since the balls in $\mathcal{B}_{X\times Y}$ are of the form $B((x,y),\epsilon) = B(x,\epsilon) \times B(y,\epsilon)$, which are in $\mathcal{B}_X \times \mathcal{B}_Y$, we see that $\mathcal{B}_{X\times Y}$ induces coarser topology than $\mathcal{B}_X \times \mathcal{B}_Y$. On the other hand, to prove $\mathcal{B}_X \times \mathcal{B}_Y$ induces coarser topology than $\mathcal{B}_{X\times Y}$, it suffices to show that $id : (X \times Y, d_{X\times Y}) \to (X \times Y, \text{product topology})$ is continuous. By Exercise 5.47, this is equivalent to the projections $(X \times Y, d_{X\times Y}) \to (X, d_X)$ and $(X \times Y, d_{X\times Y}) \to (Y, d_Y)$ being continuous. It follows from Exercise 2.44 and $d_X(x_1,x_2) \le d_{X\times Y}((x_1,y_1),(x_2,y_2))$ that the first projection is indeed continuous. Similarly, the second projection is also continuous.

The equivalence of the two topologies illustrates the compatibility between metric spaces and product topology.

Finally, we note that there are other natural ways of introducing product metric, such as $d_{X\times Y}((x_1,y_1),(x_2,y_2)) = \sqrt{d_X(x_1,x_2)^2 + d_Y(y_1,y_2)^2}$. But these metrics usually induce the same topology as the metric above.

5.5 Quotient

The sphere is obtained by collapsing the edges of a disk to a point. By gluing the two opposing edges of a rectangle, we get either a cylinder or Möbius[9] band. If we further glue the two ends of the cylinder together, we get either a torus or a Klein[10] bottle.

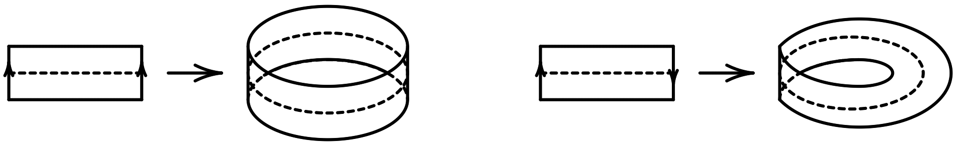


Figure 54: glue two ends of the rectangle together in two ways

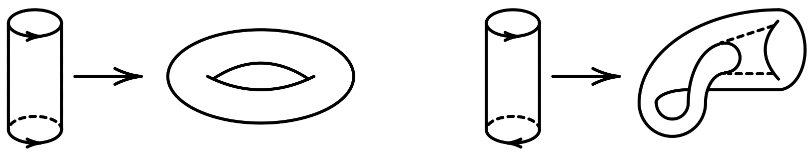


Figure 55: glue two ends of the cylinder together in two ways

We would like to construct topologies for the sphere, cylinder, Möbius band, etc, by making use of the usual topologies on the disks and the rectangles. The key question is: Given an *onto* map $f : X \to Y$ and a topology on X, construct a topology on Y.

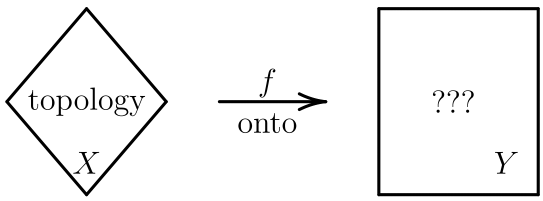


Figure 56: find reasonable topology on Y

Naturally, the topology on Y should make f a continuous map. This means that if $U \subset Y$ were open, then $f^{-1}(U) \subset X$ should also be open. This leads to the following definition.

Definition 5.11 Given a topological space X and an onto map $f : X \to Y$, the *quotient topology* on Y is

$$\mathcal{T}_Y = \{U \subset Y : \quad f^{-1}(U) \text{ is open in } X\}.$$

Strictly speaking, we need to verify that $\mathcal{T}_Y$ satisfies the three conditions for topology. This follows easily from $f^{-1}(\cup U_i) = \cup f^{-1}(U_i)$ and $f^{-1}(\cap U_i) = \cap f^{-1}(U_i)$.

For practical purpose, it is useful to charcaterizethe quotient topology in terms of topological bases.

[9]Born November 17, 1790 in Schulpforta, Saxony (now Germany); Died September 26, 1868 in Leipzig, Germany. Möbius discovered his band in 1858.

[10]Born April 25, 1849 in Düsseldorf, Prussia (now Germany); Died June 22, 1925 in Göttingen, Germany.

Lemma 5.12 *If $\mathcal{B}$ is a topological basis of X and $f : X \to Y$ is an oonto map, then $U \subset Y$ is open in the quotient topology if and only if the following holds*

$$f(x) \in U \quad \Longrightarrow \quad \text{there is } B \in \mathcal{B} \text{ such that } x \in B \text{ and } f(B) \subset U$$

Proof: The openness of $U \subset Y$ means

$$x \in f^{-1}(U) \quad \Longrightarrow \quad x \in B \subset f^{-1}(U) \quad \text{for some } B \in \mathcal{B}.$$

The left side is equivalent to $f(x) \in U$. The right side is equivalent to $x \in B$ and $f(B) \subset U$ for some $B \in \mathcal{B}$.

□

Figure 57 shows some open subsets in the quotient topology on the sphere and the cylinder. You can see they are consistent with the intuition. One may similarly consider the topologies on the Möbius band, the torus, and the Klein bottle, and find out that they are also consistent with the intuition.

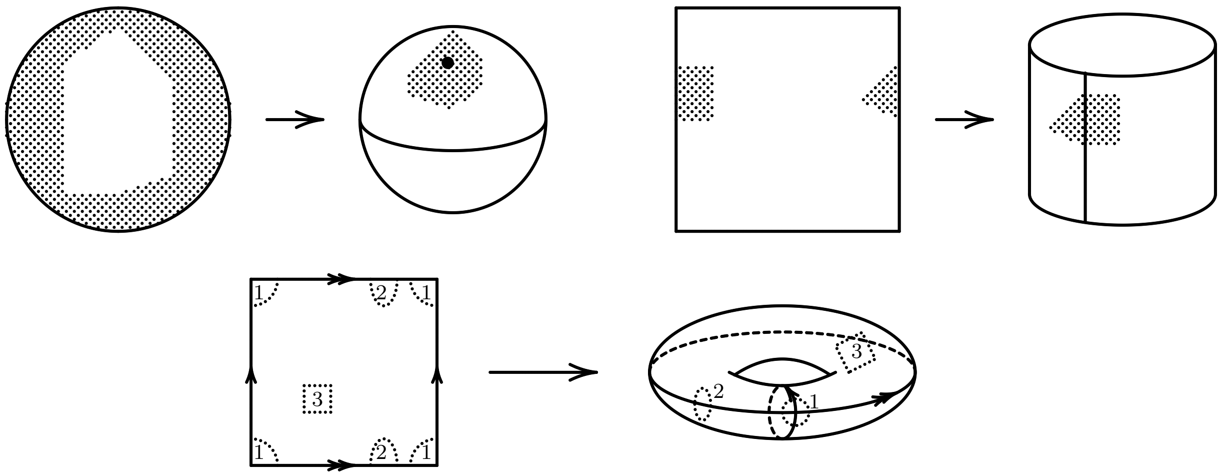


Figure 57: open subsets in the quotient topology

Example 5.22 Consider the map

$$f(x) = \begin{cases} - & x < 0 \\ 0 & x = 0 \\ + & x > 0 \end{cases} : \quad \mathbf{R} \to \{+, 0, -\}.$$

If $\mathbf{R}$ has the usual topology, then the quotient topology is given in Figure 58.

Example 5.23 In Example 5.4, we have seen that the summation map $\sigma(x, y) = x + y : \mathbf{R}^2 \to \mathbf{R}$ is continuous, if $\mathbf{R}^2$ has the product of lower limit topologies and $\mathbf{R}$ has the lower limit topology.

Now we keep the same topology on $\mathbf{R}^2$ and ask for the quotient topology on $\mathbf{R}$. The conclusion of Example 5.4 tells us that for any open subset $U \subset \mathbf{R}_{\text{lower limit}}$, $\sigma^{-1}(U)$ is open. Therefore by definition of quotient topology, U is also open in $\mathbf{R}_{\text{quotient}}$. We thus conclude that $\mathbf{R}_{\text{quotient}}$ is finer than $\mathbf{R}_{\text{lower limit}}$. On the other hand, we consider the x-axis in $\mathbf{R}^2$, which is given by the map $i(x) = (x, 0) : \mathbf{R}_{\text{lower limit}} \to \mathbf{R}^2$. By Example 5.16, i is an embedding and is continuous in particular. Therefore $\sigma \circ i(x) = x : \mathbf{R}_{\text{lower limit}} \to \mathbf{R}_{\text{quotient}}$ is continuous. This implies that $\mathbf{R}_{\text{quotient}}$ is coarser than $\mathbf{R}_{\text{lower limit}}$. Combined with earlier conclusion, we see that the quotient topology is the lower limit topology.

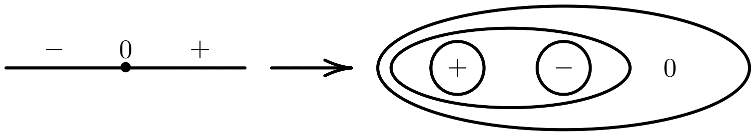


Figure 58: a quotient topology on three points

Example 5.24 Let $\mathbf{R}' = (\mathbf{R} - \{0\}) \cup \{0_+, 0_-\}$ be the *line with two origins*. We have an obvious onto map $f : \mathbf{R} \coprod \mathbf{R} \to \mathbf{R}'$ that sends the origin of the first $\mathbf{R}$ to 0_+ and sends the origin of the second $\mathbf{R}$ to 0_-. We give $\mathbf{R} \coprod \mathbf{R}$ the obvious usual topology: The open subsets are of the form $U \coprod V$, with U and V being open subsets of respective copies of $\mathbf{R}_{\text{usual}}$.

Let us study the openness of a subset $U \subset \mathbf{R}'_{\text{quotient}}$. We may write $U = V \cup O$, where V is a subset of $\mathbf{R} - \{0\}$ and $O \subset \{0_+, 0_-\}$. If $O = \emptyset$ (so that $U = V$), then $f^{-1}(U) = V \coprod V$. Therefore U is open if and only if V is open. If $O = \{0_+\}$, then $f^{-1}(U) = (V \cup \{0\}) \coprod V$. Therefore U is open if and only if $V \cup \{0\}$ is open. Similar argument can be made for the cases $O = \{0_-\}$ or $O = \{0_+, 0_-\}$.

In conclusion, the open subsets in $\mathbf{R}'_{\text{quotient}}$ are either open subsets of $\mathbf{R} - \{0\}$, or open subsets of $\mathbf{R}$ that include at least one (or both) of 0_+ and 0_-.

Figure 59: open subsets in real line with two origins

Example 5.25 We try to find the quotient topology for the onto map $f(x) = x^2 : \mathbf{R}_{\text{lower limit}} \to [0, \infty)$. According to Lemma 5.12, a subset $U \subset [0, \infty)$ is open means

$$f(x) \in U \implies f[x, x + \epsilon) \subset U \quad \text{for some } \epsilon > 0.$$

Note that for any $a \in U$, $a \neq 0$, we have a positive square root $x = \sqrt{a}$ and a negative one $-x$. For $0 < \epsilon < x$, the statement above becomes

$$x^2 = a \in U \implies [a, (x+\epsilon)^2) \quad \text{and} \quad ((-x+\epsilon)^2, a] \subset U \quad \text{for some } x > \epsilon > 0.$$

We note that the right side means $(a - \epsilon_1, a + \epsilon_2) \subset U$, where $\epsilon_1 = \epsilon(2x - \epsilon)$ and $\epsilon_2 = \epsilon(2x + \epsilon)$. Similarly, if $0 \in U$, then we have

$$0 \in U \implies [0, \epsilon^2) \subset U \quad \text{for some } \epsilon > 0.$$

Thus we see that the openness of U in the quotient topology is the same as the openness in the usual topology.

Exercise 5.56 What are the quotient topologies of the discrete and the trivial topologies?

Exercise 5.57 Consider the onto map $[0, 1] \to S^1, t \mapsto (\cos 2\pi t, \sin 2\pi t)$ obtained by identifying 0 and 1. If we give $[0, 1]$ the usual topology, prove that the quotient topology on S^1 is also the usual one.

Exercise 5.58 In Example 5.22, what is the quotient topology on $\{+, 0, -\}$ if $\mathbf{R}$ has the topology generated by a topological basis in Example 4.1?

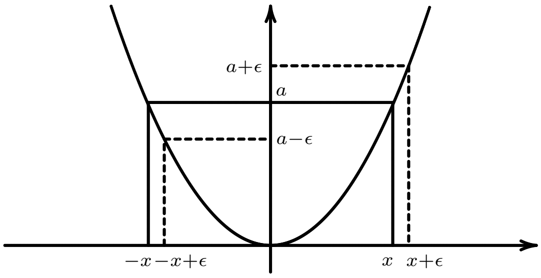


Figure 60: quotient topology induced by x^2

Exercise 5.59 In Example 5.23, what is the quotient topology on $\mathbf{R}$ if the summation is replaced by difference $\delta(x, y) = x - y : \mathbf{R}^2 \to \mathbf{R}$? What if we change the topology on $\mathbf{R}^2$ by the topology generated by the topological basis $\mathcal{B}_5$ in Example 4.2?

Exercise 5.60 The evaluation $E(f) = (f(0), f(\frac{1}{2}), f(1)) : C[0,1]_{\text{pt conv}} \to \mathbf{R}^3$ at three points is an onto map. Prove that the quotient topology on $\mathbf{R}^3$ is the usual one.

Hint: You may imitate the discussion in Example 5.23. The embedding i can be substituted by the map P in Example 5.17.

Exercise 5.61 In Example 5.25, what is the quotient topology if $\mathbf{R}$ is the usual topology?

Exercise 5.62 Prove that the quotient topology is the finest topology such that the *quotient map* f is continuous.

Exercise 5.63 Suppose $\mathcal{B}$ is a topological basis of X, and $f : X \to Y$ is an onto map. Show that $f(\mathcal{B}) = \{f(B) : B \in \mathcal{B}\}$ is not necessarily a topological basis of the quotient topology.

Exercise 5.64 Prove that if $f : X \to Y$ is an onto continuous map, and the image of any open subset of X is open in Y, then Y has the quotient topology. What if the open subsets are replaced by closed subsets?

The following tells us a useful relation between the continuity and the quotient topology. Also compare with Proposition 1.3.

Lemma 5.13 *Suppose g is a map that factors through an onto map f:*

$$g = hf : \quad \begin{array}{ccc} X & \xrightarrow{g} & Z \\ {}_f\searrow & & \nearrow_h \\ & Y & \end{array}$$

Then with the quotient topology on Y, g is continuous $\Leftrightarrow$ h is continuous.

Proof: For any open $U \subset Z$, we have

$$\begin{aligned} & h^{-1}(U) \subset Y \text{ is open} \\ \Leftrightarrow\ & f^{-1}h^{-1}(U) \subset Y \text{ is open (definition of quotient topology)} \\ \Leftrightarrow\ & g^{-1}(U) \subset Y \text{ is open (by } f^{-1}h^{-1}(U) = g^{-1}(U)) \end{aligned}$$

□

Example 5.26 Applying Lemma 5.13 to Example 1.23, we see that a continuous periodic map $f : \mathbf{R}_{\text{usual}} \to Y$ with period 1 is equivalent to a continuous map $\bar{f} : S^1 \to Y$, where S^1 has the quotient topology.

It is easy to see that the open arcs $\{e^{i\theta} : \theta_1 < \theta < \theta_2\}$ form a topological basis of the quotient topology. Moreover, these are exactly the intersection of open balls in $\mathbf{R}^2_{\text{usual}}$ with the circle. Therefore the quotient topology on S^1 is the usual one, the one obtained by thinking of S^1 as a subspace of $\mathbf{R}^2_{\text{usual}}$.

Exercise 5.65 Let $f : X \to Y$ be an onto continuous map, such that for any topological space Z and map $h : Y \to X$, h is continuous $\Leftrightarrow$ $hf : X \to Z$ is continuous. Is it necessarily true that Y has the quotient topology?

Note: This is the converse of Lemma 5.13.

Exercise 5.66 Suppose X is a topological space, and $f : X \to Y$, $g : Y \to Z$ are onto maps. Then f induces a quotient topology on Y, and with this quotient topology, g further induces a quotient topology on Z. On the other hand, the composition $gf : X \to Z$ induces a quotient topology on Z. Prove that the two topologies on Z are the same.

Note: In other words, the quotient topology of a quotient topology is still a quotient topology.

An important application of the quotient topology is the topology on spaces obtained by attaching one space to another. Specifically, let X and Y be topological spaces. Let $A \subset Y$ be a subset and $f : A \to X$ be a continuous map. Then we may construct a set $X \cup_f Y$ by starting with the disjoint union $X \coprod Y$ and then identifying $a \in A \subset X$ with $f(a) \in Y$. You may imagine this as *attaching* (or *gluing*) Y to X along A.

For example, let X and Y be two copies of the disk B^n. Let $A = S^{n-1} \subset Y$ be the boundary sphere. Let $f(a) = a : A \to X$ be the inclusion of the boundary sphere into X. Then $X \cup_f Y$ is the sphere S^{n-1}. For another example, let X be a single point and $Y = B^n$. Let $A = S^{n-1} \subset Y$ be the boundary sphere. Let $f(a) = a : A \to X$ be constant map to the point. Then $X \cup_f Y$ is also the sphere S^{n-1}.

Strictly speaking, we need to describe the set $X \cup_f Y$ as the quotient set of $X \coprod Y$. In $X \coprod Y$, define $a \sim f(a)$ for any $a \in X$. This is not yet an equivalence relation because we should also have $f(a) \sim a$, and $x \sim x$, $y \sim y$. More importantly, if $a, b \in A$ satisfy $f(a) = f(b)$, then we should have $f(a) \sim f(b)$. After adding all these additional relations, then we get an equivalence relation on $X \coprod Y$, call the *equivalence relation induced by* $a \sim f(a)$. Now we can define $X \cup_f Y$ as the quotient set of this equivalence relation.

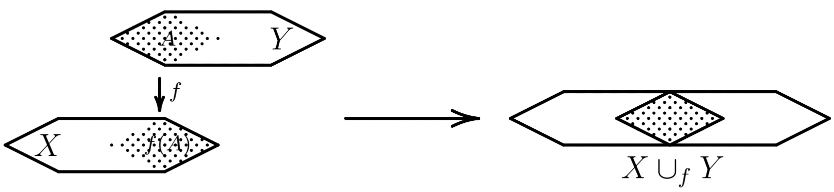


Figure 61: glue Y to X along A

Now we would like to put a topology on $X \cup_f Y$. One natural choice would be the quotient topology of the following natural topology on $X \coprod Y$: Open subsets are of the form $U \coprod V$, where $U \subset X$ and $V \subset Y$ are open. Indeed in our two examples about spheres, if we take the natural topologies on X and Y, then the quotient topology on the sphere is also the natural one. We need to be a little bit more careful in choosing the quotient topology, as illustraed by the example below. For the applications in the subsequent sections, we do not need to worry much because X and Y are often bounded and closed subsets of Euclidean spaces.

Example 5.27 Let X and Y be subsets of a topological space Z, and $Z = X \cup Y$. Then X and Y have the subspace topologies. We think of $A = X \cap Y$ as a subset of Y and let $f : A \to X$ be the inclusion. Then $X \cup_f Y = X \cup_A Y$ is easily identified with Z.

We have the quotient topology on Z on the one hand, and the original topology on the other hand. For the two topologies to be the same, we need $U \subset Z$ open in the original topology $\Leftrightarrow U \cap X \subset X$, $U \cap Y \subset Y$ open in respective subspace topologies. The statement is indeed true if both X and Y are open subsets of Z. Noting that the statement is equivalent to the similar statement with open replaced by closed, it is also true if both X ad Y are closed subspaces of Z.

In general, however, the statement is not true. Consider $Z = \mathbf{R}_{\text{usual}}$, $X = (-\infty, 0]$, $Y = (0, \infty)$, $A = \emptyset$. Then $U \subset \mathbf{R}_{\text{quotient}}$ is open if and only if $U \cap (-\infty, 0]$ is open in the usual topology of $(-\infty, 0]$, and $U \cap (0, \infty)$ is opne in the usual topology of $(0, \infty)$. In particular, $U = (-1, 0]$ is open in $\mathbf{R}_{\text{quotient}}$, but not in $\mathbf{R}_{\text{usual}}$.

Exercise 5.67 Prove that in Example 5.27, the original topology on Z is coarser than the quotient topology.

6 Complex

The theory of point set topology (open, closed, continuity, connected, compact, etc.) lays down the rigorous mathematical foundation for problems of topological nature. However, the theory only gives general characterizations of topological objects and does not capture enough internal structure of specific objects. To actually solve many topological problems, one has to develop another type of theory. We have seen one such example in the theory of graphs.

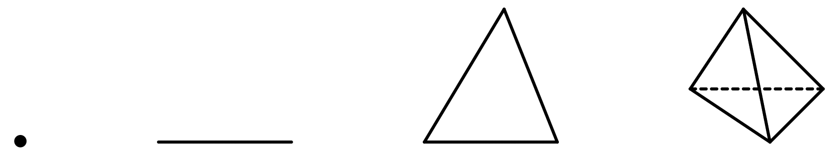
A very general strategy for solving (mathematical or nonmathematical) problems is to decompose the objects into simple pieces. By understanding the simple pieces and how they are put together to form the whole object, we may find a way of solving our problems.

For example, every integer can be decomposed into a product of prime numbers. In this way, we consider prime numbers as simple pieces and product as the method for putting simple pieces together. For another example, to study the world economy, one may study the economies of individual regions/countries and then study the trade between different regions/countries.

In a simplicial complex, we decompose a topological space into simplices (triangles at various dimensions). In a CW-complex, we decompose a topological space into cells (open balls). Such decompositions lead to the definition of Euler number in general, and more importantly, the homology theory. Other important topological decompositions, which we will not discuss in this course, include handlebody and stratification.

6.1 Simplicial Complex

Figure 62 shows some low dimensional simplices.



Δ^0=point Δ^1=interval Δ^2=triangle Δ^3=tetrahedron

Figure 62: simplices up to dimension 3

In general, suppose $v_0, v_1, \cdots, v_n$ are affinely independent vectors in a Euclidean space. In other words, the vectors $v_1 - v_0, v_2 - v_0, \cdots v_n - v_0$ are linearly independent. Then the *n-simplex* with $v_0, v_1, \cdots, v_n$ as vertices is

$$\sigma^n(v_0, v_1, \cdots, v_n) = \{\lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_n v_n : \quad 0 \le \lambda_i \le 1, \lambda_0 + \lambda_1 + \cdots + \lambda_n = 1\}.$$

For example, the vectors $0, e_1, e_2, \cdots, e_n$ in $\mathbf{R}^n$ are affinely independent. For small n, the simplices with these as vertices are shown in Figure 63.

We note that a simplex contains many simplices of lower dimension. For any subset $\{v_{i_0}, v_{i_1}, \cdots, v_{i_k}\}$ of $\{v_0, v_1, \cdots, v_n\}$, we call the simplex $\sigma(v_{i_0}, v_{i_1}, \cdots, v_{i_k})$ a *face* of the simplex $\sigma(v_0, v_1, \cdots, v_n)$. Figure 64 shows all the faces of a 2-simplex.

Exercise 6.1 How many k-dimensional faces does an n-dimensional simplex have?

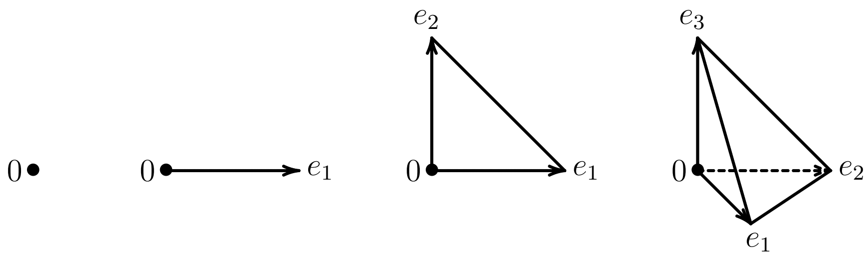


Figure 63: simplex with $0, e_1, e_2, \cdots$, as vertices

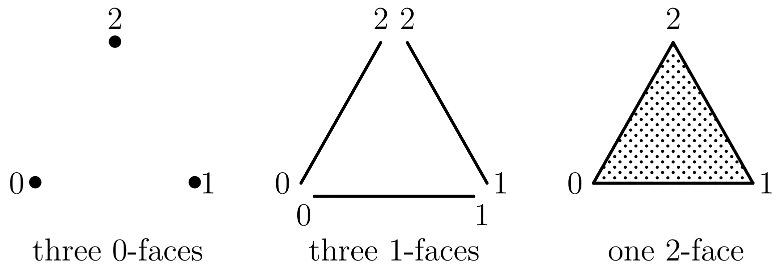


Figure 64: faces of a 2-simplex

Definition 6.1 A *simplicial complex* is a collection K of simplices in a Euclidean space $\mathbf{R}^N$, such that

1. $\sigma \in K \Rightarrow$ faces of $\sigma \in K$;
2. $\sigma, \tau \in K \Rightarrow \sigma \cap \tau$ is either empty or one common face of σ and τ;
3. K is locally finite: Every point of $\mathbf{R}^N$ has a neighborhood that intersects with only finitely many simplices in K.

Among the three conditions, the second one is the most important. It requires the simplices to be arranged nicely. Figure 65 shows some examples that do not satisfy the second condition. Another simple counterexample is given by two edges meeting at both ends. The first condition can be met by adding all the faces of existing simplices to the collection. The third condition is always satisfied when the collection is finite. In this course, we will always assume simplicial complexes are finite.

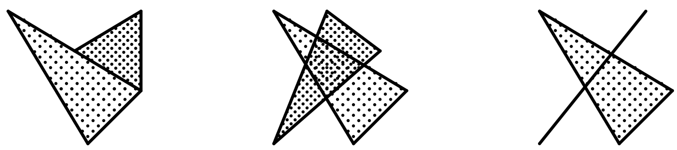


Figure 65: counterexamples of simplicial complex

Definition 6.2 Given a simplicial complex K, the union of all its simplices is called the *geometrical realization* of K and is denoted $|K|$. Conversely, suppose a subset X of $\mathbf{R}^N$ is the union of simplices of a simplicial complex K, then we call X a *polyhedron* and call K a *triangulation* of X.

A simplex Δ^n and all its faces form a simplicial complex. The simplices in this simplicial complex are in one-to-one correspondence with nonempty subsets of $\{0, 1, \cdots, n\}$. The geometrical realization of this simplicial complex is Δ^n. Therefore Δ^n is a polyhedron. Figure 66 shows many more sophisticated examples of simplicial complexes.

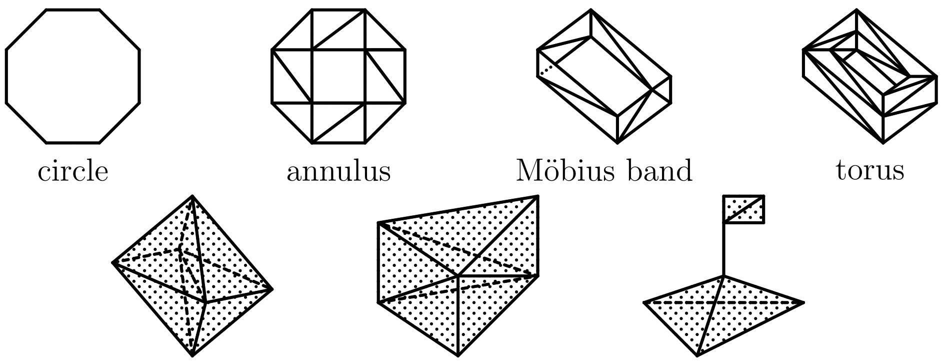


Figure 66: examples of triangulation

Note that we usually envision spaces such as circle, annulus, Möbius band, etc., as curved. To make them into polyhedra, we need to "straighten" them. See Figure 67. Since straightening is a homeomorphism, we may also think of the curved shape as polyhedra without damaging the topological study.

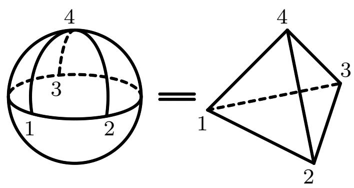


Figure 67: sphere is a polyhedron

A polyhedron may have different triangulations. For example, a sphere can be the boundary of either tetrahedron or octahedron. Figure 68 shows three different triangulations of the square polyhedron.

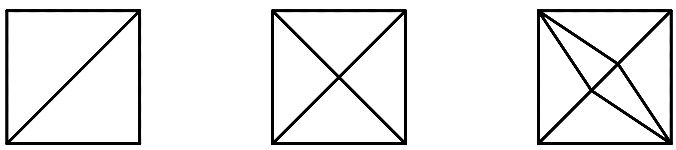


Figure 68: three triangulations of the square

Now we consider triangulations of the torus from another angle. Recall that the torus is obtained by identifying the opposite edges of a square as in Figure 69. Therefore we may construct triangulations of the torus by considering the triangulations of the square.

Specifically, we try to determine which triangulation of the square in Figure 70 give trianglations of the torus. Before detailed discussion, we need to worry about one subtle issue

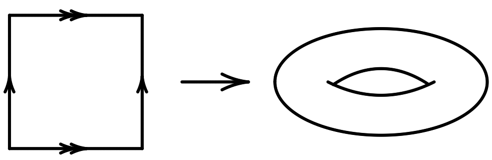


Figure 69: get a torus from a square

when triangulations are mixed with some identifications. It follows from the original definition of simplices that the vertices of a simplex must all be distinct. Therefore in a triangulation involving some identifications, we have to make sure that for any simplex, the vertices are not identified. Thus we need to verify two key points[11]: One is the second condition in the definition of the simplicial complex. The other is that vertices of a simplex are all distinct.

In Figure 70, the first triangulation fails both key points: The vertices of σ are not distinct, and the intersection of σ and τ consists of one edge and one vertex 2. The second triangulation appears to satisfy all the conditions in the definition of simplicial complex. However, it is not a triangulation of the torus because the vertices of σ are identified. The third triangulation satifies the second key point. However, it fails the first key point because the intersection of σ and τ consists of one edge and one vertex 2. The similar problem happens for the intersection of σ and ρ, which consists of two edges. Only the fourth one is a triangulation of the torus.

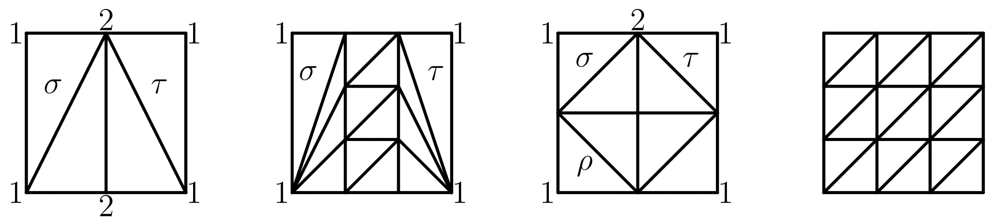


Figure 70: which is a triangulation of the torus

Exercise 6.2 In Figure 71, which are triangulations of the torus?

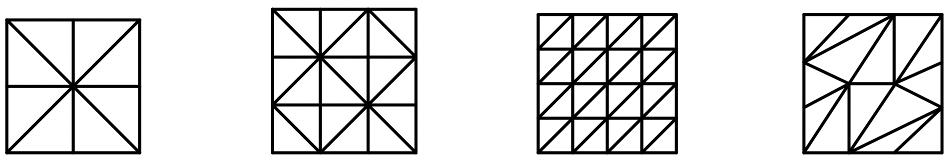


Figure 71: which are triangulations of the torus

Exercise 6.3 The Klein bottle is obtained by identifying the opposite edges of a square as in Figure 72. Find a triangulation of the square that gives a triangulation of the Klein bottle.

Exercise 6.4 The real projective space P^2 is obtained by identifying the opposite points of the boundary circle (see Figure 73). Find a triangulation of P^2.

[11]Strictly speaking, we need to prove that if the two key points are satisfied, then we can "straighten" these simplices in a Euclidean space of sufficiently high dimension. The fact is true and the detailed proof is omitted.

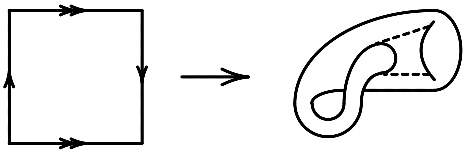


Figure 72: get Klein bottle from a square

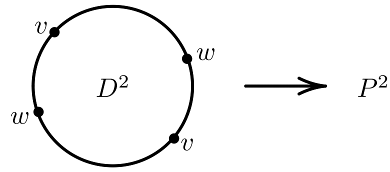


Figure 73: identifying the opposite points of the boundary circle to get a projective space

6.2 CW-Complex

Simplices are the simplest pieces you can use as the building block for topological spaces. However, simplicity comes at a cost: the decomposition often gets very complicated. Here is the analogy: The economy of individual person is much simpler than the world economy. If you use individual person as the building block for the world economy, the whole system would become so complicated that you cannot reasonably study the world economy. On the other hand, if you use countries as the (more sophisticated) building blocks for the world economy, then the whole system is more manageable, and you are able to study the world economy.

The concept of CW-complexes addresses this problem. In a CW-complex, the building blocks are *cells*, which are parts of the space homeomorphic to the closed ball B^n. Thus a *CW-structure* on a space X is a sequence of subspaces

$$X^0 \subset X^1 \subset X^2 \subset \cdots \subset X^n \subset \cdots \subset X = \cup X^n,$$

such that X^n is obtained from X^{n-1} by *attaching* (or *gluing*) some n-dimensional cells, and certain point-set topological conditions are satisfied[12]. If $X = X^n$, then we say X is an *n-dimensional CW-complex*.

A graph G can be considered as a 1-dimensional CW-complex. G^0 is all the vertices, and $G = G^1$ is obtained by attaching the end points of edges to the vertices. The sphere S^2 is a 2-dimensional CW-complex, with one 0-cell and one 2-cell. The torus T^2 is a 2-dimensional CW-complex, with one 0-cell, two 1-cells, and one 2-cell.

Exercise 6.5 Find CW-structures for the Möbius band, the Klein Bottle and the projective space P^2.

Strictly speaking, the process of attaching a cell is illustrated in Figure 75. Given a topological space X, and a continuous map $f : S^{n-1} \to X$, we construct a set ($\coprod$ = disjoint union)

$$X \cup_f B^n = X \coprod \dot{B}^n, \quad \dot{B}^n = B^n - S^{n-1} = \text{interior of unit ball},$$

[12]We have an obvious map from the disjoint union of cells *onto* X. In case the number of cells is finite, the topology on X is the quotient topology. In the infinite case, the topology is more subtle, which we will not worry about in this course.

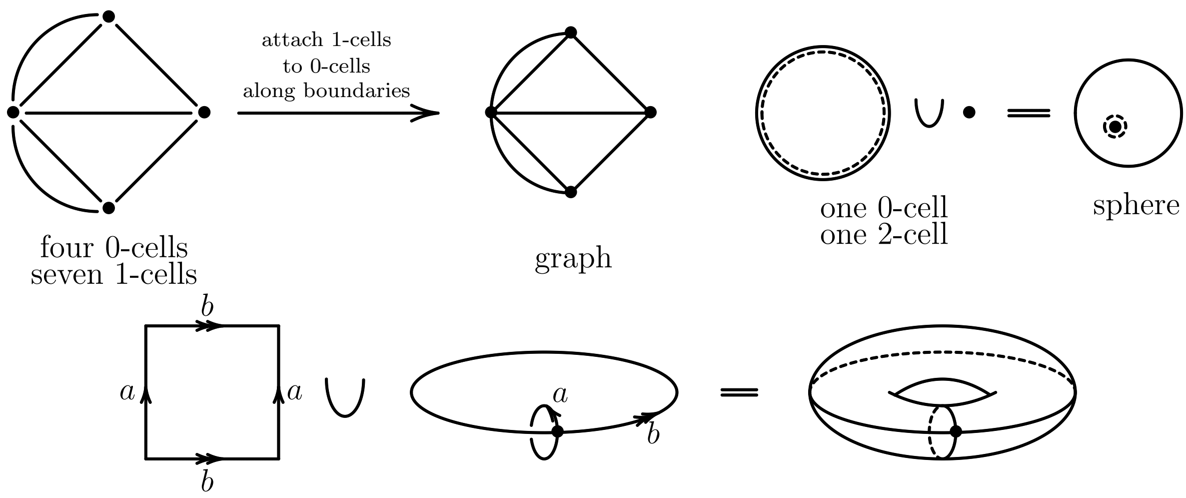


Figure 74: the CW-structures of a graph, the sphere, and the torus

where we note that the unit sphere S^{n-1} is the boundary of the unit ball B^n. We also have an onto map

$$F(x) = \begin{cases} a & a \in X \\ f(a) & a \in S^{n-1} \\ a & a \in \dot{B}^n \end{cases} \quad : \quad X \coprod B^n \to X \cup_f B^n.$$

On the disjoint union $X \coprod B^n$ we have the quotient topology

$$\{U \cup V : \quad U \subset X \text{ and } V \subset B^n \text{ are open}\}.$$

Then onto map F induces the quotient topology on $X \cup_f B^n$.

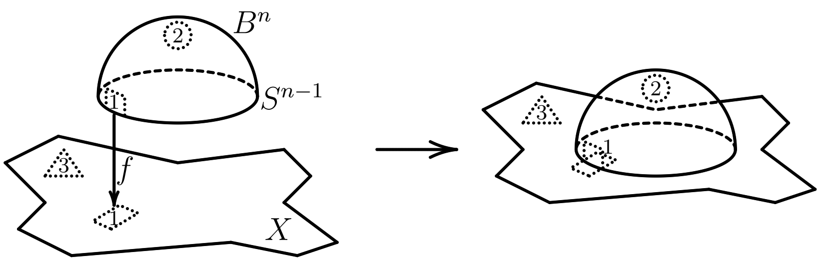


Figure 75: glue a disk and the open subsets after gluing

More generally, if we have several continuous maps $f_i : S^{n-1} \to X$, $i = 1, \cdots, k$, then we can similarly construct $X \cup_{\{f_i\}} (\coprod_{i=1}^k B^n)$. We say that this space, equipped with the quotient topology, is obtained by *attaching* k n-cells to X along the *attaching maps* $f_1, \cdots, f_k$.

Thus a *finite CW-complex* is constructed in the following steps. Start with finitely many discrete points X^0 (0-skeleton). Then attach finitely many intervals to X^0 along the ends of the intervals. The result is a graph X^1 (1-skeleton). Then attach finitely many 2-disks to X_1 along the maps from the boundary circles of the 2-disks to X^1. The result is X^2, a 2-dimensional CW-complex called 2-skeleton. We keep going and each time provide the new skeleton with the quotient topology. After finitely many steps, we have constructed the finite CW-complex (as a topological space).

The process also works for infinite CW-complexes. However, some local finiteness condition is needed in order to successfully carry out the whole construction.

Finally, we note that any triangulation is a CW-structure. The simplices in a triangulation are the cells in the CW-structure.

6.3 Projective Spaces

The n-dimensional *real projective space* is

$$P^n = \{\text{all straight lines in } \mathbf{R}^{n+1} \text{ passing through the origin}\}.$$

For example, since there is only one straight line in $\mathbf{R}^n$ passing through the origin, we see that P^0 is a single point. Moreover, the straight lines in $\mathbf{R}^2$ passing through the origin are in one-to-one correspondence with points in the upper half of the unit circle, with two ends of the upper half identified (both representing the x-axis). Clearly, this shows that P^1 is a circle.

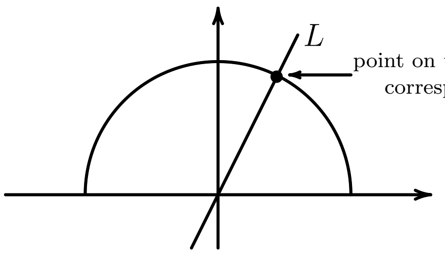


Figure 76: why P^1 is a circle

There are several useful models for the real projective space. Note that any *nonzero* vector v in $\mathbf{R}^n$ span a straight line passing through the origin (i.e., 1-dimensional linear subspace). Thus we have a map

$$f(v) = [v] = \text{span}\{v\} : \quad \mathbf{R}^{n+1} - \{0\} \to P^n,$$

which is clearly onto. We have the natural topology on $\mathbf{R}^{n+1} - \{0\}$ as a subspace of $\mathbf{R}^{n+1}$. We may use this to give P^n the quotient topology. Note that two nonzero vectors u and v span the same line if and only if $u = \lambda v$ for some $\lambda \in \mathbf{R}^* = \mathbf{R} - \{0\}$. Thus we also denote

$$P^n = \frac{\mathbf{R}^{n+1} - \{0\}}{\mathbf{R}^*}.$$

Another model is derived from the similar idea, with the additional observation that using unit vectors is sufficient to generate all straight lines. Since the unit vectors in $\mathbf{R}^{n+1}$ form the unit sphere S^n, we are led to the onto map

$$\pi(v) = [v] = \text{span}\{v\} : \quad S^n \to P^n$$

Since $u, v \in S^n$ span the same line if and only if $u = \pm v$, we also denote this process as a quotient set

$$P^n = \frac{S^n}{v \sim -v}.$$

In other words, the real projective space is obtained by identifying the antipodal points on the unit sphere.

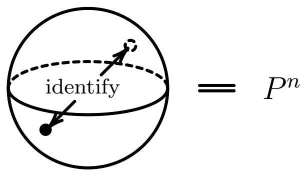


Figure 77: identifying antipodal points on the sphere gives the projective space

We also have the quotient topology on P^n induced from the subspace topology on S^n. This topology on P^n is the same as the one induced from $\mathbf{R}^{n+1} - \{0\}$. The key reason is that $\mathbf{R}^{n+1} - \{0\}$ is homeomorphic to $S^n \times (0, \infty)$.

By using the sphere model, we deduce from $S^0 = \{-1, 1\}$ that P^0 is a single point. Figure 78 shows that P^1 is homeomorphic to S^1, and the quotient map $S^1 \to P^1$ is "wrapping S^1 twice on itself". If we think of S^1 as consisting of complex numbers z of norm 1, then the quotient map is $z \to z^2$.

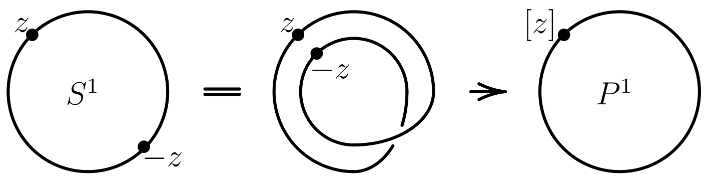


Figure 78: $S^1 \to P^1$ is wrapping S^1 twice on itself

What about P^2? The psychological difficulty in visualizing P^2 is that one has to imagine two points on S^2 as one point. We divide S^2 into upper half and lower half, and observe that any point on the lower half is identified with a point on the upper half. In other words, the map

$$\pi : \quad \text{upper } S^2 \to P^2$$

is still onto. The only identification needed is the antipodal points on the *equator* S^1. Since we know already that identifying antipodal points on S^1 will simply yield P^1, P^2 is obtained by gluing the boundary S^1 of B^2 to P^1 along the canonical projection $S^1 \to P^1$.

The discussion about P^2 applies to higher dimensional cases. In order to obtain P^n, it is sufficient to consider only the upper half B^n of the sphere S^n. We only need to identify the antipodal points of the equator S^{n-1} (which is the boundary of the upper half). However, this identification gives nothing but P^{n-1} from the equator. Therefore we see that

$$P^n = P^{n-1} \cup_\pi B^n$$

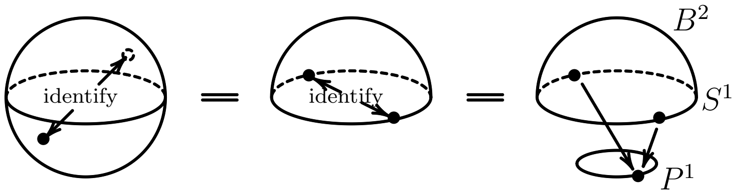


Figure 79: a model of P^2

is obtained by attaching an n-cell to P^{n-1} along the canonical projection $\pi : S^{n-1} \to P^{n-1}$. Incidentally, the discussion gives us a CW-structure of P^n. We have

$$P^0 = \{pt\} \subset P^1 = S^1 \subset P^2 \subset \cdots \subset P^{n-1} \subset P^n,$$

and P^k is obtained by attaching one k-cell to P^{k-1}.

Real projective spaces may be generalized in two ways. One is the complex (and even quaternionic) projective spaces. The other is the Grassmannians, which replaces the straight lines by linear subspaces of some fixed dimension. Here we only briefly describe the *complex projective space* CP^n. Sometimes to distinguish the real and complex projective spaces, we also denote the real projective spaces by RP^n.

Similar to the real projective space, we define

$$CP^n = \{\text{all (complex) straight lines in } \mathbf{C}^{n+1} \text{ passing through the origin}\}.$$

We have the model

$$CP^n = \frac{\mathbf{C}^{n+1} - \{0\}}{\mathbf{C}^*}, \quad \mathbf{C}^* = \mathbf{C} - \{0\}$$

given by the fact that complex straight lines are the spans of nonzero vectors. We also have the model

$$CP^n = \frac{S^{2n+1}}{S^1} = \frac{S^{2n+1}}{v \sim \lambda v, \lambda \in S^1}, \quad S^1 = \{\lambda \in \mathbf{C}, |\lambda| = 1\}$$

where we use the fact that S^{2n+1} is the unit ball in $\mathbf{C}^{n+1}$, and two unit vectors span the same complex linear subspace if and only if one is a multiple of the other by a complex number of norm 1.

In terms of the sphere model, the real projective space is obtained by thinking of (antipodal) pairs of points as single points, while the complex projective space is by thinking of circles of points as single points. Note that since S^1 is only one circle, CP^0 is a single point. As for CP^1, we may start with

$$S^3 = \{(x, y) \subset \mathbf{C}^2 : \quad |x|^2 + |y|^2 = 1\}.$$

Then the circle that (x, y) belongs to is $[x, y] = \{(\lambda x, \lambda y) : |\lambda| = 1\}$, which represents one point in CP^1. By this description, we get a map

$$h[x, y] = \begin{cases} x/y & y \neq 0 \\ \infty & y = 0 \end{cases} : \quad CP^1 \to S^2 = \mathbf{C} \cup \{\infty\}.$$

It is easy to see that $h[x, y] = h[z, w]$ if and only if (x, y) and (z, w) belong to the same circle. Therefore h is a one-to-one correspondence. It turns out that h is a homeomorphism (this can be verified either directly or by using the fact that both CP^1 and S^2 are compact Hausdorff spaces). Thus we conclude that $CP^1 = S^2$.

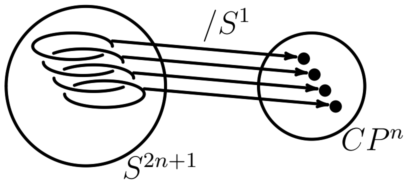


Figure 80: CP^n is obtained from S^{2n+1} by identifying circles to points

The general complex projective space has CW-structure similar to the real one. We have

$$CP^0 = \{pt\} \subset CP^1 = S^2 \subset CP^2 \subset \cdots \subset CP^{n-1} \subset CP^n,$$

and

$$CP^k = CP^{k-1} \cup_\pi B^{2k}$$

is obtained by attaching a $2k$-cell to CP^{k-1} along the canonical projection $\pi : S^{2k-1} \to CP^{k-1}$.

6.4 Euler Number

In Chapter 3, we have seen some applications of the Euler formula for the sphere and the plane. The formula is about the alternating sum of the numbers of vertices, edges, faces, etc. Note that from the CW-complex viewpoint, the vertices, edges, faces are nothing but 0-, 1-, and 2-dimensional simplices. Therefore we may generalize the Euler formula by introducing the following concept.

Definition 6.3 The *Euler number* of a finite CW-complex X is

$$\chi(X) = \sum (-1)^i (\text{number of } i\text{-cells in } X).$$

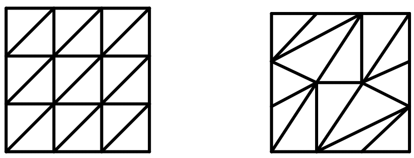
Since simplicial complexes are special cases of CW-complexes we also have the Euler number for a simplicial complex K:

$$\chi(K) = \sum (-1)^i (\text{number of } i\text{-simplices in } K).$$

If X is finitely many points, then X is a 0-dimensional CW-complex, and $\chi(X)$ =number of points in X. The closed interval has a CW-structure given by two 0-cells (two ends) and one 1-cell (the interval itself). Therefore $\chi(\text{closed interval}) = 2 - 1 = 1$. More generally, in a graph each vertex is a 0-cell, and each edge is a 1-cell. Therefore $\chi(\text{graph})$ =(number of vertices)−(number of edges). The following are more examples of CW-complexes and their Euler numbers.

CW-complex	**0-cell**	**1-cell**	**2-cell**	**k-cell**	χ
circle S^1	1	1	0	0	0
n-sphere S^n	1	0	0	$\begin{cases} 1 & k = 0, n \\ 0 & k \neq 0, n \end{cases}$	$1 + (-1)^n$
rectangle, disk, etc.	1	1	1	0 for $k \geq 3$	1
annulus	2	3	1	0 for $k \geq 3$	0
torus	1	2	1	0 for $k \geq 3$	0
real proj. space RP^n	1	1	1	1 for all $0 \leq k \leq n$	$\begin{cases} 1 & \text{even } n \\ 0 & \text{odd } n \end{cases}$
complex proj. space CP^n	1	0	1	$\begin{cases} 1 & \text{even } k \\ 0 & \text{odd } k \end{cases}$	$n + 1$

Strictly speaking, because a space (polyhedron) can have many different CW-structures (simplicial structures), the Euler number should be independent of the choice of the structure in order for the definition to be really rigorous. For example, the different triangulations of the torus in Figure 81 give the same Euler number 0. Moreover, Theorem 3.5 essentially says that the Euler number of a region on the plane without holes always have the Euler number 1, no matter how it is divided into smaller pieces by a graph. You may try more examples and convince yourself that the Euler number is indeed independent of the choice of the CW-structure.A rigorous proof of the independence of the Euler number on the choice of the CW-structure in general requires some more advanced topology theory, such as the *homology theory*. We will not develop such a theory in this course.



9−27+18=0 7−21+14=0

Figure 81: the Euler number of the torus is always 0

The following are some useful properties of the Euler number.

Lemma 6.4 *If X and Y are finite CW-complexes, then*

$$\chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(X \cap Y),$$

and

$$\chi(X \times Y) = \chi(X)\chi(Y).$$

Proof: By $X \cup Y$, we mean a CW-complex, such that X, Y, and $X \cap Y$ are unions of cells in $X \cup Y$ (thus X, Y, and $X \cap Y$ are called *CW-subcomplexes* of $X \cup Y$). Clearly, if we denote by $f_i(X)$ the number of i-cells in X, then we have

$$f_i(X \cup Y) = f_i(X) + f_i(Y) - f_i(X \cap Y).$$

Taking alternating sum of both sides over all $i = 0, 1, 2, \cdots$, we get the first equality.

By $X \times Y$, we mean the product CW-complex, in which the cells are the the products of the cells in X and in Y. The key observation here is that a product of balls is still a ball. Clearly, we have

$$f_i(X \times Y) = \sum_{j+k=i} f_j(X) f_k(Y).$$

Thus

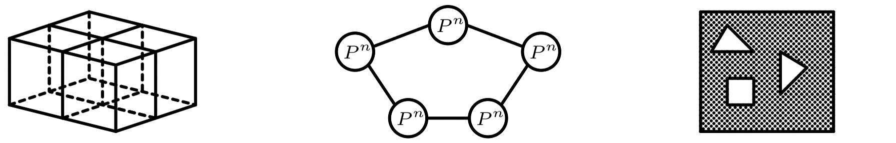
$$\begin{aligned} \chi(X \times Y) &= \sum_i (-1)^i \sum_{j+k=i} f_j(X) f_k(Y) = \sum_{j,k} (-1)^{j+k} f_j(X) f_k(Y) \\ &= \left(\sum_j (-1)^j f_j(X) \right) \left(\sum_k (-1)^k f_k(Y) \right) = \chi(X)\chi(Y). \end{aligned}$$

□

Exercise 6.6 Find the Euler number of Möbius band, Klein bottle, the n-ball B^n, and the complex projective spaces.

Exercise 6.7 Find the Euler number of the n-simplex Δ^n.

Exercise 6.8 Find the Euler number of the spaces in Figure 82.



union of 4 boxes

ring of five P^n connected by lines

square with n holes

Figure 82: find the Euler numbers

7 Surface

7.1 Manifold

Consider the spaces in Figure 83 (as subspaces of $\mathbf{R}^2$). We see that the spaces on the first row are more "regular" than the ones on the second row. In fact, we can pinpoint exactly where the irregularities are in the spaces in the second row. Such irregularities are indicated by the question marks.

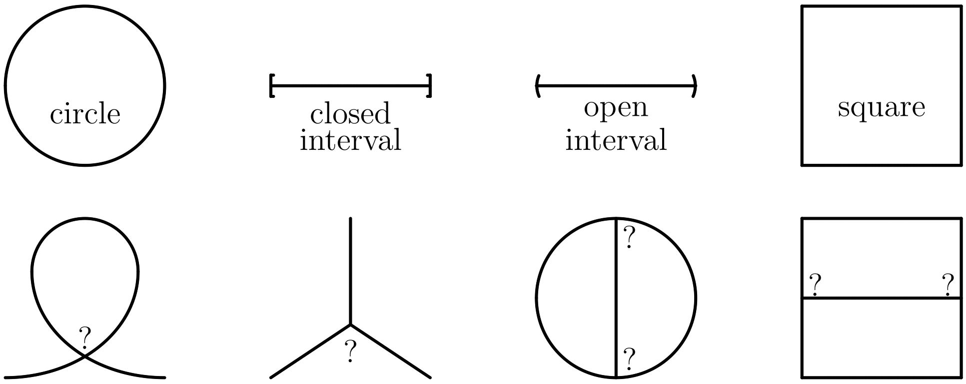


Figure 83: "regular" and "irregular" spaces

Mathematically, we may describe our observation as follows. If we pick any point in any space in the first row, we can always find a small neighborhood U around the point, such that U is either homeomorphic to $\mathbf{R}$, or $\mathbf{R}_+ = \{x : x \geq 0\}$. Evidently, such a property fails at the questioned points in the spaces on the second row.

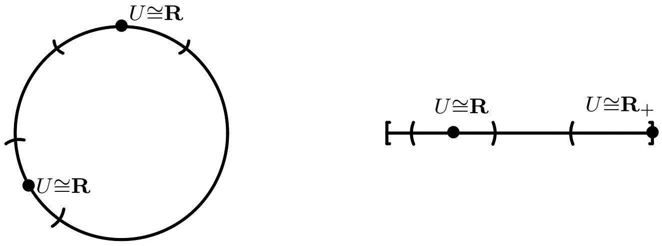


Figure 84: what "regular" means

The similar case can be made at higher dimensions. Figure 85 contains some 2-dimensional examples. In general, an *n-dimensional (topological) manifold* is a Hausdorff[13] topological space M satisfying the following properties: For any point $x \in M$, there is an open subset $U \subset M$, such that $x \in U$ and U is homeomorphic to either the Euclidean space $\mathbf{R}^n$ or the half Euclidean space $\mathbf{R}^n_+ = \{(x_1, \cdots, x_n) : x_n \geq 0\}$.

[13]Born November 8, 1869 in Breslau, Germany (now Wroclaw, Poland); Died January 26, 1942 in Bonn, Germany. In his landmark book *Grundzüge der Mengenlehre*, published in 1914, Hausdorff introduced the modern axioms of topology that we use today.

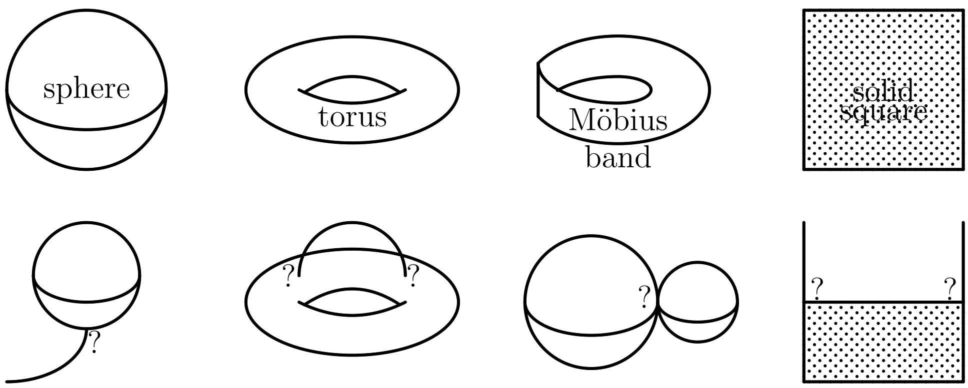


Figure 85: “regular” and “irregular” spaces at dimension 2

From the examples in one and two dimensions, it is easy to see that the points corresponding to $x_n = 0$ in case $U \cong \mathbf{R}^n_+$ form the so-called *boundary* of the manifold, which is usually denoted as ∂M. A *closed* manifold is a *compact* manifold without boundary (in other words, we always have $U \cong \mathbf{R}^n$). The circle, the sphere, and the torus are all closed manifolds.

0-dimensional manifolds are isolated points. The circle and the real line (which we recall is homeomorphic to any open interval) are examples of 1-dimensional manifolds *without* boundary. The closed interval is a 1-dimensional manifold with two end points as the boundary. The sphere and the torus are examples of closed 2-dimensional manifolds. The Möbius band and the square are 2-dimensional manifolds with one circle as the boundary (note that the square is homeomorphic to the disk). 2-dimensional manifolds are also called *surfaces*.

The unit ball

$$S^{n-1} = \{(x_1, \cdots, x_n) : \quad x_1^2 + \cdots + x_n^2 = 1\}$$

in the Euclidean space $\mathbf{R}^n$ is an $(n-1)$-dimensional closed manifold. The real projective space P^{n-1} is a closed manifold of dimension $(n-1)$, and the complex projective space CP^{n-1} is a closed manifold of dimension $(2n-2)$.

The Euclidean space $\mathbf{R}^n$ is a manifold without boundary. The unit ball

$$B^n = \{(x_1, \cdots, x_n) : \quad x_1^2 + \cdots + x_n^2 \leq 1\}$$

is an n-dimensional manifold with $\partial B^n = S^{n-1}$.

Exercise 7.1 Which alphabets are manifolds? Which are not?

Exercise 7.2 Find a homeomorphism from the Möbius band to itself that reverses the direction of the boundary circle.

Exercise 7.3 Prove that if M and N are manifolds, then $M \times N$ is a manifold. Moreover, we have $\partial(M \times N) = (\partial M \times N) \cup (M \times \partial N)$.

A fundamental question in topology is the classification of manifolds. Since a space is a manifold if and only if its *connected components* are manifolds, the problem is really about *connected* manifolds.

The only 0-dimensional connected manifold is a single point. It can also be shown that the only 1-dimensional connected and closed manifold is the circle. The main purpose of this chapter is the classification of connected and closed surfaces.

7.2 Surface

The sphere S^2 is a closed surface without holes. The torus T^2 is a closed surface with one hole. There are also closed surfaces with many holes. The number of holes is called the *genus* of the surface and is denoted by $g(S)$.

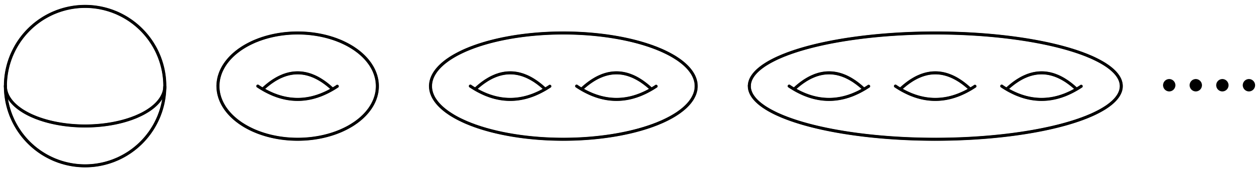


Figure 86: holes in closed surfaces

The list of closed (and connected) surfaces in Figure 86 is not complete, because it does not include the projective space P^2. Here we provide yet another way of constructing P^2. Since the boundaries of the Möbius band and the disk are circles, we may glue the two boundaries together. The result is a surface without boundary, and this is P^2.

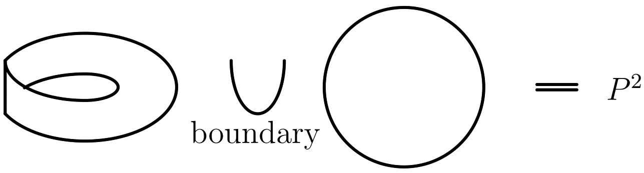


Figure 87: a way of constructing P^2

The idea for constructing P^2 may be applied to other situations. For example, we may take two Möbius bands and glue along their boundary circles. The result is the Klein bottle K^2 in Figure 88.

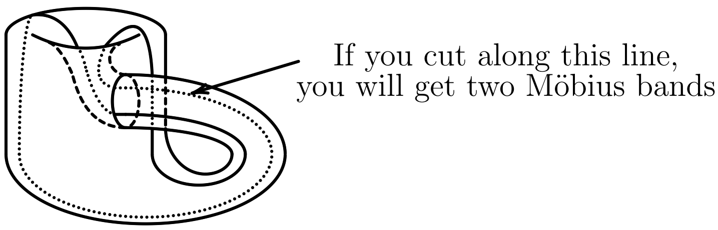


Figure 88: Klein bottle is the union of two Möbius bands

In general, for any two closed connected surfaces S_1 and S_2, we may delete one disk from each surface, and then glue $S_1 - \dot{B}^2$ and $S_2 - \dot{B}^2$ together along the boundary circles ($\dot{B}$ is the interior of B). The result is the *connected sum*

$$S_1 \# S_2 = (S_1 - \dot{B}^2) \cup_{S^1} (S_2 - \dot{B}^2)$$

of S_1 and S_2. For example, from Figure 87, we see $P^2 - \dot{B}^2$ is the Möbius band. Therefore the Klein bottle $K^2 = P^2 \# P^2$.

Strictly speaking, we need to justify the definition of the connected sum by showing that the surface $S_1 \# S_2$ is, up to homeomorphism, independent of the following choices.

1. The size of the disks to be deleted.
2. The location of the disks to be deleted.
3. The way the two boundary circles are glued together.

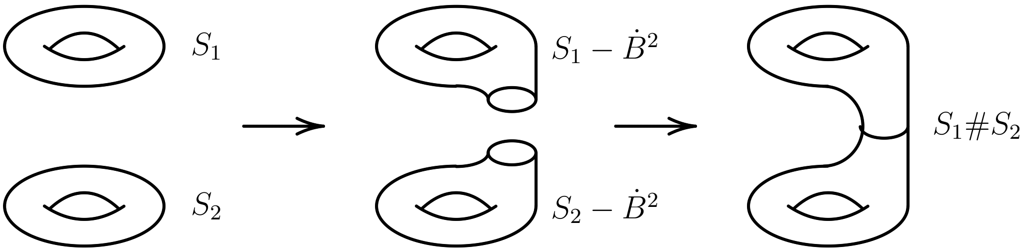


Figure 89: connected sum

To see what happens when we change the size of the disk. Consider two disks B^2_{small} and B^2_{big} in a surface S such that B^2_{small} is contained in the interior of B^2_{big}. We may find a disk B^2_{biggest} in S that is slightly bigger than B^2_{big}. It is easy to construct (see Exercise 7.4) a homeomorphism ϕ from $B^2_{\text{biggest}} - \dot{B}^2_{\text{big}}$ (with boundary $\partial B^2_{\text{biggest}} \cup \partial B^2_{\text{big}}$) to $B^2_{\text{biggest}} - \dot{B}^2_{\text{small}}$ (with boundary $\partial B^2_{\text{biggest}} \cup \partial B^2_{\text{small}}$) such that the restriction on $\partial B^2_{\text{biggest}}$ is identity. Then we have a homeomorphism

$$\begin{array}{rcccc} S - \dot{B}^2_{\text{big}} = & (S - \dot{B}^2_{\text{biggest}}) & \cup & (B^2_{\text{biggest}} - \dot{B}^2_{\text{big}}) \\ & \downarrow id & & \downarrow \phi \\ S - \dot{B}^2_{\text{small}} = & (S - \dot{B}^2_{\text{biggest}}) & \cup & (B^2_{\text{biggest}} - \dot{B}^2_{\text{small}}) \end{array}$$

This implies that the connected sum is independent of the size of the disks.

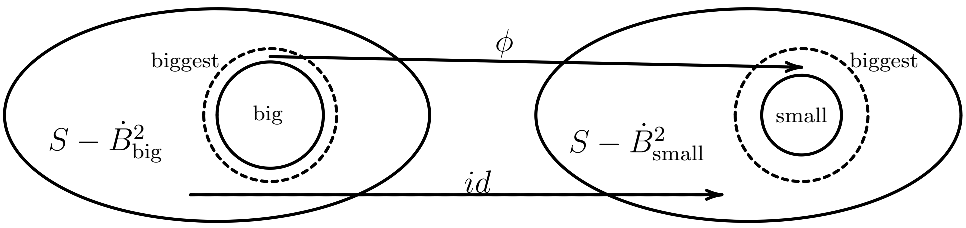


Figure 90: $S - \dot{B}^2_{\text{big}}$ and $S - \dot{B}^2_{\text{small}}$ are homeomorphic

Exercise 7.4 The triple $(B^2_{\text{biggest}}, B^2_{\text{big}}, B^2_{\text{small}})$ is homeomorphic to $(B^2(r), B^2(1), B^2(s))$, where $r > 1 > s$ and $B^2(r)$ is the disk in $\mathbf{R}^2$ centered at the origin and with radius r. Construct a homeomorphism ψ : $B^2(r) \to B^2(r)$ such that $\psi(x) = x$ for $x \in \partial B^2(r)$ (a circle of radius r) and $\psi(B^2(1)) = B^2(s)$. Then ψ induces a homeomorphism $B^2(r) - \dot{B}^2(1) \cong B^2(r) - \dot{B}^2(s)$. Show that this homeomorphism translates into a homeomorphism $B^2_{\text{biggest}} - \dot{B}^2_{\text{big}} \cong B^2_{\text{biggest}} - \dot{B}^2_{\text{small}}$ with the desired property.

Now we cosider what happens when the location of the disk is changed. Since the size does not matter, it is sufficient to consider two tiny disks B_x^2 and B_y^2 in S that contain points x and y in the interiors. We find a path connecting x to y without self crossing. Then we thicken the path a little bit an get a longish region $D \subset S$ that contains both tiny disks B_x^2 and B_y^2 in the interior. It is easy to construct (see Exercise 7.5) a homeomorphism ϕ from D to itself, such that the restriction on the boundary ∂D is identity and $\phi(B_x^2) = B_y^2$. Then we have a homeomorphism

$$\begin{array}{rccc} S - \dot{B}_x^2 = & (S - \dot{D}) & \cup & (D - \dot{B}_x^2) \\ & \downarrow id & & \downarrow \phi \\ S - \dot{B}_y^2 = & (S - \dot{D}) & \cup & (D - \dot{B}_y^2) \end{array}$$

This implies that the connected sum is independent of the location of the disks.

We remark that the argument involves a path from x to y. This requires S to be connected.

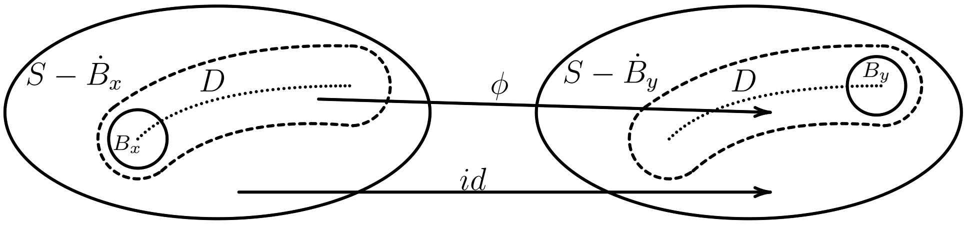


Figure 91: $S - \dot{B}_x^2$ and $S - \dot{B}_y^2$ are homeomorphic

Exercise 7.5 The triple (D, B_x^2, B_y^2) is homeomorphic to $(B^2((0,0),4), B^2((-2,0),1), B^2((2,0),1))$, where $B^2((a,b),r)$ is the disk in $\mathbf{R}^2$ centered at the point (a,b) and with radius r. Construct a homeomorphism $\psi : B^2((0,0),4) \to B^2((0,0),4)$ such that $\psi(x) = x$ for $x \in \partial B^2((0,0),4)$ (a circle of radius 4) and $\psi(B^2((-2,0),1)) = B^2((2,0),1)$. Then ψ translates into a homeomorphism $D \cong D$ with the desired property.

Exercise 7.6 Is the connected sum $S_1 \# S_2$ well-defined if we drop the condition that S_1 and S_2 are connected?

Now we turn to the way the two boundary circles are glued together. If we assign directions to the two boundary circles, then there are two possibilities for the glueing map: preserving the directions, and reversing the directions. Specifically, let $f : \partial(S_1 - \dot{B}^2) \to \partial(S_2 - \dot{B}^2)$ be a homeomorphism used in constructing a connected sum $S_1 \#_f S_2$. We identify $\partial(S_1 - \dot{B}^2)$ with $S^1 = \{z : z \in \mathbf{C}, |z| = 1\}$ and introduce a homeomorphism $\rho : \partial(S_1 - \dot{B}^2) \to \partial(S_1 - \dot{B}^2)$ corresponding to the complex conjugation map $z \to \bar{z} : S^1 \to S^1$. Then the other connected sum $S_1 \#_{f\rho} S_2$ is given by the glueing map $f\rho : \partial(S_1 - \dot{B}^2) \to \partial(S_2 - \dot{B}^2)$.

If we can extend the homeomorphism ρ from the boundary to a homeomorphism $\phi : S_1 - \dot{B}^2 \to S_1 - \dot{B}^2$ of the whole, then we have a homeomorphism

$$\begin{array}{rccc} S_1 \#_{f\rho} S_2 = & (S_1 - \dot{B}^2) & \cup_{f\rho} & (S_2 - \dot{B}^2) \\ & \downarrow \phi & & \downarrow id \\ S_1 \#_f S_2 = & (S_1 - \dot{B}^2) & \cup_f & (S_2 - \dot{B}^2) \end{array}$$

Thus different ways of glueing the boundary circles together give us homeomorphic surfaces. For the cases $S_1 = T^2$ and P^2, such an extension ϕ can be constructed explicitly (see Exercises

7.2 and 7.7). For general surfaces, the existence of ϕ can be proved by making use of the classification Theorem 7.5 (also see Exercise 7.25).

Exercise 7.7 Let M be obtained by deleting a disk from the torus T^2. Describe a homeomorphism from M to itself such that the restriction on the boundary reverses the boundary circle.

Connected sum can also be defined for manifolds of other dimensions. By essentially the same reason, the connected sum of connected manifolds is independent of the choice of the disks. However, a suitable extension homeomorphism ϕ may not always exist in general. Therefore in the definition of the connected sum at higher dimensions, the orientation of the boundary spheres must be specified.

The connected sum construction can be used repeatedly. For example, the surface in Figure 92 is the connected sum of g copies of the torus, which we denote by gT^2. We may also take the connected sum of several copies of P^2 to get a closed surface gP^2.

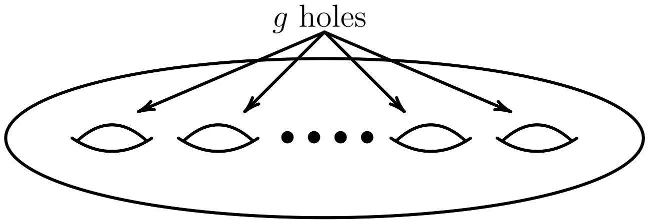


Figure 92: a closed surface with g holes

Lemma 7.1 *The connected sum of (connected and closed) surfaces has the following properties.*

1. *Commutative:* $S_1 \# S_2 = S_2 \# S_1$;
2. *Associative:* $(S_1 \# S_2) \# S_3 = S_1 \# (S_2 \# S_3)$;
3. *Identity:* $S \# S^2 = S$.

The connected sum with copies of the sphere S^2, the torus T^2, and the projective space P^2 produces many connected and closed surfaces. Using the Lemma above and the fact that $T^2 \# P^2 = P^2 \# P^2 \# P^2$ (proved later in Theorem 7.6), all the surfaces thus obtained are listed below.

1. sphere S^2;
2. connected sums of tori: $T^2, T^2 \# T^2, T^2 \# T^2 \# T^2, \cdots$;
3. connected sums of projective spaces: $P^2, P^2 \# P^2, P^2 \# P^2 \# P^2, \cdots$.

In the classification Theorem 7.5, we will show that this is the complete list of connected and closed surfaces.

Exercise 7.8 Prove that if M and N are manifolds and we have a homeomorphism $\partial M \cong \partial N$, then by gluing the two manifolds together along the homeomorphism of the boundaries, we get a manifold $M \cup_\partial N$ without boundary.

Exercise 7.9 Prove that the construction of P^2 in Figure 87 is consistent with the constructions in Section 6.3.

Exercise 7.10 By choosing special disk, special glueing, and making use of the fact $S^2 - \dot{B}^2 \cong B^2$, prove the properties of the connected sum in Lemma 7.1.

Exercise 7.11 What do you get if you shrink the boundary circle of the Möbius band to a point.

Exercise 7.12 We can construct "self-connected sum" as follows: For any (connected) surface S, remove two disjoint disks from S and then glue the two boundary circles together. What do you get?

Note: The answer may depend on the way (there are two) the circles are glued together.

7.3 Simplicial Surface

Our current definition of a surface without boundary is not easy to work with. To classify connected and closed surfaces, we need a more constructive way of describing them. Thus we take a simplicial approach and ask how can each point in a simplicial complex K have a neighborhood homeomorphic to $\mathbf{R}^2$? Note that K must be 2-dimensional, meaning that K contains only vertices (0-simplices), edges (1-simplices), and faces (2-simplices). Moreover, we have three cases to consider according to the location of the point.

(1) Suppose x is in the interior of a face. Then the interior of the face is a neighborhood of x homeomorphic to $\mathbf{R}^2$. In other words, the manifold condition is satisfied for such x.

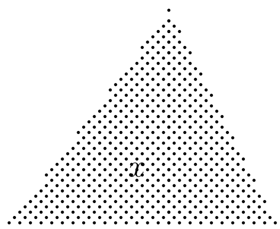


Figure 93: the (open) face x is in is homeomorphic to $\mathbf{R}^2$

(2) Suppose x is in the interior of an edge. Then the neighborhood of x in K involves all the faces with the edge as part of the boundary. From Figure 94, we get the following condition for finding a neighborhood homeomorphic to $\mathbf{R}^2$.

A. *Each edge is shared by exactly two faces.*

(3) Suppose x is a vertex. Then the neighborhood of x involves all the edges and faces containing this vertex. From Figure 95, we get the following condition for finding a neighborhood homeomorphic to $\mathbf{R}^2$.

B. *For each vertex v, all the faces with v as a vertex can be cyclically ordered as $\sigma_1, \sigma_2, \cdots, \sigma_n$, such that σ_i and σ_{i+1} have exactly one edge in common ($\sigma_{n+1} = \sigma_1$).*

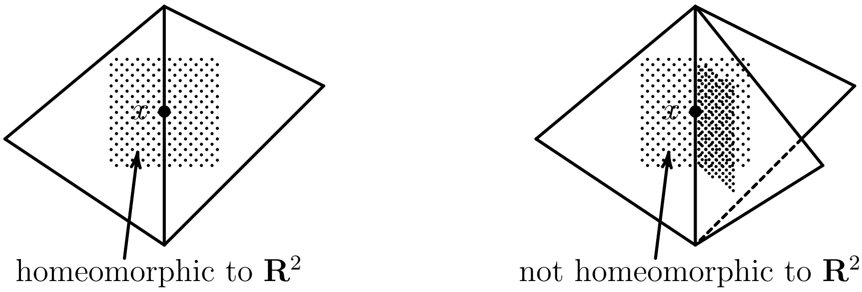


Figure 94: the neighborhood is homeomorphic to $\mathbf{R}^2$ only when there are exactly two faces

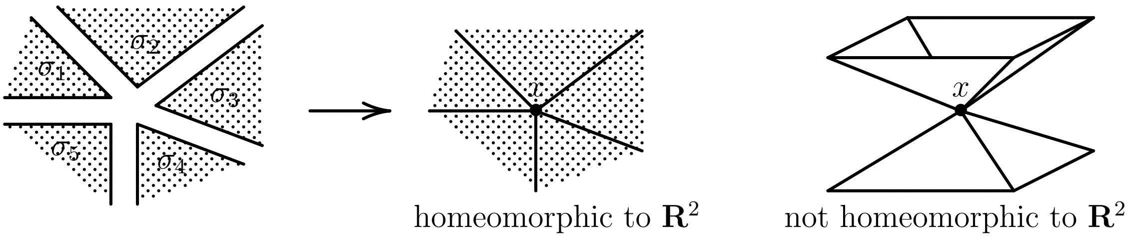


Figure 95: when the neighborhood of a vertex is homeomorphic to $\mathbf{R}^2$

In the subsequent sections, we will work with the following definition of surfaces: A closed surface is a finite 2-dimensional simplicial complex satisfying conditions **A** and **B**. Strictly speaking, we need to justify this by showing that this definition is equivalent to the definition in the last section. This was done by T. Radó[14] in 1925. We will not discuss the proof here.

7.4 Planar Diagram and CW-Structure of Surface

A CW-structure of the sphere S^2 consists of two 0-cells v and w, one 1-cell a, and one 2-cell B^2. The boundary of the 2-cell is divided into two parts (edges). By identifying the two edges, we get S^2.

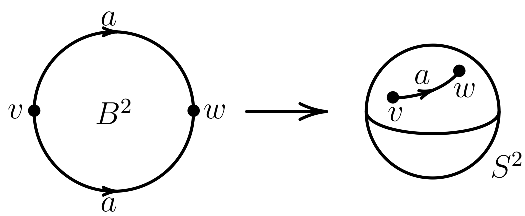


Figure 96: glueing the boundary of the disk to get a sphere

A CW-structure of the torus T^2 consists of one 0-cell v, two 1-cells a and b, and one 2-cell B^2. The boundary of the 2-cell is divided into four edges. By identifying these edges as indicated in Figure 97, we get T^2. Note that the identification of the edges forces the four vertices at the corners of the square to become the same one in T^2.

[14]Born June 2, 1895 in Budapest, Hungary; Died December 12, 1965 in New Smyrna Beach, Florida, USA

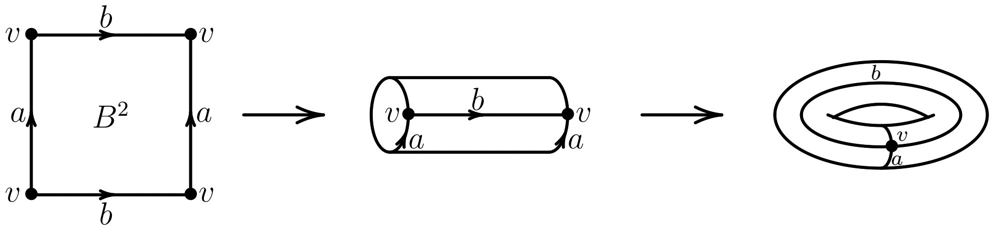


Figure 97: glueing the boundary of the disk to get a torus

A CW-structure of the projective space P^2 consists of one 0-cell v, one 1-cell a, and one 2-cell B^2. The boundary of the 2-cell is divided into two edges. We identify the two edges in a different way from the case of sphere. Since the identification is the same as identifying the antipodal points of the boundary circle, the result is the real projective space P^2. We also note that the identification of the two edges forces the two vertices to become the same one in P^2.

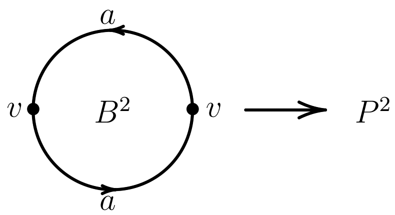


Figure 98: glueing the boundary of the disk to get a projective space

A CW-structure of K^2 consists of one 0-cell v, two 1-cells a and b, and one 2-cell B^2. The boundary of the 2-cell is divided into four edges. By identifying the edges in a different way from the torus, we get K^2. Note that the identification of the edges forces the four vertices to become the same one in K^2.

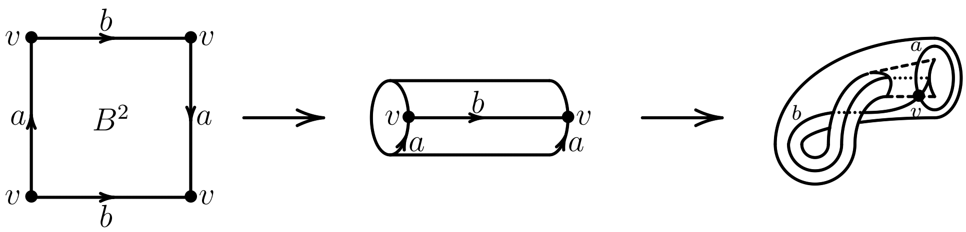


Figure 99: glueing the boundary of the disk to get a Klein bottle

Exercise 7.13 In Figure 100 is another way of gluing edges of a square together to form a surface. What is this surface?

All the examples shown above fit into the following pattern.

1. The boundary of the disc B^2 is divided into several edges. In other words, we think of B^2 as a polygon;

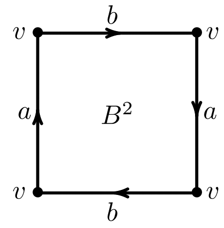


Figure 100: what surface do you get

2. The edges are grouped into pairs and assigned arrows;
3. The (connected and closed) surface is obtained by identifying all the pairs of edges in the way indicated by the assigned arrows.

The polygon with indicated pairs and arrows is called the *planar diagram* of the surface. Note that the planar diagram also gives us a CW-structure for the surface. The following technical result enables us to study surfaces by considering only the planar diagrams.

Lemma 7.2 *Any connected and closed surface is given by a planar diagram.*

Proof: We use the combinatorial description of a surface S as a 2-dimensional simplicial complex satisfying conditions **A** and **B**. We try to arrange the faces into a sequence $\sigma_1, \sigma_2, \cdots, \sigma_n$, such that σ_i intersects $\cup_{j<i}\sigma_j$ (the union of all the previous faces) along some edges. We do this inductively.

Suppose $\sigma_1, \sigma_2, \cdots, \sigma_k$ have been found and satisfy the desired condition. Denote the remaining faces by $\tau_1, \tau_2, \cdots, \tau_l$. By the connectivity of S, we can always find some σ_i and τ_j sharing a vertex v. Let $\sigma_{i_1}, \sigma_{i_2}, \cdots, \sigma_{i_p}$, $\tau_{j_1}, \tau_{j_2}, \cdots, \tau_{j_q}$ be all the faces with v as a vertex. We note that $p, q \geq 1$. Then these faces can be arranged in the cyclic way as described in condition **B**. The cyclic way implies that we can find a face σ_{i_α} and a face τ_{j_β} that are "adjacent" in the cycle. Then the two faces must share at least one edge. In particular, the intersection $\tau_{j_\beta} \cap [\cup_{j \leq k}\sigma_j]$ contains at least one edge. Thus if we denote $\tau_\beta = \sigma_{k+1}$, then $\sigma_1, \sigma_2, \cdots, \sigma_k, \sigma_{k+1}$ still satisfy the desired condition.

Now for the sequence $\sigma_1, \sigma_2, \cdots, \sigma_n$ satisfying the said condition, we take the union of the faces one by one, in the order of the sequence and each time *along exactly one edge*. For example, in the triangulation of the torus T^2 in Figure 101, the intersections $\sigma_2 \cap \sigma_1$, $\sigma_3 \cap (\sigma_1 \cup \sigma_2)$, $\cdots$, $\sigma_5 \cap (\sigma_1 \cup \cdots \cup \sigma_4)$ are all just single edges, and the union we construct is along this edge. However, $\sigma_6 \cap (\sigma_1 \cup \cdots \cup \sigma_5)$ consists of two edges: One is between σ_6 and σ_5; The other is between σ_6 and σ_1. In constructing the planar diagram, we have chosen the gluing to be along the edge between σ_6 and σ_5, and the other pair is no longer glued together. Similar choices are made when we glue σ_8, σ_{10}, σ_{11}, σ_{12}, σ_{13}, σ_{14} to the previous faces.

Suppose after the k-th step, we get a disk $B(k)$. Then all the edges in the interior of the disk have been identified. By the condition **A**, the common edge(s) between σ_{k+1} and $\cup_{j \leq k}\sigma_j$ must lie on the boundary of $B(k)$. Therefore if we glue along only one such common edge, we still get a disk $B(k+1)$. Thus we have inductively proved that the end result of our union is a disk $B(n)$.

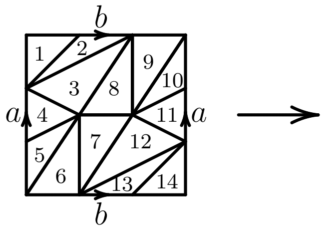


Figure 101: from triangulation to planar diagram

Of course, the edges on the boundary of $B(n)$ must still be glued together to form S. By condition **A**, each edge is identified with exactly one other edge. This makes $B(n)$ into a planar diagram of S.

□

We introduce the following simple notation for planar diagrams: First we give names to the edge pairs. Then we start from some vertex and traverse along the boundary. Our trip gives us a sequence of names, from which we make a word by using "name" when the direction of our trip is the same as the assigned arrow and using "name^{-1}" when the direction of our trip is opposite to the assigned arrow. The examples at the beginning of this section show that S^2, T^2, P^2, K^2 are given by the planar diagrams aa^{-1}, $aba^{-1}b^{-1}$, aa, $abab^{-1}$, respectively.

Lemma 7.3 *Given planar diagrams of two surfaces, the planar diagram for the connected sum is given as in Figure 102.*

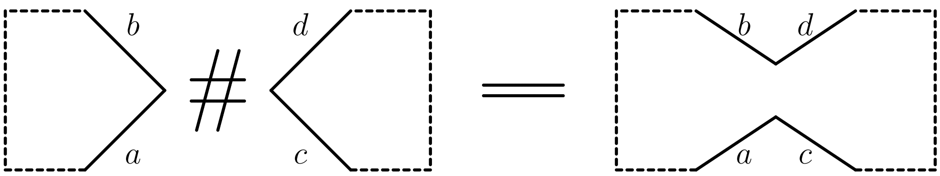


Figure 102: connected sum of planar diagrams

Proof: The proof is illustrated in Figure 103. The disks used for constructing the connected sum can be chosen anywhere. In particular, we may choose them near vertices, as indicated by the shaded regions in the figure. Since the connected sum is constructed by identifying the boundaries of the disks, we denote the boundaries of the disks by the same name e. After identifying e, we get the planar diagram for the connected sum.

□

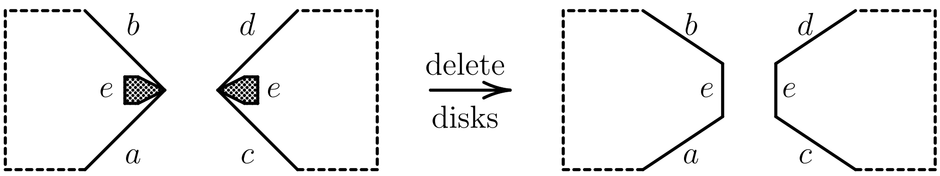


Figure 103: construct connected sum of planar diagrams

As a consequence of the lemma, we have the *standard planar diagrams* in Figure 104.

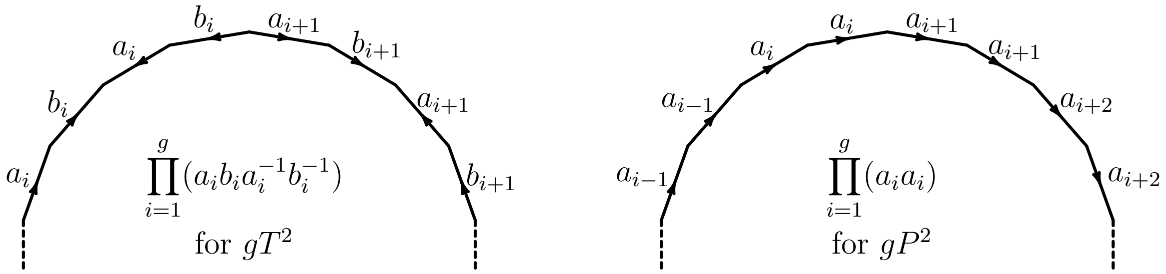


Figure 104: standard planar diagrams

7.5 Cut and Paste

Different planar diagrams may give the same surface. We may rigorously justify this by the *cut and paste* process. Such process is the key technique in the proof of the classification theorem in the next section. In this section, we practice cut and paste with some simple examples.

Lemma 7.4 *The planar diagrams $aabb$ and $abab^{-1}$ give the same surfaces.*

Since aa gives the projective space P^2, by Lemma 7.3, $aabb$ gives the connected sum $P^2\#P^2$. From Figure 72, $abab^{-1}$ gives the Klein bottle. Thus the Lemma basically says $K^2 = P^2\#P^2$. The fact was illustrated in Figure 88.

Proof: The following process shows how we start with the planar diagram $abab^{-1}$ and manipulate the glueing process and get the planar diagram $aabb$ at the end. The manipulation is taken to make sure that the surfaces represented by the diagrams are the same.

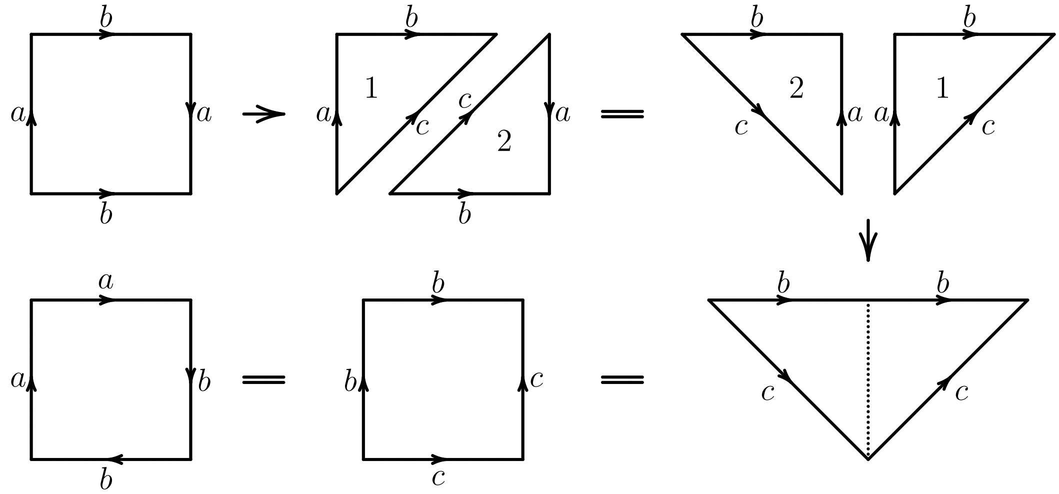


Figure 105: $abab^{-1}$ is equivalent to $aabb$

Here we explain in detail what is really going on. The first arrow means that the square may be considered as obtained by gluing two triangles together along the edge pair c. In this way, the surface is obtained by gluing two triangles together along three edge pairs a, b, and c.

The first equality is simply another way of drawing the pair of triangles. We keep the first triangle and flip the second triangle upside down.

In the second arrow, we glue the edge pair a first and leave the other two pairs alone.

The second equality means that the triangle is homeomorphic to the square, with edge pairs b and c unchanged.

The last equality simply renames the edge pairs b to a and c to b^{-1}.

□

Exercise 7.14 From the way the Klein bottle is constructed in Figure 99, show that the Klein bottle is obtained by gluing two Möbius bands together along their boundaries.

Exercise 7.15 There appears to be six different square diagrams. Find these square diagrams and identify each of them with one of S^2, P^2, T^2 and K^2.

7.6 Classification of Surfaces

The following classification theorem shows that the list of surfaces in Section 7.2 is complete. The theorem was first established by Dehn[15] and Heegaard[16] in 1907.

Theorem 7.5 *Every connected closed surface is homeomorphic to a sphere, a connected sum of several tori, or a connected sum of several projective spaces.*

Proof: By Lemma 7.2, a connected closed surface is given by a planar diagram. There are two types of edge pairs in a planar diagram: *opposing* and *twisted*, as indocated in Figure 106. The dotted lines indicate the other edges. In terms of the words, an opposing pair appears as $\cdots a \cdots a^{-1} \cdots$, and a twisted pair appears as $\cdots a \cdots a \cdots$.

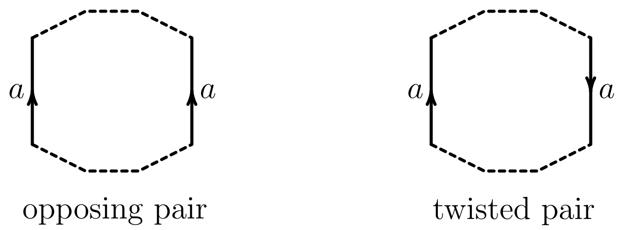


Figure 106: two possible edge pairs

Now we try to simplify the planar diagram. First we carry out the following preliminary simplifications.

1. Combine identical strings (see Figure 107);
2. Eliminate adjacent opposing pairs (see Figure 108);
3. Reduce the number of vertices to one.

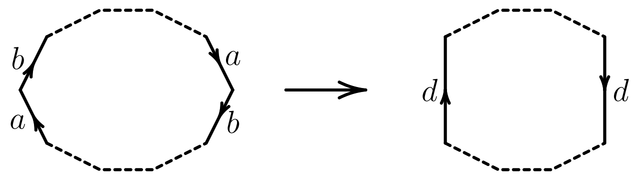


Figure 107: combining identical strings

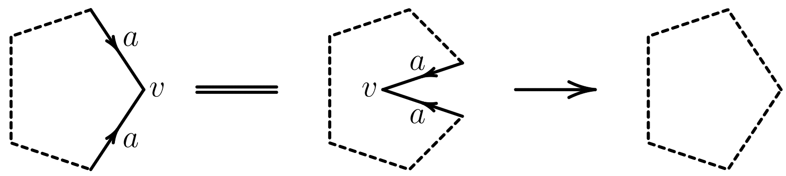


Figure 108: eliminating adjacent opposing pair

If the first and the second simplifications lead us to the adjacent opposing pair in Figure 109, then all edge pairs are eliminated, and the surface is S^2 (see Figure 96). Otherwise we continue with further simplification.

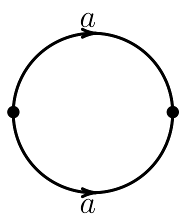


Figure 109: aa^{-1} gives the sphere

As for the reduction of the number of vertice, note that the identification of the edge pairs in a planar diagram forces some vertices in the diagram to be identified. For example, in Figure 110, six vertices are identified as v, and the remaining two are identified as w.

The cut and paste process in Figure 111 shows that if a vertex v appears at least twice, then the number of appearance may be reduced by one. We repeat the process until the number of the appearance is reduced to exactly once. Then we must have an adjacent opposing pair on the two sides of v, as in Figure 108. Thus v can be eliminated and the number of vertices is reduced by one. Repeating the process will eventually reduce the number of vertices to one.

After the preliminary simplification, we try to simplify the diagram to become a product of aa and $aba^{-1}b^{-1}$. Since aa comes from a twisted pair and $aba^{-1}b^{-1}$ comes from (two) opposing pairs, this involves two processes.

1. For a twisted pair $\cdots a \cdots a \cdots$, we bring the pair together and form $\cdots aa \cdots$;
2. For an opposing pair $\cdots a \cdots a^{-1} \cdots$, we find another opposing pair b located as $\cdots a \cdots b \cdots a^{-1} \cdots b^{-1} \cdots$. Then we bring the two pairs together and form $\cdots aba^{-1}b^{-1} \cdots$.

[15]Born November 13, 1878 in Hamburg, Germany; Died June 27, 1952 in Black Mountain, North Carolina, USA

[16]Born November 2, 1871 in Copenhagen, Denmark; Died February 7, 1948 in Oslo, Norway

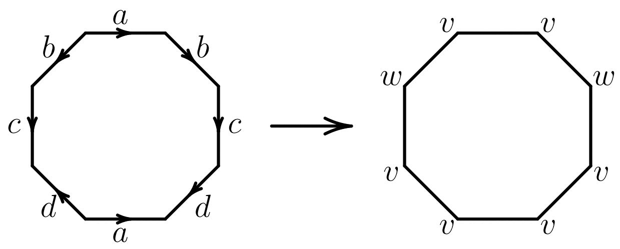


Figure 110: the number of vertices must be two

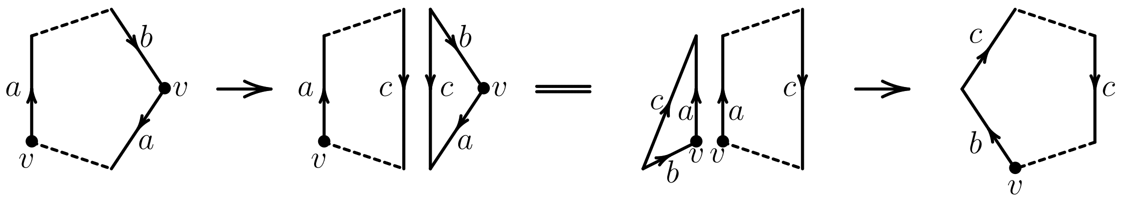


Figure 111: reduce the number of times a vertex appears

The first process is shown in Figure 112 (the notation b in the final diagram may be replaced by a). We do this for all twisted pairs so that all the twisted pairs are adjacent. Moreover, we note that the process also proved the following equalities

$$axay \sim yx^{-1}aa, \quad yaxa \sim yx^{-1}aa, \tag{15}$$

where x and y are any strings and $\sim$ means producing the same surface.

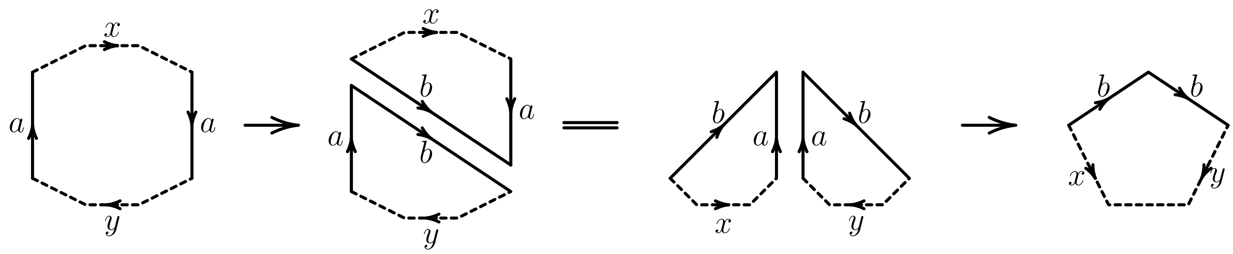


Figure 112: bring twisted pairs together

Now we turn to the second process. For any one opposing pair a, we first need to find another opposing pair b, so that locations of a and b are as described in Figure 113. The claim may be proved by counting the number of vertices. Since we have already eliminated adjacent opposing pairs in the preliminary step, the pair a must separate the boundary into two parts A and B. Since all twisted pairs have been brought together by Figure 112, any twisted pair lies completely in A or B. Suppose any opposing pair other than a also lies completely in A or B. Then any edge pair (opposing as well as twisted) lies completely in A or B. As a result, any vertex in A is identified with vertices in A only, and the same applies to vertices in B. In particular, there are at least two vertices in the surface. This contradicts with that fact that the number of vertices has been reduced to one. Thus we must have at least one opposing pair b of which one edge is in A and another edge is in B. This is exactly the situation described in

Figure 113. Then the simplification in Figure 114 further bring the pairs a and b together to from $\cdots aba^{-1}b^{-1}\cdots$.

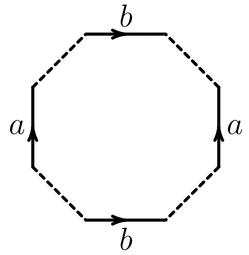


Figure 113: opposing pairs must appear as double pairs

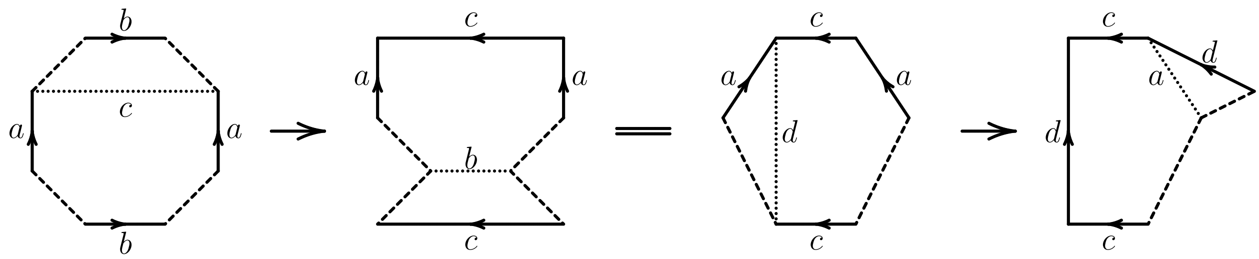


Figure 114: bring double opposing pairs together

Now the planar diagram is reduced to a product of aa and $aba^{-1}b^{-1}$. By Lemma 7.3, this means that the surface is a connected sum of copies of P^2 and T^2. Finally we use Lemma 7.1 and the equality $T^2\#P^2 = P^2\#P^2\#P^2$ (proved in Theorem 7.6) to conclude the following.

1. If there is at least one P^2 appearing in the connected sum, then (one by one) each T^2 may be replaced by two P^2. The result is a connected sum of P^2 only.

2. If P^2 does not appear in the connected sum, then we have a connected sum of T^2 only.

This completes the proof of the classification theorem.

□

Theorem 7.6 $T^2\#P^2 = P^2\#P^2\#P^2$.

Proof: We start with the standard diagram $aba^{-1}b^{-1}cc$ for $T^2\#P^2$ and use the equalities (15).

$$\begin{array}{rcll} aba^{-1}b^{-1}cc & \sim & cbab^{-1}ca & (\text{take } x = bab^{-1} \text{ and } y = a \text{ in } cxcy \sim yx^{-1}cc) \\ & \sim & cbc^{-1}baa & (\text{take } x = b^{-1}c \text{ and } y = cb \text{ in } yaxa \sim yx^{-1}aa) \\ & \sim & ccbbaa & (cbc^{-1}b \text{ is the Klein bottle; also use Lemmas 7.3 and 7.4}) \end{array}$$

The result $ccbbaa$ is the planar diagram for $P^2\#P^2\#P^2$.

□

Exercise 7.16 Prove that $T^2\#K^2 = K^2\#K^2$.

Exercise 7.17 According to the classification theorem, $mT^2\#nP^2$ must be either kT^2 or lP^2. What is k or l in terms of m and n?

Exercise 7.18 What are the surfaces given by the following planar diagrams?

1. $abab$;
2. $abcdec^{-1}da^{-1}b^{-1}e^{-1}$;
3. $ae^{-1}a^{-1}bdb^{-1}ced^{-1}c^{-1}$;
4. $abc^{-1}d^{-1}ef^{-1}fe^{-1}dcb^{-1}a^{-1}$;
5. $abcdbeafgd^{-1}g^{-1}hcife^{-1}ih^{-1}$.

7.7 Recognition of Surfaces

Theorem 7.5 gives us a complete list of closed surfaces. However, two important problems remain:

1. Is there any duplication in the list? For example, it appears that $2T^2 \neq 4T^2$. However, this needs to be proved;
2. How to recognize a closed surface in an efficient way? The problem is illustrated by the Exercise 7.18 and the problem in Figure 115.

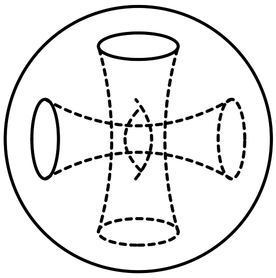


Figure 115: is this $3T^2$

The answer to the question comes from two invariants: Euler number and orientability. The Euler number can be computed from the planar diagram. For example, we consider the surface given by the planar diagram $abcdec^{-1}da^{-1}b^{-1}e^{-1}$ in Figure 116. First we decide from the diagram that there is only one 0-cell (i.e., vertex) v. Since we have five identification pairs of edges, the number of 1-cells is 5. Moreover, there is always one 2-cell in any planar diagram. Therefore the Euler number $\chi(S) = 1 - 5 + 1 = -3$.

By applying the idea to the standard planar diagrams at the end of Section 7.4, we conclude the following.

Lemma 7.7 $\chi(S^2) = 2$, $\chi(gT^2) = 2 - 2g$, $\chi(gP^2) = 2 - g$.

This lemma and Theorem 7.5 further imply the following.

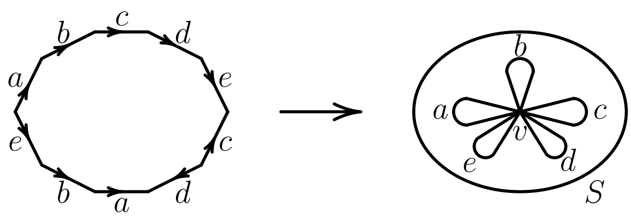


Figure 116: compute the Euler number

Corollary 7.8 *Let S be a connected closed surface. Then*

1. $\chi(S) = 2 \Longrightarrow S = S^2$;
2. $\chi(S)$ *is odd* $\Longrightarrow S = (2-\chi)P^2$.

For example, the surface given by the planar diagram $abcdec^{-1}da^{-1}b^{-1}e^{-1}$ must be $5P^2$.

Exercise 7.19 Redo Exercise 7.18 in a more efficient way.

Exercise 7.20 Prove that $\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$ and use this to prove Theorem 7.7.

Exercise 7.21 Let v, e, f be the numbers of vertices, edges, and faces in a triangulation of a closed surface S.

1. Prove that $2e = 3f$;
2. Use the Euler formula and the first part to express e and f in terms of v;
3. Prove that $v \geq \frac{1}{2}(7 + \sqrt{49 - 24\chi(S)})$.

The number ($[x]$ =largest integer $\leq x$)

$$H(S) = \left[\frac{1}{2}\left(7 + \sqrt{49 - 24\chi(S)}\right)\right]$$

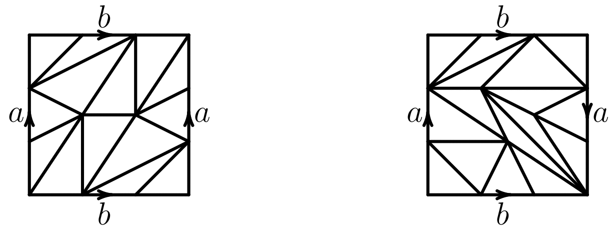
is called the *Heawood*[17] *number* of the surface S. Jungerman and Ringel proved[18] that, with the exceptions of K^2, $2T^2$, and $3P^2$, it is always possible to find a triangulation of S with $H(S)$ number of vertices. In the three exceptional cases, the minimal number is $H + 1$.

The Heawood number is a very important invariant that frequently appears in the combinatorial studies on surfaces. In 1890, Heawood proved that $H(S)$ number of colors is sufficient to color any map on S. He also proved $H(T^2) = 7$ number of colors is necessary for T^2 and conjectured that $H(S)$ number of colors is always necessary for any S. It is also not difficult to show the conjecture is true for P^2. Ringel and Youngs proved the conjecture in case $\chi < 0$ in 1968. With the help of computer, Appel and Haken solved the conjecture for S^2 (the famous *four colors problem*). This leaves the case of Klein bottle. Franklin[19] proved in 1934 that only *six* colors are needed for maps on K^2, while $H(K^2) = 7$. Therefore Heawood's conjecture is true for all surfaces except the Klein bottle.

[17] Born September 8, 1861 in Newport, Shropshire, England; Died January 24, 1955 in Durham, England

[18] Ringel proved the nonorientable case in 1955. Jungerman and Ringel proved the orientable case in 1980

[19] Born October 5, 1898 in New York, USA; Died January 27, 1965 in Belmont, Massachusetts, USA



$v = 7, e = 21, f = 14$ $v = 8, e = 24, f = 16$

Figure 117: the most efficient triangulations of T^2 and K^2

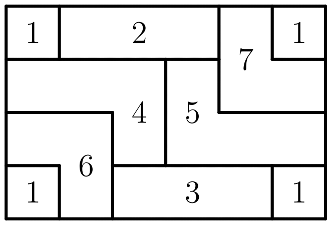


Figure 118: a torus map that requires seven colors

Note that gT^2 and $2gP^2$ have the same Euler number. However, gT^2 and $2gP^2$ should be different surfaces (for $g = 1$, we are comparing T^2 and K^2). To distinguish between gT^2 and $2gP^2$, therefore, another property/invariant (in addition to the Euler number) is needed. Such a property/invariant is the *orientability*.

The plane has two orientations, represented by a circle with one of two possible (clockwise or counterclockwise) directions. Since a surface S is locally homeomorphic to the plane, there are two possible orientations in a neighborhood of any point $x \in S$. We may indicate any choice of the orientation by a small circle around x with a choice of directions, and we denote this orientation by o_x. Note that for any path on S from x to y, we are able to move o_x to an orientation o_y at y along the path.

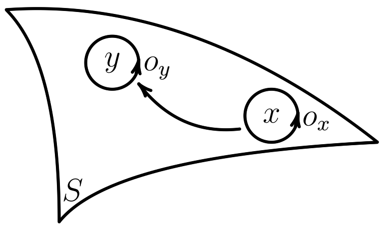


Figure 119: moving orientation from x to y along a path

Now we start from an orientation o_x at a point x and try to move o_x along all paths all over the surface. There are two possible outcomes:

1. No matter which path we choose from x to y, moving o_x along the path always gives us the same orientation at y;

2. There are two paths from x to y, such that moving o_x along the two paths gives us different orientations at y.

In the first case, we get a compatible system of orientations at all points of the surfaces. Such a system is a (global) *orientation* of the surface. In this case, we call the surface *orientable*. For example, the sphere and the torus are orientable. In the second case, we do not have compatible system of orientations at all points of the surfaces. Such surfaces are *nonorientable*. For example, the Möbius band is not orientable.

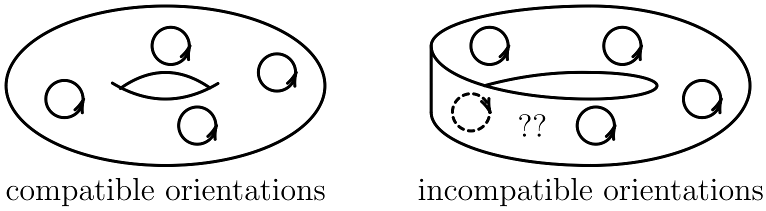


Figure 120: orientable and nonorientable surfaces

The nonorientable situation is equivalent to the following: There is a path from x to x (called a *loop* at x), such that moving o_x along the loop and back to x gives us the orientation different from o_x. In fact, we can assume the loop has no self-intersection. Now the region swept by (the little circle) o_x as it moves along the loop is a strip surrounding the loop. The fact that o_x comes back to become the opposite of o_x means exactly that the strip is a Möbius band. Therefore we have the following characterization of nonorientable surfaces.

Lemma 7.9 *A surface is nonorientable if and only if it contains the Möbius band.*

As a consequence of the lemma, P^2 is not orientable.

Exercise 7.22 Show that a connected sum $S_1 \# S_2$ is orientable if and only if both S_1 and S_2 are orientable.

Now we try to determine the orientability of a surface from its planar diagram. We fix an orientation for the interior of the planar diagram. Then we need to determine the compatibility of the local orientations when we glue the boundary edges together. Figure 121 shows that we have incompatibility if and only if there are twisted pairs.

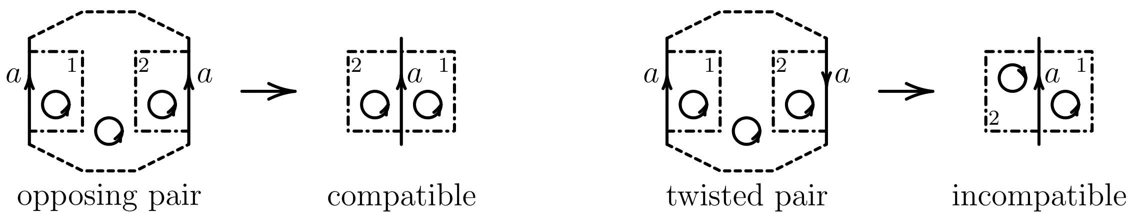


Figure 121: edge pairs versus compatibility of orientations

Therefore a planar diagram represents an oriented surface if and only if all the edge pairs are opposing pairs. In terms of the word, this means that if a appears in the word, then a^{-1} also appears in the word. For example, $abcdec^{-1}da^{-1}b^{-1}e^{-1}$ represents a nonorientable surface because d appears but d^{-1} does not appear. Applying this principle to the standard planar diagrams, we conclude the following.

Lemma 7.10 *S^2 and gT^2 are orientable. gP^2 is not orientable.*

Combining Lemma 7.7 and Lemma 7.10, we conclude the following method for determining whether two closed surfaces are the same.

Theorem 7.11 *Two closed surfaces S_1 and S_2 are homeomorphic if and only if*

1. $\chi(S_1) = \chi(S_2)$;
2. *S_1 and S_2 are either both orientable or both nonorientable.*

Exercise 7.23 What are the surfaces given by the following planar diagrams?

1. $abcdeac^{-1}edb^{-1}$;
2. $abc^{-1}d^{-1}ef^{-1}fe^{-1}dcba^{-1}$;
3. $abc^{-1}db^{-1}ea^{-1}fgd^{-1}g^{-1}hci^{-1}f^{-1}e^{-1}ih^{-1}$.

Exercise 7.24 What are the surfaces given by the following planar diagrams?

1. $a_1a_2\cdots a_na_1^{-1}a_2^{-1}\cdots a_n^{-1}$;
2. $a_1a_2\cdots a_na_1a_2\cdots a_n$.

Exercise 7.25 Use Theorem 7.11 to prove that the construction of the connected sum $S_1\#S_2$ of connected and closed spaces is independent of the choices of the size and location of the disks deleted and the way the boundary circles are identified.

8 Topological Properties

8.1 Hausdorff Space

Definition 8.1 A topological space X is *Hausdorff* if for any $x \neq y$ in X, there are open subsets U and V, such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

The Hausdorff property means that any two points can be "separated" by disjoint open subsets. Note that we may replace the open subsets U and V by topological basis.

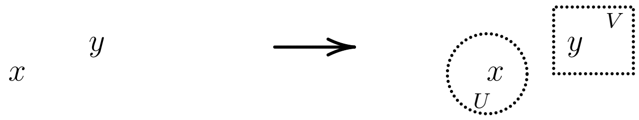


Figure 122: Hausdorff property

The Hausdorff property is clearly a topological property. More precisely, if X is a Hausdorff space, then any topological space homeomorphic to X is Hausdorff.

Example 8.1 Among the four topologies on the two point space $\{1, 2\}$ in Example 4.14, only the discrete topology $\mathcal{T}_4$ is Hausdorff. The other three are not Hausdorff.

Example 8.2 Among the topological bases in Example 4.1, only $\mathcal{B}_5$, $\mathcal{B}_6$, and $\mathcal{B}_9$ induce non-Hausdorff topologies.

Example 8.3 By taking $U = B(x, \frac{d(x,y)}{2})$ and $V = B(y, \frac{d(x,y)}{2})$, we see any metric space is Hausdorff.

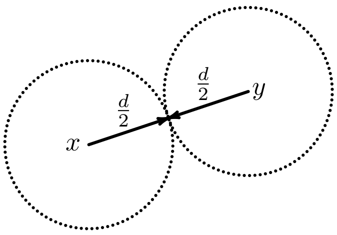


Figure 123: metric spaces are Hausdorff

Example 8.4 The Hausdorff property is defined for two distinct points. What about three distinct points?

Consider a Hausdorff space X and pairwise distinct points $x, y, z \in X$. From the Hausdorff property, we can find three pairs (U_1, V_1), (V_2, W_2), (U_3, W_3) of disjoint open subsets, such that

$$x \in U_1, y \in V_1; \quad y \in V_2, z \in W_2; \quad x \in U_3, z \in W_3.$$

Then $U = U_1 \cap U_3$, $V = V_1 \cap V_2$, $W = W_1 \cap W_2$ are pairwise disjoint open subsets, such that $x \in U$, $y \in V$, and $z \in W$.

Example 8.5 We prove the following interesting interpretation: X is Hausdorff $\Leftrightarrow$ The diagonal (see Example 5.20) $\Delta(X)$ is a closed subset of the product space $X \times X$.

The problem is to find out the meaning that $X \times X - \Delta(X)$ is open. A point (x, y) in $X \times X - \Delta(X)$ means $x \neq y$ in X. The openness of $X \times X - \Delta(X)$ means that we can find a topological basis $U \times V$ of the product topology, such that $(x, y) \in U \times V \subset X \times X - \Delta(X)$. Then we translate the meaning of the statement as follows:

1. In the topological basis of $X \times X$, U, V are open subsets of X;
2. $(x, y) \in U \times V$ means $x \in U$ and $y \in V$;
3. $U \times V \subset X \times X - \Delta(X)$ means U and V are disjoint.

The translation indicates that the openness of $X \times X - \Delta(X)$ is the same as X being Hausdorff.

Exercise 8.1 Prove that in a Hausdorff space, any single point is a closed subset. Moreover, find a non-Hausdorff space in which any single point is closed.

Exercise 8.2 Prove that on a finite set, the only Hausdorff topology is the discrete topology.

Exercise 8.3 Prove that if X is infinite, then the finite complement topology on X is not Hausdorff.

Exercise 8.4 Prove that every subspace of a Hausdorff space is Hausdorff.

Exercise 8.5 Suppose $\mathcal{T}$ is a finer topology than $\mathcal{T}'$. Is it true that $\mathcal{T}$ Hausdorff $\Rightarrow$ $\mathcal{T}'$ Hausdorff? What about the reversed direction?

Exercise 8.6 Prove that if X and Y are nonempty, then $X \times Y$ is Hausdorff $\Leftrightarrow$ X and Y are Hausdorff.

Exercise 8.7 Let $f, g : X \to Y$ be two continuous maps. Prove that if Y is Hausdorff, then $A = \{x : f(x) = g(x)\}$ is a closed subset of X.

Exercise 8.8 Let $f : X \to Y$ be a continuous map. Prove that if Y is Hausdorff, then the graph $\Gamma_f = \{(x, f(x)) : x \in X\}$ is a closed subset of $X \times Y$.

Exercise 8.9 Prove that the pointwise convergence topology in Example 4.7 is Hausdorff.

8.2 Connected Space

Intuitively, a space is connected if it consists of only "one piece". To make the intuition more precise, we study the meaning of the opposite notion: The separation of a space.

In the definition of Hausdorff property, we separate distinct points by disjoint open subsets. The idea also applies to separation of subsets: Two subsets $A, B \subset X$ are *separated* if we can find disjoint open subsets U and V, such that $A \subset U$ and $B \subset V$.

Next, we ask when X consists of two separated pieces. In other words, we consider the case $X = A \cup B$. Then we have

$$X = A \cup B, \quad A \subset U, \quad B \subset V, \quad U \cap V = \emptyset, \quad U, V \subset X \text{ are open.}$$

Note that this implies $A = U$ and $B = V$. Thus the condition leads to the following definition.

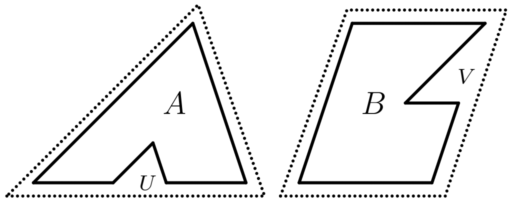


Figure 124: separating A and B

Definition 8.2 A *separation* of a topological space X is $X = A \cup B$, where A and B are disjoint nonempty open subsets. X is *not connected* if has a separation. X is *connected* if it has no separation.

Since A and B are disjoint and $X = A \cup B$, we have $A = X - B$ and $B = X - A$. Therefore the openness of A and B is the same as their closedness. Thus the following are equivalent to a separation:

1. $X = A \cup B$, where A and B are disjoint nonempty closed subsets;
2. There is an open and closed subset $A \subset X$, such that $A \neq \emptyset, X$.

As the opposite of separation, the following are equivalent to the connectedness of X:

1. $X = A \cup B$, A and B are disjoint and open $\Rightarrow$ either A or B is empty;
2. $X = A \cup B$, A and B are disjoint and closed $\Rightarrow$ either A or B is empty;
3. The only open and closed subsets of X are $\emptyset$ and X.

The connectedness is clearly a topological property. More precisely, if X is connected, then any topological space homeomorphic to X is connected.

Example 8.6 Among the four topologies on the two point space $\{1, 2\}$ in Example 4.14, only the discrete topology $\mathcal{T}_4$ is not connected. The other three are connected.

Example 8.7 Let X be an infinite set with finite complement topology. Let $A \subset X$ is open and closed. Since X is infinite, one of A and $X - A$ (which are closed subsets) is infinite. In the finite complement topology, however, the only closed infinite subsets is X. Thus we conclude that one of A and $X - A$ is X, which means that $A = \emptyset$ or X. By the third criterion for the connectedness, we see that X is connected.

Exercise 8.10 Is the topology on $\{1, 2, 3, 4\}$ in Example 4.15 connected? Which of the 3 point subspaces are connected?

Exercise 8.11 Find a separation of $\mathbf{R}_l$. This shows the lower limit topology is not connected.

Exercise 8.12 Suppose $\mathcal{T}$ is a finer topology than $\mathcal{T}'$. Is it true that $\mathcal{T}$ connected $\Longrightarrow$ $\mathcal{T}'$ connected? What about the reversed direction?

The most important example of connected spaces is given by the following result.

Theorem 8.3 *Intervals in* $\mathbf{R}_{\text{usual}}$ *are connected.*

Proof: Let $\emptyset \neq A \subset [0,1]$ be open and closed. We may assume $0 \in A$ and try to prove $A = [0,1]$. Otherwise, we may use $[0,1] - A$ in place of A in the subsequent argument. Denote

$$a = \sup\{x : \quad [0,x] \subset A\}.$$

Our problem becomes the proof of $a = 1$.

Since $0 \in A$ and A is open, we have $[0,\epsilon) \subset A$ for some $\epsilon > 0$. This implies $a > 0$.

Next by the definition of a, we have an increasing sequence x_n, such that $[0,x_n] \subset A$ and $\lim x_n = a$. Since A is closed, we conclude $a \in A$.

Finally, if $a < 1$, then by $a \in A$ and A open, we have $(a-\delta, a+\delta) \subset A$ for some $1-a > \delta > 0$. Combined with $[0,a] \in A$, we have $[0, a+\delta/2] \subset A$. By the definition of a, we have $a \geq a+\delta/2$. The contradiction implies that $a = 1$.

The proof for the connectedness of the other types of intervals is similar. One may also use the properties of connected spaces in Theorem 8.4 to prove the other cases.

□

In the next theorem, we assemble some properties of connected spaces.

Theorem 8.4 *Connectedness has the following properties.*

1. *A is connected and $A \subset B \subset \bar{A} \Rightarrow B$ is connected;*
2. *$A_i \subset X$ are connected subsets, and $\cap A_i \neq \emptyset \Rightarrow \cup A_i$ is connected;*
3. *X is connected and $f : X \to Y$ is continuous $\Rightarrow f(X)$ is connected;*
4. *If X and Y are nonempty, then X and Y are connected $\Leftrightarrow X \times Y$ is connected.*

Proof: We only prove the first and the third properties. The other two properties are left as exercises.

In the first property, we consider an open and closed (with regard to the subspace topology of B) subset $C \subset B$. The connectedness of B means that we need to show $C = B$ or $C = \emptyset$.

Since $C \cap A$ is open and closed in A, and A is connected, we see that either $C \cap A = A$, or $C \cap A = \emptyset$. Assume $C \cap A = A$, i.e., $A \subset C$. Then we have $\bar{A} \subset \bar{C}$ (the closure in taken in the whole space X), and it follows from $B \subset \bar{A}$ that $B \cap \bar{C} = B$. Since $B \cap \bar{C}$ is the closure of C in the subspace topology of B (Prove this! See Exercise 5.42), and C was assumed to be closed in B, we see that $C = B \cap \bar{C}$. Combined with $B \cap \bar{C} = B$, we conclude that $C = B$.

In the other case $C \cap A = \emptyset$, we may repeat the argument above, with $B - C$ in place of C. Then we conclude $B - C = B$ in the end. In other worlds, $C = \emptyset$.

This completes the proof of the first property. Next we prove the third property.

Let $A \subset f(X)$ be an open and closed subset. Then by the continuity of f, $f^{-1}(A)$ is open and closed in X. Since X is connected, we then have either $f^{-1}(A) = X$ or $f^{-1}(A) = \emptyset$. Since $A \subset f(X)$ implies $A = f(f^{-1}(A))$, we see that either $A = f(X)$ or $A = \emptyset$. This proves that $f(X)$ is connected.

□

Example 8.8 Since the interval is connected, the square, cube, etc., are all connected (as products of intervals). The balls are also connected because they are homeomorphic to cubes. In fact, any subset of $\mathbf{R}^n$ sandwiched between an open ball and its closure is connected, according to the first property. Moreover, the spaces in Figure 66 are also connected, after repeatedly applying the second property to the union of spaces homeomorphic to balls at various dimensions.

Sphere, torus, projective space, Klein bottle, and their connected sums are connected because they are continuous images of the disk via the planar diagram.

Example 8.9 The invertible real $n \times n$ matrices form a subset $GL(n, \mathbf{R})$ of $\mathbf{R}^{n^2}$. The determinant function $\det : GL(n, \mathbf{R}) \to \mathbf{R}$ is a continuous map, and has $\mathbf{R} - \{0\}$ as the image. Since $\mathbf{R} - \{0\}$ is not connected, it follows from the third property that $GL(n, \mathbf{R})$ is not connected.

Exercise 8.13 Show that if $A \subset \mathbf{R}$ is not an interval, then A is not connected.

Exercise 8.14 Show that any subset $A \subset \mathbf{R}_{\text{lower limit}}$ containing at least two points is not connected.

Exercise 8.15 Prove the following *fixed point theorem*: For any continuous map $f : [0, 1] \to [0, 1]$, there is $x \in [0, 1]$, such that $f(x) = x$. What if $[0, 1]$ is replaced by $(0, 1)$?

Hint: If f has no fixed point, then for the function $g(x) = f(x) - x$, the image $g[0, 1]$ cannot be an interval

Exercise 8.16 Prove the second property in Theorem 8.4.

Exercise 8.17 Prove the fourth property in Theorem 8.4 as follows:

1. For $\Leftarrow$, use the third property and the continuity of projections;
2. For $\Rightarrow$, use the second property to show that for any $x \in X$ and $y \in Y$, $T_{x,y} = x \times Y \cup X \times y$ is connected. Then use the second property and $X \times Y = \cup_{y \in Y} T_{x,y}$ to show that $X \times Y$ is connected.

Exercise 8.18 Show that the space $O(n, \mathbf{R}) = \{U : U^T U = I\} \subset \mathbf{R}^{n^2}$ of orthogonal matrices is not connected.

Exercise 8.19 Suppose $A_1, A_2, \cdots \subset X$ are connected subsets, such that $A_i \cap A_{i+1} \neq \emptyset$. Prove that $\cup A_i$ is connected.

Exercise 8.20 Suppose $A, A_i \subset X$ are connected subsets, and $A \cap A_i \neq \emptyset$ for each i. Prove that $A \cup (\cup A_i)$ is connected.

8.3 Path Connected Space

Theorems 8.3 and 8.4 may be combined to give a simple method for showing connectedness.

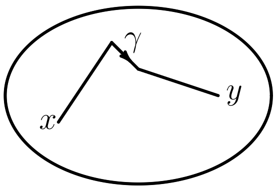


Figure 125: path connected

Definition 8.5 A *path* in a topological space X connecting x to y is a continuous map $\gamma : [0, 1] \to X$, such that $\gamma(0) = x$ and $\gamma(1) = y$. The space X is *path connected* if any two points can be joined by a path.

Lemma 8.6 *path connected $\Rightarrow$ connected.*

Proof: Let X be path connected. We fix a point $x \in X$. Then the path connectedness implies

$$X = \cup_{\gamma(0)=x}\gamma[0,1],$$

i.e., X is the union of the images of the paths starting from x. By Theorem 8.3 and the third property of Theorem 8.4, $\gamma[0,1]$ is connected. Moreover, x is a common point of all $\gamma[0,1]$ in the union. Thus it follows from the second property of Theorem 8.4 that X is connected.

□

Example 8.10 The sphere, torus, projective space, Klein bottle, and their connected sums are connected because they are clearly path connected.

If we take finitely many points away from $\mathbf{R}^n$ ($n \geq 2$), sphere, torus, projective space, Klein bottle, etc., we still get a path connected space. If we take a circle away from $\mathbf{R}^n$ ($n \geq 3$), we still get a path connected space.

Example 8.11 Consider the following subsets of $\mathbf{R}^2$ with the usual topology

$$A = (0,1] \times 0 \cup \left(\bigcup_{n=1}^{\infty} \frac{1}{n} \times [0,1]\right); \quad B = A \cup \{(0,1)\}.$$

A is connected because A is path connected. B is connected because $A \subset B \subset \bar{A}$. However, B is not path connected because there is no path connecting the point $(0,1)$ to the other points of B.

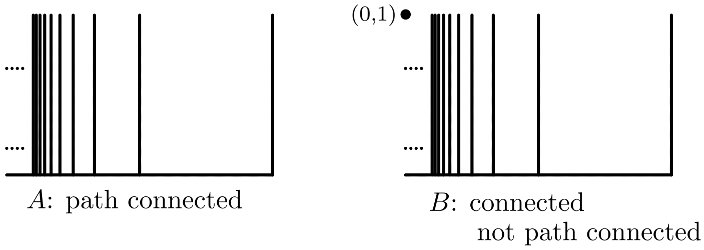


Figure 126: connected does not imply path connected

Example 8.12 We may use (path) connectedness to show $\mathbf{R}^2$ and $\mathbf{R}$ are not homeomorphic. Suppose there is a homeomorphism given by $f : \mathbf{R}^2 \to \mathbf{R}$ and $g = f^{-1} : \mathbf{R} \to \mathbf{R}^2$. Then the restrictions $f_| : \mathbf{R}^2 - \{g(0)\} \to \mathbf{R} - \{0\}$ and $g_| : \mathbf{R} - \{0\} \to \mathbf{R}^2 - \{g(0)\}$ is a homeomorphism between $\mathbf{R}^2 - \{g(0)\}$ and $\mathbf{R} - \{0\}$. However, since $\mathbf{R}^2 - \{g(0)\}$ is path connected and $\mathbf{R} - \{0\}$ is not connected, this is a contradiction.

Example 8.13 In Example 8.9, we have shown that the real invertible matrices do not form a connected subset. Our argument was based on the fact that the determinant may take any value in $\mathbf{R} - \{0\}$, which is not connected. Now, if we take only those invertible matrices with positive determinants:

$$GL_+(n, \mathbf{R}) = \{M : \quad M \text{ is } n \times n \text{ matrix}, \det M > 0\},$$

then there is a hope that $GL_+(n, \mathbf{R})$ is a connected subset of $\mathbf{R}^{n^2}$. We will show that $GL_+(n, \mathbf{R})$ is path connected. Then by Lemma 8.6, $GL_+(n, \mathbf{R})$ is indeed connected.

Any matrix M can be written as $M = AU$, where A is a positive definite matrix[20] and U is an orthogonal matrix[21]. Moreover, if $\det M > 0$, then we have $\det U = 1$.

Next, we will construct a *positive matrix path* $A(t)$ and an *orthogonal matrix path* $U(t)$, $t \in [0, 1]$, such that $A(0) = I = U(0)$, $A(1) = A$, $U(1) = U$. Then the *matrix path* $M(t) = A(t)U(t)$ is a path in $GL_+(n, \mathbf{R})$ joining I to $M(1) = M$. This implies that $GL_+(n, \mathbf{R})$ is path connected.

Recall that any symmetric matrix can be diagonalized. In our case, we have $A = PDP^{-1}$, where D is a diagonal matrix, with eigenvalues of A along the diagonal. Since A is positive definite, the diagonal of D consists of positive numbers. In particular, we may write

$$D = \begin{pmatrix} e^{\mu_1} & 0 & \cdots & 0 \\ 0 & e^{\mu_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & e^{\mu_n} \end{pmatrix}, \quad D(t) = \begin{pmatrix} e^{t\mu_1} & 0 & \cdots & 0 \\ 0 & e^{t\mu_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & e^{t\mu_n} \end{pmatrix}.$$

Then $A(t) = PD(t)P^{-1}$ is the positive matrix path joining I to A.

Recall that any orthogonal matrix is a combination of rotations, identities, and reflections with respect to an orthonormal basis. In our case, we have $U = QRQ^{-1}$, where Q is an orthogonal matrix, and

$$R = \begin{pmatrix} R_{\theta_1} & 0 & \cdots & 0 & 0 \\ 0 & R_{\theta_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -I_m & 0 \\ 0 & 0 & \cdots & 0 & I_n \end{pmatrix}, \quad R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Moreover, the condition $\det M > 0$ implies that $\det U = \det R = 1$. Therefore m must be even, and we may think of $-I_m$ as several blocks of the form

$$R_\pi = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then

$$U(t) = QR(t)Q^{-1}, \quad R(t) = \begin{pmatrix} R_{t\theta_1} & & & & & \\ & R_{t\theta_2} & & & & \\ & & \ddots & & & \\ & & & R_{t\pi} & & \\ & & & & \ddots & \\ & & & & & I_n \end{pmatrix},$$

is the orthogonal matrix path joining I to U.

Exercise 8.21 Prove that if there is $x \in X$, such that for any $y \in X$, there is a path joining x to y, then X is path connected.

Exercise 8.22 Which statements in Theorem 8.4 are still true for path connected spaces?

Exercise 8.23 Prove that for $n \geq 2$, $\mathbf{R}^n$ is not homeomorphic to $\mathbf{R}$.

Exercise 8.24 Prove that for $n \geq 2$, $\mathbf{R}^n$ is not homeomorphic to S^1.

[20] A positive definite matrix is a symmetric matrix with all eigenvalues positive.

[21] A square matrix U is orthogonal if $U^T U = I$. We have $\det U = \pm 1$ for any orthogonal matrix U.

Exercise 8.25 Prove that for $n \geq 1$, S^n is not homeomorphic to $\mathbf{R}$.

Exercise 8.26 Prove that if A is a countable subset of $\mathbf{R}^2$, then $\mathbf{R}^2 - A$ is connected.

Exercise 8.27 Prove that for open subsets of $\mathbf{R}^n$, we have connected $\Rightarrow$ path connected.

Hint: For a nonempty open subset $U \subset \mathbf{R}^n$ and $x \in U$, the subset

$$\{y \in U : \quad \text{there is a path in } U \text{ connecting } x \text{ to } y\}$$

is open and closed in U.

8.4 Connected Components

Let X be a topological space. A *connected component* A of X is a maximal connected subset of X. In other words, A satisfies the following conditions:

1. A is connected;
2. If $A \subset B \subset X$ and B is connected, then $A = B$.

One may similarly define *path connected components*.

If $A, B \subset X$ are connected components, then we have either A and B disjoint, or $A = B$. The reason is the following: Suppose $A \cap B \neq \emptyset$. Then by the second property of Theorem 8.4, $A \cup B$ is connected. Since A and B are maximal connected, we must have $A = A \cup B = B$.

For any $x \in X$, consider

$$C_x = \cup\{A \subset X : A \text{ is connected and } x \in A\}.$$

Then by the second property of Theorem 8.4, C_x is connected. In fact, it is easy to see that C_x is a connected component. Thus we conclude the following.

Lemma 8.7 *Any topological space is a disjoint union of connected components. Moreover, the connected components are closed subsets.*

The closedness of the connected components follows from the first property of Theorem 8.4.

The number of connected components is clearly a topological property. In other words, homeomorphic spaces should have the same number of connected components.

Example 8.14 Consider the topology on $\{1, 2, 3, 4\}$ in Example 4.15. Note that $\{1, 2, 3\}$ and $\{4\}$ are connected. Since $\{1, 2, 3\} \cup \{4\}$ is a separation of the whole space, we conclude that $\{1, 2, 3\}$ and $\{4\}$ are the two connected components.

Example 8.15 We have $GL(n, \mathbf{R}) = GL_+(n, \mathbf{R}) \cup GL_-(n, \mathbf{R})$, where GL, GL_+, GL_- respectively denote invertible matrices, matrices with positive determinant, and matrices with negative determinant. From Example 8.13, we saw that $GL_+(n, \mathbf{R})$ is connected. Let D be the diagonal matrix with $\{-1, 1, \cdots, 1\}$ on the diagonal. Then $M \mapsto DM$ is clearly a homeomorphism between $GL_+(n, \mathbf{R})$ and $GL_-(n, \mathbf{R})$. Therefore $GL_-(n, \mathbf{R})$ is also connected. Now we further recall from Example 8.9 that $GL(n, \mathbf{R})$ is not connected. Therefore we conclude that $GL_+(n, \mathbf{R})$ and $GL_-(n, \mathbf{R})$ are the two connected components of $GL(n, \mathbf{R})$.

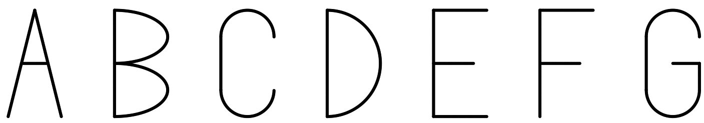


Figure 127: alphabets

Example 8.16 We try to classify the alphabets A through G by homeomorphism.

Consider the following properties.

- P_n: The number of points of $x \in X$, such that $X - x$ is not connected, is n.
- $P_{m,n}$: The number of points of $x \in X$, such that $X - x$ has m connected components, is n.

These are clearly topological properties. It is easy to see that E and F are homeomorphic, and C and G are also homeomorphic. Then we examine all the P-properties the alphabets satisfy.

Alphabet	Property P_m	Property $P_{m,n}$
A	P_∞	$P_{1,\infty}, P_{2,\infty}, P_{>2,0}$
B	P_1	$P_{1,\infty}, P_{2,1}, P_{>2,0}$
C, G	P_∞	$P_{1,2}, P_{2,\infty}, P_{>2,0}$
D	P_0	$P_{1,0}, P_{>1,0}$
E, F	P_∞	$P_{1,3}, P_{2,\infty}, P_{3,1}, P_{>3,0}$

By looking at the property P_m, we find B and D are not homeomorphic to the others. By looking at $P_{1,n}$, we also find that A, C, E are not homeomorphic to each other.

□

Exercise 8.28 Let $P_{k,m,n}$ be the following property: The number of choices of k distinct points $x_1, \cdots, x_k \in X$, such that $X - \{x_1, \cdots, x_k\}$ has m connected components, is n. Study the property for the alphabets A through G.

Exercise 8.29 Classify all alphabets by homeomorphism.

Exercise 8.30 Prove that if X has finitely many connected components, then any connected component is an open subset.

Exercise 8.31 Are path connected components closed?

8.5 Compact Space

Historically, the concept of compactness was quite an elusive one. It was understood for a long time that compactness for subsets of Euclidean spaces means closed and bounded. It was also understood that compactness for metric spaces means any sequence has a convergent subsequence. However, straightforward adoptions of these properties (and other related ones) to general topological spaces are either not appropriate or yield non-equivalent definitions of compactness. The dilemma was even more serious in the days when the concept of topology was still ambiguous.

Since then, mathematicians have gained a lot of insights and settled on the following definition.

Definition 8.8 A topological space is *compact* if any open cover has a finite subcover.

Please do not worry if you feel completely lost at the meaning of the definition. I experienced the same feeling when I first saw this. We simply need to unwrap the definition bit by bit, with the help of some examples.

First of all, by an *open cover* of X, we mean a collection $\mathcal{U}$ of open subsets, such that $X = \cup_{U\in\mathcal{U}} U$. We may also talk about open covers of subsets: A collection $\mathcal{U}$ of open subsets of X is an open cover of $A \subset X$ if $A \subset \cup_{U\in\mathcal{U}} U$.

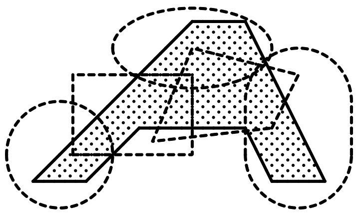


Figure 128: open cover

Example 8.17 Let X be a metric space. Then for any fixed $\epsilon > 0$, the collection $\{B(x,\epsilon) : x \in X\}$ of open balls of radius ϵ is an open cover.

More generally, suppose $\mathcal{B}$ is a topological basis of X and $A \subset X$ is a subset. If for each $a \in A$, we choose $B_a \in \mathcal{B}$, such that $a \in B_a$, then $\mathcal{U} = \{B_a : a \in A\}$ is an open cover of A.

Example 8.18 The following are open covers of $\mathbf{R}$:

$$\begin{aligned}\mathcal{U}_1 &= \{(-n,n) : \quad n \in \mathbf{N}\};\\ \mathcal{U}_2 &= \{(a,a+1) : \quad a \in \mathbf{Q}\};\\ \mathcal{U}_3 &= \{(n,\infty) : \quad n \in \mathbf{Z}\}.\end{aligned}$$

The following are open covers of $[0,1]$:

$$\begin{aligned}\mathcal{V}_1 &= \{(a,a+0.1) : \quad -1 \le a \le 1\};\\ \mathcal{V}_2 &= \{(a,a+0.1) : \quad a = -0.1, 0, 0.1, 0.2, \cdots, 1\};\\ \mathcal{V}_3 &= \{(1/n,1] : \quad n \in \mathbf{N}\} \cup \{[0,0.0001)\}.\end{aligned}$$

Note that in $\mathcal{V}_1$ and $\mathcal{V}_2$, we are thinking of $[0,1]$ as a subset of $\mathbf{R}$. In $\mathcal{V}_3$, we are thinking of $[0,1]$ itself as the whole topological space. The following are open covers of $(0,1]$:

$$\begin{aligned}\mathcal{W}_1 &= \{(a,a+0.1) : \quad -1 \le a \le 1\};\\ \mathcal{W}_2 &= \{(a,a+0.1) : \quad a = -0.1, 0, 0.1, 0.2, \cdots, 1\};\\ \mathcal{W}_3 &= \{(1/n,1] : \quad n \in \mathbf{N}\}.\end{aligned}$$

Example 8.19 The collection $\{B((m,n),\epsilon) : m,n \in \mathbf{Z}\}$ (open balls with integer center and radius ϵ) is an open cover of $\mathbf{R}^2$ if and only if $\epsilon > 1/\sqrt{2}$.

Now we try to understand the second part of the definition: Any open cover has a finite subcover. The statement means the following: If $\mathcal{U}$ is an open cover of X, then we can find finitely many $U_1, U_2, \cdots, U_n \in \mathcal{U}$, such that

$$X = U_1 \cup U_2 \cup \cdots \cup U_n,$$

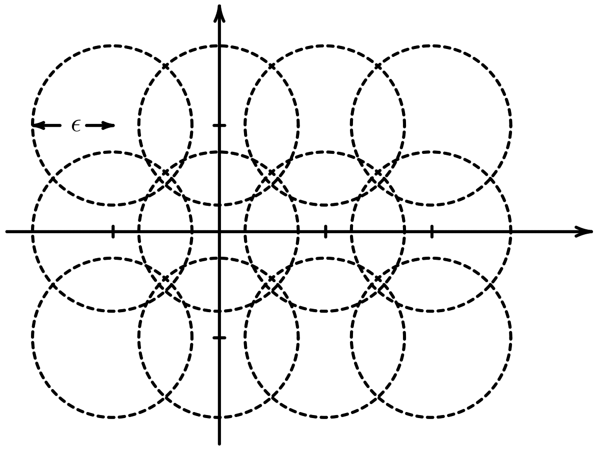


Figure 129: open cover of $\mathbf{R}^2$ in case $\epsilon > \frac{1}{\sqrt{2}}$

i.e., the finite *subcollection* $\{U_1, U_2, \cdots, U_n\}$ still covers X. For the compactness of a subset $A \subset X$, we have

$$A \subset U_1 \cup U_2 \cup \cdots \cup U_n.$$

Example 8.20 Any finite topological space is compact. The trivial topology is always compact.

Example 8.21 Consider the open cover $\mathcal{V}_1$ in Example 8.18. Although the cover is not finite, we can find a finite subcover $\mathcal{V}_2$. Similarly, the infinite open cover $\mathcal{V}_3$ has the following finite subcover

$$\mathcal{V}_4 = \{(1/n, 1] : \quad n = 1, 2, \cdots, 10001\} \cup \{[0, 0.0001)\}.$$

Although we have two open covers of $[0, 1]$ with finite subcovers, it is a grave error to immediately conclude that $[0, 1]$ is compact. For the compactness, we have to show the existence of finite subcover for *all* open covers.

Now we rigorously prove $[0, 1]$ is indeed compact. Suppose $\mathcal{U}$ is an open cover of $[0, 1]$, such that $[0, 1]$ cannot be covered by finitely many open subsets in $\mathcal{U}$. Then at least one of the intervals $[0, 1/2]$ and $[1/2, 1]$ of length $\frac{1}{2}$ cannot be covered by finitely many open subsets in $\mathcal{U}$. Denote this interval by I_1. Next we divide I_1 into two intervals of length $\frac{1}{4}$. Since I_1 cannot be covered by finitely many open subsets in $\mathcal{U}$, at least one of the two intervals cannot be covered by finitely many open subsets in $\mathcal{U}$. Denote this interval (of length $\frac{1}{4}$) by I_2. We may then further divide I_2 into two intervals of length $\frac{1}{8}$ and keep going. Eventually, we find a sequence of closed intervals $I_1, I_2, \cdots, I_n, \cdots$, satisfying (i) $I_{n+1} \subset I_n$; (ii) The length of I_n is $1/2^n$; and (iii) None of I_n is covered by finitely many open subsets in $\mathcal{U}$.

We choose one point x_n in each interval I_n. Then the properties (i) and (ii) imply that the sequence $\{x_n\}$ has a limit x. Moreover, the property (i) and the closedness of I_n implies that $x \in I_n$ for all n. Now since $\mathcal{U}$ is an open cover, we have $x \in U$ for some $U \in \mathcal{U}$. The openness of U implies that there is N, such that $(x - 1/2^N, x + 1/2^N) \subset U$. Then the fact $x \in I_N$ and the property (ii) imply that $I_N \subset U$. In other words, I_N is covered by one open subset in $\mathcal{U}$. This is in contradiction with the property (iii).

Thus we conclude that the original assumption that $\mathcal{U}$ has no finite subcover is wrong. This proves the compactness of $[0, 1]$.

Example 8.22 None of the open covers $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$ in Example 8.18 has finite subcover. Each one is a reason for $\mathbf{R}$ to be non-compact.

Although the open cover $\mathcal{W}_1$ in Example 8.18 has $\mathcal{W}_2$ as a finite subcover. The other open cover $\mathcal{W}_3$ of $(0, 1]$ has no finite subcover. Therefore $(0, 1]$ is also non-compact.

Exercise 8.32 Show that $\{1/n : n \in \mathbf{N}\} \cup \{0\}$ is a compact subset of $\mathbf{R}$.

Exercise 8.33 Show that $(0, 1)$ and $\{1/n : n \in \mathbf{N}\}$ are not compact.

Exercise 8.34 Prove that the finite complement topology is always compact.

Exercise 8.35 Prove that finite union of compact subsets is compact.

Exercise 8.36 Suppose $\mathcal{T}$ is a finer topology than $\mathcal{T}'$. Is it true that $\mathcal{T}$ compact $\Rightarrow \mathcal{T}'$ compact? What about the reversed direction?

The compactness is clearly a topological property. More precisely, if X is compact, then any topological space homeomorphic to X is compact.

Theorem 8.9 *Compactness has the following properties.*

1. *Any closed subset of a compact space is compact;*
2. *Any compact subspace of a Hausdorff space is closed;*
3. *X is compact and $f : X \to Y$ is continuous $\Rightarrow f(X)$ is compact;*
4. *If X is compact, Y is Hausdorff, then any bijective continuous map $f : X \to Y$ is a homeomorphism;*
5. *If X and Y are nonempty, then X and Y are compact $\Leftrightarrow X \times Y$ is compact;*
6. *A compact subset of a metric space is bounded;*
7. *A subset of $\mathbf{R}^n$ is compact $\Leftrightarrow$ the subset is bounded and closed.*

Proof: We prove the first, the second, the third, and the fifth properties. The other properties are left as exercises.

Let X be compact and $A \subset X$ be closed. Let $\mathcal{U}$ be a collection of open subsets of X covering A. Then $\mathcal{U} \cup \{X - A\}$ is an open cover of X. Since X is compact, we have $U_1, U_2, \cdots, U_n \in \mathcal{U} \cup \{X - A\}$ covering X. If we delete $X - A$ from the collection $\{U_1, U_2, \cdots, U_n\}$, then we get a finite subcollection of $\mathcal{U}$ that covers A. This completes the proof of the first property.

Let X be Hausdorff and $A \subset X$ be closed. We would like to show $X - A$ is open. Suppose $x \in X - A$. Then for any $a \in A$, we have $x \neq a$. Since X is Hausdorff, we have disjoint open subsets U_a and V_a, such that $x \in U_a$ and $a \in V_a$. By applying the compactness of A to the open cover $\{U_a : a \in A\}$ of A, we find a finite subcover $\{U_{a_1}, U_{a_2}, \cdots, U_{a_n}\}$ of A. Then $A \subset U_{a_1} \cup U_{a_2} \cup \cdots \cup U_{a_n}$ is disjoint from $V = V_{a_1} \cap V_{a_2} \cap \cdots \cap V_{a_n}$. Thus we have $x \in V \subset X - A$ for some open subset V. See Figure 130. This proves the openness of $X - A$, and completes the proof of the second property.

Let X be compact and $f : X \to Y$ be continuous. Let $\mathcal{U}$ be an open cover of $f(X)$. Then by the continuity of f, $\{f^{-1}(U) : U \in \mathcal{U}\}$ is an open cover of X. Since X is compact, we have a finite subcover $\{f^{-1}(U_1), f^{-1}(U_2), \cdots, f^{-1}(U_n)\}$ of X. This implies $\{U_1, U_2, \cdots, U_n\}$ is a finite subcover of $f(X)$ and completes the proof of the third property.

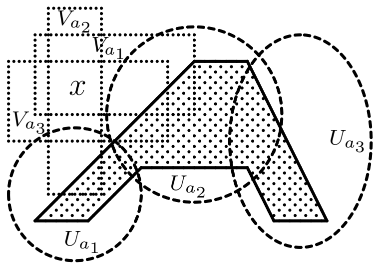


Figure 130: the intersection of V_{a_i} separates x from A

Finally, the $\Leftarrow$ direction of the fifth property follows from the third property and the continuity of the projections. For the $\Rightarrow$ direction, let X and Y be compact and $\mathcal{W}$ be an open cover of $X \times Y$. We first claim the following: For any $x \in X$, there is an open $U(x) \subset X$, such that $x \in U(x)$ and $U(x) \times Y$ is covered by finitely many open subsets in $\mathcal{W}$. To prove the claim, we fix $x \in X$ and note that for any $y \in Y$, there is $W_y \in \mathcal{W}$, such that $(x, y) \in W_y$. Since W_y is open, we have open $U_y \subset X$ and $V_y \subset Y$, such that $(x, y) \in U_y \times V_y \subset W_y$. Then the collection $\{V_y : y \in Y\}$ is an open cover of Y. Since Y is compact, we conclude that there are finitely many $V_{y_1}, V_{y_2}, \cdots, V_{y_n}$ covering Y. Let $U(x) = U_{y_1} \cap U_{y_2} \cap \cdots \cap U_{y_n}$. Then $U(x) \times V_{y_i} \subset U_{y_i} \times V_{y_i} \subset W_{y_i}$, and $U(x) \times Y = (U(x) \times V_{y_1}) \cup (U(x) \times V_{y_2}) \cup \cdots \cup (U(x) \times V_{y_n}) \subset W_{y_1} \cup W_{y_2} \cup \cdots \cup W_{y_n}$.

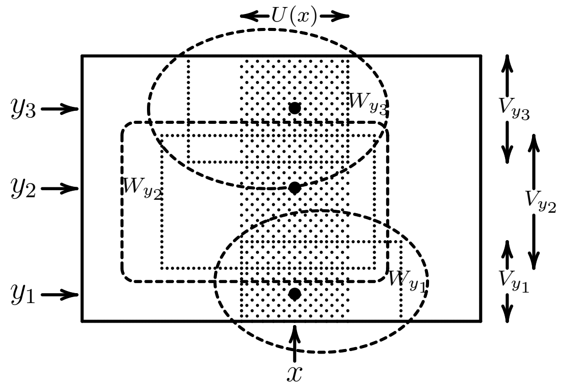


Figure 131: $x \in U(x)$ and $U(x) \times Y$ is finitely covered

Now for each $x \in X$, we find an open $U(x) \subset X$ as described in the claim. Then $\{U(x) : x \in X\}$ is an open cover of X. Since X is compact, we have finitely many $U(x_1), U(x_2), \cdots, U(x_k)$ covering X. It then follows from $X \times Y = (U(x_1) \times Y) \cup (U(x_2) \times Y) \cup \cdots \cup (U(x_k) \times Y)$ and each $U(x_j) \times Y$ being covered by finitely many open subsets in $\mathcal{W}$, that $X \times Y$ is covered by finitely many open subsets in $\mathcal{W}$. This completes the proof of the compactness of $X \times Y$.

□

Example 8.23 The spheres are compact because it is a bounded and closed subset of $\mathbf{R}^n$. As continuous images of the spheres, the real and complex projective spaces are also compact.

The torus and the Klein bottle are compact because they are continuous images of the closed square, which, as the product of two $[0, 1]$ with itself, is compact. By similar reason, we may conclude the compactness of many other surfaces.

Example 8.24 Consider the following subsets of $\mathbf{R}^3$:

$$\begin{aligned} A_1 &= \{(x_1,x_2,x_3): \quad x_1^4+x_2^4+2x_3^4=1\} \\ A_2 &= \{(x_1,x_2,x_3): \quad x_1^4+x_2^4+2x_3^4\le 1\} \\ A_3 &= \{(x_1,x_2,x_3): \quad x_1^4+x_2^4+2x_3^4< 1\} \\ A_4 &= \{(x_1,x_2,x_3): \quad x_1^4+x_2^4+2x_3^4\ge 1\} \end{aligned}$$

Let $f(x_1,x_2,x_3)=x_1^4+x_2^4+2x_3^4$. Then we have $A_1=f^{-1}\{1\}$, $A_2=f^{-1}[0,1]$. By the continuity of f, A_1 and A_2 are closed. Clearly, the points in A_1 and A_2 satisfy $|x_1|\le 1$, $|x_2|\le 1$, and $|x_3|\le 1$. Therefore A_1 and A_2 are are also bounded. By the seventh property of Theorem 8.9, we conclude that A_1 and A_2 are compact.

On the other hand, we see from $A_3=f^{-1}(-1,1)$ that A_3 is open. If A_3 were closed, then by the connectedness of $\mathbf{R}^3$, A_3 must be either empty or the whole $\mathbf{R}^3$. Since this is not the case, we see that A_3 is not closed. Consequently, A_3 is not compact.

Finally, A_4 is not compact because it is clearly not bounded.

Example 8.25 As a subset of $\mathbf{R}^{n^2}$, $GL(n,\mathbf{R})$ (see Example 8.9) is not compact, because it is unbounded.

On the other hand, the orthogonal matrices $O(n)=f^{-1}(I)$, where $f(M)=M^TM$ is a continuous map from $n\times n$ matrices to $n\times n$ matrices. Therefore $O(n)$ is a closed subset of $\mathbf{R}^{n^2}$. Since the columns of an orthogonal matrix are vectors of length 1, $O(n)$ is bounded. Thus we conclude that $O(n)$ is compact.

Exercise 8.37 Prove the fourth property of Theorem 8.9 by using the first and second properties to show f maps closed subsets to closed subsets.

Exercise 8.38 Prove the sixth property of Theorem 8.9 by considering the open cover $\{B(x,\epsilon):\epsilon>0\}$ of all open balls with a fixed center x.

Exercise 8.39 Prove the seventh property of Theorem 8.9 by using the second, fifth, and sixth properties, and the compactness of bounded closed intervals (proved in Example 8.21).

Exercise 8.40 Prove that if A and B are disjoint compact subsets in a Hausdorff space, then there are disjoint open subsets U and V, such that $A\subset U$ and $B\subset V$.

Hint: The case $A=\{x\}$ is the second property of Theorem 8.9

Exercise 8.41 Prove that if the topology is induced by a topological basis, then in the definition of compactness, we may consider only those covers by subsets in the topological basis. Moreover, use this to determine which topological bases in Example 4.1 induce compact topology.

Exercise 8.42 Prove that the compactness is equivalent to the following *finite intersection property*: If $\mathcal{C}$ is a collection of closed subsets, such that for any finitely many $C_1,C_2,\cdots,C_n\in\mathcal{C}$, the intersection $C_1\cap C_2\cap\cdots\cap C_n\ne\emptyset$, then $\cap_{C\in\mathcal{C}}C\ne\emptyset$.

Exercise 8.43 Suppose $C_1\supset C_2\supset\cdots\supset C_n\supset\cdots$ consists of nonempty closed subsets in a compact space. Show that the intersection $\cap C_n\ne\emptyset$.