Notes on the Topology of Complex Singularities

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Introduction

The algebraic varieties have played a very important role in the development of geometry. The lines and the conics where the first to be investigated and moreover, the study of equations leads naturally to algebraic geometry.

The past century has witnessed the introduction of new ideas and techniques, notably algebraic topology and complex geometry. These had a dramatic impact on the development of this subject. There are several reasons which make algebraic varieties so attractive. On one hand, it is their abundance and the wealth of techniques available to study them and, on the other hand, there are the often unexpected conclusions. These conclusions lead frequently to new research questions in other directions.

The gauge theoretic revolution of the past two decades has only increased the role played by these objects. More recently, Simon Donaldson has drawn attention to Lefschetz' old techniques of studying algebraic manifolds by extending them to the much more general context of symplectic manifolds. I have to admit that I was not familiar with Lefschetz' ideas and this gave me the impetus to teach a course on this subject and write up semi-formal notes. The second *raison d'être* of these notes is my personal interest in the isolated singularities of complex surfaces.

Loosely speaking, Lefschetz created a holomorphic version of Morse theory when the traditional one was not even born. He showed that a holomorphic map f from a complex manifold M to the complex projective line $\mathbb{P}^1$ which admits only nondegenerate critical points contains a large amount of nontrivial topological information about M. This information can be recovered by understanding the behavior of the smooth fibers of f as they approach a singular one.

Naturally, one can investigate what happens when f has degenerate points as well and, unlike the real case, there are many more tools at our disposal when approaching this issue in the holomorphic context. This leads to the local study of isolated singularities.

These notes cover the material I presented during the graduate course I taught at the University of Notre Dame in the spring of 2000. This course emphasized two subjects, Lefschetz theory and isolated singularities, relying mostly on basic algebraic topology covered by a regular first year graduate course.[1] Due to obvious time constraints these notes barely scratch this subject and yes, I know, I have left out many beautiful things. You should view these notes as an invitation to further study.

The first seven chapters cover Lefschetz theory from scratch and with many concrete and

[1]The algebraic topology known at the time Lefschetz created his theory would suffice. On the other hand, after reading parts of [23] I was left with the distinct feeling that Lefschetz' study of algebraic varieties lead to new results in algebraic topology designed to serve his goals.

I hope relevant examples. The main source of inspiration for this part was the beautiful but dense paper [21]. The second part is an introduction to the study of isolated singularities of holomorphic maps. We spend some time explaining the algebraic and the topological meaning of the Milnor number and we prove Milnor fibration theorem. As sources of inspiration we used the classical [3, 29].

I want to tank my friends and students for their comments and suggestions. In the end I am responsible for any shortcomings. You could help by e-mailing me your comments, corrections, or just to say hello.

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Contents

Chapter 1

Complex manifolds

We assume basic facts of complex analysis such as the ones efficiently surveyed in [14, Sec.0.1]. We will denote the imaginary unit $\sqrt{-1}$ by $\mathbf{i}$.

§1.1 Basic definitions

Roughly speaking, a n-dimensional *complex manifold* is obtained by holomorphically gluing open sets of $\mathbb{C}^n$. More rigorously, a n-dimensional complex manifold is a locally compact, Hausdorff topological space X together with a n-dimensional *holomorphic atlas*. This consists of the following objects.

- An open cover (U_α) of X.
- *Local charts*, i.e. homeomorphism $h_\alpha : U_\alpha \to \mathcal{O}_\alpha$ where $\mathcal{O}_\alpha$ is an open set in $\mathbb{C}^n$.

They are required to satisfy the following compatibility condition.

- All the *change of coordinates maps* (or gluing maps)

$$F_{\beta\alpha} : h_\alpha(U_{\alpha\beta}) \to h_\beta(U_{\alpha\beta}), \;\; (U_{\alpha\beta} := U_\alpha \cap U_\beta)$$

defined by the commutative diagram

$$\begin{tikzcd} & U_{\alpha\beta} \arrow[dl, "h_\alpha"'] \arrow[dr, "h_\beta"] & \\ h_\alpha(U_{\alpha\beta}) \subset \mathbb{C}^n \arrow[rr, "F_{\beta\alpha}"] & & h_\beta(U_{\alpha\beta}) \subset \mathbb{C}^n \end{tikzcd}$$

are *biholomorphic.*

For a point $p \in U_\alpha$, we usually write

$$h_\alpha(p) = (z_{1,\alpha}(p), \cdots, z_{n,\alpha}(p))$$

or $(z_1(p), \cdots, z_n(p))$ if the choice α is clear from the context.

From the definition of a complex manifold it is clear that all the local holomorphic objects on $\mathbb{C}^n$ have a counterpart on any complex manifold. For example a function $f : X \to \mathbb{C}$ is said to be *holomorphic* if

$$f_\alpha \circ h_\alpha^{-1} : \mathcal{O}_\alpha \subset \mathbb{C}^n \to \mathbb{C}, \;\; (f_\alpha := f\,|_{U_\alpha})$$

is holomorphic. The holomorphic maps $X \to \mathbb{C}^m$ are defined in the obvious fashion.

If Y is a complex m-dimensional manifold with a holomorphic atlas $(V_i; g_i)$ and $F : X \to Y$ is a continuous map, then F is *holomorphic* if for every i the map

$$g_i \circ F : F^{-1}(V_i) \to g_i(U_i) \subset \mathbb{C}^m \qquad \begin{tikzcd} F^{-1}(V_i) \subset X \arrow[r, "F"] \arrow[dr, "g_i \circ F"'] & V_i \subset Y \arrow[d, "g_i"] \\ & \mathbb{C}^m \end{tikzcd}$$

is holomorphic.

Definition 1.1. Suppose X is a complex manifold and $x \in X$. By local coordinates near x we will understand a biholomorphic map from a neighborhood of x onto an open subset of $\mathbb{C}^n$.

Remark 1.2. The complex space $\mathbb{C}^n$ with coordinates $(z_1, \cdots, z_n)$, $z_k := x_k + \mathbf{i}y_k$ is equipped with a canonical orientation given by the volume form

$$dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dz_n = \left(\frac{\mathbf{i}}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n,$$

and every biholomorphic map between open subsets preserves this orientation. This shows that *every complex manifold is equipped with a natural orientation.*

If $F : X \to \mathbb{C}^m$, $F = (F_1, \cdots, F_m)$ is a holomorphic map then a point $x \in X$ is said to be *regular* if there exist local coordinates $(z_1, \cdots, z_n)$ near m such that the Jacobian matrix

$$\left(\frac{\partial F_i}{\partial z_j}(m)\right)_{1 \leq i \leq m, 1 \leq j \leq n}$$

has maximal rank $\min(\dim X, m)$. This definition extends to holomorphic maps $F : X \to Y$. A point $x \in X$ which is not regular is called *critical.* A point $y \in Y$ is said to be a *regular value* of F if the fiber $F^{-1}(y)$ consist only of regular points. Otherwise y is called a *critical value* of F.

If $\dim Y = 1$ then, a critical point $x \in X$ is said to be nondegenerate if there exist local coordinates $(z_1, \cdots, z_n)$ near x and a local coordinate u near $F(x)$ such that F can be locally described as a function $u(\vec{z})$ and the the *Hessian*

$$\mathrm{Hess}_x(F) := \left(\frac{\partial^2 u}{\partial z_i \partial z_j}(x)\right)_{1 \leq i,j \leq n}$$

is nondegenerate, i.e.

$$\det \mathrm{Hess}_x(F) \neq 0.$$

Definition 1.3. A holomorphic map

$$F : X \to Y$$

is said to be a *Morse map* if

- $\dim Y = 1$.
- All the critical points of F are *nondegenerate*.
- If $y \in Y$ is a critical value, then the fiber $F^{-1}(y)$ contains an *unique* critical point.

Example 1.4. The function $f : \mathbb{C}^n \to \mathbb{C}$, $f = z_1^2 + \cdots + z_n^2$ is Morse.

§1.2 Basic examples

We want to describe a few fundamental constructions which will play a central role in this course.

Example 1.5. (The projective space) The N-dimensional *complex projective space* $\mathbb{P}^N$ can be regarded as the compactification of $\mathbb{C}^N$ obtained by adding the point at infinity on each (complex) line through the origin. Equivalently, we can define this space as the points at infinity (the "horizon") of $\mathbb{C}^{N+1}$. We will choose this second interpretation as starting point of the formal definition.

Each point $(z_0, z_1, \cdots, z_N) \in \mathbb{C}^{N+1} \setminus \{0\}$ determines a unique one dimensional subspace (line) which we denote by $[z_0 : \cdots : z_N]$. As a set, the projective space $\mathbb{P}^N$ consists of all these lines. To define a topological structure, note that we can define $\mathbb{P}^N$ as the quotient of $\mathbb{C}^{N+1} \setminus \{0\}$ modulo the equivalence relation

$$\mathbb{C}^{N+1} \setminus \{0\} \ni \vec{u} \sim \vec{v} \in \mathbb{C}^{N+1} \setminus \{0\} \Longleftrightarrow \exists \lambda \in \mathbb{C}^*; \;\; \vec{v} = \lambda \vec{u}.$$

The natural projection $\pi : \mathbb{C}^{N+1} \setminus \{0\} \to \mathbb{P}^N$ can be given the explicit description

$$\vec{z} = (z_0, z_1, \cdots, z_N) \mapsto [\vec{z}] := [z_0 : \cdots : z_N].$$

A subset $U \subset \mathbb{P}^N$ is open iff $\pi^{-1}(U)$ is open in $\mathbb{C}^{N+1}$. The *canonical holomorphic atlas* on $\mathbb{P}^N$ consists of the open sets

$$U_i := \Big\{ [z_0 : \cdots z_N]; \;\; z_i \neq 0 \Big\}, \;\; i = 0, \cdots, N$$

and local coordinates

$$\zeta = \zeta_i : U_i \to \mathbb{C}^N$$

$$[z_0 : \cdots : z_N] \mapsto (\zeta_1, \cdots, \zeta_N)$$

where

$$\zeta_k = \begin{cases} z_{k-1}/z_i & \text{if} \quad k \leq i \\ z_k/z_i & \text{if} \quad k > i \end{cases}.$$

Clearly, the change of coordinates maps are biholomorphic. For example, the projective line $\mathbb{P}^1$ is covered by two coordinates charts U_0 and U_1 with coordinates $z = z_1/z_0$ and respectively $\zeta = z_0/z_1$. The change of coordinates map is

$$z \mapsto \zeta = 1/z$$

which is clearly holomorphic.

Observe that each of the open sets U_i is biholomorphic to $\mathbb{C}^N$. Moreover, the complement

$$\mathbb{P}^N \setminus U_i = \Big\{[z_0 : \cdots z_N];\ \ z_i = 0\Big\}$$

can be naturally identified with $\mathbb{P}^{N-1}$= "horizon" of $\mathbb{C}^N$. Thus $\mathbb{P}^N$ decomposes as $U_0 \cong \mathbb{C}^N$ plus the "horizon", $\mathbb{P}^{N-1}$.

Example 1.6. (Submanifolds) Suppose X is a complex n-dimensional manifold. A *codimension k submanifold* of X is a closed subset $Y \subset X$ with the following property.

For every point $y \in Y$ there exists an open neighborhood $U_y \subset X$ and local holomorphic coordinates $(z_1, \cdots, z_n)$ on U_y such that

- $z_1(y) = \cdots = z_N(y) = 0$.
- $y' \in U_y \cap Y \Longleftrightarrow z_1(y') = \cdots = z_k(y') = 0$.

The codimension k submanifolds are complex manifolds of dimension $n-k$. There is a simple way of producing submanifolds.

Theorem 1.7. (Implicit function theorem) *If $F : X \to Y$ is a holomorphic map, $\dim Y = k$ and $y \in Y$ is a regular value of F then the fiber $F^{-1}(y)$ is a codimension k submanifold of X.*

Regular values exist in rich supply. More precisely, we have the following result. For a proof we refer to [28].

Theorem 1.8. (Sard Theorem) *If $F : X \to Y$ is a holomorphic map then the set of critical points has measure zero.*

Thus, most fibers $F^{-1}(y)$ are smooth submanifolds. We say that the *generic* fiber is smooth. In this course we will explain how to extract topological information about a complex manifold by studying the holomorphic maps

$$f : X \to T,\ \ \dim T = 1$$

and their critical points. We can regard X as an union of the fibers $F^{-1}(t),\ \ t \in T$. Most of them are smooth hypersurfaces with the possible exception of the fibers corresponding to the critical values. We will show that a good understanding of the changes in the topology and geometry of the fiber $F^{-1}(t)$ as t approaches a critical value often leads to nontrivial conclusions.

Example 1.9. (Algebraic manifolds) An *algebraic manifold* is a compact submanifold of some projective space $\mathbb{P}^N$. To construct such examples of complex manifolds consider the space $\mathcal{P}_{d,N}$ of degree d homogeneous polynomials in the variables $z_0, \cdots, z_N$. This is a complex vector space of dimenison $\binom{d+N}{d}$. We denote its projectivization by $\mathbb{P}(d, N)$.

To any $P \in \mathcal{P}_{d,N}$ we can associate a closed subset $V_P \subset \mathbb{P}^N$ defined by

$$V_P = \Big\{ [z_0; \cdots : z_N] \in \mathbb{P}^N; \;\; P(z_0, \cdots, z_N) = 0 \Big\}.$$

V_P is called a *hypersurface of degree d*. This depends only on the image $[P]$ of the polynomial P in $\mathbb{P}(d, N)$. We claim that for most P the hypersurface V_P is a codimension-1 submanifold.

We will use a standard transversality trick. Consider the complex manifold

$$X := \Big\{ ([\vec{z}], [P]) \in \mathbb{P}^N \times \mathbb{P}(d, N); \;\; P(\vec{z}) = 0 \Big\}.$$

A simple application of the implicit function theorem shows that X is a smooth submanifold. The hypersurface V_P can be identified with the fiber $F^{-1}(P)$ of the natural holomorphic map

$$F : X \to \mathbb{P}(d, N), \;\; ([\vec{z}], [P]) \mapsto [P].$$

According to Sard's theorem most fibers are smooth.

The special case $d = 1$ deserves special consideration. The zero set of a linear polynomial P is called a *hyperplane*. In this case the hyperplane V_P completely determines the image of P in $\mathbb{P}(1, N)$ and that is why $\mathbb{P}(1, N)$ can be identified with the set of hyperplanes in $\mathbb{P}^N$. The projective space $\mathbb{P}(1, N)$ is called the *dual* of $\mathbb{P}^N$ and is denoted by $\check{\mathbb{P}}^N$.

We can consider more general constructions. Given a set $(P_s)_{s \in S}$ of homogeneous polynomials in the variables $z_0, \cdots, z_N$ we can define

$$V(S) := \bigcap_{s \in S} V_{P_s}$$

$V(S)$ is called an projective variety. Often it is a smooth submanifold. The celebrated Chow theorem states that all algebraic manifolds can be obtained in this way. We refer to [14, Sec. 1.3] for more details. In this course we will describe some useful techniques of studying the topology of algebraic manifolds and varieties.

We conclude this section by discussing a special class of holomorphic maps.

Example 1.10. (Projections) Suppose X is a smooth, degree d curve in $\mathbb{P}^2$, i.e it is a codimension-1 smooth submanifold of $\mathbb{P}^2$ defined as the zero set of a degree d polynomial $P \in \mathcal{P}_{d,2}$. A hyperplane in $\mathbb{P}^2$ is a complex projective line. Fix a point $C \in \mathbb{P}^2$ and a line $L \subset \mathbb{P}^2 \setminus \{C\}$. For any point $p \in \mathbb{P}^2 \setminus \{C\}$ we denote by $[Cp]$ the unique projective line determined by C and p and by $f(p)$ the intersection of $[Cp]$ and L. The ensuing map

$$f : \mathbb{P}^2 \setminus \{C\} \to L$$

is holomorphic and it is called the *projection from C to L*. C is called the *center of the projection*. By restriction this induces a holomorphic map

$$f : X \setminus \{C\} \to L.$$

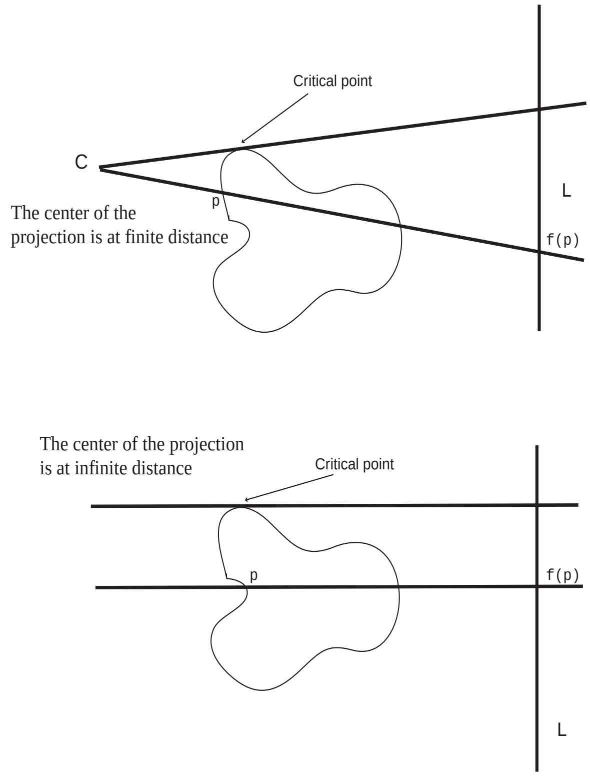

Figure 1.1: *Projecting from a point to a line*

(see Figure 1.1). Its critical points are the points $p \in X$ such that $[Cp]$ is tangent to X. The center C can be chosen at ∞ i.e. on the line $z_0 = 0$ in $\mathbb{P}^2$. The lines through C can now be visualized as lines parallel to a fixed direction in $\mathbb{C}^2$, corresponding to the point at ∞.

Suppose $C \notin X$. The projection is a well defined map $f : X \to L$. Since X has degree d every line in $\mathbb{P}^2$ intersects X in d points, counting multiplicities. In fact, by Sard's theorem a generic line will meet X in D distinct points. Since the holomorphic maps preserve the orientation we deduce that the degree of f is d (see [28] for more details about the degree of a smooth map).

The number of critical points of this map is related to a classical birational invariant of X. To describe it we need to introduce a few duality notions.

The *dual* of the center C is the line $\check{C} \in \check{\mathbb{P}}^2$ consisting of all hyperplanes (lines) in $\mathbb{P}^2$ passing through C. The *dual* of X is the closed set $\check{X} \subset \check{\mathbb{P}}^2$ consisting of all the lines in $\mathbb{P}^2$ tangent to X. $\check{X}$ is a (possibly) singular curve in $\mathbb{P}^2$, i.e. it can be describe as the zero locus of a homogeneous polynomial.

A critical point of the projection map $f : X \to L$ corresponds to a line trough C (point in $\check{C}$) which is tangent to X (which belongs to $\check{X}$). Thus the expected number of critical points is the expected number of intersection points between the curve $\check{X}$ and the line $\check{C}$. This is precisely the degree of $\check{X}$ classically known as the *class* of X.

Remark 1.11. Historically, the complex curves appeared in mathematics under a different guise, namely as multi-valued algebraic functions. For example the function

$$y = \pm\sqrt{x(x-1)(x-t)}$$

is 2-valued and its (2-sheeted) graph is the affine curve

$$y^2 = x(x-1)(x-t).$$

We can identify the complex affine plane $\mathbb{C}^2$ with the region $z_0 \neq 0$ of $\mathbb{P}^2$ using the correspondence

$$z = z_1/z_0, \quad y = z_2/z_0.$$

This leads to the homogenization

$$z_2^2 z_0 = z_1(z_1 - z_0)(z_1 - tz_0).$$

This is a cubic in $\mathbb{P}^2$ which can be regarded as the closure in $\mathbb{P}^2$ of the graph of the above algebraic function.

Chapter 2

The critical points contain nontrivial information

We want to explain on a simple but important example the claim in the title. More concretely we will show that the critical points determine most of the topological properties of a holomorphic map

$$f : \Sigma \to T$$

where Σ and T are complex curves, i.e. compact, connected, 1-dimensional complex manifolds.

§2.1 Riemann-Hurwitz theorem

Before we state and prove this important theorem we need to introduce an important notion.

Consider a holomorphic function $f : D \to \mathbb{C}$ such that $f(0) = 0$, where D denotes the unit open disk centered at the origin of the complex line $\mathbb{C}$. Since f is holomorphic it has a Taylor expansion

$$f(0) = \sum_{n \geq 0} a_n z^n$$

which converges uniformly on the compacts of D. Since $f(0) = 0$ we deduce $a_0 = f(0) = 0$ so that we can write

$$f(z) = z^k(a_k + a_{k+1}z + \cdots), \quad k > 0.$$

The integer k is called the *multiplicity* of $z_0 = 0$ in the fiber $f^{-1}(0)$. If additionally, $z_0 = 0$ happens to be a critical point as well, $f'(0) = 0$ then $k \geq 2$ and the integer $k-1$ is called the *Milnor number* (or the *multiplicity*) of the critical point. We denote it by $\mu(f, 0)$. Observe that 0 is a nondegenerate critical point iff it has Milnor number $\mu = 1$. For uniformity, define the Milnor number of a regular point to be zero.

Lemma 2.1. (Baby version of Tougeron's determinacy theorem) *Let $f : D \to \mathbb{C}$ be as above. Set $\mu = \mu(f, 0) > 0$. Then there exist small open neighborhoods U, Z of $0 \in D$*

and a biholomorphic map $U \to Z$ described by

$$U \ni u \mapsto z = z(u) \in Z$$

such that

$$f(z(u)) = u^{\mu+1}, \quad \forall u \in U.$$

Proof If $\mu = 0$ then $f'(0) \neq 0$ and the lemma follows from the implicit function theorem. In fact the biholomorphic map is $z = f^{-1}(u)$. Suppose $\mu > 0$.

We can write

$$f(z) = z^{\mu+1} g(z),$$

where $g(0) \neq 0$. We can find a small open neighborhood V of 0 and a holomorphic function $r : V \to \mathbb{C}$ so that

$$g(z) = (r(z))^{\mu+1} \Longleftrightarrow r(z) = \sqrt[\mu+1]{g(z)}, \quad \forall z \in V.$$

The map

$$z \mapsto u := zr(z)$$

satisfies $u(0) = 0$, $u'(0) \neq 0$ so that it defines a biholomorphism $Z \to U$ between two small open neighborhoods Z and U of 0. We see that $f(z) = u^{\mu+1}$, for all $z \in Z$. ■

The power map $u \to u^k$ defines *k-sheeted branched cover* of the unit disk D over itself. It is called *cover* because, off the bad point 0, it is a genuine k sheeted cover

$$D \setminus \{0\} \ni u \mapsto u^k \in D \setminus \{0\}.$$

There is a *branching at zero* meaning that the fiber over zero, which consists of a single geometric point, is substantially different from the generic fiber, which consists of k-points (see Figure 2.1). We see that the Milnor number $k-1$ is equal to the number of points in a general fiber (k) minus the number of points in the singular fiber(1).

If X and Y are one dimensional complex manifolds, then by choosing coordinates any holomorphic function $f : X \to Y$ can be locally described as a holomorphic function $f : D \to \mathbb{C}$ so we define the Milnor number of a critical point (see [31, Sec. II.4] for a proof that the choice of local coordinates is irrelevant). According to Lemma 2.1, the type of branching behavior described above occurs near each critical point. Moreover, the critical points are isolated so that if X is compact the (nonconstant) map f has only finitely many critical points. In particular, only finitely many Milnor numbers $\mu(f, x)$, $x \in X$ are nonzero.

Suppose now that Σ and T are two compact complex curves and $f : \Sigma \to T$ is a nonconstant holomorphic map. Topologically, they are 2-dimensional closed, oriented manifolds, Riemann surfaces. Their homeomorphism type is completely determined by their Euler

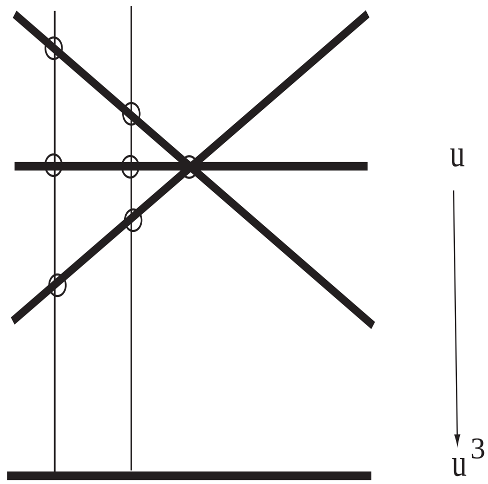

Figure 2.1: *Visualizing the branched cover $u \mapsto u^3$*

characteristics. Suppose $\chi(T)$ is known. Can we determine $\chi(\Sigma)$ from properties of f? The Riemann-Hurwitz theorem states that this is possible provided that we have some *mild global* information (the degree) and some *detailed local* information (the Milnor numbers of its critical points).

Theorem 2.2. (Riemann-Hurwitz) *Suppose* $\deg f = d > 0$. *Then*

$$\chi(\Sigma) = d\chi(T) - \sum_{p \in \Sigma} \mu(f, p).$$

Proof Denote by $t_1, \cdots, t_n \in T$ the critical values of f. Fix a triangulation $\mathcal{T}$ of T containing the critical values amongst its vertices. Denote by V, E, F the set of vertices, edges and respectively faces of this triangulation. Hence

$$\chi(T) = \#V - \#E + \#F.$$

For each $t \in T$ set

$$\mu(t) := \sum_{f(p)=t} \mu(f, p).$$

Observe that $\mu(t) = 0$ iff t is regular value. Moreover, a simple argument (see Figure 2.2) shows that

$$\mu(t_0) = \lim_{t \to t_0} \#f^{-1}(t) - \#f^{-1}(t_0) = d - \#f^{-1}(t_0), \ \ \forall t_0 \in T. \tag{2.1}$$

The map f is onto (*why ?*) and we can lift the triangulation $\mathcal{T}$ to a triangulation $\tilde{\mathcal{T}} = f^{-1}(\mathcal{T})$ of Σ. Denote by $\tilde{V}$, $\tilde{E}$ and $\tilde{F}$ the sets of vertices, edges and respectively faces of this triangulation. Since the set of critical points of f is discrete (finite) we deduce

$$\#\tilde{E} = d\#E, \ \ \#\tilde{F} = d\#F.$$

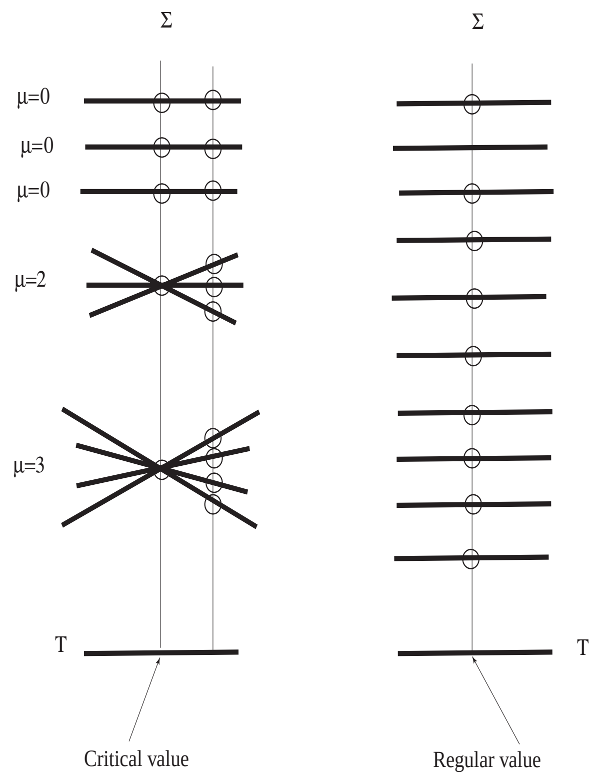

Figure 2.2: *A degree* 10 *cover*

Moreover, using (2.1) we deduce

$$\#\tilde{V} = d\#T - \sum_{t\in T}\mu(t) = d\#T - \sum \mu(f,p).$$

Thus

$$\chi(\Sigma) = \#\tilde{V} - \#\tilde{E} + \#\tilde{F} = d(\#V - \#E + \#F) - \sum \mu(f,p). \;\blacksquare$$

Corollary 2.3. *Suppose $f:\Sigma\to\mathbb{P}^1$ is a holomorphic map which has only nondegenerate critical points. If ν is their number and $d=\deg f$ then*

$$\chi(\Sigma) = 2d - \nu.$$

§2.2 Genus formula

We will illustrate the strength of Riemann-Hurwitz theorem on a classical problem. Consider a degree d plane curve curve, i.e. the zero locus in $\mathbb{P}^2$ of a homogeneous polynomial $P\in\mathcal{P}_{d,2}$, $X=V_P$. As we have explained in Chapter 1, for generic P, the set V_P is a compact, one dimensional submanifold manifold of $\mathbb{P}^2$. Its topological type is completely described by its Euler characteristic, or equivalently by its genus. We have the following formula due to Plücker. (We refer to [31, Sec. II.2] for a more general version.)

Theorem 2.4. (Genus formula) *For generic $P \in \mathcal{P}_{d,2}$ the curve V_P is a Riemann surface of genus*

$$g(V_P) = \frac{(d-1)(d-2)}{2}.$$

Proof We will use Corollary 2.3. To produce holomorphic maps $V_P \to \mathbb{P}^1$ we will use projections. Fix a line $\mathbb{L} \subset \mathbb{P}^2$ and a point $C \in \mathbb{P}^2 \setminus V_P$. We get a projection map $f : X \to \mathbb{L}$. This is a degree d holomorphic map. Modulo a linear change of coordinates we can assume all the critical points are situated in the region $z_0 \neq 0$ and C is the point at infinity $[0 : 1 : 0]$. In the affine plane $z_0 \neq 0$ with coordinates $x = z_1/z_0$, $y = z_2/z_0$, the point C corresponds to the lines parallel to the x-axis ($y = 0$). In this region the curve V_P is described by the equation

$$F(x, y) = 0$$

where $F(x, y) = P(1, x, y)$ is a degree d inhomogeneous polynomial. The critical points of the projection map are the points (x, y) on the curve $F(x, y) = 0$ where the tangent is horizontal

$$0 = \frac{dy}{dx} = -\frac{F'_x}{F'_y}.$$

Thus the critical points are solutions of the system of polynomial equations

$$\left\{ \begin{array}{l} F(x, y) = 0 \\ F'_x(x, y) = 0 \end{array} \right. .$$

The first polynomial has degree d while the second polynomial has degree $d-1$. For generic P this system will have exactly $d(d-1)$ distinct solutions. The corresponding critical points will be nondegenerate. Thus

$$2 - 2(g(V_P) = \chi(V_P) = 2d - d(d-1)$$

so that

$$g(V_P) = \frac{(d-1)(d-2)}{2}. \quad \blacksquare$$

Chapter 3

Further examples of complex manifolds

§3.1 Holomorphic line bundles

A *holomorphic line bundle* formalizes the intuitive idea of a holomorphic family of complex lines (1-dimensional complex vector spaces). The simplest example is that of a *trivial* family

$$\underline{\mathbb{C}}_M := \mathbb{C} \times M$$

where M is a complex manifold. Another nontrivial example is the family of lines tautological parametrized by a projective space $\mathbb{P}^N$.

More generally, a holomorphic line bundle consists of three objects.

• The *total space* (i.e. the disjoint union of all lines in the family) which is a complex manifold L.
• The *base* (i.e the space of parameters) which is a complex manifold M.
• The *natural projection* (i.e. the rule describing how to label each line in the family by a point in M) which is a holomorphic map $\pi : L \to M$.

(L, π, M) is called a line bundle if for every $x \in M$ there exist

• an open neighborhood U of x in M;
• a biholomorphic map $\Psi : \pi^{-1}(U) \to \mathbb{C} \times U$

such that the following hold.

• Each *fiber* $L_m := \pi^{-1}(m)$ $(m \in M)$ has a structure of complex, one dimensional vector space.
• The diagram below is commutative

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Psi"] \arrow[dr, "\pi"'] & & \mathbb{C} \times U \arrow[dl, "\mathbf{proj}"] \\ & U & \end{tikzcd} .$$

• The induced map $\Psi(m) : L_m \to \mathbb{C} \times \{m\}$ is a linear isomorphism.

The map Ψ is called a *local trivialization* of L (over U).

From the definition of a holomorphic line bundle we deduce that we can find an open cover $(U_\alpha)_{\alpha \in A}$ of M and trivializations Ψ_α over U_α. These give rise to *gluing maps* on the overlaps $U_{\alpha\beta} := U_\alpha \cap U_\beta$. These are holomorphic maps

$$g_{\beta\alpha} : U_{\alpha\beta} \to \operatorname{Aut}(\mathbb{C}) \cong \mathbb{C}^*$$

determined by the commutative diagram

$$\begin{tikzcd}
\mathbb{C} \times \{m\} \arrow[rr, "g_{\beta\alpha}(m)"] & & \mathbb{C} \times \{m\} \\
 & L_m \arrow[ul, "\Psi_\alpha(m)"] \arrow[ur, "\Psi_\beta(m)"'] &
\end{tikzcd}.$$

The gluing maps satisfy the *cocycle condition*

$$g_{\alpha\gamma}(m) \cdot g_{\gamma\beta}(m) \cdot g_{\beta\alpha}(m) = \mathbf{1}_{\mathbb{C}}, \;\; \forall \alpha, \beta, \gamma \in A, \;\; m \in U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma.$$

We can turn this construction on its head and recover a line bundle from the associated cocycle of gluing maps $g_{\beta\alpha}$. In fact, it is much more productive to think of a line bundles in terms of *gluing cocycles*. Observe that the total space L can be defined as a quotient

$$\Bigl(\coprod_{\alpha \in A} \mathbb{C} \times U_\alpha\Bigr) / \sim$$

where $\sim$ is the equivalence relation

$$\mathbb{C} \times U_\alpha \ni (z_\alpha, m_\alpha) \sim (z_\beta, m_\beta) \in \mathbb{C} \times U_\beta \Longleftrightarrow m_\alpha = m_\beta =: m, \;\; z_\beta = g_{\beta\alpha}(m) z_\alpha.$$

Definition 3.1. A **holomorphic section** of a holomorphic line bundle $L \xrightarrow{\pi} M$ is a holomorphic map

$$u : M \to L$$

such that $u(m) \in L_m$ for all $m \in M$. We denote by $\mathcal{O}_M(L)$ the space of holomorphic sections of $L \to M$.

Every line bundle admits at least one section, the *zero section* which associates to each $m \in M$ the origin of the vector space L_m. Observe that if a line bundle L is given by a gluing cocycle $g_{\beta\alpha}$, then a section can be described by a collection of holomorphic functions

$$f_\alpha : U_\alpha \to \mathbb{C}$$

satisfying the compatibility equations

$$f_\beta = g_{\beta\alpha} f_\alpha.$$

Example 3.2. The tautological line bundle. Intuitively, the tautological line bundle over $\mathbb{P}^N$ is the family of lines parameterized by $\mathbb{P}^N$. We will often denote its total space by τ_N. It is defined by the incidence relation

$$\tau_N := \Big\{[z,\ell] \in \mathbb{C}^{N+1} \times \mathbb{P}^N;\ \ z \in \ell\Big\}.$$

Notice that we have a tautological projection

$$\pi : \tau_N \to \mathbb{P}^N,\ \ [z,\ell] \mapsto \ell.$$

To show that τ_N is a complex manifold and $(\tau_N, \pi, \mathbb{P}^N)$ is a holomorphic line bundle we need to construct holomorphic charts on τ_N and to construct local trivializations. We will achieve both goals simultaneously. Consider the canonical open sets

$$U_i = \Big\{[z_0 : \cdots : z_N] \in \mathbb{P}^N;\ \ z_i \neq 0\Big\} \cong \mathbb{C}^N,\ \ i = 0, \cdots, N.$$

Denote by $(\zeta_1, \cdots, \zeta_N)$ the natural coordinates on this open set

$$\zeta_k = \zeta_k(\ell) := \begin{cases} z_{k-1}/z_i & k \leq i \\ z_k/z_i & k > i \end{cases}. \tag{3.1}$$

We can use these coordinates to introduce local coordinates $(u_0, u_1, \cdots, u_N)$ on

$$\pi^{-1}(U_i) \cong \Big\{(z_0, \cdots, z_N;\ \ell) \in \mathbb{C}^{N+1} \times U_i;\ \ (z_0, \cdots, z_N) \in \ell\Big\}.$$

More precisely, we set

$$u_0 := z_i,\ \ u_k := \zeta_k(\ell),\ \ k = 1, \cdots, N.$$

Observe that the equalities (3.1) lead to the fundamental equalities

$$z_k = u_{k+1}u_0,\ \ 0 \leq k < i,\ \ z_i = u_0,\ \ z_k = u_k u_0,\ k > i. \tag{3.2}$$

We define a trivialization

$$\pi^{-1}(U_i) \to \mathbb{C} \times U_i$$

by

$$\pi^{-1}(U_i) \ni (z_0, \cdots, z_N; \ell) \mapsto (z_i; \zeta_1, \cdots, \zeta_N) = (u_0; u_1, \cdots, u_N).$$

From this description it is clear that the gluing cocycle is given by

$$g_{ji}([z_0; \cdots; z_N]) = z_j/z_i.$$

The zero section of this bundle is the holomorphic map

$$u : \mathbb{P}^N \cong \{0\} \times \mathbb{P}^N \hookrightarrow \tau_N \subset \mathbb{C}^{N+1} \times \mathbb{P}^N.$$

The image of the zero section is a hypersurface of τ_N which in the local coordinates (u_k) on $\pi^{-1}(U_i)$ is described by the equation

$$u_0 = 0.$$

Observe that the complement of the zero section in τ_N is naturally isomorphic to $\mathbb{C}^{N+1} \setminus \{0\}$. The isomorphism is induced by the natural projection

$$\beta_N : \mathbb{P}^N \setminus \tau_N(\mathbb{P}^N) \twoheadrightarrow \mathbb{C}^{N+1} \setminus \{0\}.$$

The equation (3.2) represents a local coordinate description of the *blowdown map* β_N.

To understand the subtleties of the above constructions it is instructive to consider the special case of the tautological line bundle over $\mathbb{P}^1$. The projective line can be identified with the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$. The two open sets U_0 and U_1 on $\mathbb{P}^1$ correspond to the canonical charts

$$U_0 = V_N := S^2 \setminus \text{South Pole}, \;\; U_1 := V_S := S^2 \setminus \text{North Pole}$$

with coordinates $z = z_1/z_0$ (on V_N) and $\zeta = z_0/z_1$ (on V_S) related by $\zeta := 1/z$. On the overlap

$$U_{01} = S^2 \setminus \{\text{North and South Pole}\}$$

with coordinate z, the transition function g_{10} is given by

$$g_{10}(z) = g_{SN}(z) = z_1/z_0 = z.$$

The total space is covered by two coordinate charts

$$W_N = \pi^{-1}(U_N), \;\; W_S := \pi^{-1}(V_N)$$

with coordinates given by

$$z_0 = u, z_1 = uv, \;\text{ on }\; W_N$$

and

$$z_0 = uv, z_1 = u \;\text{ on }\; W_S.$$

There are several functorial operations one can perform on line bundles. We will describe some of them by explaining their effect on gluing cocycles.

Suppose we are given two holomorphic line bundles $L, \tilde{L} \to M$ defined by the open cover (U_α) and the holomorphic gluing cocycles

$$g_{\beta\alpha}, \; \tilde{g}_{\beta\alpha} : U_{\alpha\beta} \to \mathbb{C}^*.$$

The *dual* of L is the holomorphic line bundle L^* defined by the holomorphic gluing cocycle

$$1/g_{\beta\alpha} : U_{\beta\alpha} \to \mathbb{C}^*.$$

The *tensor product* of the line bundle $L, \tilde{L}$ is the line bundle $L \otimes \tilde{L}$ defined by the gluing cocycle $g_{\beta\alpha}\tilde{g}_{\beta\alpha}$.

A *bundle morphism* $L \to \tilde{L}$ is a holomorphic section of $\tilde{L} \otimes L^*$. Equivalently, a bundle morphism is a holomorphic map $L \to \tilde{L}$ such that for every $m \in M$ we have $\phi(L_m) \subset \tilde{L}_m$ and the induced map $L_m \to \tilde{L}_m$ is linear. The notion of bundle isomorphism is defined in an obvious fashion. We denote by $\mathrm{Pic}(M)$ the set of isomorphism classes of holomorphic line bundles over M The tensor product induces an *Abelian group* structure on $\mathrm{Pic}(M)$. The trivial line bundle $\underline{\mathbb{C}}_M$ is the neutral element while the inverse of a line bundle is given by its dual.

A notion intimately related to the notion of line bundle is that of *divisor*. Roughly speaking, a divisor is a formal linear combination over $\mathbb{Z}$ of codimension-1 subvarieties. We present a few examples which will justify the more formal definition to come.

Example 3.3. (a) Suppose $f : D \to \mathbb{C}$ is a holomorphic function defined on the unit open disk in $\mathbb{C}$ such that $f^{-1}(0) = \{0\}$. The origin is a codimension one subvariety and so it defines a divisor (0) on D. We define the zero divisor of f by

$$(f)_0 = n(0)$$

where n is the multiplicity of 0 as a root of f. ($n=$ Milnor number of f at zero $+1$.)
(b) Suppose $f : D \to \mathbb{C}\cup\{\infty\}$ is meromorphic suppose its zeros are (ζ_i) with multiplicities n_i while its poles are (μ_j) of orders (m_j). The *zero divisor* of f is the formal linear combination

$$(f)_0 = \sum_i n_i \zeta_i$$

while the *polar divisor* is

$$(f)_\infty = \sum_j m_j \mu_j.$$

The *principal divisor* defined by f is

$$(f) = (f)_0 - (f)_\infty = (f)_0 - (1/f)_0.$$

Observe that if $g : D \to \mathbb{C}$ is a holomorphic, nowhere vanishing function, then $(gf) = (f)$.
(c) More generally, if M is a complex manifold and $f : M \to \mathbb{C} \cup \{\infty\}$ is a meromorphic function, i.e. a holomorphic map $f : M \to \mathbb{P}^1$, then the principal divisor associated to f is the formal combination of subvarieties

$$(f) = (f^{-1}(0)) - (f^{-1}(\infty)).$$

What's hidden in this description is the notion of multiplicity which needs to be incorporated.
(d) A codimension 1 submanifold V of a complex manifold M defines a divisor on M.

In general, a divisor is obtained by patching the principal divisors of a family of *locally defined* meromorphic functions. Concretely a divisor is described by an open cover (U_α) and a collection of meromorphic functions

$$f_\alpha : U_\alpha \to \mathbb{C} \cup \{\infty\}$$

such that on the overlaps $U_{\alpha\beta}$ the ratios f_α/f_β are *nowhere vanishing holomorphic functions.* This means that on the overlaps f_α and f_β have zeros/poles of the same order.

A divisor is called *effective* if the defining functions f_α are holomorphic. A meromorphic function $f : M \to \mathbb{C} \cup \{\infty\}$ defines a divisor (f) called the *principal divisor* determined by f. We denote by $\emptyset$ the divisor determined by the constant function 1. We denote by $\mathbf{Div}(M)$ the set of divisors on M and by $\mathbf{PDiv}(M)$ the set of principal divisors.

To a divisor D with defining functions f_α one can associate a line bundle $[D]$ described by the gluing cocycle

$$g_{\beta\alpha} = f_\beta/f_\alpha.$$

If D, E are two divisors described by the defining functions f_α and respectively g_α, we denote by $D+E$ the divisor described by $f_\alpha g_\beta$. Also, denote by $-D$ the divisor described by $(1/f_\alpha)$. Observe that

$$D + (-D) = \emptyset$$

and $(\mathbf{Div}(M), +)$ is an abelian group, and $\mathbf{PDiv}$ is a subgroup. Since

$$[D+E] = [D] \otimes [E], \;\; [-D] = [D]^* \text{ in } \operatorname{Pic}(M)$$

the map

$$\mathbf{Div}(M) \ni D \mapsto [D] \in \operatorname{Pic}(M)$$

is a morphism of Abelian groups. Its kernel is precisely $\mathbf{PDiv}(M)$ and thus we obtain an injective morphism

$$\mathbf{Div}(M)/\mathbf{PDiv}(M) \to \operatorname{Pic}(M).$$

A theorem of Hodge-Lefschetz states that this map is an isomorphism when M is an algebraic manifold (see [14, Sec. I.2]).

Example 3.4. Consider the tautological line bundle $\tau_N \to \mathbb{P}^N$. Its dual is called the *hyperplane line bundle* and is denoted by H_N. If $(U_i)_{i=0,\cdots,N}$ is the standard atlas on $\mathbb{P}^n$ we see that H_N is given by the gluing cocycle

$$g_{ji} = z_i/z_j.$$

We claim that any linear function

$$A : \mathbb{C}^{N+1} \to \mathbb{C}, \;\; (z_0, z_1, \cdots, z_N) \mapsto a_0 z_0 + \cdots + a_N z_N$$

defines a section of H. More precisely define

$$A_i : U_i \to \mathbb{C}, \;\; A_i([z_0 : \cdots : z_N]) = \frac{1}{z_i} A(z_0, \cdots, z_N).$$

Clearly

$$A_j = (z_i/z_j) A_i = g_{ji} A_i$$

which proves the claim.

Similarly, any degree d homogeneous polynomial P in the variables $z_0, \cdots, z_N$ defines a holomorphic section of H^d. We thus have constructed an injection

$$\mathcal{P}_{d,N} \hookrightarrow \mathcal{O}_{\mathbb{P}^N}(H^d).$$

In fact, this map is an isomorphism (see [14, Sec. I.3]).

§3.2 The blowup construction

To understand this construction consider the following ideal experiment. Suppose we have two ants A_1, A_2 walking along two fibers of the tautological line bundle τ_N towards the image of the zero section. The ants have "shadows", namely the points $\beta_N(A_i) \in \mathbb{C}^{N+1}$ by S_i, $i = 1, 2$. These shadows travel towards the origin of $\mathbb{C}^{N+1}$ along two different lines. As the shadows get closer and closer to the origin, in reality, the ants are far apart, approaching the distinct points of $\mathbb{P}^N$ corresponding to the two lines described by the shadows. This separation of trajectories is the whole point of the blowup construction which we proceed to describe rigorously.

Suppose M is complex manifold of dimension N and m is a point in M. The *blowup of M at m* is the complex manifold $\hat{M}_m$ constructed as follows.

1. Choose a small open neighborhood U of M biholomorphic to the open unit ball $B \subset \mathbb{C}^N$. Set

$$\hat{U}_m := \beta_{N-1}^{-1}(B) \subset \tau_{N_1}.$$

2. The blowdown map β_{N-1} establishes an isomorphism

$$\hat{U}_m \setminus \mathbb{P}^{N-1} \cong B \setminus \{0\} \cong U \setminus \{m\}.$$

Now glue $\hat{U}_m$ to $M \setminus \{m\}$ using the blowdown map to obtain $\hat{M}_m$.

Observe that there exists a natural holomorphic map $\beta : \hat{M}_m \to M$ called the *blowdown map*. The fiber $\beta^{-1}(m)$ is called the *exceptional divisor* and it is a hypersurface isomorphic to $\mathbb{P}^{N-1}$. It is traditionally denoted by E. Observe that the map

$$\beta : \hat{M}_m \setminus E \to M \setminus \{m\}$$

is biholomorphic.

Example 3.5. τ_{N-1} is precisely the blowup of $\mathbb{C}^N$ at the origin

$$\tau_{N-1} \cong \hat{\mathbb{C}}^N_0.$$

Exercise 3.6. Prove that the blowup of the complex manifold M at a point m is diffeomeorphic in an orientation preserving fashion to the connected sum

$$M \# \bar{\mathbb{P}}^N$$

where $\bar{\mathbb{P}}^N$ denotes the oriented smooth manifold obtained by changing the canonical orientation of $\mathbb{P}^N$.

Definition 3.7. Suppose $m \in M$ and S is a closed subset in M. The **proper transform** of S in $\hat{M}_m$ is the closure of $\beta^{-1}(S \setminus \{m\})$ in $\hat{M}_m$. We will denote it by $S_m^\flat$.

The following examples describes some of the subtleties of the proper transform construction.

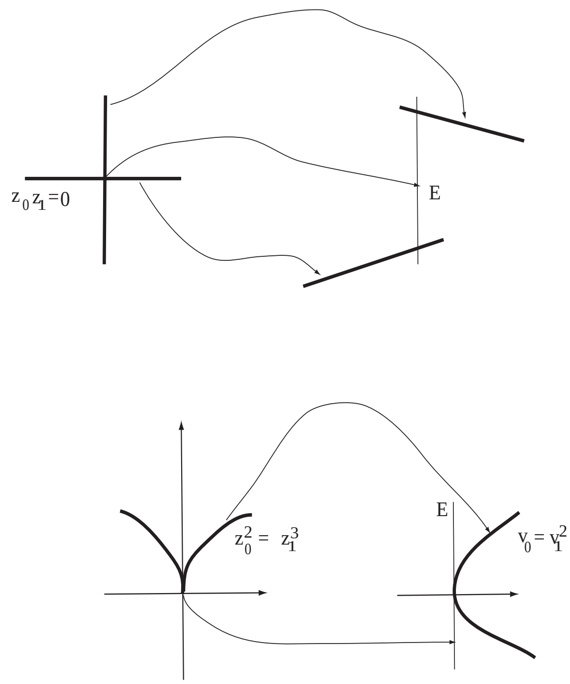

Figure 3.1: *Proper transforms of singular curves*

Example 3.8. (a) Consider the set

$$S = \{z_0 z_1 = 0\} \subset M := \mathbb{C}^2.$$

It consists of the two coordinate axes. We want to describe $S_0^\flat \subset \hat{M}_0$.

The blowup $\hat{M}_0$ is covered by two coordinate charts

$$\Big\{V_0, (u_0, u_1);\ \ z_0 = u_0,\ z_1 = u_0 u_1\Big\}$$

and

$$\Big\{V_1, (v_0, v_1);\ \ z_1 = v_0, z_0 = v_0 v_1\Big\}.$$

Inside V_0, the set $S^\flat \setminus E = \beta^{-1}(S \setminus 0)$ has the description

$$u_0^2 u_1 = 0,\ \ u_0 \neq 0 \Longleftrightarrow u_1 = 0,\ \ u_0 \neq 0$$

while inside V_1 it has the description

$$v_0^2 v_1 = 0,\ \ v_0 \neq 0 \Longleftrightarrow v_1 = 0,\ \ v_0 \neq 0.$$

On the overlap $V_0 \cap V_1$ we have the transition rules

$$u_0 = z_0 = v_0 v_1, \quad u_1 = z_1/z_0 = 1/v_1.$$

We see that $S^\flat \setminus E \cap (V_0 \cap V_1) = \emptyset$. The proper transform of S consists of two fibers of the tautological line bundle $\tau_1 \to \mathbb{P}^1$, namely the fibers above the poles (see Figure 3.1).

(b) Consider

$$S = \{z_0^2 = z_1^3\} \subset M := \mathbb{C}^2.$$

Inside V_0 the set $S^\flat \setminus E$ has the description

$$S_0 : \;\; u_0^2(1 - u_0 u_1^3) = 0, \;\; u_0 \neq 0$$

while inside V_1 it has the description

$$S_1 : \;\; v_0^2(v_0 - v_1^2) = 0, \;\; v_0 \neq 0.$$

Observe that the closure of S_0 in V_0 does not meet the exceptional divisor. The closure of S_1 inside V_1 is the parabola $v_0 = v_1^2$ which is tangent to the exceptional divisor at the point $v_0 = 0 = v_1$ (see Figure 3.1).

Chapter 4

Linear systems

§4.1 Some fundamental constructions

Loosely speaking, a *linear system* is a holomorphic family of divisors parametrized by a projective space. Instead of a formal definition we will analyze a special class of examples. For more information we refer to [14].

Suppose $X \hookrightarrow \mathbb{P}^N$ is a compact submanifold of dimension n. Each $P \in \mathcal{P}_{d,N} \setminus \{0\}$ determines a (possibly singular) hypersurface

$$V_P := \Big\{[z_0 : \cdots : z_N] \in \mathbb{P}^N; P(z_0, \cdots, z_N) = 0\Big\}.$$

The intersection

$$X_P := X \cap V_P$$

is a degree-d hypersurface (thus a divisor) on X. Observe that V_P and X_P depend only on the image $[P]$ of P in the projectivization $\mathbb{P}(d, N)$ of $\mathcal{P}_{d,N}$.

Each projective subspace $U \subset \mathbb{P}(d, N)$ defines a family $(X_P)_{[P]\in U}$ of hypersurfaces on X. This is a *linear system.* When $\dim U = 1$, i.e. U is a projective line, we say that the family $(X_P)_{P\in U}$ is a *pencil.* The intersection

$$B = B_U := \bigcap_{P\in U} X_P$$

is called the *base locus* of the linear system. The points in B are called *basic points.*

Any point $x \in X \setminus B$ determines a hyperplane $H_x \in U$ described by the equation

$$H_x := \{P \in U;\ \ P(x) = 0\}.$$

The hyperplanes of U determine a projective space $\check{U}$, the dual of U. (Observe that if U is 1-dimensional then $U = \check{U}$.) We see that a linear system determines a holomorphic map

$$f_U : X_* := X \setminus B \to \check{U},\ \ x \mapsto H_x.$$

We define the *modification of* X determined by the linear system $(X_P)_{P\in U}$ to be the variety

$$\hat{X} = \hat{X}_U = \Big\{(x, H) \in X \times \check{U};\ \ P(x) = 0,\ \ \forall P \in H \subset U\Big\}.$$

When $\dim U = 1$ this has the simpler description

$$\hat{X} = \hat{X}_U = \Big\{(x, P) \in X \times U;\ \ P(x) = 0 \Longleftrightarrow x \in V_P\Big\}.$$

We have a pair of holomorphic maps induced by the natural projections

$$\begin{tikzcd} & \hat{X}_U \subset X \times \check{U} \arrow[dl, "\pi_X"'] \arrow[dr, "\hat{f}_U"] & \\ X \arrow[rr, dashed, "f_U"] & & \check{U} \end{tikzcd}.$$

Observe that π_X induces a biholomorphic map $\hat{X}_* := \pi_X^{-1}(X_*) \to X_*$ and we have a commutative diagram

$$\begin{tikzcd} & \hat{X}_* \arrow[dl, "\pi_X"'] \arrow[dr, "\hat{f}_U"] & \\ X_* \arrow[ur] \arrow[rr, "f_U"] & & \check{U} \end{tikzcd}$$

In general, B and $\hat{X}_U$ are not smooth objects. Also, observe that when $\dim U = 1$ the map $\hat{f} : \hat{X} \to \check{U}$ can be regarded as a map to U.

Example 4.1. (Pencils of cubics) Consider two homogeneous cubic polynomials $A, B \in \mathcal{P}_{3,2}$ (in the variables z_0, z_1, z_2). For generic A, B these are smooth, cubic curves in $\mathbb{P}^2$ and the genus formula tells us they are homemorphic to tori. By Bézout's theorem, these two general cubics meet in 9 distinct points, $p_1, \cdots, p_9$. For $\mathbf{t} := [t_0 : t_1] \in \mathbb{P}^1$ set

$$C_{\mathbf{t}} := \{[z_0 : z_1 : z_2] \in \mathbb{P}^2;\ \ t_0 A(z_0, z_1, z_2) + t_1 B(z_0, z_1, z_2) = 0\}.$$

The family $C_{\mathbf{t}}$, $\mathbf{t} \in \mathbb{P}^1$, is a pencil on $X = \mathbb{P}^2$. The base locus of this system consists of the nine points $p_1, \cdots, p_9$ common to all the cubics. The modification

$$\hat{X} := \Big\{([z_0, z_1, z_2], \mathbf{t}) \in \mathbb{P}^2 \times \mathbb{P}^1;\ \ t_0 A(z_0, z_1, z_2) + t_1 B(z_0, z_1, z_2) = 0\Big\}$$

is isomorphic to the blowup of X at these nine points

$$\hat{X} \cong \hat{X}_{p_1, \cdots, p_9}.$$

For general A, B the induced map $\hat{f} \to \mathbb{P}^1$ is Morse, and its generic fiber is a torus (or equivalently, an elliptic curve). The manifold $\hat{X}$ is a basic example of elliptic fibration. It is usually denoted by $E(1)$.

Exercise 4.2. Prove the claim in the above example that

$$\hat{X} \cong \hat{X}_{p_1, \cdots, p_9}.$$

Remark 4.3. When studying linear systems defined by projective subspaces $U \subset \mathbb{P}(d, N)$ it suffices to consider only the case $d = 1$, i.e. linear systems of hyperplanes. This follows easily using the *Veronese embedding*

$$\mathcal{V}_{d,N} : \mathbb{P}^N \hookrightarrow \mathbb{P}(d, N), \ \ [\vec{z}] \mapsto [(z_\omega)] := [(\vec{z}^\omega)_{|\omega|=d}]$$

where $\vec{z} \in \mathbb{C}^{N+1} \setminus \{0\}$

$$\omega = (\omega_0, \cdots, \omega_N) \in \mathbb{Z}_+^{N+1}, \ \ |\omega| = \sum_{i=0}^{N} \omega_i, \ \ \vec{z}^\omega = \prod_{i=0}^{N} z_i^{\omega_i} \in \mathcal{P}(|\omega|, N).$$

Any $P = \sum_{|\omega|=d} p_\omega \vec{z}^\omega \in \mathcal{P}(d, N)$ defines a hyperplane in $\mathbb{P}(d, N)$

$$H_P = \{ \sum_{|\omega|=d} p_\omega z_\omega = 0 \}.$$

Observe that

$$\mathcal{V}(V_P) \subset H_P$$

so that

$$\mathcal{V}(X \cap V_P) = \mathcal{V}(X) \cap H_P.$$

Definition 4.4. A **Lefschetz pencil** on $X \hookrightarrow \mathbb{P}^N$ is a pencil determined by a one dimensional projective subspace $U \hookrightarrow \mathbb{P}(d, N)$ with the following properties.

(a) The base-locus B is either empty or it is a **smooth**, codimension 2-submanifold of X.
(b) $\hat{X}$ is a **smooth** manifold.
(c) The holomorphic map $\hat{f} : \hat{X} \to U$ is a **Morse function**.

If the base locus is empty, then $\hat{X} = X$ and the Lefschetz pencil is called a **Lefschetz fibration**.

We have the following genericity result. Its proof can be found in [21, Sec.2].

Theorem 4.5. *Fix a compact submanifold $X \hookrightarrow \mathbb{P}^N$. Then for any* generic *projective line $U \subset \mathbb{P}(d, N)$, the pencil $(X_P)_{P \in U}$ is Lefschetz.*

§4.2 Projections revisited

According to Remark 4.3, it suffices to consider only pencils generated by degree 1 polynomials. In this case, the pencils can be given a more visual description.

Suppose $X \hookrightarrow \mathbb{P}^N$ is a compact complex manifold. Fix a $N - 2$ dimensional projective subspace $A \hookrightarrow \mathbb{P}^N$ called the *axis*. The hyperplanes containing A form a line in $U \subset \check{\mathbb{P}}^N \cong \mathbb{P}(1, N)$. It can be identified with any line in $\mathbb{P}^N$ which does not intersect A. Indeed if S is such a line (called *screen*) then any hyperplane H containing A intersects S a single point $s(H)$. We have thus produced a map

$$U \ni H \mapsto s(H) \in S.$$

Conversely, any point $s \in S$ determines an unique hyperplane $[As]$containing A and passing through s. The correspondence

$$S \ni s \mapsto [As] \in U$$

is the inverse of the above map; see Figure 4.1. The base locus of the linear system

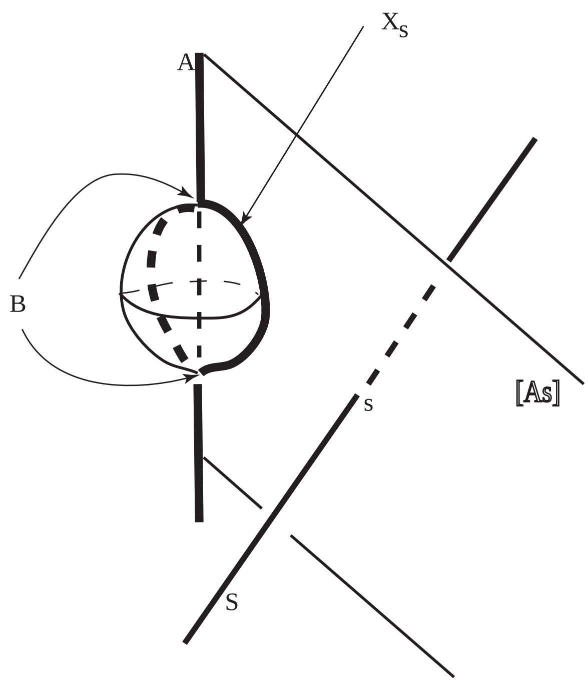

Figure 4.1: *Projecting onto the "screen" S*

$$X_s = [As] \cap X)_{s \in S}$$

is $B = X \cap A$. All the hypersurfaces X_s pass through the base locus B. For generic A this is a smooth, codimension 2-submanifold of X. We have a natural map

$$f : X \setminus B \to S, \;\; x \mapsto [Ax] \cap S.$$

We can now define the elementary modification of X to be

$$\hat{X} := \Big\{ (x,s) \in X \times S; \;\; x \in X_s \Big\}.$$

The critical points of $\hat{f}$ correspond to the hyperplanes through A which contain a tangent (projective) plane to X. We have a similar diagram

$$\begin{tikzcd} & \hat{X} \arrow[dl, "\pi"'] \arrow[dr, "\hat{f}"] & \\ X \arrow[rr, dashed, "f"] & & S \end{tikzcd}.$$

We define

$$\hat{B} := \pi^{-1}(B).$$

Observe that

$$\hat{B} := \Big\{(b,s) \in B \times S;\;\; b \in [As]\Big\} = B \times S,$$

and the natural projection $\pi : \hat{B} \to B$ coincides with the projection $B \times S \twoheadrightarrow B$. Set

$$\hat{X}_s := \hat{f}^{-1}(s).$$

The projection π induces a homeomorphism $\hat{X}_s \to X_s$.

Example 4.6. Observe that when $N = 2$ then A is a point. Assume that $X \hookrightarrow \mathbb{P}^2$ is a degree d smooth curve as $A \notin X$. We have used the above construction in the proof of the genus formula. There we proved that, generically, every Lefschetz pencil on X has exactly $d(d-1)$ critical points.

Example 4.7. Suppose X is the plane

$$\{z_3 = 0\} \cong \mathbb{P}^2 \hookrightarrow \mathbb{P}^3.$$

Assume A is the line $z_1 = z_2 = 0$ and S is the line $z_0 = z_3 = 0$. The base locus consists of the single point $B = [1:0:0:0] \in X$. The pencil obtained in this fashion consists of all lines passing through B.

Observe that $S \subset X$. Moreover, the line S can be identified with the line at ∞ in $\mathbb{P}^2$. The map $f : X \setminus \{B\} \to S$ determined by this pencil is simply the projection onto the line at ∞ with center B. The modification of X defined by this pencil is precisely the blowup of $\mathbb{P}^2$ at B.

Chapter 5

Topological applications of Lefschetz pencils

The existence of a Lefschetz pencil imposes serious restrictions on the topology of an algebraic manifold. In this lecture we will discuss some of them. Our presentation follows closely [21].

§5.1 Topological preliminaries

Before we proceed with our study of Lefschetz pencil we need to isolate a few basic facts of algebraic topology. An important technical result in the sequel will be *Ehresmann fibration theorem.*

Theorem 5.1. ([12, Ehresmann]) *Suppose $\Phi : E \to B$ is a smooth map between two smooth manifolds such that*

- *Φ is* **proper***, i.e. $\Phi^{-1}(K)$ is compact for every compact $K \subset B$.*
- *Φ is a* **submersion***, i.e. $\dim E \geq \dim B$ and F has no critical points.*
- *If $\partial E \neq \emptyset$ then the restriction $\partial\Phi$ of Φ to ∂E continues to be a submersion.*

Then $\Phi : (E, \partial E) \to B$ is a smooth **fiber bundle***, i.e. there exists a smooth manifold F, called the* **standard fiber** *and an open cover $(U_i)_{i \in I}$ of B with the following property. For every $i \in I$ there exists a diffeomorphism*

$$\Psi_i : \Phi^{-1}(U_i) \to F \times U_i$$

such that the diagram below is commutative.

$$\begin{tikzcd}
\Big(\Phi^{-1}(U_i), (\partial\Phi)^{-1}(U_i)\Big) \arrow[rr, "\Psi_i"] \arrow[dr, "\Phi"'] & & (F, \partial F) \times U_i \arrow[dl, "\mathbf{proj}"] \\
& U_i &
\end{tikzcd}.$$

The above result implies immediately that the fibers of Φ are all compact manifolds diffeomorphic to the standard fiber F.

Exercise 5.2. Use Ehresmann fibration theorem to show that if $X \hookrightarrow \mathbb{P}^N$ is an n-dimensional algebraic manifold and $P_1, P_2 \in \mathcal{P}_{d,N}$ are two generic polynomials then

$$V_{P_1} \cap X \cong_{diffeo} X \cap V_{P_2}.$$

Hint: *Consider the set*

$$\mathcal{Z} := \{(x, [P]) \in X \times \mathbb{P}(d, N);\ \ P(x) = 0\}$$

Show it is a complex manifold and analyze the map

$$\pi : \mathcal{Z} \to \mathbb{P}(d, N),\ \ (x, [P]) \mapsto [P].$$

Prove that the set of its regular values is open and **connected** *and then use Ehresmann fibration theorem.*

In the sequel we will frequently use the following consequence of the excision theorem for singular homology.

Suppose $f(X, A) \to (Y, B)$ is a continuous mapping between pairs of compact Euclidean neighborhood retracts (ENR's), such that

$$f : X \setminus A \to Y \setminus B$$

is a homeomorphism. Then f induces an isomorphism

$$f_* : H_*(X, A; \mathbb{Z}) \to H_*(Y, B; \mathbb{Z}).$$

Instead of rigorously defining the notion of ENR let us mention that the zero set of a smooth map $F : \mathbb{R}^n \to \mathbb{R}^m$ is an ENR. We refer to [4, Appendix E] for more details about ENR's.

Exercise 5.3. Prove the above excision result.

In the sequel, unless otherwise stated, $H_*(X)$ (resp. $H^*(X)$) will denote the integral singular homology (resp. cohomology) of the space X. For every compact oriented, m-dimensional manifold M denote by PD_M the Poincaré duality map

$$H^q(M) \to H_{m-q}(M),\ \ u \mapsto u \cap [M].$$

The orientation conventions for the $\cap$-product are determined by the equality

$$\langle v \cup u, c\rangle = \langle v, u \cap c\rangle,$$

where $\langle \bullet, \bullet\rangle$ denotes the Kronecker pairing $H^* \times H_* \to \mathbb{Z}$. This convention is compatible with the fiber-first orientation convention for bundles. Recall that this means that if $F \hookrightarrow E \twoheadrightarrow B$ is a smooth fiber bundle, with oriented base B and standard fiber F then the total space is equipped with the orientation

$$\mathbf{or}(E) = \mathbf{or}(F) \wedge \mathbf{or}(B).$$

§5.2 The set-up

Suppose $X \hookrightarrow \mathbb{P}^N$ is an n-dimensional algebraic manifold, and $S \subset \mathbb{P}(d, N)$ is a one dimensional projective subspace defining a Lefschetz pencil $(X_s)_{s\in S}$ on X. As usual, denote by B the base locus

$$B = \bigcap_{s\in S} X_s$$

and by $\hat{X}$ the modification

$$\hat{X} = \Big\{(x, s) \in X \times S;\ \ x \in X_s\Big\}.$$

We have an induced *Lefschetz fibration* $\hat{f} : \hat{X} \to S$ with fibers

$$\hat{X}_s := \hat{f}^{-1}(s)$$

and a surjection

$$p : \hat{X} \to X$$

which induces homeomorphisms $\hat{X}_s \to X_s$. Observe that $\deg p = 1$. Set

$$\hat{B} := p^{-1}(B).$$

Observe that we have a tautological diffeomorphism

$$\hat{B} \to\cong B \times S,\ \ \hat{B} \ni (x, s) \mapsto (x, s) \in B \times S.$$

Since $S \cong S^2$ we deduce from Künneth theorem that we have an isomorphism

$$H_q(\hat{B}) \cong H_q(B) \oplus H_{q-2}(B)$$

and a natural injection

$$H_{q-2}(B) \to H_q(\hat{B}),\ \ H_{q-2}(B) \ni c \mapsto c \times [S] \in H_q(\hat{B}).$$

Using the inclusion map $\hat{B} \to \hat{X}$ we obtain a natural morphism

$$\kappa : H_{q-2}(B) \to H_q(\hat{X}).$$

Lemma 5.4. *The sequence*

$$0 \to H_{q-2}(B) \stackrel{\kappa}{\to} H_q(\hat{X}) \stackrel{p_*}{\to} H_q(X) \to 0 \tag{5.1}$$

is exact and splits for every q. In particular, $\hat{X}$ is connected iff X is connected.

Proof The proof will be carried out in several steps.

Step 1 p_* *admits a natural right inverse.* Consider the Gysin morphism

$$p^! : H_q(X) \to H_q(\hat{X}),\ \ p^! = PD_{\hat{X}} p^* PD_X^{-1},$$

i.e. the diagram below is commutative.

$$\begin{tikzcd} H^{2n-q}(X) \arrow[r, "\bullet\cap[X]"] \arrow[d, "p^*"] & H_q(X) \arrow[d, "p^!"] \\ H^{2n-q}(\hat{X}) \arrow[r, "\bullet\cap[\hat{X}]"'] & H_q(\hat{X}) \end{tikzcd}.$$

We will show that $p_*p^! = \mathbf{1}$. Let $c \in H_q(X)$ and set $u := PD_X^{-1}(x)$, $u \cap [X] = c$. Then

$$p^!(c) = p^*(u) \cap [\hat{X}].$$

Then

$$p_*p^!(c) = p_*p^*(u) \cap p_*([\hat{X}]) = u \cap p_*([\hat{X}]) = \deg p(u \cap [X]) = c.$$

Step 2. *Conclusion* We use the long exact sequences of the pairs $(\hat{X}, \hat{B})$, (X, B) and the morphism between them induced by p_*. We have the following commutative diagram

$$\begin{tikzcd} H_{q+1}(\hat{X}) \arrow[r] \arrow[d, "p_*"] & H_{q+1}(\hat{X}, \hat{B}) \arrow[r, "\partial"] \arrow[d, "p'_*"] & H_q(B) \oplus H_{q-2}(B) \arrow[r] \arrow[d, "\mathbf{pr}"] & H_q(\hat{X}) \arrow[r] \arrow[d, "p_*"] & H_q(\hat{X}, \hat{B}) \arrow[d, "p'_*"] \\ H_{q+1}(X) \arrow[r] & H_{q+1}(X, B) \arrow[r, "\partial"] & H_q(B) \arrow[r] & H_q(X) \arrow[r] & H_q(X, B) \end{tikzcd}$$

The excision theorem shows that the morphisms p'_* are isomorphisms. Moreover, p_* is surjective. The conclusion in the lemma now follows by diagram chasing. ■

Exercise 5.5. Complete the diagram chasing argument.

Decompose now the projective line S into two closed hemispheres

$$S := D_+ \cup D_-,\ \ S^1 = D_+ \cap D_-,\ \ \hat{X}_\pm := \hat{f}^{-1}(D_\pm),\ \ \hat{X}_0 := \hat{f}^{-1}(S^1)$$

such that all the critical values of $\hat{f} : \hat{X} \to S$ are contained in the interior of D_+. Choose a point $\bullet$ on the Equator $\partial D_+ \cong S^1$. Denote by r the number of critical points (= the number of critical values) of the Morse function $\hat{f}$. In the remainder of this chapter we will assume the following fact. Its proof is deferred to Chapter 7.

Lemma 5.6. (Key Lemma).

$$H_q(\hat{X}_+, \hat{X}_\bullet) \cong \begin{cases} 0 & \text{if} \quad q \neq n = \dim X \\ \mathbb{Z}^r & \text{if} \quad q = n \end{cases} .$$

Remark 5.7. The number of r of nondegenerate singular points of a Lefschetz pencil defined by *linear polynomials* is a projective invariant of X. Its meaning when X is a plane curve was explained in Chapter 1 and we computed it explicitly in Chapter 2. A similar definition holds in higher dimensions as well; see [21].

§5.3 Lefschetz Theorems

All of the results in this section originate in the remarkable work of S. Lefschetz [23] in the 1920's. We follow the modern presentation in [21].

Using Ehresmann fibration theorem we deduce

$$\hat{X}_- \cong \hat{X}_\bullet \times D_-, \quad \hat{X}_0 \cong \hat{X}_\bullet \times S^1$$

so that

$$(\hat{X}_-, \hat{X}_0) \cong \hat{X}_\bullet \times (D_-, S^1).$$

$\hat{X}_\bullet$ is a deformation retract of $\hat{X}_-$. In particular, the inclusion

$$\hat{X}_\bullet \hookrightarrow \hat{X}_-$$

induces isomorphisms

$$H_*(\hat{X}_\bullet) \cong H_*(\hat{X}_-).$$

Using excision and Künneth formula we obtain the sequence of isomorphisms

$$H_{q-2}(\hat{X}_\bullet) \stackrel{\times[D_-]}{\longrightarrow} H_q(\hat{X}_\bullet \times (D_-, S^1)) \cong H_q(\hat{X}_-, \hat{X}_0) \stackrel{excis}{\longrightarrow} H_q(\hat{X}, \hat{X}_+). \tag{5.2}$$

Consider now the long exact sequence of the triple $(\hat{X}, \hat{X}_+, \hat{X}_\bullet)$,

$$\cdots \to H_{q+1}(\hat{X}_+, \hat{X}_\bullet) \to H_{q+1}(\hat{X}, \hat{X}_\bullet) \to H_{q+1}(\hat{X}, \hat{X}_+) \stackrel{\partial}{\to} H_q(X_+, \hat{X}_\bullet) \to \cdots$$

If we use the **Key Lemma** and the isomorphism (5.2) we deduce that we have the isomorphisms

$$L : H_{q+1}(\hat{X}, \hat{X}_\bullet) \to H_{q-1}(\hat{X}_\bullet), \quad q \neq n, n-1, \tag{5.3}$$

and the 5-term exact sequence

$$0 \to H_{n+1}(\hat{X}, \hat{X}_\bullet) \to H_{n-1}(\hat{X}_\bullet) \to H_n(\hat{X}_+, \hat{X}_\bullet) \to H_n(\hat{X}, \hat{X}_\bullet) \to H_{n-2}(\hat{X}_\bullet) \to 0 \tag{5.4}$$

Here is a first nontrivial consequence.

Corollary 5.8. *If X is connected and $n = \dim X > 1$ then the generic fiber $\hat{X}_\bullet \cong X_\bullet$ is connected.*

Proof Using (5.3) we obtain the isomorphisms

$$H_0(\hat{X}, \hat{X}_\bullet) \cong H_{-2}(\hat{X}_\bullet) = 0, \;\; H_1(\hat{X}, \hat{X}_\bullet) \cong H_{-1}(\hat{X}_\bullet) = 0.$$

Using the long exact sequence of the pair $(\hat{X}, \hat{X}_\bullet)$ we deduce that

$$H_0(\hat{X}_\bullet) \cong H_0(\hat{X}).$$

Since X is connected, Lemma 5.4 now implies $H_0(\hat{X}) = 0$ thus proving the corollary. ■

Remark 5.9. The above connectivity result is a holomorphic phenomenon and it is a special case of Zariski's Connectedness Theorem, [32], or [37, vol. II]. The level sets of a smooth function on a smooth manifold may not be connected. The proof of the corollary does not overtly uses the holomorphy assumption. This condition is hidden in the proof of the **Key Lemma**.

The next result generalizes the Riemann-Hurwitz theorem for Morse maps

$$f : \Sigma \to \mathbb{P}^1, \;\; \Sigma \text{ complex algebraic curve.}$$

Corollary 5.10.

$$\chi(\hat{X}) = 2\chi(\hat{X}_\bullet) + (-1)^n r,$$

$$\chi(X) = 2\chi(X_\bullet) - \chi(B) + (-1)^n r.$$

Proof From (5.1) we deduce

$$\chi(\hat{X}) = \chi(X) + \chi(B).$$

On the other hand, the long exact sequence of the pair $(\hat{X}, \hat{X}_\bullet)$ implies

$$\chi(\hat{X}) - \chi(\hat{X}_\bullet) = \chi(\hat{X}, \hat{X}_\bullet).$$

Using (5.3), (5.4) and the **Key Lemma** we deduce

$$\chi(\hat{X}, \hat{X}_\bullet) = \chi(\hat{X}_\bullet) + (-1)^n r.$$

Thus

$$\chi(\hat{X}) = 2\chi(\hat{X}_\bullet) + (-1)^n r$$

and

$$\chi(X) = 2\chi(\hat{X}_\bullet) - \chi(B) + (-1)^n r. \; ■$$

Example 5.11. Consider again two cubic polynomials $A, B \in \mathcal{P}_{3,2}$ defining a Lefschetz pencil on $\mathbb{P}^2 \hookrightarrow \mathbb{P}^3$. We can use the above corollary to determine the number r of singular points of this pencil. More precisely we have

$$\chi(\mathbb{P}^2) = 2\chi(X_\bullet) - \chi(B) + r.$$

We have seen that B consists of 9 distinct points. According to the genus formula the generic fiber, which is a degree 3 curve, must be a torus, so that $\chi(X_\bullet) = 0$. Finally, $\chi(\mathbb{P}^1) = 3$. We deduce $r = 12$ so that the generic elliptic fibration

$$\hat{\mathbb{P}}^2_{p_1, \cdots, p_9} \to \mathbb{P}^1$$

has 12 singular fibers.

Exercise 5.12. Suppose X is an algebraic surface ($\dim X = 2$) and $(X_s)_{s \in \mathbb{P}^1}$ defines a Lefschetz fibration with generic fiber X_s of genus g. Express the number of singular fibers of X in terms of topological invariants of X and X_s.

Exercise 5.2 shows that the diffeomorphism type of a hypersurface $V_P \cap X$ is independent of the generic polynomial P of *fixed degree*. Moreover, for general P, the hypersurface can be included in a Lefschetz pencil. Hence, studying the topological properties of the embedding

$$V_P \cap X \hookrightarrow X$$

is equivalent to studying the topological properties of the embedding $X_\bullet \hookrightarrow X$.

Theorem 5.13. (Lefschetz Theorem on hypersurface sections) *The inclusion*

$$X_\bullet \hookrightarrow X$$

induces isomorphisms

$$H_q(X_\bullet) \to H_q(X)$$

if $q < \frac{1}{2} \dim_{\mathbb{R}} X_\bullet = n - 1$ *and an epimorphism if* $q = n - 1$*. Equivalently, this means*

$$H_q(X, X_\bullet) = 0, \ \ \forall q \leq n - 1.$$

Proof We will used an argument similar to the one in the proof of (5.3), (5.4). More precisely, we will analyze the long exact sequence of the triple $(\hat{X}, \hat{X}_+ \cup \hat{B}, \hat{X}_\bullet \cup \hat{B})$.

Using excision we deduce

$$H_q(\hat{X}, \hat{X}_+ \cup \hat{B}) = H_q(\hat{X}, \hat{X}_+ \cup B \times D_-) \cong H_q(\hat{X}_-, \hat{X}_0 \cup B \times D_-)$$

(use Ehresmann fibration theorem)

$$\cong H_q\big((X_\bullet, B) \times (D_-, S^1)\big) \cong H_{q-2}(X_\bullet, B).$$

Using the Excision theorem again we obtain an isomorphism

$$p_* : H_q(\hat{X}, \hat{X}_\bullet \cup \hat{B}) \cong H_q(X, X_\bullet).$$

Finally, we have an isomorphism

$$H_*(\hat{X}_+ \cup \hat{B}, \hat{X}_\bullet \cup \hat{B}) \cong H_*(\hat{X}_+, \hat{X}_\bullet). \tag{5.5}$$

Indeed, excise $B \times \mathrm{Int}\,(D_-)$ from both terms of the pair $(\hat{X}_+ \cup \hat{B}, \hat{X}_\bullet \cup \hat{B})$. Then

$$\hat{X}_+ \cup \hat{B} \setminus (B \times \mathrm{Int}\,(D_-)\ ) = \hat{X}_+$$

and, since $\hat{X}_\bullet \cap \hat{B} = \{\bullet\} \times B$, we deduce

$$\hat{X}_\bullet \cup \hat{B} \setminus (B \times \mathrm{Int}\,(D_-)\ ) = \hat{X}_\bullet \cup \Big(D_+ \times B\Big).$$

Observe that

$$\hat{X}_\bullet \cap \Big(D_+ \times B\Big) = \{\bullet\} \times B$$

and $D_+ \times B$ deformation retracts to $\{\bullet\} \times B$. Hence $\hat{X}_\bullet \cup \Big(D_+ \times B\Big)$ is homotopically equivalent to $\hat{X}_\bullet$ thus proving (5.5).

The long exact sequence of the triple $(\hat{X}, \hat{X}_+ \cup \hat{B}, \hat{X}_\bullet \cup \hat{B})$ can now be rewritten

$$\cdots \to H_{q-1}(X_\bullet, B) \stackrel{\partial}{\to} H_q(\hat{X}_+, \hat{X}_\bullet) \to H_q(X, X_\bullet) \to H_{q-2}(X_\bullet, B) \stackrel{\partial}{\to} \cdots$$

Using the **Key Lemma** we obtain the isomorphisms

$$L' : H_q(X, X_\bullet) \to H_{q-2}(X_\bullet, B), \ \ q \neq n, n+1 \tag{5.6}$$

and the 5-term exact sequence

$$0 \to H_{n+1}(X, X_\bullet) \to H_{n-1}(X_\bullet, B) \to H_n(\hat{X}_+, \hat{X}_\bullet) \to H_n(X, X_\bullet) \to H_{n-2}(X_\bullet, B) \to 0. \tag{5.7}$$

We now argue by induction over n. The result is obviously true for $n = 1$. Observe that B is a hypersurface in $X_\bullet$, $\dim_{\mathbb{C}} X_\bullet = n - 1$, and thus, by induction, the map

$$H_q(B) \to H_q(X_\bullet)$$

is an isomorphism for $q \leq n-2$ and an epimorphism for $q = n-2$. Using the long exact sequence of the pair $(X_\bullet, B)$ we deduce that

$$H_q(X_\bullet, B) = 0, \ \ \forall q \leq n-2$$

Using (5.6) we deduce

$$H_q(X, X_\bullet) \cong H_{q-2}(X_\bullet, B) \cong 0, \ \ \forall q \leq n-1.$$

We can now conclude the proof using the long exact sequence of the pair $(X, X_\bullet)$. ■

Corollary 5.14. *If X is a hypersurface in $\mathbb{P}^n$ then*

$$b_k(X) = b_k(\mathbb{P}^n), \quad \forall k \le n-2.$$

In particular, if X is a hypersurface in $\mathbb{P}^3$ then $b_1(X) = 0$.

Consider the connecting homomorphism

$$\partial : H_n(\hat{X}_+, \hat{X}_\bullet) \to H_{n-1}(\hat{X}_\bullet)$$

Its image

$$V := \partial\Big(H_n(\hat{X}_+, \hat{X}_\bullet) \Big) \subset H_{n-1}(\hat{X}_\bullet) = H_{\dim_{\mathbb{C}} \hat{X}_\bullet}(\hat{X}_\bullet)$$

is called the module of *vanishing cycles*. Using the long exact sequences of the pairs $(\hat{X}_+, \hat{X}_\bullet)$ and $(X, X_\bullet)$ and the **Key Lemma** we obtain the following commutative diagram

$$\begin{tikzcd}
H_n(\hat{X}_+, \hat{X}_\bullet) \arrow[r, "\partial"] \arrow[d, two heads, "p_1"] & H_{n-1}(\hat{X}_\bullet) \arrow[r] \arrow[d, "p_2", "\cong"'] & H_{n-1}(\hat{X}_+) \arrow[r] \arrow[d, "p_3", "\cong"'] & 0 \\
H_n(X, X_\bullet) \arrow[r, "\partial"] & H_{n-1}(X_\bullet) \arrow[r] & H_{n-1}(X) \arrow[r] & 0
\end{tikzcd}.$$

All the vertical morphisms are induced by the map $p : \hat{X} \to X$. The morphism p_1 is onto because it appears in the sequence (5.7) where $H_{n-2}(X_\bullet, B) = 0$ by Lefschetz hypersurface section theorem. p_2 is clearly an isomorphism since p induces a homeomorphism $\hat{X}_\bullet \cong X_\bullet$. Using the five lemma we conclude that p_3 is an isomorphism. The above diagram shows that

$$V = \ker\Big(i_* : H_{n-1}(X_\bullet) \to H_{n-1}(X) \Big) = \text{Image}\Big(\partial : H_n(X, X_\bullet) \to H_{n-1}(X_\bullet) \Big), \tag{5.8a}$$

$$rk\, H_{n-1}(X_\bullet) = rk\, V + rk\, H_{n-1}(X). \tag{5.8b}$$

These observations have a cohomological counterpart

$$\begin{tikzcd}
H^n(\hat{X}_+, \hat{X}_\bullet) & H^{n-1}(\hat{X}_\bullet) \arrow[l, "\delta"'] & H^{n-1}(\hat{X}_+) \arrow[l] & 0 \arrow[l] \\
H^n(X, X_\bullet) \arrow[u, hook, "mono"'] & H^{n-1}(X_\bullet) \arrow[l, "\delta"'] \arrow[u, "\cong"'] & H^{n-1}(X) \arrow[l, "i^*"'] \arrow[u] & 0 \arrow[l]
\end{tikzcd}.$$

This diagram shows that

$$I^* := \ker\Big(\delta : H^{n-1}(\hat{X}_\bullet) \to H^n(\hat{X}_+, \hat{X}_\bullet) \Big) \cong \ker\Big(\delta : H^{n-1}(X_\bullet) \to H^n(X, X_\bullet) \Big)$$

$$\cong \operatorname{Im}\Big(i^* : H^{n-1}(X) \to H^{n-1}(X_\bullet)\Big).$$

Define the module of *invariant cycles* to be the Poincaré dual of I^*

$$I := \Big\{u \cap [X_\bullet];\ \ u \in I^*\Big\} \subset H_{n-1}(X_\bullet)$$

or equivalently

$$I = \operatorname{Im}\Big(i^! : H_{n+1}(X) \to H_{n-1}(X_\bullet)\Big),\ \ i^! := PD_{X_\bullet} i^* PD_X^{-1}.$$

Since i^* is $1-1$ on $H^{n-1}(X)$ we deduce $i^!$ is $1-1$ so that

$$rk\, I = rk H_{n+1}(X) = rk\, H_{n-1}(X). \tag{5.9}$$

The last equality implies the following result.

Theorem 5.15. (Weak Lefschetz Theorem)

$$rk\, H_{n-1}(X_\bullet) = rk\, I + rk\, V.$$

Using the **Key Lemma**, the universal coefficients theorem and the equality

$$I^* = \ker\Big(\delta : H^{n-1}(\hat{X}_\bullet) \to H^n(\hat{X}_+, \hat{X}_\bullet)\Big),$$

we deduce

$$I^* = \Big\{\omega \in H^{n-1}(\hat{X}_\bullet);\ \ \langle \omega, v\rangle = 0,\ \ \forall v \in V\Big\}.$$

Observe that $n-1 = \frac{1}{2}\dim \hat{X}_\bullet$ and thus, the Kronecker pairing on $H_{n-1}(X_\bullet)$ is given by the intersection form. This is nondegenerate by Poincaré duality. Thus

$$I := \Big\{y \in H_{n-1}(X_\bullet);\ \ y \cdot v = 0,\ \ \forall v \in V\Big\}. \tag{5.10}$$

* * *

Let us summarize the facts we have proved so far. We defined

$$V := \text{image}\Big(\partial : H_n(X, X_\bullet) \to H_{n-1}(X_\bullet)\Big),$$

$$I := \text{image}\Big(i^! : H_{n+1}(X) \to H_{n-1}(X_\bullet)\Big)$$

and we showed that

$$V = \ker\Big(i_* : H_{n-1}(X_\bullet) \to H_{n-1}(X)\Big),$$

$i^! : H_{n+1}(X) \to H_{n-1}(X_\bullet)$ is $1-1$,

$$I = \Big\{y \in H_{n-1}(X_\bullet);\ \ y \cdot v = 0,\ \ \forall v \in V\Big\},$$

$$rk\, I = rk H_{n+1}(X) = rk\, H_{n-1}(X),$$

$$rk\, H_{n-1}(X_\bullet) = rk\, I + rk\, V.$$

Chapter 6

The Hard Lefschetz theorem

The last theorem in the previous section is only the tip of the iceberg. In this chapter we enter deeper into the anatomy of an algebraic manifold and try to understand the roots of the weak Lefschetz theorem. In this chapter, unless specified otherwise, $H_*(X)$ denotes the homology with coefficients in $\mathbb{R}$. Also, assume for simplicity that the pencil $(X_s)_{s\in\mathbb{P}^1}$ consists of *hyperplane* sections. (We already know this does not restrict the generality.) We continue to use the notations in Lecture 5. Denote by $\omega \in H^2(X)$ the Poincaré dual of the hyperplane section $X_\bullet$, i.e.

$$[X_\bullet] = \omega \cap [X].$$

§6.1 The Hard Lefschetz Theorem

For any cycle $c \in H_q(X)$, its intersection with $X_\bullet$ is a new cycle in $X_\bullet$ of codimension 2 in c, i.e. a $(q-2)$-cycle. This intuitive yet unrigorous operation can be formally described as the cap product with ω

$$\omega\cap : H_q(X) \to H_{q-2}(X)$$

which factors through $X_\bullet$

$$\begin{tikzcd} H_q(X) \arrow[r, "i^!"] \arrow[dr, "\omega\cap"'] & H_{q-2}(X_\bullet) \arrow[d, "i_*"] \\ & H_{q-2}(X) \end{tikzcd} .$$

Proposition 6.1. *The following statements are equivalent.*

$$V \cap I = 0. \tag{$\mathbf{HL}_1$}$$

$$V \oplus I = H_{n-1}(X_\bullet). \tag{$\mathbf{HL}_2$}$$

$$i_* : H_{n-1}(X_\bullet) \to H_{n-1}(X) \text{ maps } I \text{ isomorphically onto } H_{n-1}(X). \qquad (\mathbf{HL}_3)$$

$$\text{The map } \omega\cap : H_{n+1}(X) \to H_{n-1}(X) \text{ is an isomorphism.} \qquad (\mathbf{HL}_4)$$

The restriction of the intersection form on $H_{n-1}(X_\bullet)$ to V remains nondegenerate. $(\mathbf{HL}_5)$

The restriction of the intersection form to I remains nondegenerate. $(\mathbf{HL}_6)$

.

Proof The weak Lefschetz theorem shows that $(\mathbf{HL}_1) \Longleftrightarrow (\mathbf{HL}_2)$.
$(\mathbf{HL}_2) \Longrightarrow (\mathbf{HL}_3)$. Use the fact established in Chapter 5 that

$$V = \ker\Big(i_* : H_{n-1}(X_\bullet) \to H_{n-1}(X)\Big).$$

$(\mathbf{HL}_3) \Longrightarrow (\mathbf{HL}_4)$. In Chaper 5 we proved that $i^! : H_{n+1}(X) \to H_{n-1}(X_\bullet)$ is a monomorphism with image I. By $(\mathbf{HL}_3)$, $i_* : I \to H_{n-1}(X)$ is an isomorphism.

$(\mathbf{HL}_4) \Longrightarrow (\mathbf{HL}_3)$ If $i_* \circ i^! = \omega\cap : H_{n+1}(X) \to H_{n-1}(X)$ is an isomorphism then we conclude that $i_* : \operatorname{Im}(i^!) = I \to H_{n-1}(X)$ is onto. In Lecture 5 we have shown that $\dim I = \dim H_{n-1}(X)$ so that $i_* : H_{n-1}(X_\bullet) \to H_{n-1}(X)$ must be $1-1$.

$(\mathbf{HL}_2) \Longrightarrow (\mathbf{HL}_5)$, $(\mathbf{HL}_2) \Longrightarrow (\mathbf{HL}_6)$. This follows from the fact established in the previous lecture that I is the orthogonal complement of V with respect to the intersection form.

$(\mathbf{HL}_5) \Longrightarrow (\mathbf{HL}_1)$ and $(\mathbf{HL}_6) \Longrightarrow (\mathbf{HL}_1)$. Suppose we have a cycle $c \in V \cap I$. Then

$$c \in I \Longrightarrow c \cdot v = 0, \ \ \forall v \in V$$

while

$$c \in V \Longrightarrow c \cdot z = 0, \ \ \forall z \in I.$$

When the restriction of the intersection from to either V or I is nondegenerate the above equalities imply $c = 0$ so that $V \cap I = 0$. ■

Theorem 6.2. (The Hard Lefschetz Theorem) *The equivalent statements ($\mathbf{HL}_1$)-($\mathbf{HL}_6$) above are true (for the homology with real coefficients).*

This is a highly nontrivial result. Its complete proof requires a sophisticated analytical machinery (Hodge theory) and is beyond the scope of these lectures. We refer the reader to [14, Sec.0.7] for more details. In the remainder of this chapter we will discuss other topological facets of this remarkable theorem.

§6.2 Primitive and effective cycles

Set

$$X_0 := X, \;\; X_1 := X_\bullet, \;\; X_2 := B$$

so that X_{i+1} is a generic smooth hyperplane section of X_i. We can iterate this procedure and obtain a chain

$$X_0 \supset X_1 \supset X_2 \supset \cdots \supset X_n \supset \emptyset.$$

so that $\dim X_q = n - q$, and X_q is a generic hyperplane section of X_{q-1}. Denote by $I_q \subset H_{n-q}(X_q)$ the module of invariant cycles,

$$I_q = \mathrm{Image}\Big(i^! : H_{n-q+2}(X_{q-1}) \to H_{n-q}(X_q)\Big)$$

and is Poincaré dual

$$I_q^* = \mathrm{Image}\Big(i^* : H^{n-q}(X_{q-1}) \to H^{n-q}(X_q)\Big) = PD_{X_q}^{-1}(I_q).$$

The Lefschetz hyperplane section theorem implies that the morphisms

$$i_* : H_k(X_q) \to H_k(X_j), \;\; j \leq q$$

are isomorphisms for $k + q < n$. We conclude by duality that

$$i^* : H^k(X_j) \to H^k(X_q), \;\; (j \leq q)$$

is an isomorphism if $k + q < n$.

Using ($\mathbf{HL}_3$) we deduce that

$$i_* : I_q \to H_{n-q}(X_{q-1})$$

is an isomorphism. Using the above version of the Lefschetz hyperplane section theorem we conclude that

$$i_* \text{ maps } I_q \text{ isomorphically onto } H_{n-q}(X). \tag{$\dagger$}$$

Now observe that

$$I_q^* = \mathrm{Image}(i^* : H^{n-q}(X_{q-1} \to H^{n-1}(X_q))$$

and, by Lefschetz hyperplane section theorem we have the isomorphisms

$$H^{n-q}(X_0) \stackrel{i^*}{\to} H^{n-q}(X_1) \stackrel{i^*}{\to} \cdots \stackrel{i^*}{\to} H^{n-q}(X_{q-1}).$$

Using Poincaré duality we obtain

$$i^! \text{ maps } H_{n+q}(X) \text{ isomorphically onto } I_q. \tag{$\dagger\dagger$}$$

Iterating ($\mathbf{HL}_6$) we obtain

The restriction of the intersection form of $H_{n-q}(X)$ to I_q remains non-degenerate. (†††)

The isomorphism i_* carries the intersection form on I_q to a nondegenerate form on $H_{n-q}(X) \cong H_{n+q}(X)$. When $n-q$ is odd this a skew-symmetric form, and thus the non-degeneracy assumptions implies

$$\dim H_{n-q}(X) = \dim H_{n+q}(X) \in 2\mathbb{Z}.$$

We have thus proved the following result.

Corollary 6.3. *The the odd dimensional Betti numbers $b_{2k+1}(X)$ of X are even.*

Remark 6.4. The above corollary shows that not all even dimensional manifolds are algebraic. Take for example $X = S^3 \times S^1$. Using Künneth formula we deduce

$$b_1(X) = 1.$$

This manifold is remarkable because it admits a complex structure, yet it is not algebraic! As a complex manifold it is known as the *Hopf surface* (see [7, Chap.1]).

The q-th exterior power ω^q is Poincaré dual to the fundamental class

$$[X_q] \in H_{2n-2q}(X)$$

of X_q. Therefore we have the factorization

$$\begin{tikzcd} H_k(X) \arrow[r, "i^!"] \arrow[dr, "\omega^q\cap"'] & H_{k-2q}(X_q) \arrow[d, "i_*"] \\ & H_{k-2q}(X) \end{tikzcd} .$$

Using (††) and (†) we obtain the following generalization of ($\mathbf{HL}_4$).

Corollary 6.5. *For $q = 1, 2, \cdots, n$ the map*

$$\omega^q \cap : H_{n+q}(X) \to H_{n-q}(X)$$

is an isomorphism.

Clearly, the above corollary is equivalent to the **Hard Lefschetz Theorem**. In fact, we can formulate and even more refined version.

Definition 6.6. (a) An element $c \in H_{n+q}(X)$, $0 \leq q \leq n$ is called **primitive** if

$$\omega^{q+1} \cap c = 0.$$

We will denote by $P_{n+q}(X)$ the subspace of $H_{n+q}(X)$ consisting of primitive elements.
(b) An element $z \in H_{n-q}(X)$ is called **effective** if

$$\omega \cap z = 0.$$

We will denote by $E_{n-q}(X)$ the subspace of effective elements.

Observe that

$$c \in H_{n+q}(X) \ \text{ is primitive} \Longleftrightarrow \omega^q \cap c \in H_{n-q}(X) \text{ is effective.}$$

Roughly speaking, a cycle is effective if it does not intersect the "part at infinity of X", $X \cap$ *hyperplane*.

Theorem 6.7. (Lefschetz decomposition) *(a) Every element $c \in H_{n+q}(X)$ decomposes uniquely as*

$$c = c_0 + \omega \cap c_1 + \omega^2 \cap c_2 + \cdots \tag{6.1}$$

where $c_j \in H_{n+q+2j}(X)$ are primitive elements.
(b) Every element $z \in H_{n-q}(X)$ decomposes uniquely as

$$z = \omega^q \cap z_0 + \omega^{q+1} \cap z_1 + \cdots \tag{6.2}$$

where $z_j \in H_{n+q+2j}(X)$ are primitive elements.

Proof Observe that because the above representations are unique and since

$$(6.2) = \omega^q \cap (6.1)$$

we deduce that Corollary 6.5 is a consequence of the Lefschetz decomposition.

Conversely, let us show that (6.1) is a consequence of Corollary 6.5. We will use a descending induction starting with $q = n$. Clearly, a dimension count shows that

$$P_{2n}(X) = H_{2n}(X), \ \ P_{2n-1}(X) = H_{2n-1}(X)$$

and (6.1) is trivially true for $q = n, n-1$. For the induction step it suffices to show that every element $c \in H_{n+q}(X)$ can be written uniquely as

$$c = c_0 + \omega c_1, \ \ c_1 \in H_{n+q+2}(X), \ \ c_0 \in P_{n+q}(X).$$

According to Corollary 6.5 there exists an unique $z \in H_{n+q+2}(X)$ such that

$$\omega^{q+2} \cap z = \omega^{q+1} \cap c$$

so that

$$c_0 := c - \omega \cap z \in P_{n+q}(X).$$

To prove uniqueness, assume

$$0 = c_0 + \omega \cap c_1, \ \ c_0 \in P_{n+q}(X).$$

Then

$$0 = \omega^{q+1} \cap (c_0 + \omega \cap c_1) \Longrightarrow \omega^{q+2} \cap c_1 = 0 \Longrightarrow c_1 = 0 \Longrightarrow c_0 = 0. \ \blacksquare$$

The Lefschetz decomposition shows that the homology of X is completely determined by its primitive part. Moreover, the above proof shows that

$$0 \leq \dim P_{n+q} = b_{n+q} - b_{n+q+2} = b_{n-q} - b_{n-q-2}$$

which imply

$$1 = b_0 \leq b_2 \leq \cdots \leq b_{2\lfloor n/2 \rfloor}, \;\; b_1 \leq b_3 \leq \cdots \leq b_{2\lfloor (n-1)/2 \rfloor + 1},$$

where $\lfloor x \rfloor$ denotes the integer part of x. These inequalities introduce additional topological restrictions on algebraic manifolds. For example, the sphere S^4 cannot be an algebraic manifold because $b_2(S^4) = 0 < b_0(S^4) = 1$.

Chapter 7

The Picard-Lefschetz formulæ

In this lecture we finally take a look at the **Key Lemma** and try to elucidate its origins. We will continue to use the notations in the previous two lectures. This time however $H_*(-)$ will denote the homology with $\mathbb{Z}$-coefficients.

§7.1 Proof of the Key Lemma

Recall that the function $\hat{f} : \hat{X} \to \mathbb{P}^1$ is Morse and its critical values $t_1, \cdots, t_r$ are all in D_+. We denote its critical points by $p_1, \cdots, p_r$, so that

$$\hat{f}(p_j) = t_j, \ \ \forall j.$$

We will identify D_+ with the unit disk at $0 \in \mathbb{C}$. Let us introduce some notations. Let $j = 1, \cdots, r$.

▼ Denote by D_j a closed disk of very small radius ρ centered at $t_j \in D_+$. If $\rho \ll 1$ these disks are pairwise disjoint.
▼ Connect $\bullet \in \partial D_+$ to $t_j + \rho \in \partial D_j$ by a smooth path ℓ_j such that the resulting paths $\ell_1, \cdots, \ell_r$ are disjoint (see Figure 7.1). Set $k_j := \ell_j \cup D_j$, $\ell = \bigcup \ell_j$ and $k = \bigcup k_j$.
▼ Denote by B_j a small ball in $\hat{X}$ centered at p_j.

The proof of the **Key Lemma** will be carried out in several steps.

Step 1 *Localizing around the singular fibers.* Set $L := f^{-1}(\ell)$ and $K := \hat{f}^{-1}(k)$. We will show that $\hat{X}_\bullet$ is a deformation retract of L and K is a deformation retract of $\hat{X}_+$ so that the inclusions

$$(\hat{X}_+, \hat{X}_\bullet) \hookrightarrow (\hat{X}_+, L) \hookleftarrow (K, L)$$

induce isomorphisms of all homology (and homotopy) groups.

Observe that k is a strong deformation retract of D_+ and $\bullet$ is a strong deformation retract of ℓ. Using Ehresmann fibration theorem we deduce that we have fibrations

$$f : L \to \ell, \ \ \hat{f} : \hat{X}_+ \setminus \hat{f}^{-1}\{t_1, \cdots, t_r\} \to D_+ \setminus \{t_1, \cdots, t_r\}.$$

Using the homotopy lifting property of fibrations (see [16, §4.3] we obtain strong deformation retractions

$$L \to \hat{X}_\bullet, \ \ \hat{X}_+ \setminus \hat{f}^{-1}\{t_1, \cdots, t_r\} \to K \setminus \hat{f}^{-1}\{t_1, \cdots, t_r\}.$$

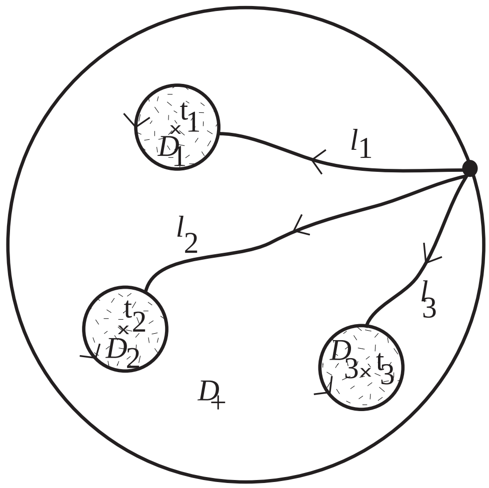

Figure 7.1: *Isolating the critical values*

Step 2 *Localizing near the critical points.* Set $T_j := \hat{f}^{-1}(D_j) \cap B_j$, $F_j := f^{-1}(t_j + \rho) \cap B_j$

$$T := \bigcup T_j, \quad F := \bigcup F_j.$$

The excision theorem shows that the inclusion

$$(B, F) \to (K, L)$$

induces an isomorphism

$$\bigoplus_{j=1}^{r} H_*(T_j, F_j) \to H_*(K, L) \cong H_*(\hat{X}_+, \hat{X}_\bullet).$$

Step 3 *Conclusion* We will show that for every $j = 1, \cdots, r$ we have

$$H_q(T_j, F_j) = \begin{cases} 0 & \text{if} \quad q \neq \dim_{\mathbb{C}} X = n \\ \mathbb{Z} & \text{if} \quad q = n. \end{cases}$$

At this point we need to use the nondegeneracy of p_j. To simplify the presentation, in the sequel we will drop the subscript j.

We can regard B as the unit open ball B centered at $0 \in \mathbb{C}^n$ and $\hat{f}$ as a function $B \to \mathbb{C}$ such that $\hat{f}(0) = 0$ and $0 \in B$ is a nondegenerate critical point of $\hat{f}$. By making B even smaller we can assume the origin is the only critical point. At this point we want to invoke the following classical result. It is a consequence of the more general *Tougeron finite determinacy theorem* which will be proved later in this course. For a direct proof we refer to the classical source [27].

Lemma 7.1. (Morse Lemma) *There exist local holomorphic coordinates* $(z_1, \cdots, z_n)$ *in an open neigborhood* $0 \in U \subset B$ *such that*

$$\hat{f}\mid_U = z_1^2 + \cdots + z_n^2.$$

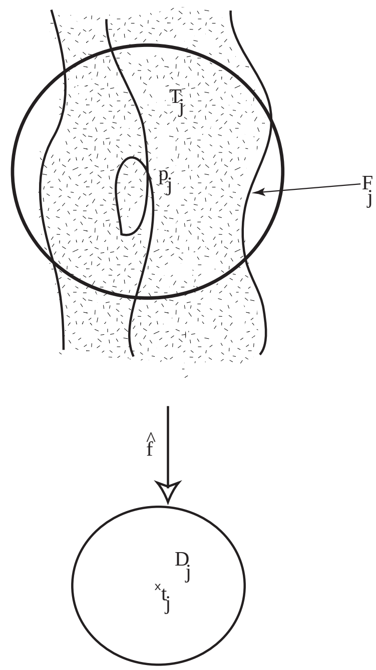

Figure 7.2: *Isolating the critical points*

By making B even smaller we can assume that it coincides with the neighborhood U postulated by Morse Lemma. Now observe that T and F can be given the explicit descriptions

$$T := \Big\{(z_1, \cdots, z_n);\ \sum_i |z_i|^2 \leq \varepsilon^2,\ \ \big|\sum_i z_i^2\big| < \rho\Big\} \tag{7.1}$$

$$F = F_\rho := \Big\{z \in T;\ \sum_i z_i^2 = \rho\Big\}.$$

The description (7.1) shows that T can be contracted to the origin. This shows that the connecting homomorphism

$$H_q(T, F) \to H_{q-1}(F)$$

is an isomorphism for $q \neq 0$. Moreover $H_0(T, F) = 0$. The **Key Lemma** is now a consequence of the following result.

Lemma 7.2. *F_ρ is diffeomeorhic to the disk bundle of the tangent bundle TS^{n-1}.*

Proof Set $z_j := u_j + \mathbf{i}v_j$, $\vec{u} = (u_1, \cdots, u_n)$, $\vec{v} = (v_1, \cdots, v_n)$, $|\vec{u}|^2 := \sum_j u_j^2$, $|\vec{v}|^2 := \sum_j v_j^2$. The fiber F has the description

$$|\vec{u}|^2 = \rho + |\vec{v}|^2,\ \ \vec{u} \cdot \vec{v} = 0 \in \mathbb{R}$$

$$|\vec{u}|^2 + |\vec{v}|^2 \leq \varepsilon^2.$$

Now let

$$\vec{\xi} := (\rho + |v|^2)^{-1/2}\vec{u} \in \mathbb{R}^n.$$

In the coordinates ξ, $\vec{v}$ the fiber F has the description

$$|\vec{\xi}|^2 = 1, \;\; \vec{\xi} \cdot \vec{v} = 0, \;\; 2|\vec{v}|^2 \leq \varepsilon^2 - \rho.$$

The first equality describes the unit sphere $S^{n-1} \subset \mathbb{R}^n$. The condition

$$\vec{\xi} \cdot \vec{v} \Longleftrightarrow \vec{v} \perp \vec{\xi}$$

shows that $\vec{v}$ is tangent to S^{n-1} at ξ. The last inequality shows that the tangent vector $\vec{v}$ has length $\leq \sqrt{(\varepsilon^2 - \rho)/2}$. It is now obvious that F is the disk bundle of TS^{n-1}. This completes the proof of the **Key Lemma**. ■

§7.2 Vanishing cycles, local monodromy and the Picard-Lefschetz formula

We want to analyze in greater detail the picture evolving from the proof of Lemma 7.2. Denote by B a small closed ball centered at $0 \in \mathbb{C}^n$ and consider

$$f : B \to \mathbb{C}, \;\; f(z) = z_1^2 + \cdots + z_n^2.$$

We have seen that the regular fiber of $F = F_\rho = f^{-1}(\rho)$ $(0 < \rho \ll 1)$ is diffeomorphic to a disc bundle over a $n-1$-sphere S_ρ of radius $\sqrt{\rho}$. This sphere is defined by the equation

$$S_\rho := \{\vec{v} = 0\} \cap f^{-1}(\rho) \Longleftrightarrow \{\vec{v} = 0, \;\; |\vec{u}|^2 = \rho\}.$$

As $\rho \to 0$, i.e. we are looking at fibers closer and closer to the singular one $F_0 = f^{-1}(0)$, the radius of this sphere goes to zero , while for $\rho = 0$ the fiber is locally the cone $z_1^2 + \cdots + z_n^2 = 0$. The homology class in F carried by this collapsing sphere generates $H_{n-1}(F)$. This homology class was named *vanishing cycle* by Lefschetz. We will denote it by Δ (see Figure 7.3). The proof of the **Key Lemma** in the previous section shows that Lefschetz' vanishing cycles coincide with what we previously named vanishing cycles.

Observe now that since $\partial : H_n(B, F) \to H_{n-1}(F)$ is an isomorphism, there exists a relative n-cycle $Z \in H_n(B, F)$ such that

$$\partial Z = \Delta.$$

Z is known as the *thimble* determined by the vanishing cycle Δ. It is filled in by the family (S_ρ) of shrinking spheres. In Figure 7.3 it is represented by the shaded disk.

Exercise 7.3. Find an equation describing the thimble.

Denote by $D_r \subset \mathbb{C}$ the open disk of radius r centered at the origin and by $B_r \subset \mathbb{C}^n$ the ball of radius r centered at the origin. We will use the following technical result whose proof is left to the reader as an exercise.

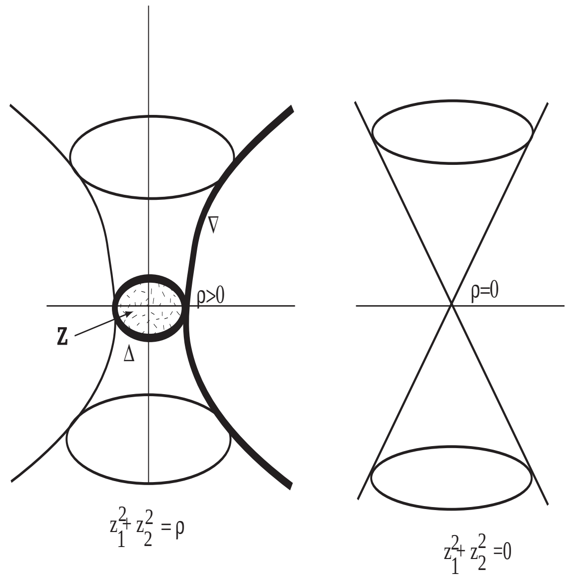

Figure 7.3: *The vanishing cycle for functions of $n = 2$ variables*

Lemma 7.4. *For any $\rho, r > 0$ such that $r^2 > \rho$ the maps*

$$f : X_{\varepsilon,\rho} = \Big(B_r \setminus F_0\Big) \cap f^{-1}(D_\rho) \to D_\rho \setminus \{0\} =: D_\rho^*,$$

$$f_\partial : \partial X_{r,\rho} = \partial B_\varepsilon \cap f^{-1}(D_\rho) \to D_\rho$$

are proper, surjective, *submersions.*

Exercise 7.5. Prove the above lemma.

Set $r = 2$, $\rho = 1 + \varepsilon$, $(0 < \varepsilon \ll 1)$, $X = X_{r=2,\rho=1+\varepsilon}$, $B = B_2$, $D = D_{1+\varepsilon}$. According to the Ehresmann fibration theorem we have the fibrations

$$\begin{array}{ccc} F & \hookrightarrow & X \\ & & \downarrow \\ & & D \setminus \{0\} \end{array}$$

with standard fiber the manifold with boundary $F \cong f^{-1}(1) \cap \bar{B}_2$ and

$$\begin{array}{ccc} \partial F & \hookrightarrow & \partial X \\ & & \downarrow \\ & & D \end{array}$$

with standard fiber $\partial F \cong f^{-1}(1) \cap \partial \bar{B}$. We deduce that ∂X is a trivial bundle

$$\partial X \cong \partial F \times D.$$

We can describe one such trivialization explicitly. Denote by $\mathbb{M}$ the standard model for the fiber, incarnated as the *unit* disk bundle determined by the tangent bundle of the *unit* sphere $S^{n-1} \hookrightarrow \mathbb{R}^n$. $\mathbb{M}$ has the algebraic description

$$\mathbb{M} = \Big\{ (\vec{u}, \vec{v}) \in \mathbb{R}^n \times \mathbb{R}^n; \;\; |\vec{u}| = 1, \;\; \vec{u} \cdot \vec{v} = 0, \;\; |\vec{v}| \leq 1 \Big\}.$$

Note that

$$\partial\mathbb{M} = \Big\{ (\vec{u}, \vec{v}) \in \mathbb{R}^n \times \mathbb{R}^n; \;\; |\vec{u}| = 1 = |\vec{v}|, \;\; \vec{u} \cdot \vec{v} = 0 \Big\}.$$

As in the previous section we have

$$F = f^{-1}(1) = \Big\{ \vec{z} = \vec{x} + \mathbf{i}\vec{y} \in \mathbb{C}^n; \;\; |\vec{x}|^2 + |\vec{y}|^2 \leq 4, \;\; |\vec{x}|^2 = 1 + |\vec{y}|^2, \;\; \vec{x} \cdot \vec{y} = 0 \Big\}$$

and a diffeomorphism

$$\Phi : F \to \mathbb{M}, \quad F \ni \vec{z} = \vec{x} + \mathbf{i}\vec{y} \mapsto \begin{cases} \vec{u} = \frac{1}{(1+|\vec{y}|^2)^{1/2}} \vec{x} \\ \\ \vec{v} = \alpha \vec{y} \end{cases}, \quad \alpha = \sqrt{\frac{2}{3}}.$$

Its inverse is given by

$$\mathbb{M} \ni (\vec{u}, \vec{v}) \overset{\Phi^{-1}}{\mapsto} \begin{cases} \vec{x} = (1 + |\vec{v}|^2/\alpha^2)^{1/2} \vec{u} \\ \vec{y} = \frac{1}{\alpha} \vec{v} \end{cases}.$$

Set

$$F_w := f^{-1}(w) \cap \bar{B}, \;\; 0 \leq |w| \leq 1 + \varepsilon.$$

Note that

$$\partial F_{a+\mathbf{i}b} = \Big\{ \vec{x} + \mathbf{i}\vec{y}; \;\; |\vec{x}|^2 = a + |\vec{y}|^2, \;\; 2\vec{x} \cdot \vec{y} = b, \;\; |\vec{x}|^2 + |\vec{y}| 2 = 4 \Big\}.$$

For every $w = a + \mathbf{i}b \in \bar{D}_1$ define

$$\Gamma_w : \partial F_w \to \partial\mathbb{M}, \quad \partial F_w \ni \vec{x} + \mathbf{i}\vec{y} \mapsto \begin{cases} \vec{u} = c_1(w)\vec{x} \\ \\ \vec{v} = c_3(w)\big(\vec{y} + c_2(w)\vec{x}\big) \end{cases},$$

where

$$c_1(w) = \left(\frac{2}{4+a}\right)^{1/2}, \;\; c_2(w) = -\frac{b}{4+a}, \;\; c_3(w) = \left(\frac{8+2a}{16 - a^2 - b^2}\right)^{1/2}. \tag{7.2}$$

The family $(\Gamma_w)_{|w|<1+\varepsilon}$ defines a trivialization $\partial X \to \partial\mathbb{M} \times D$.

Fix once and for all this trivialization and a metric h on ∂F. We now equip ∂X with the product metric $g_\partial := h \oplus h_0$ where h_0 denotes the Euclidean metric on D_1. Now extend

g_∂ to a metric on X and denote by H the sub-bundle of TX consisting of tangent vectors g-orthogonal to the fibers of f. The differential f_* produces isomorphisms

$$f_* : H_p \to T_{f(p)} D_1^*, \ \ \forall x \in X_{\varepsilon,\rho}.$$

In particular, any vector field V on D_1^* admits a unique horizontal lift, i.e. a smooth section V^h of H such that $f_*(V^h) = V$.

Fix a point $* \in \partial D^*$ and suppose $w : [0,1] \to D^*$ is a closed path beginning and ending at $*$

$$w(0) = w(1) = *.$$

Using the horizontal lift of $\dot{w}$ we obtain for each $p \in f^{-1}(*)$ a smooth path $\tilde{w}_p : [0,1] \to X$ which is tangent to the horizontal sub-bundle H and it is a lift of w starting at p, i.e. the diagram below is commutative

$$\begin{tikzcd} & (X,p) \arrow[d, "f"] \\ ([0,1],0) \arrow[ur, "\tilde{w}_p"'] \arrow[r, "w"] & (D^*,*) \end{tikzcd}.$$

We get in this fashion a map

$$h_w : F = f^{-1}(*) \to f^{-1}(*), \ \ p \mapsto \tilde{w}_p(1).$$

The standard results on the smooth dependence of solutions of ODE's on initial data show that h_w is a smooth map. It is in fact a diffeomorphism of F with the property that

$$h_w \mid_{\partial F} = \mathbf{1}_{\partial F}.$$

The map h_w is not canonical because it depends on several choices: the choice of trivialization $\partial X \cong \partial F \times D_*$, the choice of metric h on F and the choice of the extension g of g_∂.

We say that two diffeomorphisms $G_0, G_1 : F \to F$ such that $G_i \mid_{\partial F} = \mathbf{1}_{\partial F}$ are isotopic if there exists a homotopy

$$G : [0,1] \times F \to F$$

connecting them such that for each t the map $G_t = G(t, \bullet) : F \to F$ is a diffeomorphism satisfying $G_t \mid_{\partial F} = \mathbf{1}_{\partial F}$ for all $t \in [0,1]$.

The isotopy class of $h_w : F \to F$ is independent of the various choices listed above and in fact depends only on the image of w in $\pi_1(D^*, *)$. The induced map

$$h_w : H_*(F) \to H_*(F)$$

is called the *monodromy along the loop* w. The correspondence

$$h : \pi_1(D_*, *) \ni w \mapsto h_w \in \operatorname{Aut}\big(H_*(F)\big)$$

is a group morphism called *the local monodromy*. Since $h_w\mid_{\partial F} = \mathbf{1}_{\partial F}$ we obtain another morphism

$$h : \pi_1(D^*, *) \to \mathrm{Aut}\,\big(H_*(F, \partial F)\,\big)$$

which will continue to call *local monodromy*.

Observe that if $z \in H_*(F, \partial F)$ is a relative cycle (i.e. z is a chain such that $\partial z \in \partial F$) then for every $\gamma \in \pi_1(D^*_\rho, *)$ we have

$$\partial z = \partial h_\gamma z \Longrightarrow \partial(z - h_w z) = 0$$

so that $(z - h_w z)$ is a cycle in F. In this fashion we obtain a map

$$\mathbf{var} : \pi_1(D^*, *) \to \mathrm{Hom}\,(H_{n-1}(F, \partial F) \to H_{n-1}(F)\,),\ \ \mathbf{var}_\gamma(z) = h_\gamma z - z$$

($z \in H_{n-1}(F, \partial F)$, $\gamma \in \pi_1(D^*, *)$) called the *variation map*.

The vanishing cycle $\Delta \in H_{n-1}(F)$ is represented by the zero section of $\mathbb{M}$ described in the $(\vec{u}, \vec{v})$ coordinates by $\vec{v} = 0$. It is oriented as the unit sphere $S^{n-1} \hookrightarrow \mathbb{R}^n$. Let

$$\vec{u}_\pm = (\pm 1, 0, \cdots, 0) \in \Delta,\ \ P_\pm = (\vec{u}_\pm, \vec{0}) \in \mathbb{M}.$$

The standard model $\mathbb{M}$ admits a natural orientation as the total space of a fibration where we use the fiber-first convention of Chapter 6

$$\mathbf{or}(\text{total space}) = \mathbf{or}(\text{fiber}) \wedge\ \mathbf{or}(\text{base}).$$

We will refer to this orientation as the *bundle orientation*.

Near $P_+ \in \mathbb{M}$ we can use as local coordinates the pair $(\vec{\xi}, \vec{\eta})$, $\vec{\xi} = (u_2, \cdots, u_n)$, $\vec{\eta} = (v_2, \cdots, v_n)$. The orientation of Δ at $\vec{u}_+$ is given by

$$du_2 \wedge \cdots \wedge du_n \stackrel{\Phi}{\longleftrightarrow} dx_2 \wedge \cdots \wedge dx_n.$$

The orientation of the fiber over $\vec{u}_+$ is given by

$$dv_2 \wedge \cdots \wedge dv_n \stackrel{\Phi}{\longleftrightarrow} dy_2 \wedge \cdots \wedge dy_n.$$

Thus

$$\mathbf{or}_{bundle} = dv_2 \wedge \cdots \wedge dv_n \wedge du_2 \wedge \cdots \wedge du_n \longleftrightarrow dy_2 \wedge \cdots \wedge dy_n \wedge dx_2 \wedge \cdots \wedge dx_n$$

On the other hand, F has a natural orientation as a complex manifold. We will refer to it as the *complex orientation*. The collection $(z_2, \cdots, z_n)$ defines holomorphic local coordinates on F near Φ^{-1} so that

$$\mathbf{or}_{complex} = dx_2 \wedge dy_2 \wedge \cdots \wedge dx_n \wedge dy_n.$$

We see that [1]

$$\mathbf{or}_{complex} = (-1)^{n(n-1)/2}\mathbf{or}_{bundle}.$$

[1]This sign is different from the one in [3] due to our use of the fiber-first convention. This affects the appearance of the Picard-Lefschetz formulæ. The fiber-first convention is employed in [21] as well.

Any orientation **or** on F defines an intersection pairing

$$H_{n-1}(F) \times H_{n-1}(F) \to \mathbb{Z}$$

formally defined by the equality

$$c_1 *_{\mathbf{or}} c_2 = \langle PD_{\mathbf{or}}^{-1}(i_*(c_1)), c_2 \rangle$$

where $i_* : H_{n-1}(F) \to H_{n-1}(F, \partial F)$ is the inclusion induced morphism,

$$PD_{\mathbf{or}} : H^{n-1}(F) \to H_{n-1}(F, \partial F), \quad u \mapsto u \cap [F]$$

is the Poincaré-Lefschetz duality defined by the orientation **or** and $\langle -, - \rangle$ is the Kronecker pairing. More concretely, to compute the self-intersection number of the generator $\Delta \in H_{n-1}(F)$ slightly perturb inside F the sphere S representing Δ,

$$S \to S'$$

so that[2] $S_\rho \pitchfork S'_\rho$, and then count the intersection points with appropriate signs determined by the chosen orientation. For that reason, the self-intersection number of Δ is

$$\begin{aligned} \Delta \circ \Delta = (-1)^{n(n-1)/2} \Delta * \Delta = \mathbf{e}(TS^{n-1})[S^{n-1}] = \chi(S^{n-1}) \\ = \begin{cases} 0 & \text{if} \quad n \text{ is even} \\ 2 & \text{if} \quad n \text{ is odd} \end{cases} . \end{aligned} \tag{7.3}$$

Above $\mathbf{e}$ denotes the Euler class of TS^{n-1}.

Observe also that there exists an intersection pairing

$$H_{n-1}(F, \partial F) \times H_{n-1}(F) \to \mathbb{Z}, \quad u *_{\mathbf{or}} v = \langle PD_{\mathbf{or}}^{-1}(u), v \rangle$$

which produces a morphism

$$H_{n-1}(F, \partial F) \to \operatorname{Hom}(H_{n-1}(F), \mathbb{Z}), \quad z \mapsto z *_{\mathbf{or}} \quad .$$

Let us observe this is an isomorphism. Denote by $\nabla \in H_{n-1}(F, \partial F)$ the relative cycle carried by an oriented fiber of the disk bundle of TS^{n-1} (see Figure 7.3) so that

$$\nabla \circ \Delta = 1.$$

Hence, the image of ∇ in $\operatorname{Hom}(H_{n-1}(F), \mathbb{Z}) \cong \mathbb{Z}$ is a generator. On the other hand, by Poincaré-Lefschetz duality we have

$$H_{n-1}(F, \partial F) \cong H^{n-1}(F) \cong H^{n-1}(S^{n-1}) \cong \mathbb{Z}$$

so that ∇ must be a generator of $H_{n-1}(F, \partial F)$.

The variation map is thus completely understood if we understand its effect on ∇ (see Figure 7.4). At this point we can be much more explicit. The loop

[2] $\pitchfork$=transverse intersection

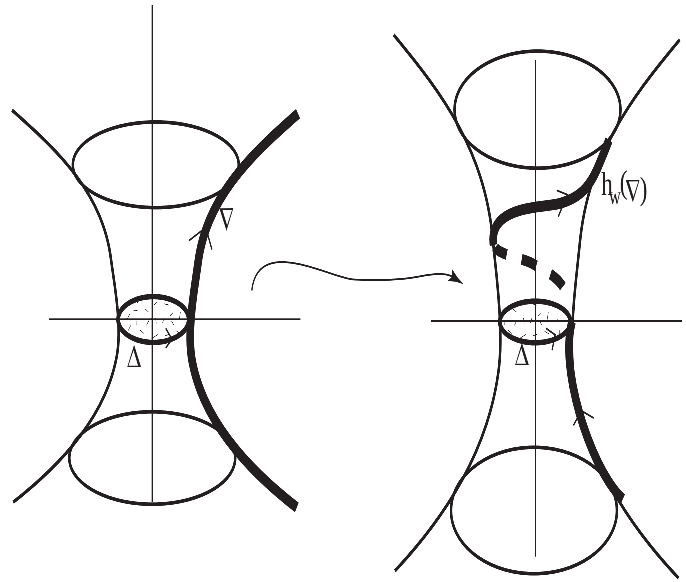

Figure 7.4: *The effect of monodromy on* ∇

$$\gamma_0 : \quad [0,1] \ni t \mapsto w(t) := e^{2\pi \mathbf{i} t}$$

generates the fundamental group of D^*. We denote by $\mathbf{var} : H_{n-1}(F, \partial F) \to H_{n-1}(F)$ the variation along this loop. Observe that

$$\mathbf{var}\,(\nabla) = m \cdot \Delta$$

where

$$m := \nabla \circ \mathbf{var}\,(\nabla).$$

We want to describe the integer m explicitly. We have the following fundamental result.

Theorem 7.6. (Local Picard-Lefschetz formulæ)

$$m = (-1)^n$$

$$\mathbf{var}_{\gamma_0}(\delta) = (-1)^n(\delta \circ \Delta)\Delta = (-1)^{n(n+1)/2}(\delta * \Delta)\Delta, \quad \forall \delta \in H_{n-1}(F, \partial F).$$

Proof ([18, Hussein-Zade]) The proof consists of a three-step reduction process. Set

$$E := f^{-1}(\partial D_1) \cap \bar{B}.$$

E is a smooth manifold with boundary

$$\partial E = f^{-1}(\partial D_1) \cap \partial \bar{B}_2$$

It fibers over $\partial \bar{D}_1$ and the restriction $\partial E \to S^1$ is equipped with the trivialization $(\Gamma_w)_{|w|=1}$. Observe that $\Phi\,|_{\partial F} = \Gamma_1$. Fix a vector field V on E such that

$$f_*(V) = 2\pi \partial_\theta \ \text{ and } \ V\,|_{\partial E = \partial F \times S^1} := 2\pi \partial_\theta.$$

Denote by μ_t the time t-map of the flow determined by V. Observe that μ_t defines a diffeomorphism

$$\mu_t : F \to F_{w(t)}$$

compatible with the chosen trivialization Γ_w of ∂E. More explicitly, this means that the diagram below is commutative.

$$\begin{tikzcd} \partial F \arrow[r, "\Gamma_1"] \arrow[d, "\mu_t"'] & \partial\mathbb{M} \arrow[d, "\mathbf{1}_{\partial\mathbb{M}}"] \\ \partial F_{w(t)} \arrow[r, "\Gamma_{w(t)}"] & \partial\mathbb{M} \end{tikzcd}$$

Consider also the flow Ω_t on E given by

$$\Omega_t(\vec{z}) = \exp(\pi\mathbf{i}t)\vec{z} = (\cos(\pi t)\vec{x} - \sin(\pi t)\vec{y}) + \mathbf{i}(\sin(\pi t)\vec{x} + \cos(\pi t)\vec{y}).$$

This flow is periodic, satisfies

$$\Omega_t(F) = F_{w(t)}$$

but it is **not compatible** with the chosen trivialization of ∂E.

We pick two geometric representatives $T_\pm$ of ∇. In the standard model $\mathbb{M}$ the representative consists of the fiber over $\vec{u}_+$ and is given by the equation

$$\vec{u} = \vec{u}_+.$$

it is oriented by $dv_2 \wedge \cdots \wedge dv_n$. Its image in F via Φ^{-1} is described by the equation

$$\vec{x} = (1 + |\vec{y}|^2/\alpha^2)^{1/2}\vec{u}_+ \Longleftrightarrow x_1 > 0, \ \ x_2 = \cdots = x_n = 0,$$

and is oriented by $dy_2 \wedge \cdots \wedge dy_n$.

The representative T_- is described in $\mathbb{M}$ as the fiber over $\vec{u}_-$. The orientation of S^{n-1} at $\vec{u}_-$ is determined by the outer-normal-first convention and we deduce that it is given by $-du_2 \wedge \cdots \wedge du_n$. This implies that T_- is oriented by $-dv_2 \wedge \cdots \wedge dv_n$. Inside F the the chain T_- is described by

$$\vec{x} = (1 + |\vec{y}|^2/\alpha^2)^{1/2}\vec{u}_+ \Longleftrightarrow x_1 < 0, \ \ x_2 = \cdots = x_n = 0,$$

and is oriented by $-dy_2 \wedge \cdots \wedge dy_n$. Note that $\Omega_1 = -1$ so that, taking into account the orientations, we have

$$\Omega_1(T_+) = (-1)^n T_- = (-1)^n \nabla.$$

Step 1. $m = (-1)^n\Omega_1(T_+) \circ \mu_1(T_+)$. Note that

$$m = \nabla \circ \big(\mu_1(T_+) - T_+\big) = T_- \circ \big(\mu_1(T_+) - T_+\big).$$

Observe that the manifolds T_+ and T_- in F are disjoint so that

$$m = T_- \circ \mu_1(T_+) = (-1)^n \Omega_1(T_+) \circ \mu_1(T_+).$$

Step 2. $\Omega_1(T_+) \circ \mu_1(T_+) = \Omega_t(T_+) \circ \mu_t(T_+)$, $\forall t \in (0,1]$. To see this observe that the manifolds $\Omega_1(T_+)$ and $\mu_t(T_+)$ have disjoint boundaries if $0 < t \leq 1$. Hence the deformations $\Omega_1(T_+) \to \Omega_{1-s(1-t)}(T_+)$ $\mu_1(T_+) \to \mu_{1-s(1-t)}(T_+)$ do not change the intersection numbers.

Step 3. $\Omega_t(T_+) \circ \mu_t(T_+) = 1$ if $t > 0$ is sufficiently small. Set

$$A_t := \Omega_t(T_+), \;\; B_t = \mu_t(T_+).$$

Denote by C_ε the arc

$$C_\varepsilon = \Big\{ \exp(2\pi \mathbf{i} t); \;\; 0 \leq t \leq \varepsilon \Big\}.$$

Extend the trivialization $\Gamma : \partial E \mid_{C_\varepsilon} \to \partial \mathbb{M} \times C_\varepsilon$ to a trivialization

$$\tilde{\Gamma} : E \mid_{C_\varepsilon} \to \mathbb{M} \times C_\varepsilon$$

such that

$$\tilde{\Gamma} \mid_F = \Phi.$$

For $t \in [0, \varepsilon]$ we can view Ω_t and μ_t as diffeomorphisms $\omega_t, h_t : \mathbb{M} \to \mathbb{M}$ such that the diagrams below are commutative.

$$\begin{array}{ccc} F & \xrightarrow{\tilde{\Gamma}_1} & \mathbb{M} \\ {\scriptstyle \Omega_t}\downarrow & & \downarrow{\scriptstyle \omega_t} \\ F_{w(t)} & \xrightarrow{\tilde{\Gamma}_{w(t)}} & \mathbb{M} \end{array}$$

$$\begin{array}{ccc} F & \xrightarrow{\tilde{\Gamma}_1} & \mathbb{M} \\ {\scriptstyle \mu_t}\downarrow & & \downarrow{\scriptstyle h_t} \\ F_{w(t)} & \xrightarrow{\tilde{\Gamma}_{w(t)}} & \mathbb{M} \end{array}$$

We will think of A_t and B_t as submanifolds in $\mathbb{M}$

$$A_t = \omega_t(T_+), \;\; B_t = h_t(T_+)$$

Observe that $h_t \mid_{\partial \mathbb{M}} = \mathbf{1}_{\mathbb{M}}$ so that $B_t(T_+)$ is homotopic to T_+ via homotopies which are trivial along the boundary. Such homotopies do not alter the intersection number and we have

$$A_t \circ B_t = A_t \circ T_+.$$

Observe now that along $\partial\mathbb{M}$ we have

$$\omega_t = S_t := \Omega_t \circ \Gamma_{w(t)} \circ \Gamma_1^{-1}. \tag{7.4}$$

Choose $0 < r < 1/2$. For t sufficiently small the manifold B_t lies in neighborhood

$$U_r := \Big\{ (\vec{\xi}, \vec{\eta});\ \ |\xi| < r,\ \ |\vec{\eta}| \leq 1 \Big\}$$

of the point $P_+ \in \mathbb{M}$, where we recall that $\xi = (u_2, \cdots, u_n)$ and $\vec{\eta} = (v_2, \cdots, v_n)$ denote local coordinates on $\mathbb{M}$ near P_+. More precisely if $P = (\vec{u}, \vec{v})$ is a point of $\mathbb{M}$ near P_+ then its $(\vec{\xi}, \vec{\eta})$ coordinates are $\mathbf{pr}(\vec{u}, \vec{v})$, where $\mathbf{pr} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ is the orthogonal projection

$$(\vec{u}, \vec{v}) \mapsto (u_2, \cdots, u_n; v_2, \cdots, v_n).$$

We can now rewrite (7.4) entirely in terms of the local coordinates $(\vec{\xi}, \vec{\eta})$ as

$$\omega_t(\vec{\xi}, \vec{\eta}) = S_t := \mathbf{pr} \circ \Omega_t \circ \Gamma_{w(t)} \circ \Gamma_1^{-1} \Big(u(\vec{\xi}, \vec{\eta}), \vec{v}(\vec{\xi}, \vec{\eta}) \Big).$$

Now observe that S_t is the restriction to ∂F of a (real) linear operator

$$L_t : \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}.$$

More precisely,

$$L_t \begin{bmatrix} \vec{\xi} \\ \vec{\eta} \end{bmatrix} = C(t) R(t) C(0)^{-1} \cdot \begin{bmatrix} \vec{\xi} \\ \vec{\eta} \end{bmatrix},$$

where

$$C(t) := \begin{bmatrix} c_1(t) & 0 \\ c_3(t) c_2(t) & c_3(t) \end{bmatrix}, \quad R(t) := \begin{bmatrix} \cos(\pi t) & -\sin(\pi t) \\ \sin(\pi t) & \cos(\pi t) \end{bmatrix}$$

and $c_k(t) := c_k(w(t))$, $k = 1, 2, 3$. The exact description of $c_k(w)$ is given in (7.2). We can thus replace $A_t = \omega_t(T_+)$ with $L_t(T_+)$ for all t sufficiently small without affecting the intersection number. Now observe that for t sufficiently small

$$L_t = L_0 + t\dot{L}_0 + O(t^2), \quad \dot{L}_0 := \frac{d}{dt}\,|_{t=0}\, L_t.$$

Now observe that

$$\dot{L}_0 = \dot{C}(0) C(0)^{-1} + C(0) J C(0)^{-1}, \quad J = \dot{R}(0) = \pi \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Using (7.2) with $a = \cos(2\pi t)$, $b = \sin(2\pi t)$ we deduce

$$c_1(0) = \sqrt{\frac{2}{5}} > 0, \quad c_2(0) = 0, \quad c_3(0) = \sqrt{\frac{2}{3}} > 0$$

$$\dot{c}_1(0) = \dot{c}_3(0) = 0, \;\; \dot{c}_2(0) = -\frac{2\pi}{25}.$$

Thus

$$\dot{C}(0) = -\frac{2\pi}{25}\begin{bmatrix} 0 & 0 \\ c_3(0) & 0 \end{bmatrix}, \;\; C(0)^{-1} = \begin{bmatrix} \frac{1}{c_1(0)} & 0 \\ 0 & \frac{1}{c_3(0)} \end{bmatrix}$$

$$\dot{C}(0)C(0)^{-1} = -\frac{2\pi}{25}\begin{bmatrix} 0 & 0 \\ \frac{c_3(0)}{c_1(0)} & 0 \end{bmatrix}$$

Next

$$C(0)JC(0)^{-1} = \pi\begin{bmatrix} c_1(0) & 0 \\ 0 & c_3(0) \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{c_1(0)} & 0 \\ 0 & \frac{1}{c_3(0)} \end{bmatrix}$$

$$= \pi\begin{bmatrix} c_1(0) & 0 \\ 0 & c_3(0) \end{bmatrix}\begin{bmatrix} 0 & -\frac{1}{c_3(0)} \\ \frac{1}{c_1(0)} & 0 \end{bmatrix} = \pi\begin{bmatrix} 0 & -\frac{c_1(0)}{c_3(0)} \\ \frac{c_3(0)}{c_1(0)} & 0 \end{bmatrix}$$

The upshot is that the matrix $\dot{L}_0$ has the form

$$\dot{L}_0 = \begin{bmatrix} 0 & -a \\ b & 0 \end{bmatrix}, \;\; a, b > 0.$$

For t sufficiently small we can now deform A_t to $(L_0 + t\dot{L}_0)(T_+)$ such that during the deformation the boundary of A_t does not intersect the boundary of T_+. Such deformation again do not alter the intersection number. Now observe that $\Sigma_t := (L_0 + t\dot{L}_0)(T_+)$ is the portion inside U_r of the $(n-1)$-subspace

$$\vec{\eta} \mapsto (L_0 + t\dot{L}_0)\begin{bmatrix} 0 \\ \vec{\eta} \end{bmatrix} = \begin{bmatrix} -ta\vec{\eta} \\ \vec{\eta} \end{bmatrix}.$$

It carries the orientation given by

$$(-tadu_2 + dv_2) \wedge \cdots \wedge (-tadu_n + dv_n).$$

Observe that Σ_t intersects the $(n-1)$-subspace T_+ given by $\vec{\xi} = 0$ transversely at the origin so that

$$\Sigma_t \circ T_+ = \pm$$

The sign coincides with the sign of the real number ν defined by

$$\nu dv_2 \wedge \cdots \wedge dv_n \wedge du_2 \wedge \cdots \wedge du_n$$

$$= (-tadu_2 + dv_2) \wedge \cdots \wedge (-tadu_n + dv_n) \wedge dv_2 \wedge \cdots \wedge dv_n$$

$$= (-ta)^{n-1} du_2 \wedge \cdots \wedge du_n \wedge dv_2 \wedge \cdots \wedge dv_n$$

$$= (-1)^{(n-1)+(n-1)^2} dv_2 \wedge \cdots \wedge dv_n \wedge du_2 \wedge \cdots \wedge du_n$$

Since $(n-1)+(n-1)^2$ is even we deduce that ν is positive so that

$$1 = \Sigma_t \circ T_t = \Omega_t(T_+) \circ \mu_t(T_+), \quad \forall 0 < t \ll 1.$$

This completes the proof of the local Picard-Lefschetz formula. ■

Remark 7.7. For a different proof we refer to [25]. For a more conceptual proof of the Picard-Lefschetz formula we refer to [3, Sec.2.4]. We will analyze this point of view a bit later.

§7.3 Global Picard-Lefschetz formulæ

Let us return to the setting at the beginning of Section 7.1. Recall that the function

$$\hat{f} : \hat{X} \to S \cong \mathbb{P}^1$$

is Morse and its critical values $t_1, \cdots, t_r$ are all in D_+. We denote its critical points by $p_1, \cdots, p_r$, so that

$$\hat{f}(p_j) = t_j, \quad \forall j.$$

We will identify D_+ with the unit disk at $0 \in \mathbb{C}$. Let us introduce some notations. Let $j = 1, \cdots, r$.

▼ Denote by D_j a closed disk of very small radius ρ centered at $t_j \in D_+$. If $\rho \ll 1$ these disks are pairwise disjoint.
▼ Connect $\bullet \in \partial D_+$ to $t_j + \rho \in \partial D_j$ by a smooth path ℓ_j such that the resulting paths $\ell_1, \cdots, \ell_r$ are disjoint (see Figure 7.1). Set $k_j := \ell_j \cup D_j$, $\ell = \bigcup \ell_j$ and $k = \bigcup k_j$.
▼ Denote by B_j a small ball in $\hat{X}$ centered at p_j.

Denote by γ_j the loop in $D_+ \setminus \{t_1, \cdots, t_r\}$ based at $\bullet$ obtained by traveling along ℓ_j from $\bullet$ to $t_j + \rho$ and then once, counterclockwise around ∂D_j and then back to $\bullet$ along ℓ_j. The loops γ_j generate the fundamental group

$$G := \pi_1(S^*, \bullet), \quad S^* := \mathbb{P}^1 \setminus \{t_1, \cdots, t_r\}.$$

Set

$$\hat{X}_+^* := \hat{f}^{-1}(S^*).$$

We have a fibration

$$\hat{f} : \hat{X}_+^* \to S^*$$

and, as in the previous section, we have an action

$$\mu : G \to \operatorname{Aut}\left(H_*(\hat{X}_\bullet)\right)$$

called *the monodromy of the Lefschetz pencil.*

From the above considerations we deduce that for each critical point p_j of $\hat{f}$ there exists a cycle $\Delta_j \in H_{n-1}(\hat{X}_\bullet)$ corresponding to the vanishing cycle in a fiber near p_j. It is represented by an embedded S^{n-1} with normal bundle isomorphic (up to orientation) to TS^{n-1}. In fact, using (7.3) we deduce

$$\Delta_j \cdot \Delta_j = (-1)^{n(n-1)/2}\Big(1 + (-1)^{n-1}\Big) = \begin{cases} 0 & \text{if} \quad n \text{ is even} \\ -2 & \text{if} \quad n \equiv -1 \mod 4 \\ 2 & \text{if} \quad n \equiv 1 \mod 4 \end{cases}.$$

This cycle bounds a *thimble*, $\tau_j \in H_n(\hat{X}_+, \hat{X}_\bullet)$ which is described as this sphere shrinks to p_j. Using the localization procedure in the first section and the local Picard-Lefschetz formulæ we obtain the following important result.

Theorem 7.8. (Global Picard-Lefschetz formulæ) *(a) For $q \neq n-1 = \dim_{\mathbb{C}} \hat{X}_\bullet$, the action of G on $H_q(\hat{X}_\bullet)$ is trivial i.e.*

$$\mathbf{var}_{\gamma_j}(z) := \mu_{\gamma_j}(z) - z = 0, \ \ \forall z \in H_q(\hat{X}_\bullet).$$

(b) If $z \in H_{n-1}(\hat{X}_\bullet)$ then

$$\mathbf{var}_{\gamma_j}(z) := \mu_{\gamma_j}(z) - z = (-1)^{n(n+1)/2}(z \cdot \Delta_j)\Delta_j$$

Exercise 7.9. Complete the proof of the global Picard-Leftschetz formula.

Hint Set $B := \cup_j B_i$, $F_j := f^{-1}(t_j + \rho) \cap \bar{B}_j$. Use the long exact sequence of the pair $(\hat{X}_\bullet, \hat{X}_\bullet \backslash B)$ and the excision property of this pair to obtain the natural short exact sequence

$$0 \to H_{n-1}(\hat{X}_\bullet \setminus B) \to H_{n-1}(\hat{X}_\bullet) \to \bigoplus_{j=1}^r H_{n-1}(F_j, \partial F_j)$$

where the last arrow is given by

$$z \mapsto \bigoplus_j (z \cdot \Delta_j)\mho_j.$$

Definition 7.10. The **monodromy group** of the Lefschetz pencil is the subgroup of $\mathfrak{G} \subset \operatorname{Aut}\big(H_{n-1}(X_\bullet)\big)$ generated by the monodromies μ_{γ_j}.

Remark 7.11. Suppose n is odd so that

$$\Delta_j \cdot \Delta_j = 2(-1)^{(n-1)/2}.$$

Denote by q the intersection form on $L := H_{n-1}(\hat{X}_\bullet)$. It is a symmetric bilinear form because $n-1$ is even. An element $u \in L$ defines the orthogonal reflection

$$R_u : L \otimes \mathbb{R} \to L \otimes \mathbb{R}$$

uniquely determined by the requirements

$$R_u(x) = x + t(x)u, \quad q\big(u, x + \frac{t(x)}{2}u\big) = 0, \quad \forall x \in L \otimes \mathbb{R}$$

$$\Longleftrightarrow R_u(x) = x - \frac{2q(x,u)}{q(u,u)}u$$

We see that the reflection defined by Δ_j is

$$R_j(x) = x + (-1)^{(n+1)/2} q(x, \Delta_j)\Delta_j.$$

This is precisely the monodromy along γ_j. This shows that the monodromy group $\mathfrak{G}$ is a Coxeter group.

Chapter 8

The Hard Lefschetz theorem and monodromy

We now return to the Hard Lefschetz theorem and establish its connection to monodromy. The results in this lecture are essentially due to Pierre Deligne. We will follow closely the approach in [21]. We refer to [26] for a nice presentation of Deligne's generalization of the Hard Lefschetz theorem and its intimate relation with monodromy.

§8.1 The Monodromy Theorem

In the proof of the **Key Lemma** we learned the reason why the submodule

$$V : \text{Image}\Big(\partial : H_n(\hat{X}_+, \hat{X}_\bullet) \to H_{n-1}(\hat{X}_\bullet)\Big) \subset H_{n-1}(\hat{X}_\bullet)$$

is called the vanishing submodule: it is spanned by the vanishing cycles Δ_j. We can now re-define the sub-module I by

$$I := \{y \in H_{n-1}(\hat{X}_\bullet);\ \ y \cdot \Delta_j = 0,\ \ \forall j\}$$

(use the global Picard-Lefschetz formulæ)

$$= \{y \in H_{n-1}(\hat{X}_\bullet);\ \ \mu_{\gamma_j} y = y,\ \ \forall j\}.$$

We have thus proved the following result.

Proposition 8.1. *I consist of the cycles invariant under the action of the monodromy group $\mathfrak{G}$.*

Theorem 8.2. *(***The Monodromy Theorem***) For the homology with real coefficients the following statements are equivalent.*

(a) The Hard Lefschetz Theorem (see Lecture 6).
(b) $V = 0$ or V is a **nontrivial simple** *$\mathfrak{G}$-module.*
(c) $H_{n-1}(\hat{X}_\bullet)$ is a **semi-simple** *$\mathfrak{G}$-module.*

Proof $(b) \Longrightarrow (c)$. Consider the submodule $I \cap V$ of V. Since V is simple we deduce

$$I \cap V = 0 \text{ or } I \cap V = V \neq 0.$$

The latter condition is impossible because $\mathfrak{G}$ acts nontrivially on V. Using the weak Lefschetz theorem

$$\dim I + \dim V = \dim H_{n-1}(\hat{X}_\bullet)$$

we deduce that $H_{n-1}(\hat{X}_\bullet) = I \oplus V$ so that $H_{n-1}(\hat{X}_\bullet)$ is a semi-simple $\mathfrak{G}$-module.

$(c) \Longrightarrow (a)$. More precisely, we will show that (c) implies that the restriction of the intersection form q on $H_{n-1}(\hat{X}_\bullet)$ to I is nondegenerate.

Denote by $\check{I}$ the dual module of I. We will show that the natural map

$$I \to \check{I}, \;\; z \mapsto q(z, -)$$

is onto. Let $u \in \check{I}$. Since $H_{n-1}(\hat{X}_\bullet)$ is semi-simple the $\mathfrak{G}$-module I admits a complementary $\mathfrak{G}$-submodule M such that

$$H_{n-1}(\hat{X}_\bullet) = I \oplus M.$$

We can extend u to a linear functional U on $H_{n-1}(\hat{X}_\bullet)$ by setting it $\equiv 0$ on M. Since q is nondegenerate on $H_{n-1}(\hat{X}_\bullet)$ there exists $z \in H_{n-1}(\hat{X}_\bullet)$ such that

$$U(x) = q(z, x+y), \;\; \forall x \oplus y \in I \oplus M.$$

If $g \in \mathfrak{G} \subset \operatorname{Aut}(H_{n-1}(\hat{X}_\bullet), q)$ then, since $\mathfrak{G}$ acts trivially on I and $\mathfrak{G}M \subset M$ we deduce

$$q(gz, x+y) = q(z, g^{-1}(x+y)) = q(z, x + g^{-1}y) = U(x), \;\; \forall x \oplus y \in I \oplus M.$$

Thus $\mathfrak{G}z = z \Longrightarrow z \in I$. This proves that the above map $I \to \check{I}$ is onto.

$(a) \Longrightarrow (b)$. More precisely, we will show that if the restriction of q to V is nondegenerate, then $V = 0$ or V is a nontrivial simple $\mathfrak{G}$-module. We will use the following auxiliary result whose proof is deferred to the next section.

Lemma 8.3. *(a) The elementary monodromies $\mu_1 := \mu_{\gamma_1}, \cdots, \mu_r := \mu_{\gamma_r}$ are pairwise conjugate in $\mathfrak{G}$ that is, for any $i, j \in \{1, \cdots, r\}$ there exists $g = g_{ij} \in \mathfrak{G}$ such that*

$$\mu_i = g \mu_j g^{-1}.$$

(b) For every $i, j \in \{1, \cdots, r\}$ there exists $g = g_{ij} \in \mathfrak{G}$ such that

$$\pm \Delta_i = g \Delta_j.$$

Suppose $F \subset V$ is a $\mathfrak{G}$-invariant subspace and $x \in F \setminus \{0\}$. Since q is nondegenerate on $\operatorname{span}\{\Delta_j\} = V$ we deduce there exists Δ_i such that

$$q(x, \Delta_i) \neq 0.$$

Now observe that

$$\mu_i \cdot x = x \pm q(x, \Delta_i)\Delta_i$$

so that

$$\Delta_i = \mp \frac{1}{q(x, \Delta_i)} \Big(\mu_i \cdot x - x \Big) \in F.$$

Thus $\operatorname{span}(\mathfrak{G}\Delta_i) \subset F$. From Lemma 8.3 (b) we deduce

$$\operatorname{span}(\mathfrak{G}\Delta_i) = V. \ \blacksquare$$

§8.2 Zariski's Theorem

The proof of Lemma 8.3 relies on a nontrivial topological result of Oskar Zariski. We will present only a weaker version and we refer to [15] for a proof and more information.

Proposition 8.4. *If Y is a (possibly singular) hypersurface in $\mathbb{P}^N$ then for any generic projective line $L \hookrightarrow \mathbb{P}^N$ the inclusion induced morphism*

$$\pi_1(L \setminus Y) \to \pi_1(\mathbb{P}^N \setminus Y)$$

is onto.

Remark 8.5. The term generic should be understood in an algebraic-geometric sense. More precisely, a subset S of a complex algebraic variety X is called generic if its complement $X \setminus S$ is contained in the support of a divisor on X. In the above theorem, the family $\mathcal{L}_N$ of projective lines in $\mathbb{P}^N$ is an algebraic variety isomorphic to the complex Grassmanian of 2-planes in $\mathbb{C}^{N+1}$. The above theorem can be rephrased as follows. There exists a hypersurface $\mathcal{W} \subset \mathcal{L}_N$, such that for any line $L \in \mathcal{L}_N \setminus \mathcal{W}$ the morphism

$$\pi_1(L \setminus Y) \to \pi_1(\mathbb{P}^N \setminus Y)$$

is onto.

Observe that a generic line intersects a hypersurface along a finite set of points of cardinality equal to the degree d of the hypersurface Y. Thus $L \setminus Y$ is homeomorphic to a sphere S^2 with d points deleted. Fix a base point $b \in L \setminus Y$. Any point $p \in L \cap Y$ determines an element

$$\gamma_p \in \pi_1(L \setminus Y, b)$$

obtained by traveling in L from b to a point $p' \in L$ very close to p along a path ℓ then going once, counterclockwise around p along a loop λ and then returning to b along ℓ. Thus, we can write

$$\gamma_p = \ell \lambda \ell^{-1}.$$

Lemma 8.6. *Suppose now that*

- *Y is a, possibly singular, degree d connected hypersurface.*
- *L_0, L_1 are two generic projective lines passing through the same point $b \in \mathbb{P}^N \setminus Y$ and*
- *$p_i \in L_i \cap Y$, $i = 0, 1$.*

Then the loops

$$\gamma_{p_i} \in \pi_1(L_i \setminus Y, b) \to \pi_1(\mathbb{P}^N \setminus Y, b)$$

are conjugate in $\pi_1(\mathbb{P}^N \setminus Y, b)$.

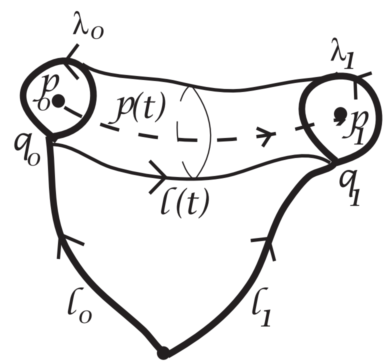

Figure 8.1: *The fundamental group of the complement of a hypersurface in $\mathbb{P}^N$*

Sketch of proof For each $y \in Y$ denote by L_y the projective line determined by the points b and y. The set

$$Z = \Big\{ y \in Y; \;\; \pi_1(L_y \setminus Y, b) \to \pi_1(\mathbb{P}^N \setminus Y, b) \text{ is not onto} \Big\}$$

is a complex (possibly singular) subvariety of codimension ≤ 1. In particular $Y^* := Y \setminus Z$ is connected.

Denote by U a small open neighborhood of $Y \hookrightarrow \mathbb{P}^N$. (When Y is smooth U can be chosen to be a tubular neighborhood of Y in $\mathbb{P}^N$.) Connect the point p_0 to p_1 using a generic path $p(t)$ in Y^* and denote by U_0 a small tubular neighborhood of this path inside U. We write

$$\gamma_{p_i} = \ell_i \lambda_i \ell_i^{-1}, \;\; i = 0, 1$$

and we assume that the endpoint q_i of ℓ_i lives on $L_{p_i} \cap U$. Now connect q_0 to q_1 by a smooth path along ∂U_0 and set $w = \mathbf{l}_0 \ell \ell_1^{-1}$ (see Figure 8.1). By inspecting this figure we obtain the following homotopic identities

$$w\gamma_{p_1} w^{-1} = \mathbf{l}_0 \ell \lambda_1 \ell^{-1} \mathbf{l}_0^{-1} = \mathbf{l}_0 \lambda_0 \mathbf{l}_0^{-1} = \gamma_{p_0}. \;\blacksquare$$

Proof of Lemma 8.3 (outline) Assume for simplicity that the pencil $(X_s)_{s\in S}$ on X consists of hyperplane sections. Recall that the dual $\check{X} \subset \check{\mathbb{P}}^N$ of X is defined by

$$\check{X} = \Big\{ H \in \check{\mathbb{P}}^N;\ \text{ all the projective lines in } H \text{ are either disjoint or tangent to } X \Big\}.$$

More rigorously consider the variety

$$W = \{(x, H) \in X \times \check{\mathbb{P}}^N;\ \ x \in H \Big\}.$$

equipped with the natural projections

$$\begin{tikzcd} & W \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & \check{\mathbb{P}}^N \end{tikzcd}$$

The $\check{X}$ is the **discriminant locus** of π_2, i.e. it consists of all the critical values of π_2. One can show (although it is not trivial) that $\check{X}$ is a (possible singular) hypersurface in $\check{\mathbb{P}}^N$ (see [21, Sec 2] or [38] for a more in depth study of discriminants. In Chapter 9 we will explicitly describe the discriminant locus in a special situation.)

The Lefschetz pencil $(X_s)_{s\in S}$ is determined by a line $S \subset \check{\mathbb{P}}^N$. The critical points of the map $\hat{f} : \hat{X} \to S$ are precisely the intersection points $S \cap \check{X}$. The fibration

$$\hat{X} \to S^* = S \setminus \{t_1, \cdots, t_r\} = S \setminus \check{X}$$

is the restriction of the fibration

$$\pi_2 : W \setminus \pi_2^{-1}(\check{X}) \to \check{\mathbb{P}}^N \setminus \check{X}$$

to $S \setminus \check{X}$. We see that the monodromy representation

$$\mu : \pi_1(S^*, \bullet) \to \mathrm{Aut}\,\big(H_{n-1}(X_\bullet) \big)$$

factors trough the monodromy

$$\tilde{\mu} : \pi_1(\check{\mathbb{P}}^N \setminus \check{X}, \bullet) \to \mathrm{Aut}\,\big(H_{n-1}(X_\bullet) \big)$$

Using Lemma 8.6 we deduce that the fundamental loops $\gamma_i, \gamma_j \in \pi_1(S^*, \bullet)$ are conjugate in $\pi_1(\check{\mathbb{P}}^N \setminus \check{X}, \bullet)$. Since the morphism

$$i_* : \pi_1(S^*, \bullet) \to \pi_1(\check{\mathbb{P}}^N \setminus \check{X}, \bullet)$$

is onto, we deduce that there exists

$$g \in \pi_1(S^*, \bullet)$$

such that

$$i_*(\gamma_i) = i_*(g\gamma_j g^{-1}) \in \pi_1(\check{\mathbb{P}}^N \setminus \check{X}, \bullet)$$

Hence

$$\mu(\gamma_i) = \mu(g\gamma_j g^{-1}) \in \mathfrak{G}. \tag{8.1}$$

This proves the first part of Lemma 8.3.

To prove the second part we use the global Picard-Lefschetz formulæ to rewrite the equality

$$\mu(\gamma_i g) = \mu(g\gamma_j) \in \mathfrak{G}$$

as

$$(x \cdot \Delta_j)g(\Delta_j) = (g(x) \cdot \Delta_i)\Delta_i, \ \ \forall x \in H_{n-1}(X_\bullet; \mathbb{R}). \tag{8.2}$$

By Poincaré duality, the intersection pairing is nondegenerate so that either $\Delta_j = 0$ (so $\Delta_i = 0$) or $x \cdot \Delta_i \neq 0$ which implies

$$g(\Delta_j) = c\Delta_i, \ \ c \in \mathbb{R}^*.$$

To determine the constant c we use the above information in (8.2).

$$c(x \cdot \Delta_j)\Delta_i = (g(x) \cdot \Delta_i)\Delta_i = (x \cdot g^{-1}(\Delta_i)\,)\Delta_i = \frac{1}{c}(x \cdot \Delta_j)\Delta_i.$$

Hence $c^2 = \pm 1$ so that

$$g(\Delta_j) = \pm\Delta_i. \ \blacksquare$$

Chapter 9

Basic facts about holomorphic functions of several variables

Up to now we have essentially investigated the behavior of a holomorphic function near a *nondegenerate* critical point. To understand more degenerate situations we need to use more refined techniques. The goal of this chapter is to survey some of these techniques. In the sequel, all rings will be commutative with 1.

§9.1 The Weierstrass preparation theorem and some of its consequences

The Weierstrass preparation theorem can be regarded as generalization of the implicit function theorem to degenerate situations. To state it we need to introduce some notations.

For any complex manifold M and any open set $U \subset M$ we denote by $\mathcal{O}_M(U)$ the ring of holomorphic functions $U \to \mathbb{C}$.

Denote by $\mathcal{O}_{n,p}$ the ring of germs at $p \in \mathbb{C}^n$ of holomorphic functions. More precisely, consider the set $\mathcal{F}_p$ of functions holomorphic in a neighborhood of p. Two such functions f,g are equivalent if there exists a neighborhood U of p contained in the domains of both f and g such that

$$f\mid_U = g\mid_U$$

The germ of $f \in \mathcal{F}_p$ at p is then the equivalence class of f, and we denote it by $[f]_p$. Thus $\mathcal{O}_{n,p} = \Big\{[f]_p;\ \ f \in \mathcal{F}_p\Big\}$.

For simplicity we set $\mathcal{O}_n := \mathcal{O}_{n,0}$. The unique continuation principle implies that we can identify $\mathcal{O}_n$ with the ring $\mathbb{C}\{z_1, \cdots, z_n\}$ of power series in the variables $z_1, \cdots, z_n$ convergent in a neighborhood of $0 \in \mathbb{C}^n$. For a function f holomorphic in a neighborhood of 0, its germ at zero is described by the Taylor expansion at the origin.

Lemma 9.1. *The ideal*

$$\mathfrak{M}_n = \Big\{f \in \mathcal{O}_n;\ \ f(0) = 0\Big\}$$

is the unique *maximal ideal of $\mathcal{O}_n$. In particular, $\mathcal{O}_n$ is a local ring.*

Proof Observe that $f \in \mathcal{O}_n$ is invertible iff

$$f(0) \neq 0 \Longleftrightarrow f \in \mathcal{O}_n \setminus \mathfrak{M}_n$$

which proves the claim in the lemma. ■

Lemma 9.2. (Hadamard Lemma) *Suppose* $f \in \mathbb{C}\{z_1, \cdots, z_n; w_1, \cdots, w_m\}$ *satisfies*

$$f(0, \cdots, 0; w_1, \cdots, w_m) = 0.$$

Then there exist $f_1, \cdots, f_n \in \mathbb{C}\{z_1, \cdots, z_n; w_1, \cdots, w_m\}$ *such that*

$$f(z; w) = \sum_{j=1}^{n} z_j f_j(z, w).$$

Notice that the above lemma implies that $\mathfrak{M}_n$ is generated by $z_1, \cdots, z_n$.
Proof

$$f(z; w) = f(z; w) - f(0; w) = \int_0^1 \frac{df(tz; w)}{dt} dt = \sum_{i=1}^{n} z_i \int_0^1 \frac{\partial f}{\partial z_i}(tz; w) dt =: \sum_{i=1}^{n} z_i f_i(z; w)$$

where the functions f_i are clearly holomorphic in a neighborhood of 0. This completes the proof of Hadamard's Lemma. ■

An important tool in local algebra is *Nakayama Lemma*. For the reader who, like the author, is less fluent in commutative algebra we present below several typical applications of this important result.

Proposition 9.3. (Nakayama Lemma) *Suppose R is a local ring with maximal ideal* $\mathfrak{m}$. *If E is a* finitely generated *R-module such that*

$$E \subset \mathfrak{m} \cdot E$$

then $E = 0$.

Proof Pick generators $u_1, \cdots, u_n$ of E. We can now find $a^i_j \in \mathfrak{m}$, $i, j = 1, \cdots, n$ such that

$$u_j = \sum_i a^i_j u_i, \ \ \forall j = 1, \cdots, n.$$

We denote by A the $n \times n$ matrix with entries a^i_j and by U the $n \times 1$ matrix with entries in E

$$U = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}.$$

We have

$$(1 - A)U = 0.$$

Since $\det(1 - A) \in 1 + \mathfrak{m}$ is invertible we conclude that $U \equiv 0$. ■

Corollary 9.4. *Suppose R is a local ring with maximal ideal $\mathfrak{m}$, E, F R-modules such that F is finitely generated. Then*

$$F \subset E \Longleftrightarrow F \subset E + \mathfrak{m}F.$$

Proof The implication $\Longrightarrow$ is trivial. To prove the converse notice that

$$F \subset F \cap E + \mathfrak{m}F \Longrightarrow F/(F \cap E) \subset \mathfrak{m}\Big(F/(F \cap E)\Big).$$

The desired conclusion follows by applying Nakayama Lemma to the module $F/(F \cap E)$. ■

Corollary 9.5. *Suppose R is a local ring with maximal ideal $\mathfrak{m}$, F is a finitely generated ideal and $x_1, \cdots, x_n \in F$. Then $x_1, \cdots, x_n$ generate F if and only if they generate $F/\mathfrak{m}F$.*

Proof Use Corollary 9.4 for the submodule E of F generated by $x_1, \cdots, x_n$. ■

We will present the Weierstrass preparation theorem in a form suitable for the applications we have in mind.

Definition 9.6. An **analytic algebra** is a $\mathbb{C}$-algebra R isomorphic to a quotient ring $\mathcal{O}_n/\mathfrak{A}$ where $\mathfrak{A} \subset \mathcal{O}_n$ is a finitely generated ideal.

Note that an analytic algebra $R = \mathcal{O}_n/\mathfrak{A}$ is a local $\mathbb{C}$-algebra whose maximal ideal $\mathfrak{M}_R$ is the projection of $\mathfrak{M}_n \subset \mathcal{O}_n$. Hadamard's lemma shows that the maximal ideal $\mathfrak{M}_R$ is generated by the images of the coordinate germs $z_1, \cdots, z_n$.

A **morphism of analytic algebras** R, S is a morphism of $\mathbb{C}$-algebras $u : R \to S$.

Exercise 9.7. Prove that a morphism of analytic algebras $u : R \to S$ is local, i.e.

$$u(\mathfrak{M}_R) \subset \mathfrak{M}_S.$$

A morphism of analytic algebras $u : R \to S$ is called **finite** if S, regarded as an R-module via u is finitely generated. In other words, there exist $s_1, \cdots, s_m \in S$ such that for any $s \in S$ there exist $r_1, \cdots r_n \in R$ so that

$$s = u(r_1)s_1 + \cdots + u(r_n)s_n.$$

Example 9.8. A holomorphic map $F : \mathbb{C}^n \to \mathbb{C}^m$ induces a morphism of analytic algebras

$$F^* : \mathcal{O}_m \to \mathcal{O}_n, \;\; f \mapsto f \circ F.$$

If F^* is finite then $m = n$, F is proper, and maps neighborhoods of 0 to neighborhoods of 0.

A morphism of analytic algebras $u : R \to S$ is called **quasi-finite** if the morphism

$$\bar{u} : R/\mathfrak{M}_R \cong \mathbb{C} \to S/\langle u(\mathfrak{M}_R)\rangle$$

induces a **finite dimensional** $\mathbb{C}$-vector space structure on $S/\langle u(\mathfrak{M}_R)\rangle$, where $\langle u(\mathfrak{M}_R)\rangle$ denotes the ideal generated by $u(\mathfrak{M}_R)$. Clearly, a finite morphism is quasi-finite.

Lemma 9.9. *$u : R \to S$ is quasi-finite if and only if there exists $r \in \mathbb{Z}_+$ such that*

$$\mathfrak{M}_S^r \subset \langle u(\mathfrak{M}_R) \rangle.$$

Proof Suppose $S = \mathcal{O}_m/\mathfrak{A}$. We denote the coordinates on $\mathbb{C}^m$ by $\xi_1, \cdots, \xi_m$. Suppose u is quasi-finite. Then for some $p \gg 0$ the germs $1, \xi_j, \cdots, \xi_j^p$ are linearly dependent modulo $\langle u(\mathfrak{M}_R) \rangle$, $\forall 1 \leq j \leq m$. For all $1 \leq j \leq m$ there exist

$$a_{j0}, a_{j1}, \cdots, a_{jp} \in \mathbb{C},$$

not all equal 0, such that

$$a_{j0} + a_{j1}\xi_j + \cdots, +a_{jp}\xi_j^p \in \langle u(\mathfrak{M}_R) \rangle.$$

If $a_{j0}, \cdots, a_{j(\nu-1)} = 0$ and $a_{j\nu} \neq 0$ ($\nu = \nu(j)$) then we deduce that for some $\sigma \in \mathfrak{M}_S$ we have

$$\xi_j^\nu (1 + \sigma) \in \langle u(\mathfrak{M}_R) \rangle.$$

This implies $\xi_j^{\nu(j)} \in \langle u(\mathfrak{M}_R) \rangle$. Thus

$$u \text{ quasi} - \text{finite} \Longrightarrow \exists r > 0 : \mathfrak{M}_S^r \subset \langle u(\mathfrak{M}_R) \rangle.$$

The converse is obvious. ■.

We have the following fundamental result. For a proof we refer to [19, §3.2], [33, Chap2].

Theorem 9.10. (Genral Weierstrass Theorem) *A morphism of analytic algebras $u : R \to S$ is a finite if and only if it is quasi-finite.*

Let us present a few important consequences of this theorem.

Corollary 9.11. *Suppose $u : R \to S$ is a homomorphism of analytical algebras. Then $s_1, \cdots, s_p \in S$ generate S as an R-module if and only if their images $\bar{s}_1, \cdots, \bar{s}_p$ in $S/\langle u(\mathfrak{M}_R) \rangle$ generate the $\mathbb{C}$-vector space $S/\langle u(\mathfrak{M}_R) \rangle$.*

Proof Suppose $\bar{s}_i$ generate $S/\langle u(\mathfrak{M}_R) \rangle$. Then u is quasi-finite. Now the elements s_i generate S modulo $\mathfrak{M}_S$ so that by Nakayama lemma they must generate the R-module S.

■

Definition 9.12. Consider a holomorphic map $F : U \subset \mathbb{C}_x^n \to \mathbb{C}_y^n$

$$(x_1, x_2, \cdots, x_n) \mapsto (y_1, y_2, \cdots, y_n) = (F_1(x), F_2(x), \cdots, F_n(x))$$

such that $F(p_0) = 0$. The **ideal of** F at p_0 is the ideal $I_F = I_{F,p_0} \subset \mathcal{O}_{n,p_0}$ generated by the germs of $F_1, \cdots, F_n$ at p_0. The **local algebra** at p_0 of F is the quotient

$$Q_F = Q_{F,p_0} := \mathcal{O}_{n,p_0}/I_F.$$

F is called **finite** at p_0 if the morphism

$$F^* : \mathcal{O}_{n,F(p_0)} \to \mathcal{O}_{n,p_0}, \ \ f \mapsto F \circ f$$

is finite. The integer $\mu = \mu(F, p_0) := \dim_{\mathbb{C}} Q_{F,p_0}$ is called the **multiplicity** of F at p_0.

Corollary 9.13. *Consider a holomorphic map*

$$F : U \subset \mathbb{C}^n \to \mathbb{C}^n, \;\; x \mapsto (F_1(x), \cdots, F_n(x))$$

such that $F(0) = 0$. Then the following are equivalent.
(i) F is finite at 0.
(ii) $\dim_{\mathbb{C}} Q_F < \infty$.
(iii) There exists a positive integer μ such that $\mathfrak{M}_n^{\mu} \subset I_F$. (We can take $\mu = \dim_{\mathbb{C}} Q_F$.)

Proof We only have to prove (ii) $\Longrightarrow$ (iii). Set $\mu := \dim_{\mathbb{C}} Q_F$. We will show that given $g_1, \cdots, g_\mu \in \mathfrak{M}_n$ then

$$g_1 \cdots g_\mu \in I_F.$$

The germs $1, g_1, g_1g_2, \cdots, g_1g_2\cdots g_\mu$ are linearly dependent in Q_F so there exist $c_0, c_1, \cdots, c_\mu \in \mathbb{C}$, not all equal to zero, such that

$$c_0 + c_1g_1 + \cdots + c_\mu g_1 \cdots g_\mu \in I_F.$$

Let c_r be the first coefficient different from zero. Then

$$g_1 \cdots g_r \Big(c_r + c_{r+1}g_{r+1} + \cdots + c_\mu g_{r+1} \cdots g_\mu \Big) \in I_F.$$

The germ within brackets is invertible in $\mathcal{O}_n$ so that $g_1 \cdots g_r \in I_F$. ■

Definition 9.14. Suppose $0 \in \mathbb{C}^n$ is a critical point of $f \in \mathfrak{M}_n \subset \mathbb{C}\{z_1, \cdots, z_n\}$. Then the **Jacobial ideal** of f at 0, denoted by $\mathfrak{J}(f)$, is the ideal generated by the first order partial derivatives of f. Equivalently

$$\mathfrak{J}(f, 0) = \mathfrak{J}(f) := I_{df}$$

where df is the gradient map

$$df : \mathbb{C}^n \to \mathbb{C}^n, \;\; \mathbb{C}^n \ni z \mapsto \Big(\frac{\partial f}{\partial z_1}(z), \cdots, \frac{\partial f}{\partial z_n}(z) \Big).$$

The **Milnor number** of 0 is the multiplicity at 0 of the gradient map. We denote it by $\mu(f, 0)$.

Exercise 9.15. Show that 0 is a nondegenerate critical point of $f \in \mathfrak{M}_n$ if and only if $\mu(f, 0) = 1$.

Corollary 9.16. *Suppose $f \in \mathbb{C}\{z_1, \cdots, z_n\}$ is regular in the z_n-direction, i.e.*

$$g(z_n) := f(0, \cdots, 0, z_n) \neq 0 \in \mathbb{C}\{z_n\}.$$

Denote by p the order of vanishing of $g(z_n)$ at 0 *so that*

$$g(z_n) = z_n^p h(z_n), \;\; h(0) \neq 0.$$

We have the following.

(Weierstrass Division Theorem) *For every $\varphi \in \mathbb{C}\{z_1, \cdots, z_n\}$ there exist*

$$q \in \mathbb{C}\{z_1, \cdots, z_n\}, \;\; b_1, \cdots, b_p \in \mathbb{C}\{z_1, \cdots, z_{n-1}\}$$

such that

$$\varphi = qf + \sum_{i=1}^{p} b_i z_n^{p-i}.$$

(Weierstrass Preparation Theorem) *There exists an invertible $u \in \mathbb{C}\{z_1, \cdots, z_n\}$ and*

$$a_1, \cdots, a_p \in \mathbb{C}\{z_1, \cdots, z_{n-1}\}$$

such that $a_j(0) = 0$ and

$$f = u \cdot (z_n^p + \sum_{j=1}^{p} a_j z_n^{p-j}).$$

Definition 9.17. A holomorphic germ $P \in \mathcal{O}_n$ of the form

$$P = z_n^p + \sum_{j=1}^{p} a_j z_n^{p-j}, \;\; a_q \in \mathcal{O}_{n-1}$$

such that $a_q(0) = 0$ is called a **Weierstrass polynomial**.

Proof Let

$$R := \mathbb{C}\{z_1, \cdots, z_{n-1}\}, \;\; S := \mathbb{C}\{z_1, \cdots, z_n\}/(f).$$

We denote by $u : R \to S$ the composition of the inclusion

$$\mathbb{C}\{z_1, \cdots, z_{n-1}\} \hookrightarrow \mathbb{C}\{z_1, \cdots, z_n\}$$

followed by the projection

$$\mathbb{C}\{z_1, \cdots, z_n\} \twoheadrightarrow \mathbb{C}\{z_1, \cdots, z_n\}/(f).$$

The Weierstrass division theorem is then equivalent to the fact that the elements $1, z_n, \cdots, z_n^{p-1}$ generate S over R. By Corollary 9.11 it suffices to show that their images in $S/\langle u(\mathfrak{M}_R)\rangle$ generate this quotient as a complex vector space. Now observe that

$$f(z_1, z_2, \cdots, z_n) - f(0, \cdots, 0, z_n) = \sum_{k=0}^{\infty} a_k(z_1, \cdots, z_{n-1}) z_n^k, \;\; a_k \in \mathfrak{M}_R$$

so that

$$f(z_1, z_2, \cdots, z_n) - f(0, \cdots, 0, z_n) \in \langle u(\mathfrak{M}_R)\rangle$$

Thus

$$S/\langle u(\mathfrak{M}_R)\rangle = \mathbb{C}\{z_1, \cdots, z_{n-1}, z_n\}/(z_1, \cdots, z_{n-1}, f)$$

$$= \mathbb{C}\{z_1, \cdots, z_{n-1}, z_n\}/(z_1, \cdots, z_{n-1}, g(z_n)) = \mathbb{C}\{z_n\}/(g(z_n) = \mathbb{C}\{z_n\}/(z_n^p).$$

Clearly the images of $1, z_n, \cdots, z_n^{p-1}$ generate $\mathbb{C}\{z_n\}/(z_n^p)$.

Let us now apply the Weierstrass division theorem to $\varphi = z_n^p$. Then there exists $u \in \mathbb{C}\{z_1, \cdots, z_n\}$ and $a_j \in \mathbb{C}\{z_1, \cdots, z_{n-1}\}$. such that

$$z_n^p = u \cdot f - \sum_{j=1}^p a_j z_n^{p-j}.$$

If we set $z_1 = \cdots = z_{n-1} = 0$ we obtain

$$z_n^p = u(0, \cdots, 0, z_n)g(z_n) - \sum_{j=1}^p a_j(0)z_n^{p-j} = u(0, \cdots, 0, z_n)z_n^p h(z_n) - \sum_{j=1}^p a_j(0)z_n^{p-j}$$

where $h(0) \neq 0$. Observe that $u(0) = 1/h(0)$ so that u is invertible in $\mathcal{O}_n$. Hence

$$f = u^{-1}\Big(z_n^p + \sum_{j=1}^p a_j z_n^{p-j}\Big).$$

Note that if $a_j(0) \neq 0$ for some j then the order of vanishing of $f(0, 0, \cdots, 0, z_n)$ at $z_n = 0$ would be strictly smaller than p. ■

Remark 9.18. The preparation theorem is actually equivalent to Theorem 9.10; see [33].

Corollary 9.19. (Implicit function theorem) *Suppose $f \in \mathbb{C}\{z_1, \cdots, z_n\}$ is such that $f(0) = 0$ and $\frac{\partial f}{\partial z_n}(0) \neq 0$. Then there exists $g \in \mathbb{C}\{z_1, \cdots, z_{n-1}\}$ such that the zero set*

$$V(f) := \{z;\ \ f(z) = 0\}$$

coincides in a neighborhood of 0 *with the graph of the function g,*

$$\Gamma_g := \{z;\ \ z_n = g(z_1, \cdots, z_{n-1})\}.$$

Proof Observe that

$$f(0, \cdots, 0, z_n) = z_n h(z_n),\ \ h(0) \neq 0.$$

From the Weierstrass Preparation Theorem we deduce

$$f(0) = u(z_n - g)$$

where $u \in \mathcal{O}_n$ is invertible and $g \in \mathbb{C}\{z_1, \cdots, z_{n-1}\}$. It is now clear that $V(f) = \Gamma_g$ near 0.

■

Corollary 9.20. *The ring $\mathcal{O}_n$ is Noetherian (i.e. every ideal is finitely generated) and factorial (i.e. each $f \in \mathcal{O}_n$ admits an unique prime decomposition).*

Proof The proof uses induction. Observe that every element $f \in \mathcal{O}_1$ admits an unique decomposition

$$f = u \cdot z_1^p$$

where $u \in \mathcal{O}_1$ is invertible. It follows immediately that $\mathcal{O}_1$ is a PID (principal ideal domain) so that it is both Noetherian and factorial.

Assume now that $\mathcal{O}_k$ is Noetherian and factorial for all $1 \leq k < n$. We will prove that $\mathcal{O}_n$ is Noetherian and factorial. The Hilbert basis theorem and the Gauss lemma imply that the polynomial ring $\mathcal{O}_{n-1}[z_n] \subset \mathcal{O}_n$ is both Noetherian and factorial.

Suppose now that $I \subset \mathcal{O}_n$ is an ideal and $0 \neq f \in I$. According to the preparation theorem we may assume that after a possible re-labeling of the variables we have

$$f = \Big(z_n^p + \sum_{j=1}^{p} a_j z_n^{p-j} \Big), \;\; a_j \in \mathcal{O}_{n-1}, \;\; u \in \mathcal{O}_n \text{ is invertible.}$$

Set $I' := I \cap \mathcal{O}_{n-1}[z_n]$. I' is finitely generate by $p_1, \cdots, p_m \in \mathcal{O}_{n-1}[z_n]$. We will show that I is generated by $f, p_1, \cdots, p_m$. Indeed, let us pick $g \in I$. Using the division theorem we have

$$g = qf + r, \;\; r \in \mathcal{O}_{n-1}[z_n]$$

which shows that

$$g \in (f, p_1, \cdots, p_m).$$

To show that is factorial we can use the preparation theorem to show that up to an invertible factor and/or a linear change of coordinates, each element in $\mathcal{O}_n$ is a Weierstrass polynomial, i.e it belongs to $\mathcal{O}_{n-1}[z_n]$. The factoriality of $\mathcal{O}_n$ now follows from the factoriality of the ring of Weierstrass polynomials. ■

Exercise 9.21. Complete the proof of factoriality of $\mathcal{O}_n$.

§9.2 Fundamental facts of complex analytic geometry

We now want to present a series of basic objects and results absolutely necessary in the study of singularities. For details and proofs we refer to our main sources of inspiration [11, 14, 19, 35].

The building bricks of complex analytic geometry are the analytic subsets of complex manifolds.

Definition 9.22. A set A is called **analytic** if it can be described as the zero set of a finite collections of holomorphic function defined on an open subset $U \subset \mathbb{C}^n$. For every open subset $V \subset U$ define

$$\mathcal{I}_A(V) := \Big\{ f : U \to \mathbb{C}; \;\; f \text{ holomorphic}, \;\; A \cap U \subset f^{-1}(0) \Big\}.$$

Suppose X is a Hausdorff space. Every point $p \in X$ defines an equivalence relation on 2^X

$$A \sim_p B \Longleftrightarrow \exists \text{ neighborhood } U \text{ of } p \in X \text{ such that } A \cap U = B \cap U.$$

The equivalence class of a set A is called the **germ** of A at p and is denoted by $\hat{A}_p$ or (A, p). Note that if p does not belong to the closure of A then $(A, p) = (\emptyset, p)$. The set-theoretic operations $\cup$ and $\cap$ have counterparts on the space of germs. We denote these new operations by the same symbols. If A is an analytic subset of a complex manifold and $p \in \bar{A}$, then the germ (A, p) is called an analytic germ.

Definition 9.23. An analytic germ (A, p) is called **reducible** if it is a finite union of distinct analytic germs. An analytic germ is called **irreducible** if it is not reducible.

Given $f \in \mathcal{O}_n$ such that $f(0) = 0$ (i.e. $f \in \mathfrak{M}_n$) we denote by $\hat{V}(f)$ the germ at 0 of the analytic set $V(f) = \{z;\ f(z) = 0\}$ at $0 \in \mathbb{C}^n$. For any ideal $I \subset \mathcal{O}_n$ we set

$$\hat{V}(I) := \bigcap_{f \in I} \hat{V}(f).$$

Since $\mathcal{O}_n$ is Noetherian, every ideal is finitely generated, so that the germs $\hat{V}(I)$ are analytic germs. Note that every analytic germ has this form.

Example 9.24. Consider the germ $f := z_1 z_2 \in \mathcal{O}_2$. Then $\hat{V}(f)$ is reducible because it decomposes as $\hat{V}(z_1) \cup \hat{V}(z_2)$. On the other hand if $g = y^2 - x^3 \in \mathcal{O}_2$ then $\hat{V}(g)$ is irreducible.

Theorem 9.25. *(a) The germ $\hat{V}(I)$ is irreducible if and only if I is a prime ideal of $\mathcal{O}_n$.*
(b) Every reducible germ is a finite union of irreducible ones.

The local (and global) properties of analytic sets are best described using the language of sheaves.

Definition 9.26. (a) Suppose X is a paracompact Hausdorff space. A **presheaf** of rings (groups, modules etc.) on X is a correspondence $U \mapsto \mathcal{S}(U)$ U open set in X, $\mathcal{S}(U)$ commutative ring (group, module etc.) such that for every open sets $U \subset V$ there exists a ring morphism $r_{UV} : \mathcal{S}(V) \to \mathcal{S}(U)$ such that if $U \subset\subset W$ we have

$$r_{UW} = r_{UV} \circ r_{VW}.$$

We set $f\,|_U := r_{UV}(f)$, $\forall U \subset V$, $f \in \mathcal{S}(V)$.

(b) A presheaf $\mathcal{S}$ is called a **sheaf** if it satisfies the following additional property. For any open set $U \subset X$, any open cover $(U_\alpha)_{\alpha \in A}$ of U, and any family $\{f_\alpha \in \mathcal{S}(U_\alpha)\}_{\alpha \in A}$ such that

$$f_\alpha\,|_{U_{\alpha\beta}} = f_\beta\,|_{U_{\alpha\beta}}, \quad \forall \alpha, \beta, \ \ (U_{\alpha\beta} := U_\alpha \cap U_\beta),$$

then there **exists** an **unique** element $f \in \mathcal{S}(U)$ such that $f\,|_{U_\alpha} = f_\alpha$, $\forall \alpha \in A$

(c) Let $\mathcal{S}_0$ and $\mathcal{S}_1$ be two (pre)sheaves of rings (groups, modules etc.) A **morphism** $\phi : \mathcal{S}_0 \to \mathcal{S}_1$ is a collection of morphisms of rings (groups, modules etc.)

$$\phi_U : \mathcal{S}_0(U) \to \mathcal{S}_1(U), \;\; U \;\; \text{open}$$

compatible with the restriction maps, i.e for every $V \subset U$ the diagram below is commutative.

$$\begin{tikzcd} \mathcal{S}_0(U) \arrow[r, "\phi_U"] \arrow[d, "r_U^V"'] & \mathcal{S}_1(U) \arrow[d, "r_U^V"] \\ \mathcal{S}_0(V) \arrow[r, "\phi_V"] & \mathcal{S}_1(V) \end{tikzcd} .$$

An isomorphism of sheaves is defined in an obvious fashion.

Example 9.27. (a) If X is a paracompact space and for every open set $U \subset X$ we denote by $C(U)$ the ring of complex valued continuous functions on U then the correspondence $U \mapsto C(U)$ is a sheaf on X.

(b) If A is an analytic subset of $\mathbb{C}^n$ then the correspondence $V \mapsto \mathcal{I}_A(V)$ is a sheaf on M called the **ideal sheaf** of A. Observe that $\mathcal{I}_A(U)$ is an ideal of $\mathcal{O}_M(U)$.

(c) For every open set $V \subset M$ set

$$\mathcal{O}_A(V \cap A) := \mathcal{O}_{\mathbb{C}}(V)/\mathcal{I}_A(V).$$

The correspondence $V \cap A \mapsto \mathcal{O}_A(V \cap A)$ is a sheaf on $\mathbb{C}^n$ called the **structural sheaf**. The elements of $\mathcal{O}_A(V \cap A)$ should be regarded as holomorphic functions $U \cap A \to \mathbb{C}$.

Definition 9.28. Suppose A_i are analytic subsets of $\mathbb{C}^n$, $i = 0, 1$. A continuous map $F : A_0 \to A_1$ is called **holomorphic** if for every open set $U_0 \subset \mathbb{C}^n$ there exists a holomorphic map $\tilde{F} : U_0 \to \mathbb{C}^n$ such that $\tilde{F}\mid_{U_0 \cap A_0} = F$. A biholomorphic map is homeorphism $F : A_0 \to A_1$ such that both F and F^{-1} are biholomorphic. A holomorphic map $A_0 \to \mathbb{C}$ is called a **regular function**.

Suppose A is an analytic set. For every open subset $U \subset A$ we denote by $\mathcal{R}_A(U)$ the ring of regular functions $U \to \mathbb{C}$. We obtain in this fashion a sheaf on A. For every open set $U \subset M$ we have a natural map

$$\mathcal{O}_M(U) \to \mathcal{R}_A(U \cap A)$$

whose kernel is the ideal $\mathcal{I}_A(U)$ so that we have an induced map

$$\mathcal{O}_A(U \cap A) \to \mathcal{R}_A(U \cap A).$$

This is clearly an isomorphism of sheaves. For this reason we will always think of the structural sheaf of $\mathcal{O}_A$ as a subsheaf of the sheaf of continuous functions on A. Note that the stalk $\mathcal{O}_{A,x}$ is an analytic algebra, and conversely, every analytic algebra is the stalk at some point of the structural sheaf of some analytic set.

Suppose $\mathcal{S}$ is a presheaf of rings on a paracompact space X. If U, V are open sets containing x and $f \in \mathcal{S}(U)$, $g \in \mathcal{S}(V)$ then we say that f is equivalent to g near x, and we write this $f \sim_x g$, if there exists an open set W such that

$$x \in W \subset U \cap V, \;\; f\,|_W = g\,|_W\,.$$

The $\sim_x$ equivalence class of f is called the **germ** of f at x and is denoted by $[f]_x$. The set of germs at x is denoted by $\mathcal{S}_x$ and is called the **stalk** of $\mathcal{S}$ at x. The stalk has a natural ring structure.

Given a presheaf $\mathcal{S}$ on a paracompact Hausdorff space X we form the disjoint union

$$\tilde{\mathcal{S}} := \coprod_{x \in X} \mathcal{S}_x.$$

For every open set $U \subset X$ and any $f \in \mathcal{S}(U)$ we get a map

$$\tilde{f} : U \to \tilde{\mathcal{S}}, \;\; u \mapsto [f]_u \in \tilde{\mathcal{S}}.$$

Observe that we have a natural projection $\pi : \tilde{\mathcal{S}} \to X$. Define

$$W_{U,f} := \tilde{f}(U) = \Big\{ [f]_u; \;\; u \in U \Big\} \subset \tilde{\mathcal{S}}.$$

Observe that the family $\mathcal{B} := \Big\{ W_{U,f} \Big\}$ of subsets of $\tilde{\mathcal{S}}$ satisfies the conditions

$$\forall W_1, W_2 \in \mathcal{B}, \;\; \exists W_3 \in \mathcal{B} : \;\; W_3 \subset W_1 \cap W_2, \;\; \bigcup_{W \in B} W = \tilde{\mathcal{S}}.$$

These show that $\mathcal{B}$ is a basis of a topology on $\tilde{\mathcal{S}}$. The natural projection $\pi : \tilde{\mathcal{S}} \to X$ is continuous, and moreover, for every germ $[f]_x \in \tilde{\mathcal{S}}$ there exists a neighborhood $W \in \mathcal{B}$ such that the restriction of π to W is a homeomorphism[1] onto $\pi(W)$.

Denote by $\tilde{\mathcal{S}}(U)$ the space of continuous sections $f : U \to \tilde{\mathcal{S}}$ of π, i.e. i.e. continuous functions $f : \to \tilde{\mathcal{S}}$ such that $f(u) \in \mathcal{S}_u$. The correspondence

$$U \mapsto \tilde{\mathcal{S}}(U)$$

is a sheaf of rings on X called the *sheafification* of $\mathcal{S}$. Since every $f \in \mathcal{S}(U)$ tautologically defines a continuous section of $\pi : \tilde{\mathcal{S}} \to X$ we deduce that we have a natural morphism of presheaves $i : \mathcal{S} \to \tilde{\mathcal{S}}$. When $\mathcal{S}$ is a sheaf then $\mathcal{S} = \tilde{\mathcal{S}}$.

Example 9.29. The presheaf of *bounded* continuous functious on $\mathbb{R}$ is not a sheaf. Its sheafification is the sheaf of continuous functions.

If $F : X \to Y$ is a continuous map between paracompact, Hausdorff spaces, and $\mathcal{S}$ is a presheaf on X, then we get a presheaf $F_*\mathcal{S}$ on Y described by

$$(F_*\mathcal{S})(U) = \mathcal{S}(F^{-1}(U)).$$

[1] A continuous map with such a property is called an *étale* map. Étale maps resemble in many respects covering maps.

If $\mathcal{S}$ is a sheaf then so is $F_*\mathcal{S}$. If $\mathcal{T}$ is a sheaf on Y, and $\pi : \mathcal{T} \to Y$ denotes the natural projection then define

$$F^{-1}\mathcal{T} = \mathcal{T} \times_Y X = \Big\{(s,x) \in \mathcal{T} \times X;\ \ \pi(s) = F(x)\Big\}.$$

There is a natural projection $F^{-1}\mathcal{T} \to X$ and as above we can define a sheaf by using continuous sections of this projection. Note that

$$(F^{-1}\mathcal{T})_x = \mathcal{T}_{F(x)}.$$

When $\mathcal{T}$ is a subsheaf of the sheaf of continuous functions on Y and U is an open subset in X then we can define $F^{-1}\mathcal{T}(U)$ as consisting of pullbacks $g \circ F$ where g is the restriction of a continuous function defined on an open neighborhood of $F(U)$ in Y.

Example 9.30. Consider the holomorphic map $F : \mathbb{C} \to \mathbb{C}$, $z \mapsto z^n$. Denote by $\mathcal{O}$ the sheaf of holomorphic functions in one variable. Then $F^{-1}\mathcal{O}$ is also a sheaf on $\mathbb{C}$ and

$$F^{-1}\mathcal{O}_{z_0} \cong \begin{cases} \mathbb{C}\{z\} & \text{if} \quad z_0 \neq 0 \\ \\ \mathbb{C}\{z^n\} & \text{if} \quad z_0 = 0 \end{cases}.$$

If D_r is the disc of radius r centered at the origin then

$$(F^{-1}\mathcal{O})(D_r) = \Big\{f(z^n);\ \ f : D_{r^n} \to \mathbb{C} \ \text{ is holomorphic}\Big\}.$$

Suppose $F : A \to B$ is a holomorphic map between two analytic sets, and let $p \in A$. Then F induces a natural morphism of sheaves

$$F^* : F^{-1}\mathcal{O}_B \to \mathcal{O}_A.$$

In particular, for every $p \in A$, there is an induced morphism of analytic algebras

$$F^* : \mathcal{O}_{B,F(p)} \to \mathcal{O}_{A,p}.$$

This suggest that one could interpret the morphisms of analytic algebras as germs of holomorphic maps. Exercise 9.34 shows that this is an accurate intuition.

Theorem 9.31. (Noether normalization) *Suppose (A,x) is the germ of an analytic set. Then there exists a positive integer d and a finite injective morphism of analytic algebras*

$$\varphi : \mathbb{C}\{z_1, \cdots, z_d\} \to \mathcal{O}_{A,x}$$

This theorem has a very simple geometric interpretation. If we think of A as an analytic subset in $\mathbb{C}^n$, then the normalization theorem essentially states that we can find a system of linear coordinates $z_1, \cdots, z_n$ such that the restriction to A of the natural projection $(z_1, \cdots, z_n) \mapsto (z_1, \cdots, z_d)$ is a finite-to-one holomorphic map. The integer d can be interpreted as the dimension of A.

It is useful to present the proof of this result in the special case of hypersurfaces because it highlights the geometric meaning of this important theorem, and explains the central role played by the Weierstrass division theorem.

Suppose (A, x) is the germ of a hypersurface A in $\mathbb{C}^n$. We assume $x = 0$. This means there exists $f \in \mathcal{O}_n$ such that $f(0) = 0$ and

$$(A, 0) = (V(f), 0).$$

Assume for simplicity that f is irreducible. We can choose linear coordinates $(z_1, \cdots, z_n)$ on $\mathbb{C}^n$ such that f is regular in the z_n-direction. By Weierstrass preparation theorem we can write

$$f = u(z)P(z_n)$$

where $u \in \mathcal{O}_n$, $u(0) \neq 0$, and $P_n \in \mathcal{O}_{n-1}[z_n]$ is a Weierstrass polynomial of degree q

$$P_n = z_n^q + \sum_{k=0}^{q-1} a_k(z')z_n^k, \quad z' := (z_1, \cdots, z_{n-1}).$$

The natural projection

$$\pi : \{P(z', z_n) = 0\} \ni (z', z_n) \mapsto z' \in \mathbb{C}^{n-1}$$

induces a one-to-one morphism $\mathcal{O}_{n-1} \to \mathcal{O}_n/(P)$, which factors trough the natural inclusion

$$\mathcal{O}_{n-1} \hookrightarrow \mathcal{O}_n.$$

Weierstrass division theorem implies that this map is finite.

We can think of the hypersurface $P = 0$ as the graph of the multivalued function $\phi = \phi(z')$ defined by the algebraic equation

$$\phi^q + \sum_{k=0}^{q-1} a_k(z')\phi^k = 0.$$

Equivalently, we can think of P as a family of degree q polynomials parameterized by $z' \in \mathbb{C}^{n-1}$, $P = P_{z'}$. Then $\phi(z')$ can be identified with the set of roots of $P_{z'}$. For most values of z' this set consists of q-distinct roots. Denote by $\Delta \subset \mathbb{C}^{n-1}$ the subset consisting of those z' for which $P_{z'}$ has multiple roots. Δ is known as the **discriminant locus** of π for reasons which will become apparent later. Note that $0 \in \Delta$. The subset $A' := \{P = 0\} \setminus \pi^{-1}(\Delta)$ is a smooth hypersurface in $\mathbb{C}^n$ and the projection

$$\pi : A' \to \mathbb{C}^{n-1} \setminus \Delta$$

is a genuine $q : 1$ covering map. The Noether normalization theorem thus says that locally, a hypersurface can be represented as a **finite** branched cover over a hyperplane. The branching locus is precisely the discriminant locus (see Figure 9.1). Moreover, when F is irreducible the nonsingular part A' is *connected.*

As we have mentioned before, the discriminant locus consists of those z' for which the polynomial $P_{z'} \in \mathbb{C}[z_n]$ has multiple roots, i.e. $P_{z'}(z_n)$ its z_n-derivative $P'_{z'}(z_n)$ have a root in common. This happens if and only if the discriminant of $P_{z'}$ is zero (see [22, Chap. IV])

$$\Delta(a_1(z'), \cdots, a_q(z')) = 0.$$

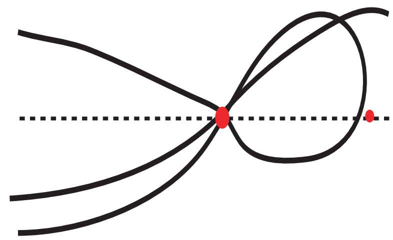

Figure 9.1: *A* $3 \to 1$ *branched cover and its discriminant locus (in red).*

Example 9.32. Suppose $F(x)$ is a poynomial of one complex variable such that $F(0) = 0$. Then the hypersurface in $\mathbb{C}^2$ given by

$$C := \{y^2 = F(x)\}$$

can be viewed as the graph of the 2-valued function $y = \pm\sqrt{F(x)}$. The natural projection π onto the x axis displays C as a double branched cover of $\mathbb{C}$,

$$C \ni (x, y) \mapsto x.$$

The branching locus is described in this case by the zero set of F which coincides with the zero set of the discriminant of the quadratic polynomial $P(y) = y^2 - F$.

The next result is an immediate consequence of the normalization theorem and is at the root of the rigidity of analytic sets.

Corollary 9.33. (Krull intersection theorem) *Suppose R is an analytic algebra, $R = \mathcal{O}_n/I$. Then*

$$\bigcap_{k \geq 1} \mathfrak{M}_R^k = (0).$$

Proof Using Noether normalization we can describe I as a finite extension of $\mathbb{C}$-algebras

$$i : \mathbb{C}\{z_1, \cdots, z_d\} \hookrightarrow R.$$

Using Lemma 9.9 we deduce that there exists $r > 0$ such that

$$\mathfrak{M}_R^r \subset \mathfrak{M}_d \subset \mathcal{O}_d \Longrightarrow \bigcap_{k \geq 1} \mathfrak{M}_R^k \subset \bigcap_{k \geq 1} \mathfrak{M}_d^k.$$

On the other hand we have the strong unique continuation property of holomorphic functions, which states that if all the partial derivatives of a holomorphic function vanish at a point then the holomorphic function must vanish in a neighborhood of that point. Another way of stating this is $\bigcap_{k \geq 1} \mathfrak{M}_d^k = 0$. ■

Exercise 9.34. Suppose $(X, x) \subset \mathbb{C}^n$ and $(Y, y) \subset \mathbb{C}^m$ are germs of analytic sets and $u : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is a morphism of analytic algebras. Then there exist neighborhoods $\mathbb{C}^n \supset U \ni x$ and $\mathbb{C}^m \supset V \ni y$ and a holomorphic map

$$F : U \to V$$

such that $F(x) = y$ and $F^* = u : \mathcal{O}_{Y,F(x)} \to \mathcal{O}_{X,x}$. (**Hint:** Use Krull intersection theorem.)

Exercise 9.35. Noether normalization shows that any analytic algebra is a finite extension of some $\mathcal{O}_d$. Prove that the converse is also true, i.e. if the $\mathbb{C}$-algebra is a finite extension of some $\mathcal{O}_d$ then it must be an analytic algebra, i.e. it is the quotient of some $\mathcal{O}_n$ by some ideal I.

▶ Let us summarize what we have established so far. We have shown that we can identify the germs of analytic sets with analytic algebras, i.e. quotients of the algebras $\mathcal{O}_n$, $n = 1, 2, \cdots$. This correspondence is in fact functorial. To any morphism of germs of analytic sets we can associate a morphism of analytic algebras, and any morphism of algebras can be obtain in this way. This shows that two analytic germs are isomorphic (i.e. biholomorphic) if and only if their associated local analytic algebras are isomorphic.
▶ We have have also seen that any analytic germ is a finite branched cover of some piece of an affine space, and any such branched cover defines an analytic germ. Algebraically, this means than the category of analytic algebras coincides with the category of $\mathbb{C}$-algebras which are finite extensions of some $\mathcal{O}_n$.

Given the analytic germ $\hat{V} = \hat{V}(I)$, $I \subset \mathcal{O}_n$ we define

$$\mathfrak{J}(\hat{V}) := \{f \in \mathcal{O}_n;\ \hat{V} \subset \hat{V}(f)\}.$$

Observe that $\mathfrak{J}(\hat{V})$ is an ideal of $\mathcal{O}_n$. To formulate our next result we need to remind a classical algebraic concept.

Definition 9.36. The **radical** of an ideal I of a ring R is the ideal $\sqrt{I}$ defined by

$$\sqrt{I} := \{r \in R;\ \exists n \in \mathbb{Z}_+ :\ r^n \in I\}.$$

We have the following important nontrivial result.

Theorem 9.37. (Analytical Nullstellensatz) *For every ideal $I \subset \mathcal{O}_n$ we have*

$$\mathfrak{J}(\hat{V}(I)) = \sqrt{I}.$$

Equivalently this means that a function $f \in \mathcal{O}_n$ vanishes on the zero locus $V(I)$ of the ideal I if an only if a power f^m of f belongs to the ideal I.

Example 9.38. Consider $f = z^n \in \mathcal{O}_1$. Then $\hat{V}(f)$ is the germ at 0 of the set $A = \{0\}$. Note that $\mathfrak{J}(\hat{V}(f)) = (z) = \mathfrak{M}_1$.

The above theorem implies that the finiteness of the Milnor number of a critical point is tantamount to the isolation of that point. More precisely, we have the following result[2].

Proposition 9.39. *Suppose $F : 0 \in U \subset \mathbb{C}^n \to \mathbb{C}^n \in \mathbb{C}^n \to \mathbb{C}^n$ and $F(0) = 0$. Then the following are equivalent.*
(i) 0 *is an isolated solution of* $F(z) = 0$.
(ii) $\mu(F, 0) < \infty$.

[2]We refer to [2, Sec. 5.5] for a very elegant proof of this fact not relying on Nullstellensatz.

Proof Denote by $\hat{V}$ the germ of analytic subset generated by the ideal $I_F \subset \mathcal{O}_n$. If 0 is isolated then $\hat{V} = 0$ and by analytical Nullstellensatz we have

$$\sqrt{I_F} = \mathfrak{M}_n$$

Hence there exists $k > 0$ such that $\mathfrak{M}_n^k \subset I_F$ which implies $\dim_{\mathbb{C}} \mathcal{O}_n / I_F < \infty$.

Conversely, if $\mu = \dim_{\mathbb{C}} \mathcal{O}_n / I_F < \infty$ then $\mathfrak{M}_n^{\mu} \subset I_F$ so that $\hat{V} = 0$, i.e. 0 is an isolated solution of $F(z) = 0$. ■

Inspired by the above result we will say that a critical point p of a holomorphic function f is **isolated** if $\mu(f, p) < \infty$.

§9.3 Tougeron's finite determinacy theorem

The Morse Lemma (which we have not proved in these lectures) has played a key role in the classical Picard-Lefschetz theory. It states that if 0 is a nondegenerate critical point of $f \in \mathfrak{M}_n$ then, we can holomorphically change coordinates to transform f into a polynomial of degree 2. The change in coordinates requirements can be formulated more conceptually as follows.

Definition 9.40. Denote by $\mathfrak{G}_n$ the space of germs of holomorphic maps $G : 0 \in U \subset \mathbb{C}^n \to \mathbb{C}^n$ such that $G(0) = 0$ and the differential of G at 0 is an invertible linear map $DG(0) : \mathbb{C}^n \to \mathbb{C}^n$.

The elements in $\mathfrak{G}_n$ can be regarded as local holomorphic changes of coordinates near $0 \in \mathbb{C}$. $\mathfrak{G}_n$ is a group. There is a right action of $\mathfrak{G}_n$ on $\mathcal{O}_n$ defined by

$$\mathcal{O}_n \ni f \mapsto f \circ G, \;\; \forall G \in \mathfrak{G}_n.$$

Two germs $f, g \in \mathcal{O}_n$ are said to be *right equivalent*, $f \sim_r g$, if they belong to the same orbit of $\mathfrak{G}_n$. In more intuitive terms, this means that g can be obtained from f by a local change of coordinates.

Definition 9.41. The k-jet at 0 of $f \in \mathcal{O}_n$ is the polynomial $j_k(f) = j_k(f, 0) \in \mathbb{C}[z_1, \cdots, z_n]$ obtained by removing from the Taylor expansion of f at 0 the terms of degree $> k$. More precisely

$$j_k(f) := \sum_{|\alpha| \leq k} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial z^{\alpha}}(0) z^{\alpha},$$

where for any nonnegative multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ we set

$$|\alpha| := \alpha_1 + \cdots + \alpha_n, \;\; \alpha! := \alpha_1! \cdots \alpha_n!,$$

$$z^{\alpha} := z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \;\; \frac{\partial^{|\alpha|} f}{\partial z^{\alpha}} := \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}.$$

Exercise 9.42. Suppose $I \subset \mathcal{O}_n$ is a proper ideal and $f \in \mathcal{O}_n$ is a holomorphic germ such that for every $k \geq 1$ there exists $f_k \in I$ so that

$$j_k(f - f_k) = 0.$$

Prove that $f \in I$. We can interpret this result by saying that if f can be approximated to any order by functions in I then f must be in I. In more geometric terms, this means that if two analytic sets have contact at a point of arbitrarily high order then they must coincide in a neighborhood of that point. This is a manifestation of the coherence of the sheaf of holomorphic functions. (**Hint:** Use Krull intersection theorem.)

Morse Lemma can now be rephrased by saying that if 0 is a critical point of f with $\mu(f,0) = 1$ then f is right equivalent to its second jet. The next result is a considerable generalization of Morse's Lemma. We refer to [2] for an even more general statement.

Theorem 9.43. (Tougeron finite determinacy theorem) *Suppose* 0 *is an isolated critical point of* $f \in \mathfrak{M}_n$ *with Milnor number* μ. *Then* f *is right equivalent to* $j_{\mu+1}(f)$.

Proof We follow the strategy in [2, Sec.63] or [34, Sec. 5] which is based on the so called *homotopy method.*

Roughly speaking our goal is to construct a local biholomorphism which will "kill" the terms of order $> \mu + 1$ of f. One of the richest sources of biholomorphisms is via (time dependent) flows of vector fields. Our local biholomorphism will be described as the *time*-1 *map* of a flow defined by a time dependent vector field.

We will prove that for every $\varphi \in \mathfrak{M}_n^{\mu+2}$ the germ $f+\varphi$ is right equivalent to f. We have an affine path $f + t\varphi$. We seek a one parameter family $G_t \in \mathfrak{G}_n$ such that

$$(f + t\varphi)(G_t z) \equiv f(z), \;\; G_0(z) = z, \;\; G_t(0) = 0. \tag{9.1}$$

Define the time dependent vector field

$$V_\tau(z) := \frac{d}{dt}\,|_{t=\tau}\, G_t(z).$$

Differentiating (9.1) with respect to t we obtain the infinitesimal version of (9.1) known as the *homology equation*

$$V_t(G_t(z)) \cdot (f + t\varphi) = -\varphi(G_t(z)) \in \mathfrak{M}_n^{\mu+2}. \tag{9.2}$$

The proof will be completed in two steps: solve the homology equation and then integrate its solution with respect to t.

Step 1. For every $\alpha \in \mathfrak{M}^{\mu+1}$ there exists a time dependent holomorphic vector field $V_t(z) = V_{t,\alpha}(z)$ defined in a neighborhood of 0 depending smoothly on $t \in [0,1]$ such that

$$V_t(0) = 0 \; \forall t \in [0,1] \tag{9.3a}$$

$$V_{t,\alpha}(z) \cdot (f + t\varphi)(z) = \alpha(z), \;\; \forall t \in [0,1], \; \forall |z| \ll 1. \tag{9.3b}$$

Step 2. The equation (9.1) has at least one solution.

Lemma 9.44. *(a) The equation (9.3b) has at least one solution $V_{t,\alpha}$ for every $\alpha \in \mathfrak{M}^\mu$.*
(b) The "initial value" problem (9.3a) + (9.3b) has at least one solution for every $\alpha \in \mathfrak{M}^{\mu+1}$.

Proof Consider all the monomials $M_1, \cdots, M_N \in \mathbb{C}[z_1, \cdots, z_n]$ of degree μ. We will first explain how to solve (9.3b) when $\alpha = M_j$. We already know that

$$\mathfrak{M}^\mu \subset \mathfrak{J}(f)$$

so that there exist $h_{ij} \in \mathcal{O}_n$ such that

$$M_j = \sum_i h_{ij} \frac{\partial f}{\partial z_i}.$$

Hence

$$M_j = \sum_i h_{ij} \frac{\partial (f + t\varphi)}{\partial z_i} - t \sum_i h_{ij} \frac{\partial \varphi}{\partial z_i}$$

Next observe that since $\varphi \in \mathfrak{M}^{\mu+2}$ we have[3]

$$\frac{\partial \varphi}{\partial z_i} \in \mathfrak{M}^{\mu+1}.$$

We can therefore write

$$\sum_i h_{ij} \frac{\partial \varphi}{\partial z_i} = \sum_p a_{jp} M_p, \quad a_{pj} \in \mathfrak{M}$$

and

$$M_j = \sum_i h_{ij} \frac{\partial (f + t\varphi)}{\partial z_i} - t \sum_p a_{jp} M_p.$$

As in the proof of Nakayama Lemma we can consider this as a linear system for the row vector $\vec{M} := (M_1, \cdots, M_N)$. More precisely we have

$$\vec{M}(\mathbf{1} + tA) = \vec{B}_t$$

where

$$B_{\ell,t} = \sum_i h_{i\ell} \frac{\partial (f + t\varphi)}{\partial z_i} \in \mathfrak{J}(f + t\varphi)$$

and the entries of $A = A(z)$ are in $\mathfrak{M}$. The matrix $(I + tA(z))$ is invertible for all $t \in [0, 1]$ and all sufficiently small z. We denote by $K_t(z)$ its inverse. We deduce

$$\vec{M} = \vec{B}_t K_t$$

[3]This is the only place where the assumption $\varphi \in \mathfrak{M}^{\mu+2}$ is needed. The rest of the proof uses only the milder condition $\varphi \in \mathfrak{M}^{\mu+1}$.

or more explicitly

$$M_j = \sum_\ell K_{\ell j,t} B_{\ell,t} = \sum_{\ell,i} K_{\ell j,t} h_{i\ell} \frac{\partial (f + t\varphi)}{\partial z_i}.$$

Thus,

$$V_{t,j}(z) := \sum_i \Bigl(\sum_\ell h_{i\ell} K_{\ell j,t} \Bigr) \frac{\partial}{\partial z_i}$$

solves (9.3b) for $\alpha = M_j$. Observe that this vector field *need not satisfy (9.3a).*

Any $\alpha \in \mathfrak{M}^\mu$ can be represented as a linear combination

$$\alpha = \sum_j \alpha_j M_j, \quad \alpha_j \in \mathcal{O}_n.$$

Then

$$V_{t,\alpha} := \sum_j \alpha_j V_{t,j}$$

is a solution of (9.3b). If moreover $\alpha \in \mathfrak{M}^{\mu+1}$ so that $\alpha_j(0) = 0$ then this $V_{t,\alpha}$ also satisfies the "initial condition" (9.3a). ■

Since $\varphi \in \mathfrak{M}^{\mu+2} \subset \mathfrak{M}^{\mu+1}$ we can find a solution $V_{t,-\varphi}$ of (9.3a) + (9.3b) with $\alpha = -\varphi$. To complete the proof of Tougeron's theorem we need to find a solution $G_t(z) \in \mathfrak{G}_n$ of the equation

$$\frac{d}{dt}(f + t\varphi)(G_t z) = 0, \quad G_t(0) = 0, \quad \forall t \in [0,1], \quad \forall |z| \ll 1.$$

Such a solution can be obtained by solving the differential equation

$$\frac{dG_t}{dt} = V_{t,-\varphi}(G_t(z)), \quad G_t(0) = 0. \ \blacksquare$$

Chapter 10

Singularities of holomorphic functions of two variables

To get an idea of the complexity of the geometry of an isolated singularity we consider in greater detail the case of isolated singularities of holomorphic functions of two variables. This is is a classical subject better which plays an important role in the study of plane algebraic curves. For more details we refer to [5, 6, 19] from which this chapter is inspired. We begin by considering a few guiding examples.

§10.1 Examples

As we have indicated in the previous chapter, all the information about the local structure of an analytic set near a point is entirely contained in the analytic algebra associated to that point. In particular, if $P \in \mathbb{C}[z_1, \cdots, z_n]$ is a polynomial such that $P(0) = 0$, and 0 is an isolated critical point of P, then the local structure of the hypersurface $P = 0$ near $0 \in \mathbb{C}^n$ contains a lot of information about the critical point 0.

Example 10.1. (Nodes) Consider the polynomial $P(x, y) = xy \in \mathbb{C}\{x, y\}$. Then the origin $0 \in \mathbb{C}^2$ is a nondegenerate critical point of P, i.e. $\mu(P, 0) = 1$. Near 0 the hypersurface A defined by $P = 0$ has the form in Figure 10.1. The analytic algebra of $(A, 0)$ is given by

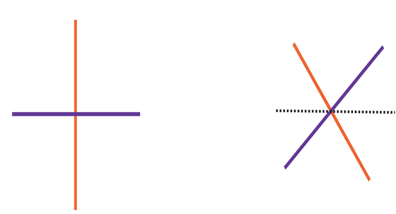

Figure 10.1: *The* **node** $xy = 0$ *in* $\mathbb{C}^2$.

the quotient $\mathcal{O}_{A,0} := \mathbb{C}\{x, y\}/(xy)$. Note that $\mathcal{O}_{A,0}$ is not an integral domain. If we rotate the figure by 45 degrees we see that A is a double branched cover of a line.

Example 10.2. (Cusps) Consider the polynomials $P(x, y) = y^2 - x^3 \in \mathbb{C}\{x, y\}$, and $Q(x, y) = y^2 - x^5 \in \mathbb{C}\{x, y\}$. Their zero sets $V(P)$ and $V(Q)$ are depicted in Figure

10.2. This figure also shows that both curves are double branched covers of the affine line.

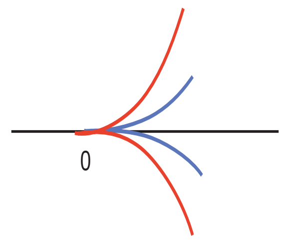

Figure 10.2: *The* **cusps** $y^2 = x^3$ *in red, and* $y^2 = x^5$, *in blue.*

The Jacobian ideal of P at 0 is $\mathfrak{J}(P, 0) = (x^2, y)$, while the Jacobian ideal of Q at 0 is $\mathfrak{J}(Q, 0) = (x^4, y)$. We deduce that $\mu(P, 0) = 2$ while $\mu(Q, 0) = 4$. This shows that the functions P and Q ought to have different behaviors near 0.

The analytic algebra of $V(P)$ at zero is $R(P) := \mathbb{C}\{x, y\}/(y^2 - x^3)$, and the analytic algebra of $V(Q)$ near zero is $R(Q) := \mathbb{C}\{x, y\}/(y^2 - x^5)$. Both are integral domains so that none of the them is isomorphic to the analytic algebra of the node in Figure 10.1. This suggests that the behavior near these critical points ought to be different from the behavior near a nondegenerate critical point.

We can ask whether $R(P) \cong R(Q)$. Intuitively, this should not be the case, because $\mu(P, 0) \neq \mu(Q, 0)$. The problem with the Milnor number μ is that it is an *extrinsic* invariant, determined by the way these two curves sit in $\mathbb{C}^2$, or equivalently, determined by the defining equations of these two curves. We cannot decide this issue this topologically because $V(P)$ and $V(Q)$ are locally homeomorphic near 0 to a two dimensional disk. We have to find an *intrinsic* invariant of curves which distinguishes these two local rings.

First, we want to provide a more manageable description of these two rings. Define

$$\varphi_3 : \mathbb{C}\{x, y\} \to \mathbb{C}\{t\},\ \ x \mapsto t^2,\ \ y \mapsto t^3,$$

and

$$\varphi_5 : \mathbb{C}\{x, y\} \to \mathbb{C}\{t\},\ \ x \mapsto t^2,\ \ y \mapsto t^5.$$

Observe that $(y^2 - x^k) \subset \ker \varphi_k$, $k = 3, 5$. Let us now prove the converse, $\ker \varphi_k \subset (y^2 - x^k)$. We consider only the case $k = 3$. Suppose $f(x, y) \in \ker \varphi_3$. We write

$$f = \sum_{m,n \geq 0} A_{mn} x^m y^n.$$

Then

$$0 = \varphi_3(f) = \sum_{m,n \geq 0} A_{mn} t^{2m+3n} = \sum_{k=0}^{\infty} \Big(\sum_{2m+3n=k} A_{mn} \Big) t^k = 0. \tag{10.1}$$

Consider the quasihomogeneous polynomial $\Phi_k = \sum_{2m+3n=k} A_{mn} x^m y^n$. We want to show that $(y^2 - x^3) | \Phi_k$. Set

$$S_k := \Big\{ (m, n) \in \mathbb{Z}_+^2;\ \ 2m + 3n = k \Big\}.$$

We denote by $\pi : \mathbb{Z}^2 \to \mathbb{Z}$ the natural projection $(m,n) \mapsto m$, and we set $m_0 = \max \pi(S_k)$. Then there exists a unique $n_0 \in \mathbb{Z}_+$ such that $(m_0, n_0) \in S_k$. We can then represent

$$S_k := \Big\{ (m_0 - 3s, n_0 + 2s);\ \ 0 \le s \le \lfloor m_0/3 \rfloor \Big\},$$

and

$$\Phi_k = x^{m_0} y^{n_0} \sum_{s=0}^{\lfloor m_0/3 \rfloor} C_s u^s,\ \ u := \frac{y^2}{x^3},\ \ C_s := A_{(m_0-3s),(n_0+2s)}.$$

The condition

$$\sum_{s=0}^{\lfloor m_0/3 \rfloor} C_s = 0$$

implies that $u = 1$ is a root of the polynomial $p(u) = \sum_s C_s u^s$. Hence we have

$$\Phi_k = x^{m_0} y^{n_0} (u-1) \sum_{j=0}^{\lfloor m_0/3 \rfloor - 1} D_j u^j = (y^2 - x^3) \sum_{j=0}^{\lfloor m_0/3 \rfloor - 1} D_j x^{m_0 - 3j} y^{n_0 + 2j}.$$

This shows that φ_3 induces an one-to-one morphism $\varphi_3 : R(P) \to \mathbb{C}\{t\}$ We denote by $R_{2,3}$ its image. We conclude similarly that φ_5 induces an one-to one morphism $R(Q) \to \mathbb{C}\{t\}$ and we denote by $R_{2,5}$ its image. We will now show that the rings $R_{2,3}$ and $R_{2,5}$ are not isomorphic.

Consider for $k = 3, 5$ the morphisms of semigroups

$$\pi_{2,k} : (\mathbb{Z}_+^2, +) \to (\mathbb{Z}_+, +),\ \ (m,n) \mapsto 2m + kn.$$

The image of $\pi_{2,k}$ is a sub-semigroup of $(\mathbb{Z}_+, +)$ which we denote by E_k. Observe that

$$E_3 = \{0, 2, 3, 4, 5, \cdots\},\quad E_5 = \{0, 2, 4, 5, 6, \cdots\}.$$

For each $f = \sum_{n \ge 0} a_n t^n \in \mathbb{C}\{t\}$ define $e(f) \in \mathbb{Z}_+$ by the equality

$$e(f) := \min\{n;\ \ a_n \ne 0\}.$$

We get surjective morphisms of semigroups

$$e : (R_{2,k}, \cdot) \to (E_k, +),\ \ f \mapsto e(f).$$

Suppose we have a ring isomorphism $\Phi : R_{2,3} \to R_{2,5}$. Set $A = \Phi(t^2)$, $B = \Phi(t^3)$, $a = e(A)$, and $b = e(B)$. Observe that $a, b > 0$ and $e(\Phi(t^{2m+3n})) = am + bn \in E_5$. We have thus produced a surjective morphism of semigroups

$$\Psi : E_3 \to E_5,\ \ (2m + 3n) \mapsto am + bn.$$

Since $2 = \min E_3 \setminus \{0\}$ we deduce either $a = 2$, or $b = 2$, Assume $a = 2$. Since Ψ is surjective, we deduce that $b = 5$. To get a contradiction it suffices to produce two pairs $(m_i, n_i) \in \mathbb{Z}_+^2$, $i = 1, 2$ such that

$$2m_1 + 3n_1 = 2m_2 + 3n_2 \ \text{ and } \ 2m_1 + 5n_1 \ne 2m_2 + 5n_2.$$

For example $11 = 2 \cdot 1 + 3 \cdot 3 = 2 \cdot 4 + 3 \cdot 1$ but $2 \cdot 1 + 5 \cdot 3 = 17 \ne 13 = 2 \cdot 4 + 5 \cdot 1$.

Exercise 10.3. Consider an additive sub-monoid[1] $S \subset (\mathbb{Z}, +)$. Suppose that S is *asymptotically complete*, i.e. there exists $\nu = \nu_S > 0$ such that $n \in S$, $\forall n \geq \nu_S$.
(a) Prove that S is finitely generated.
(b) Let

$$r = \min\Big\{g \in \mathbb{Z}_+;\ \ \exists\ \ s_1, \cdots, s_g \in S \ \text{ which generate } \ S\Big\}.$$

Suppose G_1 and G_2 are two sets of generators such that $|G_1| = |G_2| = r$. Then $G_1 = G_2$. In other words, S has a unique minimal set of generators. This finite set of positive integers is therefore an invariant of S.

We have discussed Example 10.2 in great detail for several reasons. First, we wanted to convince the reader that by reducing the study of the local structure of a singularity to a purely algebraic problem does by no means lead to an immediate answer. As we saw, deciding whether the two rings $R_{2,3}$ and $R_{2,5}$ are isomorphic is not at all obvious. The technique used in solving this algebraic problem is another reason why we consider Example 10.2 very useful. Despite appearances, this technique works for the isolated singularities of any holomorphic function of two variables. In the next section we describe one important algebraic concept hidden in the above argument.

§10.2 Normalizations

Suppose $f \in \mathbb{C}\{x, y\}$ is a holomorphic function defined in a neighborhood of $0 \in \mathbb{C}$ such that 0 is an isolated critical point. Assume it is an irreducible Weierstrass y-polynomial in $f \in \mathbb{C}\{x\}[y]$.

Denote by $Z = Z(f)$ the zero set of f, and by $\mathcal{O}_{Z,0} = \mathbb{C}\{x, y\}/(f)$ the local ring of the germ $(Z, 0)$. It is an integral domain. Following Example 10.2, we try to embed $\mathcal{O}_{Z,0}$ in $\mathbb{C}\{t\}$, such that $\mathbb{C}\{t\}$ is a finite $\mathcal{O}_{Z,0}$-module. More geometrically, Z is a complex 1-dimensional analytic set in $\mathbb{C}^2$, better known as a *plane (complex) curve* . A *normalization* is a then a germ of a finite map $\mathbb{C} \to \mathbb{Z}$ with several additional properties to be discussed later.

Observe that when $f = y^2 - x^3$ we get such an embedding by setting $x = t^2$, $y = t^3$ so that $t = y/x$. We see that in this case we can obtain $\mathbb{C}\{t\}$ as a simple extension of $\mathcal{O}_{Z,0}$, more precisely,

$$\mathbb{C}\{t\} \cong \mathcal{O}_{Z,0}[y/x].$$

Similarly, when $f = y^2 - x^5$ we set $x = t^2$, $y = t^5$ so that $t = y/x^2$, and

$$\mathbb{C}\{t\} \cong \mathcal{O}_{Z,0}[y/x^2].$$

Note that in both cases t is an integral element over $\mathcal{O}_{Z,0}$, i.e. it satisfies a polynomial equation of the form

$$t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0 = 0, \quad a_k \in \mathcal{O}_{Z,0}.$$

[1]An additive monoid is a commutative semigroup $(S, +)$ with 0, satisfying the cancelation law, $a + x = b + x \Longleftrightarrow a = b$.

In both cases we have $t^2 - x = 0$. This shows that in both cases $\mathbb{C}\{t\}$ is a finite $\mathcal{O}_{Z,0}$-module. To analyze the general case we need to introduce some terminology.

Definition 10.4. Suppose R is an integral domain, and $K \supset R$ is a field.
(a) An element $x \in K$ is said to be **integral** over R over R if there exists a polynomial $P \in R[T]$ with leading coefficient 1 such that $P(x) = 0$. We denote by $\tilde{R}_K \subset K$ the set of integral elements in K. When K is the field of fractions $Q(R)$ of R we write $\tilde{R}$ instead of $\tilde{R}_{Q(R)}$.
(b) R is said to be **integrally closed** in K if $R = \tilde{R}_K$. The ring R is called integrally closed, if it is integrally closed in its field of fractions $Q(R)$.

Exercise 10.5. (a) Prove that $x \in Q(R)$ is integrally closed if and only if the R-module $R[x]$ is finitely generated.
(b) Suppose $R \subset S$ as a finite extension of integral *Noetherian* domains (i.e. S is finitely generated as an R-module), and K is a field containing S. If an element $\alpha \in K$ is integral over S, then it is also integral over R.
(b) Prove that $\tilde{R}$ is a ring.

Exercise 10.6. Prove that any unique factorization domain is integrally closed.

The set $\tilde{R}$ is a subring of $Q(R)$ called the **normalization of** R . Observe that $\tilde{R}$ is integrally closed.

Example 10.7. Let $f = y^2 - x^3 \in \mathbb{C}\{x, y\}$, and $Z = \{f = 0\} \subset \mathbb{C}^2$. The isomorphism $\mathbb{C}\{t\} \cong \mathcal{O}_{Z,0}[y/x]$ shows that we can view $\mathbb{C}\{t\}$ as a subring of the field of fractions of $\mathcal{O}_{Z,0}$.

We have the following fundamental result.

Theorem 10.8. *Suppose $f \in \mathbb{C}\{x, y\}$ is irreducible, $f(0) = 0$. Assume y is regular in the y-direction and set $R_f := \mathbb{C}\{x, y\}/(f)$. Then the normalization $\tilde{R}_f$ of R_f is isomorphic to $\mathbb{C}\{t\}$.*

There are several essentially equivalent ways of approaching this theorem, which states a fact *specific only to dimension* 1. It is thus not surprising that the concept of dimension should play an important role in any proof. We will present a proof which combines ideas from [11, 19], and assumes only the geometric background presented so far. For a more algebraic proof we refer to [19].

Sketch of proof We can assume f is a Weierstrass polynomial of degree q,

$$f(x, y) = y^q + a_1(x)y^{q-1} + \cdots + a_q(x) \in \mathbb{C}\{x\}[y], \quad a_j(0) = 0, \ \ \forall j = 1, \cdots q.$$

Denote by Z_1 a small open disk in $\mathbb{C}$centered at 0 , and by Z_f a small neighborhood of $(0,0)$ in $f^{-1}(0)$. We denote by x the local coordinate on Z_1. The natural projection

$$\mathbb{C}^2 \to \mathbb{C}, \quad (x, y) \mapsto x$$

induces a degree q cover $\pi_f : Z_f \to Z_1$, branched over the zero set of the discriminant $\Delta(x)$ of f. We can assume Z_1 is small enough so that $\Delta^{-1}(0) \cap Z_1 = \{0\}$. We set

$$Z_f^* := Z_f \setminus \{(0,0)\} = \pi^{-1}(Z_1^*), \quad Z_1^* := Z_1 \setminus \{0\}.$$

Since f is irreducible we deduce Z_f^* is a *connected*, smooth one dimensional complex manifold, and $\pi : Z_f^* \to Z_1$ is q-sheeted cover of the punctured disk Z_1^*. Thus we can find a small disk D in $\mathbb{C}$ centered at 0, and a bi-holomorphic map

$$\phi : D^* \to Z_f^*$$

such that the diagram below is commutative

$$\begin{tikzcd} (D^*, t) \arrow[rr, "\phi"] \arrow[dr, "\pi_q : t \mapsto x = t^q"'] & & Z_f^* \arrow[dl, "\pi_f"] \\ & (Z_1^*, x) & \end{tikzcd}.$$

The holomorphic functions x, y on Z_f define by pullback bounded holomorphic functions on the punctured disk D^*, and they extend to holomorphic functions $x(t), y(t)$ on D. Moreover, $x(t) = t^q$. We can view the coordinate t as a holomorphic function on D. It induces by pullback via ϕ^{-1} a *bounded* holomorphic function $t = t(x, y)$ on Z_f^*. We want to prove that is the restriction of a meromorphic function on Z_f. More precisely, we want to prove that, by eventually shrinking the size of Z_f we have

$$t = \frac{A(x,y)}{B(x,y)} \mid_{Z_f}, \;\; A, B \in \mathbb{C}\{x, y\}.$$

Denote by K_1 the field of meromorphic functions in the variable x, i.e. K_1 is the field of fractions of $\mathbb{C}\{x\}$. Denote by K_f the field of fractions of R_f. K_f is a degree q extension of K_1 and in fact, it is a primitive extension. As primitive element we can take the restriction of the function y to Z_f. Denote by $\Delta(x)$ the discriminant of f.

For every point $p \in Z_1^*$ there exists a small neighborhood U_p, and q holomorphic functions

$$r_j = r_{j,p}(x) : U_p \to \mathbb{C}$$

such that

$$\pi^{-1}(U_p) = \bigcup_{j=1}^{q} \Bigl\{ \bigl(x, r_j(x)\bigr); \;\; x \in U_p \Bigr\}.$$

In particular, this means that for each $\underline{x} \in U_p$ the roots of the polynomial

$$f(\underline{x}, y) = y^q + a_1(\underline{x}) y^{q-1} + \cdots + a_q(\underline{x}) \in \mathbb{C}[y]$$

are $r_1(\underline{x}), \cdots, r_1(\underline{x})$, so that

$$\Delta(\underline{x}) = \prod_{i \neq j} \bigl(r_i(\underline{x}) - r_j(\underline{x})\bigr).$$

For $(x, y) \in \pi^{-1}(U_p)$ define the Lagrange interpolation polynomial

$$R(x,y) := \frac{1}{\Delta(x)} \sum_{j=1}^{q} t(x, r_j(x)) \bigl(y - r_j(x)\bigr).$$

Note that $R(x,y) = t(x,y)$, $\forall (x,y) \in \pi^{-1}(U_p)$. Observe that $\Delta(x)R(x,y)$ is a polynomial in y with coefficients holomorphic functions in the variable $x \in U_p$. The coefficients of this polynomial do not depend on p and are in fact *bounded* holomorphic functions on Z_1^* and thus they extend to genuine holomorphic functions on X. We have thus proved the existence of q holomorphic functions $b_1, \cdots, b_q$ on Z_1 such that

$$\Delta(x)t(x,y) = y^q + b_1(x)y^{q-1} + \cdots + b_q(x)$$

which shows that $t \in K_f$ as claimed.

We thus get a map $\Phi : R_f \to \mathbb{C}\{t\}$, defined by $x \mapsto t^q$, $y = y(t)$. Let us first show it is an injection. Indeed, if $P(t^q, y(t)) = 0$ for some $P \in \mathbb{C}\{x,y\}$ then $P \mid_{Z_f}$ is irreducible we deduce from the analytical Nullstellensatz that $P \in (f)$. This is also a finite map because it is quasifinite,

$$t^q = x \in \Phi(\mathfrak{M}_{R_f}).$$

We can thus regard R_0 as a finite extension of R_f. Now observe that $R_0 \in Q(R_f)$ because $t \in Q(R_f)$. Since $\mathbb{C}\{t\}$ is integrally closed we deduce that $\mathbb{C}\{t\}$ is precisely the normalization of R_f. ■

The holomorphic map $\phi : D \to Z_f$ constructed above, which restricts to a biholomorphism $\phi : D^* \to Z_f^*$ is called a **resolution of the singularity** of the germ of the curve $f = 0$ at the point $(0,0)$. We have proved that we can resolve the singularities of the irreducible germs. The reducible germs are only slightly more complicated. One has to resolve each irreducible branch separately.

Definition 10.9. Suppose $(C, 0)$ is an irreducible germ of a plane curve defined by an equation $f(x,y) = 0$. Then a *resolution* of $(C,0)$ is a pair $(\tilde{C}, \pi)$ where $\tilde{C}$ is a smooth curve and $\pi : \tilde{C} \to C$ is a holomorphic map with the following properties

(a) $\pi^{-1}(0)$ consists of a single point $\{p\}$.
(b) $\pi : \tilde{C} \setminus \pi^{-1}(0) \to C \setminus \{0\}$ is biholomorphic.

A resolution defines a finite morphism $\pi^* : \mathcal{O}_{C,0} \to \mathcal{O}_{\tilde{C},p}$ called the *normalization* and we set

$$\delta(C,0) := \dim_{\mathbb{C}} \mathcal{O}_{\tilde{C},p}/\pi^*\mathcal{O}_{C,0}.$$

The integer $\delta(C,0)$ is called the *delta invariant* of the singularity. Later on we will prove that it indeed is an *invariant* of the singularity.

Exercise 10.10. Prove that the polynomial $y^2 - x^{2k+1}$ is irreducible as an element in $\mathbb{C}\{x,y\}$.

Example 10.11. Let $C_{2,k}$ denote the germ at 0 of the curve $y^2 - x^{2k+1}$. As in Example 10.2 we see that $\mathcal{O}_{C,0} \cong \mathbb{C}\{S_k\}$ where S_k is the sub-monoid of $(\mathbb{Z}_+, +)$ generated by 2 and $2k+1$, and $\mathbb{C}\{S_k\} \subset \mathbb{C}\{t\}$ is the subring defined by

$$\mathbb{C}\{S_k\} = \Big\{ f = \sum_{m \in S_k} a_m t^m \Big\}.$$

Then

$$\delta(C_k, 0) = \dim \mathbb{C}\{t\}/\mathbb{C}\{S_k\} = \#(\mathbb{Z}_+ \setminus S_k) = k.$$

We will next present a constructive description of the resolution of an irreducible singularity based on *Newton polygons*, and then we will discuss a few numerical invariants of an isolated singularity of a curve.

§10.3 Puiseux series and Newton polygons

The resolution described in the proof of Theorem 10.8 has a very special form

$$x = t^q, \;\; y = y(t) \in \mathbb{C}\{t\}, \;\; f(t^q, y(t)) = 0.$$

If we think, as the classics did, that y is an algebraic function of x implicitly defined by the equation $f(x, y) = 0$, we can use the above resolution to produce a power series description of $y(x)$. More precisely, we set $t = x^{1/q}$ and we see that

$$y = \sum_{k \geq 0} y_k x^{k/q}.$$

Such a description is traditionally known as a *Puiseux series expansion.*

The above argument is purely formal, since the the function $z \mapsto z^{1/q}$ is a multivalued function. Denote by $\mathbb{C}((z))$ the field of fractions of the ring of formal power series $\mathbb{C}[[z]]$. It can be alternatively described as the ring of formal Laurent series in the variable z. Denote by $\mathbb{C}((z^{1/n}))$ the finite extension of $\mathbb{C}((z))$ defined by

$$\mathbb{C}((z^{1/n})) := \frac{\mathbb{C}((z))[t]}{(t^n - z)}.$$

Observe that if $m|n$ then we have a natural inclusion

$$\imath_{nm} : \mathbb{C}((z^{1/m})) \hookrightarrow \mathbb{C}((z^{1/n})), \;\; z^{1/m} \mapsto (z^{1/n})^{n/m},$$

or more rigorously,

$$\frac{\mathbb{C}((z))[t]}{(t^m - z)} \hookrightarrow \frac{\mathbb{C}((z))[s]}{(s^n - z)}, \;\; t \mapsto s^{n/m}.$$

The inductive limit of this family of fields is denoted by $\mathbb{C}\langle\langle z \rangle\rangle$. The elements of this field are called *Puiseux-Laurent series* and can be *uniquely* described as formal series

$$f = \sum_{k \geq d} a_k z^{k/n}, \;\; d, n \in \mathbb{Z}, \;\; n > 0, \;\; \text{g.c.d.}\Big(\{n\} \cup \{k; \;\; a_k \neq 0\}\Big) = 1.$$

Define

$$S(f) := \{k; \;\; a_k \neq 0\}, \;\; o_z(f) := \frac{\min S(f)}{n}.$$

$S(f)$ is called the *support* of f, $o_z(f)$ is called the *order* of f and n is called the *polydromy order*. We denote it by $\nu(f)$. A *Puiseux series* is then a Puiseux-Laurent series f such that $o_z(f) \geq 0$. It is convenient to describe a Puiseux series f of polydromy order n in the form

$$f(z) = g(z^{1/n}), \quad g \in \mathbb{C}[[x]].$$

Observe that g is uniquely determined by the identity

$$g(x) = f(x^n).$$

We say that g is the *power series expansion associated to* f. The Puiseux series f is called convergent if the associated power series is convergent.

Theorem 10.8 shows that if $f \in \mathbb{C}\{x\}[y]$ is a Weierstrass polynomial, irreducible as an element in $\mathbb{C}\{x, y\}$, then there exists a (convergent) Puiseux series $y = y(x)$ such that

$$f(x, y(x)) = 0.$$

Moreover, the polydromy order of $y(x)$ is equal to the y-degree of the Weierstrass polynomial $f(x, y)$. In particular, this means that the polynomial in y has a root in the extension $\mathbb{C}\langle\langle x \rangle\rangle$ of $Q(\mathbb{C}\{x\})$.

The Galois group of the extension $\mathbb{C}((x)) \hookrightarrow \mathbb{C}((x^{1/n}))$ is a cyclic group of G_n order n. Fix a generator ρ of G_n. Then there exists a primitive n-th root ϵ of 1 such that

$$(\rho f)(x^{1/n}) = f(\epsilon x^{1/n}), \quad \forall f \in \mathbb{C}((x^{1/n})).$$

We conclude immediately that if $f(x, y) \in \mathbb{C}\{x\}[y]$ is a Weierstrass polynomial of degree n, irreducible as an element of $\mathbb{C}\{x, y\}$, and $y \in \mathbb{C}((x^{1/n}))$ is a convergent Puiseux series resolving the singularity at 0 of $f(x, y) = 0$ then

$$f(x, y) = \prod_{\epsilon^n = 1} \big(y - y(\epsilon x^{1/n})\big).$$

A natural question arises. How do we *effectively* produce a Puiseux series expansion for an algebraic function $y(x)$ defined by an irreducible equation $f(x, y) = 0$? In the remainder of this section we will outline a classical method, based on Newton polygons.

Definition 10.12. Let $f = \sum_\alpha a_\alpha X^\alpha \in \mathbb{C}\{x, y\}$, where $\alpha \in \mathbb{Z}_+^2$, and $X^\alpha := x^{\alpha_1} y^{\alpha_2}$. The the *support* of f is the set

$$S(f) := \Big\{ \alpha \in \mathbb{Z}_+^2; \;\; a_\alpha \neq 0 \Big\}.$$

Definition 10.13. The *Newton polygon* associated to $S \subset \mathbb{Z}_+^2$ is the convex hull of the set $S + \mathbb{Z}_+^2$. We denoted it by $\Gamma(S)$. The *Newton polygon* of $f \in \mathbb{C}\{x, y\}$ is the Newton polygon of its support. We set $\Gamma(f) := \Gamma(S(f))$.

The Newton polygon of a set $S \subset \mathbb{Z}_+^2$ is noncompact. It has finitely many vertices $P_0, P_1, \cdots, P_r \in \mathbb{Z}_+^2$, which we label in decreasing order of their heights, i.e. if P_i has coordinates (a_i, b_i) then

$$b_0 > b_1 > \cdots > b_q \geq 0.$$

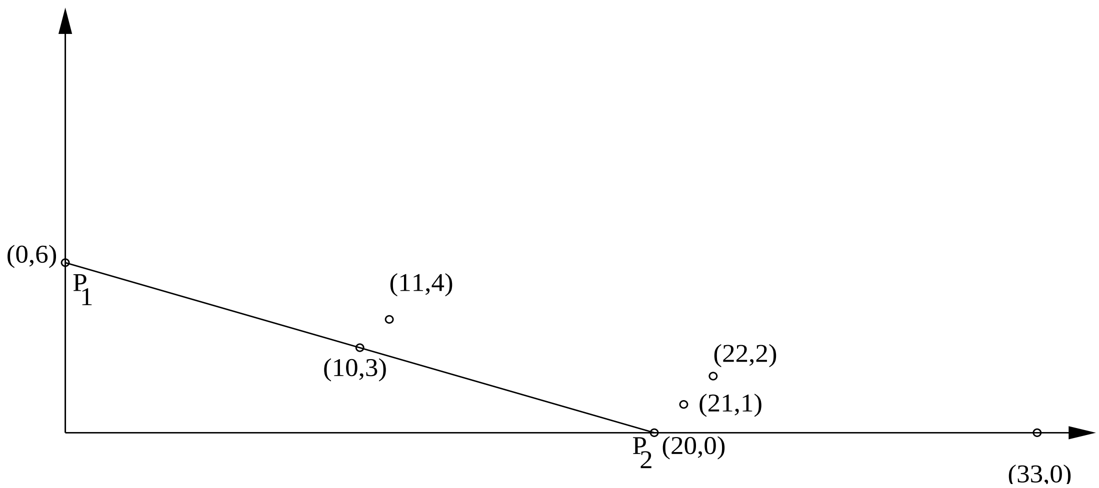

Figure 10.3: *The Newton polygon of* $f = y^6 + 3x^{11}y^4 + 2x^{10}y^3 - 3x^{22}y^2 + 6x^{21}y + x^{33} - x^{20}$.

Note that $0 \leq a_0 < a_1 < \cdots < a_q$. Define the *height* of a Newton Polynomial to be $ht(\Gamma(S)) = b_0 - b_1$, and the *width* to be $wd(\Gamma(S)) = a_q - a_0$. A Newton polygon is called *convenient* P_0 is on the vertical axis, and P_r is on the horizontal axis, i.e. $a_1 = 0$, $b_r = 0$. In Figure 10.3 is depicted a Newton polygon with two vertices.

In general, a Newton polygon has a finite number of vertices $r+1$, and a finite number of finite edges, r. It has two infinite edges, a vertical one, and a horizontal one. Note that f is y-regular of order m (i.e. $y \mapsto f(0,y)$ has a zero of order m at $y=0$) if and only if the first vertex of its Newton polygon is the point $(0,m)$ on the vertical axis.

The following elementary result offers an indication that the Newton polygon captures some nontrivial information about the geometry of a planar curve.

Exercise 10.14. If $f \in \mathbb{C}\{x,y\}$ is irreducible then its Newton polygon is convenient and has a single finite edge.

It is not always easy or practical to draw the picture of the Newton polygon of a given polynomial, so we should have of understanding its basic geometric characteristics without having to draw it. This can be achieved using basic facts of convex geometry.

Set $V := \mathbb{R}^2$, $L := \mathbb{Z}^2 \subset V$, $L_+ := \mathbb{Z}^2_+ \subset L$, and denote by $V^\sharp$ the dual of V. Suppose $S \subset L_+$. The *polar* of $\Gamma(S)$ is the convex set

$$\Gamma(S)^\sharp = \Bigl\{\chi \in V^\sharp;\ \ \langle \chi, v\rangle \geq 0,\ \ \forall v \in \Gamma(S)\Bigr\}.$$

The restriction of any linear functional $\chi \in \Gamma(S)^\sharp$ to $\Gamma(S)$ achieves its minimum either at a vertex of $\Gamma(S)$ or along an entire edge of $\Gamma(S)$. For uniformity, we will use the term *face* to denote vertices and *finite* edges. A vertex is a 0-face, and an edge is a 1-face. For each $\chi \in \Gamma(S)^\sharp$ we denote by ϕ_χ the face of $\Gamma(S)$ along which χ achieves its minimum. We say that $\phi(\chi)$ is the *trace* of χ along $\Gamma(S)$. Define the *supporting function* of $\Gamma(S)$ to be

$$\ell_S : \Gamma(S)^\sharp \to \mathbb{R} \to \mathbb{R}_+,\ \ \ell_S(\chi) = \min\Bigl\{\langle \chi, v\rangle;\ \ v \in \Gamma(S)\Bigr\}.$$

Observe that

$$\phi(\chi) = \Big\{ v \in \Gamma(S);\ \ \langle \chi, v \rangle = \ell_S(\chi) \Big\}.$$

Two linear functionals $\chi_1, \chi_2 \in \Gamma(S)^\sharp$ are called *equivalent* if $\phi(\chi_1) = \phi(\chi_2)$. We denote by C_χ the closure of the equivalence class of χ. We have the following elementary result.

Exercise 10.15. For each $\chi \in \Gamma(S)^\sharp$ the set C_χ is *rational* cone, i.e. it is a closed, convex cone generated over $\mathbb{R}$ by a finite collection of vectors in the dual lattice

$$L^\flat := \Big\{ \chi \in V^\sharp;\ \ \langle \chi, v \rangle \in \mathbb{Z},\ \ \forall v \in L \Big\}$$

Definition 10.16. A 2-dimensional *fan* is a finite collection $\mathcal{F}$ of rational cones in $V^\sharp$ with the following properties.

(a) A face of a cone in $\mathcal{F}$ is a cone in $\mathcal{F}$.
(b) The intersection of two cones in $\mathcal{F}$ is a cone in $\mathcal{F}$.

We have the following result.

Proposition 10.17. *Consider a set $S \subset L_+$ and its Newton polygon $\Gamma(S)$. Then the collection*

$$\Phi(S) := \Big\{ C_\chi;\ \ \chi \in \Gamma(S)^\sharp \Big\}$$

is a fan.

Proof Denote by $\{P_0, P_1, \cdots, P_r\}$ the vertices of $\Gamma(S)$ arranged in decreasing order of their heights. For $i = 1, \cdots, r$, denote by λ_i the line trough the origin perpendicular to $P_{i-1}P_i$. Observe that

$$0 < \mathrm{slope}\,(\lambda_1) < \mathrm{slope}\,(\lambda_2) < \cdots < \mathrm{slope}\,(\lambda_r) < \infty.$$

These rays partition the first quadrant V_+ of V into a fan consisting of the origin, the nonnegative parts of the horizontal and vertical axes, the parts of the rays λ_i inside the first quadrant, and the angles formed by these rays. If we identify V and $V^\sharp$ using the Euclidean metric, then we can a fan in $V^\sharp$ consisting precisely of the cones C_χ, $\chi \in \Gamma(S)^\sharp$. ■

We see that the one dimensional cones in $\Phi(S)$ correspond to the one dimensional faces of $\Gamma(S)$ (see Figure 10.4). We denote by $\Delta = \Delta_S$ the steepest 1-face of $\Gamma(S)$. It is the first 1-face, counting from left to right. It corresponds to the first (least inclined) non-horizontal ray in the associated fan.

For each 1-face ϕ of $\Gamma(S)$, the corresponding one-dimensional cone C_ϕ in $\Phi(S)$ contains an additive monoid $C_\phi \cap L^\flat$ generated by the lattice vector on C_ϕ closest to the origin. We denote this vector by $\vec{w}_\phi$, and we will refer to it as the *weight* of the one-dimensional face ϕ. Observe that the face ϕ is the trace of $\vec{w} = (w_1, w_2)$ on $\Gamma(S)$, and the coordinates (w_1, w_2) are coprime integers. The quantity

$$d(\phi) := \ell_S(\vec{w}_\phi) = \langle \vec{w}, v \rangle,\ \ v \in \phi$$

is called the *(weighted) degree* of the the face ϕ. The weight of $\Gamma(S)$ is defined to be the weight of the first face Δ.

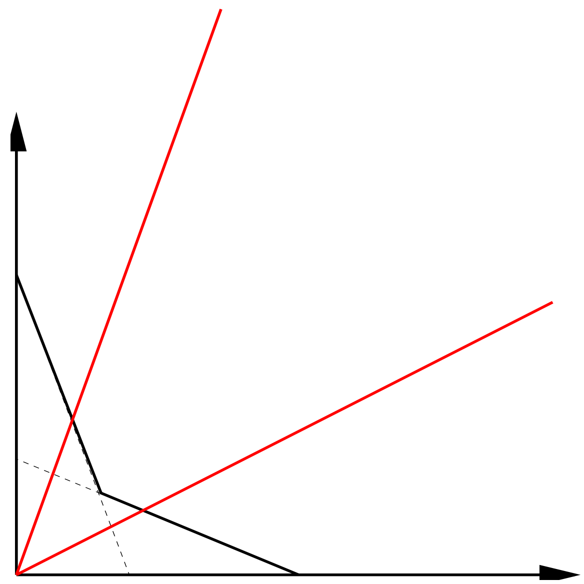

Figure 10.4: *A Newton polygon (in black) and its associated fan (in red).*

Definition 10.18. Suppose $f = \sum_\alpha a_\alpha X^\alpha \in \mathbb{C}\{x, y\}$, and Δ is the first face of $\Gamma(f)$. We set

$$f_\Delta = \sum_{\alpha \in \Delta} a_\alpha X^\alpha.$$

The function f is called *nondegenerate* if it is y-regular and the weight of Δ has the form $(1, w)$, $w \in \mathbb{Z}_{>0}$.

Example 10.19. The Newton polygon in Figure 10.3 has an unique 1-face ϕ described by the equation

$$\phi: \quad \frac{x}{20} + \frac{y}{6} = 1, \quad x, y \geq 0.$$

The corresponding cone in the associated fan is the ray

$$C_\phi: \quad y = \frac{10}{3} x, \quad x \geq 0.$$

We deduce that the weight of this face is $\vec{w} = (3, 10)$ and its degree is 60.

Consider now an irreducible $f \in \mathbb{C}\{x, y\}$. We assume it is a Weierstrass polynomial in y. Its Newton polygon Γ_f has only one face. We denote by $\vec{w}$ the weight of the unique finite 1-face of Δ and by d_0 its degree. For each $\alpha \in L_+$ we denote by $\deg_w(X^\alpha)$ its $\vec{w}$-weighted degree,

$$\deg_w(X^\alpha) = \langle \vec{w}, \alpha \rangle = w_1\alpha_1 + w_2\alpha_2.$$

We can write

$$f := \sum_{\alpha \in L_+} a_\alpha X^\alpha$$

where $a_\alpha = 0$ if $\alpha \notin S(f)$. Then

$$f = \sum_{d \geq d_0} f_d, \text{ where } f_d := \sum_{\deg_w \alpha = d} a_\alpha X^\alpha, \ \ (\deg_w \alpha := \langle \vec{w}, \alpha \rangle).$$

To find a Puiseux series expansion of the algebraic function $y = x(x)$ defined by $f(x,y) = 0$ we will employ a method of successive approximation. We first look for a Puiseux series of the form

$$x := x_1^{w_1}, \ : y = x_1^{w_2}(a_0 + y_1), \ \ c_0 \in \mathbb{C}, \ \ y_1 = y_1(x_1) \in \mathbb{C}\{x_1\}, \ \ y_1(0) = 0.$$

Observe that

$$f_d\big(x_1^{w_1}, x_1^{w_2}(a_0 + x_1^{w_2} y_1)\big) = x_1^d \sum_{\deg_w \alpha = d} a_\alpha (c_0 + y_1)^{\alpha_2},$$

so that

$$f\big(x_1^{w_1}, x_1^{w_2}(a_0 + x_1^{w_2} y_1)\big) = O(x_1^{d_0}), \ \ \forall y_1.$$

We want to find c_0 such that

$$f\big(x_1^{w_1}, x_1^{w_2}(a_0 + x_1^{w_2} y_1)\big) = O(x_1^{d_0+1}), \ \ \forall x_2.$$

We see that this is possible iff c_0 is a root of the polynomial equation

$$f_{d_0}(1, c_0) = f_\Delta(1, c_0) = \sum_{\deg_w \alpha = d_0} a_\alpha c_0^{\alpha_2} = 0.$$

Now define a new function

$$f_1(x_1, y_1) := \frac{1}{x_1^{d_0}} f(x_1^{w_1}, x_1^{w_2}(c_0 + y_1)) \in \mathbb{C}\{x_1, y_1\}.$$

We believe it is more illuminating to illustrate the above construction on a concrete example, before we proceed with the next step.

Example 10.20. Consider the Weierstrass polynomial

$$f(x,y) = y^6 + 3x^{11}y^4 + 2x^{10}y^3 - 3x^{22}y^2 + 6x^{21}y + x^{33} - x^{20} \in \mathbb{C}\{x\}[y]$$

with elementary Newton polynomial depicted in Figure 10.3. The weight of the unique finite 1-face is according to Example 10.19, $\vec{w} = (3, 10)$, and its degree is 60. Then

$$f_\Delta = f_{60}(x,y) := y^6 + 2x^{10}y^3 - x^{20}.$$

so that $f_\Delta(1,y) = y^6 + 2y^3 - 1$. It has six roots determined by

$$r^3 = -1 \pm \sqrt{2}.$$

Pick one of them, and denote it by c_0. Then

$$\frac{1}{x^{60}} f\big(\, x^3, x^{10}(c_0+y)\,\big)$$

$$= (c_0+y)^6 + 3x^{13}(c_0+y)^4 + 2(c_0+y)^3 - 3x^{26}(c_0+y)^2 + 6x^{13}(c_0+y) + x^{39} - 1$$

$$= y^6 + 6c_0y^5 + \Big(3x^{13} + 15c_0^2\Big)y^4 + \Big(12c_0x^{13} + 2 + 20c_0^3\Big)y^3$$

$$+\Big(15c_0^4 + 6c_0 + 18c_0^2x^{13} - 3x^{26}\Big)y^2 + \Big(6c_0^5 + 6c_0^2 + 12c_0^3x^{13} - 6c_0x^{26}\Big)y$$

$$+(13c_0^4 + 6c_0)x^{13} - 3c_0^2x^{26} + x^{39}.$$

This looks ugly! However, here is a bit of good news. The height of the Newton polygon of f_1 is substantially smaller. It is equal to one, and it is due to the presence of the nontrivial monomial $(6c_0^5 + 6c_0^2)y$ which shows that the multi-exponent $(0,1)$ lies in the support of f_1. Let us phrase this in different terms. The condition that $(0,1) \in S(f_1)$ is equivalent to $\frac{\partial f_1}{\partial y}(0,0) \neq 0$. Using the holomorphic implicit function theorem we deduce that we can express y as a *holomorphic* function of x, $y = y_1(x)$. Note that $y_1(x)$ vanishes up to order 13 at $x=0$, i.e. it has a Taylor expansion of the form

$$y_1(x) = c_1x^{13} + \text{ higher order terms.}$$

We now have a Puiseux expansion

$$x = x_1^3,\ \ y = x_1^{10}(c_0 + y_1(x_1)) = c_0x_1^{10} + c_1x_1^{23} + \cdots$$

All the other monomials arising in the power series expansion of $y_1(x_1)$ can be (theoretically) determined inductively from the implicit equation $f_1(x_1, y_1) = 0$. Practically, the volume of computation can be overwhelming.

The above example taught us some valuable lessons. First, the passage from f to f_1 reduces the complexity of the problem (in a sense yet to be specified). We also see that once we reach a Newton polygon of height one the problem is essentially solved, because we can invoke the implicit function theorem. The transformation $f \mapsto f_1$ produces a hopefully simpler curve gem $(C_1, 0)$ and holomorphic map $(C_1, 0) \mapsto (C, 0)$.

Let us formalize this construction. Denote by $\mathfrak{R}\{x,y\}$ the set of holomorphic functions $f \in \mathbb{C}\{x,y\}$ y-regular. Define a multivalued transformation

$$\mathfrak{P} : \mathfrak{R}\{x,y\} \ni f \mapsto \mathfrak{P}(f) \subset \mathfrak{R}\{x_1,y_1\}$$

where

$$\hat{f}(x_1,y_1) \in \mathfrak{P}(f) \Longleftrightarrow \hat{f}(x_1,y_1) = \frac{1}{x_1^{d_0}} f(x_1^{w_1}, x_1^{w_2}(r+y_1))$$

where $r \in \mathbb{C}$ is a root of the polynomial equation

$$f_\Delta(1,r) = 0$$

We first need to show that $\mathfrak{P}$ is well defined.

Lemma 10.21. *Let $f \in \mathfrak{R}\{x,y\}$ be y-regular of order m. Then* $\deg f_\Delta(1,y) = m$ *and any $\hat{f} \in \mathfrak{P}f$ is y_1-regular of order $m_1 \leq m$, where m_1 is the multiplicity of the root r of the degree m polynomial $f_\Delta(1,y) \in \mathbb{C}[y]$. Moreover, $m_1 = m$ iff $f_\Delta(1,y) = c(y-r)^m$ in which case the weight (w_1, w_2) of Δ is nondegenerate, i.e. $w_1 = 1$.*

Proof Denote by $\vec{w} = (w_1, w_2)$ the weight of $\Gamma(f)$ and by d_0 the weighted degree of f_Δ. Denote by P and Q the endpoints of the first edge Δ of $\Gamma(f)$, $P = P(0,m)$ and $Q = Q(a,b)$, $b < m$. Pick a root r of $f_\Delta(1,y) = 0$ and set

$$\hat{f}(x_1, y_1) = \frac{1}{x_1^{d_0}} f(x_1^{w_1}, x_1^{w_2}(r + y_1)).$$

We write

$$f = \sum_{d \geq d_0} f_d(x,y),$$

where $f_d \in \mathbb{C}[x,y]$, $\deg_w f_d = d$. Then

$$\frac{1}{x_1^{d_0}} f(x_1, x_1^{w_2}(r+y_1)) = f_\Delta(1, r+y_1) + \sum_{k=1}^{\infty} x_1^k f_d(1, r+y_1).$$

On the other hand

$$f_\Delta(1,y) = c\prod_{j=1}^{m}(y - r_j), \quad c \in \mathbb{C}^*$$

so that, if r is a root of of $f_\Delta(1,y)$ of multiplicity m_1 we get

$$f_\Delta(1, r+y_1) = cy_1^{m_1} \prod_{j=m_1+1}^{m} (y_1 + r - r_j).$$

The lemma is now obvious. ■

The above lemma shows that for every y-regular germ f there exists $k_0 > 0$ such that for all $k \geq k_0$ all the germs $g \in \mathfrak{P}^k(f)$ are nondegenerate. To understand what is happening let us consider a few iterations $f_n \mapsto f_{n+1} \in \mathfrak{P}(f_n)$, $n = 0, 1$.

$$f_1(x_1, y_1) = \frac{1}{x_1^{d_0}} f(x_1^{w_{11}}, x_1^{w_{21}}(y_1 + r_1))$$

We can formally set $x_1 = x^{1/w_{11}}$ and

$$y = x^{w_{21}/w_{11}}(y_1 + r_1).$$

At the second iteration we get $x_2 = x_1^{1/w_{12}} = x^{1/w_{11}w_{12}}$ and

$$y_1 = x_1^{w_{22}/w_{12}}(y_2 + r_2) \Longrightarrow y = x^{w_{21}/w_{11}}\Big(x^{w_{22}/w_{11}w_{12}}(y_2 + r_2) + r_1\Big)$$

$$= r_1 x^{w_{21}/w_{11}} + r_2 x^{\frac{w_{21}}{w_{11}}+\frac{w_{22}}{w_{11}w_{12}}} + y_2 x^{\frac{w_{21}}{w_{11}}+\frac{w_{22}}{w_{11}w_{12}}}.$$

In the limit we will obtain a power series of the form

$$y = \sum_{k=1}^{\infty} r_k x^{a_k},$$

where the rational exponents a_k are determined inductively from

$$a_{k+1} = a_k + \frac{w_{2(k+1)}}{w_{1(k+1)}} \cdot \frac{1}{w_{11}w_{12}\cdots w_{1k}}$$

where $\vec{w}_k = (w_{1k}, w_{2k})$ is the weight of f_k. For $k \geq k_0$ we have $w_{1k} = 1$ so that the denominators of the exponents a_k will not increase indefinitely. In the limit we obtain a formal Puiseux series $y = y(x) \in \mathbb{C}((x^{1/N}))$ which solves

$$f(x, y(x)) = 0.$$

Assume for simplicity that f is irreducible in $\mathbb{C}\{x, y\}$. Then we can write

$$f(x,y) = \prod_{k=0}^{N-1} (y - y(\epsilon x)), \quad \epsilon = \exp(\frac{2\pi \mathbf{i}}{N}).$$

On the other hand, we know that $f(x, y)$ admits a *convergent* Puiseux series expansion which must coincide with one of the formal Puiseux expansions $y(\epsilon x)$. This shows that all the Formal Puiseux expansions must be convergent, and in particular, any formal Puiseux expansion obtained my the above iterative method must be convergent.

Example 10.22. Consider the germ $f(x, y) = y^3 - x^5 - x^7$. It has weight $\vec{w} = (3, 5)$ and

$$f_\Delta = y^3 - x^5, \quad \deg_w f_\Delta = 15$$

so that $f_\Delta(1, y) = y^3 - 1$, which has $r_1 = 1$ as a root of multiplicity 1. Then

$$f_1(x_1, y_1) = \frac{1}{x_1^{15}} f(x_1^3, x_1^5(1 + y_1)) = (1 + y_1)^3 - 1 - x_1^6 = y_1^3 + 3y_1^2 + y_1 - x_1^6.$$

The germ f_1 has weight $(1, 6)$ and

$$(f_1)_\Delta = y_1 - x_1^6, \quad \deg_w = 6.$$

The degree 1-polynomial $f_1(1, y_1) = y_1 - 1$ has only one root $r_2 = 1$, and we can define

$$f_2(x_2, y_2) = \frac{1}{x_2^6} f_1(x_2, x_2^6(1 + y_2)) = \frac{1}{x_2^6}\left\{ \Big(1 + x_2^6(1 + y_2)\Big)^3 - 1 - x_2^6 \right\}$$

$$= 3(1 + x_2^6)y_2 + 3x_2^6(1 + y_2)^2 + x_2^{12}(1 + y_2)^3 - 1.$$

In this case $\vec{w} = (1, 6)$, $(f_2)_\Delta = 3y_2 - 3x_2^6$, so we can take $y_2 = x_2^6(1 + y_3)$. We deduce

$$y_2 = x_2^6(1 + y_3), \quad y_1 = x_2^6(1 + y_2), x_2 = x_1 \Longrightarrow y_1 = x_1^6 + x_1^{12} y_3$$

$$y = x_1^5(1 + y_1), \quad x_1 = x^{1/3} \Longrightarrow y = x^{5/3} + x^{2+5/3} + x^{4+5/3} y_3 = x^{5/3} + x^{11/3} + x^{17/3} y_3.$$

The above computations suggest that computing the whole Puiseux series expansion may be a computationally challenging task. A natural question arises.

How many terms of the Puiseux series do we need to compute to capture all the relevant information about the singularity?

The term *"relevant information"* depends essentially on what type of questions we are asking. There are essentially two types of questions: topological and analytical (geometrical).

Definition 10.23. Consider two irreducible Weierstrass polynomials $f_i \in \mathbb{C}\{x\}[y]$, $i = 0, 1$, such that $f_0(0,0) = 0$. Set $C_i := \{f_i = 0\}$.

(a) The germ $(C_0, 0)$ is *topologically equivalent* to $(C_1, 0)$ if there exist neighborhoods U_i of 0 in $\mathbb{C}^2$ and a homeomorphism

$$\Phi : U_0 \to U_1$$

such that $\Phi(C_0 \cap U_0) = C_1 \cap U_1$.
(b) The germ $(C_0, 0)$ is *analytically equivalent* to $(C_0, 0)$ if the local rings $\mathcal{O}_{C_0,0}$ and $\mathcal{O}_{C_1,0}$ are isomorphic.

Using Exercise 9.34 we deduce that the germs C_i are analytically equivalent if and only if there exist neighborhoods U_i of 0 in $\mathbb{C}^2$ and a biholomorphic map $\Psi : U_0 \to U_1$ such that

$$\Psi(C_0 \cap U_0) = C_1 \cap U_1.$$

Before we explain how to extract the relevant information (topological and/or analytic) we want to discuss some arithmetical invariants of Puiseux series. Fix an irreducible Weierstrass polynomial $f \in \mathbb{C}\{x\}[y]$ and Puiseux series expansion

$$x = t^N, \;\; y(t) = \sum_{k>0} a_k t^k$$

Define the *set of exponents* of f by

$$E_1 = E_1(f) := \left\{ \frac{k}{N}; \;\; a_k \neq 0 \right\} \subset \frac{1}{N}\mathbb{Z}_{>0}.$$

Let $\kappa_1 := \min E_1 \setminus \mathbb{Z}$. We write

$$\kappa_1 := \frac{n_1}{m_1}, \;\; (m_1, n_1) = 1.$$

We then define

$$E_2 = E_1 \setminus \left\{ \frac{q}{m_1}; \;\; q \in \mathbb{Z}_+ \right\}$$

and, if $E_2 \neq \emptyset$ we set $\kappa_2 = \frac{r_2}{m_1} = \min E_2$,

$$r_2 = \frac{n_2}{m_2} \in \mathbb{Q}_{>0}, \;\; m_2 > 1, \;\; (n_2, m_2) = 1.$$

If the pairs $(m_1,n_1),\cdots(m_j,n_j)$ have been selected then define

$$E_{j+1}=E_1\setminus\Big\{\frac{q}{m_1\cdots m_j};\;\;q\in\mathbb{Z}_+\Big\}.$$

If $E_{j+1}=\emptyset$ we stop, and if not, we set $\kappa_{j+1}=\min E_{j+1}$

$$\kappa_{j+1}=r_{j+1}\cdot\frac{1}{m_1\cdots m_j},\;\;r_{j+1}=\frac{n_{j+1}}{m_{j+1}},\;\;m_{j+1}>1,\;\;(n_{j+1},m_{j+1})=1.$$

The process stops in $g<\infty$ steps because $(m_1\cdots m_j)|N$, $\forall j$. The sequence

$$\Big\{(m_1,n_1),(m_2,n_2),\cdots,(m_g,n_g)\Big\}$$

called the sequence of *Puiseux pairs* of f. The integer $m_1\cdots m_g$ coincides with the *polydromy* of N of the Puiseux series expansion. Define the *characteristic exponents* of E_1 by

$$k_0=k_0(E_1):=m_1\cdots m_g,\;\;k_j=k_j(E_1):=k_0\cdot\kappa_j=n_1\cdots n_jm_{j+1}\cdots m_g,\;\;j=1,\cdots g. \tag{10.2}$$

The Puiseux series expansion can be obtained by applying Newton's algorithm so we can write

$$y(x)=\sum_{k=1}^{L}c_kx^{r_k},\;\;L\in\mathbb{Z}_+\cup\infty,\;\;c_k\in\mathbb{C}^*,\;\;r_k\in\mathbb{Q}_{>0},$$

where

$$r_k=r_{k-1}+\frac{w_{2k}}{w_{1k}}\cdot\frac{1}{w_{11}\cdots w_{1(k-1)}},\;\;r_{k-1}=\frac{\alpha_{k-1}}{w_{11}\cdots w_{1(k-1)}},\;\;\gcd(w_{1k},w_{2k})=1.$$

There are only finitely many $w_{1j}>1$. The Puiseux pairs are exactly the pairs (w_{1j},n_j) such that $w_{1j}>1$, where

$$n_j=\alpha_{j-1}w_{11}\cdots w_{1(j-1)}w_{1j}+w_{2j}.$$

We have the following classical result.

Theorem 10.24. *Suppose $f_i(x,y)$, $i=0,1$, are two irreducible Weierstrass polynomials in y. Set $C_i:=\{f_i=0\}\subset\mathbb{C}^2$. Then the germs $(C_i,0)$ are topologically equivalent if and only if they have the same sequences of Puiseux pairs.*

This theorem essentially says that if we want to extract all the topological information about the singularity we only need to perform the Newton algorithm until $w_{11}\cdots w_{1m}=N$, where N is the poydromy order of the Puiseux expansion. If f is a Weierstrass y-polynomial, then $N=\deg_y f$.

We refer to [5] for a detailed, clear and convincing explanation of the geometric intuition behind this fact. We content ourselves with a simple example which we hope will shed some light on the topological information carried by the Puiseux pairs.

Example 10.25. (Links of singularities of plane curves.) Consider the singular plane curve

$$C = \{(x,y) \in \mathbb{C}^2;\ y^2 = x^3\},$$

It has a single Puiseux pair $(2,3)$. The link of the singularity at $(0,0)$ is by definition

$$K_C = K_{C,r} := C \cap \partial B_r$$

where B_r is the sphere of radius r centered at the origin. We will see later in Chapter 12 that for all sufficiently small r the link is a compact, smooth one-dimensional submanifold of the 3-sphere ∂B_r. In other words, K_C is a link, which has as many components as irreducible components of the *germ* of C at $(0,0)$. In our case the germ of C at the origin is irreducible so that K_C is a knot. Its isotopy type is independent of r and thus can be viewed as a topological invariant of the singularity.

To understand this knot consider the polydisk

$$D_r^2 = \{(x,y) \in \mathbb{C}^2;\ |x| \le r^2,\ \ |y| \le r^3\}.$$

Note that ∂D_r^2 is homeomorphic to the 3-sphere and it describes an explicit decomposition of the 3-sphere as an union of two (linked) solid tori

$$\partial D_r^2 = H_x \cup H_y = \big(\partial D^x_{r^2} \times \times D^y_{r^3}\big) \cup \big(D^x_{r^2} \times \partial D^y_{r^3}\big)$$

$$:= \{|x| = r^2,\ \ |y| \le r^3\} \cup \{|x| \le r^2,\ \ |y| = r^3\}.$$

The core of $D^x_{r^2}$ is an unknot K_0 situated in the plane $y = 0$, parametrized by

$$S^1 \ni \zeta \mapsto (r\zeta, 0).$$

One can show that the knot $C \cap \partial D_r$ is isotopic to $C \cap \partial B_r$. Moreover there is an embedding

$$\phi : \{|z| = r\} \subset \mathbb{C}^* \to \partial B_r,\ \ z \mapsto (x,y) = (z^2, z^3)$$

whose image is precisely the link of the singularity. It lies on the torus

$$T_r = \partial D^x_{r^2} \times \partial D^y_{r^3}.$$

and carries the homology class

$$2[\partial D^x_{r^2}] + 3[\partial D^y_{r^3}] \in H_1(T_r, \mathbb{Z}).$$

We say that it is a $(2,3)$-torus knot. It is isotopic to the trefoil knot depicted in Figure 12.1. Observe that this link is completely described by the Puiseux expansion corresponding to this singularity. More generally, the link of the singularity described by $y^p = x^q$, $\gcd(p,q) = 1$, is a (p,q)-torus knot.

Suppose now that we have a singularity with Puiseux series

$$y = x^{3/2} + x^{7/4}.$$

Then the Puiseux pairs are $(2,3),(2,7)$. To construct its link consider a polydisk

$$D^2 = \{(x,y) \in \mathbb{C}^2;\ \ |x| \le r,\ \ |y| \le s\}.$$

so that ∂D decomposes again into an union of solid tori

$$\partial D^2 = H_x \cup H_y,\ \ H_x = \{|x| = r\} \times \{|y \le s\},\ \ H_y = \{|x| \le r\} \times \{|y| = s\}.$$

We will use the Puiseux expansion to describe the link as an embedding $S^1 \to H_x$. Consider the map

$$\phi : S^1 = \{|z| = r^{1/4}\} \to \mathbb{C}^2,\ \ z \mapsto (x,y) = (z^4, z^7 + z^7).$$

Observe that if $r^{3/2} + r^{7/4} \le s$, then $\phi(S^1) \subset H_x$ and the image of ϕ is precisely the link of the singularity.

To understand the knot $\phi(S^1)$ we will adopt a successive approximations approach. Set

$$\phi_1, \phi_2 : S^1 \to H_x,\ \ \phi_1(z) = (z^2, z^3),\ \ \phi_2(z) = \phi_1(z^2) + (0, z^7) = \phi(z).$$

The image of ϕ_1 is a $(2,3)$-torus knot K_1. We see that for $r \ll 1$ we have $|\phi_2(z) - \phi_1(z^2)| \ll 1$ and the image of ϕ is a knot which winds around K_1 2 times in one direction and 7 times in the other direction. Here we need to be more specific abound the winding. This can be unambiguously defined using the *cabling* operations on *framed* knots.

A *framing* of a knot K is a homotopy class of nowhere vanishing sections $\vec{\nu} : K \to \nu_K$, where $\nu_K \to K$ is the normal bundle of the embedding $K \hookrightarrow S^3$. We can think of $\vec{\nu}$ as a vector field along K which is nowhere tangent to K. As we move around the knot the vector $\vec{\nu}$ describes a ribbon bounded on one side by the knot, and on the other side by the parallel translation of the knot $K \mapsto K + \vec{\nu}$ given by the vector field $\vec{\nu}$.

Alternatively, we can think of the translate $K + \vec{\nu}$ as lieing on the boundary of a tubular neighborhood U_K of the knot in S_3. Topologically, U_K is a solid torus and the cycles K and $K + \vec{\nu}$ carry the same homology class in $H_1(U_K, \mathbb{Z})$. They define a generator of this infinite cyclic group. We denote it by λ_K and we call it the *longitude* of the framed knot.

The boundary ∂U_K of this tubular neighborhood carries a canonical 1-cycle called the *meridian of the knot*, μ_K. It is a generator of the kernel of the inclusion induced morphism

$$i_* : H_1(\partial U_K, \mathbb{Z}) \to H_1(U_K, \mathbb{Z}).$$

Since this kernel is an infinite cyclic group, it has two generators. Choosing one is equivalent to fixing an orientation. In this case, if we orient K, then the meridian is oriented by the right hand rule. Once we pick a framing $\vec{\nu}$ of a knot we have an integral basis of $H_1(\partial U_K, \mathbb{Z})$, (μ_K, λ_K). We thus have a third interpretation of a framing, that of a completion of μ_K to an integral basis of $H_1(\partial U_K, \mathbb{Z})$. In particular, we see that each framing defines a homeomorphism

$$U_K \to S^1 \times D^2,\ \ K \to S^1 \times \{0\},\ \ \mu_K \mapsto \{1\} \times \partial D^2,\ \ \lambda_K \mapsto S^1 \times \{1\}.$$

This homeomorphism is unique up to an isotopy.

If $(K, \vec{\nu})$ is a framed knot then a (p, q)-*cable* of K is a knot disjoint from K, situated in a tubular neighborhood U_K of K, and which is homologous to $q[\mu_K] + p[\lambda_K]$ in $H_1(\partial U_K, \mathbb{Z})$.

Each knot bounds a (Seifert) surface in S^3, and a tubular neighborhood of a knot inside this surface is a ribbon, and thus defines a framing, called the *canonical framing*. This is not the only way to define framings. Another method, particularly relevant in the study of singularities, goes as follows.

Suppose $(K_0, \vec{\nu}_0)$ is a framed knot, so that we can identify a tubular neighborhood U_{K_0} with $S^1 \times D^2$. Denote by π the natural (radial) projection $S^1 \times D^2 \mapsto S^1 \times \{0\}$. A *cable* of K_0 is a knot K with the following properties.

- $K_0 \subset U_{K_0}$ and $K \cap K_0 = \emptyset$.
- The restriction $\pi : K \to K_0$ is a regular cover, of degree $k > 0$.

Fix a thin tubular neighborhood U_K of K contained in $U_{K_0} = S^1 \times D^2$. Then U_K intersects each slice $S_t = \{t\} \times D^2$, $t \in S^1$, in k-disjoint disks, $\Delta_1, \cdots, \Delta_k$, centered at $p_1(t), \cdots, p_k(t)$. Fix a vector $\mathbf{u} \in D^2$ and then parallel transport it at each of the points $p_1(t), \cdots, p_k(t)$, $t \in S^1$ as in Figure 10.5. In this fashion we obtain a smooth vector field along K which is nowhere tangent to K. Its homotopy class is independent of the choice $\mathbf{u}$. In this fashion we have associated a framing to each cable of a framed knot. In $H_1(\partial U_{K_0}, \mathbb{Z})$ we have an equality

$$[K] = q[\mu_{K_0}] + k\lambda_{K_0}.$$

We have thus shown that *the cable of a framed knot is naturally framed itself.*

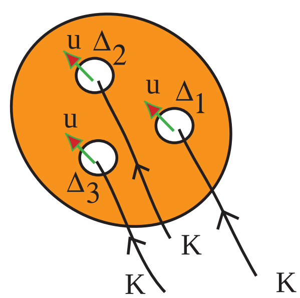

Figure 10.5: *A punctured slice*

Suppose we are given two vectors $\vec{m}, \vec{n} \in \mathbb{Z}^r$ such that $\gcd(m_i.n_i) = 1$, $\forall i = 1, \cdots, r$. An *iterated torus knot* of type $(\vec{m}; \vec{n})$ is a knot K such that there exist *framed* knots $K_0, \cdots, K_{r-1}, K_r = K$ with the following properties.

- K_0 is the unknot with the obvious framing.
- K_i is the (m_i, n_i)-cable of K_{i-1} equipped with the framing described above.

If $(C, 0)$ is the germ at 0 of a planar (compplex) curve, and its Puiseux pairs are $(m_1, n_1), \cdots, (m_r, n_r)$ then the link $C \cap B_r(0)$ is an iterated torus knot of the type

$$\Big(m_1, \cdots, m_r;\ n_1, \cdots, n_r\Big).$$

In particular, for the singularity given by the Puiseux series $t \mapsto (t^4, t^6 + t^7)$ the link is an iterated torus of type $(2,2;3,7)$. The link of the singular germ $y^5 = x^3$ is the $(3,5)$-torus knot depicted in Figure 10.6.

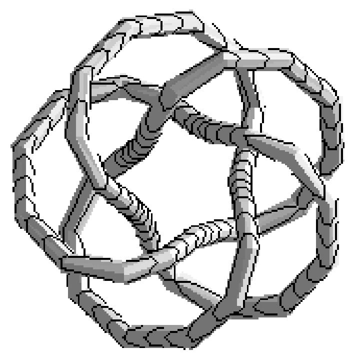

Figure 10.6: *A $(3,5)$-torus knot generated with MAPLE.*

The Puiseux expansion $(x(t), y(t))$ produces an embedding

$$R_f := \mathbb{C}\{x,y\}/(f) \to \mathbb{C}\{t\}.$$

We have a morphism of semigroups

$$\mathbf{ord}_t : \Big(\mathbb{C}\{t\}^*, \cdot\Big) \to \Big(\mathbb{Z}_+, +\Big)$$

uniquely defined by

$$\mathbf{ord}_t(t^k) = k, \ \ \mathbf{ord}_t(u) = 0, \ \ \forall u \in \mathbb{C}\{t\}, \ \ u(0) \neq 0.$$

The image of R_f^* in $\mathbb{Z}_+$ is a monoid $\Gamma(f) \subset (\mathbb{Z}_+, +)$. Define the *conductor*

$$c(f) := \min\Big\{n; \ \ n + \mathbb{Z}_+ \subset \Gamma(f)\Big\}.$$

$\Gamma(f)$ is called the *monoid* associated to the singularity of a plane curve.

Example 10.26. (a) Consider again the polynomial f Example 10.20. Then

$$E_1 = \Big\{\frac{10}{3}, \frac{23}{3}\Big\} \cup A$$

where

$$A \subset \left\{\frac{m}{3};\;\; m \geq 23\right\}$$

Then $\kappa_1 = \frac{10}{3}$ so that $(m_1, n_1) = (3, 10)$,

$$E_2 = \left\{\frac{23}{3}\right\} \cup \left(A \setminus \left\{\frac{n}{3};\;\; n \in \mathbb{Z}_+\right\}\right) = \emptyset$$

It is now clear that $k_0 = 3$ and $k_1 = 10$. $\Gamma(f)$ contains the semigroup $\langle 3, 10\rangle_+$ generated by $3, 10$ and it happens that $A \subset \langle 3, 10\rangle_+$. Hence

$$\Gamma(f) = \left\{3, 6, 9, 10, 12, 13, 15, 16, 18, 19, 20, 21, 22, 23, \cdots\right\}.$$

Hence $c(f) = 18$.

(b) Consider the polynomial $f = y^3 - x^5 - x^7$ in Example 10.22. Then

$$E_1(f) = \left\{\frac{5}{3}, \frac{11}{3}\right\} \cup A,\;\; A \subset \frac{1}{3}(17 + \mathbb{Z}_+).$$

Then $k_0 = 3$, $\kappa_1 = \frac{5}{3}$ so that $(m_1, n_1) = (3, 5)$ and $k_1 = 5$. The semigroup generated by 3 and 5 is

$$\langle 3, 5\rangle_+ = \left\{3, 5, 6, 8, 9, 10, 11, \cdots\right\}$$

This shows $\Gamma(f) = \langle 3, 5\rangle_+$ and $c(f) = 8$.

Let us say a few words about analytical equivalence which is rather subtle issue. More precisely we have the following result of Hironaka.

Theorem 10.27. *Suppose we are given two irreducible germs of plane curves with Puiseux expansions*

$$C_1 :\;\; t \mapsto (t^{n_1}, \sum_{j \geq 1} a_j t^j),\;\; C_2 :\;\; t \mapsto (t^{n_2}, \sum_{j \geq 1} b_j t^j).$$

Then the two germs are analytically equivalent if and only if

$$n_1 = n_2,\;\; \delta(C_1, 0) = \delta(C_2, 0) =: \delta,$$

and

$$a_j = b_j,\;\; \forall j = 1, \cdots, 2\delta.$$

We see that the analytical type is determined by a discrete collection of invariants and a continuous family of invariants. Later in this chapter we will see that the Puiseux pairs determine the delta -invariant, and the monoid $\Gamma(f)$ as well. To see this we need a new technique for understanding singularities.

§10.4 Very basic intersection theory

We interrupt a bit the flow of arguments to describe a very important concept in algebraic geometry, that of intersection number. We will consider only a very special case of this problem, namely the intersection problem for plane algebraic curves.

Suppose we have two irreducible holomorphic functions $f, g \in \mathbb{C}\{x, y\}$ defined on some open neighborhood U of the origin, such that $f(0,0) = g(0,0) = 0$. In other words, the curves

$$C_f : \ \{f = 0\} \ \text{ and } \ C_g : \ \{g = 0\}$$

intersect at $(0,0)$ (and possibly other points). We want to consider only the situation when $(0,0)$ is an isolated point of the intersection. This means, that there exists a ball B_r in $\mathbb{C}^2$ centered at $(0,0)$ such that the origin is the only intersection point inside this ball. We can rephrase this as follows. Consider the holomorphic map

$$F : U \subset \mathbb{C}^2 \to \mathbb{C}^2, \ \ (x, y) \mapsto (f(x, y), g(x, y)).$$

Then $C_f \cap C_g = F^{-1}(0)$ and saying that $(0,0)$ is an isolated intersection point is equivalent to the fact that the origin is an isolated zero of F. Using Proposition 9.39 we deduce the following fact.

Proposition 10.28. *The origin is an isolated intersection point if and only if*

$$\mu(C_f \cap C_g, 0) := \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/(f, g) < \infty.$$

The integer $\mu(C_f \cap C_g, 0)$ is called the *multiplicity* of the intersection $C_f \cap C_g$ at the origin, or the *local intersection number* of the two curves at the origin. If C and D are two plane curves such that $C \cap D$ is a finite set we define the *intersection number* of C and D to be

$$C \cdot D = \sum_{p \in C \cap D} \mu(C \cap D, p).$$

The intersection number generalizes in an obvious fashion to curves on smooth surfaces. To justify this terminology we consider a few examples.

If two *distinct* lines L_1, L_2 intersect at the origin it is natural to consider their intersection number to be zero. By changing coordinates it suffices to assume L_1 is the x-axis an L_2 is the y-axis. Then

$$\mathbb{C}\{x, y\}/(x, y) = \mathbb{C}$$

so that $\mu(L_1 \cap L_2, 0) = 1$ which agrees with the geometric intuition.

Suppose that C_f and C_g intersect transversally at the origin, meaning that the covectors $df(0,0)$ and $dg(0,0)$ are linearly independent over $\mathbb{C}$. In particular, the origin is a *smooth* point on each of the curves C_g and C_f. Then the inverse function theorem implies that we can find a holomorphic change of coordinates near the origin such that $f = x$ and $g = y$.

(Geometrically, this means that the curves are very well approximated by their tangents at the origin.) In this case it is natural to say that the origin is a simple (multiplicity one) intersection point. This agrees with the above algebraic definition.

Let us look at more complicated situations. Suppose

$$C_f: \quad y=0, \quad C_g: \quad y=x^2.$$

Thus C_f is the x axis, and C_g is a parabolla tangent to this axis at the origin. In this case we should consider the origin to be a multiplicity 2 intersection point. This choice has the following "dynamical" interpretation (see Figure 10.7).

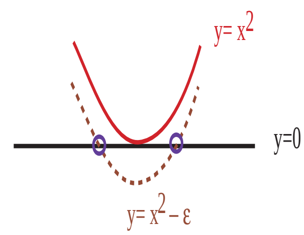

Figure 10.7: *A dynamical computation of the intersection number*

To justify this choice consider the curve $C_{g,\varepsilon}$ given by $y = x^2 - \varepsilon$, $0 < |\varepsilon| \ll 1$. It intersects the x axis at two points $P_\varepsilon^\pm$ which converge to the origin as $\varepsilon \to 0$.

This dynamical description is part of a more general principle called the *conservation of numbers principle.*

Theorem 10.29. *There exist $\varepsilon > 0$ and $r > 0$ such that for any two functions $f_\varepsilon, g_\varepsilon$ holomorphic in a ball B_r of radius r centered at the origin of $\mathbb{C}^2$ such that*

$$\sup_{p\in B_r}\Bigl(|f(p)-f_\varepsilon(p)|+|g(p)-g_\varepsilon(p)|\Bigr)<\varepsilon$$

we have

$$\mu(C_f\cap C_g,0)=\sum_{p\in C_{f_\varepsilon}\cap C_{g_\varepsilon}\cap B_r}\mu(C_{f_\varepsilon}\cap C_{g_\varepsilon},p).$$

We do not present here a proof of this result since we will spend the next two chapters discussing different proofs and generalizations of this result.

The intersection number is particularly relevant in the study of the monoid determined by an isolated singularity. More precisely, we have the following result.

Proposition 10.30. (Halphen-Zeuhten formula) *Suppose $f, g \in \mathbb{C}\{0,0\}$ are two irreducible holomorphic functions defined on a neighborhood U of $0 \in \mathbb{C}^2$ such that f is a y-Weierstrass polynomial of degree n and the origin is an isolated point of the intersection $C_f \cap C_g$. Consider a Puiseux expansion of the germ $(C_f, 0)$,*

$$\pi: \quad t \mapsto (x,y)=(t^n,\chi(t)), \quad \chi(t)\in\mathbb{C}\{t\}.$$

Then

$$\mu(C_f \cap C_g, 0) = \mathbf{ord}_t \pi^*(g) = \mathbf{ord}_t g(t^n, \chi(t)).$$

Proof π is a resolution morphism

$$\pi : (\tilde{C}_f, 0) \cong (\mathbb{C}, 0) \to C_f$$

where $\tilde{C}_f$ is a smooth curve. We have the commutative diagram

$$\begin{array}{ccc} \mathbb{C}\{t\} \cong \mathcal{O}_{\tilde{C}_f,0} & \xrightarrow{\times \pi^*(g)} & \mathcal{O}_{\tilde{C}_f,0} \cong \mathbb{C}\{t\} \\ \uparrow \pi^* & & \uparrow \pi^* \\ \mathcal{O}_{C_f,0} & \xrightarrow{\times g} & \mathcal{O}_{C_f,0} \end{array}.$$

Then

$$\mathbf{ord}_t \pi^*(g) = \dim_{\mathbb{C}} \operatorname{coker}(\times \pi^*(g)), \quad \mu(C_f \cap C_g, 0) = \dim_{\mathbb{C}} \operatorname{coker}(\times g).$$

The equality $\dim \operatorname{coker}(\times g) = \dim \operatorname{coker}(\times \pi^*(g))$ follows from the following elementary linear algebra result.

Lemma 10.31. *Suppose V is a vector space, $U \subset V$ is a subspace and $T : V \to V$ is a linear map such that $T(U) \subset U$. Then there exists a natural isomorphism* $\operatorname{coker} T \to \operatorname{coker} T|_U$.

Proof of the lemma We think of $T : V \to V$ as defining a co-chain complex

$$K_V : \ 0 \to V \xrightarrow{T} V \to 0 \to 0 \cdots .$$

Then $H^0(K_V) = \ker T = 0$ (since f and g are irreducible), $H^1(K_V) = \operatorname{coker} T$. The condition $T(U) \subset U$ implies that

$$K_U : \ 0 \to U \xrightarrow{T} U \to 0 \to 0 \cdots$$

is a subcomplex of K_V. Moreover the quotient complex is K_V/K_U is irreducible. The lemma now follows from the long exact sequence determined by

$$0 \to K_U \to K_V \to K_V/K_U \to 0. \quad \blacksquare$$

Proposition 10.30 has the following consequence.

Corollary 10.32. *Suppose $f \in \mathbb{C}\{x, y\}$ is an irreducible Weierstrass y-polynomial such that $f(0,0) = 0$. Then the monoid Γ_f determined by the germ $(C_f, 0)$ can be described as*

$$\Gamma_f = \Big\{ \mu(C_f \cap C_g, 0); \ \ g \in \mathbb{C}\{x, y\}, \ \ g(0,0) = 0, \ \ g \notin (f) \Big\}.$$

Traditionally, the intersection numbers are defined in terms of *resultants*. We outline below this method since we will need it a bit later. For more details we refer to [5].

Recall (see [22, IV,§8], or [39, §27]) that if f and g are polynomials in the variable y with coefficients in the commutative ring R then their resultant is a polynomial $\mathcal{R}_{f,g}$ in the coefficients of f and g with the property that $\mathcal{R}_{f,g} \equiv 0$ if and only if f and g have a nontrivial common divisor. More precisely, if

$$f = \sum_{k=0}^{n} a_k y^k, \;\; a_n \neq 0, \;\; g = \sum_{j=1}^{m} b_j y^j, \;\; b_m \neq 0,$$

then $\mathcal{R}_{f,g}$ is described by the determinant of the $(m+n)\times(m+n)$ matrix

$$\begin{bmatrix} a_n & a_{n-1} & \cdots & a_0 & 0 & 0 & \cdots & \cdots \\ 0 & a_n & \cdots & a_1 & a_0 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & a_n & a_{n-1} & \cdots & \cdots & a_0 \\ b_m & b_{m-1} & \cdots & b_0 & 0 & 0 & \cdots & \cdots \\ 0 & b_m & \cdots & b_1 & b_0 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & b_m & b_{m-1} & \cdots & b_0 \end{bmatrix}$$

Suppose now that f, g are Weierstrass y-polynomials,

$$f = \sum_{k=0}^{n} a_k(x) y^k, \;\; a_n(0) \neq 0, \;\; g = \sum_{j=1}^{m} b_j(x) y^j, \;\; b_m(0) \neq 0,$$

Their resultant of f and g is then a holomorphic function of x

$$\mathcal{R}_{f,g} \in \mathbb{C}\{x\}.$$

We then have the following result

Proposition 10.33. *Let $C_f := \{f = 0\}$, $C_g := \{g = 0\}$. Then*

$$\mu(C_f \cap C_g, O) = \mathbf{ord}_x \mathcal{R}_{f,g}(x)$$

Exercise 10.34. Prove Proposition 10.33. (**Hint**: Use Halphen-Zheuten formula.)

§10.5 Embedded resolutions and blow-ups

The link of an irreducible germ $(C, 0)$ of plane curve is a circle so its topology is not very interesting. However, the link is more than a circle. It is a circle together with an embedding in S^3. As explained above, the topological type of this *embedding* completely determines the topological type of the singularity. This suggests that the manner in which $(C, 0)$ sits inside $(\mathbb{C}^2, 0)$ carries nontrivial information. The resolution of singularities by Puiseux expansion produces a smooth curve $\tilde{C}$ and a holomorphic map $\pi : \tilde{C} \to C$. Topologically, the smooth curve $\tilde{C}$ is uninteresting. All the topological information is contained in the holomorphic map π. We want now to describe another method of resolving the singularity which produces an embedded resolution.

More precisely, an *embedded resolution* of the curve $C \hookrightarrow \mathbb{C}^2$ is a triplet $(X, \tilde{C}, \pi)$ where X is a smooth complex surface, $\tilde{C} \subset X$ is a smooth curve, $\pi : X \to \mathbb{C}^2$ is a holomorphic map such that $\pi(\tilde{C}) = C$, the induced map $\pi : \tilde{C} \to C$ is a resolution of C and the set $\pi^{-1}(0)$ is a curve in X with only mild singularities (nodes). We will produce embedded resolutions satisfying a bit more stringent conditions using the blowup construction in Chapter 3.

We recall that if M is a smooth complex surface and $p \in M$ then the blowup of M at p is a smooth complex surface $\tilde{M}_p$ together with a holomorphic map $\beta = \beta_p : \tilde{M}_p \to M$ (called the blowdown map) such that

$$\beta : \tilde{M}_p \setminus \beta^{-1}(p) \to M \setminus \{p\}$$

is biholomorphic and there exists a neighborhood $\tilde{U}_p$ of $E_p := \beta^{-1}(p)$ in $\tilde{M}_p$ biholomorphic to a neighborhood $\tilde{V}$ of $\mathbb{P}^1$ inside the total space of the tautological line bundle $\tau_1 \to \mathbb{P}^1$ such that the diagram below is commutative.

$$\begin{array}{ccc}
(\tilde{U}_p, E_p) & \longrightarrow & (\tilde{V}, \mathbb{P}^1) \\
\Big\downarrow{\scriptstyle \beta_p} & & \Big\downarrow{\scriptstyle \beta_1} \\
(U_p, p) & \longrightarrow & (V, 0)
\end{array} ,$$

where $U_p := \beta(U_p)$ is a neighborhood of p in M and $V = \beta_1(\tilde{V})$ is a neighborhood of the origin in $\mathbb{C}^2$. This implies that we can find local coordinates z_1, z_2 on $U_p \subset M$, and an open cover $\tilde{U}_p$

$$\tilde{U}_p = \tilde{U}_p^1 \cup \tilde{U}_p^2$$

with the following properties.

- $z_i(p) = 0$, $i = 1, 2$.
- There exists coordinates u_1, u_2 on $\tilde{U}_p^i$, $i = 1, 2$ such that

$$\tilde{U}_p^i \cap E_p = \{u_i = 0\}$$

- Along $\tilde{U}_p^1$ the blowdown map β has the description

$$(u_1, u_2) \mapsto (z_1, z_2) = (u_1, u_1 u_2).$$

- Along $\tilde{U}_p^2$ the blowdown map β has the description

$$(u_1, u_2) \mapsto (z_1, z_2) = (u_1 u_2, u_2).$$

We will sometime denote the blowup of M at p by

$$(M, p) \dashrightarrow \tilde{M}.$$

The proper transform of C is the closure in $\tilde{M}$ of $\beta^{-1}(C \setminus \{0\})$.

Example 10.35. Consider again the germ $f(x,y) = y^3 - x^5 - x^7$ we analyzed in Example 10.22. We blowup $\mathbb{C}^2$ at the origin and we want to describe the proper transform of the curve C defined by $f = 0$. Conside the blowup $(\mathbb{C}^2, O) \dashrightarrow M$, denote by $\hat{C}$ the proper transform of C, and by U a neighborhood of the exceptional divisor E. We have an open cover of U by coordinate chartes

$$U = U_1 \cup U_2$$

We denote by (u,v) the coordinates on U_1 so that $x = u$, $y = uv$, $E \cap U_1 = \{u = 0\}$. Then we have

$$\beta^* f(u, uv) = (uv)^3 - u^3 - u^5 = u^3(v^3 - 1 - u^2).$$

This shows that the part of $\hat{C}$ in U_1 intersects the exceptional divisor E in three points $P_k(u_k, v_k)$, $k = 0, 1, 2$ given by

$$u_k = 0, \quad v_k = \exp(\frac{2\pi \mathbf{i} k}{3}), \quad k = 0, 1, 2.$$

Moreover

$$\mu(\hat{C} \cap E, P_k) = 1,$$

and each of the points P_k are *smooth* points of the curve $\hat{C}$ (see Figure 10.8).

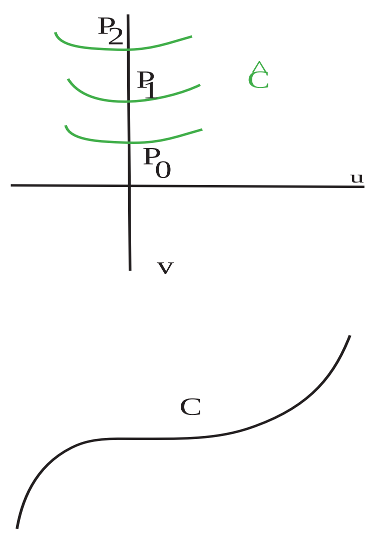

Figure 10.8: *Blowing up* $y^3 = x^5 + x^7$

On the chart U_2 we have

$$\beta^* f(uv, v) = v^3 - (uv)^5 - (uv)^7 = v^3(1 - u^5v^2 - u^5v^4)$$

Clearly $\hat{C} \cap E \cap U_2 = \emptyset$. This shows that

$$\hat{C} \cdot E = 3.$$

To explain some of the phenomena revealed in the above example we need to introduce a new notion.

Definition 10.36. Suppose C is a plane curve defined near $O = (0,0) \in \mathbb{C}^2$ by an equation $f(x,y) = 0$, where $f \in \mathbb{C}\{x,y\}$, $f(0,0) = 0$. The *multiplicity of O on the curve C* is the integer $e_C(O)$ defined by

$$e_C(O) = \min_{k \geq 1} \Big\{ f_k(x,y) \neq 0 \Big\}$$

where $f_d(x,y)$ denotes the degree d homogeneous part of f. The principal par $f_e(x,y)$, $e = e_C(O)$ decomposes into linear factors

$$f_e = \prod_{j=1}^{e} (a_j x + b_j y).$$

and the lines L_j described by $a_j x + b_j y = 0$ are called the principal tangents of C at O.

Exercise 10.37. Prove that

$$e_C(O) = \min_D \mu(C \cap D, O),$$

where the minimum is taken over all the plane curves D such that O is an isolated point of the intersection $C \cap D$.

For example, the multiplicity of O on the curve $C:\ \{y^3 = x^5 + x^7\}$ is 3. The multiplicity can be determined from Puiseux expansions.

Proposition 10.38. *Suppose C is a plane curve such that the germ (C, O) is irreducible. If C admits near O the puiseux expansion*

$$x = t^n, \ \ y = at^m + \cdots, \ \ a \neq 0,$$

then

$$e_C(O) = \min(m, n).$$

Exercise 10.39. Prove Proposition 10.38.

The computation in Example 10.35 shows that

$$e_C(O) = \hat{C} \cdot E.$$

This is a special case of the following more general result

Proposition 10.40. *Suppose C is a plane curve. Denote by $\hat{C}$ the proper transform of C in the blowup at $\mathbb{C}^2$ at O, and by E the exceptional divisor. Then*

$$e_C(O) = \hat{C} \cdot E = \sum_{p \in \hat{C} \cap E} \mu(\hat{C} \cap E, p).$$

Exercise 10.41. Prove Proposition 10.40.

The computations in Example 10.35 show something more, namely that the proper transform of a plane curve is better behaved than the curve itself. The next results is a manifestation of this principle.

Proposition 10.42. *Suppose $f \in \mathbb{C}\{x, y\}$ is a holomorphic function such that O is a point of multiplicity $m > 0$ on the curve $C = \{f = 0\}$. Then the proper trasform of C intersects the exceptional divisor at precisely those points in $\mathbb{P}^1$ corresponding to the principal tangents. In particular, the blowup separates distinct principal tangents.*

To formulate our next batch of results we need to introduce some terminology.

Definition 10.43. (a) If $\tilde{M}_p$ is the blow-up of the smooth complex surface M at the point p, then the exceptional divisor $E \hookrightarrow \tilde{M}_p$ is called the *first infinitesimal neighborhood* of p.
(b) An *iterated blowup* of M is a a sequence of complex manifolds

$$M_0, M_1, \cdots, M_k$$

with the following properties

- $M = M_0$.
- M_i is the blowup of M_{i-1} at a point p_{i-1}, $i = 1, \cdots, k-1$. We denote by E_i the exceptional divisor in M_i.
- $p_i \in E_i$, $\forall i = 1, \cdots, k-1$.

The exceptional divisor E_j is called the *j-th infinitesimal neighborhood* of p_0. We will denote the iterated blowups by

$$(M_0, p_0) \dashrightarrow (M_1, p_1) \dashrightarrow \cdots \dashrightarrow (M_{k-1}, p_{k-1}) \dashrightarrow M_k.$$

Given a plane curve through $O \in \mathbb{C}^2$, and an iterated blowup

$$(\mathbb{C}^2, O) \dashrightarrow (M_1, p_1) \dashrightarrow \cdots \dashrightarrow M_k$$

we get a sequence of proper transforms $C_{(1)} = \hat{C}$, $C_{(j)} = \hat{C}_{(j-1)}$, $j = 2, \cdots, k$. The points $C_{(j)} \cap E_j$ are called *j-th order infinitesimal points* of the germ (C, O). To minimize the notation, we will denote by E_j all the proper transforms of E_j in $M_{j+1}, M_{j+2}, \cdots, M_k$.

Suppose $f \in \mathbb{C}\{x, y\}$ is irreducible and O is a point on $f = 0$ of multiplicity N. Then, after a linear change of coordinates we can assume that f is a Weierstrass polynomial in y such that $\deg_y f = N$.

Proposition 10.44. *Suppose $f \in \mathbb{C}\{x, y\}$ is an irreducibe Weierstrass y-polynomial, and $\deg_y f = N = e_C(O)$, $C = \{f = 0\}$. Assume that near O the cuve C is tangent at O to the x-axis, so that it has the Puiseux expansion*

$$y = y(x) = \sum_{j \geq N} a_j x^{j/N}.$$

Then the proper transform of $\hat{C}$ intersects the exceptional divisor at a single point p and the germ $(\hat{C},p)$ is irreducible. Moreover, with respect to the coordinates (u,v) near p defined by $u=x$, $v=y/x$ we have $p=(0,a_N)$ and the germ $(\hat{C},p)$ has the Puiseux expansion

$$v-a_N=\sum_{j>N} a_j u^{(j-N)/N}. \tag{10.3}$$

Proof The fact that $\hat{C}$ intersects the exceptional divisor at a single point is immediate. The expansion (10.3) follows immediately from the equality $v=y/x$. The irreducibility follows from the Puiseux expansion (10.3). ■

Suppose now that $C\subset\mathbb{C}^2$ is a plane curve such that the germ (C,O) is irreducible. We can choose linear coordinates on $\mathbb{C}^2$ such that near O the curve C has a Puiseux expansion

$$x=t^p;\ y=at^q+\cdots,\ \ p=e_C(O)<q.$$

After 1-blowup the proper transform $C_{(1)}$ will intersect the exceptional divisor at a point p_1 and the germ $(C_{(1)},p_1)$ has a Puiseux expansion

$$x=t^p,\ \ y=at^{q-p}+\cdots.$$

In particular, we deduce

$$e_{C_{(1)}}(p_1)=\min\{(q-p),p\}.$$

If $q-p<p$ we conclude that the infinitesimal point p_1 has strictly smaller multiplicity than O. In general, we have

$$q=pm+r,\ \ 0\le r<p$$

Blowing up m times we deduce that $C_{(m)}$ intersects the m-th infinitesimal neighborhood of O at a point p_m and

$$e_{C_{(m)}}(p_m)=r<p.$$

We conclude that by performing an iterated blowup we can reduce the multiplicity. In particular, we can perform iterated blowups until some infinitesimal point of C has multiplicity one. We can thus conclude that there exists an iterated blowup with respect to which the proper transform of C is smooth.

We want to show there is a more organized way of doing this provided we require a few additional conditions. The next example will illustrate some things we would like to avoid.

Example 10.45. Consider the curve $y^4=x^{11}$. The singular point O has multiplicity 4. By making the changes in coordinates

$$x\to x,\ \ y\to xy$$

we deduce that the proper transform of C after the first blowup has the local description near the first order infinitesimal point $p_1=(0,0)$ given by

$$x^4(y^4-x^7)=0.$$

The exceptional divisor E_1 has the equation $x = 0$ so the multiplicity of p_1 is 4.

We blowup again, and using the same change in coordinates as above we deduce that the new exceptional divisor is described by $x = 0$, and the second proper transform of C takes the form

$$x^4(y^4 - x^3) = 0.$$

The second order infinitesimal point p_2 on C has coordinates $(0,0)$ so that it has multiplicity 3. To understand the proper transform of E_1 we need to use the other change in coordinates

$$x \to xy, \quad y \to y$$

in which the exceptional divisor is described by $y = 0$. The proper transform of E_1 intersects E_2 at ∞. We can also see this in Figure 10.9. The curves $C_{(1)}$ and E_1 have no principal tangents in common so a blowup will separate them.

We perform the change in coordinates $x \to xy$, $y \to y$ near p_2, i.e. we blow up for the third time at p_2. The exceptional divisor E_3 is described by $y = 0$, and the 3-rd proper transform of the curve C has the description

$$y^3(y - x^3) = 0$$

near the third infinitesimal point $p_3 = (0,0)$. The proper transform of E_2 is described by $y = 0$ so that p_3 is a nonsingular point of $C_{(3)}$.

Figure 10.9 describes various transformations as we perform the blowups. As we have mentioned, $C_{(3)}$ is already smooth but the situation is not optimal. More precisely, three different curves intersect on the third infinitesimal point p_3. It will be very convenient to avoid this situation. We can separate E_3 and E_2 by one blowup (see Figure 10.10).

We perform the change in coordinates

$$x \to x, \quad y \to xy.$$

The exceptional divisor E_4 is given by $x = 0$, the proper transform of E_3 is described by $y = 0$, and $C_{(4)}$is given by

$$y = x^2.$$

Still, the situation is not perfect because $C_{(4)}$, E_3 and E_4 have a point in common, p_4 We blowup at p_4 using the change in coordinates

$$x \to x, \quad y \to xy.$$

E_3 and E_4 separate but now the proper transform $C_{(5)}$ goes through the intersection point of E_5 and the (second order) proper transform of E_3. Moreover, near the fifth order infinitesimal point p_5 the curve $C_{(5)}$ has the linear form. A final blowup will separate $C_{(5)}$, E_3 and E_5 (see Figure 10.10).

Motivated by the above example we introduce the following concept.

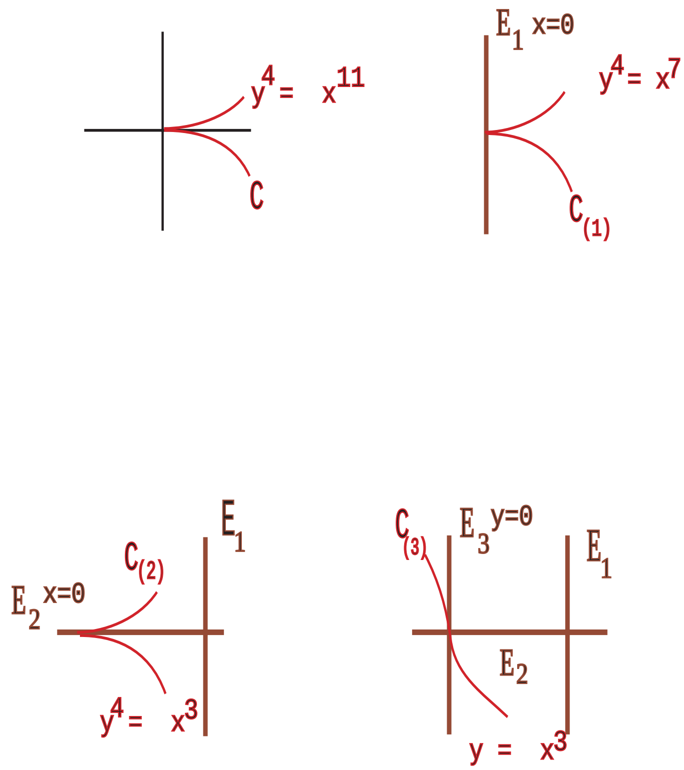

Figure 10.9: *Resolving $y^4 = x^{11}$ by an iterated blowup.*

Definition 10.46. Let $(C, O) \subset (\mathbb{C}^2, O)$ be an irreducible germ of plane curve. An iterated blowup

$$(\mathbb{C}^2, O) \dashrightarrow (M_1, p_1) \dashrightarrow \cdots \dashrightarrow (M_{n-1}, p_{n-1}) \dashrightarrow M_n$$

is called a *standard resolution* of (C, O) if either (C, O) is smooth and $n = 0$ or for $k = 1, \cdots, n-1$ either

(a) $C_{(k)} \subset M_k$ has one singular point p_k or
(b) $C_{(k)}$ is smooth but the intersection with E_k at p_k is not transverse or
(c) $C_{(k)}$ is smooth, intersects E_k transversally at p_k, but does intersect (also at p_k) some other E_j, $j < k$, and
(d) C_n is smooth and intersects E_n transversally, and intersects no other E_k.

We denote by m_k the multiplicity of $C_{(k)}$ at p_k,

$$m_k = e_{C_{(k)}}(p_k).$$

We also set $m_0 := e_C(O)$. The sequence $(m_0, m_1, \cdots, m_{n-1})$ is called the *multiplicity sequence* of the resolution.

Arguing as in Example 10.45 one can prove that each irreducible germ of planar curve admits a standard resolution. In this example we have constructed a standard resolution of

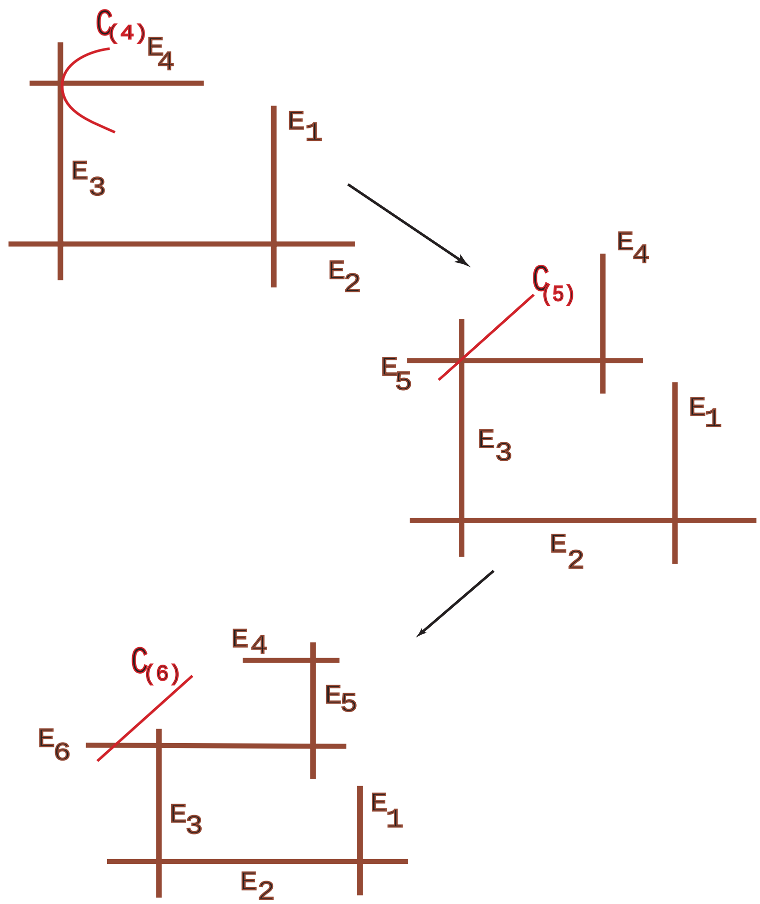

Figure 10.10: *Improving the resolution of* $y^4 = x^{11}$.

the germ $y^4 = x^{11}$. The multiplicity sequence is $(4, 4, 3, 1, 1, 1)$. This example shows that the multiplicity sequence can be determined from the Puiseux series. In fact, the multiplicity sequence completely determines the topological type of a singularity. More precisely, we have the following result.

Theorem 10.47. (Enriques-Chisini) *The Puiseux pairs are algorithmically determined by the multiplicity sequence, and conversely, the multiplicity sequence can be determined from the Puiseux series.*

For a tedious but fairly straightforward proof of this result we refer to [5, Sec. 8.4, Them. 12] or [19, Thm. 5.3.12]. We include below the algorithm which determines the multiplicity sequence from the Puiseux pairs. Suppose the Puiseux pairs are

$$(m_1, n_1), \cdots, (mg, n_g).$$

Form the characteristic exponents

$$k_0 = m_1 \cdots m_g = N, \;\; \frac{k_j}{k_0} = \frac{n_j}{m_j} \cdot \frac{1}{m_1 \cdots m_{j-1}}.$$

Perform the sequence of Euclidean algorithms, $i = 1, \cdots, g$ for χ_1^i and q_1^i,

$$\begin{array}{rcl} \chi_1^i & = & \mu_1^i \cdot q_1^i + q_2^i \\ q_1^i & = & \mu_2^i \cdot q_2^i + q_3^i \\ \vdots & \vdots & \vdots \\ q_{\ell(i)-1}^i & = & \mu_{\ell(i)}^i \cdot q_{\ell(i)}^i \end{array}$$

where $\chi_1^1 = k_1$, $q_1^1 = k_0 = N$,

$$\chi_1^i = k_i - k_{i-1}, \quad q_1^i = q_{\ell(i-1)}^{i-1}, \quad i = 2, \cdots, g.$$

Then in the multiplicity sequence the multiplicity q_j^i appears μ_j^i times, $i = 1, \cdots, q$, $j = 1, \cdots \ell(i)$.

Example 10.48. (a) Consider the germ given by the Puiseux expansion

$$y = x^{11/4}$$

In this case there is only one Puiseux pair, $(4, 11)$. The characteristic exponents are

$$k_0 = 4, \quad k_1 = 11.$$

We have

$$11 = 2 \cdot 4 + 3, \quad 4 = 1 \cdot 3 + 1, \quad 3 = 3 \cdot 1.$$

We conclude that the multiplicity sequence is

$$(4, 4, 3, 1, 1, 1)$$

as seen before from the standard resolution.

(b) Consider the germ with Puiseux expansion

$$y = x^{3/2} + x^{7/4}.$$

Its Puiseux pairs are $(2, 3)$, $(2, 7)$. Using the equality (10.2) we deduce that the characteristic exponents are

$$k_0 = 4, \quad k_1 = 6, \quad k_2 = 21.$$

Then $\chi_1^1 = k_1 = 6$, $q_1^1 = 4$,

$$6 = 1 \cdot 4 + 2, \quad 4 = 2 \cdot 2.$$

Hence $\ell(1) = 2$, $\chi_1^2 = 15$, $q_1^2 = q_2^1 = 2$

$$15 = 7 \cdot 2 + 1, \quad 2 = 2 \cdot 1.$$

We deduce $\ell(2) = 3$. The multiplicity sequence is

$$(4, 2, 2; \; 2, 2, 2, 2, 2, 2, 2, 1, 1).$$

The standard resolution of an irreducible germ can be geometrically encoded by the *resolution graph.*. Suppose

$$(C,O) \dashrightarrow (M_1,p_1) \dashrightarrow \cdots \dashrightarrow (M_{n-1},p_{n-1}) \dashrightarrow M_n$$

is the standard resolution. Then the resolution graph has $n+1$ vertices, $1,\cdots,n,*$. Two vertices $i<j$ are connected if the divisors E_i and E_j intersect. Finally, we connect n and $*$ since $\hat{C}$ intersects E_n.

From Figure 10.10 we deduce that the resolution graph of the singularity $y^4=x^{11}$ is the one depicted in Figure 10.11. One can prove (see [5, 19]) that the resolution graph can

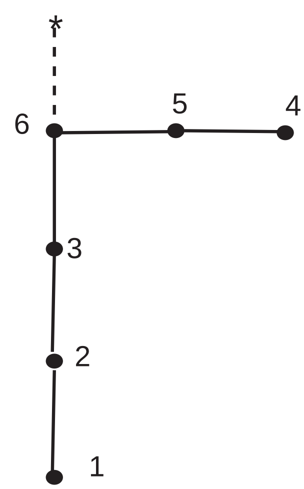

Figure 10.11: *The resolution graph of $y^4=x^{11}$.*

be algorithmically constructed from the multiplicity sequence.

§10.6 Intersection multiplicities, the δ-invariant and the Milnor number

We have so far described several equivalent collections of complete *topological* invariants of an irreducible germ of plane curve. Namely they are

- The Puiseux pairs, $(m_1,n_1),\cdots,(m_g,n_g)$.
- The characteristic exponents $k_0=m_1\cdots m_g$, $k_1,\cdots,k_g$.
- The multiplicity sequence.
- The resolution graph.

In this section we explain how to use these invariants to compute a few other frequently used quantities.

TO BE CONTINUED

Chapter 11

The Milnor number of an isolated singularity

Tougeron theorem states that when interested in the local geometric properties of a holomorphic function f near one of its critical points p_0, we can neglect most of its Taylor expansion at that point, except the terms of order $\leq \mu + 1$, where μ is the Milnor number of that critical point. The Milnor number $\mu(f, p_0)$ has a very elegant but quite opaque definition

$$\mu(f, p_0) := \dim_{\mathbb{C}} \mathcal{O}_{n,p_0}/\mathfrak{J}(f, p_0).$$

The goal of this Chapter is to unveil some of the rich geometric content of this number.

§11.1 The index of a critical point and morsifications

Definition 11.1. Consider a holomorphic map $F : 0 \in U \subset \mathbb{C}^n \to \mathbb{C}^n$. If $z_0 \in U$ is an isolated solution of the equation

$$F(z) = 0$$

we define the index of F at z_0 as the integer

$$\mathbf{ind}\,(F, z_0) := \lim_{\varepsilon \searrow 0} \deg \nu_{F,\varepsilon}$$

where

$$\nu_{F,\varepsilon} : \{|z - z_0| = \varepsilon\} \to \{|w| = 1\} \subset \mathbb{C}^n$$

is the smooth map defined by

$$\nu_{F,\varepsilon}(z) = \frac{1}{|F(z)|} F(z).$$

When F is the gradient df of a holomorphic function f then $\mathbf{ind}\,(df, z_0)$ is called the **index** of the critical point 0 and will be denoted by $\mathbf{ind}\,(f, z_0)$

Example 11.2. Suppose 0 is a regular value of the holomorphic map $F : U \subset \mathbb{C}^n \to \mathbb{C}^n$. If $F(0) = 0$ then

$$\mathbf{ind}\,(F, 0) = 1.$$

Indeed, we can write

$$F(z) = Az + R(z)$$

where $A \in GL_n(\mathbb{C})$, and

$$R(z) = O(|z|)^2 \text{ as } |z| \to 0.$$

Since the group $GL_n(\mathbb{C})$ is connected we can find a continuous deformation $(A_t)_{t\in[0,1]}$ of A inside $GL_n(\mathbb{C})$ such that $A_0 = A$, $A_1 := \mathbf{1}$. Note that

$$|A_t z + (1-t)R(z)| \geq \frac{1}{\|A_t^{-1}\|} \cdot |z| - C|z|^2$$

so that there exists $\varepsilon_0 > 0$ so that

$$|A_t z + (1-t)R(z)| \neq 0, \ \ \forall 0 < |z| < \varepsilon_0, \ \ t \in [0, 1].$$

This shows that for $\varepsilon < \varepsilon_0$ we have a homotopy

$$\nu_{F,\varepsilon,t}(z) := \frac{1}{|A_t z + (1-t)R(z)|}(A_t z + (1-t)R(z))$$

between

$$\nu_{F,\varepsilon} : \{|z| = \varepsilon\} \to \{|w| = 1\}$$

and the rescaling map

$$\{|z| = \varepsilon\} \to \{|w| = 1\}, \ \ w = \frac{1}{\varepsilon}z$$

which has degree 1.

The computation in the above example is a special case of a more general result concerning the topological degree of a continuous map.

Proposition 11.3. *Suppose $F : \{|z| < r\} \to \mathbb{C}^n$ is a holomorphic map such that 0 is a regular value. Then for every $\varepsilon > 0$ such that F has no zero on the sphere $S_\varepsilon := \{|z| = \varepsilon\}$ we have*

$$\deg \nu_{F,\varepsilon} = \#\{z;\ F(z) = 0,\ |z| < \varepsilon\}.$$

Proof This result is an immediate consequence of the cobordism invariance of the degree. Here are the details.

Denote by $z_1, \cdots, z_m$ the zeros of F in $\{|z| < \varepsilon\}$. Isolate them using tiny disjoint, open balls $B_i = B(z_i, r_i)$. Set $B := \bigcup_i B_i$ and

$$M := \{|z| \leq \varepsilon\} \setminus B.$$

Since $\partial M = S_\varepsilon \cup \partial B$ can regard M as an oriented cobordism between S_ε and $\partial B = \bigcup_i \partial B_i$. The map F has no zero on M and we thus get

$$\tilde{\nu}_F : M \to S_1 := \{|w| = 1\}, \;\; z \mapsto \frac{1}{|F(z)|} F(z).$$

Thus

$$(\deg \nu_{F,\varepsilon}) \cdot [S_1] = \nu_{F,\varepsilon}([S_\varepsilon]) = \tilde{\nu}_F([\partial B]) = \sum_i \tilde{\nu}_F([\partial B_i])$$

$$= \sum_i \mathbf{ind}\,(F, z_i) \cdot [S_1] = m[S_1]$$

where at the last step we have used the computation in Example 11.2. ■

Corollary 11.4. *Suppose $0 \in \mathbb{C}^n$ is an isolated zero of the holomorphic map*

$$F : \{|z| < r\} \subset \mathbb{C}^n \to \mathbb{C}^n.$$

Then there for any for any sufficiently small ε and any sufficiently small holomorphic map

$$\Phi_\varepsilon : \{|z| < r\} \to \mathbb{C}^n$$

such that 0 is a regular value of $F_\varepsilon := F + \Phi_\varepsilon$ we have

$$\mathbf{ind}\,(F, 0) = \#\{|z| < \varepsilon; \;\; F_\varepsilon(z) = 0\}.$$

In particular, for any $0 < \varepsilon \ll 1$ there exists $r(\varepsilon) > 0$ such that for any regular value c of F satisfying $|c| < r(\varepsilon)$ we have

$$\mathbf{ind}\,(F, 0) := \#\{|z| < \varepsilon; \;\; F(z) = c\}.$$

Proof Observe that if Φ_ε is small $\nu_{F+t\Phi_\varepsilon,\varepsilon}$ is a homotopy from $\nu_{F,\varepsilon}$ to $\nu_{F_\varepsilon,\varepsilon}$. The index of the latter counts the number of small solutions of $F_\varepsilon(z) = 0$. ■

We have the following highly nontrivial result whose proof will be discussed in the next section.

Theorem 11.5. *Suppose $F : U \subset \mathbb{C}^n \to \mathbb{C}^n$ is a holomorphic map and $0 \in \mathbb{C}^n$ is an isolated solution of the equation $F(z) = 0$. Then*

$$\mu(F, 0) = \mathbf{ind}\,(F, 0).$$

This theorem provides a simple way of computing the Milnor number of an isolated critical point using the technique of *morsification*.

Definition 11.6. Suppose the holomorphic function $f \in \mathbb{C}\{z_1, \cdots, z_n\}$ has an unique critical point in a small ball $\{|z| < r\}$. A **local morsification** of f is a small perturbation

$$f + \varphi : \{|z| < \varepsilon \ll r\} \to \mathbb{C}$$

where $\varphi : \{|z| < |\varepsilon|\} \to \mathbb{C}$ holomorphic which has only nondegenerate critical points.

If f is a holomorphic function as in the above definition and $f+\varphi$ is a local morsification then using Theorem 11.5 and Corollary 11.4 we deduce

$$\mu(f, 0) = \mathbf{ind}\,(df, 0) = \#\{|z| < \varepsilon;\ \ d(f + \varphi)(z) = 0\}.$$

Thus when f is slightly (and generically) perturbed by φ the (possibly degenerate) critical point 0 of f "disintegrates" into the μ nondegenerate critical points of $f + \varphi$.

Example 11.7. (Brieskorn-Pham singularities) Consider the function

$$f(z_1, \cdots, z_n) = \sum_{k=1}^{n} z_k^{\alpha_k} \in \mathcal{O}_n,\ \ 2 \le \alpha_k \in \mathbb{Z}.$$

0 is an isolated critical point of f. To morsify f we can perturb it by a generic linear function

$$f \to f_\varepsilon := \sum_{k=1}^{n} (z_k^{\alpha^k} - \varepsilon_k z_k)$$

which has only nondegenerate critical points described by the system

$$n_k z_k^{\alpha_k - 1} = \varepsilon_k,\ \ 1 \le k \le n.$$

This system has

$$\mu := \prod_{k=1}^{n} (\alpha_k - 1)$$

distinct solutions. The above μ is precisely the Milnor number of the Brieskorn-Pham singularity, i.e. the complex dimension of the local algebra

$$Q_{df} = \mathcal{O}_n/(\partial_{z_1} f, \cdots, \partial_{z_n} f) = \mathbb{C}\{z_1, \cdots, z_n\}/(z_1^{\alpha_1 - 1} = 0, \cdots, z_n^{\alpha_n - 1} = 0)$$

§11.2 Proof that the Milnor number equals the index

As promised, we present a proof of Theorem 11.5. We will follow the elementary geometric approach in [2, Chap. 5]. For an analytical proof we refer to [14, 5.1,5.2]. For a mostly algebraic proof we refer to [34] or [37, IV.1].

Here is a rough outline of the proof. We will define an equivalence relation on the space of germs at 0 of maps $\mathbb{C}^n \to \mathbb{C}^n$ such that both the Milnor number and the index do not vary within an equivalence class. We will then find a complete set of representatives of this equivalence relation for which both the Milnor number and the index can be shown to be equal by elementary computations.

Denote by $\mathfrak{F}_n$ the space of germs at 0 of holomorphic maps $F : U \subset \mathbb{C}^n \to \mathbb{C}^n$, such that $F(0) = 0$ and $\mu(F, 0) < \infty$.

Definition 11.8. Two germs $F, G \in \mathfrak{F}_n$ are called **algebraically equivalent** (or A-equivalent for brevity) and we write this

$$F \sim_A G$$

if there exists a holomorphic map

$$A = A(z) : 0 \in U \subset \mathbb{C}^n \to GL_n(\mathbb{C})$$

such that $F(z) = A(z)G(z)$.

Definition 11.9. For every $\vec{m} = (m_1, \cdots, m_n) \in \mathbb{Z}_+^n$ we define $\Phi_{\vec{m}} \in \mathfrak{F}_n$ by the equations

$$w_1 = z_1^{m_1}, \cdots, w_n = z_n^{m_n}.$$

$\Phi_{\vec{m}}$ is called a **Pham map**.

Theorem 11.5 is a consequence of the following facts.

Proposition 11.10. *For every $F \in \mathfrak{F}_n$ there exists a Pham map $\Phi_{\vec{m}}$ such that*

$$F \sim_A \Phi_{\vec{m}} + tF, \quad \forall t \neq 0.$$

Proposition 11.11. *For every multi-index $\vec{m}$ we have*

$$\mu(\Phi_{\vec{m}}, 0) = \mathbf{ind}\,(\Phi_{\vec{m}}, 0).$$

Proposition 11.12.

$$F \sim_A G \Longrightarrow \mathbf{ind}\,(F, 0) = \mathbf{ind}\,(G, 0).$$

Proposition 11.13.

$$F \sim_A G \Longrightarrow \mu(F,0) = \mu(G,0).$$

Proposition 11.14. *Suppose $F_\lambda : \{|z| < r\} \subset \mathbb{C}^n \to \mathbb{C}^n$ is a holomorphic map depending holomorphically on $\lambda \in \mathbb{C}^k$, $|\lambda| < \varepsilon$. Suppose the equation $F_0(z) = 0$ has only the solution $z = 0$. Then*

$$\mathbf{ind}\,(F_0,0) = \sum_{F_\lambda(\zeta)=0} \mathbf{ind}\,(F_\lambda,\zeta) \tag{11.1}$$

$$\mu(F_0,0) \geq \sum_{F_\lambda(\zeta)=0} \mu(F_\lambda,\zeta) \tag{11.2}$$

Proposition 11.15.

$$\mathbf{ind}\,(F,0) \leq \mu(F,0), \quad \forall F \in \mathfrak{F}_n.$$

Let us first explain why the above facts imply Theorem 11.5. Let $F \in \mathfrak{F}_n$. Choose a Pham map Φ such that

$$F \sim_A \Phi_\lambda := \Phi + \lambda F, \quad \forall \lambda \neq 0.$$

Fix a small neighborhood U of 0 and denote by $\zeta_i(\lambda)$ the zeros of Φ_λ in U. We obtain the following relations

$$\left\{\begin{array}{cc} \mu(\Phi,0) \geq \sum_i \mu(\Phi_\lambda,\zeta_i(\lambda)\,) & \text{by (11.2)} \\ \mu(\Phi_\lambda,\zeta_i(\lambda)) \geq \mathbf{ind}\,(\Phi_\lambda,\zeta_i(\lambda)), \ \ \forall i & \text{by Proposition 11.15} \\ \sum_i \mathbf{ind}\,(\Phi_\lambda,\zeta_i(\lambda)) = \mathbf{ind}\,(\Phi,0) & \text{by (11.1)} \\ \mathbf{ind}\,(\Phi,0) = \mu(\Phi,0) & \text{by Proposition 11.11} \end{array}\right. .$$

It is now clear that all of the above inequalities are equalities and since 0 is one of the roots of Φ_λ we deduce

$$\mu(\Phi_\lambda,0) = \mathbf{ind}\,(\Phi_\lambda,0).$$

Now, since $F \sim_A \Phi_\lambda$ we deduce

$$\mu(F,0) = \mu(\Phi_\lambda,0) = \mathbf{ind}\,(\Phi_\lambda,0) = \mathbf{ind}\,(F,0).$$

Let us now prove Proposition 11.10 -11.15.

Proof of Proposition 11.12 Observe first that if $F \sim_A G$ then, arguing exactly as in Example 11.2, we conclude that there exist a small sphere $S_\varepsilon\{|z| = \varepsilon\}$ and a continuous deformation $H_t \in \mathfrak{F}_n$, $t \in [0,1]$, such that $H_0 = F$, $H_1 = G$ and

$$H_t(S_\varepsilon) \cap \{0\} = \emptyset.$$

More precisely. H_t is a two-step deformation. If $G(z) = A(z)F(z)$ we first deform

$$F(z) \to F_t(z) = B_t F(z) \to U(z) = A(0)F(z)$$

where $t \to B_t \in GL_n(\mathbb{C})$ is a smooth path from $\mathbf{1}$ to $A(0)$. Next we deform

$$U(z) \to U_t(z) = A(tz)F(z) \to G(z).$$

The homotopy H_t produces a homotopy $\nu_{H_t,\varepsilon}$ connecting $\nu_{F,\varepsilon}$ to $\nu_{G,\varepsilon}$ thus proving Proposition 11.12. ∎

The proof of the additivity relation (11.1) is identical to the proof of Proposition 11.3.

Exercise 11.16. Fill in the missing details in the proof of (11.1).

Proposition 11.13 follows from the observation

$$F \sim_A G \Longrightarrow I_F = I_G \Longrightarrow Q_F = Q_G.$$

Proof of Proposition 11.10 Let $F \in \mathfrak{F}_n$. We will use the following auxiliary result.

Lemma 11.17. *Let $\mu := \mu(F,0)$. If $G \in \mathfrak{F}_n$ is a perturbation of F such that $j_\mu(F) = j_\mu(G)$ then $F \sim_A G$.*

Proof of the lemma We know that $\mathfrak{M}_n^\mu \subset I_F$ so there exist $a_{ij} \in \mathfrak{M}_n$ such that

$$\mathfrak{M}_n^{\mu+1} \ni G_i - F_i = \sum_j a_{ij} F_j.$$

Thus

$$G = \Big(\mathbf{1} + A\Big)F$$

where $A(z) = (a_{ij}(z))_{1 \leq i,j \leq n}$. Clearly $B(z) = \mathbf{1} + A(z) \in GL_n(\mathbb{C})$ for sufficiently small z. This proves $F \sim_A G$. ∎

Consider $\vec{m} = (\mu+1, \cdots, \mu+1) \in \mathbb{Z}^m$ and set $\Phi = \Phi_{\vec{m}}$, $\Phi_\lambda = \Phi + \lambda F$. Then

$$\lambda F \sim_A \lambda F, \;\; \forall \lambda \neq 0$$

while $j_\mu(\lambda F) = j_\mu(\Phi_\lambda)$ so that by the above lemma

$$\lambda F \sim_A \Phi_\lambda.$$

This completes the Proof of Proposition 11.10. ∎

Let us now observe that Proposition 11.15 follows from (11.2). To see this consider the perturbation $F_\lambda = F - \lambda$. If λ is a regular value of F then for every $\zeta \in F^{-1}(\lambda)$ we have

$$\mu(F_\lambda, \zeta) = \mathbf{ind}\,(F_\lambda, \zeta) = 1$$

because, by the inverse function theorem, up to a change in coordinates the germ of F at zero is the germ of the identity map. Hence

$$\mathbf{ind}\,(F, 0) = \sum_{\zeta \in F^{-1}(\lambda)} \mathbf{ind}\,(F_\lambda, \zeta) = \sum_{\zeta \in F^{-1}(\lambda)} \mu(F_\lambda, \zeta) \leq \mu(F, 0).$$

The subadditivity property (11.2) is much more subtle and requires a more in depth study. We need to introduce additional terminology.

For any open set $U \subset \mathbb{C}^n$ we denote by $\mathcal{O}(U)$ the space of holomorphic functions $f : U \to \mathbb{C}$ and by $\mathcal{P}(U) \subset \mathcal{O}(U)$ the subspace consisting of polynomials. If

$$F = (F_1, \cdots, F_n) : U \to \mathbb{C}^n$$

is a holomorphic map then we denote by $I_F(U)$ the ideal of $\mathcal{O}(U)$ generated by $(F_1, \cdots, F_n)$. We set

$$Q_F(U) := \mathcal{O}(U)/I_F(U), \;\; Q_F^0(U) := \mathcal{P}(U)/I_F(U) \subset Q_F(U).$$

$Q_F(U)$ is called the *algebra of F on U* while $Q_F^0(U)$ is called the *polynomial subalgebra*. The subadditivity property is an immediate consequence of the following two auxiliary results of independent interest.

Lemma 11.18. *For every holomorphic map $G : U \subset \mathbb{C}^n \to \mathbb{C}^n$ such that $\dim_{\mathbb{C}} Q_G^0(U) < \infty$ we have*

$$\sum_{\zeta \in G^{-1}(0)} \mu(G, \zeta) \leq \dim_{\mathbb{C}} Q_G^0(U).$$

Lemma 11.19. *For every deformation F_λ of F there exists a neighborhood U of $0 \in \mathbb{C}^n$ such that for all sufficiently small $\lambda \in \mathbb{C}^k$ we have*

$$\dim_{\mathbb{C}} Q_{F_\lambda}^0(U) \leq \mu(F, 0).$$

Proof of Lemma 11.18 The proof will be carried out in several steps.

Step 1. $k := \dim_{\mathbb{C}} Q_G^0(U) < \infty \Longrightarrow \mu(G, \zeta) < \infty, \;\; \forall \zeta \in G^{-1}(0)$. We will show that

$$\mathfrak{M}_{n,\zeta}^k \subset I_{G,\zeta}.$$

Assume for simplicity $\zeta = 0$. Suppose $M = z_{j_1} \cdots z_{j_k}$ is a degree k monomial. Observe that the family of monomials

$$1, z_{j_1}, z_{j_1} z_{j_2}, \cdots, z_{j_1} \cdots z_{j_k}$$

is linearly dependent in $Q^0_G(U)$ so that there exist $a_0, \cdots, a_k \in \mathcal{O}(U)$ such that

$$a_0 + a_1 z_{j_1} + \cdots + a_k z_{j_1} \cdots z_{j_k} \in I_G(U).$$

Denote by m the first index i such that $a_i(0) \neq 0$. We deduce

$$z_{j_1} \cdots z_{j_i}(a_i + X) \in I_G(U), \quad X \in \mathcal{O}(U), \ \ X(0) = 0.$$

The function $\rho = a_i + X$ produces an invertible germ in $\mathcal{O}_{n,\zeta}$ so that

$$z_{j_1} \cdots z_{j_i} \in I_{G,\zeta} \Longrightarrow z_{j_1} \cdots z_{j_k} \in I_{G,\zeta} \Longrightarrow \mathfrak{M}^k_{n,\zeta} \subset I_{G,\zeta}.$$

Step 2. $\#G^{-1}(0) \leq \dim_{\mathbb{C}} Q^0_G(U)$. Suppose there exists m distinct solutions $\zeta_1, \cdots, \zeta_m$ of $G(z) = 0$. There exist polynomials P_i such that

$$P_i(\zeta_j) = \delta_{ij}$$

where δ_{ij} denotes the Kronecker symbol. The images of these polynomials in the polynomial algebra $Q^0_G(U)$ are linearly independent showing that $m \leq k$.

We define the **multi-local algebra** of G by

$$\Lambda_G(U) := \bigoplus_{\zeta \in G^{-1}(0)} Q_{G,\zeta}.$$

Observe that there exists a natural projection

$$\pi : \mathcal{O}(U) \to \Lambda_G(U).$$

Step 3. $\Lambda_G(U) = \pi(\mathcal{P}(U))$. Denote by $\zeta_1, \cdots, \zeta_m$ the roots of G. (By **Step 2** $m \leq k < \infty$). By **Step 1** each of them has finite multiplicity

$$\mu_i = \mu(G, \zeta_i).$$

Since $\mathfrak{M}^{\mu_i}_{n,\zeta_i} \subset I_{G,\zeta_i}$ we deduce that if $f, g \in \mathcal{O}(U)$ have the sane jets of order μ_i at ζ_i then $\pi(f) = \pi(g)$. The desired conclusion follows from the fact that we can find polynomials with arbitrarily specified jets at a finite number of points.

Step 4. Conclusion

$$\sum_{\zeta \in G^{-1}(0)} \mu(G, \zeta) := \dim \Lambda_G(U) \overset{\textbf{Step3}}{\leq} \dim_{\mathbb{C}} Q^0_G(U). \ \blacksquare$$

Proof of Lemma 11.19 Assume (only for notational simplicity) that the perturbation parameter is one dimensional $\lambda \in \mathbb{C}$ (as opposed to $\lambda \in \mathbb{C}^k$). Consider the holomorphic deformation F_λ, $\lambda \in \mathbb{C}$. It defines a map-germ

$$\hat{F} : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}^n \times \mathbb{C}, 0), \quad (z, \lambda) \mapsto (F_\lambda(z), \lambda).$$

Step 1. $Q_{F,0} \cong Q_{\hat{F},0}$. *Moreover if the function* $e_1, \cdots, e_\mu$ *define a basis of* $Q_{F,0}$ *then they also form a basis of* $Q_{\hat{F},0}$.

This isomorphism is induced by the obvious map

$$\mathbb{C}\{z_1, \cdots, z_n\} \to \mathbb{C}\{z_1, \cdots, z_n, \lambda)$$

$$\mathbb{C}\{z_1, \cdots, z_n\} \ni f(z_1, \cdots, z_n) \mapsto f(z_1, \cdots, z_n) \in \mathbb{C}\{z_1, \cdots, z_n, \lambda).$$

The map $\hat{F}$ induces a quasi-finite morphism ($x = (z_1, \cdots, z_n, \lambda)$, $y = (w_1, \cdots, w_n, \lambda)$)

$$\hat{F}^* : \mathcal{O}_{n+1}(y) \to \mathcal{O}_{n+1}(x), \;\; f(y) \mapsto f(\hat{F}(x)).$$

By **Step 1** we can find $e_1(z), \cdots, e_\mu(z) \in \mathcal{O}_n$ such that any $\varphi \in \mathcal{O}_{n+1}(x)$ has a *Weierstrass decomposition*. More precisely, this means that for any $\varphi \in \mathcal{O}_{n+1}(x)$ there exist $\varphi_i(y) \in \mathcal{O}_{n+1}(y)$ so that

$$\varphi(x) = \sum_i e_i(z)(\hat{F}^*\varphi_i)(x) = \sum_i e_i(z)\varphi_i(\hat{F}(x)). \tag{11.3}$$

In fact we can be much more precise about the domains of convergence of the germs $\varphi_i(y)$.

Step 2. *There exist open neighborhoods* U_1, U_2 *of* 0 *in the target space (y-coordinates) and source space (x-coordinates) of* $\hat{F}$ *such that, for any polynomial* $\varphi = \varphi(x)$, *its Weierstrass decomposition is well defined over* U_1 *and* U_2. *More precisely, this means that the corresponding holomorphic functions* $\varphi_i(y)$ *are defined over* U_1 *and* $\hat{F}(U_2) \subset U_1$.

Consider the finite set of germs

$$\Phi := \Big\{1, \;\; x_j e_k(z), \;\; 1 \le j \le n+1, \;\; 1 \le k \le \mu\Big\}.$$

We define U_1 to be an open poly-disk centered at $y = 0$ so that all the φ_i entering into the Weierstrass decompositions (11.3) of the function $\varphi \in \Phi$ converge on U_1. We then define U_2 as the sub-domain of $\hat{F}^{-1}(U_1)$ where the all germs $e_k(z)$ converge.

We proceed by induction on the degree. Every polynomial P can be put in the form

$$P = \sum_j x_j Q_j + c \cdot 1, \;\; \deg Q_j < \deg P.$$

Q_j has a Weierstrass decomposition over (U_1, U_2)

$$Q_j = \sum_k q_{kj}(y) e_k(z)$$

so that

$$P = \sum_j \sum_k q_{kj} x_j e_k(z) + c \cdot 1. = \sum_{\varphi \in \Phi} q_\varphi(y)\varphi(z).$$

We can now use the Weierstrass decomposition of $\varphi \in \Phi$ over (U_1, U_2) to obtain a Weierstrass decomposition of P over (U_1, U_2).

Step 3. Conclusion *There exists an open neighborhood U of $0 \in \mathbb{C}^n$ such that for all sufficiently small λ we have*

$$\operatorname{span}_{\mathbb{C}}\{e_1, \cdots, e_\mu\}/I_{F_\lambda}(U) \supset Q^0_{F_\lambda}(U).$$

In particular $\mu \geq \dim Q^0_{F_\lambda}(U)$.

Using **Step 2** we can find a neighborhood of zero $U \times V \subset \mathbb{C}^n \times \mathbb{C}$ and a ball B in the target space of $\hat{F}$ such that $\hat{F}(U \times V) \subset B$ and over $U \times V$ any polynomial $P = P(z)$ can be represented as

$$P(z) = \sum_i \varphi_i(w, \lambda) e_i(z), \quad w = F_\lambda(z). \tag{$*$}$$

Using Hadamard Lemma we obtain the representations

$$\varphi_i(w, \lambda) = \varphi_i(0, \lambda) + \sum_j w_j \varphi_{ij}(w, \lambda).$$

We substitute these decompositions in $(*)$ and we get a representation $(c_i(\lambda) = \varphi_i(0, \lambda))$

$$P(z) = \sum c_i(\lambda) e_i(z) + \sum h_j(z, \lambda) w_j, \quad w_j = F_{\lambda, j}(z). \tag{$**$}$$

Observe that the second sum belongs to $I_{F_\lambda}(U)$. This completes the proof of Lemma 11.19.

■

Chapter 12

The link and the Milnor fibration of an isolated singularity

In this chapter we will enter deeper into the structure of an isolated singularity and we will introduce several very useful topological invariants.

§12.1 The link of an isolated singularity

Suppose $f \in \mathbb{C}\{z_1, \cdots, z_n\}$ is a holomorphic function defined on an open neighborhood of 0 in $\mathbb{C}^n$ such that $f(0) = 0$, 0 is a critical point of f of finite multiplicity, i.e.

$$\mu := \dim_{\mathbb{C}} \mathcal{O}_n/\mathfrak{J}(f) < \infty$$

where we recall that $\mathfrak{J}(f) \in \mathcal{O}_n$ denotes the Jacobian ideal of f, i.e. the ideal generated by the first order partial derivatives of f. According to Tougeron theorem we may as well assume that f is a polynomial of degree $\leq \mu + 1$ in the variables $z_1, \cdots, z_n$.

The origin of $\mathbb{C}^n$ is an isolated critical point of f and, according to the results in Chapter 10, for every sufficiently small $r > 0$ and every generic small vector $\vec{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_n) \in \mathbb{C}^n$ the perturbation

$$g = f + \frac{1}{2}\sum_j \varepsilon_j z_j^2 : B_r := \{|z| \leq r\} \to D_\rho := \{|w| \leq \rho\}$$

has exactly μ, nondegenerate critical points $p_1, \cdots, p_\mu$ and the same number of critical values, $w_1, \cdots, w_\mu$, that is g is a Morse function. Moreover, the generic fiber $F_g = g^{-1}(w)$, $0 < |w| \ll 1$ is a smooth manifold with boundary. We can now invoke the arguments in Chapter 7 (proof the **Key Lemma**) to conclude that if we set

$$X_g := g^{-1}\Big(D_\rho\Big) \cap B_r$$

then $F_g \subset X_g$ and

$$H_k(X_g, F_g; \mathbb{Z}) \cong \begin{cases} 0 & k \neq n \\ \mathbb{Z}^\mu & k = n \end{cases}. \tag{12.1}$$

We can actually produce a basis of $H_n(X_g, F_g; \mathbb{Z})$ by choosing a point $\bullet$ on ∂D_ρ and joining it by non-intersecting paths $u_1, \cdots, u_\mu$ inside D_ρ, to the critical values $w_1, \cdots, w_\mu$. Each critical point p_j generates a vanishing cycle Δ_j thought as a cycle in the fiber over $\bullet$. By letting this vanishing cycle collapse to the critical point p_j along the path u_j we obtain the thimble $T_j \in H_n(X_g, F_g)$. Clearly, the special form of g played no special role. Only the fact that g is a morsification of f is relevant. To proceed further we need the following consequence of Sard theorem.

Lemma 12.1. *There exists $r_0 > 0$ such that, for all $r \in (0, r_0]$, the restriction of the function*

$$\nu : \mathbb{C}^n \to \mathbb{R}, \quad \vec{z} \mapsto |\vec{z}|^2$$

to $f^{-1}(0) \cap (B_r \setminus 0)$ has no critical points.

If we set

$$L_r(f) := f^{-1}(0) \cap \partial B_r = \nu^{-1}(r^2) \cap f^{-1}(0)$$

we deduce that $L_r(f) \cong L_{r_0}(f)$. This diffeomorphism is given by the descending gradient flow of ν along $f^{-1}(0)$. For this reason we will set

$$L_f := L_r(f), \quad 0 < r \ll 1.$$

This smooth manifold is called the *link of the isolated singularity* of f at 0. It has codimension 2 in the sphere ∂B_r and thus is a manifold of dimension $(2n-1)$. The function f defines a natural family of neighborhoods of $L_r(f) \hookrightarrow \partial B_r$,

$$U_{r,c}(f) := \{\vec{z} \in \partial B_r;\ \ |f(\vec{z})| \le c\}, \quad 0 < \delta \ll 1.$$

$U_{r,c}$ could be regarded as a fattening of the link $L_r(f)$.

Example 12.2. (a) If $n = 2$ and $f = f(z_1, z_2)$ then L_f is a one dimensional submanifold of the 3-dimensional sphere ∂B_r, i.e a knot or a link in the 3-sphere ∂B_r. The (knots) links obtained in this fashion are called *algebraic knots (links)*. For example, if $f = z_1^2 + z_2^3$, then L_f is the celebrated trefoil knot (see Figure 12.1). It also known as a torus $(2,3)$-knot. To visualize consider the line

$$\ell_{2,3} = \{3y = 2x\} \subset \mathbb{R}^2$$

and project it onto the torus $\mathbb{R}^2/\mathbb{Z}^2$. It goes 2-times in one angular direction and 3 times the other.

(b) If $n = 3$ and $f(z_1, z_2, z_3) = z_1^{a_1} + z_2^{a_2} + z_3^{a_3}$ then the link of f at zero is a 3-manifold. It is usually denoted by $\Sigma(a_1, a_2, a_3)$ and is referred to as a *Brieskorn manifold.* If the exponents a_i are pairwise coprime, then $\Sigma(a_1, a_2, a_3)$ is a homology sphere. For example $\Sigma(2,3,5)$ is known as the *Poincaré sphere.* It was the first example of 3-manifold with the same homology as the 3-sphere but not diffeomorphic to it. ■

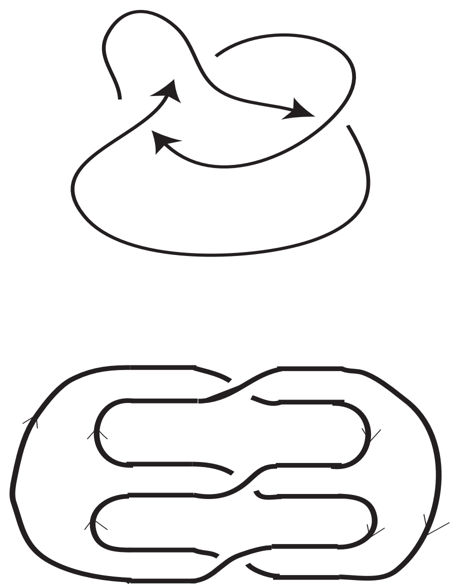

Figure 12.1: *Two equivalent diagrams for the trefoil knot*

§12.2 The Milnor fibration

Let us now return to the general situation. Since 0 is an isolated critical point, there exists $\varepsilon_0 > 0$ such that all the values $0 < |w| < \varepsilon_0$ are regular values of f. Restriction of f to the sphere ∂B_r vanishes in the complement of the link $L_r(f) \subset \partial B_r$ and thus defines a smooth map

$$\Theta = \Theta_{f,r} : \partial B_r \setminus L_r(f) \to S^1, \ \ p \mapsto \frac{1}{|f(p)|} f(p).$$

We have the following important result.

Theorem 12.3. (Milnor fibration theorem. Part I) *There exists $r_0 = r_0(f) > 0$ such that for all $r \in (0, r_0)$ the map $\Theta_{f,r}$ has no critical points and defines a fibration*

$$\Theta_{f,r} : \partial B_r \setminus L_r(f) \to S^1$$

called the **Milnor fibration**. *Its fiber is a real, $2n-2$-dimensional manifold called the* **Milnor fiber**. *We will denote it by $\Phi_r(f)$.*

Remark 12.4. Let $\langle \bullet, \bullet \rangle$ denote the Hermitian inner product on $\mathbb{C}^n$, conjugate linear in the second variable,

$$\langle \vec{u}, \vec{v} \rangle = \sum_{j=1}^{n} u_j \bar{v}_j.$$

We can think of $\mathbb{C}^n$ as a real vector space as well. As such, it is equipped with a **real** inner product

$$(\bullet, \bullet) = \mathfrak{Re} \langle \bullet, \bullet \rangle.$$

Any (complex) linear functional $L : \mathbb{C}^n \to \mathbb{C}$ has a Hermitian dual $L^\dagger \in \mathbb{C}^n$ uniquely determined by the requirement

$$\langle v, L^\dagger \rangle = L(v), \;\; \forall v \in \mathbb{C}^n.$$

The real part of L defines a (real) linear functional $\mathfrak{Re} L : \mathbb{C}^n \to \mathbb{R}$. It has a dual $L^\flat \in \mathbb{C}^n$ with respect to the real metric uniquely defined by the condition

$$(v, L^\flat) = \mathfrak{Re}\, L(v), \;\; \forall v \in \mathbb{C}^n.$$

It is easy to see that $L^\flat = L^\dagger$.

Proof of the Milnor fibration theorem We follow closely [29, Chap. 4]. Define the *gradient* of a holomorphic function $h(z_1, \cdots, z_n)$ to be the dual of the differential df with respect to the canonical Hermitian metric on $\mathbb{C}^n$. More precisely

$$\nabla h := (dh)^\dagger = \Bigl(\overline{\frac{\partial h}{\partial z_1}}, \cdots, \overline{\frac{\partial h}{\partial z_n}}\Bigr).$$

By definition,

$$dh(v) = \langle v, \nabla h \rangle, \;\; \forall v \in \mathbb{C}^n.$$

Let us first explain how to recognize the critical points of Θ.

Lemma 12.5. *The critical points of $\Theta_{f,r}$ are precisely those points $\vec{z} \in \partial B_r \setminus L_r(f)$ such that the complex vectors*

$$\mathbf{i}\nabla \log f \;\; \textit{and} \;\; \vec{z}$$

are linearly dependent over $\mathbb{R}$.

Proof of Lemma 12.5 $\vec{z} \in \partial B_r \setminus L_r(f)$ is a critical point of f if and only if the differential $d\Theta$ vanishes along $T_{\vec{z}}\partial B_r$, i.e.

$$d\Theta(v) = 0, \;\; \forall v \in \mathbb{C}^n \;\; \text{such that} \;\; \mathfrak{Re}\langle \vec{z}, v \rangle = 0.$$

If we locally write

$$f = |f| \exp(\mathbf{i}\theta)$$

then we can identify

$$\Theta = \theta = -\mathbf{i}\Bigl(\log f - \log(|f|)\Bigr).$$

Since $|f|^2 = f\bar{f}$ we deduce

$$d|f| = \frac{1}{2|f|}(d|f|^2) = \frac{1}{2|f|}(\bar{f}df + fd\bar{f})$$

and

$$d\theta = -\mathbf{i}\Big(d\log f - d\log|f|\Big) = -\mathbf{i}\Big(\frac{df}{f} - \frac{d|f|}{|f|}\Big)$$

$$= -\mathbf{i}\left(\frac{df}{f} - \frac{1}{2}\Big(\frac{df}{f} + \frac{d\bar{f}}{\bar{f}}\Big)\right) = \frac{1}{2}\left(-\mathbf{i}\frac{df}{f} + \overline{\left(-\mathbf{i}\frac{df}{f}\right)}\right) = -\mathfrak{Re}(\mathbf{i}d\log f).$$

Hence $d\Theta(v) = 0$ for all $v \in \mathbb{C}^n$ such that $\mathfrak{Re}\langle v, \vec{z}\rangle = 0$ implies that $(\mathbf{i}d\log f)^\flat$, the dual of $\mathfrak{Re}\,(\mathbf{i}d\log f)$ with respect to the **real** inner product on $\mathbb{C}^n$ is colinear to $\vec{z}$. Lemma 12.5 is now a consequence of Remark 12.4. ■

To prove Milnor fibration theorem we first need to show that if $|\vec{z}|$ is sufficiently small then the vectors $\mathbf{i}\nabla\log f$ and $\vec{z}$ are linearly independent over $\mathbb{R}$. We will rely on the following technical result.

Lemma 12.6. *Suppose $\vec{z} : [0, \varepsilon) \to \mathbb{C}^n$ is a real analytic path with $\vec{z}(0) = 0$ such that for all $t > 0$ $f(\vec{z}(t)) \neq 0$ and $\nabla\log f(\vec{z}(t))$ is a complex multiple of $\vec{z}(t)$*

$$\nabla\log f(\vec{z}(t)) = \lambda(t)\vec{z}(t),\ \ \lambda(t) \in \mathbb{C}^*.$$

Then

$$\lim_{t\searrow 0}\frac{\lambda(t)}{|\lambda(t)|} = 1.$$

Proof of Lemma 12.6 We have the Taylor expansions

$$\vec{z}(t) = \sum_{\nu\geq \ell_0}\vec{z}_\nu t^\nu,\ \ \vec{z}_{\ell_0} \neq 0$$

$$f(\vec{z}(t) = \sum_{\nu\geq m_0}a_\nu t^\nu,\ \ a_{m_0} \neq 0$$

and

$$\nabla f(\vec{z}(t)) = \sum_{\nu\geq n_0}\vec{u}_\nu t^\nu,\ \ \vec{u}_{n_0} \neq 0.$$

The equality $\nabla\log f(\vec{z}(t)) = \lambda(t)\vec{z}(t)$ is equivalent to

$$\nabla f(\vec{z}(t)) = \lambda(t)\vec{z}(t)\bar{f}(\vec{z}(t)).$$

Using the above Taylor expansions we get

$$\sum_{\nu\geq n_0}\vec{u}_\nu t^\nu = \lambda(t)\cdot\Big(\sum_{\nu\geq\ell_0}\vec{z}_\nu t^\nu\Big)\cdot\Big(\sum_{\mu\geq m_0}\bar{a}_\mu t^\mu\Big)$$

We deduce that $\lambda(t)$ has a Laurent expansion near $t = 0$

$$\lambda(t) = t^{r_0}\Big(\sum_{k\geq 0}\lambda_k t^k\Big),$$

where

$$r_0 := n_0 - m_0 - \ell_0, \;\; \vec{u}_{n_0} = \lambda_0 \bar{a}_{m_0}\vec{z}_{\ell_0}.$$

Thus, as $t \searrow 0$ we have $\lambda \approx \lambda_0 t^{r_0}$ and we need to show that λ_0 is real and positive.

Using the identity

$$\frac{df}{dt} = \Big\langle \frac{\vec{z}(t)}{dt}, \nabla f(\vec{z}(t))\Big\rangle = \Big\langle \frac{\vec{z}(t)}{dt}\,,\, \lambda(t)\vec{z}(t)\bar{f}(\vec{z}(t))\Big\rangle$$

we obtain

$$(m_0 a_{m_0} t^{m_0-1} + \cdots) = \Big\langle \Big(\ell_0 \vec{z}_{\ell_0} t^{\ell_0 - 1} + \cdots\Big), \Big(\lambda_0 \vec{z}_{\ell_0}\bar{a}_{m_0} t^{r_0+\ell_0+m_0} + \cdots\Big)\Big\rangle$$

so that

$$m_0 a_{m_0} = \ell_0 |\vec{z}_{\ell_0}|^2 a_{m_0}.$$

This shows $\lambda_0 \in (0,\infty)$ as claimed. ■

Lemma 12.7. *There exists $\varepsilon_0 > 0$ such that for all $\vec{z} \in \mathbb{C}^n \setminus f^{-1}(0)$ with $|\vec{z}| < \varepsilon_0$ the vectors $\vec{z}$ and $\nabla \log f(\vec{z})$ are either linearly independent over $\mathbb{C}$ or*

$$\nabla \log f(\vec{z}) = \lambda \vec{z},$$

where the argument of the complex number $\lambda \in \mathbb{C}^*$ is in $(-\pi/4, \pi/4)$.

Proof of Lemma 12.7 Set

$$\mathfrak{Z} := \Big\{|\vec{z}| \in \mathbb{C}^n;\;\; \vec{z} \text{ and } \nabla \log f(\vec{z}) \text{ are linearly dependent over } \mathbb{C}\Big\}.$$

The above linear dependence condition can be expressed in terms of the 2×2 minors of the $2 \times n$ matrix obtained from the vectors $\vec{z}$ and $\nabla \log f(\vec{z}) = \frac{1}{f}(\nabla f)$. Thus $\mathfrak{Z}$ is a closed, real algebraic subset of $\mathbb{C}^n$.

A point $\vec{z} \in \mathbb{C}^n \setminus f^{-1}(0)$ belongs to $\mathfrak{Z}$ if and only if there exists $\lambda \in \mathbb{C}^*$ such that

$$\nabla f(\vec{z}) = \lambda \bar{f}(\vec{z})\vec{z}.$$

Taking the inner product by $\bar{f}(\vec{z})\vec{z}$ we obtain

$$\mu(\vec{z}) := \Big\langle \nabla f(\vec{z}), \bar{f}(\vec{z}\vec{z})\Big\rangle = \lambda |\bar{f}(\vec{z})|^2.$$

This shows that λ has the same argument as $\mu(\vec{z})$. Since

$$|\arg(\zeta)| < \pi/4 \Longleftrightarrow \mathfrak{Re}\Big(\,(1 \pm \mathbf{i})\zeta\Big) > 0$$

we set

$$\Xi_\pm := \Big\{\vec{z};\ \mathfrak{Re}\,(\,(1\pm\mathbf{i})\mu(\vec{z})\,) < 0\Big\},\ \ \Xi := \Xi_+ \cup \Xi_-.$$

Assume 0 is an accumulation point of $W := \mathfrak{Z} \cap \Xi$ (or else there is nothing to prove). Set $W_\pm := \mathfrak{Z} \cap \Theta_\pm$.

The Curve Selection Lemma in real algebraic geometry[1] implies that there exists a real analytic path $\vec{z}(t)$, $0 \leq t < \varepsilon$ such that$\vec{z}(0) = 0$ and either $\vec{z}(t) \in W_+$ for all $t > 0$ or $\vec{z}(t) \in W_-$ for all $t > 0$. In either case we obtain a contradiction to Lemma 12.6 which implies that

$$\lim_{t\searrow 0} \arg\mu(\vec{z}(t)) = 0$$

while $|\arg\mu(\vec{z}(t))| > \pi/4$.

This contradiction does not quite complete the proof of Lemma 12.7. It is possible that the set $\mathfrak{Z} \setminus f^{-1}(0)$ contains points $\vec{z}$ arbitrarily close to 0 such that either $\mu(\vec{z}) = 0$ or $|\arg\mu(\vec{z})| = \pi/4$. In this case we reach a contradiction to Lemma 12.6 using the Curve Selection Lemma for the open set in the algebraic variety

$$\mathfrak{Re}\,(\,(1+\mathbf{i})\mu(\vec{z})\,)\mathfrak{Re}\,(\,(1-\mathbf{i})\mu(\vec{z})\,) = 0$$

defined by the polynomial inequality $|f(\vec{z})|^2 > 0$. ■

We have thus proved that

$$\Theta_{f,r} : \partial B_r \setminus L_r(f) \to S^1,\ \ \vec{z} \mapsto \frac{1}{|f(\vec{z})|} f(\vec{z})$$

has no critical points.

We could not invoke Ehresmann fibration theorem because $\partial B_r \setminus L_r(f)$ is not compact. Extra work is needed.

Lemma 12.8. *For all $r > 0$ sufficiently small there exists a vector field $\mathbf{v}$ tangent to $\partial B_r \setminus f^{-1}(0)$ such that*

$$\zeta(z) := \langle \mathbf{v}(\vec{z}), \mathbf{i}\nabla \log f(\vec{z})\rangle \neq 0\ \ \textit{and}\ \ |\arg\zeta(\vec{z})| < \pi/4. \tag{12.2}$$

---

[1] **Curve Selection Lemma:** Suppose $V \subset \mathbb{R}^m$ is a real algebraic set and $U \subset \mathbb{R}^m$ is described by finitely many inequalities,

$$U = \{x \in \mathbb{R}^m;\ \ g_1(z) > 0, \cdots, g_k(x) > 0\}$$

where g_i are real polynomials. If 0 is an accumulation point of $U \cap V$ then we can reach o following a real analytic path. This means there exists a real analytic curve $p : [0,1) \to \mathbb{R}^n$ such that $p(0) = 0$ and $p(t) \in U \cap V$ for all $t > 0$.

Proof The vector field will be constructed from local data using a partition of unity. Consider $\vec{z}_0 \in \partial B_r \setminus f^{-1}(0)$. We distinguish two cases.

A. *The vectors $\vec{z}_0$ and $\nabla \log f(\vec{z}_0)$ are linearly independent over $\mathbb{C}$.* In this case the linear system

$$\begin{cases} \big\langle \mathbf{v}, \vec{z}_0 \big\rangle & = & 0 \\ \big\langle \mathbf{v}, \mathbf{i}\nabla \log f(\vec{z}_0)\big\rangle & = & 1 \end{cases}$$

has a solution $\mathbf{v} = \mathbf{v}(\vec{z}_0)$. The first equation guarantees that $\mathfrak{Re}\langle \mathbf{v}, \vec{z}_0\rangle = 0$ so that $\mathbf{v}$ is tangent to ∂B_r.

B. $\nabla \log f(\vec{z}_0) = \lambda \vec{z}_0$, $\lambda \in \mathbb{C}$. In this case we set $\mathbf{v}(\vec{z}_0) := \mathbf{i}\vec{z}_0$. Clearly $\mathfrak{Re}\langle v, \vec{z}_0\rangle = 0$ and, according to Lemma 12.7, the complex number

$$\big\langle \mathbf{v}, \mathbf{i}\nabla \log f(\vec{z}_0)\big\rangle = \big\langle \mathbf{i}\vec{z}_0, \mathbf{i}\nabla \log f(\vec{z}_0)\big\rangle = \bar{\lambda}|\vec{z}_0|^2$$

has argument less than $\pi/4$ in absolute value.

Extend $\mathbf{v}(\vec{z}_0)$ to a tangent vector field $\mathbf{u}_{\vec{z}_0}$ defined along a tiny neighborhood $U_{\vec{z}_0}$ of $\vec{z}_0$ in $\partial B_r \setminus f^{-1}(0)$ and satisfying the (open) condition (12.2). Choose a partition of unity $(\eta_k) \subset C_0^\infty(\partial B_r \setminus f^{-1}(0))$ subordinated to the cover $(U_{\vec{z}})$ and set

$$\mathbf{v} := \sum_k \eta_k \mathbf{u}_{\vec{z}_k}.$$

This vector field satisfies all the conditions listed in Lemma 12.8. ■

Normalize

$$\mathbf{w}(\vec{z}) := \frac{1}{\mathfrak{Re}\langle \mathbf{v}(\vec{z}), \mathbf{i}\nabla \log f(\vec{z})\rangle}\mathbf{v}(\vec{z}).$$

The vector field satisfies two conditions.

• The real part of the inner product

$$\big\langle \mathbf{w}(\vec{z}), \mathbf{i}\nabla \log f(\vec{z})\big\rangle \tag{12.3}$$

is identically 1.

• The imaginary part satisfies

$$\Big|\mathfrak{Re}\big\langle \mathbf{w}(\vec{z}), \nabla \log f(\vec{z})\big\rangle\Big| < 1. \tag{12.4}$$

(This follows from the argument inequality (12.2).)

Lemma 12.9. *Given any $\vec{z}_0 \in \partial B_r \setminus f^{-1}(0)$ there exists a unique smooth path*

$$\gamma : \mathbb{R} \to \partial B_r \setminus f^{-1}(0)$$

such that

$$\gamma(0) = \vec{z}_0, \;\; \frac{d\gamma}{dt} = \mathbf{w}(\gamma(t)).$$

In other words, all the integral curves of $\mathbf{w}$ *exist for all moments of time.*

Proof Denote by γ the maximal integral curve of $\mathbf{w}$ starting at $\vec{z}_0$. Denote its maximal existence domain by (T_-, T_+). To show that $T_\pm = \pm\infty$ we will argue by contradiction. Suppose $T_+ < \infty$. This means that as $t \nearrow T_+$ the point $\gamma(t)$ approaches the frontier of $\partial B_r \setminus f^{-1}(0)$,or better yet

$$|f(\gamma(t))| \searrow 0 \Longleftrightarrow \log|f(\gamma(t))| \searrow -\infty \Longleftrightarrow \mathfrak{Re}\log f(\gamma(t)) \searrow -\infty. \tag{12.5}$$

On the other hand,

$$\frac{d}{dt}\mathfrak{Re}\log f(\gamma(t)) = \mathfrak{Re}\Big\langle\frac{d\gamma}{dt}, \nabla\log f\Big\rangle = \mathfrak{Re}\big\langle\mathbf{w}(\gamma(t)), \nabla\log f\big\rangle \overset{(12.4)}{>} -1$$

so that

$$\mathfrak{Re}\log f(\gamma(t)) > \mathfrak{Re}\log f(\vec{z}_0) - t.$$

This contradicts the blow-up condition (12.5) and concludes the proof of Lemma 12.9. ■

Suppose $\gamma(t)$ is an integral curve of $\mathbf{w}$. We can write

$$f(\gamma(t)) = |f(\gamma(t))|\exp(\mathbf{i}\theta(t))$$

and

$$\frac{d\theta(t)}{dt} = \frac{1}{\mathbf{i}}\frac{d}{dt}\log f(\gamma(t)) = \mathfrak{Re}\langle\frac{d\gamma}{dt}, \mathbf{i}\nabla\log f(\gamma)\rangle \overset{(12.3)}{=} 1.$$

Hence $\theta(t) = t + \mathit{const}$ and thus the path $\gamma(t)$ projects under Θ_r to a path which winds around the unit circle in the positive direction with unit velocity. Clearly the point $\gamma(t)$ depends smoothly on the initial condition $\vec{z}_0 := \gamma(0)$ and we will write this as

$$\gamma(t) = H_t(\vec{z}_0).$$

H_t is a diffeomorphism of $\partial B_r \setminus f^{-1}(0)$ to itself, and mapping the fiber $\Theta_r^{-1}(e^{\mathbf{i}\theta})$ diffeomorphically onto $\Theta_r^{-1}(e^{\mathbf{i}(\theta+t)})$. This completes the proof of the Fibration Theorem. ■

Chapter 13

The Milnor fiber and local monodromy

We continue to use the notations in the previous chapter.

§13.1 The Milnor fiber

We want to first show that the function $|f| : \partial B_r \setminus L_r(f)$ has no critical values accumulating to zero. In fact, a much more precise statement is true. For each angle $\theta \in [-\pi, \pi]$ we denote by $\Phi_r(\theta) = \Phi_r(f, \theta)$ the fiber of $\Theta_{f,r}$ over $e^{\mathbf{i}\theta}$.

Proposition 13.1. *Fix an angle $\theta \in [-\pi, \pi]$. There exists $r_0 > 0$ with the following property. For every $0 < r < r_0$ there exists $c = c(r) > 0$ such that*

$$\Phi_{r,c}(\theta) := \Phi_r(\theta) \cap \Big\{ |f| > c \Big\} = \Phi_r(f) \setminus U_{r,c}(f)$$

is diffeomorphic to the Milnor fiber.

Proof We will prove a slightly stronger result namely that for every sufficiently small r there exists $c = c(r, \theta) > 0$ such that the function $|f| : \Phi_r(f, \theta) \to \mathbb{R}_+$ has no critical values $< c(r)$. Then the diffeomorphism in the proposition is given by the gradient flow of $|f|$.

We first need a criterion to recognize the critical points of $|f|$, or which is the same, the critical points of $\log |f|$.

Lemma 13.2. *Fix an angle $\theta \in [-\pi, \pi]$. The critical points of $\log |f|$ along the Milnor fiber $\Phi_r(\theta)$ are those points $\vec{z}$ such that $\nabla \log f(\vec{z})$ is a complex multiple of $\vec{z}$.*

Proof Set $h(\vec{z}) := \log |f(\vec{z})| = \mathfrak{Re} \log f(z)$. Observe that for every vector $\vec{v} \in \mathbb{C}^n$ we have

$$dh(\vec{v}) = \mathfrak{Re} \langle \vec{v}, \nabla \log f(\vec{z}) \rangle.$$

Thus $\vec{z}$ is critical for h restricted to the Milnor fiber if and only if $\nabla \log f(\vec{z})$ is orthogonal to the tangent space $T_{\vec{z}}\Phi$ of $\Phi_r(\theta)$ at $\vec{z}$. The fiber is described as the intersection of two hypersurfaces

$$\Big\{ |\vec{z}| = r \Big\} \cap \Theta_{f,r}^{-1}(\theta)$$

so that the orthogonal complement is of $T_{\vec{z}}\Phi$ in $\mathbb{C}^n$ is spanned (over $\mathbb{R}$) by $\nabla|\vec{z}|^2$ and $\nabla\Theta_r = \mathbf{i}\nabla\log f(z)$. Thus $\vec{z}$ is a critical point if and only if there exists a linear relation between the vectors $\vec{z}$, $\nabla\log f(z)$ and $\mathbf{i}\nabla\log f(\vec{z})$. This proves Lemma 13.2. ■

As in the previous chapter, set

$$\mathfrak{Z} := \Big\{\vec{z} \text{ and } \nabla\log f(\vec{z}) \text{ are linearly dependent over } \mathbb{C}\Big\}, \ \ \mathfrak{Z}_\theta := \mathfrak{Z}\cap\Phi_r(\theta).$$

Both $\mathfrak{Z}$ and $\mathfrak{Z}_\theta$ are real algebraic varieties and we have to show that $\mathfrak{Z}\cap f^{-1}(0)$ contains no accumulation points of $\mathfrak{Z}_\theta$. We argue by contradiction. If $\vec{z}_0\in\mathfrak{Z}\cap f^{-1}(0)$ is an accumulation point of $\mathfrak{Z}_\theta$ then there would exist a real analytic path $\vec{z}:[0,\varepsilon)\to\mathfrak{Z}$ such that $\vec{z}(0)=\vec{z}_0$ and $\vec{z}(t)\in\mathfrak{Z}_\theta$, $\forall t>0$. Clearly $\log|f(\vec{z})|$is constant along this path so that $|f(z)|$ is constant as well. This constant can only be $|f(\vec{z}_0)|=0$ which is clearly impossible: $|f|>0$ on $\Phi_r(\theta)$. This concludes the proof of Proposition 13.1. ■

The Milnor fiber can be given a simpler description, which will show that it is equipped with a natural complex (even Stein) structure.

Proposition 13.3. *Consider a very small complex number $c=|c|e^{\mathbf{i}\theta}\neq 0$. The intersection of the hypersurface $f^{-1}(c)$ with the small open ball B_r,*

$$M_r(f)=M_{r,c}(f):=f^{-1}(c)\cap B_r$$

is diffeomorphic to the portion $\Phi_{r,|c|}(\theta)\subset\Phi_r(\theta)$ of the Milnor fiber.

Proof Using the same local patching argument as in the proof of Lemma 2.6 of Lecture 11 we can find a vector field $\mathbf{v}(\vec{z})$ on $\bar{B}_r\setminus f^{-1}(0)$ so that the Hermitian inner product

$$\Big\langle\mathbf{v}(\vec{z}),\nabla\log f(\vec{z})\Big\rangle\in\mathbb{R}_+,\ \ \forall\vec{z}\in\bar{B}_r\setminus f^{-1}(0) \tag{13.1}$$

and the inner product

$$\mathfrak{Re}\Big\langle\mathbf{v}(\vec{z}),\vec{z}\Big\rangle>0 \tag{13.2}$$

has positive real parts. Now consider the flow determined by this vector field,

$$\frac{d\vec{z}}{dt}=\mathbf{v}(\vec{z})$$

on $\bar{B}_r\setminus f^{-1}(0)$. The condition

$$\Big\langle\frac{d\vec{z}}{dt},\nabla\log f(\vec{z})\Big\rangle\in\mathbb{R}_+$$

shows that the argument of $f(\vec{z}(t))$ is constant and that $|f(\vec{z})|$ is monotone increasing function of t. The condition

$$2\mathfrak{Re}\Big\langle\frac{d\vec{z}}{dt},\vec{z}(t)\Big\rangle=\frac{d|\vec{z}(t)|^2}{dt}>0$$

guarantees that $t \mapsto |\vec{z}(t)|$ is strictly increasing.

Thus, starting at any point $\vec{z}$ of $\bar{B}_r \setminus f^{-1}(0)$ and following the flow line starting at $\vec{z}$ we travel away from the origin, in a direction increasing $|f|$, until we reach a point $\vec{\zeta}$ on $\partial B_r \setminus f^{-1}(0)$ such that

$$\arg f(\vec{z}) = \arg(\vec{\zeta}).$$

The correspondence

$$\vec{z} \mapsto \vec{\zeta}$$

provides the diffeomorphism $f^{-1}(c) \cap B_r \mapsto \Phi_{r,|c|}(\arg c)$ claimed in the proposition. ■

Definition 13.4. Fix a sufficiently small number $\varepsilon > 0$. A (ε-) **Milnor vector field** for f is a vector field on B_{ρ_0} which

(a) satisfies (13.1) on B_{ρ_0} and
(b) both conditions (13.1) and (13.2) on $B_{\rho_0} \cap \{|f| > \varepsilon\}$.

The above proof shows that f admits Milnor vector fields.

Corollary 13.5. ([29, Milnor]) *For sufficiently small $c > 0$ the fibration*

$$f^{-1}(\partial B_c) \to \partial D_c := \Big\{ce^{\mathbf{i}\theta};\ \ |\theta| \leq \pi\Big\},\ \ \vec{z} \mapsto f(\vec{z})$$

is diffeomorphic to the Milnor fibration

$$\partial B_r \setminus f^{-1}(D_c) \to S^1,\ \ \vec{z} \mapsto \frac{1}{|f(\vec{z})|} f(\vec{z}).$$

We thus see that the Milnor fibration is a fibration over S^1 with fibers manifolds with boundary. Such a fibration is classified by a gluing diffeomorphism

$$\Gamma_f : \Phi_r(f) \to \Phi_r(f).$$

Theorem 13.6. (Milnor fibration theorem. Part II) *Suppose $f \in \mathbb{C}[z_1, \cdots, z_n]$ is a polynomial such that $0 \in \mathbb{C}^n$ is an isolated singularity of the hypersurface*

$$Z_f := f^{-1}(0) \subset \mathbb{C}^n,$$

i.e. $f(0) = 0$, $df(0) = 0$. Denote by μ the Milnor number of this singularity. Then there exist $\rho_0 > 0$, $\varepsilon_0 > 0$ and $\delta_0 > 0$ with the following properties.

(a) f has no critical values $0 < |w| < \varepsilon_0$ and every morsification g of f such that

$$\sup_{\vec{z} \in B_{\rho_0}} |f(\vec{z}) - g(\vec{z})| < \delta_0$$

has exactly μ critical points $p_1, \cdots, p_\mu \in B_{\rho_0}(0) \subset \mathbb{C}^n$ and exactly μ critical values $w_j = f(p_j)$, $|w_j| < \varepsilon_0$.

(b) For every $0 < \varepsilon < \varepsilon_0$ and $0 < r < \rho_0$ the fibrations

$$f : f^{-1}(\partial D_\varepsilon) \cap \bar{B}_r \to \partial D_\varepsilon, \;\; \Theta_{f,r} : \partial B_r \setminus f^{-1}(D_\varepsilon) \to S^1$$

are isomorphic.

(c) If $w \in D_\varepsilon \setminus \{0\}$ is a regular value of g then the Milnor fiber $M_{r,w}(g) = g^{-1}(w) \cap B_r$ of g is diffeomorphic to $M_r(f)$

(d) The Milnor number $\mu = \mu(f, 0)$ is equal to the middle Betti number of the Milnor fiber. More precisely,

$$H_k(M_r(f), \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^\mu & k = n-1 (= \frac{1}{2} \dim_{\mathbb{R}} M_r(f)) \\ & \\ 0 & k \neq n-1 \end{cases} \tag{13.3}$$

Proof Part (a) is essentially the content of Chapter 10.

(b) The isomorphism

$$f : f^{-1}(\partial D_\varepsilon) \cap \bar{B}_r \to \partial D_\varepsilon \Longleftrightarrow \Theta_{f,r} : \partial B_r \setminus f^{-1}(D_\varepsilon) \to S^1$$

follows from Corollary 13.5.

(c) This follows from the fact that g approximates f very well and thus the regular fibers of g ought to approximate well the regular fibers of f.

(d) To prove this define as in the beginning of Chapter 11

$$X_r(g) := g^{-1}(\bar{D}_\varepsilon) \cap \bar{B}_r, \;\; X_g := X_{\rho_0}(g), \;\; M_g := M_{\rho_0}(g).$$

Then

$$H_k(X_g, M_g; \mathbb{Z}) \cong \begin{cases} 0 & k \neq n \\ \mathbb{Z}^\mu & k = n \end{cases}. \tag{13.4}$$

Next, observe the following fact.

Lemma 13.7. *The manifold with corners X_g is contractible.*

Sketch of proof The idea of proof is quite simple. Since g is very close to f we deduce that $X_r(g)$ is homotopic to $X_r(f)$ for all $r \leq \rho_0$. Next, using the backwards flow of a Milnor vector field $\mathbf{v}(\vec{z})$ of f, we observe that $X_{\rho_0}(f)$ is homotopic to $X_r(f)$ so that

$$X_{\rho_0}(g) \simeq X_r(g), \;\; \forall r < \rho_0.$$

We can choose $r > 0$ sufficiently small so that g has at most one critical point in B_r and, in case it exists it is the origin and is nondegenerate. By choosing B_r even smaller we can use Morse lemma to change the coordinates so that g is a polynomial of degree ≤ 2. The contractibility of $X_r(g)$ now follows from the local analysis involved in the Picard-Lefschetz formula (see Chapter 7). ■

The equality (13.3) now follows from (13.4), the contractibility of X_g and the long exact sequence of the pair (X_g, M_g). ■

§13.2 The local monodromy, the variation operator and the Seifert form of an isolated singularity

The above results show that the Milnor fibration, is a fibration in manifolds with boundary $M_r(f)$ classified by a gluing map Γ_f. The total space of the boundary fibration is the intersection of the hypersurface. This extends to a fibration

$$Y_c := \{|f| = c\}$$

with the sphere ∂B_r. The Milnor fibration is then simply described by the map f. Because $|f| \,|_{\partial B_r}$ does not have critical values accumulating at zero we deduce that this fibration extends to a fibration over a small disk

$$\Theta_{f,r} := \Big\{|\vec{z}| = r;\ \ |f(\vec{z})| \leq c\Big\}\Big(= U_{r,c}(f) \cap \partial B_r\Big) \to \Big\{|w| \leq c\Big\},\ \ \vec{z} \mapsto f(\vec{z}).$$

Thus the fibration $f : Y_c \to \{|w| = c\}$ is trivializable

$$U_{rc}(f) \cap \partial B_r \cong \{|w| \leq c\} \times L - r(f)$$

so that the restriction of Γ_f is homotopic to the identity. For simplicity we assume

$$\Gamma_f \,|_{\partial M_r(f)} \equiv \mathbf{1}.$$

Definition 13.8. The **local monodromy** of the isolated singularity of f at 0 is the automorphism

$$(\Gamma_f)_* : H_{n-1}(M_r(f), \mathbb{Z}) \to H_{n-1}(M_r(f), \mathbb{Z}),$$

induced by the gluing map Γ_f. Whenever no confusion is possible, we will write Γ_f instead of $(\Gamma_f)_*$.

Suppose $\mathbf{z} \in H_{n-1}(M_r(f), \partial M_r(f); \mathbb{Z}$ is a relative cycle. Since Γ_f acts as $\mathbf{1}$ on $\partial M_r(f)$ we deduce that

$$\partial(\mathbf{1} - \Gamma_f)\mathbf{z} = 0$$

so that $z - \Gamma_f \mathbf{z} \in H_{n-1}(M_r(f); \mathbb{Z})$. The morphism

$$H_{n-1}(M_r(f), \partial M_r(f); \mathbb{Z}) \mapsto H_{n-1}(M_r; \mathbb{Z}),\ \ \mathbf{z} \mapsto \mathbf{z} - \Gamma_f \mathbf{z}$$

is called **the variation operator** of the singularity and is denoted by $\mathbf{var}_f$. We see that the Picard-Lefschetz formula is nothing but an explicit description of the variation operator of the simplest type of singularity.

Before we proceed further we need to discuss one useful topological invariant, namely, the *linking number*. (For more details we refer to the classical [24].)

Suppose $\mathbf{a}$ and $\mathbf{b}$ are $(n-1)$-dimensional cycles inside the $(2n-1)$-sphere ∂B_r. (When $n = 1$ we will assume the cycles are also homotopic to zero.) We can then choose a n-chain $\mathbf{A}$ bounding $\mathbf{a}$. The intersection number $\mathbf{A} \cdot \mathbf{b}$ is independent of the choice of $\mathbf{A}$. The resulting integer is called the *linking number* of $\mathbf{a}$ and $\mathbf{b}$ and is denoted by $\mathbf{lk}(\mathbf{a}, \mathbf{b})$.

The computation of the linking number can be alternatively carried as follows. Choose two n-chains $\mathbf{A}$ and $\mathbf{B}$ bounding $\mathbf{a}$ and $\mathbf{b}$, which, except their boundaries, lie entirely inside B_r. We then have

$$\mathbf{lk}(\mathbf{a}, \mathbf{b}) = (-1)^n \mathbf{A} \cdot \mathbf{B}.$$

In particular, we deduce

$$\mathbf{lk}(\mathbf{a}, \mathbf{b}) = (-1)^n \mathbf{lk}(\mathbf{b}, \mathbf{a}).$$

To prove the first equality is suffices to choose the chains $\mathbf{A}$ and $\mathbf{B}$ in a clever way. Choose $\mathbf{B}$ as the cone over $\mathbf{b}$ centered at 0

$$\mathbf{B} = \Big\{ t\vec{z};\ \ \vec{z} \in \mathbf{b},\ \ t \in [0,1] \Big\}.$$

Next, choose a chain $\mathbf{A}_0 \subset \partial B_r$ bounding $\mathbf{a}$ and then define

$$\mathbf{A} = \Big\{ \frac{1}{2}\vec{z};\ \ \vec{z} \in \mathbf{A}_0 \Big\} \cup \Big\{ t\vec{z};\ \ \vec{z} \in \mathbf{a},\ \ t \in [\frac{1}{2}, 1] \Big\}$$

(see Figure 13.1.)

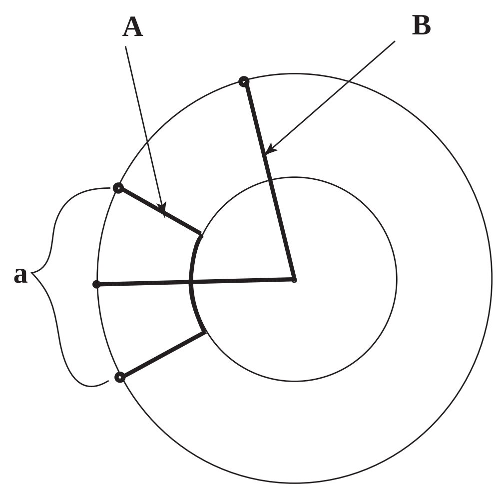

Figure 13.1: *Linking numbers*

Fix a sufficiently small number $\varepsilon > 0$ and set for simplicity

$$\Phi_r(\theta) := \Theta_{f,r}(e^{\mathbf{i}t}) \cap \{|f| \geq \varepsilon\},\ \ T := \{|f| \leq \varepsilon\} \cap \partial B_r.$$

Consider a family of diffeomorphisms of

$$Y_t : \Phi_r(0) \to \Phi_r(2\pi t),\ \ t \in [0,1]$$

which lifts the homotopy $t \mapsto \exp(2\pi\mathbf{i}t)$, $Y_0 \equiv \mathbf{1}$ and agrees with a fixed trivialization of the boundary fibration. Observe two things.

- Y_1 can be identified with Γ_f.
- If $\mathbf{a},\ \mathbf{b} \in H_{n-1}(\Phi_r(0); \mathbb{Z})$ then $Y_{1/2} \in \Phi_r(\pi)$ and thus the cycles $\mathbf{a}$ and $Y_{1/2}\mathbf{b}$ in ∂B_r are **disjoint**.

Definition 13.9. The **Seifert form** of the singularity f is the bilinear form L_f on $H_{n-1}(\Phi_r(0);\mathbb{Z})$ defined by the formula

$$L_f(\mathbf{a},\mathbf{b}) = \mathbf{lk}(\mathbf{a}, Y_{1/2}\mathbf{b}).$$

Proposition 13.10. *Consider two cycles* $\mathbf{a} \in H_{n-1}(\Phi_r(0), \partial\Phi_r(0);\mathbb{Z})$ *and* $\mathbf{b} \in H_{n-1}(\Phi_r(0);\mathbb{Z})$. *Then*

$$L_f(\mathbf{var}_f(a), b) = \mathbf{a} \cdot \mathbf{b}$$

where the dot denotes the intersection number of $(n-1)$*-cycles inside the* $(2n-2)$*-manifold* $\Phi_r(0)$.

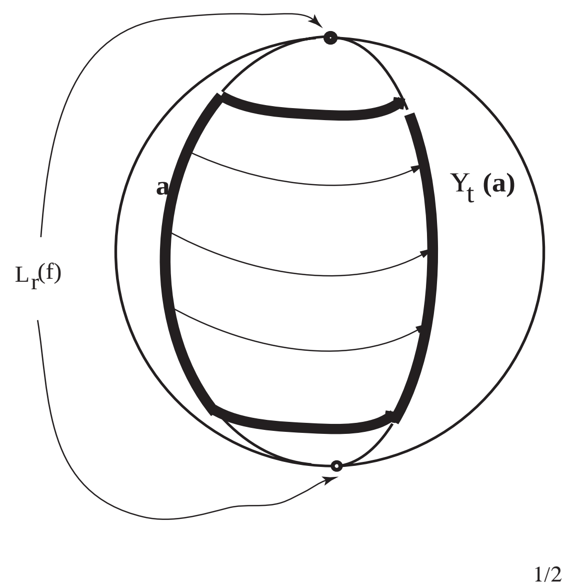

Figure 13.2: *The variation operator*

Proof Consider the map

$$Y : [0,1] \times \mathbf{a} \to \partial B_r, \quad (t, \vec{z}) \mapsto Y_t(\vec{z}).$$

The image of Y is an n-chain $C \subset \partial B_r$ whose boundary consists of two parts: the variation of $\mathbf{a}$

$$\mathbf{var}_f(\mathbf{a}) = Y_1\mathbf{a} - \mathbf{a},$$

which lies inside $\Phi_r(0)$, and the cylinder $Y([0,1] \times \partial\mathbf{a})$, which lies entirely inside on ∂T (see Figure 13.2). We have a natural identification

$$\partial T \cong \{|w| \le \varepsilon\} \times L_r(f)$$

obtained by fixing a trivialization of the boundary of the Milnor fibration. Note that $Y_t(\partial\mathbf{a})$ corresponds via this identification to the cycle $\{\varepsilon e^{2\pi\mathbf{i}t}\} \times \partial\mathbf{a}$. Now flow this cycle along the radii to the t-independent cycle $\{0\} \times \partial\mathbf{a}$.

We thus have extended the cylinder $Y([0,1]\times\mathbf{a})$ to a chain $\mathbf{A}$ in ∂B_r whose boundary represents $\mathbf{var}_f(\mathbf{a})\subset\Phi_r(0)$. The intersection of the chain $\mathbf{A}$ with $Y_{1/2}\mathbf{b}$ is the same as the intersection of the cycles $Y_{1/2}\mathbf{a}$ and $Y_{1/2}\mathbf{b}$ in the fiber $\Phi_r(\pi)$. Hence

$$L_f\big(\,\mathbf{var}_f(\mathbf{a},\mathbf{b})\,\big)=(Y_{1/2}(\mathbf{a})\cdot Y_{1/2}\mathbf{b})_{\Phi_r(\pi)}=(\mathbf{a}\cdot\mathbf{b})_{\Phi_r(0)}.\ \blacksquare$$

Proposition 13.11. *The Seifert form is nondegenerate, i.e. it induces an isomorphism from $H_{n-1}(\Phi_r(0);\mathbb{Z})$ to its dual.*

Proof The Alexander duality theorem (see [24]) asserts that the linking pairing

$$\mathbf{lk}:H_{n-1}(\Phi_r(0),\mathbb{Z})\times H_{n-1}(\partial B_r\setminus\Phi_r(0);\mathbb{Z})\to\mathbb{Z}$$

is a duality, (i.e. nondegenerate). A bit of soul searching shows that the middle fiber $\Phi_r(\pi)$ is a deformation retract of $\partial B_r\setminus\Phi_r(0)$. Consequently, we have an isomorphism

$$H_{n-1}(\partial B_r\setminus\Phi_r(0);\mathbb{Z})\cong H_{n-1}(\Phi_r(\pi);\mathbb{Z}).$$

The proposition now follows from the fact that $Y_{1/2}$ induces an isomorphism

$$H_{n-1}(\Phi_r(0);\mathbb{Z})\to H_{n-1}(\Phi_r(\pi);\mathbb{Z}),\ \blacksquare$$

By Poincaré-Lefschetz duality, the intersection pairing

$$H_{n-1}(\Phi_r(0),\partial\Phi_r(0);\mathbb{Z})\times H_{n-1}(\Phi_r(0);\mathbb{Z})\to\mathbb{Z}$$

is nondegenerate. Proposition 13.10, 13.11 have the following remarkable consequence.

Corollary 13.12. *The variation operator of the singularity f is an isomorphism of homology groups*

$$H_{n-1}(\Phi_r(0),\partial\Phi_r(0);\mathbb{Z})\to H_{n-1}(\Phi_r(0);\mathbb{Z}).$$

Moreover

$$L_f(a,b)=(\mathbf{var}_f^{-1}\mathbf{a})\cdot\mathbf{b},\ \ \forall\mathbf{a},\mathbf{b}\in H_{n-1}(\Phi_r(0)). \tag{13.5}$$

Corollary 13.13. *For* $\mathbf{a},\mathbf{b}\in H_{n-1}(\Phi_r(0);\mathbb{Z})$

$$\mathbf{a}\cdot\mathbf{b}=-L_f(\mathbf{a},\mathbf{b})+(-1)^nL_f(\mathbf{b},\mathbf{a}).$$

Proof Observe first that

$$\mathbf{var}_f\mathbf{a}\cdot\mathbf{var}_f\mathbf{b}+\mathbf{a}\cdot\mathbf{var}_f\mathbf{b}+\mathbf{var}_f\mathbf{a}\cdot\mathbf{b}=0. \tag{13.6}$$

If we set $\mathbf{a}_0:=\mathbf{var}_f\mathbf{a}$, $\mathbf{b}_0:=\mathbf{var}_f\mathbf{b}$ we deduce

$$\mathbf{a}_0\cdot\mathbf{b}_0=-\mathbf{var}_f^{-1}\mathbf{a}_0\cdot\mathbf{b}_0-\mathbf{a}_0\cdot\mathbf{var}_f^{-1}\mathbf{b}_0\overset{(13.5)}{=}-L_f(\mathbf{a}_0,\mathbf{b}_0)+(-1)^nL_f(\mathbf{b}_0,\mathbf{a}_0).\ \blacksquare$$

§13.3 Picard-Lefschetz formula revisited

We want to outline a computation the variation operator of the simplest singularity,

$$f = z_1^2 + \cdots + z_n^2.$$

The answer is the Picard-Lefschetz formula discussed in great detail in Chapter 7 using a more artificial method.

The Milnor fibration of this quadratic singularity is given by the formula

$$\partial B_r \setminus \Bigl\{ \sum_j z_j^2 = 0 \Bigr\} \ni \vec{z} \mapsto \frac{1}{|\sum_j z_j^2|} \sum_j z_j^2.$$

The vanishing cycle corresponds in the fiber $\Phi_r(0)$ to the cycle Δ defined by the equations

$$\sum_j z_j^2 = 1, \;\; \mathfrak{Im}\, z_j = 0.$$

We have

$$\mathbf{var}^{-1}\Delta \cdot \Delta = \mathbf{lk}(\Delta, Y_{1/2}\Delta) = L(\Delta, \Delta) = (-1)^n \mathbf{A} \cdot \mathbf{B}$$

where $\mathbf{A}$ and $\mathbf{B}$ are cycles in B_r with boundaries Δ and respectively $Y_{1/2}\Delta$. To calculate the linking number $\mathbf{lk}(\Delta, Y_{1/2}\Delta)$ it is possible to use the family of diffeomorphisms

$$\Psi_t : \Phi_r(0) \to \Phi_r(2\pi t), \;\; (z_1, \cdots, z_n) \mapsto (e^{\pi \mathbf{i} t} z_1, \cdots, e^{\pi \mathbf{i} t} z_n).$$

The reason is very simple. $\Psi_{1/2}\Delta$ and $Y_{1/2}\Delta$ are homologous *inside* the Milnor fiber $\Phi_r(1/2)$ so that they have the same linking number with Δ.

The cycle $\Psi_{1/2}\Delta$ is determined by the equations

$$\sum_j z_j^2 = -1, \;\; \mathfrak{Re}\, z_j = 0.$$

We can take as $\mathbf{A}$ and $\mathbf{B}$ the chains determined by the equation $\mathfrak{Im}\, z_j$ and respectively $\mathfrak{Re}\, z_j = 0$. Their intersection is ± 1 with the sign which can be determined following the rules in Chapter 7.

Chapter 14

Clemens's generalization of the Picard-Lefschetz formula

The Picard-Lefschetz formula explains the topological implications of a nondegenerate critical point of a holomorphic function. In this chapter we want to approach the general case and try to understand the monodromy of such a critical point. The description will be in terms of a resolution of that singularity. Our presentation is greatly inspired from the work of H. Clemens [8, 9, 10] and N. A'Campo, [1].

§14.1 Functions with ordinary singularities

Suppose $P \in \mathbb{C}[z_0, z_1, \cdots, z_n]$ is a polynomial in $n+1$-variables such that the origin $0 \in \mathbb{C}^{n+1}$ is an isolated critical point. A famous result of H. Hironaka [17] implies that P admits an embedded resolution of singularities. This means that there exists a $n+1$-dimensional Kähler manifold X and a holomorhic map$\pi : X \to \mathbb{C}^{n+1}$ with the following properties.

• The restriction of π to $X \setminus \pi^{-1}(0)$ is a biholomorphism onto a punctured polydisk of $\mathbb{C}^{n+1}$ of the form

$$0 < |z_k| < r_k, \;\; k = 0, \cdots, n.$$

• The exceptional set $E = \pi^{-1}(0)$ consists of smooth divisors in X intersecting transversely.
• If $H = \{P = 0\} \subset \mathbb{C}^{n+1}$ then the closure of $\pi^{-1}(H \setminus 0)$ in X is a smooth divisor.

The composition $f := P \circ \pi : X \to \mathbb{C}$ is now a holomorphic function on the Kähler manifold X such that the fiber $f^{-1}(0)$ has better controlled singularities. Note that for any $t \in \mathbb{C}$, $0 < |t| \ll 1$ the fibers $f^{-1}(t)$ and $P^{-1}(t)$ are diffeomorphic. As $t \to 0$ the Milnor fiber "collapses" onto the singular fiber $f^{-1}(0)$ and thus we can expect that this singular fiber carries a considerable amount of information about the generic nearby fibers. This is the type of problem we intend to address in this chapter.

Suppose X is an (open) Kähler manifold of complex dimension $n+1$, Δ is the unit open disk in $\mathbb{C}$, and $f : X \to \Delta$ with the following properties.

• $0 \in \Delta$ is the unique critical value of f. Set $X_t := f^{-1}(t)$

• $X_0 = \bigcup_{j=1}^{\nu} D_j$, where D_j are *smooth* divisors in X. For any set $U \subset X$ we define

$$I(U) := \Big\{ i;\ \ D_i \cap U \neq \emptyset \Big\}.$$

For simplicity we set $I(x) = I(\{x\})$, $\forall x \in X$.

• For any subset $I \subset \overline{1,\nu} := \{1, \cdots, \nu\}$ the divisors $\{D_i\}_{i\in I}$ intersect transversely. We set

$$D_I := \bigcap_{i \in I} D_i.$$

Note that D_I is either empty, or it is a codimension $|I|$ complex submanifold of X.

The above requirements imply that there exist holomorphic line bundles

$$L_1, \cdots, L_\nu \to X,$$

positive integers $m_1, \cdots, m_\mu$ (the multiplicities of the divisors) and holomorphic sections $s_i \in \Gamma(L_i)$, $i = 1, \cdots, \nu$, of $L_1^{m_1} \otimes L_\nu^{m_\nu}$ is the trivial holomorphic line bundle and

$$D_k = s_k^{-1}(0),\ \ f(x) = u(x)s_1^{m_1} \cdots, s_\nu^{m_\nu}$$

where $u : X \to \mathbb{C}$ is a nowhere vanishing holomorphic function. We fix once and for all a hermitian metric h_i on L_i and we define

$$r_i : X \to [0, \infty),\ \ r_i(x) = |s_i(x)|_{h_i}. \tag{14.1}$$

We will assume $u \equiv 1$.

Example 14.1. (Working example. Part I.) Consider the function $f(x, y) = y^2 - x^5$. By resolving the singularity at $(0, 0)$ we obtain a two dimensional manifold X (which is an iterated blowup of $\mathbb{C}^2$ and a map $\hat{f} : X \to \mathbb{C}$ which satisfies all the above conditions. We want to determine all the relevant invariants.

By using the substitution $x \to x$, $y \to xy$ we see that

$$f = (y^2 - x^5) \to f_1 = x^2(y^2 - x^3)$$

where the exceptional divisor E_1 is given by $x = 0$. f_1 has order 2 along E_1. Next we make the substitution $x \to x$, $y \to xy$ to get

$$f_1 \to f_2 = x^4(y^2 - x)$$

where the exceptional divisor E_2 is given by $x = 0$. f_2 has order 4 along E_2. The substitution $x \mapsto xy$, $y \to y$ leads to

$$f_2 \to f_3 = y^5x^4(y - x)$$

where the exceptional divisor E_3 is given by $y = 0$. f_3 has order 5 along E_3 A final blowup $x \mapsto x$, $y \to xy$ leads to

$$f_3 \to \hat{f} = f_4 = y^5x^{10}(y - 1)$$

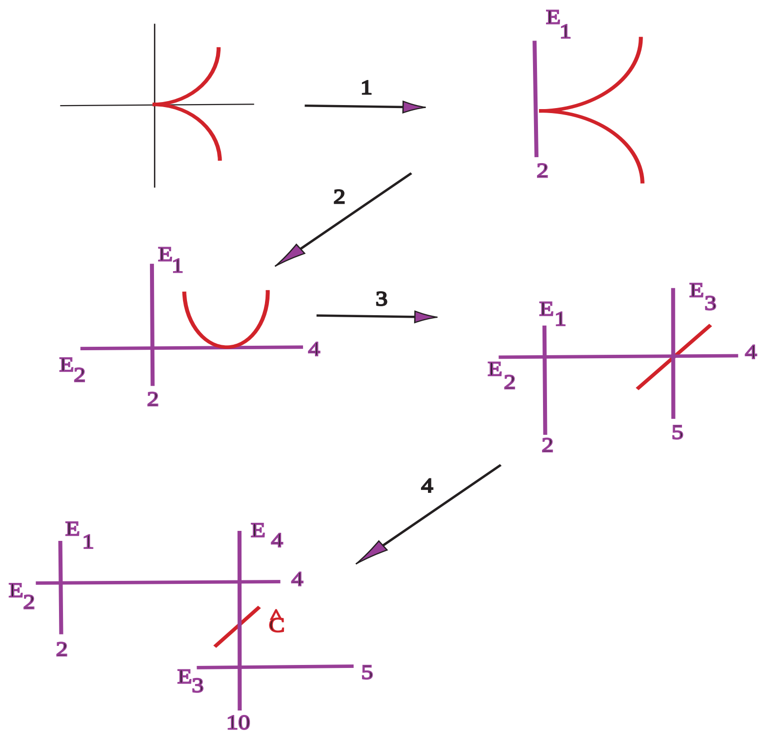

Figure 14.1: *The resolution of singularity* $y^2 = x^5$

where E_4 is described by $x = 0$. $\hat{f}$ has order 10 along E_4. These transformations are depicted in Figure 14.1 where we have also kept track of the multiplicities of the exceptional divisor. We denote by $\hat{C}$ the proper transform of the germ $C = \{y^2 - x^5 = 0\}$. In this case we can take $\nu = 5$ and set

$$D_j = E_j, \;\; 1 \leq j \leq 4, \;\; D_5 = \hat{C}.$$

Then $D_J = \emptyset$ if $|J| \geq 3$ and the only nonempty D_J with $|J| = 2$ correspond to

$$J = \{1, 2\}, \; \{2, 4\}, \; \{3, 4\}, \; \{4, 5\}.$$

In all these cases D_J consists of a single point.

§14.2 Local behavior

To proceed further, we need to collect more information about the (local) topology of the overlaps D_J. Consider the family of hypersurfaces

$$S_t := \mathcal{Z}(\vec{z}^{\vec{m}} - t) \subset \mathbb{C}^q$$

where $\vec{z} = (z_1, \cdots, z_q)$, $\vec{m} = (m_1, \cdots, m_q) \in \mathbb{Z}_+^q$ and

$$\vec{z}^{\vec{m}} := z_1^{m_1} \cdots z_q^{m_q}.$$

Define

$$T_t := \Big\{ \vec{z} \in S_t;\ \ |z_1|^{m_1} = \cdots = |z_q|^{m_q} \Big\},$$

$$\mathcal{R} := \Big\{ \vec{z} \in S_1;\ \ \mathfrak{Re}\, z_j > 0,\ \ \mathfrak{Im}\, z_k = 0,\ \ \forall j, k = 1, \cdots, q \Big\}$$

$$= \Big\{ \vec{x} \in (0, \infty)^q;\ \ x_1^{m_1} \cdots x_q^{m_q} = 1 \Big\}.$$

We set $d := \gcd(m_1, \cdots, m_q)$. The connected components of S_t and T_t are parametrized by the d-th roots of 1. This follows from the factorization of the function $\vec{z}^{\vec{m}} - t$

$$\vec{z}^{\vec{m}} - t = \prod_{j=1}^{d} (\vec{z}^{\vec{n}} - t_0 \zeta^j)$$

where

$$\vec{n} = \frac{1}{d}\vec{m},\ \ \zeta := \exp(\frac{2\pi \mathbf{i}}{m}),$$

and t_0 is a fixed d-th root of t. When t is real we assume $t_0 \in \mathbb{R}_+$. In the sequel, for simplicity, we will denote by S_t by T_t the connected component of S_t and respectively T_t labeled by the trivial d-th root of 1.

Lemma 14.2. *For $t \neq 0$ we have*

$$S_t \cong \mathcal{R} \times T_t.$$

The hypersurface S_1 lies in $(\mathbb{C}^*)^q$, and the universal cover of this space is $\mathbb{C}^q$. We will denote the complex coordinates on the universal cover by $\zeta_j = \rho_j + \mathbf{i}\theta_j$, $j = 1, \cdots, q$. As covering projection $\mathbb{C}^q \to (\mathbb{C}^*)^q$ we pick the map

$$\Psi : \mathbb{C}^q \to (\mathbb{C}^*)^q,\ \ \zeta_i \mapsto z_i = e^{\frac{1 - q\zeta_i}{m_i}},\ \ i = 1, \cdots, q.$$

Ψ is invariant under the action by translation of the lattice

$$\Lambda_{\vec{m}} = \bigoplus_{j=1}^{q} 2\pi\hbar m_j \mathbf{i}\mathbb{Z},\ \ \hbar = \frac{1}{q}.$$

Note the restriction of Ψ to $\mathfrak{Re}\,\mathbb{C}^q$ is one-to-one. Its inverse is

$$\Phi : (\mathbb{R}^*)^q \to \mathfrak{Re}\,\mathbb{C}^q,\ \ r_i \mapsto \frac{1}{q}(1 - m_i \log r_i),\ \ i = 1, \cdots, q$$

and

$$\mathcal{R} = \Psi(Q), \;\; Q := \Big\{ (\rho_1, \cdots, \rho_q) \in \mathbb{R}^q; \;\; \sum_{i=1}^{q} \rho_i = 1 \Big\}.$$

Define

$$\Omega : \mathbb{C}^q \to \mathbb{C}, \;\; \Omega(\vec{\rho} + \mathbf{i}\vec{\theta}) = \sum_{i=1}^{q} (\rho_i + \mathbf{i}\theta_i).$$

The universal cover of $S_{\exp(\mathbf{i}t)}$ is the complex hyperplane $L_t := \Omega^{-1}(1 - \mathbf{i}\hbar t)$. We set

$$T = \mathfrak{Re}\,\Omega, \;\; \Theta = \mathfrak{Im}\,\Omega.$$

Consider the closed polydisk

$$D_{\vec{m}} = \Big\{ \vec{z} \subset \mathbb{C}^q; \;\; |z_k|^{m_k} \leq e, \;\; k = 1, \cdots, q \Big\}.$$

The portion

$$\sigma_0 := \mathcal{R} \cap D_{\vec{m}}$$

is the image via Ψ of the standard simplex

$$\Delta := \Big\{ \vec{\rho} \in Q; \;\; \rho_i \geq 0, \;\; \forall i = 1, \cdots, q \Big\}.$$

For $j = 1, \cdots, q$ we denote by V_j the vertex of Δ defined by the condition $\rho_j = 1$. For $J \subset \{1, \cdots, q\}$ denote by Δ_J the face spanned by the vertices $\{V_j; \;\; j \in J\}$.

Fix a partition of unity $(\upsilon_i)_{0 \leq i \leq q}$ on Q subordinated to the open cover

$$\mathcal{O}_i = \Big\{ \vec{\rho} \in Q; \;\; \rho_i > 0 \Big\}.$$

This defines a retraction

$$\Upsilon : Q \to \Delta, \;\; \vec{\rho} \mapsto \Big(\upsilon_0(\vec{\rho}), \cdots, \upsilon_q(\vec{\rho}) \Big).$$

Extend υ_k to a $\Lambda_{\vec{m}}$-invariant function on $\upsilon_k : Q \oplus \mathfrak{Im}\,\mathbb{C}^q \to \mathbb{R}_+$ by setting

$$\upsilon_k(\vec{\rho} + \mathbf{i}\vec{\theta}) = \upsilon_k(\vec{\rho}).$$

The collection $(\upsilon_k)_{1 \leq k \leq q}$ descends to a collection of nonnegative functions $\tau_1, \cdots, \tau_q$ on the *real* hypersurface

$$\mathfrak{H} = \{ |z_1|^{m_1} \cdots |z_q|^{m_q} = 1 \} \subset (\mathbb{C}^*)^q.$$

By definition

$$\upsilon_k = \Psi^* \tau_k.$$

For every $t \geq 0$ define

$$F_t : S_1 \to S_{\exp(\mathbf{i}t)}, \quad \vec{z} \mapsto \Big(\exp\big(\frac{2\pi\mathbf{i}\tau_1(\vec{z})t}{m_1}\big)z_1, \cdots, \exp\big(\frac{2\pi\mathbf{i}\tau_q(\vec{z})t}{m_q}\big)z_q\Big).$$

We need to talk about the possible orientations of S_1. Using the map Ψ we can introduce local coordinates

$$(\rho_1, \cdots, \rho_{q-1}; \theta_1, \cdots, \rho_{q-1})$$

on S_1. S_1 has an orientation as a complex submanifold locally described by the volume form

$$\mathbf{or}_{complex} = d\rho_1 \wedge d\theta_1 \wedge \cdots \wedge d\rho_{q-1} \wedge d\theta_{q-1}.$$

On the other hand, in view of Lemma 14.2 we can view S_1 as a $q-1$-dimensional disk bundle over the $(q-1)$ torus T_1 and as such it has a fiber-first orientation

$$\mathbf{or}_{bundle} = d\rho_1 \wedge \cdots \wedge d\rho_{q-1} \wedge d\theta_1 \wedge \cdots \wedge d\theta_{q-1}.$$

Observe that

$$\mathbf{or}_{bundle} = (-1)^{\frac{(q-1)(q-2)}{2}} \mathbf{or}_{complex}.$$

Set

$$\hat{S}_1 := S_1 \cap D_{\vec{m}}.$$

We will denote by PD_c (resp. PD_b) the Poincaré duality isomorphisms

$$H^*_{DR}(\hat{S}_1) \to H^*_{DR}(\hat{S}_1, \partial\hat{S}_1)^* \cong H_*(\hat{S}_1, \partial\hat{S}_1; \mathbb{R}), \quad H^*_{DR}(\hat{S}_1, \partial\hat{S}_1) \to H^*_{DR}(\hat{S}_1)^* \cong H_*(\hat{S}_1, \mathbb{R})$$

induced by the complex (resp. bundle) orientation. More precisely, for every $\omega \in \Omega^k(\hat{S}_1)$ and every $\eta \in \Omega^{\dim \hat{S}_1 - k}(\hat{S}_1)$ such that $\omega \mid_{\partial\hat{S}_1} \equiv 0$ we have

$$\langle \mathbf{PD}(\omega), \eta \rangle = \int_{\hat{S}_1} \omega \wedge \eta = \langle \omega, \mathbf{PD}(\eta) \rangle$$

where $\langle \bullet, \bullet \rangle$ denotes the natural pairing between a vector space and its dual, and the integral should be defined in terms of the chosen orientation.

To proceed further we need the following elementary geometric fact.

Lemma 14.3. *Suppose f is a smooth function defined on the Riemann manifold (M, g) and s is a regular value of f. Then the volume form on $M_s = f^{-1}(s)$ with respect to the induced metric is given by*

$$\frac{1}{|df|_g}\Big(*df \mid_{M_s}\Big).$$

.

Now denote by $\mathbb{T}^q$ the torus $\mathfrak{Im}\,\mathbb{C}^q/\Lambda_{\vec{m}}$. Then T_1 is the codimension one subtorus of $\mathbb{T}^q$ defined by the equation

$$\Big\{\sum_{k=1}^{q}\theta_j = 0\Big\}/\Lambda_{\vec{m}}.$$

The metric on $\mathbb{T}^q$ is induced from the Euclidean metric on $\mathfrak{Im}\,\mathbb{C}^q$ and its volume is

$$\mathrm{vol}\,(\Lambda_{\vec{m}}) = (2\pi\hbar)^q\prod_{k=1}^{q} m_k.$$

Then the Euclidean volume form on $\Theta^{-1}(0)$, the universal cover of T_1, is given by

$$dv_{T_1} = \hbar^{1/2} * d\Theta = \hbar^{1/2}P(d\theta_1,\cdots,d\theta_q)$$

where $*$-denotes the Hodge $*$-operator on the Euclidean space $\mathfrak{Im}\,\mathbb{C}^q$ and P denotes the polynomial in the super-commuting variables $\lambda_1,\cdots,\lambda_q\in\Omega^*(\mathbb{C}^q)$

$$P(\lambda_1,\cdots,\lambda_q) = \sum_{k=1}^{q}(-1)^{k-1}\lambda_1\wedge\cdots\wedge\lambda_{k-1}\wedge\lambda_{k+1}\wedge\cdots\wedge\lambda_q.$$

To compute the volume of T_1 we use the following elementary result.

Lemma 14.4.

$$\mathrm{vol}\,(T_1) = \frac{1}{2\pi d\hbar^{3/2}}\omega_0,\ \ \omega_0 := \mathrm{vol}\,(\mathbb{T}^q),\ \ (\hbar = \frac{1}{q}).$$

Proof Set

$$\Lambda_0 = \Theta^{-1}(0)\cap\Lambda.$$

Consider the sublattice $\Lambda' = \Lambda_0\oplus\langle\mu\rangle\subset\Lambda_{\vec{m}}$, where

$$\mu = 2\pi m_0\hbar\underbrace{(1,1,\cdots,1)}_{q\ times}\in\Lambda_{\vec{m}}$$

and m_0 is the least common multiple of the integers $m_1,\cdots,m_q$. Denote by r the index of Λ' in $\Lambda_{\vec{m}}$. Since $\mu\perp\Lambda_0$ we deduce

$$|\mu|\cdot\mathrm{vol}\,(T_1) = |\mu|\mathrm{vol}\,(\Lambda_0) = \mathrm{vol}\,(\Lambda')$$

$$= r\mathrm{vol}\,(\Lambda_{\vec{m}}) = r\cdot\mathrm{vol}\,(\mathbb{T}^q).$$

so it suffices to determine r. We have the following commutative diagram with exact rows and columns.

$$
\begin{tikzcd}
 & 0 \arrow[d, hook] & 0 \arrow[d, hook] & 0 \arrow[d] & \\
0 \arrow[r, hook] & \Lambda_0 \arrow[r] \arrow[d, hook] & \Lambda_0 \arrow[r] \arrow[d, hook] & 0 \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r, hook] & \Lambda' \arrow[r, hook] \arrow[d, two heads, "\Theta"] & \Lambda_{\vec{m}} \arrow[r, two heads] \arrow[d, two heads, "\Theta"] & \Lambda/\Lambda' \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r, hook] & 2\pi\hbar m_0 q\mathbb{Z} \arrow[r, hook] \arrow[d] & 2\pi\hbar d\mathbb{Z} \arrow[r, two heads] \arrow[d] & \Lambda_{\vec{m}}/\Lambda' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Thus $r = m_0 q/d$ so that

$$\operatorname{vol}(T_1) = \frac{m_0 q}{d|\mu|}\operatorname{vol}(\mathbb{T}^q) = \frac{m_0 q}{d m_0 \hbar\sqrt{q}} = \frac{1}{2\pi d\hbar^{3/2}}\operatorname{vol}(\mathbb{T}^q). \quad \blacksquare$$

Observe that $Q = T^{-1}(1)$ and thus

$$dv_Q = h^{1/2} * dT = \hbar^{1/2} P(d\rho_1, \cdots, d\rho_k)$$

Additionally

$$\int_\Delta \Upsilon^* dv_Q = \int_\Delta dv_Q = \frac{\sqrt{q}}{(q-1)!} = \frac{1}{\hbar^{1/2}(q-1)!}. \tag{14.2}$$

Now observe that $\Upsilon^* dv_Q$ is supported in Δ and thus, along Q, the compactly supported $(q-1)$-form

$$\beta = \hbar^{1/2}(q-1)!\Upsilon^* dv_Q$$

represents the Poincaré dual (in Q) of a point . This form descends to a compactly supported $(q-1)$-form on S_1 representing the Thom class of the fibration

$$\sigma_0 \hookrightarrow \hat{S}_1 \twoheadrightarrow T_1.$$

The $(q-1)$-form dv_{T_1} descends to the volume form on $T_1 \hookrightarrow S_1$. Observe that the cycles $[T_1] \in H_{q-1}(\bar{S}_1)$ and $[\sigma_0] \in H_{q-1}(\hat{S}_1, \partial\hat{S}_1)$ define by integration elements

$$[T_1] \in H^{q-1}_{DR}(\hat{S}_1)^*, \;\; [\sigma_0] \in H^{q-1}_{DR}(\hat{S}_1, \partial\hat{S}_1)^*$$

Using the computation in Lemma 14.4 we deduce.

$$[\sigma_0] = PD_b\Big(\frac{2\pi d\hbar^{3/2}}{\omega_0} dv_{T_1}\Big) \tag{14.3}$$

and

$$[T_1] = PD_b(\beta) \tag{14.4}$$

Suppose m is a common multiple of $m_1, \cdots, m_q$. Observe that the pullback of

$$F_m^* dv_{T_1} \in \Omega^{q-1}(S_1)$$

to $\mathbb{C}^q$ via Ψ is the $(q-1)$-form

$$\hbar^{1/2} P(d\theta_k + 2\pi m\hbar dv_k). \tag{14.5}$$

Since the 1-forms dv_k are identically zero outside the $\Delta \times \mathfrak{Im}\,\mathbb{C}^q$ we deduce that

$$\mathrm{supp}\,(F_m^* dv_{T_1} - dv_{T_1}) \subset \Delta_{\vec{m}}$$

so that

$$F_m^* dv_{T_1} - dv_{T_1} \in H^{q-1}_{DR}(\hat{S}_1, \partial\hat{S}_1).$$

Form the degree $(q-1)$ polynomial

$$V(t) = \hbar^{1/2}\int_\Delta P(d\theta_j + 2\pi t m\hbar dv_j).$$

To proceed further we need the following combinatorial result.

Lemma 14.5. *Suppose* $R = \sum_{j=0}^k a_j t^j \in \mathbb{C}[t]$. *Then*

$$a_k = \frac{1}{k!}(\Delta^k R)(t)\mid_{t=0}$$

where Δ *denotes the finite difference operator*

$$(\Delta R)(t) = R(t+1) - R(t).$$

Proof For every polynomial $R \in \mathbb{C}[t]$ define

$$F_R = F_R[t; z] := \sum_{n\geq 0} R(t+n)z^n \in \mathbb{C}[t]\,[[z]].$$

Observe that and

$$zF_{\Delta R} = (z-1)F_R + R(t).$$

This implies that

$$F_{\Delta R} = q(z)F_R + z^{-1}R, \quad q(z) = \frac{z-1}{z}.$$

We deduce iteratively that

$$F_{\Delta^k R} = q(z)^k + \frac{1}{z}\sum_{j=0}^{k-1} q(z)^{k-1-j}\Delta^j R. \tag{14.6}$$

so that

$$z^k F_{\Delta^k R} = (z-1)^k F_R + \sum_{j=1}^{k-1}(z-1)^{k-1-j}\Delta^j R.$$

Since $\deg \Delta R = \deg R - 1$ we deduce that if $R = \sum_{j=0}^k a_j t^j$ then $\Delta^k R \equiv a_k C_k$, where C_k is an universal constant. It suffices to consider only the case $R = t^k$. Observe that

$$F_{\Delta^k t^k} = \frac{C_k}{1-z}.$$

On the other hand (see e.g.[30, p. 286])

$$F_{t^k}\mid_{t=0} = \sum_{n\geq 0} n^k z^k = \frac{z(z+1)\cdots(z+k)}{(1-z)^{k+1}}.$$

Using (14.6) with $t=0$ we deduce

$$C_k\frac{z^k}{1-z} = \frac{z(z+1)\cdots(z+k)}{(1-z)} + \sum_{j=1}^{k-1}(z-1)^{k-1-j}\Delta^j R(0)$$

so that

$$C_k z^k = z(z+1)\cdots(z+k) + \sum_{j=1}^{k-1}(z-1)^{k-j}\Delta^j R(0).$$

By setting $z=1$ we deduce $C_k = k!$. ■

We now apply Lemma 14.5 to the degree $(q-1)$ polynomial $V(s)$. Its leading coefficient is

$$a_{q-1} = (2\pi m\hbar)^{q-1}\cdot \hbar^{1/2}\int_\Delta P(dv_k;\ \ 1\leq k\leq q) = (2\pi m\hbar)^{q-1}\mathrm{vol}\,(\Delta) \overset{(14.2)}{=} \frac{(2\pi m\hbar)^{q-1}}{\hbar^{1/2}(q-1)!}$$

We deduce

$$(\Delta^{q-1}V)(0) = q(2\pi m)^{q-1}\hbar^{\frac{2q-1}{2}} = (2\pi m)^{q-1}\hbar^{\frac{2q-3}{2}}.$$

Now observe that

$$(\Delta^{q-1}V)(0) = \int_{\sigma_0}\left(\sum_{j=0}^{q-1}(-1)^j\binom{q-1}{j}F^*_{(q-1-j)m}dv_{T_1}\right).$$

where as we have explained, the integrand is a compactly supported $(q-1)$ form on $\hat{S}_1$ which, according to the Thom isomorphism theorem, has to be a multiple of the Thom class β. For obvious reasons we set

$$\mathbf{var}_m^{q-1}(dv_{T_1}) := \sum_{j=0}^{q-1}(-1)^j\binom{q-1}{j}F^*_{(q-1-j)m}dv_{T_1}.$$

We deduce that

$$\mathbf{var}_m^{q-1}(dv_{T_1}) = (2\pi m)^{q-1}\hbar^{\frac{2q-3}{2}}\beta.$$

Multiplying both sides by $\frac{2\pi d\hbar^{3/2}}{\omega_0}$, $\omega_0 = \mathrm{vol}_q(\mathbb{T}^q) = (2\pi\hbar)^q m_1\cdots m_q$ and then using (14.3) and (14.4) we deduce

$$\mathbf{var}_m^{q-1}(\mathbf{PD}_b^{-1}[\sigma_0]) = \mathbf{var}_m^{q-1}(\frac{2\pi d\hbar^{3/2}}{\omega_0}dv_{T_1}) = \frac{m^{q-1}d}{(m_1\cdots m_q}\beta = \mathbf{PD}_b^{-1}[T_1]$$

We have thus proved the following generalization of the local Picard-Lefschetz formula.

Theorem 14.6. *Orient the fiber S_1 as a complex manifold. Suppose m is a common multiple of $m_1,\cdots,m_q$. If we set*

$$\mathbf{var}_m^{q-1}([\sigma_0]) := \mathbf{PD}_c\circ\mathbf{var}_m^{q-1}\circ\mathbf{PD}_c^{-1}([\sigma_0])$$

then

$$\mathbf{var}_m^{q-1}([\sigma_0]) \in H_{q-1}(\hat{S}_1;\mathbb{R})$$

and moreover

$$\mathbf{var}_m^{q-1}([\sigma_0]) = \mathbf{PD}_b\circ\mathbf{var}_m^{q-1}\circ\mathbf{PD}_b^{-1}([\sigma_0]) = \frac{dm^{q-1}}{m_1\cdots m_q}[T_1].$$

In particular if "$\cdot$" denotes the intersection pairing with respect to the complex orientation we deduce

$$[\sigma_0]\cdot\mathbf{var}_m^{q-1}([\sigma_0]) = (-1)^{(q-1)(q-2)/2}\frac{dm^{q-1}}{m_1\cdots m_q}.$$

§14.3 Reconstructing the Milnor fiber and the monodromy

We now patch-up the local information we obtained so far. Define

$$\mathfrak{H} := \Big\{ x \in X;\ \prod_{i=1}^{\nu} r_i(x)^{m_i} = 1 \Big\}$$

where the function $r_i(x)$ is defined in (14.1) and essentially measures the distance away from D_i, $\forall i = 1, \cdots, \nu$. Upon rescaling the sections s_i we can assume that $\mathcal{R}$ lies in the neighborhood $\mathcal{N}$ of X_0 defined by

$$\mathcal{N} = \Big\{ x \in X;\ \min_{1 \leq i \leq \nu} r_i^{m_i}(x) \leq e \Big\}.$$

Define

$$\rho_i : X \setminus X_0,\ \ \rho_i = \frac{1}{\nu}\Big(1 - m_i \log r_i \Big).$$

The functions ρ_i define

$$\Phi : X \setminus X_0 \to \mathbb{R}^{\nu},\ \ x \mapsto (\rho_1(x), \cdots, \rho_{\nu}(x)).$$

Its image lies in the hyperplane

$$Q_{\nu} = \Big\{ (\rho_1, \cdots, \rho_{\nu});\ \sum_{i=1}^{\nu} \rho_i = 1 \Big\}.$$

We denote by Δ_{ν} the simplex

$$\Delta_{\nu} = \Big\{ (\rho_1, \cdots, \rho_{\nu}) \in Q_{\nu};\ \ \rho_i \geq 0,\ \ \forall i = 1, \cdots, \nu \Big\}.$$

As in the previous section we have a retraction

$$\Upsilon : Q_{\nu} \to Q_{\nu},\ \ \vec{\rho} \mapsto \Upsilon(\vec{\rho}) = (\upsilon_1(\vec{\rho}), \cdots, \upsilon_{\nu}(\vec{\rho}))$$

of Q_{ν} onto Δ_{ν} and we set

$$\tau_i := \rho_i \circ \Upsilon \circ \Phi : \mathcal{R} \to [0, 1].$$

Note that

$$\sum_{i=1}^{\nu} \tau_i = 1.$$

We get a map

$$\vec{\tau} : \mathfrak{H} \to \Delta_{\nu}$$

whose image is the simplicial subcomplex $\mathcal{K} \subset \Delta_{\nu}$ defined by

$$[i_1, \cdots, i_m] \in \mathcal{K} \Longleftrightarrow D_{i_1} \cap \cdots \cap D_{i_m} \neq \emptyset.$$

The faces of $\mathcal{K}$ are labeled by subsets of $\{1, \cdots, \nu\}$, and we will denote them by the symbols Δ_I. We will identify it with the $|I|-1$ simplex

$$\Delta_I := \Big\{ \vec{x} \in \mathbb{R}_+^I;\ \sum_{i\in I} x_i = 1 \Big\}.$$

When $\dim X = 2$, $\mathcal{K}$ coincides with the resolution graph. To each face $I \in \mathcal{K}$ we associate the closed subset

$$\mathfrak{H}_I := \Big\{ x \in \mathfrak{H};\ \ r_i(x) \leq e,\ \ \forall i \in I \Big\}$$

For $t \in \mathbb{C}$, $|t| = 1$

$$S_{I,t} := X_t \cap \mathfrak{H}_I,\ \ S_{I,t}^{\dagger} = \Big\{ x \in S_I;\ \ r_k(x)^{m_k} \geq e,\ \ \forall k \notin I \Big\},$$

$$N_I = \Big\{ x \in X;\ \ r_i(x)^{m_i} \leq e,\ \ \forall i \in I \Big\},\ \ D_I^{\dagger} = \Big\{ x \in D_I;\ \ r_k(x)^{m_k} \geq e,\ \ \forall k \notin I \Big\}.$$

For simplicity we drop the subscript t when $t = 1$. We can explicitly describe the structure $S_I^{\dagger}$. Set

$$E_I = \bigoplus_{i\in I} L_i,\ \ F_I := \bigoplus_{i\in I} L_i^{\otimes m_i},\ \ P_I := \bigotimes_{i\in I} L_i^{m_i} = \det F_I.$$

E_I and F_I are rank $|I|$ holomorphic vector bundles equipped with the hermitian metrics

$$h_I = \oplus_{i\in I} h_i,\ \ g_I = \oplus_{i\in I} h_i^{m_i}.$$

Denote by

$$\pi_I : E_I\mid_{D_I} \rightarrow D_I$$

the canonical projection. Using the metric h_I on E_I and the associated Chern connection we can smoothly identify N_I with a neighborhood U_I of the zero section of $E_I\mid_{D_I}$. We denote by $\mu_I \rightarrow N_I$ this diffeomorphism. We set

$$U_I^{\dagger} = \pi_I^{-1}(D_I^{\dagger}),\ \ N_I^{\dagger} = \mu_I(U_I^{\dagger}).$$

We have a natural holomorphic map

$$\mathfrak{p}_I : E_I\mid_{D_I} \rightarrow P_I\mid_{D_I}$$

induced by the the tensorization maps $\bullet^{\otimes m_i} : L_i \rightarrow L_i^{m_i}$. More precisely

$$\mathfrak{p}_I : E_I \ni (p;\ (z_i)_{i\in I}) \mapsto (p;\ \prod_{i\in I} z_i^{m_i}) \in P_I.$$

Above, p denotes a point in D_I while z_i denotes a local coordinate in the fiber $L_{i,p}$.

Using the diffeomorphism μ_i we can identify $s_I = \oplus_{i\in I} s_i$ with the tautological section $u_I = \oplus_{i\in I} u_i$ of $\pi_I^* E_I \to U_I$. Set

$$f_I = \prod_{i\in I} u_i^{m_i}.$$

Now observe several things.

• The restriction of P_I to $D_I^\dagger$ is trivial since it is isomorphic to the restriction of P to D_I°. Denote by $\xi_I : D_I^\dagger \to P_I$ the tautological section defined by this trivialization. We identify it with a function $\xi_I : D_I^\dagger \to \mathbb{C}^*$ via the isomorphism $P_I \cong P \cong \underline{\mathbb{C}}$ over $D_I^\dagger$.

• $\Sigma_I^\dagger := \mu_I^{-1}(S_I^\dagger) \subset U_i$ is mapped by $\mathfrak{p}_I$ onto $v_I(D_I^\dagger)$.

• The map $\mathfrak{p}_I$ factors through a map

$$\hat{\mathfrak{p}}_I : E_I \to P_I' = \bigotimes_{i\in I} L_i^{\frac{m_i}{d_I}}$$

defined similarly. The power map

$$\bullet^{\otimes d_I} : P_I' \to P_I$$

is a degree d_I cover branched over the zero section of P_I. We thus have the following commutative diagram.

$$\begin{tikzcd} E_I \arrow[r, "\hat{\mathfrak{p}}_I"] \arrow[dr, "\mathfrak{p}_I"'] & P_I' \arrow[d, "\bullet^{\otimes d_I}"] \\ & P_I \end{tikzcd}$$

We conclude that the image of $\Sigma_I^\dagger$ via $\hat{\mathfrak{p}}_I$ is a d_I-fold cover of $v_I(D_I^\dagger) \cong D_I^\dagger$. We denote this cover by $C_I^\dagger$ and by $\lambda_I : C_I^\dagger \to D_I^\dagger$ the covering projection. The computations in the preceding section show that the map

$$\hat{\mathfrak{p}}_I : \Sigma_I^\dagger \to C_I^\dagger$$

is a fibration. We have the commutative diagram.

$$\begin{tikzcd} S_I^\dagger \arrow[rr, two heads, "\hat{\mathfrak{p}}_I"] \arrow[dr, "\pi_I"'] & & C_I^\dagger \arrow[dl, "\lambda_I = d_I:1"] \\ & D_I^\dagger & \end{tikzcd} \tag{14.7}$$

The fiber of $\hat{\mathfrak{p}}_I$ over $c \in C_I^\dagger$ can be explicitly described as follows.

First, identify each point $c \in C_I^\dagger$ with a pair $(p, c(p))$, where $p = \lambda_I(c) \in D_I^\dagger$ and $c(p) \in \mathbb{C}^*$ satisfies $c(p)^{d_I} = \xi_I(p)$. Then $\hat{\mathfrak{p}}_I^{-1}(c)$ is one the connected components of

$$\Big\{ v \in E_{I,p};\ \prod_{i\in I} s_i(\mu_I(v))^{m_i} = \Big(\prod_{k\notin I} s_k(\mu_I(v))^{m_k}\Big)^{-1} \Big\},$$

namely the one parametrized by $c(p)$. Now observe that the unit circle bundles $S_1(L_i) \hookrightarrow L_i$, $i \in |I|$ define a sub-bundle $\mathcal{T}_I \hookrightarrow E_I$ with fibers $|I|$-dimensional real tori. The space $E_I \setminus D_I$ is the total space of a fibration

$$(0,\infty)^{|I|} \hookrightarrow (E_I \setminus D_I) \to \mathcal{T}_I. \tag{14.8}$$

Then the fiber $\hat{\mathfrak{p}}_I^{-1}(p)$ is the total space of the restriction of the above fibration over a certain codimension 1 sub-torus of $\mathcal{T}_{I,p}$. The diagram (14.7) can be refined as follows.

$$\begin{tikzcd}
\Delta_I \arrow[r, hook] & S^\dagger \cap E_{I,p} \arrow[r, hook] \arrow[d, two heads] & (S_I^\dagger, S_I \cap E_{I,p}) \arrow[r, two heads, "\hat{\mathfrak{p}}_I"] \arrow[dr, "\pi_I"'] & (C_I^\dagger, \{c_1, \cdots, c_{d_I}\}) \arrow[d, "\lambda_I = d_I:1"] \\
& T^{I-1} \arrow[rr, two heads] & & (D_I^\dagger, p)
\end{tikzcd} \tag{14.9}$$

where $T^{I-1} \subset (\mathbb{C}^*)^I$ denotes the union of $(|I|-1)$-dimensional tori defined by

$$\Big\{ \vec{z} \in (\mathbb{C}^*)^I;\ \ |z_i| = 1,\ \ \forall i \in I,\ \prod_{i\in I} z_i^{m_i} = 1 \Big\}.$$

Suppose now that Δ_I is a face of Δ_J. Define

$$S_{I,J} = S_I^\dagger \cap S_J^\dagger$$

$$= \Big\{ x \in S_1;\ \ r_i(x) \le e,\ \ \forall i \in I,\ \ r_j(x) = e,\ \ \forall j \in J \setminus I,\ \ r_k(x) \ge e,\ \ \forall k \notin I \Big\}$$

We denote by $\Gamma_{I,J}$ the inclusion $S_{I,J} \hookrightarrow S_I^\dagger$ and by $\Gamma_{J,I}$ the inclusion $S_{I,j} \hookrightarrow S_J^\dagger$.

To understand the nature of these gluing maps we need to better describe the structure of $S_{I,J}$. The inclusion $\Gamma_{J,I}$ is obtained by viewing $S_{I,J}$ as a subset of the total space of the bundle

$$\pi_J : E_J \mid_{D_J^\dagger} \to D_J^\dagger.$$

The neighborhood N_J of D_J in X can be identified with a neighborhood U_J of the zero section of $E_J \mid_{D_J} \to D_J$. The part of the divisor D_I which intersects N_J can be identified either

- with a part of the total space of the splitting sub-bundle $E_{J\setminus I} \hookrightarrow E_J$ or

• with the zero section of $\pi^*_{J\setminus I}E_I \to E_{J\setminus I}\mid_{D_J^\dagger}$.

Set

$$T_{I,J} = D_I^\dagger \cap D_J\dagger.$$

We view $T_{I,J}$ as the $(|J|-|I|)$-torus sub-bundle of the totally split bundle $E_{J\setminus I} \to D_J^\dagger$ defined by the equations $r_j = e$, $\forall j \in J \setminus I$. We have the following commutative diagram

$$\begin{tikzcd}
\pi^*_{J\setminus I}(E_I|_{D_J^\dagger}) \supset S_{I,J} \arrow[d, twoheadrightarrow, "\hat{\mathfrak{p}}_I"'] \arrow[dr, twoheadrightarrow, "\pi_I"] & S_{I,J} \arrow[l, hook', "\Gamma_{I,J}"'] \arrow[r, hook, "\Gamma_{J,I}"] & S_J^\dagger \subset E_J \arrow[d, "\hat{\mathfrak{p}}_J"] \arrow[dl, twoheadrightarrow, "\pi_J"'] \\
C_{(I,J)} \arrow[r, twoheadrightarrow, "d_I:1"] & T_{I,J} & C_{(J,I)} \arrow[l, twoheadrightarrow, "d_J:1"']
\end{tikzcd}$$

where

$$C_{(J,I)} = \lambda_J^{-1}(T_{I,J}),\;\; C_{(I,J)} = \lambda_I^{-1}(T_{I,J}).$$

We want to discuss a bit the coverings

$$C_{(J,I)} \xrightarrow{d_J:1} T_{I,J} \xleftarrow{d_I:1} C_{(I,J)}.$$

The covering $C_{(I,J)} \xrightarrow{d_I:1} T_{I,J}$ is obtained from the $d_I : 1$ branched cover

$$P_I' \mid_{T_{I,J}} \to P_I \mid_{T_{I,J}} \cong (P_I')^{d_I} \mid_{T_{I,J}}$$

by restricting it over the nowhere vanishing section $s_I := \bigotimes_{i\in I} s_i^{m_i}$ of $P_I \mid_{T_{I,J}}$. The covering $C_{(J,I)} \xrightarrow{d_J:1} T_{I,J}$ is obtained from the $d_J : 1$ branched covering

$$P_J' \mid_{T_{I,J}} \to P_J' \mid_{T_{I,J}}$$

by restricting it over the nowhere vanishing section s_J of P_J. Observe that we have a natural isomorphism

$$(P_I')^{d_I/d_J} \mid_{T_{I,J}} \to P_J$$

described as follows. Trivialize $P_I' \mid_{T_{I,J}}$ using the nowhere vanishing section

$$s_I' := \bigotimes_{i\in I} s_i^{\frac{m_i}{d_I}}.$$

Then the above isomorphism is defined by

$$u(x)(s_I')^{\frac{d_I}{d_J}} \;\; \mapsto \;\; u(x)\cdot s_J' := u(x)\cdot \bigotimes_{j\in J} s_j^{\frac{m_j}{d_J}}.$$

We denote by $\lambda_{J,I}$ the $\frac{d_I}{d_J}:1$ map $P'_I \to (P'_I)^{d_I/d_J} \cong P'_J$ we obtain via this isomorphism. We get the following commutative diagram.

$$\begin{tikzcd} C_{(I,J)} \arrow[rr, two heads, "\lambda_{J,I}=\frac{d_I}{d_J}:1"] \arrow[dr, "\lambda_I=d_I:1"'] & & C_{(J,I)} \arrow[dl, "\lambda_J=d_J:1"] \\ & T_{I,J} & \end{tikzcd}$$

The image of $S_{I,J}$ in $S_J^\dagger$ via the inclusion $\Gamma_{J,I}$ is is described by the equations $r_k = e$, $k \in J \setminus I$. In less rigorous but more intuitive terms, $\Gamma_{J,I}(S_{I,J})$ looks locally like

$$\underbrace{\Bigl((\Delta_I \subset \Delta_J) \times T^{|J|-1}\Bigr)}_{fiber} \times \underbrace{d_J \mathcal{O}_J}_{base},$$

where $\mathcal{O}_J$ is a small open set in $D_J^\dagger$ and $d_J\mathcal{O}_J$ denotes the disjoint union of d_J copies of $\mathcal{O}_J$. This shows that the the total space of the covering $\lambda_I : C_{(I,J)} \to T_{I,J}$ has d_J connected components. Then $\Gamma_{I,J}(S_{I,J})$ looks locally like

$$\underbrace{(\Delta_I \times T^{|I|-1})}_{fiber} \times \underbrace{d_J(T^{|J\setminus I|} \times \mathcal{O}_J)}_{base}.$$

The nature of the diffeomorphism $\Gamma_{I,J}(S_{I,J} \to \Gamma_{J,I}(S_{I,J})$ is now intuitively clear. We see that X_1 is obtained by a process similar in spirit to the operation of plumbing.

We interrupt for a while the flow of arguments to present a low dimensional example which we hope will display the strengths and limitations of the constructions we introduced so far.

Example 14.7. (Working Example. Part II.) Consider $n = 2$ and $f : \mathbb{C}^2 \to \mathbb{C}$, is the resolution of $y^2 - x^5$ we described in Example 14.1 (see Figure 14.2). Then $\nu = 5$ with $D_k = E_k$, $1 \le k \le 4$, $D_5 = \hat{C}$. The simplicial complex $\mathcal{K}$ is precisely the resolution graphs of the singularity depicted in Figure 14.3.

There are 4 crossings and near them f is equivalent to one of the monomials

$$z_1^2 z_2^4, \;\; z_1^4 z_2^5, \;\; z_1^5 z_2^{10}, \;\; z_1^{10} z_2.$$

$D_1^\dagger$ is a sphere E_1 with one hole, $D_2^\dagger$ is the sphere E_2 with two holes, $D_3^\dagger$ is the sphere E_3 with one hole, $D_4^\dagger$ is the sphere E_4 with three holes, and $D_5^\dagger$ is a disk with one hole. We can now reconstruct S_t, $0 < t \ll 1$. Consider small polydisks $\Delta_1, \cdots, \Delta_4$ centered at the crossing points as depicted in Figure 14.2. We begin by considering one by one each the five pieces $S_I^\dagger$, $|I| = 1$.

- $k = 1$. $S_1^\dagger$ is a $m_1 = 2$-cover of $D_1^\dagger$. tThus S_t^1 is a disjoint union of two disks.
- $k = 2$. $S_2^\dagger$ is a $m_2 = 4$-cover of the sphere with two holes $D_2^\dagger$. It thus consists of 1, 2 or 4 distinct cylinders. It must consist of *two* cylinders to be attached inside Δ_1 to the boundaries of the two disks which form $S_1^\dagger$.

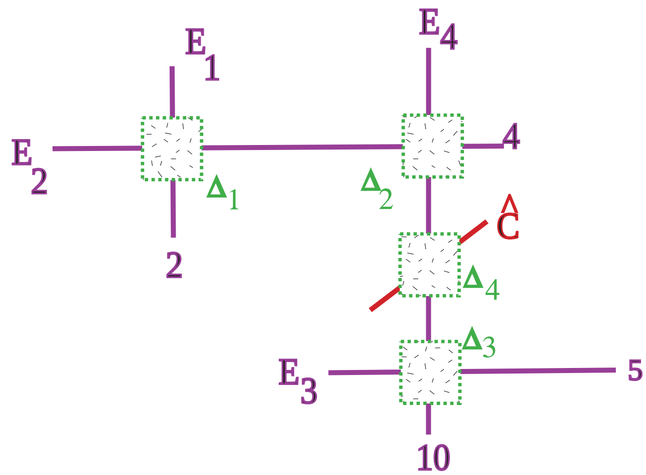

Figure 14.2: *Dissecting the resolution of* $y^2 - x^5 = 0$.

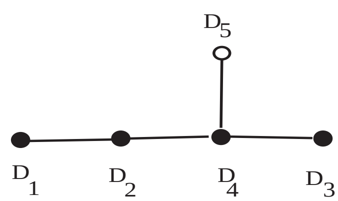

Figure 14.3: *The resolution graph of* $y^2 - x^5 = 0$.

• $k = 3$. S_t^3 is a $m_3 = 5$-cover of the sphere with one hole $D_3^\dagger$. It is thus the disjoint union of five disks.

• $k = 4$. $S_4^\dagger$ is a $m_4 = 10$-cover of the sphere E_4 with three holes. Moreover its Euler characteristic is ten times the Euler characteristic of the twice punctured disk so that $\chi(S_4^\dagger) = -10$.

• $k = 5$. $S_5^\dagger$ is diffeomorphic to the disk $\hat{C}$ with a hole around the intersection point with E_4. It is thus a disk.

There are four pieces $S_I^\dagger$, $|I| = 2$, which we label by $C_j = S_t \cap \Delta_j$, $j = 1, \cdots, 4$. C_1 consists of $\gcd(2,4) = 2$ cylinders. C_2 consists of $\gcd(4,10) = 2$ cylinders, C_3 consists of one cylinder while C_4 consists of $\gcd(10,5) = 5$ cylinders. The boundary of $S_3^\dagger$ consists of three parts. A part to be connected with the two cylinders forming C_2, a part to be glued with the disk S_t^5 and a part to be glued with the five cylinders forming C_4. We obtain the situation depicted in Figure 14.4. Observe that the genus of S_t is $2 = \frac{(2-1)(5-1)}{2}$, that is half the Milnor number of the singularity $y^2 - x^5 = 0$. This is not an accident.

As one can imagine, the situation in higher dimensions will be much more complicated. Even the determination of the multiplicities of the divisors D_i is much more involved. We will have more to say about this when we discus toric manifolds.

Exercise 14.8. Consider the an irreducible germ C of planar curve with an isolated sin-

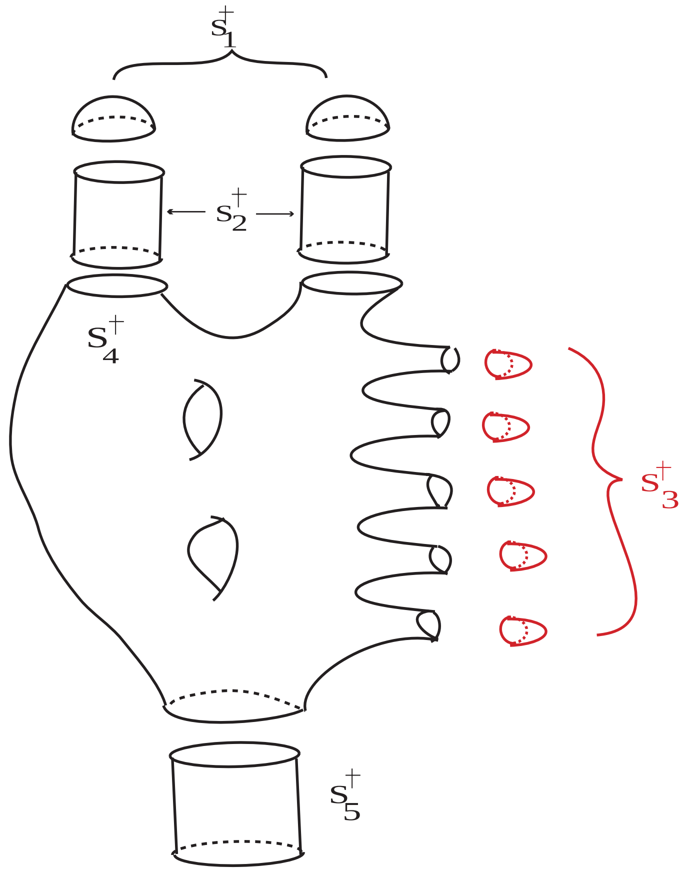

Figure 14.4: *Reconstructing the Milnor fiber.*

gularity at $(0,0)$, denote by Γ_C its resolution graph and by $\mathbf{V}(C)$ the set of its vertices. To each $v \in \mathbf{V}(C)$ it corresponds a component of the exceptional divisor with multiplicity $m(v)$. Prove that the Euler characteristic of the Milnor fiber is given by the formula

$$\sum_{v \in \mathbf{V}(C)} m(v) \cdot \Big(2 - \deg(v)\Big).$$

We now return to the general situation. We want to explain how we can obtain information about the monodromy. Again we use the identifications $\mu_I : U_I \to N_I$. The bundle $E_I\,|_{D_I}$ is equipped with a natural periodic $\mathbb{R}$-action described as follows. If

$$x = (\oplus_{i \in I} v_i, p) \in E_{I,p}, \quad , v_i \in L_{i,p}, \quad p \in D_i, \quad t \in \mathbb{R}$$

then

$$\exp(\mathbf{i}t) \cdot x = \Big(\oplus_{i \in I} \exp(\frac{2\pi \mathbf{i}t}{m_i}) v_i, \ p\Big) \in E_{I,p}.$$

Now set $w(t) = \exp(2\pi \mathbf{i}t)$ and define

$$F_{I,t} : S_I^\dagger \to S_{I,w(t)}^\dagger$$

so that if $x = \mu_I(\oplus_{i\in I} v_i, p) \in S_I^\dagger$

$$F_{I,t}(x) = \mu_I\Big(\oplus_{i\in I} \exp(\frac{2\pi\mathbf{i}t\tau_i(x)}{m_i})v_i,\ p\Big) \in E_{I,p}.$$

Let us observe that that whenever $I \subset J$ we have

$$F_{I,t}(x) = F_{J,t}(x)$$

for every x in the overlap $T_{I,J} = S_I^\dagger \cap S_{J,\dagger}$. Indeed on the overlap we have $r_k = e$, $\forall k \in J\setminus I$ so that $\tau_k(x) = 0$ for all $k \in J \setminus I$. This shows we have a well defined map

$$F_t : X_1 \to X_{w(t)}$$

The monodromy is given by F_1.

Example 14.9. (Working example. Part III.) We continue to look at the situation explained in Example 14.7. Each of the pieces $S_k^\dagger$ is a cyclic cover of $D_k^\dagger$ of degree m_k. In the interior of $S_k\dagger$ the action of F_t generates the action of the cyclic deck groups of these covers. We consider the two cases separately.

- $S_1^\dagger$, $m_1 = 2$. F_1 flips the two connected components of $S_1^\dagger$.
- $S_2^\dagger$, $m_2 = 4$. F_1 interchanges the two components of $S_2^\dagger$ but its action in the interior is not trivial (see Figure 14.5).
- $S_3^\dagger$, $m_3 = 5$. F_1 cyclically permutes the five components $C_1, \cdots, C_5$ but the transition $C_i \to C_{i+1}$ is followed by a $2\pi/5$ rotation of the disk C_{i+1} (see Figure 14.5
- $S_4^\dagger$, $m_4 = 10$. $S_4^\dagger$ is a 10-fold cover of the twice punctured disk $D_4^\dagger$ which has three boundary components which we label by $\gamma_2, \gamma_3, \gamma_5$ (see Figure 14.6). γ_k is covered by $S_4^\dagger \cap S_k^\dagger$, $k = 2, 3, 5$. The fundamental group of this twice punctured disk is a free group on two generators γ_2, γ_3. We have a monodromy representation

$$\phi : \pi_1(D_4^\dagger) \to \mathrm{Aut}\,(S_4^\dagger \stackrel{\lambda_4}{\to} D_4^\dagger) \cong \mathbb{Z}/10\mathbb{Z}.$$

We identify this automorphism group with the group of 10-th roots of 1. Fix a primitive 10-th root ζ of 1. Since $S_4^\dagger \cap S_2^\dagger$ has two components we deduce

$$\phi(\gamma_2) = \zeta^5.$$

We conclude similarly that $\phi(\gamma_3) = \zeta^2$. Since $\gamma_5 = \gamma_3\gamma_2$ we deduce $\phi(\gamma_5) = \zeta^7$) so that $\phi(\gamma_5)$ is a generator of the group of deck transformations of the covering λ_4. This also helps to explain the action of F_1 on $S_5^\dagger$ depicted at the bottom of Figure 14.5.

The most complicated to understand is the action of the deck group of λ_4. To picture it geometrically it is convenient to remember that it is part of the general fiber of the map $f(x, y) = y^2 - x^5$,

$$X_\varepsilon := \Big\{(x, y) \in \mathbb{C}^2,\ \ y^2 - x^5 = \varepsilon,\ \ ,\ \ |x|^2 + |y|^2 \le 1\Big\}.$$

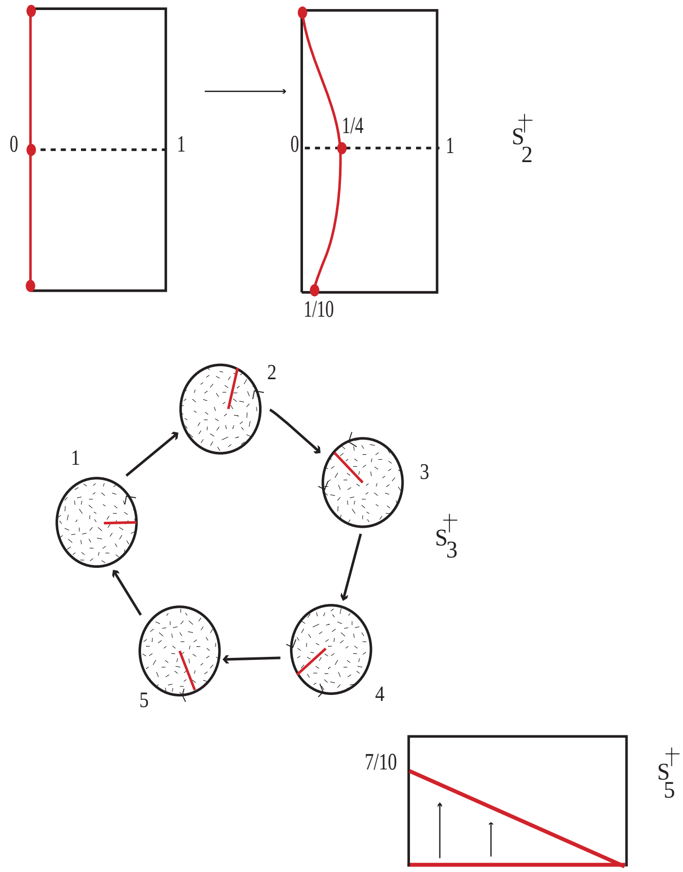

Figure 14.5: *The action of F_1 on $S_2^\dagger$, $S_3^\dagger$ and $S_5^\dagger$.*

We already know its is a Riemann surface of genus 2 with one boundary component

$$\partial X_\varepsilon = \Big\{(x, y) \in X_\varepsilon, \ \ |x|^2 + |y|^2 = 1\Big\}$$

The boundary is a nontrivially embedded $S^1 \hookrightarrow S^3$ and in fact it represents the $(2, 5)$-torus knot (see Figure 14.7). There is a natural action of the cyclic group $C_{10} := \mathbb{Z}/10\mathbb{Z}$ on X_ε given by

$$\zeta \cdot (x, y) = (\zeta^2 x, \zeta^5 y).$$

The points on the surface where $x = 0$ or $y = 0$ have nontrivial stabilizers. The hyperplane $x = 0$ intersects the surface in two points given by

$$y^2 = \varepsilon.$$

The stabilizers of these points are cyclic groups of order 5. The hyperplane $y = 0$ intersects the surface in 5 points given by

$$x^5 = \varepsilon.$$

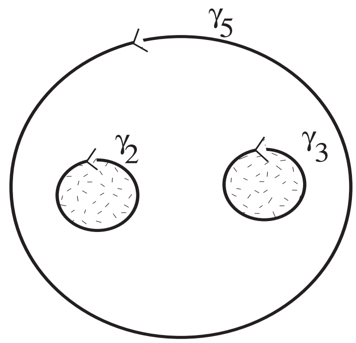

Figure 14.6: $S_4^\dagger$ *is a cyclic* 10*-fold cover of the twice punctured disk* $D_4^\dagger$.

The stabilizers of these points are cyclic groups of order 2. Now remove small C_{10}-invariant disks centered at these points. The Riemann surface we obtained is equivariantly diffeomorphic to $S_4^\dagger$. To visualize it it is convenient to think of X_ε as a Seifert surface of the $(2,5)$-torus knot. It can be obtained as follows (see [36] for an explanation).

Consider two regular 10-gons situated in two parallel horizontal planes in $\mathbb{R}^3$ so that the vertical axis is a common axis of symmetry of both polygons. Assume the projections of their vertices on the xy-plane correspond to the 10-th roots of 1 and the $180°$ rotation about the y-axis interchanges the two polygons. Label the edges of both of them with numbers from 1 to 10 so that the edges symmetric with respect to the xy-plane are labeled by identical numbers. We get five pairs of parallel edges (drawn in red in Figure 14.7) labeled by identical pairs of even numbers. To each such pair attach a band with a half-twist as depicted in Figure 14.7. Remove a small disk from the middle of each of the attached twisted bands and one disk around the center of each of the polygons. We get a Riemann surface with the desired equivariance properties.

Denote by $\mathbb{Z}[C_{10}]$ the integral group algebra of C_{10},

$$\mathbb{Z}[C_{10}] \cong \mathbb{Z}[t]/(t^{10}-1).$$

The Abelian group $G := H_1(S_4^\dagger)$ has a natural $\mathbb{Z}[C_{10}]$-module structure. To describe it we follow a very elegant approach we learned from Frank Connolly. Denote by M the algebra $\mathbb{Z}[C_{10}]$ as a module over itself. Also we denote by M_0 the trivial $\mathbb{Z}[C_{10}]$-module $\mathbb{Z}$.

First recall that G is the abelianization of $\pi_1(S_4^\dagger)$. $S_4^\dagger$ is a 10-fold cover of the twice punctured disk $D_4^\dagger$ and thus $\pi_1(S_4^\dagger)$ is the kernel of the morphism

$$\phi : \pi_1(D_4^\dagger) \to C_{10},\ \ \gamma_2 \mapsto \zeta^5,\ \ \gamma_3 \mapsto \zeta^2.$$

$\pi_1(D_4^\dagger)$ is a free group of rank 2 generated by γ_2 and γ_3. We want to pick a different set of generators

$$x = \gamma_2\gamma_3,\ \ y = \gamma_2 x^5.$$

They have the property that $\phi(y) = 1$ and $\phi(x)$ is the generator $\rho = \zeta^7$ of C_{10}. Then $K := \ker\phi$ is a free group of rank $\operatorname{rank}_{\mathbb{Z}} H_1(S_4^\dagger) = 11$. As generators of K we can pick

$$a = x^{10},\ b_j = x^j y x^{10-j},\ \ j = 0, \cdots, 9.$$

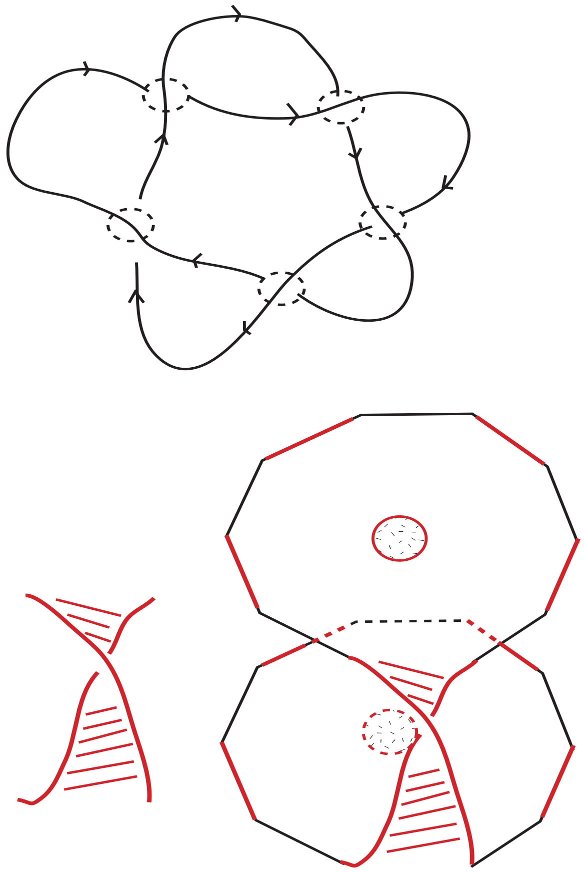

Figure 14.7: *A* $(2,5)$*-torus knot and its spanning Seifert surface.*

From the short exact sequence

$$1 \hookrightarrow K = \langle a;\ b_j,\ \ j = 0, \cdots, 9\rangle \hookrightarrow \pi_1(D_4^\dagger) = \langle x, y\rangle \twoheadrightarrow C_{10} \twoheadrightarrow 1.$$

we deduce that C_{10} acts on K by conjugation. For every $k \in K$ we denote by $[k]$ its image in the abelianization $K/[K,K] = G$. Observe that

$$\rho \cdot [a] = [x \cdot x^{10} \cdot x^{-1}] = [a],$$

and

$$\rho \cdot [b_j] = [x \cdot x^j y x^{10-j} x^{-1}] = b_{j+1},\ \ \forall j = 0, \cdots, 8.$$

Finally

$$\rho \cdot b_9 = [x^{10} y] = [x^{10} y x^{10} x^{-10}] = [a] + [b_0] - [a] = [b_0].$$

This shows that G is isomorphic as a $\mathbb{Z}[C_{10}]$-module to $M_0 \oplus M$.

TO BE CONTINUED....

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