Proof in Mathematics
An Introduction

James Franklin and Albert Daoud

This book provides a short and straightforward introduction to the essential core of mathematics: proof. The book features:

- Brief discussions of the nature and necessity of proof
- Simple explanations of the basic proof techniques
- Immediate application to familiar mathematical material
- Numerous graded exercises
- Fully worked solutions to selected exercises
- A compelling, clear presentation

Introduction to Proofs in Mathematics (Prentice Hall of Australia, 1988), by the same authors, was warmly received around the world:

"Delightfully written... "
- *Mathematics Teacher* (USA) 82, (December 1989)

"The language is easy to read and stimulating... A definite step in the right direction."
- *The Mathematical Gazette* (UK), vol. 73 (Oct 1989)

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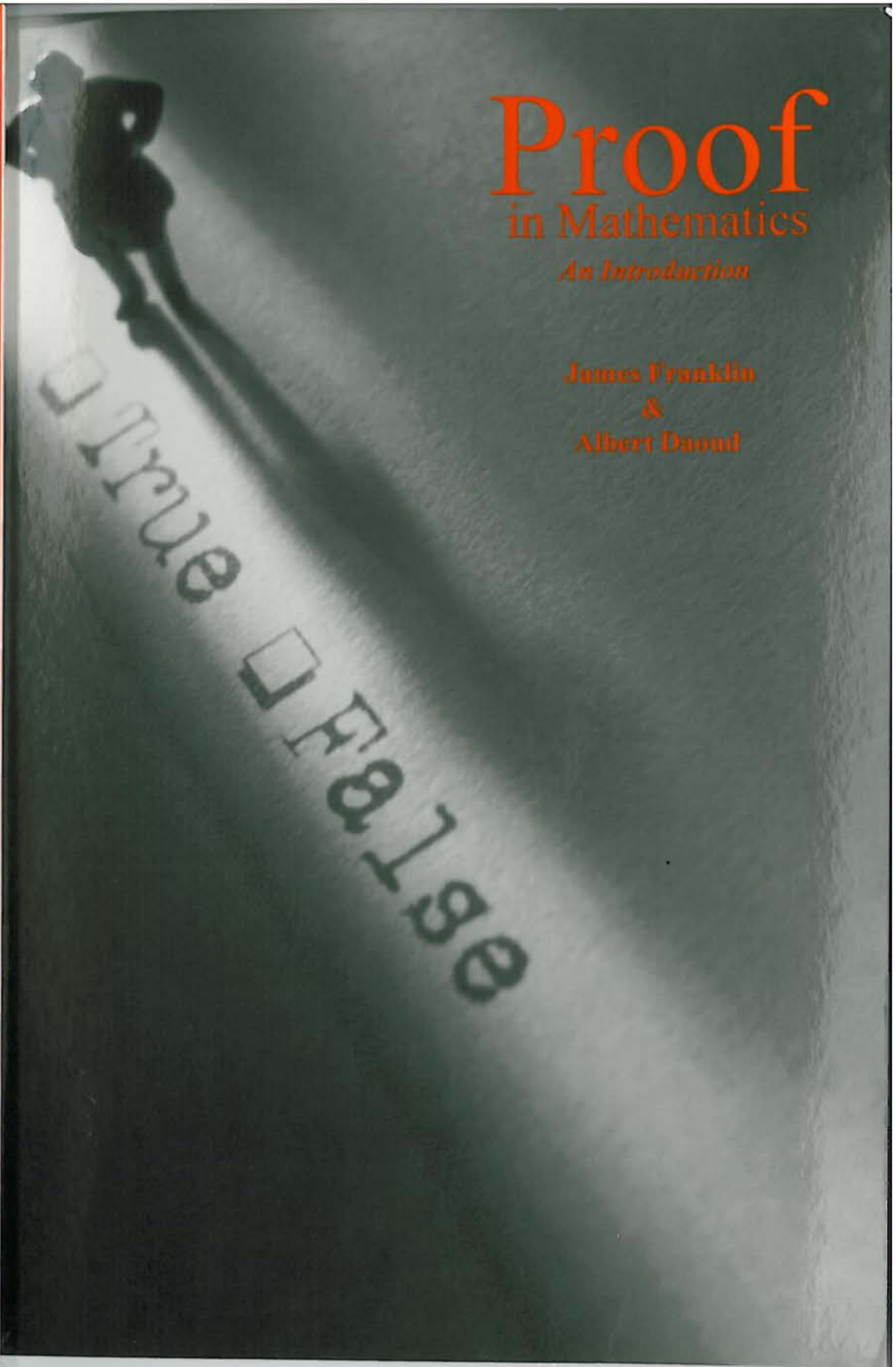

PROOF in MATHEMATICS

An Introduction

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Preface

University teachers of mathematics agree that their students' lack of knowledge is not uniformly distributed across topics, proofs being, however, usually at the head of the list of problems. Why do students take the instruction, "Prove ... " in examinations to mean, "Go to the next question"?

Students of mathematics and computing need to learn how to understand and construct proofs. Proofs are central to mathematics—and more so as the level of mathematics rises. In computer science, it has become clear with the increased importance of software that a computer program is a logical object, the understanding of which requires the same tools as mathematical proof. Proof, however, has not been well served by textbooks. While students do not have the background their teachers had in school of deductive geometry, nothing has replaced Euclid. Textbooks on mathematics for computer science are plentiful, but, as stated in *Carnegie-Mellon Curriculum for Undergraduate Computer Science* (ed. M. Shaw, Springer-Verlag, New York, 1985) on p. 81, their chief problem is "inadequate treatment of logic".

The present book aims to fill this gap. Some of its features are:

1. Techniques of proof are introduced in the context of mathematics, such as arithmetic, already familiar to the student.
2. The techniques are later applied to the usual topics of first-year university mathematics, such as linear algebra and calculus.
3. At all times, matters of context and motivation are kept in mind.
4. Issues of strategy and tactics in constructing proofs are emphasised.
5. The skills of setting out proofs to result in a convincing argument are insisted upon.
6. The continuity of proof with ordinary mathematical techniques, such as calculation, is respected.
7. There is no discussion of symbolic logic, as there are enough books on this subject already.
8. Since the propositional calculus has little importance in mathematical proof, it is barely mentioned. The quantifiers "all" and "some" are the central concern.

We owe a great debt to Jack Gray who read the manuscript in full and made many very valuable suggestions. Useful suggestions were also made by Rod James, John Loxton and David Hunt, as well as by Daniel Solow, whose book *How to Read and Do Proofs* (Wiley, New York, 1982) has broadly the same approach as this book. We would also like to thank Rose Gonzalvez for her fast and efficient typing of the manuscript.

J. Franklin
A. Daoud

1

Proof

In mathematics things are proved; in other subjects they are not.

This statement needs certain qualifications, but it does express the most obvious difference between mathematics and the other sciences. In most fields of study, knowledge is acquired the hard way—from observations, by reasoning about the results of observations and by studying the observations, methods and theories of others. Mathematics was once like this too. Ancient Egyptian, Babylonian and Chinese mathematics consisted of rules for measuring land, computing taxes, predicting eclipses, solving equations, and so on. Methods were learnt from the observations and handed down to others. Modern school mathematics is still often practised in this way.

There were changes in the approach to mathematics. The ancient Greeks found that in arithmetic and geometry it was possible to *prove* that results were true. They found that some truths in mathematics were obvious and that many of the others could be shown to follow logically from the obvious ones. Pythagoras' theorem on right-angled triangles, for example, is not obvious, but a way was found of deducing it from geometrical facts that were apparent. At first it was hoped that every subject would become like mathematics, with all the truths following from obviously true basic statements. This did not happen. Physics, biology, economics and other sciences discover general truths, but to do so they rely on observations. The theory of relativity is not *proved* true; it is tested against observations.

As a result, mathematics has always been regarded as having a different kind of certainty to that obtainable in other sciences. If a scientific theory is accepted because observations have agreed with it, there is always in principle a small doubt that a new observation will not agree with the theory, even if all previous observations have agreed with that theory. If a result is proved correctly, that cannot happen. For more than two thousand years mathematics has attracted those who valued certainty, and has served as the supreme example of certain knowledge. It has also attracted those who wanted knowledge that did not rely on the authority of others; a moment's thought will reveal how little of our knowledge is like this.

Can we be sure, however, that the steps in our reasoning are correct? Are we really sure that what seems obvious to us is true? Can we expect *all* mathematical truths to follow from the obvious ones? These questions are not easily answered, and must be left until after some examples of proofs have been provided. Let us now look at the construction of proofs. We begin with an easy and short proof.

Example 1

Show that,

$$\frac{1}{1{,}000} - \frac{1}{1{,}001} < \frac{1}{1{,}000{,}000}$$

Proof

$$\frac{1}{1{,}000} - \frac{1}{1{,}001} = \frac{1{,}001}{(1{,}000).(1{,}001)} - \frac{1{,}000}{(1{,}000).(1{,}001)}$$
$$= \frac{1}{(1{,}000).(1{,}001)} = \frac{1}{1{,}001{,}000}$$

but,

$$1{,}001{,}000 > 1{,}000{,}000$$

so,

$$\frac{1}{1{,}001{,}000} < \frac{1}{1{,}000{,}000}$$

therefore,

$$\frac{1}{1{,}000} - \frac{1}{1{,}001} < \frac{1}{1{,}000{,}000}$$

Notes

1. Creating this proof consisted of two main parts: Firstly, after noting that the result was not obviously true, we applied a standard technique to,

$$\frac{1}{1{,}000} - \frac{1}{1{,}001}$$

namely the technique for subtracting fractions. The result of this was the number $\frac{1}{1{,}001{,}000}$.

Secondly, comparing where we were with where we wanted to be, we realised that we had yet to show that,

$$\frac{1}{1{,}001{,}000} < \frac{1}{1{,}000{,}000}$$

It is clear that the reason this is true is that the bottom line on the left is greater than the bottom line on the right. By writing this down, we have completed the proof.

2. We could have checked the answer by calculating both sides using a calculator or computer and comparing the answers. Why not do this, instead of going to the trouble of finding a proof? The main reason is that mathematics looks for understanding. A calculation in a machine gives an answer, but no understanding of why that answer is right. A proof can make it clear *why* that answer is true—as it did in the example above. Not only is understanding good in itself, it also has practical advantages. For example, if we next want to know about the size of,

$$\frac{1}{2{,}000} - \frac{1}{2{,}001}$$

the calculation that checked that,

$$\frac{1}{1{,}000} - \frac{1}{1{,}001} < \frac{1}{1{,}000{,}000}$$

would be of no help at all, for we would just have to start calculating again. We can, however, see that a proof similar to the one above would show that,

$$\frac{1}{2{,}000} - \frac{1}{2{,}001} < \frac{1}{(2{,}000)^2}$$

In fact there is a similar proof for any whole number n. It would show that,

$$\frac{1}{n} - \frac{1}{n+1} < \frac{1}{n^2}$$

for *all* whole numbers n. (Proofs of such "all" statements are the subject of the Chapter 2.) Understanding, then, allows us to generalise. When we understand a proof we can examine it to see which aspects of the problem are really relevant to the argument, and which are not. In the above proof, while it was important that the reciprocals of two consecutive numbers were subtracted, it was of less importance what exactly these numbers were.

Computers have further limitations. There is a limit to the size of numbers they can work with, whereas a proof can deal with numbers of any size. Furthermore, although beginners in computing believe that everything the computer calculates is correct, this is not true. Because of such problems as the accumulation of round-off errors, it is in the nature of computers to produce wrong answers some of the time. "Garbage in, garbage out", as the experts say, but also occasionally, "Perfect figures in, garbage out". A simple example can be found in A. W. Roberts, *Elementary Linear Algebra* (2nd edn) (1985), Benjamin/ Cummings, Reading, Mass., p. 15. The art of telling how much of computer output is true requires exactly the same kind of thinking as is required for proofs (see Appendix for some examples).

3. The proof derived a result that was not obvious, namely,

$$\frac{1}{1{,}000} - \frac{1}{1{,}001} < \frac{1}{1{,}000{,}000}$$

using a sequence of steps, each of which was obviously correct. However, whatever follows logically from what is obviously true, must itself be true. In this case there is no difficulty in knowing that the steps are obvious, since we have used only easy arithmetical operations at each stage. In general, however, what counts as "obvious" and "obviously follows" is something the student has to learn from experience—though in fact most people intuitively agree about what is obvious. Still, even in advanced mathematics, there are occasionally disputes about what is obvious. There are many anecdotes of mathematics professors stopping in the middle of lectures after saying, "This step is obvious.", thinking for 15 minutes, then continuing with "Yes, it is obvious."

Disputes and mistakes about what is obvious could, in principle, be avoided by laying down for each branch of mathematics certain basic statements, called *axioms*, and agreeing that proofs must be derived from these. For example, we could insist that statements of arithmetic must be proved from the basic number laws (such as the commutative law of addition: a + b = b + a for all real numbers a and b). (Axioms are discussed further in Chapter 15.) In practice, axioms are usually not referred to explicitly. Instead, a wide variety of well-known facts and techniques, such as those used in the proof above, are taken to be obvious.

The next proof is an easy one to understand, but a difficult one to find.

Example 2

Show that the sum of the first hundred whole numbers is 5050.

Proof

We have to show that,

$$1 + 2 + 3 + 4 + 5 + \ldots + 99 + 100 = 5050$$

Instead of adding the numbers from left to right, we add the first and the last, then the second and the second to last, and so on.

So,

$$1 + 2 + 3 + 4 + \ldots + 99 + 100 = (1 + 100) + (2 + 99) + (3 + 98) + \ldots + (50 + 51)$$

Now each pair in brackets adds up to 101. So there are fifty 101s. So the sum of the first hundred numbers is,

$$50 \times 101 = 5050$$

Notes

1. Again, an answer could have been obtained on a calculator, but only after much effort. The point is, however, to find a method that makes the answer clear, and brute calculation cannot do this.
2. Adding a hundred randomly chosen numbers *would* require a calculator, but here we have the opportunity to use the fact that the numbers given exhibit a *pattern*: they are the *first* hundred numbers, with each number being one more than the one before it. The essential idea of the proof was to take advantage of this pattern by pairing the first and the last, the second and the second to last, etc. (Some other ideas are given in Exercise 1.13 below.) This step is definitely a hard one to create, with no rules that can be followed in order to find such steps. Some intellectual creativity is needed. In a sense, finding proofs is like playing chess; there are some rules, such as the number laws, and some commonly-used tactics, such as adding numbers in a different order. Overall strategy is a combination of creativity and techniques learnt from experience.
3. The following method used to derive the result $1 + 2 + 3 + \ldots + 100 = 5050$, does not count as a proof:

   "The formula for the sum of an arithmetic progression,

   $$a + (a + d) + (a + 2d) + (a + 3d) + \ldots + (a + (n - 1)d)$$

   is,

   $$na + \frac{n(n-1)}{2}d$$

   Putting,

   $$a = 1,\ d = 1,\ n = 100$$

   we get,

   $$1 + 2 + 3 + \ldots + 100 = 100.1 + \frac{100(99)}{2}.1 = 5050"$$

   The reason this is not a full proof is that it does not deduce the result to be proved from something obvious; the formula for the sum of an arithmetic progression is, if anything, less obvious than the result to be proved. This defect could be repaired by adding a proof of the formula, but in fact this proof is just a more general version of the proof given for $1 + 2 + 3 + \ldots + 100$.
4. It is again natural to look for generality. It is clear that there is no reason why the numbers to be added should stop at 100; it would, therefore, be natural to use the method of proof to derive a formula for $1 + 2 + 3 + \ldots + n$, for *any* whole number n. (See Chapter 2 for further developments.) A further analysis of the proof will also show that the numbers do not need to start at 1, and the gap between one number and the next need not be 1. This will lead to the general formula for the sum of an arithmetic progression.

In the next example, although it takes some effort to find the proof, it is possible to explain and learn the methods for finding it. It does not need a major "bright idea" as in the last example.

Example 3

Prove that, $\sqrt[8]{8!} < \sqrt[9]{9!}$ (where, for any whole number n, n!, read, "n factorial", is the product of the whole numbers up to n, that is, $1.2.3. \ldots .n$).

Finding the proof

The general strategy should be to do something to both sides that will make the problem simpler. A much easier example of a similar kind would be:

Show,

$$\sqrt{2} < \sqrt{3}$$

The reason this is true is that $2 < 3$; that is, one compares the *square* of both sides because $\sqrt{2}$ *means*, "the number whose square is two", so that to be able to deal with it, it is most natural to consider its square. The complete proof of $\sqrt{2} < \sqrt{3}$ would then start with the obviously true fact:

$$2 < 3$$

and, by taking the square root of both sides, arrive at the result to be proved:

$$\sqrt{2} < \sqrt{3}$$

The case here is similar. Because $\sqrt[8]{8!}$ means the (positive) number whose 8th power is 8!, it is natural to consider the 8th power of both sides. This would give,

$$(\sqrt[8]{8!})^8 < (\sqrt[9]{9!})^8$$

that is,

$$8! < (\sqrt[9]{9!})^8$$

If we could show this we could take the 8th root of both sides to show the original result. Unfortunately, while this step has made the left-hand side simpler, it has made the right-hand side more complicated. Similarly, if we took the 9th power of both sides to remove the $\sqrt[9]{\ }$, we would get,

$$(\sqrt[8]{8!})^9 < 9!$$

which is an advance on the right, but not so on the left. To overcome this we look at a power of both sides which will cause the roots to disappear on both sides. We take the 72nd power, since $8 \times 9 = 72$. That is, we will try to show,

$$(\sqrt[8]{8!})^{72} < (\sqrt[9]{9!})^{72}$$

which is, on simplifying,

$$(8!)^9 < (9!)^8$$

This is still not obviously true, but using the meanings of 8th and 9th powers (namely, x^8 means $x.x. \ldots$ eight times, and similarly x^9 means $x.x. \ldots$ nine times), we are trying to show,

$$\underbrace{8! \ldots 8!}_{\text{nine times}} < \underbrace{9! \ldots 9!}_{\text{eight times}}$$

As this is still not obviously true, we apply the definitions of 8! and 9!. We wish to show,

$$\underbrace{(1.2. \ldots .8)(1.2. \ldots .8) \ldots (1.2. \ldots .8)}_{\text{nine times}} < \underbrace{(1.2. \ldots .9)(1.2. \ldots .9) \ldots (1.2. \ldots .9)}_{\text{eight times}}$$

It is clear that most of the numbers appear on both sides. Each factor $(1.2. \ldots .9)$ on the right can be thought of as $(1.2. \ldots .8).9$, so $(1.2. \ldots .8)$ appears nine times on the left and eight times on the right. Dividing both sides by $(1.2. \ldots .8)^8$ gives,

$$1.2. \ldots .8 < \underbrace{9. \ldots .9}_{\text{eight times}}$$

If we could show this, we could multiply it by $(1.2. \ldots .8)^8$ to get the result above. But now we have reached a result which *is* obviously true, since both sides have eight factors, and the ones on the left are all less than the ones on the right.

The final proof is obtained by writing these steps in *reverse* order, since the aim is to derive the result to be proved *from* something obvious, not vice versa (see Note 2 below).

Proof

$1.2. \ldots .8 < \underbrace{9.9. \ldots .9}_{\text{eight times}}$ since $1, 2, \ldots, 8$ are all less than 9.

Multiplying both sides by $(1.2. \ldots .8)^8$,

$$\begin{aligned}(1..2. \ldots .8)^9 &< (1.2. \ldots .8)^8.9^8 \\ &= (1.2. \ldots .9)^8\end{aligned}$$

that is,

$$(8!)^9 < (9!)^8$$

Taking the (positive) 72nd root of both sides,

$$\sqrt[8]{8!} < \sqrt[9]{9!}$$

Notes

1. The proof as now written is designed to convince the reader of the truth of the conclusion. However, it gives no understanding of how the proof was derived. No explanation is given, for example, of why the first line should be what it is. This is the usual style in written mathematics. The advantage of such a style is that it communicates results and produces confidence in them in the shortest possible space; its disadvantage is that it covers up a certain amount of what is going on, by writing down assertions for which the motivation is totally unclear.
2. The reason for writing the steps in reverse order is that we must derive the result to be proved from something obvious, not derive something obvious from the result to be proved. The point is clear from the following fallacious argument:

   Show that $1 = 3$

   "Proof": Suppose,
   $$1 = 3$$
   so,
   $$-1 = 1$$
   (subtracting 2 from both sides).

   So,
   $$1 = 1$$
   (squaring both sides).

   This is true, so,
   $$1 = 3$$
   The fallacy of this "proof" is that it has derived correctly something obviously true (namely $1 = 1$) from the result to be proved ($1 = 3$) but has then wrongly concluded that this means the result to be proved is also true. A correct proof must *end* with the result to be proved.
3. The need to reverse steps, in some cases, means that care is needed in checking that the steps involved are in fact reversible. For example, the step,
   $$\sqrt{2} < \sqrt{3}$$
   therefore,
   $$\sqrt{2} + 2 < \sqrt{3} + 2$$
   is reversible, in that one can correctly argue,
   $$\sqrt{2} + 2 < \sqrt{3} + 2$$
   therefore,
   $$\sqrt{2} < \sqrt{3}$$
   (because 2 has been subtracted from both sides).

   Most simple arithmetical steps are reversible like this, but some are not. For example, one can correctly argue,
   $$x = 2$$
   therefore,
   $$x^2 = 4$$
   However, one cannot reverse this step to argue,
   $$x^2 = 4$$

therefore,

$$x = 2$$

Rather, it would be only correct to argue,

$$x^2 = 4$$

therefore,

$$x = 2 \quad \text{or} \quad x = -2.$$

Otherwise, fallacious arguments like the following would result:

$$\text{"}(-1)^2 = 1$$

but,

$$1 = 1^2$$

so,

$$(-1)^2 = 1^2$$

Taking the square root of both sides,

$$-1 = 1\text{"}$$

4. Could the search for a proof have started in some other way? For example, could we have started with:

$$\sqrt[8]{8!} = \sqrt[8]{1.2. \ldots .8}$$
$$= \sqrt[8]{1}.\sqrt[8]{2}. \ldots .\sqrt[8]{8}$$

In fact a proof can be found in this way, with the resulting proof not being very different to the one found above. The reader may like to try it. As in chess, one can learn by experience which moves tend to be good; in general, however, there is only a limited amount of advice which can be given. Some general guidelines which could be given are: "Do something that makes things simpler."; and "Look for some step that will make the current situation more like the goal." A technique that was used a number of times in the last example was what could be roughly described as, "expand the definitions". To deal with $\sqrt[8]{x}$, we used the definition of an 8th root, namely,

$$\sqrt[8]{x} \text{ is the number y such that } y^8 = x$$

to deal with x^9, we replaced it with,

$$\underbrace{x \ldots x}_{\text{nine times}}$$

which is what x^9 means; and to deal with n!, we replaced it by,

$$1.2. \ldots .n$$

In each of the examples above a true result was given and a proof was asked for. So, where did the result itself come from? Which came first, the proof or the result? How would anyone know that 1/1,000 − 1/1,001 was less than 1/1,000,000 unless they had already performed the calculation given in the proof above? Realistically, the proof and the result are often discovered together. If the following two questions are asked, "About how big is a number like 1/1,000 and 1/1,001. Wouldn't it be much smaller than the numbers themselves?", the natural thing would be to perform the calculation of 1/1,000 − 1/1,001 and find that the answer was smaller than 1/1,000,000. The result and the proof would be found simultaneously, and the discovery would then be presented as the result, followed by its proof. On the other

hand, it often happens that a result is found or conjectured without a proof, with a subsequently found proof giving an understanding of why the result is true. For example, someone might add up 1 + 2 + 3 + . . . + 100 with a calculator, find the answer 5050, notice this is 50 × 101, and ask why this should be so. The above proof provides the answer. (The matter of conjectures that are not yet proved will be discussed further in Chapter 14.)

Exercises

(Gradings of exercises: * easy, ** moderate, *** difficult.)

*1. Show that: $\frac{1}{1{,}000} - \frac{1}{1{,}002} < \frac{2}{1{,}000{,}000}$

*2. Show that: $1 + 2 + 3 + \ldots + 999 + 1{,}000 = 500{,}500$

*3. Show that: $\sqrt{1{,}001} - \sqrt{1{,}000} < \frac{1}{2\sqrt{1{,}000}}$

Hint: Multiply $\sqrt{1{,}001} - \sqrt{1{,}000}$ by $\frac{\sqrt{1{,}001} + \sqrt{1{,}000}}{\sqrt{1{,}001} + \sqrt{1{,}000}}$

In questions 4, 5, 8, 14 and 21 it will be necessary to assume common trig. formulas such as, $\sin(A + B) = \sin A \cos B + \cos A \sin B$. These are not obvious, but are at least well-known.

*4. Given that: $\sin A = \frac{3}{5}$ and $\sin B = \frac{5}{13}$ and that A is obtuse (i.e. between 90° and 180°) and B is acute, show that: $\cos(A + B) = -\frac{63}{65}$

*5. Show that: $\tan\frac{\pi}{12} = 2 - \sqrt{3}$

*6. Prove that: $x = 1 - \sqrt{5}$ is a solution of the equation $x^3 - 3x^2 - 2x + 4 = 0$

*7. (a) Show that: $\log_5 1^2 + \log_5 2^2 + \log_5 3^2 - 2\log_5 6 = 0$

(b) Show that: $\log_4((\sqrt{2})^9) = \frac{9}{4}$

*8. Show that: $\cos\frac{5\pi}{12} = \frac{\sqrt{3} - 1}{2\sqrt{2}}$

*9. Prove that: $\sqrt[7]{7!} > \sqrt[6]{6!}$

*10. Comment on the following reasoning: "If a pond is 1 metre across, how far is it around the edge? Answer: circumference = πd metres = 3.14159265 metres (correct to 8 decimal places)."

****11.** The Golden Ratio is defined to be the number l so that if a 1×1 square is removed from an $l \times 1$ rectangle the remaining rectangle is the same shape as the original. (See Figure 1.1.) (The ancient Greeks considered rectangles of this shape especially well proportioned, and used this shape in architecture.)

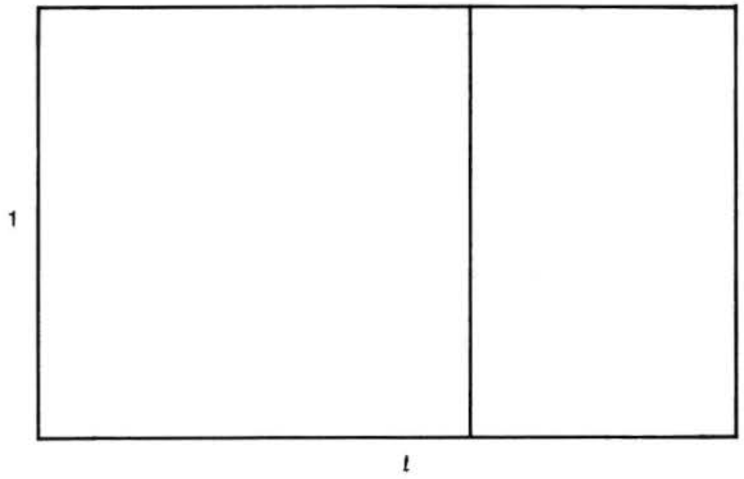

Figure 1.1

Prove that: $l^2 = l + 1$ and hence find an expression for l.

****12.** Show that: $\sqrt{2 + \sqrt{2}} + \sqrt{2 - \sqrt{2}} < 2\sqrt{2}$

****13.** Discuss whether the following proof of: $1 + 2 + 3 + \ldots + 99 + 100 = 5{,}050$ is better or worse than the one given in Example 2.
It might be useful to see how the method works with $1 + 2 + 3 + \ldots + 99$:

$$2(1 + 2 + 3 + \ldots + 99 + 100) = \begin{array}{l} 1 + 2 + 3 + \ldots + 99 + 100 \\ + 100 + 99 + 98 + \ldots + 2 + 1 \end{array}$$

(since in the second row we have just written down the numbers in reverse order)

$$= \underbrace{101 + 101 + \ldots + 101 + 101}_{100 \text{ times}}$$

(since each number in the top row plus the one below it gives 101)

$$= 100 \times 101$$

so,

$$1 + 2 + 3 + \ldots + 99 + 100 = \frac{100 \times 101}{2} = 5{,}050$$

****14.** (a) Prove that: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

(b) Hence show that $\cos\frac{2\pi}{3}$ is a root of the equation $4x^3 - 3x - 1 = 0$

****15.** Prove that: π is between 3 and 4 (where π is defined to be the ratio of the circumference of a circle to its diameter; that is, π is defined by:

$$\text{Circumference} = \pi \times \text{diameter})$$

Hint: See Figure 1.2.

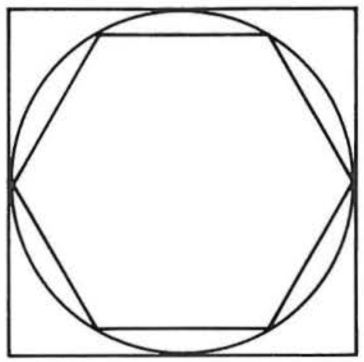

Figure 1.2

****16.** Prove that the second digit after the decimal point of $\sqrt{2}$ is 1. (Calculators not allowed.)

****17.** Show that: $\sqrt[3]{3 + \sqrt[3]{3}} + \sqrt[3]{3 - \sqrt[3]{3}} < 2\sqrt[3]{3}$

*****18.** Prove that $0 < \left(1 + \frac{1}{1000}\right)^{1000} < 3$

Hint:

$$\left(1 + \frac{1}{n}\right)^n = 1 + \frac{n}{n} + \frac{n(n-1)}{1.2}.\frac{1}{n^2} + \frac{n(n-1)(n-2)}{1.2.3}.\frac{1}{n^3} + \ldots + \frac{n!}{n!}\left(\frac{1}{n}\right)^n$$

$$< 1 + 1 + \frac{n.n}{1.2}.\frac{1}{n^2} + \frac{n.n.n}{1.2.3}\frac{1}{n^3} + \ldots + \frac{n^n}{n!}\frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}}$$

*****19.** Prove that: $10^9 < 9^{10}$

*****20.** Prove that: $99^{100} > 100^{99}$

*****21.** Show that: $\frac{\pi}{4} = \tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right)$

2

"All" statements

The results proved in Chapter 1 were *particular* statements. A particular statement either asserts that a property, or *predicate*, is true of a *subject* (e.g. "seven is prime"; "Socrates is mortal"), or asserts some *relation* between two or more things (e.g., $\sqrt[8]{8!} < \sqrt[9]{9!}$; $1/1{,}000 - 1/1{,}001 < 1/1{,}000{,}000$). In any science, but especially in mathematics, we are interested in *generalising* from particular facts. When we saw that,

$$\frac{1}{1{,}000} - \frac{1}{1{,}001} < \frac{1}{1{,}000{,}000}$$

we realised that something similar would have happened whenever we subtracted the reciprocals of two consecutive whole numbers. A similar proof would have shown, for example, that,

$$\frac{1}{203} - \frac{1}{204} < \frac{1}{(203)^2}$$

The *particular* result that,

$$\frac{1}{1{,}000} - \frac{1}{1{,}001} < \frac{1}{1{,}000{,}000}$$

is an instance of the *general* pattern: for all whole numbers n,

$$\frac{1}{n} - \frac{1}{n+1} < \frac{1}{n^2}$$

This is an example of an "all" statement, or a "universal generalisation" in the jargon of logic. (Unfortunately, logic is a subject especially given to producing jargon; a precise technical language is sometimes needed, but should not be overdone.)

Some equivalent ways to express the same result in English are:

For every whole number n,

$$\frac{1}{n} - \frac{1}{n+1} < \frac{1}{n^2}$$

For any whole number n,

$$\frac{1}{n} - \frac{1}{n+1} < \frac{1}{n^2}$$

If n is a whole number, then,

$$\frac{1}{n} - \frac{1}{n+1} < \frac{1}{n^2}$$

The difference of the reciprocals of any two consecutive whole numbers is less than the reciprocal of the square of the smaller number.

Some other examples of "all" statements are:

- All men are mortal. (This example has been traditional since the time of the ancient Greeks.)
- All multiples of ten are multiples of five.
- For all whole numbers n,

$$\sqrt[n]{n!} < \sqrt[n+1]{(n+1)!}$$

(this is the natural generalisation of the particular result in Chapter 1, Example 3: $\sqrt[8]{8!} < \sqrt[9]{9!}$).

- Every fourth power is a square (i.e. if a whole number is the fourth power of some other whole number, it is the square of some whole number). For example, 81 is a fourth power because it is 3^4 and is also a square because it is 9^2.

The general form of an "all" statement is,

All As are Bs

This states that anything that has the property of being an A also has the property of being a B. While some of the above examples cannot be put easily or naturally into the form, "All As are Bs". ("All whole numbers n are such that, $1/n - 1/(n+1) < 1/n^2$", for example, is a little awkward.) The form, "All As are Bs", is nevertheless very useful for explaining the general features of "all" statements.

To prove an "All As are Bs" statement, we must show that *anything* that is an A is also a B. So the proof of, "All As are Bs", should look like this:

Let x be an A

therefore, x is a B.

Example 1

Prove that for all whole numbers n,

$$\frac{1}{n} - \frac{1}{n+1} < \frac{1}{n^2}$$

Proof

Let n be a whole number. Then,

$$\frac{1}{n} - \frac{1}{n+1} = \frac{n+1-n}{n(n+1)}$$
$$= \frac{1}{n^2+n}$$

but,

$$n^2 + n > n^2$$

so,

$$\frac{1}{n^2+n} < \frac{1}{n^2}$$

therefore,

$$\frac{1}{n} - \frac{1}{n+1} < \frac{1}{n^2}$$

Notes

1. Although the result,

$$\frac{1}{n} - \frac{1}{n+1} < \frac{1}{n^2}$$

is much more general than,

$$\frac{1}{1{,}000} - \frac{1}{1{,}001} < \frac{1}{1{,}000{,}000}$$

its proof is no harder. If anything, it is easier, since there is no danger of being blinded by any irrelevant facts about the particular numbers chosen (e.g. their large size, which might easily lead to mistakes in comparing them).

2. It is most important to understand that, to prove an, "All As are Bs" statement, it is *not* sufficient to look at a number of particular As and check that they are Bs. There is no "proof by example". Thus, it would be incorrect to argue,

$$\frac{1}{1} - \frac{1}{2} < \frac{1}{1^2}$$

and,

$$\frac{1}{2} - \frac{1}{3} < \frac{1}{2^2}$$

and,

$$\frac{1}{3} - \frac{1}{4} < \frac{1}{3^2}$$

so,

$$\frac{1}{n} - \frac{1}{n+1} < \frac{1}{n^2}$$

for all n. Checking some examples *supports* the "all" statement, but does *not prove* it; the statement could be true for the examples checked, but false for some others. (This joke told by mathematicians about physicists illustrates the point. A physicist argued that he could show that all odd numbers are prime: "Three and five are prime," he said, "and so is seven. Nine—well, experimental error—but eleven and thirteen are prime, so all odd numbers are

prime." Underlying this joke is a real difference between mathematics and experimental sciences such as physics: in physics the "all" statements can only ultimately be confirmed by a sufficiently large number of experiments, while in mathematics they can be *proved* true.)

Example 2

Show that every whole number that is a fourth power is a square.

Proof

We should begin the proof by taking a number that is a fourth power, and writing in symbols what this means:

Let x be a whole number that is a fourth power. That is, $x = a^4$ for some number a.

We wish to conclude that x is in fact a square, so we ask how we can write a^4 as the square of something. As a^4 is $a.a.a.a$, it can be written $(a.a)^2$, that is $(a^2)^2$. So the complete proof is:

Let x be a whole number which is a fourth power. So,

$$x = a^4$$

for some whole number a

$$= a.a.a.a$$
$$= (a^2)^2$$

So x is a square.

Every fourth power is therefore a square.

Notes

1. As already pointed out, it is not sufficient to check that the result is true for some particular xs that are fourth powers, for example, 16 and 81. The proof must be general, that is, it must apply to *any* x that is a fourth power.
2. The step after, "x is a whole number which is a fourth power" expresses in symbols what it means to be a fourth power. It often happens that in a proof of, "All As are Bs", the second line (after, "Let x be an A") expresses the defining characteristic of As, that is, what it means to be an A. Similarly, the second last line (before, "therefore, x is a B") is often an expression in symbols which shows that x *is* a B. Check this in the example above.
3. The sentence, "Let x be an A", is sometimes expressed as, "Take a general (or arbitrary) A". This expression is possibly misleading, since all actual As are particular. It can be useful, however, as a reminder that in the proof of "All As are Bs", we must use only facts that are true of *all* As.

Watch for these points in the following example.

Example 3

Prove that the square of an odd number is odd. (Obviously, "of an odd number" means "of any odd number".)

Finding the proof

We are asked to prove something about *all* odd numbers, namely, that their squares are odd. So the proof must look like:

Let x be an odd number.

⋮

Therefore, x^2 is odd.

To complete the proof it is necessary to express "is odd" in symbols. A number is odd if it is of the form $2k + 1$ for some k; this is the general form of an odd number. So the proof should look like this:

Let x be an odd number.

So, $x = 2k + 1$ for some number k,

⋮

so, $x^2 = 2\,(\text{something}) + 1$

therefore, x^2 is odd.

To fill in the steps still missing, we ask how to get from $x = 2k + 1$ to something about x^2. Clearly we should square both sides. The complete proof is as follows:

Proof

Let x be an odd number.

So,

$$x = 2k + 1$$

for some number k,

$$\begin{aligned} x^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned}$$

therefore, x^2 is odd.

Note

As occasionally happens, this proof has actually proved a stronger result than was intended. We intended to show that the square of any odd number is of the form $2K + 1$; we actually showed it was of the form $4K + 1$. This is a stronger result, in the sense that the statement

"The square of any odd number is of the form $4K + 1$"

implies the statement,

"The square of any odd number is odd"

but the reverse implication is not true, since some odd numbers (e.g. 999) are not of the form $4K + 1$. The stronger result tells us that a number such as 999 cannot be the square of an odd number, while the weaker result does not tell us this.

"All As are Bs" statements are so common that many equivalent ways of expressing them have appeared. We have mentioned already the forms,

"Any A is a B"
"Every A is a B"
"If anything is an A then it is a B"

Some others are:

"Whatever is an A is a B"
"Being an A is a sufficient condition for being a B"
"Something is an A only if it is a B"

(Note that the order of A and B is the *same*: "If A then B"; "A only if B". Think of, "If anything is a horse then it is an animal"; "Something is a horse only if it is an animal".)

"Being a B is a necessary condition for being an A"

(A and B are in the opposite order here. Think of, "Being an animal is a necessary condition for being a horse"—to be a horse, something must at least be an animal.)

"The set of As is included in the set of Bs"

(See Figure 2.1.)

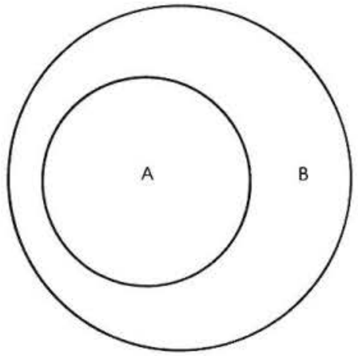

Figure 2.1

Sets will be discussed more fully in Chapter 7. Some ways of expressing "All As are Bs", using "not", will be given in Chapter 6. These statements are used in different situations in English, because they emphasise different things. But while their rhetorical force is different, their logical force is the same: "All As are Bs".

It is most important to understand that "All As are Bs" is not the same as, "All Bs are As". "All horses are animals" is true but, "All animals are horses" is false. The statement "All Bs are As" is called the *converse* of "All As are Bs", and, "If anything is a B then it is an A" is the converse of "If anything is an A then it is a B". If an "all" statement is true, its converse may or may not be true; more work is needed to decide.

Note that there can be "if" statements which are not of the form, "If an x is an A then x is a B". Any two statements p and q can be joined by "if . . . then" to form the statement "If p then q". For example, "If the squares of all odd numbers are of the form $4k + 1$, then the squares of all odd numbers are odd". An "if p then q" statement is proved by showing how q follows from p, so its proof is of the form:

p

q

In practice, however, almost all the "if . . . then" statements used in mathematics can be thought of in the form "if x is an A then x is a B".

We conclude this chapter with some examples from the theory of linear equations, linear algebra and calculus. They illustrate how important "all" statements are in every branch of mathematics.

Example 4 (Linear equations)

Show that every system of linear equations of the form,

$$\begin{aligned} x + Ay &= B \\ 2x + Cy &= D \end{aligned}$$

such that $C \neq 2A$ can be solved.

Proof

Taking the second equation minus twice the first gives,

$$(C - 2A)\,y = D - 2B$$

since $C \neq 2A$ and so $C - 2A \neq 0$ we may divide by $C - 2A$:

$$y = \frac{D - 2B}{C - 2A}$$

Then from the first equation,

$$\begin{aligned} x &= B - Ay \\ &= B - \frac{A(D - 2B)}{C - 2A} \end{aligned}$$

So the system has been solved.

Example 5 (Linear algebra)

Show that all linear combinations of the vectors (1, 1, 2) and (2, 3, 5) in $\mathbf{R}^3$ lie on the plane,

$$x + y - z = 0$$

Proof

Any linear combination of (1, 1, 2) and (2, 3, 5) is of the form, $a(1, 1, 2) + b(2, 3, 5)$

for some $a, b \in \mathbf{R}$. However, this is $(a + 2b, a + 3b, 2a + 5b)$, and this point satisfies the equation,

$$x + y - z = 0$$

since

$$\begin{aligned} (a + 2b) + (a + 3b) - (2a + 5b) &= a + a - 2a + 2b + 3b - 5b \\ &= 0 \end{aligned}$$

So every linear combination of (1, 1, 2) and (2, 3, 5) lies on the plane,

$$x + y - z = 0$$

Example 6 (Calculus)

Show that, for all $x > 0$, $x \geqslant \ln x + 1$

Proof

We use calculus methods to compare the graphs of,

$$y = x$$

and

$$y = \ln x + 1$$

The graph of,

$$y = x$$

is the straight line through (0, 0) with gradient 1. Now for $x = 1$,

$$x = \ln x + 1 = 1$$

so the two graphs meet at (1, 1). Also,

$$\frac{d}{dx}(\ln x + 1) = \frac{1}{x}$$

and for all $x > 1$,

$$\frac{1}{x} < 1$$

(and indeed $1/x \to 0$ as $x \to \infty$).

So for $x > 1$, $y = \ln x + 1$ has a gradient less than $y = x$. Thus for $x \geqslant 1$,

$$x \geqslant \ln x + 1$$

For $0 < x < 1$, $1/x$ (the gradient of $y = \ln x + 1$) is greater than 1 (the gradient of $y = x$). So here again

$$x > \ln x + 1$$

So for all $x > 0$,

$$x \geqslant \ln x + 1$$

(See Figure 2.2, which shows these graphs.)

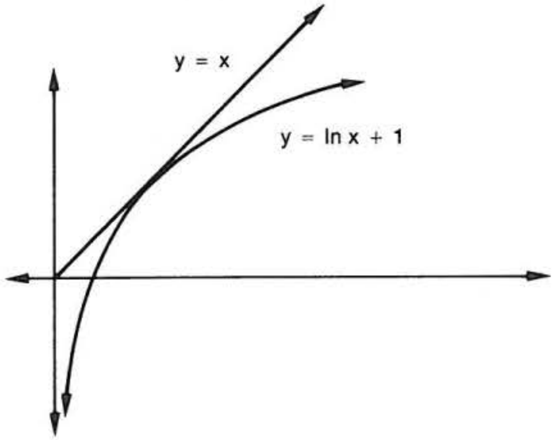

Figure 2.2

Exercises

(Gradings of exercises: * easy, ** moderate, *** difficult.)

*1. Prove that the square of any even number is even.

*2. Prove that the sum of any two consecutive numbers is odd.

*3. Prove that the product of any two odd numbers is an odd number.

*4. Give an "all" statement relating "cows" and "mammals" which illustrates that, "All As are Bs" is not logically equivalent to, "All Bs are As".

*5. Decide whether the statement:

$$x^2 - 3x + 2 < 0$$

for $1 < x < 2$ (i.e. for *all* x between 1 and 2) is true or false and prove your answer.

*6. (a) Rewrite the all statement, "All As are Bs" in the forms
(i) ". . . only if . . ."
(ii) "If . . . then . . ."
(b) A father told his son, "Only if you pass will you get a bike". The son passed but he did not get a bike. Did the father break his promise? Explain.

*7. Show that if m and n are odd integers, then $m + n$ is even.

*8. Comment on the reasoning:
(a) It takes one person two hours to mow this lawn. So it would take 6,000 people $\frac{2}{6{,}000}$ hours.
(b) If it takes four ships three days to cross the Tasman, how long will it take seven ships?

*9. Give one example of an "all" statement (universal generalisation) in the form, "All As are Bs". Then rewrite it in different forms.

*10. Show that for any whole number n, $\frac{n}{n+1} < \frac{n+1}{n+2}$

*11. Prove that $\sqrt[n]{n!} < \sqrt[n+1]{(n+1)!}$ for any whole number n (imitate the proof in Chapter 1).

*12. Prove that the product of three consecutive whole numbers, of which the middle one is odd, is divisible by 24.

*13. Let α and β be the roots of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$.

Show that:

$$\alpha + \beta = -\frac{b}{a}$$

and,

$$\alpha\beta = \frac{c}{a}$$

(Do not use the quadratic formula.)

**14. Show that: $\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)\ldots\left(1 - \frac{1}{n}\right) = \frac{1}{n}$

**15. Prove that if, $0 < x < 1$ then, $0 < x^2 < x < 1$

**16. Show that for any non-zero real numbers x and y such that $x + y = 1$,

$$\left(1 - \frac{1}{x}\right)\left(1 - \frac{1}{y}\right) = 1$$

**17. Prove that $x^2 - 4x + 5 > 0$ for all real numbers x.

**18. (a) Prove that if $a \neq 0$ then;

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

are solutions of $ax^2 + bx + c = 0$

(b) Prove that if $a \neq 0$ and x is a solution of $ax^2 + bx + c = 0$, then,

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

••19. Find a generalisation of:

$$\frac{1}{1000} - \frac{1}{1002} < \frac{2}{(1000)^2}$$

and prove it.

••20. Show that if a, b are integers such that 7 divides $a + b$ and $a^2 + b^2$, then 7 divides both a and b.

••21. Prove that for all whole numbers n, $(n + 1)(n + 2) \ldots (2n - 1)(2n) = 2^n.1.3.5 \ldots (2n - 1)$

••22. (a) What is wrong with the following "proof" of Pythagoras' theorem? (The theorem states that if a, b, c are the sides of any right-angled triangle with c the hypotenuse then $a^2 + b^2 = c^2$.)

By the cosine rule,

$$c^2 = a^2 + b^2 - 2ab\cos 90°$$

but,

$$\cos 90° = 0 \quad \text{so} \quad c^2 = a^2 + b^2$$

(b) What about this attempt?
Let θ be the angle opposite b.
Then,

$$\begin{aligned} a^2 + b^2 &= (c\cos\theta)^2 + (c\sin\theta)^2 \\ &= c^2(\cos^2\theta + \sin^2\theta) \\ &= c^2 \quad \text{since} \quad \cos^2\theta + \sin^2\theta = 1 \end{aligned}$$

••23. (a) Prove that for any positive real numbers x, y,

$$xy \leqslant \left(\frac{x + y}{2}\right)^2$$

(b) Hence show that of all rectangles with a fixed perimeter, the square has the largest area.

•••24. (a) Prove that the area, S, of a triangle of base b and altitude h is given by,

$$S = \frac{1}{2}bh$$

Hint: Draw some rectangles (make sure all shapes of triangles are accounted for). See Figure 2.3.

(b) Hence prove that the area S of a triangle is given by,

$$S = \frac{1}{2}ab\sin\theta$$

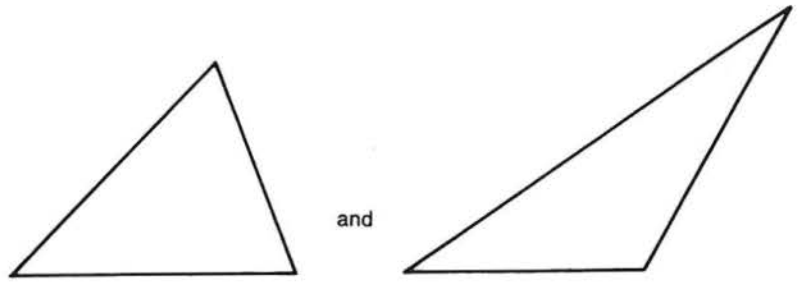

Figure 2.3

where θ is the angle between its two sides a and b.

•••25. Prove that the area A of a right circular cone is given by,

$$A = \pi r l$$

where r is the radius of the base and l is the slant height. (Assume the formula πr^2 for the area of a cirlcle.)
Hint: See Figure 2.4.

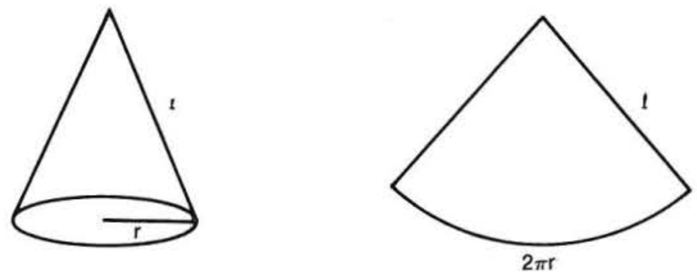

Figure 2.4

•••26. Prove Pythagoras' theorem: If a, b, c are the sides of any right-angled triangle with c the hypotenuse, then,

$$a^2 + b^2 = c^2$$

•••27. Show that for any positive integers m and n, 2 is between $\frac{m}{n}$ and $\frac{m + 2n}{m}$ (inclusive).

•••28. (a) Show that any figure in the XY plane, which is symmetrical about the X axis and symmetrical about the Y axis, is also symmetrical about the origin.
(A figure is said to be symmetrical about an axis if, for any point in the figure, the point opposite it across the axis is also in the figure. Similarly, a figure is symmetrical about a point if, for any point in the figure, the point opposite it across the point of symmetry is also in the figure. Thus a square is symmetrical about both diagonals and also about its centre.)

(b) Find some similar result in three dimensions and prove it.

Linear equations

***29.** Show that any system of equations,

$$\begin{aligned} x + y &= A \\ x - y &= B \end{aligned}$$

can be solved.

****30.** Show that if A or B is non-zero, any system of equations,

$$\begin{aligned} Ax + By &= C \\ Bx - Ay &= D \end{aligned}$$

can be solved.

****31.** Show that any system of equations of the form,

$$\begin{aligned} x + y + Az &= B \\ x - y + Cz &= D \end{aligned}$$

has infinitely many solutions.

Linear algebra

***32.** Show that any linear combination of (1, 2, 3) and (−2, −4, −6) lies on the line,

$$6x = 3y = 2z$$

****33.** Show that if:

$$x(1, 1, 0) + y(1, 2, 3) + z(3, 4, 3) = (0, 0, 0)$$

then (x, y, z) is a scalar multiple of (2, 1, −1)

Calculus

***34.** Prove that for all $x > 0$, $x > \sin x$

****35.** Show that for all $x > 4$, $2^x > x^2$

****36.** Show that any solution of $\frac{dy}{dx} = y$ is a multiple of e^x

****37.** Show that for $x > 10$, the graph of,

$$y = \frac{\sin x}{x^2}$$

lies within 0.01 of the x-axis.

*****38.** Is it true that for all $X > 0$,

$$\int_0^X \frac{\sin x}{x}\,dx > 0?$$

Prove your answer.

3

"If and only if" statements

It was explained in the last chapter that, when an "All As are Bs" statement is true, its converse, "All Bs are As", may or may not be true. If both "All As are Bs" and "All Bs are As" are true, then the As are exactly the same things as the Bs. Such a situation is usually expressed by an "if and only if" statement:

"Something is an A if and only if it is a B"

You will recall, from Chapter 2, that "Something is an A only if it is a B" is equivalent to "All As are Bs", and that "Something is an A if it is a B" is equivalent to "All Bs are As". Therefore, the "if and only if" statement, "Something is an A if and only if it is a B" is equivalent to,

"All As are Bs and all Bs are As"

"If and only if" is sometimes abbreviated to "iff" and symbolised by $\Leftrightarrow$.
In accordance with the use of "necessary" and "sufficient" introduced in Chapter 2, the above "iff" statement may also be expressed as:

"Being an A is a necessary and sufficient condition for being a B"

Some famous "if and only if" statements in mathematics are:

- A triangle has three equal sides if and only if it has three equal angles.
- A polyhedron is regular if and only if it is one of the five Platonic solids (the regular tetrahedron, cube, octahedron, dodecahedron and icosahedron). A polyhedron is said to be *regular* if all its faces are the same, all its edges are the same and all its vertices are the same, that is, at each vertex the same number of edges meet and the angles between them are all the same.
- A number is divisible by 9 if and only if the sum of its digits is divisible by 9.
- A real number is rational if and only if its decimal expansion is terminating or (eventually) repeating.

"If and only if" statements are very highly prized in mathematics. If an, "All As are Bs" statement is found, the mathematician asks, "What conditions are there on a B which will be just enough to make sure it is an A?" If such a condition C is found, the result will be an "if and only if" statement:

"Something is an A if and only if it is a C and a B"

This will completely express the connection between the concepts A and B. An important example is Euler's formula for polyhedra:

$$V - E + F = 2$$

where V is the number of vertices of a polyhedron, E the number of its edges and F the number of faces. This formula is true for most simple polyhedra. It can be checked for the Platonic solids by counting, and, in the eighteenth century, Euler proved that it was true for many others. But finding exactly *which* polyhedra it was true for, that is, finding a condition C such that,

"A polyhedron is a C if and only if $V - E + F = 2$"

proved extremely difficult, and led to many conceptual advances. The story is entertainingly told in Imre Lakatos's book, *Proofs and Refutations* (1976), Cambridge University Press, Cambridge.

Since an "if and only if" statement really makes two assertions, its proof must contain two parts. The proof of "Something is an A if and only if it is a B" will look like this:

Let x be an A,

therefore, x is a B.

Let y be B,

therefore, y is an A.

So, something is an A if and only if it is a B.

Example 1

Prove that a whole number is even if and only if its square is even.

Finding a proof

The proof should look like this:

Let x be even.

Therefore, x^2 is even.

Let y be a number such that y^2 is even.

Therefore, y is even.

So a number is even if and only if its square is even.

The first half of this proof was an exercise in the last chapter. In the second half of the proof, we begin with,

Let y^2 be even,

and then write this in symbols,

$$y^2 = 2K$$

for some whole number K.

We then look for a reason why y should be even. As 2 divides the right-hand side, it also divides the left-hand side. So 2 divides one (or both) of the factors on the left. These are both y, so 2 divides y.

Proof

The final proof (including both parts) is:

Let x be even,

so that,

$$x = 2K$$

for some whole number K.

Then,

$$\begin{aligned} x^2 &= (2K)^2 \\ &= 4K^2 \\ &= 2(2K^2) \end{aligned}$$

which is even.

Therefore, if x is even then x^2 is even.

Conversely, let y be a whole number such that y^2 is even.

So,

$$y^2 = 2K$$

for some whole number K.

As 2 divides 2K, 2 divides y^2.

So 2 divides either y or y, that is, 2 divides y.

Therefore, if y^2 is even then y is even.

Therefore, a whole number is even if and only if its square is even.

Example 2

Prove that a whole number is divisible by 9 if and only if the sum of its digits is divisible by 9.

Finding a proof

The main problem is to find a way to symbolise the number so that the sum of its digits can also be written in symbols. It will be necessary to have a symbol for each digit. If the number written in the usual decimal form is,

$$a_n a_{n-1} \ldots a_2 a_1 a_0$$

(i.e. a_0 in the units column, a_1 in the tens column, and so on), then the number itself is,

$$10^n a_n + \ldots + 100a_2 + 10a_1 + a_0$$

and the sum of the digits is,

$$a_n + \ldots + a_2 + a_1 + a_0$$

The reason why one of these is divisible by 9 if and only if the other one is, is that the difference between the two numbers is,

$$\underbrace{99\ldots9}_{\text{n nines}}a_n + \ldots + 99a_2 + 9a_1$$

which is obviously divisible by 9. The proof could be set out as below.

Proof

Let x be a number with digits $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$
if x is divisible by 9 then,

$$10^n a_n + \ldots + 100a_2 + 10a_1 + a_0$$

is divisible by 9.

So,

$$(10^n a_n + \ldots + 100a_2 + 10a_1 + a_0) - (\underbrace{99\ldots9}_{\text{n nines}}a_n + \ldots + 99a_2 + 9a_1)$$

is divisible by 9,

that is,

$$a_n + \ldots + a_2 + a_1 + a_0$$

is divisible by 9,

that is, the sum of the digits is divisible by 9.

If the sum of the digits is divisible by 9,

$$a_n + \ldots + a_2 + a_1 + a_0$$

is divisible by 9,

so,

$$(a_n + \ldots + a_2 + a_1 + a_0) + (\underbrace{99\ldots9}_{\text{n nines}}a_n + \ldots + 99a_2 + 9a_1)$$

is divisible by 9,

that is, the number itself,

$$10^n a_n + \ldots + 100a_2 + 10a_1 + a_0$$

is divisible by 9.

Therefore, a number is divisible by 9 if and only if the sum of its digits is divisible by 9.

Note

In this case, where the steps in the second half are just the reverse of those in the first half, it is possible to combine the steps in a string of, "if and only ifs":
A number with digits $a_n \ldots a_2a_1a_0$ is divisible by 9,

iff $10^n a_n + \ldots + 100a_2 + 10a_1 + a_0$ is divisible by 9,

iff $(10^n a_n + \ldots + 100a_2 + 10a_1 + a_0) - (99\ldots9a_n + \ldots + 99a_2 + 9a_1)$ is divisible by 9

iff $a_n + \ldots + a_2 + a_1 + a_0$ is divisible by 9,

iff the sum of the digits is divisible by 9.

However it is rare that this is possible, and it is usually better to keep separate the two parts of an "if and only if" proof.

Exercises

(Gradings of exercises: * easy, ** moderate, *** difficult.)

*1. Are the following two "iff" statements:
"Something is an A if and only if it is a B"
and,
"Something is a B if and only if it is an A"
equivalent?

*2. Consider the statement,
"Something is an A if and only if it is a B"
(a) Write down the two assertions made by the above "iff" statement.
(b) How many parts does the proof of the given "iff" statement contain?
(c) Does the proof of the two "all" statements:
"All As are Bs"
and
"All Bs are As"
complete the proof of the given "iff" statement?

*3. (a) Give one example of a true "iff" statement.
(b) Give one example of a false "iff" statement.

***4.** Are the following "iff" statements true or false?:
(a) "An even number is prime if and only if it is 2".
(b) "An odd number is prime if and only if it is 3".

***5.** One of the following three statements is true. State which one it is.
(a) A figure is a polygon if and only if it is a triangle.
(b) A figure is a circle if and only if it is a polygon.
(c) A figure is a polygon if it is a triangle.

***6.** Prove that a whole number is odd if and only if its square is odd.

***7.** Write the "iff" statement:
x is a non-zero real number if and only if x^2 is positive in:
(a) "necessary and sufficient form.
(b) "if . . . then . . . and conversely" form.

***8.** Rewrite the "iff" statement,
"Something is an A if and only if it is a B"
in five different forms.

****9.** Prove that a number is divisible by 4 if and only if its last two digits form a number divisible by 4.

****10.** Prove that a triangle is isosceles if and only if two of its angles are equal. (An isosceles triangle is by definition a triangle with two equal sides.)

****11.** Let m and n be two integers. Show that $m^3 - n^3$ is even if and only if $m - n$ is even.

****12.** Show that a triangle has 3 equal sides if and only if it has 3 equal angles.

****13.** Show that a number is divisible by 3 if and only if the sum of its digits is divisible by 3.

****14.** Prove that the lines,

$$Ax + By + E = 0$$

and,

$$Cx + Dy + F = 0$$

are parallel if and only if $AD - BC = 0$

****15.** Prove that the simultaneous equations,

$$Ax + By + E = 0$$

and,

$$Cx + Dy + F = 0$$

have exactly one solution if and only if $AD - BC \neq 0$.

*****16.** Prove that the lines,

$$Ax + By + E = 0$$

and,

$$Cx + Dy + F = 0$$

are perpendicular if and only if $AC + BD = 0$
(Do not assume without proof that the product of the gradients of two perpendicular lines is -1.)

*****17.** Prove that three distinct points,

$$(x_1, y_1), (x_2, y_2) \text{ and } (x_3, y_3)$$

are collinear if and only if,

$$(x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1) = 0$$

*****18.** Prove that a real number is rational if and only if its decimal expansion is terminating or (eventually) repeating.

Linear algebra

****19.** Show that a vector in $\mathbf{R}^3$ is a linear combination of (1, 1, 2) and (2, 2, 3) if and only if it lies on the plane, $x = y$.

Calculus

****20.** Show that y_1 and y_2 are solutions of,

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0$$

if and only if $y_1 + y_2$ and $y_1 - y_2$ are solutions.

4

"Some" statements

An existential generalisation, or "some" statement, states that there exists something which satisfies a certain condition. For example, "245 is a multiple of 7" says,

"There exists a whole number x such that $245 = 7 \times x$"

or,

"$245 = 7 \times x$ for some whole number x"

The expressions, "there is . . .", and "there are . . .", "there exist . . .", all mean "there is at least one . . .". For example, the statement:

"There is a real number x such that $x^2 = 2$"

is true; the fact that there are two real numbers such that $x^2 = 2$ is not relevant to the truth of the statement.

To prove a "some" statement, the usual method is to exhibit or "construct" a thing satisfying the given condition.

Example 1

Prove that 245 is a multiple of 7.

Proof

We have to prove that,

$$245 = 7 \times x$$

for some x.

But,

$$245 = 7 \times 35$$

so,

$$x = 35$$

is a solution.

Therefore, 245 is a multiple of 7.

The general form of a "some" statement is,

"Some As are Bs"

or

"Some A is a B"

(For example, "Some whole number x is such that $245 = 7 \times x$".)

As with "all" statements, there are alternative forms, for example,

- Something is both an A and a B.
- There exists something that is both an A and a B.
- There is an A that is also a B.
- There is at least one A that is a B.

There is at least one "some" statement that cannot be put in the form, "Some A are B", namely, "something exists" or, "there is something". It is difficult to see how this statement can be proved; if there is a way, then it is the business of philosophy rather than mathematics. The mathematical attitude to existence is illustrated by the following old joke: A philosopher, a physicist and a mathematician were travelling by train to Victoria. Just after crossing the border they saw a paddock containing one black sheep. "Look at that", said the physicist, "Victorian sheep are black". "But", said the mathematician, "all we've established is that there exists at least one Victorian sheep that is black". The philosopher said, "What we actually *know* is that at least one Victorian sheep is black, on at least one side".

Non-constructive existence proofs, that is, proofs of the existence of things without exhibiting them, are occasionally possible.

Example 2

Show that there is a solution of $f(x) = 0$ between $x = 0$ and $x = 1$, where,

$$f(x) = x^5 - 2x^3 + 5x^2 + x - 2$$

(There is no formula for the roots of a fifth-power equation, so one cannot straightforwardly exhibit or construct the solutions, as one could with a quadratic.)

Proof

$$f(0) = -2$$

which is negative,

while,

$$f(1) = 3$$

which is positive.

So the graph of f(x) must cross the x-axis somewhere between 0 and 1; that is, there is a solution of $f(x) = 0$ somewhere between 0 and 1.

Note

This proof depends on the fact that a continuous function which is negative somewhere and positive somewhere else must be zero at some point in between. Is this obvious enough to use as the basis of a proof? For ordinary purposes, it is generally considered adequate, but work on the foundations of calculus has shown that it is possible to prove such results from ones that are perhaps a little more obvious. (See M. Spivak, *Calculus* (1980), (2nd edn), Publish or Perish, Berkeley, p. 108 or S. R. Lay, *Analysis: An Introduction to Proof* (1986), Prentice Hall, Englewood Cliffs, NJ, p. 159.)

Non-constructive existence proofs such as the above are useful in methods of approximation in computing. Once it has been established that there exists a solution of $x^5 - 2x^3 + 5x^2 + x - 2 = 0$, there is some point in applying a method, such as Newton's method, to approximate the solution. Applying such computational techniques when the solution being sought does not in fact exist can lead to strange outputs that are difficult to interpret.

Generally, constructive existence proofs are preferred to non-constructive ones, when they are possible.

Linear algebra

A sum of scalar multiples of some vectors is called a *linear combination* of them. Thus, a vector $\underset{\sim}{v}$ is a linear combination of the vectors $\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_n$ if *there exist* scalars $\alpha_1, \ldots, \alpha_n$ such that,

$$\underset{\sim}{v} = \alpha_1\underset{\sim}{v}_1 + \ldots + \alpha_n\underset{\sim}{v}_n$$

Example 3

Show that the vector $(1, 0, -1)$ in $\mathbf{R}^3$ is a linear combination of $(1, 1, 1)$ and $(1, 2, 3)$.

Finding the proof

We want to show there exist real numbers α_1 and α_2 such that,

$$(1, 0, -1) = \alpha_1(1, 1, 1) + \alpha_2(1, 2, 3)$$

so we find α_1 and α_2 by solving these equations:

$$\begin{aligned} \alpha_1 + \alpha_2 &= 1 \\ \alpha_1 + 2\alpha_2 &= 0 \\ \alpha_1 + 3\alpha_2 &= -1 \end{aligned}$$

Solving this system we find,

$$\alpha_1 = 2, \ \alpha_2 = -1$$

Proof

For the proof it is sufficient to simply write:

$$(1, 0, -1) = 2(1, 1, 1) + (-1)(1, 2, 3)$$

so $(1, 0, -1)$ is a linear combination of $(1, 1, 1)$ and $(1, 2, 3)$.

Example 4

The vectors $\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_n$ are said to be *linearly dependent* if there exist scalars $\alpha_1, \ldots, \alpha_n$, not all zero, such that,

$$\alpha_1\underset{\sim}{v}_1 + \ldots + \alpha_n\underset{\sim}{v}_n = \underset{\sim}{0}$$

Show that the vectors $(1, 1, 0)$, $(2, 1, 1)$ and $(7, 6, 1)$ are linearly dependent.

Proof

We solve,

$$a(1, 1, 0) + b(2, 1, 1) + c(7, 6, 1) = (0, 0, 0)$$

so,

$$\begin{aligned} a + 2b + 7c &= 0 \\ a + b + 6c &= 0 \\ b + c &= 0 \end{aligned}$$

Solving, we find,

$$\begin{aligned} a &= -5c \\ b &= -c \end{aligned}$$

so there are infinitely many solutions.

So there exist a, b, c, not all zero, such that,

$$a(1, 1, 0) + b(2, 1, 1) + c(7, 6, 1) = (0, 0, 0)$$

for example, $a = -5$, $b = -1$, $c = 1$.

Therefore the vectors are linearly dependent.

Calculus

Example 5

Show that there exists a (non-zero) function, y of x such that,

$$\frac{d^2y}{dx^2} = -y \quad \text{and} \quad y(0) = y(\pi) = 0$$

Proof

As is usual with differential equations, one finds the general solution of the equation, and then asks how the initial conditions restrict the general solution. The general solution of,

$$\frac{d^2y}{dx^2} = -y$$

is,

$$y = A \sin x + B \cos x$$

Substituting $x = 0$, $y = 0$ gives,

$$0 = A.0 + B.1$$

so,

$$B = 0$$

thus,

$$y = A \sin x$$

This already satisfies the second condition $y(\pi) = 0$, so there is no further restriction. There are, therefore, many functions y satisfying the conditions of the problem, and we need only exhibit one, say $y = \sin x$.

Exercises

(Gradings of exercises: * easy, ** moderate, *** difficult.)

*1. Consider the "some" statement,

"7073 is a multiple of 643"

(a) How would you attempt to prove it?
(b) Prove it.

*2. Do there exist two integers whose sum is 73 and whose difference is 11? Prove your answer.

*3. Consider the "some" statement:

"There exists a number x such that, $x^3 + 7x^2 - x - 7 = 0$"

To prove the above statement we can exhibit one x satisfying the condition,

$$x^3 + 7x^2 - x - 7 = 0$$

(a) Verify that $x = -7$ is such a number.
(b) Does the existence of one x satisfying the given condition exclude the possibility of other such xs?
(c) Is the existence of more than one x satisfying the given condition relevant to the truth of the "some" statement?

*4. A *perfect* number is one which equals the sum of its factors (counting 1 as a factor, but not the number itself). Show that there exists a perfect number.

*5. Rewrite the "some" statement,

"63 is a multiple of 9"

in four different forms.

*6. Determine the truth or falsity of the two statements:
(a) For all $x \in \mathbf{R}$, $x^2 \geqslant 0$
(b) For some $x \in \mathbf{R}$, $x^2 > 0$
and then prove your answers.

*7. Determine the truth or falsity of the following "some" statements:
(a) For some $x \in \mathbf{Z}$, $49 = 7x$
(b) For some $x \in \mathbf{Z}$, $50 = 7x$
(c) For some $x \in \mathbf{R}$, $x^2 = 16$
(d) For some $x \in \mathbf{R}$, $x^2 \geqslant 1000$
(e) For some $x \in \mathbf{R}$, $x^2 = -8$
(f) For some $x \in \mathbf{R}$, $x^3 = -8$
(g) For some $x \in \mathbf{Z}$, $x^3 = -8$
(h) For some $x \in \mathbf{R}$, $x - 7 = x$

*8. Show that there exists a real number x such that x^2 is irrational but x^4 is rational.

*9. For each of the following statements, state whether it is true or false and then prove your answer:
(a) For some positive x, both x and $x^2 + 10x + 16$ are prime.
(b) For some positive x, both x and $x^2 + 4x + 3$ are prime.
(c) For some positive x, both x and $x^2 + x + 1$ are prime.
(d) For some positive x, both x and $x^5 + 1$ are prime.

*10. For each of the following statements state whether it is true or false, then prove your answer:
(a) There exists a positive integer x such that $2^x = x^2$
(b) There exists a positive integer x such that $2^x > x^2$
(c) There exists a positive integer x such that $2^x < x^2$

*11. Show that there is a solution of, $x^{100} + 5x - 2 = 0$ between $x = 0$ and $x = 1$.

*12. Translate,

"The line, $y = 2x$ intersects the circle, $x^2 + (y - 2)^2 = 4$"

into a "there exists" statement, and prove it.

**13. Consider the infinite geometric progression,

$$\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots + \left(\frac{1}{2}\right)^n + \ldots$$

Prove that there exists an integer N such that the sum of the first N terms of the above series differs from 1 by less than 10^{-6}.

**14. Show that the curve, $x^2 + xy + y^2 = 1$ has an axis of symmetry.

**15. Show that there exist at least three different solutions of,

$$88x^3 - 126x^2 + 41x - 2 = 0$$

between $x = 0$ and $x = 1$.

••**16.** A formula for e^x is,

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

Show that $e^3 = 20.1$, correct to 1 decimal place, by showing:

(a) For any $n \geqslant 4$,

$$\frac{3^n}{n!} < \frac{9}{2}\left(\frac{3}{4}\right)^{n-3}$$

(b) Hence, that there exists N such that,

$$\frac{3^{N+1}}{(N+1)!} + \frac{3^{N+2}}{(N+2)!} + \dots < \frac{1}{20}$$

(c) Hence, that $e^3 = 20.1$ correct to 1 decimal place.

Linear algebra

•**17.** (a) Show that (1, 0, 1) is a linear combination of (0, 0, 1), (0, 1, 1) and (1, 1, 1).

(b) Is (1, 2, 1) a linear combination of (1, 3, 0), (2, 1, 0), (−5, 1, 0) and (1, 1, 0)? Prove your answer.

(c) Show that the vectors (3, 4) and (−36, −48) are linearly dependent.

••**18.** Is any linear combination of (1, 1, 0) and (2, 1, 0), other than the zero vector, also a combination of (−1, 0, 1) and (2, 1, 1)?

••**19.** Show that two vectors are linearly dependent if and only if one is a scalar multiple of the other.

Calculus

••**20.** Does there exist $x < 10{,}000$ such that,

$$\left| \frac{x^2}{1 + x^2} \right| < 0.001?$$

•**21.** Are there any differentiable functions which are equal to their second derivative but not their first?

5

Multiple quantifiers

The last three chapters studied "all" statements and "some" statements. "All" and "some" are called *quantifiers* (the universal and the existential quantifier, respectively), since they say something about how many things satisfy some condition. Next we look at statements that contain more than one quantifier. These include many of the important theorems of mathematics (and a few statements of real life as well). They require considerable logical skill, although they do not need any techniques other than those already described for "all" and "some". A number of examples are:

- Every body attracts every other body (two universals).
- Between any two real numbers there is a rational number (two universals and one existential).
- The polynomial $x^2 - 3x + 7$ has at least two real roots, that is, there exists a real number x and there exists a real number y such that $x^2 - 3x + 7 = 0$ and $y^2 - 3y + 7 = 0$ and $x \neq y$ (two existentials).
- Everybody loves somebody sometime (one universal and two existentials).
- There are infinitely many whole numbers, that is, for every number there is a bigger number (one universal and one existential).

The techniques used to prove these are the same as for single "all" and "some" statements.

Example 1

Prove that the square of every even number is a multiple of 4.

(That is, prove that if x is even then $x^2 = 4k$ for some number k.)

This contains one "all" and one "some". The "all" comes first. Something is to be proved about *all* even numbers. So the proof must begin with,

"Let x be an even number."

For each x, we have to show that $x^2 = 4k$ for *some* k, so we must find a k such that $x^2 = 4k$ (of course, we would expect it to be a different k for each x).

Proof

Let x be an even number.

Then, $x = 2y$ for some whole number y.

So,

$$x^2 = (2y)^2 = 4y^2$$

So x^2 is a multiple of 4.

Example 2

One of the most famous proofs in mathematics is Euclid's proof that there are infinitely many prime numbers. The first step of this proof is to express the statement in logical language: For every prime number, there is a larger prime number (one universal and one existential, as in the last example). Thus, the proof begins with,

"Let p be a prime number."

The aim is to find a prime number bigger than p.

Consider one plus the product of all the prime numbers up to p, that is, in symbols, let p_i stand for the ith prime (so $p_1 = 2$, $p_2 = 3$, and so on).

Suppose p is the nth prime, p_n.

Then let,

$$X = p_1p_2 \ldots p_n + 1$$

X has at least one prime factor q, and q is not any of $p_1, p_2, \ldots, p_n$, since none of these divide X.

So q is a prime different from $p_1, \ldots, p_n$; and since these are the first n primes, q is greater than p_n

This completes the proof.

Notes

1. The explanatory comment, "So $p_1 = 2$, $p_2 = 3$, and so on" is not necessary, but it is usually better to err on the side of more, rather than less, explanation. Reading proofs is not easy.
2. Much the hardest step in the proof is the one introducing the number,

$$X = p_1p_2 \ldots p_n + 1$$

   The point is to produce a number which is not divisible by any of the prime numbers up to p. The genius of the ancient Greek mathematicians in thinking of such steps is the reason why Euclid is still remembered, while his parents are not.
3. The statement "X has a prime factor" is perhaps not completely "obvious", but it is at least "well-known" that every number can be factored into primes. Ideally, this should be proved somewhere earlier.
4. The number X itself is often prime, but not always. For example,

$$2.3.5 + 1 = 31$$

   which is prime, but,

$$2.3.5.7.11.13 + 1 = 30031$$

   which is 59×509.
5. It is usual to add some statement like "this completes the proof", even in advanced mathematics, as a kind of punctuation to help people read the text quickly. In geometry, the letters Q.E.D. (abbreviation for "quod erat demonstrandum" meaning "which was to be proved") are often added at the end of a proof of a theorem. Some books write the sign ▌ at the end of proofs. Mathematics should always be read through quickly first, to "get the idea", and then read in detail later. It is often impossible to understand the details of a proof until the overall idea or strategy of the proof is understood. Regrettably, not all authors regard it as their job to help their readers by making their strategy clear.
6. What about proving, "There are infinitely many whole numbers"? Notice that this was implicitly assumed in the above proof, since we assumed the number $p_1p_2 \ldots p_n + 1$ could always be formed and that there was no danger of "sailing over the edge" of the whole number system. It has not, in fact, been found possible to prove the statement, "There are infinitely many whole numbers" (or, "For every whole number there is a larger whole number") from anything simpler. There is a problem here, if we wish to continue the practice of trying to prove everything from what is obviously true, since it seems dangerous to regard anything about the infinite as completely obvious. The situation has to be put up with, and there seems little future in doubting that there are infinitely many numbers.

Non-constructive existence proofs again

(The rest of this chapter is a little more advanced and could be left out at a first reading.)

Non-constructive existence proofs are especially common in cases where there is an "all" as well as a "some" in the statement to be proved.

Example 3

Any region in the plane can be divided in half by some vertical line. (See Figure 5.1.)

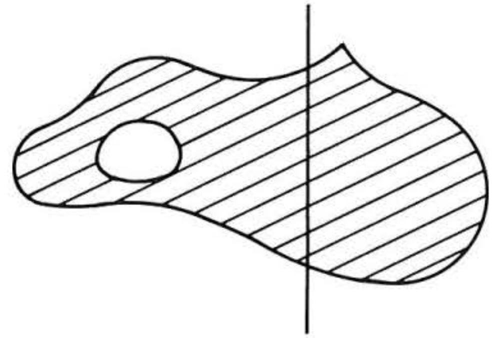

Figure 5.1

Proof

Think of a vertical line moving slowly from the left of the page to the right. As it passes over the region, the amount of shaded area to the left of the line gradually increases from zero to the whole area of the region. So somewhere in between there must be a time when the area to the left is half the whole area. ▌

Notes

1. For simple regions, such as a square or a circle, it is obvious how to construct a vertical line that divides the region in half. For regions of arbitrary shape, however, this proof gives us no method of actually drawing the line (by ruler and compass, for example).
2. The reasoning used in this proof is the same as the reasoning employed in: "The function $f(x) = x^5 + x^3 - 1$ has a zero between 0 and 1, because f(0) is negative and f(1) is positive, which means that the graph of f(x) must cross the x-axis somewhere between 0 and 1." In abstract terms, if a continuous function takes two values, it must take all the values in between as well.

One of the most famous theorems of mathematics, the *Fundamental Theorem of Algebra*, is an existence theorem with a non-constructive proof. It states that any polynomial has a root in the complex numbers. That is, for any polynomial,

$$a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$$

there exists a complex number y such that,

$$a_ny^n + a_{n-1}y^{n-1} + \ldots + a_1y + a_0 = 0$$

The proof is difficult. The first convincing proof was given by Gauss in 1799; there has been some debate as to whether his proof was sufficiently rigorous.

Symbols for quantifiers

The quantifiers are sometimes expressed in symbols: The symbol ∀ (an upside-down A, for "all") is used for the universal quantifier. It is read as, "for all", "for every" or "for each". Thus:

$$\forall x \in \mathbf{R} \quad x^2 \geqslant 0$$

is read as,

"For all x in the set of real numbers, x^2 is greater than or equal to 0"

that is, in words,

"The square of every real number is non-negative".

The symbol ∃ (a backwards E, for "exists") is used for the existential quantifier. It is read as, "for some" or "there exists". Thus,

$$\exists x \in \mathbf{R} \quad x < 0 \quad \text{and} \quad x^2 = 2$$

is read as,

"There exists x in the set of real numbers such that $x < 0$ and $x^2 = 2$"

that is,

"The square of some negative real number is 2"

One advantage of using symbols is that it makes the treatment of complicated cases clearer. For example, in statements which have more than one quantifier, it is natural to ask whether changing the order of the quantifiers makes any difference. In fact two universal quantifiers can be interchanged. Thus,

$$\forall x \in \mathbf{R} \quad \forall y \in \mathbf{R} \quad x^2 - y^2 = (x - y)(x + y)$$

is logically equivalent to,

$$\forall y \in \mathbf{R} \quad \forall x \in \mathbf{R} \quad x^2 - y^2 = (x - y)(x + y)$$

Two existential quantifiers can also be interchanged,

$$\exists n \in \mathbf{Z} \quad \exists m \in \mathbf{Z} \quad 7n + 4m = 1$$

and,

$$\exists m \in \mathbf{Z} \quad \exists n \in \mathbf{Z} \quad 7n + 4m = 1$$

both simply say, "There are two integers, m and n, such that $7n + 4m = 1$".

It is not correct, however, to interchange a universal and an existential. For example,

$$\forall n \in \mathbf{Z} \quad \exists m \in \mathbf{Z} \quad m > n$$

says that for each integer there is a bigger integer. This is true, but,

$$\exists m \in \mathbf{Z} \quad \forall n \in \mathbf{Z} \quad m > n$$

says that there is some integer that is bigger than every integer, which is false.

Once this distinction was made, one of the most famous errors in mathematics was corrected. There have been very few cases where one of the great mathematicians claimed to have proved something that later turned out to be false. One of these rare cases occurred in the early 1800s when Cauchy claimed to have proved a statement about the convergence of functions, which was soon found to be false. Cauchy's argument had used some geometrical intuitions about continuity. When these were replaced with more rigorous logical arguments (using the ideas to be described in Chapter 10) it was realised that Cauchy's argument had involved an illegitimate change in the order of quantifiers.

Linear algebra

Example 4

Show that every vector in the XY-plane in $\mathbf{R}^3$ is a linear combination of (1, 1, 0) and (2, 3, 0).

Method

We are asked to show that for *all* vectors $\underset{\sim}{v}$ in the XY-plane, *there exist* scalars a and b such that $\underset{\sim}{v} = a(1, 1, 0) + b(2, 3, 0)$.

Any vector in the XY-plane is of the form (A, B, 0) for some real numbers A and B.

We need to show how to write (A, B, 0) as $a(1, 1, 0) + b(2, 3, 0)$, so we solve,

$$(A, B, 0) = a(1, 1, 0) + b(2, 3, 0)$$

that is, we find a and b in terms of A and B,

$$a + 2b = A$$
$$a + 3b = B$$

Solving, we find $a = 3A - 2B$, $b = B - A$.

So the final proof can be written as follows.

Proof

Any vector in the XY-plane is of the form $(A, B, 0)$ for some $A, B \in \mathbf{R}$.
But,

$$(A, B, 0) = (3A - 2B)(1, 1, 0) + (B - A)(2, 3, 0)$$

so every vector in the XY-plane is a linear combination of $(1, 1, 0)$ and $(2, 3, 0)$.

Example 5

Show that a set of vectors which includes the zero vector is linearly dependent.

Proof

Let $\{\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_n\}$ be a set of vectors which includes the zero vector. That is, some $\underset{\sim}{v}_i = \underset{\sim}{0}$.
We now need to show a "there exists" statement:

"There exist scalars $a_1, \ldots, a_n$, not all zero, such that $a_1\underset{\sim}{v}_1 + \ldots + a_n\underset{\sim}{v}_n = \underset{\sim}{0}$"

We look for a particular choice of $a_1, \ldots, a_n$ that will do this, remembering that $\underset{\sim}{v}_i = \underset{\sim}{0}$. We can see that one choice is $a_i = 1$, and all other a's zero. All we need to write is:

Then,

$$0\,\underset{\sim}{v}_1 + \ldots + 0\,\underset{\sim}{v}_{i-1} + 1\,\underset{\sim}{v}_i + 0\,\underset{\sim}{v}_{i+1} + \ldots + 0\,\underset{\sim}{v}_n = \underset{\sim}{0}$$

so, $\{\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_n\}$ is linearly dependent.

(The converse of this theorem is not true—if a set of vectors is dependent, it does not follow that one of them must be zero.)

Calculus

An important use of logic is in making concepts precise through definitions. We have already seen two examples:

- "x is a multiple of y" means, "There exists z such that $x = yz$";
- "There are infinitely many primes" means, "For any prime there is a larger prime".

Such definitions not only improve precision in thinking, but prepare the ground for proofs. Logical definitions using many quantifiers are especially important in calculus. This will be treated fully in Chapter 10, but in the meantime we give a short example.

Example 6

It is well known that, "exponentials grow faster than powers of x". For example, although 2^x is less than x^{100} for many small values of x, 2^x grows faster and eventually

overtakes x^{100}. (See Figure 5.2.) How can we make this statement precise?

First, the statement is about *all* exponentials and *all* powers of x. So the statement will begin:

"For all a and for all n, . . . a^x . . . x^n . . ."

The next step is to translate the "eventually" in, "a^x is eventually greater than x^n, and stays greater after that" (the x would be different for different a and n).

The final expression is:

"For all $a > 1$ and for all positive integers n there exists x_0 such that for all $x > x_0$, $a^x > x^n$"

Note

Note that the, "for all $x > x_0$" can be placed at the end of the statement without changing its meaning, but the, "there exists y" must remain where it is, between the second and third "all". "For all $a > 1$" and "For all positive integers n" are interchangeable.

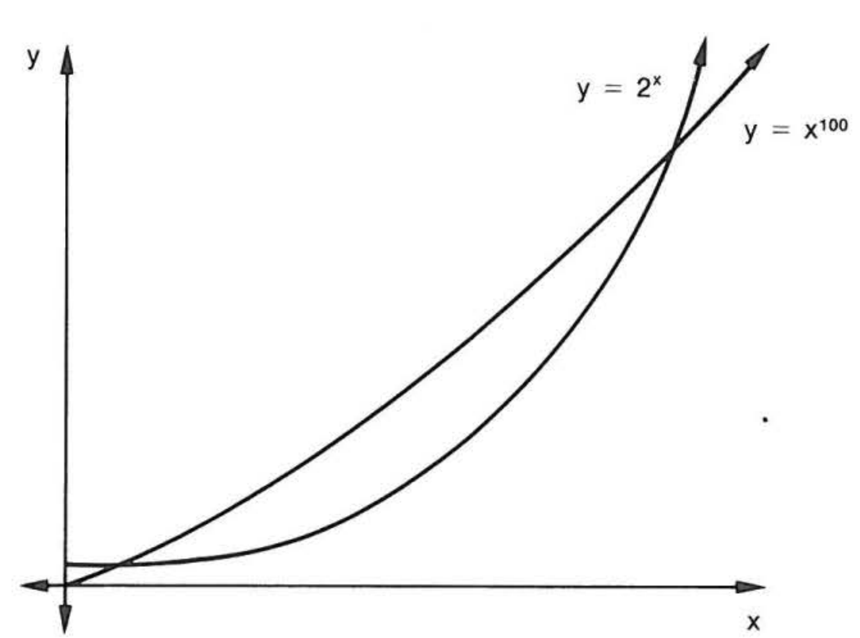

Figure 5.2

Exercises

(Gradings of exercises: * easy, ** moderate, *** difficult.)

*1. (a) How many quantifiers are there in the sentence,
"For some integers m and n, $3m + 2n = 1$"

(b) Decide whether the given statement is true or not and then prove your answer.

2. Let $f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbf{R}$.
Show that if either a or b is non-zero and if $b^2 - 4ac \geqslant 0$ then there exists $y \in \mathbf{R}$ such that $f(y) = 0$.

3. Show that for any whole number n there exists a whole number N such that,

$$\frac{1}{N} - \frac{1}{N+1} < \frac{1}{n}$$

4. (a) Show that for any real numbers a, b, c, d, e, f, if,

$$ad - bc \neq 0$$

then there exists a solution of the simultaneous equations:

$$ax + by = e$$
$$cx + dy = f.$$

(b) Is it true that if there exists a solution then $ad - bc \neq 0$?
(c) Is it true that if $ad - bc = 0$, then there does not exist a solution?

5. (a) Prove that if a and b are rational numbers with $a \neq b$ then,

$$a + \frac{1}{\sqrt{2}}(b - a)$$

is irrational.
(b) Hence prove that between any two rational numbers there is an irrational number.

6. Show that if three points in the plane have their x-coordinates all different, there is a quadratic passing through all three points.

7. Show that for any integer $n \geqslant 1$, $n^5 - n$ is divisible by 5.

8. Prove that between any two irrational numbers there is a rational number.

9. (a) Show that if a positive integer x is not divisible by any positive integer less than or equal to $\sqrt{x}$ then it is a prime number.
(b) Is the statement still true if "or equal to" is omitted?

10. Prove that if a plane figure has an axis of symmetry, and if a point on the axis is a centre of symmetry, then the figure has another axis of symmetry. (The terminology is explained in Chapter 2, Exercise 28.)

11. Every one of six points is joined to every other one by either a red or a blue line. Show that there exist three of the points joined by lines of the same colour.

Linear algebra

12. Show that any vector $(x, y, z) \in \mathbf{R}^3$ is a linear combination of (1, 0, 0), (1, 1, 0) and (1, 1, 1).

13. (a) Show that a set of vectors, one of which is a scalar multiple of another, is linearly dependent.
(b) Show that a set of vectors, one of which is the sum of two of the others, is linearly dependent.

14. Show that the vectors $\{\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_n\}$ are linearly dependent if and only if, for some i, $\underset{\sim}{v}_i$ is a linear combination of $\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_{i-1}$.

Calculus

15. Express in logical terms: n! grows faster than 2^n.

16. Does there exist a function, other than the zero function, such that,

$$f(0) = f'(0) = f''(0) = \ldots = f^{(n)}(0) = \ldots = 0?$$

6

"Not", contradiction and counterexample

"Not" and counterexamples

If "not" is put in front of a statement it negates the statement. For any statement p, either "p", or "not-p" is true and the other is false. "Not-p" can also be expressed as "p is false". It is sometimes called the *negation* (or *contradictory*) of p. For example, "$\sqrt{2}$ is irrational" means the same as "not-($\sqrt{2}$ is rational)", that is, in everyday English, "$\sqrt{2}$ is not rational". This statement is the negation of "$\sqrt{2}$ is rational".

We next consider the negations of "all" and "some" statements. The statement "all animals are horses" is false (so its negation, "not all animals are horses", is true). The reason the first statement is false is that there are some animals that are not horses.

In general, the negation of "All As are Bs" is logically equivalent to the statement "Some A is not a B". If "Not all As are Bs" is true, then "Some A is not a B" is true, and vice versa; if one is false, the other is false. Therefore, to show that a "not all" statement is true, or to show that an "all" statement is false, it is necessary to find one *counterexample*.

Example 1

Is "All multiples of 3 are multiples of 6" true or false? Prove your answer.

Answer

False, because 9 is a multiple of 3, but is not a multiple of 6. ▮

(9 is called a "counterexample" to the "all" statement "All multiples of 3 are multiples of 6".)

Example 2

Is "All prime numbers are odd" true or false? Prove your answer.

Answer

False, because 2 is a prime number and it is not odd. ▮

The above examples demonstrate that showing "All As are Bs" is false, requires a quite different strategy to that required to show that the statement is true. To prove, "All As are Bs" is true, we must show something about *every* A, and so the proof begins with "Let x be an A" and must derive "x is a B" from this. But to show "All As are Bs" is *false*, all we need to do is exhibit *one* counterexample, that is, a single A that is not a B.

Usually an "all" statement that is false has many counterexamples, but when proving it is false, it is considered stylish to exhibit just one (as in the first example above.)

"Not all As are Bs" is definitely not equivalent to, "All As are not Bs", that is, "No As are Bs". For example, "Not all animals are horses" is true, (since there are animals that are not horses, such as sloths), while "No animals are horses" is false (since some animals *are* horses). This is illustrated clearly in the Venn diagram in Figure 6.1.

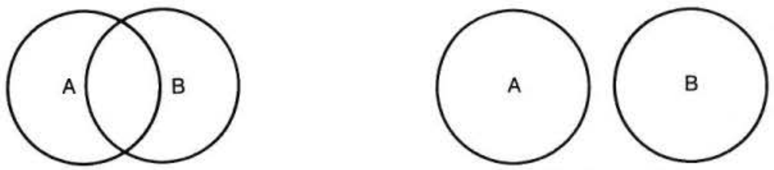

Figure 6.1 "Not all As are Bs" "No As are Bs"

(It should be noted that occasionally in everyday English "All As are not Bs" *is* used to mean "Not all As are Bs", as in "All is not well" and "All that glitters isn't gold".)

The contrapositive

It is sometimes natural to introduce "Not" into a proof to make the proof easier. In Chapter 3 we proved the "all" statement, "For all whole numbers y, if y^2 is even then y is even", using a direct proof. We could have, however, argued equally well that, "If y is *not* even, it is odd. As we can show that the square of an odd number is odd, not even, then, if the square of y is even, y itself cannot be odd. So y is even." Such an argument illustrates the following important logical equivalence. The "all" statement,

"All As are Bs"

is logically equivalent to the statement, called its *contrapositive*,

"All not-Bs are not-As"

(or, "Anything that is not a B is not an A")

(Note carefully the opposite order of B and A in the two statements.)

The equivalence of an "all" statement and its contrapositive is clear from a Venn diagram. If "All As are Bs" is true, the set of As is inside the set of Bs. (See Figure 6.2.) Anything that is not a B is outside the set of Bs, and hence is outside the set of As as well (i.e. is not an A).

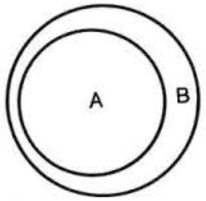

Figure 6.2

(One could also prove the "if and only if" statement, "All As are Bs if and only if all not-Bs are not-As", using the techniques of Chapter 3.) For example, all horses are animals, so the set of horses lies inside the set of animals. Therefore, whatever is not an animal is not a horse. So,

"All horses are animals"

is logically equivalent to,

"All not-animals are not-horses"

This last statement would more naturally be expressed as, "Anything that is not an animal is not a horse".

Example 3

The proof of "For all whole numbers y, if y^2 is even then y is even" can therefore be written as shown in the proof below.

Proof

Let x be an odd whole number.

Then,

$$x = 2k + 1$$

for some whole number k.

So,

$$\begin{aligned} x^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned}$$

So x^2 is odd.

We have proved that if x is odd then x^2 is odd. Therefore, if x^2 is not odd then x is not odd, since this statement is the contrapositive of the last one. Therefore, if x^2 is even then x is even.

Proof by contradiction

Proof by contradiction, or *reductio ad absurdum* proof, works by assuming the negation of the proposition to be proved and deducing a contradiction. This shows the negation is false, and hence that the original proposition is true. Example 4 is the classic proof of this kind.

Example 4

Prove that $\sqrt{2}$ is irrational.

Finding the proof

A number is rational when it is of the form m/n for some integers m and n. So we are asked to show that for all integers m and n, m/n is not $\sqrt{2}$. The most natural thing to do is to assume that m/n *is* $\sqrt{2}$, for some integers m and n, and deduce a contradiction.

The first line should be:

Suppose,

$$\frac{m}{n} = \sqrt{2}$$

for some integers m and n.

By cancelling out any common factors, we can suppose m and n have no common factor. (This is often done with fractions, but the reason for doing it here is only clear later in the proof.)

The only possible way to proceed is to apply the definition of $\sqrt{2}$: It is the number whose square is 2. So we square both sides:

$$\frac{m^2}{n^2} = 2$$

The equation becomes simpler if we multiply by n^2:

$$m^2 = 2n^2$$

At this stage some new idea is needed. Since we are dealing with integers, it may be useful to look at divisibility: The right-hand side is even (divisible by 2) so the left-hand side is too. So,

$$m^2 \text{ is even}$$

Therefore,

$$m \text{ is even}$$

So, m = 2k for some integer k.

Putting this into the equation,

$$m^2 = 2n^2$$

gives,

$$\begin{aligned} (2k)^2 &= 2n^2 \\ 4k^2 &= 2n^2 \\ 2k^2 &= n^2 \end{aligned}$$

Therefore, n^2 is even.

So n is even.

We have shown that m and n are both even, that is, they have 2 as a common factor. This contradicts the original assumption that m and n have no common factor.

Therefore $\sqrt{2}$ is not rational. The final version of this proof would normally be written as shown below:

Proof

Suppose,

$$\frac{m}{n} = \sqrt{2}$$

for some integers m, n.

By cancelling any common factors, we may suppose m and n have no common factor.

Then,

$$\frac{m^2}{n^2} = 2$$
$$m^2 = 2n^2$$

So m^2 is even and thus m is even.

So,

$$m = 2k$$

for some integer k.

So,

$$(2k)^2 = 2n^2$$
$$4k^2 = 2n^2$$
$$2k^2 = n^2$$

So n^2 is even and thus n is even.

So m and n are even, contradicting the assumption that m and n have no common factor.

So $\sqrt{2}$ is not of the form m/n, so $\sqrt{2}$ is irrational. ▮

This is a very neat proof, but it is important to understand that a large part of the thinking happens before the first line is written. This fact is covered up in the final version.

In principle, any proof that a number is irrational can be expected to be a *reductio ad absurdum*. The irrationality of e is not difficult to prove; a proof can be found in M. Spivak, *Calculus* (1980), (2nd edn) p. 353 or in R. Courant and H. Robbins, *What is Mathematics?* (1941), Oxford University Press, London, pp. 298–9. Proving the irrationality of π is however more difficult. Although π has been well-known since ancient times, its irrationality was only proved (by Lambert) around 1761. A proof can be found in M. Spivak, *Calculus* (1980), (2nd edn), Publish or Perish, Berkeley, Chap. 16.

Sometimes there is a choice about whether a proof is to be treated as a *reductio ad absurdum* or not. For example, the proof that there are infinitely many prime numbers could begin: "Suppose there is a largest prime number", and deduce a contradiction, by showing there is a larger one. Usually *reductio ad absurdum* arguments are avoided where there is a choice, since a string of statements that are all false is harder for the mind to deal with than a string of true statements, which can be understood.

"Not" and multiple quantifiers (more advanced)

At the beginning of this chapter we saw that the contradictory of an "all" statement is a "some . . . not" statement. This can be stated more formally as:

If Px is some statement involving x,

$$\text{Not } \forall x \text{ Px}$$

is logically equivalent to,

$$\exists x \text{ Not Px}$$

That is, to say that "Not all x are P" is the same as to say that "Some x is not P". Thus, "Not all animals are horses" is equivalent to "Some animal is not a horse".

Similarly,

$$\text{Not } \exists x \text{ Px}$$

is logically equivalent to,

$$\forall x \text{ Not Px}$$

These patterns can in fact be extended to negate (i.e. find the contradictory of) statements with multiple quantifiers. Consider,

$$\text{Not } \forall x\ \exists y \text{ Qxy}$$

where Qxy is some statement involving x and y. First, "not all" is "some . . . not", so this is equivalent to,

$$\exists x \text{ Not } \exists y \text{ Qxy}$$

Then, "not . . . some" is "all . . . not", so this is equivalent to,

$$\exists x\ \forall y \text{ Not Qxy}$$

Example 5

Is it true that,

$$\forall x \in \mathbf{R} \quad \exists y \in \mathbf{R} \quad y^2 = x$$

(i.e. does every real number have a real square root?)

Prove your answer.

Proof

The statement is not true.
A counterexample is,

$$x = -3$$

Because -3 is not the square of any real number, as the square of any real number is positive, this proves,

$$\forall y \in \mathbf{R} \quad y^2 \neq -3$$

So,

$$\exists x \in \mathbf{R} \quad \forall y \in \mathbf{R} \quad y^2 \neq x$$

This is equivalent to,

$$\text{Not} \quad \forall x \in \mathbf{R} \quad \exists y \in \mathbf{R} \quad y^2 = x$$

which we had to show.

Note

It is intuitively obvious that to show that "Every real number has a real square root" is false, we must find a number that does not have a real square root. The use of symbols to make this intuition explicit helps us to understand it better and allows us to generalise to more complex cases. A general rule to find the contradictory of a statement with any number of quantifiers is: Change each ∀ to ∃, and vice versa, and put the "not" after the quantifiers. For example,

$$\text{Not} \quad \forall x \quad \exists y \quad \forall z \quad Rxyz$$

is equivalent to,

$$\exists x \quad \forall y \quad \exists z \quad \text{Not} \quad Rxyz$$

Linear algebra

Example 6

Show that (1, 0, 1) is not a linear combination of (2, 3, 0) and (1, 1, 0).

Proof 1

If (1, 0, 1) were a linear combination of (2, 3, 0) and (1, 1, 0), there would exist scalars a and b such that,

$$(1, 0, 1) = a(2, 3, 0) + b(1, 1, 0)$$

So,

$$\begin{aligned} 2a + b &= 1 \\ 3a + b &= 0 \\ 0a + 0b &= 1 \end{aligned}$$

The last equation is $0 = 1$, which is a contradiction. So we have shown (using proof by contradiction) that (1, 0, 1) is not a linear combination of (2, 3, 0) and (1, 1, 0). ▌

Proof 2

All linear combinations of (2, 3, 0) and (1, 1, 0) have last co-ordinate zero; (1, 0, 1) does not, so it cannot be a linear combination of (2, 3, 0) and (1, 1, 0). ▌

Recall that vectors $\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_n$ are *linearly dependent* if there exist scalars $a_1, \ldots, a_n$, not all zero, such that,

$$a_1\underset{\sim}{v}_1 + \ldots + a_n\underset{\sim}{v}_n = \underset{\sim}{0}$$

"Linearly independent" means, "not linearly dependent". Thus vectors are linearly independent when the *only* solution of the equation,

$$a_1\underset{\sim}{v}_1 + \ldots + a_n\underset{\sim}{v}_n = \underset{\sim}{0}$$

is,

$$a_1 = \ldots = a_n = 0$$

Example 7

Show that if $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are linearly independent vectors, then $\underset{\sim}{u} + \underset{\sim}{v}$ and $\underset{\sim}{u} - \underset{\sim}{v}$ are also independent.

Proof

Suppose $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are independent.

We wish to show $\underset{\sim}{u} + \underset{\sim}{v}$ and $\underset{\sim}{u} - \underset{\sim}{v}$ are independent.

Let,

$$a(\underset{\sim}{u} + \underset{\sim}{v}) + b(\underset{\sim}{u} - \underset{\sim}{v}) = \underset{\sim}{0}$$

So,

$$(a + b)\underset{\sim}{u} + (a - b)\underset{\sim}{v} = \underset{\sim}{0}$$

But, since $\underset{\sim}{u}, \underset{\sim}{v}$ are independent, the only solution of,

$$A\underset{\sim}{u} + B\underset{\sim}{v} = \underset{\sim}{0}$$

is,

$$A = B = 0$$

Thus,

$$\begin{aligned} a + b &= 0 \\ a - b &= 0 \end{aligned}$$

Solving this system, we find $a = 0$ and $b = 0$.

Therefore, we have shown that the only solution of,

$$a(\underset{\sim}{u} + \underset{\sim}{v}) + b(\underset{\sim}{u} - \underset{\sim}{v}) = \underset{\sim}{0}$$

is,

$$a = b = 0$$

So $\underset{\sim}{u} + \underset{\sim}{v}$ and $\underset{\sim}{u} - \underset{\sim}{v}$ are linearly independent. ▌

Calculus

Example 8

Is it true that if a function f is increasing everywhere (i.e. if $x > y$ then $f(x) > f(y)$), then its derivative is always greater than zero?

Answer

This is not true. A counterexample is,

$$f(x) = x^3$$

It is increasing everywhere, but its derivative at zero is zero. ▮

Exercises

(Gradings of exercises: * easy, ** moderate, *** difficult.)

*1. What is the common feature in showing:
(i) a "'not all" statement to be true?
(ii) an "all" statement to be false?
Give an example in each case.

*2. You are told that the statement, "Not all As are Bs", is true.
(a) What can you say about the truth or falsity of the statement, "Some A is not a B"?
(b) Give an example.

*3. What is the contradictory of each of the following statements?
(a) All cars are red.
(b) All men are humans.

*4. (a) If a statement p is false what can you say about not-p?
(b) If a statement p is true what can you say about not-p?

*5. (a) Write down the contradictory of the following statement, "My car is red".
(b) Write down the contradictory of the following "all" statement, "All cars are red", in:
(i) "not-all" form.
(ii) "some" form.
(c) Write down the contradictory of the "some" statement, "For some $x \in \mathbf{R}$, $x^2 = 2$"

*6. Disprove, by counterexample, each of the following statements:
(a) "For all integers m and n, $m + n$ is positive".
(b) "If p and q are irrational numbers then $p + q$ is irrational".
(c) "For all $x \in \mathbf{R}$,

$$\frac{1}{x(x+1)} = \frac{1}{x} + \frac{1}{x+1}$$

*7. Give an example to show that, "Not all A are B" and, "Some A are B" can both be true.

*8. Is the following statement true or false? Prove your answer.

"For all real numbers a, b, c, if $ac = bc$ then $a = b$"

*9. Comment on the reasoning: "One dog has four legs, and no dog has five legs. Therefore one dog has nine legs".

*10. Use proof by counterexample to prove the truth or falsity of the following statements:
(a) "For all $x \in \mathbf{R}$, $x^2 > 0$"
(b) "Not all multiples of 6 are multiples of 9".

*11. Are the following statements true or false? Prove your answers.
(a) Let a, b, c be three integers. If a divides c and b divides c, then either a divides b or b divides a.
(b) If a, b, c, d are real numbers with $a < b$ and $c < d$, then $ac < bd$.

**12. (a) Explain, by listing the steps, how proof by contradiction (*reductio ad absurdum*) works.
(b) What steps would you use to prove by contradiction that $\sqrt{3}$ is irrational?
(c) Carry out the proof that $\sqrt{3}$ is irrational.

**13. (a) Write down the contradictory of the "all" statement, "All As are not Bs".
(b) What would a direct proof of the statement, "All As are not Bs", look like?
(c) Often it is hard to prove, "All As are not Bs" by proceeding as for any universal. Instead we resort to proof by contradiction. Explain how proof by contradiction works in proving, "All As are not Bs".
(d) Apply this to the proof of "If x^2 is even, then x is not odd".

**14. Show that no points of the form $(\cos\theta, \sin\theta)$ lie inside the triangle with vertices (0, 0), (0, 0.9) and (0.9, 0).

**15. Prove that the square of every number is not of the form $4k + 2$ or $4k + 3$. Hence show that no number of the form 100 . . . 003 is a square. (A direct proof is suggested.)

**16. Prove that each of the following is irrational:
(a) $\sqrt[3]{4}$
(b) $1 + \sqrt{3}$
(c) $\sqrt{3} - \sqrt{2}$

**17. Show that if a, b, c and d are rational and $\sqrt{b}$ and $\sqrt{d}$ are irrational and,

$$a + \sqrt{b} = c + \sqrt{d}$$

then,

$$a = c \quad \text{and} \quad b = d$$

18. Use proof by contradiction to show that the set of primes is infinite.

19. Show that if n is a whole number, then $2^n - 1$ and $2^n + 1$ are usually not both prime. What is/are the exceptional case(s)?

20. Show that if $2^n - 1$ is prime, then n is prime.
Hint: Consider the contrapositive and use the formula,

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \dots + y^{n-1})$$

21. Show that if the product of two whole numbers is even, then at least one of the numbers is even.

22. Show that the product of three consecutive positive integers is never a cube.

23. Is "All As are not Bs" logically equivalent to "All Bs are not As"? Explain.

24. Show that there do not exist three consecutive whole numbers such that the cube of the greatest equals the sum of the cubes of the other two.

25. Show that if m and b are real numbers with $m \neq 0$, and the function f is defined by,

$$f(x) = mx + b$$

then if $x \neq y$, $f(x) \neq f(y)$.

26. The points A, B, C, D are the midpoints of the sides of a square. Show that not all the angles in the octagon in the middle are equal. See Figure 6.3.

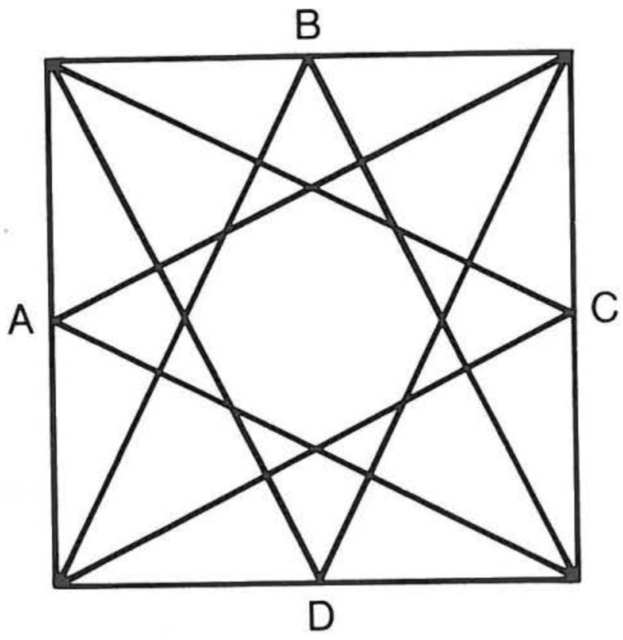

Figure 6.3

27. In plane geometry, it can be proved that the exterior angle of a triangle is equal to the sum of the two opposite interior angles. That is, in Figure 6.4, $\angle C = \angle A + \angle B$.

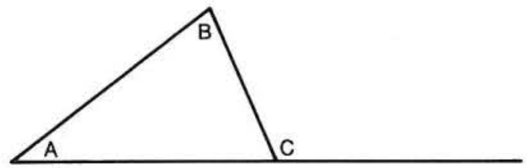

Figure 6.4

Prove by contradiction, using this fact, that if a line cuts two lines such that the alternate interior angles are equal (angles E and F in Figure 6.5) then the two lines are parallel (i.e. they do not meet).

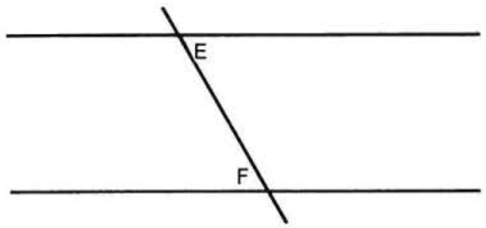

Figure 6.5

28. Prove that if three points in the plane are not collinear, there exists one, and only one, circle passing through them.

Linear algebra

29. Show that the vectors (1, 1, 1), (0, 1, 1) and (0, 0, 1) are linearly independent.

30. Is it true that, if $\underset{\sim}{v}_n$ is not a linear combination of $\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_{n-1}$, then $\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_n$ are linearly independent?

31. If $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are independent, do there exist scalars a and b such that $\underset{\sim}{u} + a\underset{\sim}{v}$ and $\underset{\sim}{u} + b\underset{\sim}{v}$ are dependent?

32. If $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are independent, do there exist scalars a, b, c, d, all non-zero, such that $a\underset{\sim}{u} + b\underset{\sim}{v}$ and $c\underset{\sim}{u} + d\underset{\sim}{v}$ are dependent?

33. Prove that a set of vectors $\underset{\sim}{v}_1, \dots, \underset{\sim}{v}_n$ is linearly independent (i.e. the only solution of $a_1\underset{\sim}{v}_1 + \dots + a_n\underset{\sim}{v}_n = 0$ is $a_1 = \dots = 0$), if and only if none of the vectors is a linear combination of the others.

Calculus

34. Is it true that every continuous function is differentiable?

35. Is it true that if the derivative of a function is always positive, then its graph crosses the x-axis?

****36.** (a) Is it true that if $f(x)$ has a minimum at $x = a$, then $f'(a) = 0$ and $f''(a) > 0$?

(b) Is it true that if $f(x)$ is differentiable and has a minimum at $x = a$, then $f'(a) = 0$ and $f''(a) > 0$?

****37.** (a) Show that every cubic has a real root.

(b) Is the same true for quartics?

(c) Generalise.

****38.** Rolle's theorem states that if $f(x)$ is a function which is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), and if $f(a) = f(b)$, then there exists c between a and b such that $f'(c) = 0$.

(a) Show that the theorem is not true if "$[a, b]$" is replaced by "(a, b)".

(b) Show that the theorem is not true if the condition "differentiable on the open interval (a, b)" is removed.

7
Sets

The language of sets has proved very convenient for expressing a wide variety of mathematical statements. It also aids mathematical thinking by allowing one to think of a collection of related things as a single entity, or set, which can then be given a name or symbol. (The mathematics student will already appreciate the importance of symbols.) The things in a set are called *members*, or *elements*. Sets may be defined by listing the elements in them. For example, $\{1, \pi, 107\}$ is the set consisting of the three numbers: 1, π, and 107. (The order in which the elements are listed is immaterial, so that $\{1, \pi, 107\} = \{107, 1, \pi\}$.) Almost always, however, we collect things together for a reason, that is, because they have some property in common. One speaks of "The set of Fs", meaning the set of all things which have the property F. This set is denoted by,

$$\{x \mid Fx\} \quad \text{or} \quad \{x : Fx\}$$

and read as,

"The set of xs such that Fx"

(There is no significance in choosing x; $\{y \mid Fy\}$ is the same set as $\{x \mid Fx\}$.)

Some sets have special names:

N is the set of natural numbers
Z is the set of integers
Q is the set of rational numbers
R is the set of real numbers
C is the set of complex numbers.

The symbol for "is a member of" is $\in$. Thus,

$$13 \in \{x \mid x \text{ is prime and } 10 < x < 20\}$$

So, "$y \in \{x \mid Fx\}$" is logically equivalent to "Fy".

Two sets are defined as the same, or equal, if they have the same members. In this respect, sets are unlike properties. For example, "having 3 angles equal" and, "having 3 sides equal" are different properties of triangles, but the sets,

$$\{x \mid x \text{ is a triangle with 3 equal angles}\}$$

and,

$$\{x \mid x \text{ is a triangle with 3 equal sides}\}$$

are the same set, since a triangle has 3 equal sides if and only if it has 3 equal angles. Thus for sets A and B, "A = B" means "$x \in A$ if and only if $x \in B$". Similarly, "$A \subset B$" (read as, "A is a subset of B") means that everything in A is also in B, that is, "if $x \in A$, then $x \in B$" or "All members of A are members of B" (the case A = B is not excluded).

Often, we define a subset of a given set by choosing those members that have some common property. Thus,

$$\{x \in \mathbf{R} \mid x^2 = 2\},$$

which is read as, "The set of x in **R** such that $x^2 = 2$", is the set of real numbers whose square is 2, that is,

$$\{\sqrt{2}, -\sqrt{2}\}.$$

Relations between sets are sometimes depicted by *Venn diagrams* (called after the nineteenth-century English logician, clergyman and historian John Venn, although actually invented by Leibniz). The elements of a set A are represented by points inside a circle. (If there are many sets, figures other than circles may be needed.) The points outside the circle represent those things not in the set A. (Figure 7.1.) The set of things not in set A is called the *complement* of A, and is denoted by $\bar{A}$ or $\tilde{A}$.

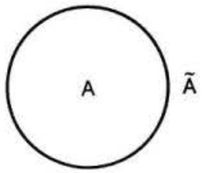

Figure 7.1

For any two sets A and B, the *intersection*, $A \cap B$, is the set of things that are in both A and B. (See Figure 7.2.) Thus,

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

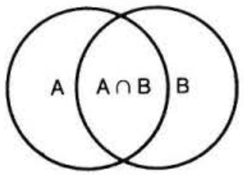

Figure 7.2

The *union* $A \cup B$ is the set of things which are in either A or B. (See Figure 7.3.) Thus,

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

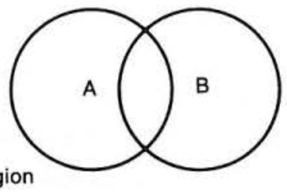

Figure 7.3 $A \cup B$ is the shaded region

Note that "or" is always used inclusively in mathematics, that is, to mean "and/or". Thus, "x is an F or x is a G" is true when x is an F, or a G, or both.

$A \subset B$ is represented in Figure 7.4.

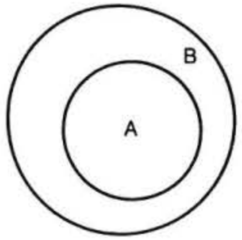

Figure 7.4

This diagram helps to clarify the reason why "All As are Bs" is logically equivalent to its contrapositive, "All not-Bs are not-As". If all points in A are in B, then obviously all points outside B are outside A, and vice versa. Any "all" statement can be represented by this diagram, since "All As are Bs" is logically equivalent to "The set of all As is a subset of the set of all Bs". Venn diagrams are not usually used in representing "some" statements.

Since "$A \subset B$" means, "All members of A are members of B" or, "If $x \in A$ then $x \in B$", the proof of a statement of the form "$A \subset B$" must look like this:

Let $x \in A$.

Therefore, $x \in B$.

Similarly, proving "A = B" involves proving an "if and only if" statement, so the proof of "A = B" will usually look like this:

Let $x \in A$.

Therefore $x \in B$.

Then, let $x \in B$.

Therefore, $x \in A$.

Example 1

Prove that the set of multiples of 4 is a subset of the set of even numbers.

Proof

Let x be an element of the set of multiples of 4.

That is, x is a multiple of 4.

So,

$$x = 4k$$

for some integer k,

$$= 2(2k)$$

Therefore, x is a multiple of 2, that is, x is even. So x is in the set of even numbers. This has shown that any member of the set of multiples of 4 is a member of the set of even numbers. So the set of multiples of 4 is a subset of the set of even numbers. ▌

Note

The example is artificially simple, in that the result to be proved does not really say any more than, "All multiples of 4 are even"; the reference to sets adds little in this case. But the example is useful for illustrating something important that happens in general: when proving an "$A \subset B$" statement using,

Let $x \in A$.

Therefore, $x \in B$.

The line after "Let $x \in A$" will usually be an explanation of what "$x \in A$" means. In the above case, "x is in the set of multiples of 4" means just "x is a multiple of 4".

The next example shows a general result about sets, rather than a result about a particular set (such as the set of even numbers). Nevertheless, the techniques of proof are the same as those described above.

Example 2

Prove that $(A \cap B) \cap C = A \cap (B \cap C)$ for any sets A, B, C.

Proof

Let,

$$x \in (A \cap B) \cap C$$

So,

$$x \in A \cap B \quad \text{and} \quad x \in C$$

So,

$$x \in A \quad \text{and} \quad x \in B \quad \text{and} \quad x \in C$$

So,

$$x \in A \quad \text{and} \quad x \in B \cap C$$

So,

$$x \in A \cap (B \cap C)$$

Therefore,

$$(A \cap B) \cap C \subset A \cap (B \cap C)$$

Now let,

$$x \in A \cap (B \cap C)$$

So,

$$x \in A \quad \text{and} \quad x \in B \cap C$$

So,

$$x \in A \quad \text{and} \quad x \in B \quad \text{and} \quad x \in C$$

So,

$$x \in A \cap B \quad \text{and} \quad x \in C$$

So,

$$x \in (A \cap B) \cap C$$

Therefore,

$$A \cap (B \cap C) \subset (A \cap B) \cap C$$

So we have shown that,

$$(A \cap B) \cap C = A \cap (B \cap C)$$ ▌

Diagrams are not considered adequate as a proof of such results, though they are often helpful. Diagrams would be impossible to understand when working with more than about four or five sets.

Example 3

For any integer n, the *ideal* $n\mathbf{Z}$ is defined to be the set,

$$\{x \in \mathbf{Z} \mid x = ny \text{ for some } y \in \mathbf{Z}\}$$

(i.e. the set of all multiples of n). Prove that $n\mathbf{Z} \subset m\mathbf{Z}$ if and only if n is a multiple of m.

Proof

The proof will look like this:

Let $n\mathbf{Z} \subset m\mathbf{Z}$.

Therefore, n is a multiple of m.

Let n be a multiple of m.

Therefore, $n\mathbf{Z} \subset m\mathbf{Z}$.

Inside the second half, there must be,

Let, $x \in n\mathbf{Z}$.

So, $x \in m\mathbf{Z}$,

in order to derive, $n\mathbf{Z} \subset m\mathbf{Z}$.

The full proof is:

Let,

$$n\mathbf{Z} \subset m\mathbf{Z}$$

Now,

$$n \in n\mathbf{Z}$$

So,

$$n \in m\mathbf{Z}$$

Therefore,

$$n = my$$

for some $y \in \mathbf{Z}$ (from the definition of $m\mathbf{Z}$).

That is, n is a multiple of m.

So we have proved that if $n\mathbf{Z} \subset m\mathbf{Z}$ then n is a multiple of m.

Now suppose n is a multiple of m,

So,

$$n = mz$$

for some $z \in \mathbf{Z}$.

Let,

$$x \in n\mathbf{Z}$$

So,

$$x = ny$$

for some $y \in m\mathbf{Z}$

$$= mzy$$

since $n = mz$

That is, x is a multiple of m.

So,

$$x \in m\mathbf{Z}$$

Thus,

$$n\mathbf{Z} \subset m\mathbf{Z}$$

So we have proved that if n is a multiple of m, then $n\mathbf{Z} \subset m\mathbf{Z}$.

Altogether we have proved that $n\mathbf{Z} \subset m\mathbf{Z}$ if and only if n is a multiple of m. ▮

Notes

1. The step:

$$n\mathbf{Z} \subset m\mathbf{Z}$$

Now,

$$n \in n\mathbf{Z}$$

So,

$$n \in m\mathbf{Z}$$

is an example of the more general argument form,

All-As are Bs
x is an A
therefore, x is a B.

That is, what is true of all As is true of any particular one. There is another example of this type of argument in Exercise 14(b) below.

2. The set with no members is called the *empty set*, and is denoted $\varnothing$. (Does the empty set really exist? Philosophically inclined readers may like to regard it as a useful fiction, like the number zero.) The empty set is considered to be a subset of *every* set. This has some technical advantages, such as making the number of subsets of a set of n elements equal to 2^n. (See Example 5 in Chapter 8.) Now,

$$\varnothing \subset A$$

is equivalent to the "if" statement "If $x \in \varnothing$ then $x \in A$". However, $x \in \varnothing$ is false (for any x), so we must make a convention about statements "If p then q" in cases where p is false. The convention is to count all such statements as true. This is consistent with counting $\varnothing \subset A$ as true. Similarly,

$$\varnothing \subset A$$

is equivalent to the "all" statement,

"All members of $\varnothing$ are members of A"

but there are no members of $\varnothing$. The convention is to count "All As are Bs" as true if there are no As.

These rather odd conventions are technically useful, for example in starting inductions where the case $n = 1$ (or $n = 0$) means an empty set is being considered. The point of the conventions is to separate the question "Does being a B follow from being an A?", from the question "Are there any As?".

Linear algebra

The *span* of a set of vectors is the set of all linear combinations of them. We write this in symbols thus:

If,

$$S = \{\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_n\}$$

then,

$$\text{span } S = \{\underset{\sim}{v} : \underset{\sim}{v} = a_1\underset{\sim}{v}_1 + \ldots + a_n\underset{\sim}{v}_n \text{ for some scalars } a_1, \ldots, a_n\}$$

For example, in $\mathbf{R}^3$,

$$\text{span } \{(1, 0, 0), (0, 1, 0)\}$$

is the xy-plane, since,

$$\begin{aligned}\text{span } \{(1, 0, 0), (0, 1, 0)\} &= \{v : v = a(1, 0, 0) + b(0, 1, 0)\} \\ &= \{(a, b, 0)\}\end{aligned}$$

that is, the set of all vectors with the z-coordinate zero, which is the xy-plane.

This example gives some sense to the word "span". The two vectors are a kind of "framework" that is enough to "support" the whole plane, in the sense that every point in the plane can be got by extending out from the two vectors (1, 0, 0) and (0, 1, 0). (See Figure 7.5.)

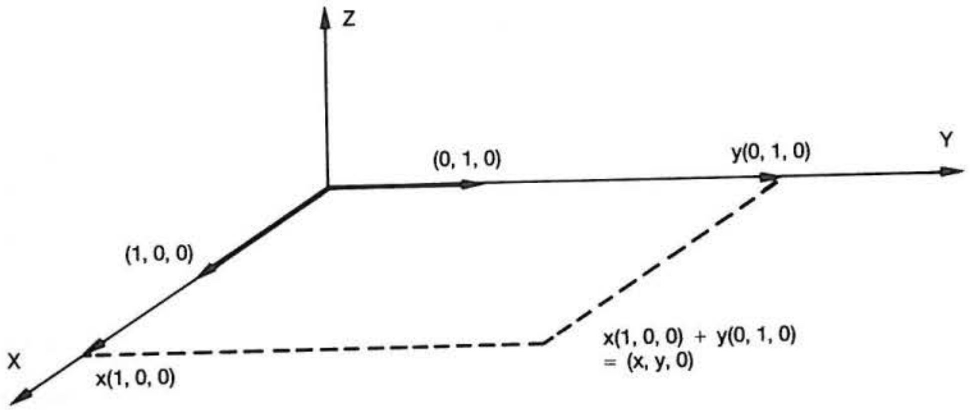

Figure 7.5

Example 4

Prove that the span of (1, 1) and (1, 2) consists of all vectors (a, b), (that is, span $\{(1, 1), (1, 2)\} = \mathbf{R}^2$).

This asks us to show that *every* point (a, b) is of the form x(1, 1) + y(1, 2) for some $x, y \in \mathbf{R}$. This statement has two quantifiers, an "all" and a "some".

Proof

Suppose (a, b) = x(1, 1) + y(1, 2). (We don't know that x, y exist yet; the aim is to show that they do.)

Then,

$$\begin{aligned}(a, b) &= (x, x) + (y, 2y) \\ &= (x + y, x + 2y)\end{aligned}$$

So,

$$a = x + y$$

and,

$$b = x + 2y$$

Subtracting the first equation from the second equation,

$$b - a = y$$

and substituting this in the first,

$$a = x + b - a$$

so,

$$2a - b = x$$

This shows that for *all* (a, b), we can find x and y (namely x = 2a − b and y = b − a), such that,

$$(a, b) = x(1, 1) + y(1, 2)$$

Therefore every (a, b) is a linear combination of (1, 1) and (1, 2).

That is,

$$\mathbf{R}^2 \subset \text{span}\{(1, 1), (1, 2)\}$$

Also, it is obvious that,

$$\text{span}\{(1, 1), (1, 2)\} \subset \mathbf{R}^2$$

since every linear combination of (1, 1) and (1, 2) is a vector in $\mathbf{R}^2$.

Therefore $\text{span}\{(1, 1), (1, 2)\} = \mathbf{R}^2$. ▮

(We have applied the techniques above: To show two sets are equal, we show that everything in the first is in the second, and everything in the second is in the first.)

Notes

1. Usually the set S is a small finite set, as in Example 4, while span S is an infinite set. However, sense can still be given to the definition if S itself is infinite.
2. The following sentences all mean the same:
   "Span {(1, 0, 0), (0, 1, 0)} is the xy-plane";
   "The span of (1, 0, 0) and (0, 1, 0) is the xy-plane";
   "(1, 0, 0) and (0, 1, 0) span the xy-plane".
   Notice in the first two, "span" is a noun while in the last it is a verb.

The following sentences are meaningless:
"(1, 0, 0) and (0, 1, 0) span";
"S is a span";
"The xy-plane is a span".

Example 5

Prove that span{(2, 4), (1, 2)} $\neq \mathbf{R}^2$.

As in Example 4, it is obvious that, span {(2, 4), (1, 2)} $\subset \mathbf{R}^2$, so we have to show that $\mathbf{R}^2$ is not a subset of span{(2, 4), (1, 2)}. That is, we have to show that span{(2, 4), (1, 2)} is not the whole of $\mathbf{R}^2$.

Proof

This is a "not all" statement: "Not all (a, b) are of the form x(2, 4) + y(1, 2)". To show this we exhibit a particular (a, b) which is not of the form x(2, 4) + y(1, 2). We can see that anything of the form x(2, 4) + y(1, 2) will have the second co-ordinate twice the first, so we choose a vector which does not have that property, say (1, 3). The proof is:

If,

$$(1, 3) = x(2, 4) + y(1, 2)$$

then,

$$\begin{aligned}(1, 3) &= (2x, 4x) + (y, 2y)\\ &= (2x + y, 4x + 2y)\end{aligned}$$

so,

$$1 = 2x + y$$

and,

$$\begin{aligned}3 &= 4x + 2y\\ &= 2(2x + y)\\ &= 2.1\\ &= 2\end{aligned}$$

but,

$$3 \neq 2$$

so (1, 3) cannot be of the form x(2, 4) + y(1, 2).

So, (1, 3) is not in the span of (2, 4) and (1, 2).

Therefore, the span of (2, 4) and (1, 2) does not include all points (a, b).

So span {(2, 4), (1, 2)} $\neq \mathbf{R}^2$. ∎

Calculus

The *general solution* of a differential equation is the set of all solutions. Thus, when we say, "The general solution of $\frac{df}{dx} = f$ is Ae^x", we mean that the set of all solutions of $df/dx = f$ is,

$$\{f : f = Ae^x \text{ for some real number } A\}$$

Example 6

If S is the set of all solutions of,

$$\frac{df}{dx} = f$$

and T is the set of all solutions of,

$$\frac{d^2f}{dx^2} = f$$

show that,

$$S \subset T$$

Method 1

Solving the differential equations, we find,

$$S = \{f : f = Ae^x \text{ for some real number } A\}$$

and,

$$T = \{f : f = Ae^x + Be^{-x} \text{ for some real numbers } A, B\}$$

and clearly,

$$S \subset T$$

Method 2

Let,

$$f \in S$$

So,

$$\frac{df}{dx} = f$$

Differentiating,

$$\frac{d^2f}{dx^2} = \frac{df}{dx}$$

but this is f, from the line above.

So f satisfies,

$$\frac{d^2f}{dx^2} = f$$

That is,

$$f \in T$$

Which method is better? See Exercise 23.

Exercises

(Gradings of exercises: * easy, ** moderate, *** difficult.)

*1. (a) Write down an "all" statement equivalent to, "$A \subset B$". Then write down your "all" statement in "if" form.
(b) Use this "if" statement to explain the form of the proof of the statement, "$A \subset B$".

*2. The truth of the relation, $B \subset (A \cup B)$, may be verified from a Venn diagram, but this does not constitute a proof. How would you prove this statement?

*3. Prove that, "If $A \subset B$ and $B \subset C$ then $A \subset C$" (without using a Venn diagram).

*4. (a) For any two sets A and B explain what is meant by:
(i) $A \cap B$
(ii) $A \cup B$
(b) Prove that, $(A \cap B) \subset B$

*5. (a) Write down an "if and only if" statement equivalent to, $A = B$.
(b) Use this to explain the form of the proof of the statement, $A = B$.

*6. (a) How would you prove the statement, $B \cup B = B$?
(b) Prove it.

*7. Prove that $(A \cap B) \subset (A \cup B)$

*8. (a) Prove that $B \cup \varnothing = B$.
(b) Simplify the following:
(i) $B \cup B$
(ii) $B \cap B$
(iii) $B \cap \bar{B}$
(iv) $B \cup \varnothing$
(v) $B \cap \varnothing$
(Check that you can prove your answers.)

*9. (a) Explain by the use of an appropriate Venn diagram that: If $A \subset B$ then $\bar{B} \subset \bar{A}$.

(b) Are the two statements, "All As are Bs", and its contrapositive, "All not-Bs are not-As" logically equivalent?

**10. Prove that, $A \cup (B \cup C) = (A \cup B) \cup C$.

**11. (a) Draw appropriate diagrams to verify the truth of the statement, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, then prove it.
(b) Prove that, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

**12. De Morgan's laws state:

$$\overline{(A \cup B)} = \bar{A} \cap \bar{B}$$
$$\overline{(A \cap B)} = \bar{A} \cup \bar{B}$$

(a) Draw appropriate Venn diagrams to verify the truth of, $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$.
(b) Prove, $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$.
(c) Prove, $\overline{(A \cap B)} = \bar{A} \cup \bar{B}$.

**13. Comment on the reasoning,

Most As are Bs
Most As are Cs

Therefore some Bs are Cs.

**14. A set of real numbers is called *bounded* if it does not "go to infinity". More precisely, S is bounded if there exist real numbers M, N such that for all $s \in S$, $M < s < N$. (For example, the set of real numbers x such that $1 < x^3 < 2$ is bounded, since for all $x \in S$, $1 < x < 1.5$.)
(a) Give an example of a set which is not bounded.
(b) Prove that if S is bounded and $T \subset S$, then T is bounded.
(c) Prove that any finite set of real numbers is bounded.
(d) Prove that if T is not bounded and $T \subset S$, then S is not bounded.
(e) Prove that if S and T are bounded, then $S \cap T$ is bounded.
(f) If S and T are bounded, is $S \cup T$ always bounded? Prove your answer.
(g) Let S and T be bounded. Let,

$$U = \{u \in \mathbf{R}: u = s + t \text{ for some } s \in S, t \in T\}$$

Show that U is bounded. It might help to calculate some examples first, say,

$$S = [0, 1],\ T = [2, 3]$$

(The set U is sometimes denoted S + T, since it is the set of all sums of something in S with something in T.)

**15. A region in the plane is called *convex* if the line segment joining any two points in the region lies wholly inside the region. For example, an ellipse, a parallelogram, a triangle and a straight line are convex, but an annulus and a star-shaped region are not. In symbols, R is convex if, for all (x_1, y_1) and (x_2, y_2) in R, $(\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2) \in R$ for all $\lambda \in [0, 1]$.
(a) Prove that if R and S are convex, then $R \cap S$ is convex.

(b) If R and S are convex, is $R \cup S$ always convex? Prove your answer.
(c) Prove that if R is convex, then the reflection of R in the x-axis is convex.
(d) If R is convex, is the set,

$$2R = \{(x, y) : (x, y) = (2x', 2y') \text{ for some } (x', y') \in R\}$$

always convex? Prove your answer and illustrate with a diagram.

16. Prove that $n\mathbf{Z} = m\mathbf{Z}$ if and only if $n = m$ or $n = -m$

17. Fill in the blank and prove the theorem:

"$A \cap (B \cup C) = (A \cap B) \cup C$ if and only if ——"

Linear algebra

18. (a) Show that $\mathbf{R}^2 = \text{span}\{(1, 1), (1, -1)\}$
(b) Show that $\mathbf{R}^2 \neq \text{span}\{(1, 1), (-3, -3)\}$

19. Show that (1, 2, 3) is not in the span of (1, 1, 1) and (1, 0, 2)

20. Show that any vector (a, b, c) such that $a + b + c = 0$ is in the span of $\{(1, 1, -2), (1, 0, -1)\}$

21. Show that the span of any set of vectors $\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_n$ in a vector space V is a subspace of V (that is, is closed under addition and scalar multiplication).

22. Show that, $\text{span}\{(a, b), (c, d)\} = \mathbf{R}^2$ if and only if $ad - bc \neq 0$.

Calculus

23. Let S be the set of all solutions of,

$$\frac{d^2f}{dx^2} = f$$

and let T be the set of all solutions of,

$$\frac{d^4f}{dx^4} = f$$

Show that $S \subset T$.

24. Show that the set of solutions of,

$$\frac{df}{dx} = 2x$$

is infinite.

25. The general solution of,

$$\frac{d^2y}{dx^2} = y$$

may be written as $Ae^x + Be^{-x}$.

(a) Check that sinh x and cosh x are solutions of this differential equation.
(b) Prove that $y = C \sinh x + D \cosh x$ is a solution, for any real numbers C and D.
(c) Is,

$$\{(y : y = C \sinh x + D \cosh x \text{ for some } C, D \in \mathbf{R}\}$$

equal to the set,

$$\{y : y = Ae^x + Be^{-x} \text{ for some } A, B \in \mathbf{R}\}?$$

8

Proof by mathematical induction

Mathematical induction is a special method of proof used to prove statements about all the natural numbers. For example,

"$n^3 - n$ is always divisible by 3"

"The sum of the first n integers is $\frac{n(n+1)}{2}$"

The first of these makes a different statement for each natural number n. It says, $1^3 - 1$, $2^3 - 2$, $3^3 - 3$, and so on, are all divisible by 3. The second also makes a statement for each n. It says,

for n = 1,

$$1 = \frac{1.2}{2}$$

for n = 2,

$$1 + 2 = \frac{2.3}{2}$$

for n = 100,

$$1 + 2 + 3 + \ldots + 100 = \frac{100.101}{2}$$

and in general, for any natural number n,

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

The idea behind proof by mathematical induction is a simple one: The natural numbers form a list, so that if we start at the beginning and keep going from one number to the next, we eventually get to any given number. Thus:

$$1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \quad \ldots \quad 100 \rightarrow 101 \quad \ldots \quad n \rightarrow n+1 \quad \ldots$$

Therefore, if we can prove that some statement involving n is true for n = 1 (the beginning of the list) and that the truth of the statement for n implies its truth for n + 1 (so we can get from any number to the next), then we have proved the statement for all n.

Example 1

Prove by mathematical induction that $n^3 - n$ is divisible by 3 for all natural numbers n.

Proof

For n = 1,

$$\begin{aligned} n^3 - n &= 1 - 1 \\ &= 0 \end{aligned}$$

which is divisible by 3.

Assume the statement is true for *some* number n, that is, $n^3 - n$ is divisible by 3. Now,

$$\begin{aligned} (n+1)^3 - (n+1) &= n^3 + 3n^2 + 3n + 1 - n - 1 \\ &= (n^3 - n) + 3(n^2 + n) \end{aligned}$$

which is $n^3 - n$ plus a multiple of 3.

Since we assumed that $n^3 - n$ was a multiple of 3, it follows that $(n+1)^3 - (n+1)$ is also a multiple of 3.

So, since the statement "$n^3 - n$ is divisible by 3" is true for n = 1, and its truth for n implies its truth for n + 1, the statement is true for all whole numbers n.

Example 2

Prove that,

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

Proof

For n = 1,

$$1 = \frac{1.2}{2}$$

This is true.

Assume the truth of the statement for some n, that is,

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

Now,

$$1 + 2 + 3 + \ldots + n + (n + 1) = \frac{n(n + 1)}{2} + (n + 1)$$

by induction assumption,

$$= (n + 1)\left(\frac{n}{2} + 1\right)$$

$$= \frac{(n + 1)(n + 2)}{2}$$

$$= \frac{(n + 1)((n + 1) + 1)}{2}$$

which is the statement with n replaced by n + 1.

So the statement is true for n = 1, and its truth for n implies its truth for n + 1.

Therefore, it is true for all n.

Notes

1. The essential steps of a proof by mathematical induction are: the proof for n = 1; and the proof of the "*if*" proposition, "If the statement is true for n it is true for n + 1." As discussed earlier, the way to prove an "if" proposition is to assume the first part and deduce the second part from it.
2. It is not necessary to start at n = 1. The results proved in Examples 1 and 2 both actually make sense, and are true, for n = 0, and the proofs could have started by showing their truth for n = 0 instead of n = 1. On the other hand, some proofs may need to start later than n = 1. For example, $2^n > n^2$ is true for $n \geqslant 5$, (but is not true for n = 2, 3 or 4). Its proof would start by showing it is true for n = 5, then showing that its truth for n implied its truth for n + 1. This would show the statement was true for all $n \geqslant 5$.
3. Sometimes, to get the truth of the statement for n + 1, it is necessary to assume its truth not just for n, but for n and n − 1, or sometimes for all numbers up to n.

Example 3

The formula for the nth term a_n of the Fibonacci sequence,

1, 1, 2, 3, 5, 8, 13, 21, 34, . . .

is given by,

$$a_n = \begin{cases} 1 \text{ for } n = 1 \text{ and } 2 \\ a_{n-2} + a_{n-1} \text{ for } n > 2 \end{cases}$$

Prove by mathematical induction that,

$$a_n = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{\sqrt{5}\, 2^n}$$

Proof

For n = 1,

$$1 = \frac{(1 + \sqrt{5}) - (1 - \sqrt{5})}{\sqrt{5}\, 2}$$

This is true.

For n = 2,

$$1 = \frac{(1 + \sqrt{5})^2 - (1 - \sqrt{5})^2}{\sqrt{5}\, 2^2}$$

This is also true.

Assume the truth of the statement for some n − 1 and n, that is,

$$a_{n-1} = \frac{(1 + \sqrt{5})^{n-1} - (1 - \sqrt{5})^{n-1}}{\sqrt{5}\, 2^{n-1}}$$

and,

$$a_n = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{\sqrt{5}\, 2^n}$$

Now,

$$a_{n+1} = a_{n-1} + a_n$$

$$= \frac{(1 + \sqrt{5})^{n-1} - (1 + \sqrt{5})^{n-1}}{\sqrt{5}\, 2^{n-1}} + \frac{(1 + \sqrt{5})^n - (1 + \sqrt{5})^n}{\sqrt{5}\, 2^n}$$

$$= \frac{4(1 + \sqrt{5})^{n-1} - 4(1 - \sqrt{5})^{n-1} + 2(1 + \sqrt{5})^n - 2(1 - \sqrt{5})^n}{\sqrt{5}\, 2^{n+1}}$$

$$= \frac{(1 + \sqrt{5})^{n-1}(4 + 2(1 + \sqrt{5})) - (1 - \sqrt{5})^{n-1}(4 + 2(1 - \sqrt{5}))}{\sqrt{5}\, 2^{n+1}}\,.$$

$$= \frac{(1 + \sqrt{5})^{n-1}(6 + 2\sqrt{5}) - (1 - \sqrt{5})^{n-1}(6 - 2\sqrt{5})}{\sqrt{5}\, 2^{n+1}}$$

$$= \frac{(1 + \sqrt{5})^{n-1}(1 + \sqrt{5})^2 - (1 - \sqrt{5})^{n-1}(1 - \sqrt{5})^2}{\sqrt{5}\, 2^{n+1}}$$

$$= \frac{(1 + \sqrt{5})^{n+1} - (1 - \sqrt{5})^{n+1}}{\sqrt{5}\, 2^{n+1}}$$

which is the statement for n + 1.

So the statement is true for n = 1 and n = 2, and its truth for n − 1 and n implies its truth for n + 1.

Therefore it is true for all n.

Example 4

Prove that every positive integer greater than 1 can be written as a product of primes.

The result was assumed "sufficiently well known" in the proof for Example 2 in Chapter 5 (i.e. that there are infinitely many primes). Now it can be proved properly. The words, "can be written" are simply a metaphor for an existential statement. In this example the statement is:

"For any integer $n > 1$ there exist primes $p_1, \ldots, p_m$ such that $n = p_1 \ldots p_m$."

Of course, some of the primes could be equal.

Proof

If n is a prime, then it is already a product of primes.

If n is not a prime, then $n = n_1n_2$ where neither n_1 or n_2 are 1.

So, n_1 and n_2 are both less than n.

We take as our induction hypothesis the statement, "Any number less than n (but greater than 1) can be written as a product of primes".

So, n_1 and n_2 are products of primes and therefore $n = n_1n_2$ is also a product of primes.

So the truth of the result for all numbers less than n implies the truth of the result for n.

The result is true for $n = 2$, so, by induction, it is true for all $n > 1$.

Fermat's Infinite Descent

The original version of proof by mathematical induction was Pierre Fermat's method of "Infinite Descent" (devised around 1650). In some ways it is a more natural way to regard the method. It is illustrated in Example 5, as follows.

Example 5

A set of n elements has 2^n subsets.

(When counting subsets, it must be remembered that, by convention, the empty set and the whole set are regarded as subsets of any set.)

Proof

Denote the number of subsets of a set of n elements by S(n).

Take a set of n elements.

Take one of the elements, say a, and consider separately the subsets which contain a and those which do not.

There are $S(n - 1)$ subsets not containing a (because they are the subsets of the $(n - 1)$ element set consisting of the original set without a).

The number of subsets which do contain a is also $S(n - 1)$, since a subset with a is formed by adding a to one of the subsets without a.

So,

$$S(n) = S(n - 1) + S(n - 1) = 2\,S(n - 1)$$

Similarly,

$$S(n - 1) = 2\,S(n - 2)$$

and so on, so that,

$$S(n) = 2\,S(n - 1) = 4\,S(n - 2) = \ldots = 2^{n-1}\,S(1)$$

and,

$$S(1) = 2$$

(since a one-element set has two subsets, namely the whole set and the empty set).

Thus,

$$S(n) = 2^n$$ ▌

Notes

1. It would be easy to recast this solution in the usual form by beginning, "Assume $S(n) = 2^n$", but it seems more natural to proceed as above, reducing the problem for n to the problem for $n - 1$, and then "descending". A different proof of this result is given in Chapter 11.
2. Fermat's statement of his method of infinite descent also included a use of proof by contradiction. To prove a statement about n, he suggested showing that if the statement was *false* for n, it would also be false for some number smaller than n. Then, by "descending", one would find the statement to be false for $n = 1$. Checking would show that the result was in fact true for $n = 1$, so it could be concluded that the result was true for any n. Later writers have preferred not to use proof by contradiction here, since it is not necessary. (C. B. Boyer, *A History of Mathematics* (1968), Wiley, New York, p. 387.)

If there is a choice between proving a statement by mathematical induction and proving it by some other method, the other method is often preferred because it gives more insight. For example, the statement in Example 1. in this Chapter, can be better proved thus:

Prove $n^3 - n$ is a multiple of 3.

Proof

$$n^3 - n = n(n^2 - 1) = (n - 1)n(n + 1)$$

which is the product of 3 consecutive integers.

One of these must be a multiple of 3, so, $n^3 - n$ is a multiple of 3. ▌

(The fact that one of any three consecutive numbers is a multiple of 3 can be assumed to be sufficiently obvious to serve as the basis of a proof. Nevertheless, if we were to ask for a proof of this fact from more basic facts about numbers, we would find ourselves using induction again. Thus, there may be an induction hidden even in this proof.)

Example 6

Prove,

$$1 + 2 + 3 + \ldots + n = \frac{n(n + 1)}{2}$$

Proof

This proof is suggested by the 2 on the bottom of the formula, which indicates it might be helpful to write down the sum twice:

$$1 + 2 + 3 + \ldots + n = \frac{1}{2}\begin{pmatrix} 1 + & 2 & + & 3 & + \ldots + (n - 1) + n + \\ n + (n - 1) & + & (n - 2) & & + \ldots + \quad 2 \quad + 1 \end{pmatrix}$$

The second sum has been written backwards and underneath the first so we can compare the numbers written together. Each pair adds up to n + 1:

$$= \frac{1}{2}\underbrace{((n + 1) + (n + 1) + \ldots + (n + 1))}_{\text{n times}}$$

$$= \frac{1}{2}(n + 1)n$$

which is the answer.

Notes

1. A great disadvantage of the method of mathematical induction is that the formula to be proved true must be known, or at least conjectured, to be true before we start. This method gives no help at all in finding the formula. However, to some extent this is a problem with any kind of proof. There is a possible way of finding the formula to be proved. Start by working out the first few cases, then try to see a pattern, and from this conjecture what the general result should be. Some examples will be given in Chapter 14.
2. "Mathematical induction" must be distinguished from "induction" in general. See Chapter 14 for further discussion.

Proofs by induction are commonly used in computing applications when establishing facts about the strings of symbols in computer languages.

Example 7

A formal language to be used for simple algebra uses the symbols,

x y z ( ) +

The *words* of the language are strings of symbols formed according to the rules:

1. x, y and z are words.
2. If A and B are words, so is (A)(B).
3. If A and B are words, so is (A) + (B).

Nothing is a word other than the words formed according to the rules, 1., 2. and 3.

For example, ((x)(z)) + (z) is a word, since x and z are words (Rule 1.) and therefore so is (x)(z) (Rule 2.) and hence ((x)(z)) + (z) (Rule 3.). However, (x) + z is not a word.

Prove that any word in this language has the same number of ('s and)'s.

Proof

We use induction on the length of words, that is, the number of symbols in them.

A word of length 1 is either x, y or z so the number of ('s and)'s in a word of length 1 is zero.

So the result is true for words of length 1.

Assume the result is true for words of lengths less than n.

A word of length n is either of the form (A) (B) or of the form (A) + (B), for some words A, B of length less than n.

By induction assumption, A and B have the same number of ('s and)'s.

Therefore, (A) (B) and (A) + (B) also have the same number of ('s and)'s.

So the result is true for words of any length.

Exercises

(Gradings of exercises: * easy, ** moderate, *** difficult.)

*1. Prove by mathematical induction the following:

(a) $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n + 1)(2n + 1)$

(b) $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n + 1)}{2}\right)^2$

(c) $2 + 4 + 6 + \ldots + 2n = n(n + 1)$

(d) $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

(e) For all positive integers n, $n^2 - n$ is even.

**2. Show that, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \ldots + (-1)^{n-1}\frac{1}{n}$ is always positive.

**3. Prove, by mathematical induction, the formula for the sum of the first n terms of an arithmetic progression:

$$a + (a + d) + (a + 2d) + (a + 3d) + \ldots + (a + (n - 1)d) = \frac{n}{2}(2a + (n - 1)d)$$

**4. Prove, by mathematical induction, the formula for the sum of the first n terms of a geometric progression:

$$a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(1 - r^n)}{1 - r}, \; r \neq 1$$

**5. (a) Prove, by mathematical induction, that if n is a whole number then,

$$n^3 + 3n^2 + 2n$$

is divisible by 6.

(b) Prove the same result without mathematical induction by first factorising $n^3 + 3n^2 + 2n$.

**6. Show that a number which consists of 3^n equal digits is divisible by 3^n.

**7. (a) Calculate,

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{n}{(n+1)!}$$

for a few small values of n.

(b) Make a conjecture about a formula for this expression.

(c) Prove your conjecture by mathematical induction.

**8. The following is a famous fallacy that uses the method of mathematical induction. Explain what is wrong with it.

"Theorem"

Everything is the same colour.

"Proof"

Let P(n) be the statement, "In every set of n things, all the things have the same colour".
We will show that P(n) is true for all n = 1, 2, 3, . . . , so that *every* set consists of things of the same colour.
Now, P(1) is true, since in every set with only one thing in it, everything is obviously of the same colour.
Now, suppose P(n) is true.
Consider any set of n + 1 things.
Take an element of the set, a. The n things other than a form a set of n things, so they are all the same colour (since P(n) is true).
Now take a set of n things out of the n + 1 which does include a.
These are also all the same colour, so a is the same colour as the rest.
Therefore P(n + 1) is true.

**9. If a sequence a_n satisfies,

$$a_{n+1} = \frac{a_n}{a_n + 1}$$

Show that,

$$a_n = \frac{a_0}{na_0 + 1}$$

***10. Suppose that we draw on a plane n lines in "general position" (i.e. with no three concurrent, and no two parallel). Let s_n be the number of regions into which these lines divide the plane, for example, $s_3 = 7$ in Figure 8.1.

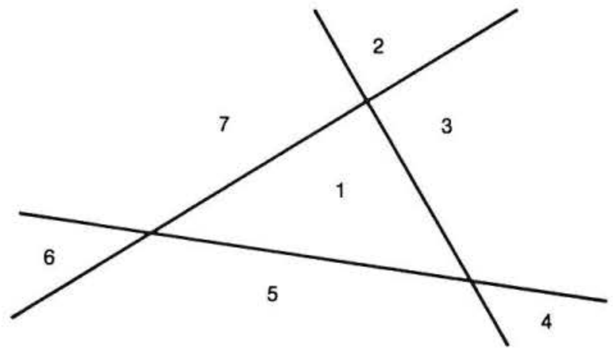

Figure 8.1

(a) By drawing diagrams, find s_1, s_2, s_3, s_4 and s_5

(b) From these results, make a conjecture about a formula for s_n

(c) Prove this formula by mathematical induction.

***11. The "pigeonhole principle" states: If n things are put in fewer than n holes, some hole has at least two things. Prove this.

***12. "Fermat's Small Theorem" states that $n^p - n$ is divisible by p for any prime p. Prove this.

***13. If a set has a binary operation • satisfying the associative law:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

show that any product of n elements is independent of the order of brackets.

Calculus

**14. Show that the nth derivative of x^n is n!.

**15. Use integration by parts to show that for $n \geq 2$,

$$\int_0^\pi \sin^n x \, dx = \frac{n-1}{n} \int_0^\pi \sin^{n-2} x \, dx$$

and hence find,

$$\int_0^\pi \sin^8 x \, dx$$

Solutions to selected exercises

Chapter 1

1. $$\frac{1}{1000} - \frac{1}{1002} = \frac{1002}{1000 \times 1002} - \frac{1000}{1000 \times 1002}$$
$$= \frac{2}{1000 \times 1002}$$
$$= \frac{2}{1002000}$$

But, 1002000 > 1000000, so,

$$\frac{2}{1002000} < \frac{2}{1000000}$$

Therefore, $\frac{1}{1000} - \frac{2}{1002} < \frac{2}{1000000}$

5. $$\tan\left(\frac{\pi}{12}\right) = \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$
$$= \frac{\tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{4}\right)}{1 + \tan\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{4}\right)}$$
$$= \frac{\sqrt{3} - 1}{1 + \sqrt{3}(1)}$$
$$= \frac{(\sqrt{3} - 1)(\sqrt{3} - 1)}{(\sqrt{3} + 1)(\sqrt{3} - 1)}$$
$$= \frac{3 - 2\sqrt{3} + 1}{3 - 1}$$
$$= \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}.$$

6. Putting $x = 1 - \sqrt{5}$ in the left hand side of the equation,
$$(1 - \sqrt{5})^3 - 3(1 - \sqrt{5})^2 - 2(1 - \sqrt{5}) + 4$$
$$= 1 - 3\sqrt{5} + 3(5) - 5\sqrt{5} - 3 + 6\sqrt{5} - 15 - 2 + 2\sqrt{5} + 4 = 0.$$

10. The accuracy of the answer is ridiculous; the diameter is given only to one significant figure, and ponds are not perfectly circular.

12. On squaring both sides we would have,
$$(2 + \sqrt{2}) + 2\sqrt{2 + \sqrt{2}}\sqrt{2 - \sqrt{2}} + (2 - \sqrt{2}) < 4(2)$$
Rearranging, we obtain,
$$2\sqrt{(2 + \sqrt{2})(2 - \sqrt{2})} < 4$$
giving,
$$\sqrt{2} < 2$$
which is true. Each step is reversible, so the final proof consists of the above steps in the reverse order.

17. The first few steps are: Dividing through by $\sqrt[3]{3}$,
$$\sqrt[3]{1 + 3^{-2/3}} + \sqrt[3]{1 - 3^{-2/3}} < 2$$
Then subtract $\sqrt[3]{1 + 3^{-2/3}}$ from both sides and consider the cube of the right-hand side.

20. $$\frac{100^{99}}{99^{100}} = \left(\frac{100}{99}\right)^{99}\frac{1}{99}$$
$$= \left(1 + \frac{1}{99}\right)^{99}\frac{1}{99}$$
$$< 3.\frac{1}{99}$$
(since, $\left(1 + \frac{1}{99}\right)^{99} < 3$, see problem 18) and this is less than 1.
So,
$$99^{100} > 100^{99}$$

Chapter 2

2. Let n and n + 1 be any pair of consecutive integers. Then,
$$n + (n + 1) = 2n + 1$$
which is odd.

5. True.
$$x^2 - 3x + 2 = (x - 1)(x - 2)$$
This is negative for all x between 1 and 2, since the factor $(x - 2)$ is negative and the factor $(x - 1)$ is positive, so their product is negative.

6. (b) No. (But fathers who use misleading language may find they have ungrateful children.)

10. The equation,

$$\frac{n}{n+1} < \frac{n+1}{n+2}$$

is equivalent to,

$$n(n+2) < (n+1)^2$$

(since n + 1 and n + 2 are positive).

That is,

$$n^2 + 2n < n^2 + 2n + 1$$

This is true, so the final proof consists of these steps in the reverse order.

12. Let the middle number be 2n + 1.

Then the product of the three consecutive numbers is 2n(2n + 1)(2n + 2). One of these three is divisible by 3; of the two consecutive even numbers 2n and 2n + 2, one is divisible by 4 and the other by 2.

So the product is divisible by 3 × 4 × 2 = 24.

15. Since x is positive, we can multiply both sides of the inequality x < 1 by x. This gives the answer.

18. (a) This can be done by substituting the two numbers given for x in $ax^2 + bx + c$ and showing that the result is 0.
(b) This can be done by completing the square.

19. One generalisation is,

$$\frac{1}{n} - \frac{1}{n+2} < \frac{2}{n^2}$$

20. Since 7 divides a + b, it divides,

$$(a+b)^2 = a^2 + 2ab + b^2$$

Hence 7 also divides the difference of this and $a^2 + b^2$, which is 2ab.

But 7 does not divide 2, so it must divide ab.

Since 7 is a prime, it must divide either a or b.

But if it divides a, it divides b as well (since it divides a + b); similarly, if 7 divides b, it also divides a.

So it divides both a and b.

22. Both the cosine rule and the equation,

$$\cos^2\theta + \sin^2\theta = 1$$

rely on Pythagoras' theorem for their proof.

So they cannot be used to prove Pythagoras' theorem, since this would make the argument circular.

26. A proof is given in Euclid's *Elements*, Book 1, proposition 47.

29. Adding the equations gives,

$$2x = A + B$$

so,

$$x = \tfrac{1}{2}(A + B)$$

Substituting this in the first equation gives,

$$y = A - x = \tfrac{1}{2}(A - B)$$

So the equations can be solved, no matter what A and B are.

33. Equating the first, second and third co-ordinates on both sides, we have the system of equations,

$$x + y + 3z = 0$$
$$x + 2y + 4z = 0$$
$$3y + 3z = 0$$

Solving these by row reduction, we find $x = -2\lambda$, $y = -\lambda$, $z = \lambda$.

So,

$$(x, y, z) = -\lambda(2, 1, -1)$$

which is a multiple of (2, 1, −1).

34. The graphs of y = x and y = sin x meet at x = 0. The gradient of y = x is 1 (always), while the gradient of y = sin x is cos x.
But, cos x ≤ 1, and is sometimes less than 1. So the graph of y = sin x lies below y = x for all x > 0.

38. True. The problem cannot be done straightforwardly by integrating (sin x)/x, since this function does not have an integral expressible in terms of elementary functions. It can be done by comparing areas; draw a rough graph first.

Chapter 3

1. Yes. (In fact we could *prove* that they are equivalent, by showing that each implies the other.)

4. (a) True.
(b) False, since 5 is an odd prime.

5. (c).

6. The proof is in two parts:

First part: To show that if x is odd then x^2 is odd.
If x is odd, then x = 2k + 1 for some integer k.
So $x^2 = 4k^2 + 4k + 1$, which is odd.

Second part: To show that if x^2 is odd then x is odd.
Suppose x^2 is odd.
If x were even, then x^2 would be even, contrary to what we have just supposed.
So x cannot be even, so it must be odd.

9. Let x be a number with digits $a_n, a_{n-1}, \ldots, a_1, a_0$. Then,

$$x = 10^n a_n + \ldots + 100a_2 + 10a_1 + a_0$$

The number formed by the last two digits is $10a_1 + a_0$, and $x - (10a_1 + a_0)$ is a multiple of 100, which is a multiple of 4.

Hence if x is divisible by 4, so is $10a_1 + a_0$, and if $10a_1 + a_0$ is divisible by 4, so is x.

14. Suppose that $AD - BC = 0$.

Case 1: Neither B nor D is zero.

Then since $AD = BC$, we obtain,

$$A/B = C/D$$

and hence,

$$-A/B = -C/D.$$

Thus, the gradients of the two lines are equal, so they are parallel.

Case 2: Either B or D is zero.

If $B = 0$, then $AD = 0$, so either $A = 0$ or $D = 0$.

If $A = 0$ then the first "line",

$$Ax + By + E = 0$$

would be $E = 0$. There is no such line so we must have $D = 0$.

Then the two lines are,

$$Ax + E = 0$$

and,

$$Cx + F = 0$$

which are both vertical, and hence parallel.

A similar argument applies if $D = 0$. Hence in both cases, the two lines are parallel.

Conversely, suppose that the lines,

$$Ax + By + E = 0$$

and,

$$Cx + Dy + F = 0$$

are parallel.

Case 1: Neither B nor D is zero.

Then the gradient of the first line is $-A/B$ and the gradient of the second line is $-C/D$.

These are equal since the lines are parallel, so,

$$-A/B = -C/D$$

and hence,

$$AD - BC = 0$$

Case 2: Either $B = 0$ or $D = 0$.

If $B = 0$ then the first line is,

$$Ax + E = 0$$

which is vertical.

So the other line is vertical too, that is, $D = 0$.

Thus $AD - BC = 0 - 0 = 0$.

Similarly, if $D = 0$ then $B = 0$ and again $AD - BC = 0$.

18. Suppose that x has a decimal expansion that is terminating or (eventually) repeating. If the expansion is terminating,

$$x = a_n a_{n-1} \ldots a_1 . b_1 \ldots b_m$$

So,

$$x = \frac{a_n \ldots a_1 b_1 \ldots b_m}{10^m}$$

which is rational. If x has an (eventually) repeating decimal,

$$x = a_n \ldots a_1 . b_1 \ldots b_{r-1} \overline{b_r \ldots b_m}$$

Then $x = y + z$, where,

$$y = a_n \ldots a_1 . b_1 \ldots b_{r-1}$$

(which is rational), and,

$$z = .00...00\overline{b_r \ldots b_m}$$

Then,

$$10^{r-1} z = 0.\overline{b_r \ldots b_m}$$

and,

$$10^{m-r} 10^{r-1} z = b_r \ldots b_m . \overline{b_r \ldots b_m}$$

Subtracting the last two equations,

$$10^{m-1} z - 10^{r-1} z = b_r \ldots b_m$$

giving,

$$z = b_r \ldots b_m / (10^{m-1} - 10^{r-1})$$

which is rational. Thus, x is rational.

Conversely, suppose x is rational, so that $x = m/n$ for some integers m and n.

Let us obtain the decimal expansion by dividing n into m.

Let the remainder produced by the division step that gives the kth decimal place be p_k.

p_k is an integer less than n.

After at most n division steps some p_k will be repeated. Once p_k reappears then the string of digits between the appearances of p_k will be repeated.

Chapter 4

1. (a) By finding an integer x such that $7073 = 643x$.
   (b) $7073 = 643 \times 11$.

2. Yes, 31 and 42.

4. 6 is a perfect number.

6. (a) True: if x is positive, x^2 is positive; if x is negative, x^2 is also positive; if $x = 0$ then $x^2 = 0$. So in all cases, $x^2 \geqslant 0$.

(b) $1^2 > 0$, so for some $x \in \mathbf{R}, x^2 > 0$.

7. (a) T (b) F (c) T (d) T (e) F (f) T (g) T (h) F.

8. $2^{1/4}$ is such a number.

9. (a) F (b) F (c) T (d) F.

10. All are true.

13. N = 20 (or any larger integer) satisfies the condition.

14. The line y = x is an axis of symmetry, since if (x, y) lies on the curve, then,

$$x^2 + xy + y^2 = 1$$

Thus,

$$y^2 + yx + x^2 = 1$$

so the point (y, x) also lies on the curve; this is the point opposite (x, y) across the line y = x.

17. (a) (1, 0, 1) = 1(0, 0, 1) + (−1)(0, 1, 1) + 1(1, 1, 1).
(b) No, because any linear combination of the four given vectors has last coordinate zero, but (1, 2, 1) does not.
(c) 12(3, 4) + 1(−36, −48) = (0, 0).

18. Yes, for example, (3, 1, 0).

21. Yes, for example, e^{-x}.

Chapter 5

1. (a) Two, both existential.
(b) True: m = 1, n = −1 is one solution.

2. This is similar to Chapter 2, Exercise 18(a), but the case a = 0 must be considered separately.

3. We want,

$$\frac{1}{N} - \frac{1}{N+1} < \frac{1}{n}$$

that is,

$$\frac{1}{N(N+1)} < \frac{1}{n}$$

Now,

$$\frac{1}{N(N+1)} < \frac{1}{N^2}$$

So if we can find N such that,

$$\frac{1}{N^2} < \frac{1}{n}$$

then,

$$\frac{1}{N(N+1)}$$

will also be less than 1/n.

But,

$$\frac{1}{N^2} < \frac{1}{n}$$

is equivalent to $N^2 > n$, that is, $N > \sqrt{n}$.

So any integer N greater than $\sqrt{n}$ will satisfy the condition.

4. (a) Suppose ad − bc ≠ 0. Then a times the second equation minus c times the first equation gives,

$$(ad - bc)y = af - ce$$

and hence,

$$y = (af - ce)/(ad - bc)$$

(the division is possible since ad − bc ≠ 0).
Similarly,

$$x = (bf - ed)/(ad - bc)$$

So there exists a solution to the system.

(b) No, it is not true that if there exists a solution then ad − bc ≠ 0.
For example, the equations x + y = 1 and 2x + 2y = 2.
(c) No; the same example applies (in fact, these two questions are logically equivalent, as will be explained in Chapter 6).

7.
$$n^5 - n = n(n^4 - 1) = n(n - 1)(n + 1)(n^2 + 1)$$

5 divides one of the five consecutive numbers n − 2, n − 1, n, n + 1, n + 2.

If it divides n − 1, n or n + 1, then it divides $n^5 - n$.

If 5 divides n − 2, then,

$$n = 5k + 2$$

for some k, so,

$$n^2 + 1 = 25k^2 + 20k + 5$$

which is a multiple of 5; thus 5 again divides $n^5 - n$.

A similar argument holds if 5 divides n + 2. (A proof by mathematical induction is also possible.)

8. For any positive real number d there exists an integer n such that n > 1/d. So 1/n < d.

We now take any two irrational numbers a and b, and take d = b − a.

So there exists an integer n such that 1/n < b − a.

We consider the set of rational numbers of the form h/n (h an integer), and show that one of them must lie between a and b.

Let h_1 be the largest integer such that $h_1/n < a$. So,

$$h_1/n < a < (h_1 + 1)/n.$$

Now,

$$(h_1 + 1)/n - a < (h_1 + 1)/n - h_1/n = 1/n < b - a$$

so,

$$a < (h_1 + 1)/n < b$$

Thus, there is a rational number between a and b.

11. Take one of the points, A. It is joined to five others, so at least three of the lines from A are of the same colour, say red. Call the points at their ends B, C and D. If one of BC, BD, CD is red, then there is a red triangle; if not, there is a blue triangle, BCD.

12. We have to find whether, for all (x, y, z), there exist scalars a, b, c such that,

$$(x, y, z) = a(1, 0, 0) + b(1, 1, 0) + c(1, 1, 1)$$

Solving, we find,

$$(x, y, z) = (x - y)(1, 0, 0) + (y - z)(1, 1, 0) + z(1, 1, 1)$$

So there exist such scalars a, b, c.

16. Yes, for example,

$$f(x) = e^{-\frac{1}{x^2}}$$

(with f(0) defined to be 0).

Chapter 6

1. Both require one counterexample to be found. For example, "Not all prime numbers are odd", is proved true by finding the counterexample 2, which is an even prime.

   The statement, "All prime numbers are odd", is proved false by exhibiting the same counterexample.

4. (a) True.
   (b) False.

5. (a) "My car is not red".
   (b) (i) "Not all cars are red".
   (ii) "Some cars are not red".
   (c) "For all $x \in \mathbf{R}$, $x^2 \neq 2$".

8. False; take $a = 1$, $b = 2$, $c = 0$. Then $ac = bc$ but $a \neq b$.

10. (a) Take $x = 0$.
    (b) Take 6 itself.

11. (b) False; $-2 < -1$ and $-4 < -3$ but $(-2)(-4) > (-1)(-3)$.

15. If n is even, then $n = 2k$ for some integer k.

    So $n^2 = 4k^2$, which is a multiple of 4.

    If n is odd,

$$n = 2k + 1$$

so,

$$n^2 = 4k^2 + 4k + 1$$

which is 1 plus a multiple of 4.

So for all integers n, n^2 is either a multiple of 4 or 1 more than a multiple of 4, so it cannot be of the form $4k + 2$ or $4k + 3$.

17. First,

$$a + \sqrt{b} - c = \sqrt{d}$$

Squaring and rearranging gives,

$$2(a - c)\sqrt{b} = d - b - (a - c)^2$$

So $a - c$ must be zero, since otherwise the left-hand side would be irrational and the right-hand side rational. Putting this in the original equation results in $\sqrt{b} = \sqrt{d}$, so $b = d$.

19. One of the three consecutive integers $2^n - 1$, 2^n, $2^n + 1$ is a multiple of 3. It cannot be 2^n, since 3 does not divide 2.

    So 3 divides either $2^n - 1$ or $2^n + 1$.

    So these are not primes, unless one of them is 3. This happens when $n = 1$ or 2.

23. Yes. Both are equivalent to, "There is nothing that is both an A and a B".

25. We prove the contrapositive: If,

$$f(x) = f(y)$$

then,

$$mx + b = my + b$$

so $mx = my$,

so $x = y$ (since $m \neq 0$).

Therefore, if $x \neq y$ then $f(x) \neq f(y)$.

26. By calculating enough angles, we find that four of the angles in the octagon are $\pi/2 + 2\tan^{-1}\frac{1}{2}$ while the other four are $\pi - 2\tan^{-1}\frac{1}{2}$. It can be checked that these angles are not equal using a proof by contradiction.

30. No; for example, take $\underset{\sim}{v}_1 = (1, 0)$, $\underset{\sim}{v}_2 = (2, 0)$, $\underset{\sim}{v}_3 = (0, 1)$.

    Then $\underset{\sim}{v}_3$ is not a linear combination of $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$, but $\underset{\sim}{v}_1$, $\underset{\sim}{v}_2$ and $\underset{\sim}{v}_3$ are dependent.

32. Yes. For example, $a = b = c = d = 1$.

34. No. For example, $f(x) = |x|$.

35. No. For example, $f(x) = e^x$.

36. (b) No. For example, $f(x) = x^4$ has a minimum at $x = 0$, but its second derivative there is zero.

Chapter 7

1. (a) "All members of A are members of B"; "If something is a member of A then it is a member of B".
   (b) Hence the proof of "$A \subset B$" must begin by assuming $x \in A$ and must deduce from this $x \in B$.

3. Suppose that $A \subset B$ and $B \subset C$.

   Let $x \in A$. Therefore $x \in B$ (since all members of A are members of B).

   Hence $x \in C$ (since all members of B are members of C).

   So we have proved that if $x \in A$ then $x \in C$.

   Therefore, $A \subset C$.

9. (b) Yes

12. (b) Let $x \in \overline{(A \cup B)}$.

    So $x \notin A \cup B$.

    So $x \notin A$ and $x \notin B$ (since if x were in either A or B, it would be in $A \cup B$).

    So $x \in \bar{A}$ and $x \in \bar{B}$.

    So $x \in \bar{A} \cap \bar{B}$.

    This shows that, $\overline{(A \cup B)} \subset \bar{A} \cap \bar{B}$.

    The proof of the converse consists of the same steps in the reverse order.

13. The reasoning is correct.

14. (a) The set of real numbers itself is not bounded.
    (b) Suppose that S is bounded and that $T \subset S$.

    Then there exist numbers M and N such that for all $s \in S$, $M < s < N$.

    Now let $t \in T$.

    Since $T \subset S$, $t \in S$, so $M < t < N$.

    So for all $t \in T$, $M < t < N$. That is, M and N are also bounds for T.

    Therefore, T is bounded.
    (c) Let $\{a_1, \ldots, a_n\}$ be a finite set of real numbers. Let a_i be the minimum of these, and a_j be the maximum.

    Then if we take $M = a_i - 1$ and $N = a_j + 1$, then $M < s < N$ for all s in the set.

    So the set is bounded.
    (d) This is logically equivalent to (b), so it is already proved.
    (e) Suppose that S and T are bounded. So there exist M and N such that for all $s \in S$, $M < s < N$.

    Now let $r \in S \cap T$. Then $r \in S$, so $M < r < N$.

    So M and N are bounds for $S \cap T$, so $S \cap T$ is bounded. (Note that the condition "T is bounded" was not used, so we may strengthen the conclusion to: "If S is bounded and T is any set of real numbers, then $S \cap T$ is bounded".)
    (f) Yes. Let S and T be bounded, so that there exist M_1 and N_1 such that $M_1 < s < N_1$ for all $s \in S$, and there exist M_2 and N_2 such that $M_2 < t < N_2$ for all $t \in T$.

Now let M_3 be the minimum of M_1 and M_2, and let N_3 be the maximum of N_1 and N_2.

Take $r \in S \cup T$. Then $r \in S$ or $r \in T$.

If $r \in S$, then $M_3 \leqslant M_1 < r < N_1 \leqslant N_3$.

If $r \in T$, then $M_3 \leqslant M_2 < r < N_2 \leqslant N_3$.

So for all $r \in S \cup T$, $M_3 < r < N_3$.

So $S \cup T$ is bounded.

(g) As in (e), let M_1 and N_1 be bounds for S, and M_2 and N_2 be bounds for T.

Then is it straightforward to show that for all $u \in U$, $M_1 + M_2 < u < N_1 + N_2$.

So U is bounded.

17. $(A \cap B) \cup C = A \cap (B \cup C)$ if and only if $C \subset A$.

20. Take a vector (a, b, c) such that $a + b + c = 0$. Then $c = -a - b$, so the vector is of the form $(a, b, -a - b)$.

    So we need to show that any vector $(a, b, -a - b)$ can be written as $x(1, 1, -2) + y(1, 0, -1)$.

    Solving the equations gives $x = b$, $y = a - b$.

    So the equations can always be solved.

24. All functions of the form $f(x) = x^2 + C$, for $C \in \mathbf{R}$, are solutions, so the set of solutions is infinite.

25. (c) Yes. Any function of the form $C \sinh x + D \cosh x$ is $((C + D)/2)e^x + ((D - C)/2)e^{-x}$, while any function $A e^x + B e^{-x}$ is $(A - B)\sinh x + (A + B)\cosh x$.
    So the sets are equal.

Chapter 8

1. (a) For $n = 1$, $1^2 = \frac{1}{6}(1)(2)(3)$, so the formula is true for $n = 1$.
   We assume the formula is true for n,

$$1^2 + 2^2 + \ldots + n^2 = \tfrac{1}{6}n(n + 1)(2n + 1)$$

Then,

$$\begin{aligned} 1^2 + 2^2 + \ldots + n^2 + (n + 1)^2 &= \tfrac{1}{6}n(n + 1)(2n + 1) + (n + 1)^2 \\ &= (n + 1)(\tfrac{1}{6}n(2n + 1) + n + 1) \\ &= (n + 1)\frac{2n^2 + n + 6n + 6}{6} \\ &= \frac{(n + 1)(n + 2)(2n + 3)}{6} \\ &= \tfrac{1}{6}(n + 1)((n + 1) + 1)(2(n + 1) + 1) \end{aligned}$$

This is the formula with n replaced by $n + 1$. So we have shown that the truth of the formula for n implies its truth for $n + 1$.

Since it is true for $n = 1$, it is true for all natural numbers n.

2. For $n = 1$, the expression is 1, which is positive.
Assume that $1 - 1/2 + 1/3 - \ldots + (-1)^{n-1} 1/n$ is positive for all n less then or equal to some number k.

If k is even, the expression for $k + 1$ is,

$$1 - \frac{1}{2} + \ldots - \frac{1}{k} + \frac{1}{k+1}$$

which is positive, since it is $1/(k + 1)$ plus something that we have assumed is positive.

If k is odd, the expression for $k + 1$ is,

$$1 - \frac{1}{2} + \ldots - \frac{1}{k-1} + \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

which is positive because $1/k - 1/(k + 1)$ is positive and we have assumed that $1 - 1/2 + \ldots - 1/(k - 1)$ is positive.

So the expression is positive for all n.

(Note that if we bracket the expression as,

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{2m-1} - \frac{1}{2m}\right)$$

It is clear that the expression must always be positive; the above inductive argument simply made the reasoning here explicit.)

7. (b)

$$\frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{n}{(n+1)!} = \frac{(n+1)! - 1}{(n+1)!}$$

8. The reasoning does not work when $n = 1$.

10. (b)

$$s_n = \tfrac{1}{2}n(n + 1) + 1$$

(c) The essential step is to prove that $s_{n+1} = s_n + n + 1$. This is true because the $(n + 1)$st line cuts all the existing n lines, and hence creates $n + 1$ new regions.

14. The result is clearly true for $n = 1$. Assume it is true for a number n. Then,

$$\begin{aligned}\frac{d^{n+1}(x^{n+1})}{dx^{n+1}} &= \frac{d^n}{dx^n}\left(\frac{d}{dx}(x\,x^n)\right)\\ &= \frac{d^n}{dx^n}(x^n + x\,n\,x^{n-1})\\ &= n! + n\left(\frac{d^n}{dx^n}x^n\right)\\ &= n! + n.n! = (n+1)n! = (n+1)!\end{aligned}$$