Solution for Chapter 16

(compiled by Guodong Wang)
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1 Problem A.(BT-16.4)

[by Guodong Wang/03]
(a). There are four dimensional variables M, ρ_∞, c_∞, G with only three independent dimensional fundamentals: mass, length and time. So by the dimensional considerations alone we cannot get a unique answer. Considering $\dot{M}$ should be proportional to ρ_∞, then

$$\dot{M} = K(\gamma)\frac{\rho_\infty (GM)^2}{c_\infty^3}, \tag{1}$$

Where $K(\gamma)$ is a dimensionless function of γ.

(b). The mass accretion rate $\dot{M}$ at radius r is $4\pi r^2 \rho v$. By mass conservation, we know it is a constant at all radii. Calculating $\dot{M}$ at $r \sim \frac{GM}{c_\infty^2}$, where it is reasonable to expect $v \sim c_\infty$:

$$\dot{M} \sim 4\pi\rho_\infty c_\infty (\frac{GM}{c_\infty^2})^2 \sim K(\gamma)\frac{\rho_\infty (GM)^2}{c_\infty^3}. \tag{2}$$

(c). The gas will finally stop on the surface of the neutron star so the flow speed has to change to subsonic near the surface. For the black hole, there is nothing to resist the flow, so it remains supersonic all the way through the horizion. Because no sound waves can propagate upstream in the supersonic region, there is no way for the flow before the shock to know that there will be a shock; so that flow is the same for the neutron star as for the black hole. Since the mass accretion rate is determined by conditions at the sonic point, which is before the shock, it must also be the same in both case.

(d). The Euler equation for $v(r)$ is

$$\rho v \frac{dv}{dr} = -\frac{dP}{dr} - \frac{GM\rho}{r^2} \tag{3}$$

The mass conservation reads

$$\frac{d\dot{M}}{dr} = 4\pi \frac{d}{dr}(r^2 \rho v) = 0 \tag{4}$$

Plugging in $c^2 = (\partial P/\partial \rho)_s$,

$$\frac{dP}{dr} = c^2 \frac{d\rho}{dr} \tag{5}$$

Inserting eq. (4) and eq. (5) into eq. (3), we get

$$(v^2 - c^2)\frac{1}{\rho}\frac{d\rho}{dr} = \frac{GM}{r^2} - \frac{2v^2}{r}. \tag{6}$$

So at the sonic point,

$$v_s^2 = c_s^2 = \frac{GM}{2r_s} \tag{7}$$

(e). Put eq. (7) into Bernoulli equation

$$\frac{1}{2}v_s^2 + \frac{v_s^2}{\gamma - 1} - \frac{GM}{r_s} = \frac{c_\infty^2}{\gamma - 1} \tag{8}$$

thus

$$c_s^2 = \frac{2c_\infty^2}{5 - 3\gamma}, \qquad r_s = \frac{(5-3\gamma)}{4}\frac{GM}{c_\infty^2}. \tag{9}$$

Calculate $\dot{M}$ at r_s (since it is constant at all r) and note $\rho^{1-\gamma}c^2 = \rho_\infty^{1-\gamma}c_\infty^2$ (BT-16.8). We get

$$\dot{M} = \frac{4\pi\lambda G^2 M^2 \rho_\infty}{c_\infty^3}, \tag{10}$$

where

$$\lambda = (\frac{1}{2})^{\frac{\gamma+1}{2(\gamma-1)}}(\frac{5-3\gamma}{4})^{\frac{3\gamma-5}{2(\gamma-1)}} \tag{11}$$

For $\gamma = 5/3$, $\lambda \simeq 0.25$ (by taking limit). So this factor is of order 1. It agrees well with $\dot{M}$ in parts (a) and (b).

(f). Let $\gamma = 5/3$, $c_\infty = \sqrt{\gamma P/\rho} = \sqrt{\gamma nkT/\rho} \simeq 10^5$ m/s, solar mass $\sim 2 \times 10^{33}$ g, Then $\dot{M} \sim 10^{10}$ g/s. Integrating eq. (10), to double the mass, it takes $M/2\dot{M} \simeq 10^{24}s \sim 3 \times 10^{16}$ year. So it is too slow to double the hole's mass by this way in the age of the universe (1.5×10^{10} year).

2 Problem B.(BT-16.6)

[by Guodong Wang/03]
(a).

$$\frac{\partial h}{\partial t} + \frac{\partial(hv)}{\partial x} = 0 \tag{12}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + g_e\frac{\partial h}{\partial x} = 0 \tag{13}$$

(13)$\pm\sqrt{\frac{g_e}{h}}$ (12), we get

$$\frac{\partial(v \pm 2\sqrt{g_e h})}{\partial t} + (v \pm \sqrt{g_e h})\frac{\partial(v \pm 2\sqrt{g_e h})}{\partial x} = 0 \tag{14}$$

So the Riemann invariants are

$$J_\pm \equiv v \pm 2\sqrt{g_e h} \tag{15}$$

with the characteristic speeds

$$V_\pm \equiv v \pm \sqrt{g_e h} \tag{16}$$

The conresponding conservation equations are

$$(\frac{\partial}{\partial t} + V_\pm \frac{\partial}{\partial x})J_\pm = 0 \tag{17}$$

(b). The argument is the same as in BT-16.4.1. Here $v = \sqrt{2g_e h} - constant$, so the water at the peak of the wave moves faster than the water in the bottom. This causes the leading edge of the wave to steepen.

(c).

$$J_+ = v + 2\sqrt{g_e h} = 2\sqrt{g_e h_0} \tag{18}$$

As in BT-16.42, Solve the leftward characteristics C_-, we obtain

$$C_- : \qquad x = (v - \sqrt{g_e h})t = (2\sqrt{g_e h_0} - 3\sqrt{g_e h})t \tag{19}$$

So

$$h(x,t) = \frac{h_0}{9}(2 - \frac{x/t}{\sqrt{g_e h_0}})^2 \tag{20}$$

and

$$v(x,t) = \frac{2}{3}(x/t + \sqrt{g_e h_0}) \tag{21}$$

at $-\sqrt{g_e h_0}t < x < 2\sqrt{g_e h_0}t$. The water is unperturbed outside this range.

3 Problem C.(BT-16.7)

[by Kip Thorne/03]

(i). Behind the wave, at height z,

$$P_2 = \rho g(h_2 - z) \tag{22}$$

and ahead the wave, under the water the pressure is

$$P_1 = \rho g(h_1 - z) \tag{23}$$

Analyze the flow in the rest frame of the breaking wave (the "shock wave"), so the speed of upstream water (water closes to the breaker) is $v_1 = v_{break}$. Let v_2 be the downstream flow speed away from the breaker. Then

$$\rho h_2 v_2 = \rho h_1 v_1, \tag{24}$$

and momentum conservation is

$$\rho v_1^2 h_1 + \int_0^{h_1} P_1 dz = \rho v_2^2 h_2 + \int_0^{h_2} P_2 dz \tag{25}$$

Combines Eqs. (24) and (25) we get

$$v_1 = v_{break} = \sqrt{\frac{g(h_1 + h_2)h_2}{2h_1}} \tag{26}$$

and the flow speed v_2 relative to the breaker (away from the breaker):

$$v_2 = \frac{h_1}{h_2} v_{break} \tag{27}$$

(ii). As for a shock wave in a gas, so also here, some of the external energy of the incoming flow goes into disordered, turbulent energy and thermal heat in the breaker. The incoming external energy flux $\frac{1}{2}\rho v_1^2 + \int_0^{h_1} \rho gzdz$ is greater than the outgoing external energy flux $\frac{1}{2}\rho v_2^2 + \int_0^{h_2} \rho gzdz$.

(iii). The small amplitude gravity wave speed in the long wavelength limit is $v_w = \sqrt{gh}$ (BT-15.2.2). In front of the breaker,(the upstream flow), $v_w = \sqrt{gh_1} < v_1 = v_{break}$. So the waves cannot avoid being caught by the breaker. (The flow is "supersonic"). Behind the breaker (the downstream flow), $v_w = \sqrt{gh_2} > v_2$, so waves can move into the breaker (The flow is "subsonic").

4 Problem D.(BT-16.11)

[by Alexei Dvoretskii/00]

(i) Equation of continuity

$$\partial_t \rho + \frac{2}{r}(\rho v) + \partial_r(\rho v) = 0$$

$$\rho = \frac{\gamma+1}{\gamma-1}\rho_0 g$$

$$v = \frac{2}{\gamma+1}\dot{R}h$$

Rewriting the equation in terms of ξ and R obtain

$$(\frac{\gamma+1}{\gamma-1})(-\frac{\xi\dot{R}}{R})\rho_0 g' + \frac{2}{\xi R}\frac{2}{(\gamma-1)}\rho_0\dot{R}gh + \frac{1}{R}\partial_\xi(\frac{2}{\gamma-1}\rho_0\dot{R}gh) = 0$$

Simplifying

$$-\xi g' + \frac{4}{(\gamma+1)}\frac{1}{\xi}gh + \frac{2}{(\gamma+1)}\partial_\xi(gh) = 0$$

So far, complete generality has been retained. Now let's take $h = \xi$.

$$-\xi g' + \frac{4}{(\gamma+1)}g + \frac{2}{(\gamma+1)}g'\xi + \frac{2}{(\gamma+1)}g = 0$$

This is readily solved to give

$$g = \xi^{\frac{6}{\gamma-1}},$$

where the integration constant has been fixed by the demand that $g = 1$ at $\xi = 1$.

We see that for $\gamma = 7$ the solution takes a simple form

$$g = \xi$$

.

(ii) Equation of motion

$$\partial_t v + v\partial_r v + \frac{1}{\rho}\partial_r P = 0$$

$$P = \frac{2}{\gamma+1}\rho_0 \dot{R}^2 f$$

Rewriting the equation in terms of ξ and R obtain

$$-\frac{\xi\dot{R}}{R}\partial_\xi(\frac{2}{\gamma+1})\dot{R}h+\dot{R}\partial_R(\frac{2}{\gamma+1})\dot{R}h+(\frac{2}{\gamma+1})\dot{R}h\frac{1}{R}\partial_\xi(\frac{2}{\gamma+1})\dot{R}h+\frac{\gamma-1}{\gamma+1}\rho_0^{-1}g^1\frac{1}{R}\partial_\xi(\frac{2}{\gamma+1})\rho_0\dot{R}^2 f = 0$$

Using $\partial_R \dot{R} = -\frac{3}{2}\frac{\dot{R}}{R}$ get

$$-\xi h'\frac{2}{\gamma+1} + h\frac{2}{\gamma+1}(-\frac{3}{2}) + \frac{4}{(\gamma+1)^2}hh' + 2\frac{\gamma-1}{(\gamma+1)^2}\frac{f'}{g} = 0$$

Substituting $h = \xi$ and $g = \xi$ and simplifying get

$$f' = 3\xi^2$$

And therefore

$$f = \xi^3$$

(iii) Entropy equation
Using

$$\partial_t + v\partial_r = -\frac{3}{4}\frac{\dot{R}}{R}\xi\partial_\xi + \dot{R}\partial_R$$

and

$$\frac{P}{\rho^\gamma} \propto \dot{R}^2\xi^{-4}$$

it is straightforward to verify that the above solution also satisfies the entropy equation.
(iv) It's trivial to take the integral to get

$$E = \frac{2\pi R^5\rho_0}{225t^2}$$

(v) The Sedov-Taylor solution will be more or less valid until the velocity of the shock front $\dot{R}$ falls to about the speed of sound in water.

$$\dot{R} \approx (\frac{E}{\rho_0 R^3})^{\frac{1}{2}}$$

so it will happen when

$$R \approx (\frac{E}{\rho_0 c_0^2})^{\frac{1}{3}}$$

Using $E = 10^{10}$J, $\rho_0 = 10^3$kgm^{-3} and $c_0 = 1.5 \times 10^3$ms^{-1} get

$$R \approx 20\text{m}$$

5 Problem D.(BT-16.13)

[by Guodong wang/03]
Let $\xi = \frac{x/t}{c_0}$. then $\frac{\partial}{\partial t} = -\frac{x}{c_0 t^2}\frac{d}{d\xi}$ and $\frac{\partial}{\partial x} = \frac{x}{c_0 t}\frac{d}{d\xi}$. Inserting these relations into the two partial differential equations (BT-16.32) with $\int dP/\rho c = 2c/(\gamma-1)$,

$$(\frac{\partial}{\partial t} + (v \pm c)\frac{\partial}{\partial x})(v \pm \frac{2c}{\gamma - 1}) = 0 \tag{28}$$

becomes a pair of ordinary differential equations,

$$((v - c_0\xi) \pm c)(v' \pm \frac{2c'}{\gamma - 1}) = 0 \tag{29}$$

Then it is straightforward to get the non-trival solutions,

$$c' = -(\frac{\gamma - 1}{\gamma + 1})c_0 \Rightarrow c = Constant - (\frac{\gamma - 1}{\gamma + 1})\frac{x}{t} \tag{30}$$

$$v = c_0\xi + c \tag{31}$$

The integration constant can be determined by matching v and c at the boundary with the unperturbed fluid: $v \to 0$, $c \to c_0$ when $-x/t \to c_0$. we thereby obtain

$$c = \frac{2c_0}{\gamma + 1} - (\frac{\gamma - 1}{\gamma + 1})\frac{x}{t}, \qquad v = \frac{2}{\gamma + 1}(c_0 + \frac{x}{t}) \tag{32}$$

at $-c_0 t < x < \frac{2c_0}{\gamma - 1}t$.