Solution for Chapter 6

(compiled by Xinkai Wu, revised by Kip Thorne)

A.

1. Ex. 6.2 Gaussian Wave-Packet and its Spreading [by Xinkai Wu/02]

Taylor expanding $\Omega(k)$ to $O((k-k_0)^2)$ and noticing that $V_g = \frac{d\Omega}{dk}|_{k_0}$, one finds $\Omega = \omega_0 + V_g\kappa + (dV_g/dk)\kappa^2/2$.

(a) Plugging the Taylor expansions of $A(k), \alpha(k), \omega(k)$ into the expression for $\psi(x,t)$, one gets eqn. (6.16)

(b) Using Mathematica, one finds the following integral formula

$$\int_{-\infty}^{\infty} d\kappa\; exp[-(a^2\kappa^2 + ib\kappa + ic\kappa^2)] = \left(\frac{\pi}{a^2+ic}\right)^{1/2} exp\left[-\frac{b^2(a^2-ic)}{4(a^4+c^2)}\right]$$

where a, b, c are real, and $a > 0$.

In our case, $a^2 = \frac{1}{2(\Delta k)^2}$, $b = -(x - x_0 - V_g t)$, $c = \frac{1}{2}\frac{dV_g}{dk}t$. So we find

$$|\psi| \propto exp\left[-\frac{b^2a^2}{4(a^4+c^2)}\right] = exp\left[-\frac{(x-x_0-V_gt)^2}{2L^2}\right]$$

$$\text{where } L = \frac{1}{\Delta k}\sqrt{1 + \left[\frac{dV_g}{dk}(\Delta k)^2 t\right]^2}$$

(c) At $t = 0$, we find $L = \frac{1}{\Delta k}$. Recalling that $L \sim \Delta x$(the width of the wave packet), we obtain the "uncertainty principle" $\Delta x \Delta k \sim 1$.

(d) $L(t=0) = \frac{1}{\Delta k}$. Solving $L(t) = 2L(0)$, we find $t = \frac{\sqrt{3}}{\frac{dV_g}{dk}(\Delta k)^2}$. Now let's consider a wave packet travelling from Hawaii to California. For the spreading to be less than a factor of 2, we must have, $\left|\frac{dV_g}{dk}\right|\frac{1}{(\Delta x)^2}t < \sqrt{3}$, where we've used the fact the the initial width of the wave packet $\Delta x = \frac{1}{\Delta k}$. The dispersion relation for waves on the surface of a deep body of water should be used: $\omega = \sqrt{gk}$ (see eqn.(6.5) in the text). Aslo $t = \frac{D}{V_g}$ with D being the distance between Hawaii and CA which we take to be $3 \times 10^3 km$. $k_0 = \frac{2\pi}{\lambda_0}$ and let's take $\lambda_0 = 100m$. We find it's necessary that

$$\Delta x > \sqrt{\frac{D\lambda_0}{4\pi\sqrt{3}}} \approx 4km = 40\lambda_0$$

2. Ex. 6.4 Gravitational Waves from a Spinning, Deformed Neutron Star [by Xinkai Wu/02]

(a) Using the expression for the phase $\phi(\mathbf{x},t) = \omega_0\tau exp[(r_*(r) - t)/\tau]$, we find,

$$\omega \equiv -\frac{\partial\phi}{\partial t} = \omega_0 e^{(r_*-t)/\tau}$$

$$\mathbf{k} \equiv \nabla\phi = \omega_0 e^{(r_*-t)/\tau}\frac{r}{r-2M}\mathbf{e}_r$$

and we see that the frequency is slowly decreasing.

(b) We have

$$\begin{aligned}\frac{1}{n}k &= \frac{1}{1+\frac{2M}{r}}\omega_0 e^{(r_*-t)/\tau}\frac{r}{r-2M} \\ &= \frac{1}{1-\left(\frac{2M}{r}\right)^2}\omega_0 e^{(r_*-t)/\tau} \\ &= \omega\left\{1+O\left[\left(\frac{M}{r}\right)^2\right]\right\}\end{aligned}$$

We are working with a weak gravitational field, thus we can neglect the $O\left[\left(\frac{M}{r}\right)^2\right]$ term and say that $\omega = \frac{1}{n}k$.

(c) For this simple dispersion relation, $\mathbf{V}_g = \mathbf{V}_{ph} = \frac{1}{n}\hat{\mathbf{k}}$. In general, we have

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial t} + \mathbf{V}_g\cdot\nabla\phi = -\omega + \mathbf{V}_g\cdot\mathbf{k} = -\mathbf{V}_{ph}\cdot\mathbf{k} + \mathbf{V}_g\cdot\mathbf{k} = (\mathbf{V}_g - \mathbf{V}_{ph})\cdot\mathbf{k}$$

The above equation makes it clear that if there's no dispersion, i.e. $\mathbf{V}_g = \mathbf{V}_{ph}$, then $\frac{d\phi}{dt} = 0$, the phase is constant along the ray. Now the ray is given by $\frac{d\mathbf{x}}{dt} = \mathbf{V}_g = \mathbf{V}_{ph}$, whose tangent is always normal to the surfaces of equal phase. The surfaces of equal phase are just spheres so we conclude $\{\theta, \phi\} = constant$ along the ray. Also the phase ϕ is a function of $t - r_*$ and we've shown above that the phase is a constant along the ray, thus $t - r_* = constant$. We have seen explicitly that both the frequency and the phase are functions of $t - r_*$, which tells us $t - r_*$ can be regarded as the retarded time for these waves.

(d) Let's reproduce eqn. (6.35) below

$$\frac{dA}{dt} \equiv \frac{\partial A}{\partial t} + \mathbf{V}_g\cdot\nabla A = -\frac{1}{2}\left[\frac{c}{nk}\nabla\cdot\mathbf{k} + \frac{\partial ln(\omega n^2)}{\partial t}\right]A$$

Carrying out the partial differentiations, we get

$$\begin{aligned}\frac{\partial A}{\partial t} &= A\frac{2(r_*-t)}{\tau^2\left[1+(r_*-t)^2/\tau^2\right]} \\ \mathbf{V}_g\cdot\nabla A &= -\frac{A}{n}\left[\frac{1}{r} + \frac{\frac{2(r_*-t)}{\tau^2}\frac{r}{r-2M}}{1+(r_*-t)^2/\tau^2}\right] \\ \left[-\frac{1}{2}\frac{c}{nk}\nabla\cdot\mathbf{k}\right]A &= -\frac{A}{2n}\left[\frac{1}{\tau}\frac{r}{r-2M} + \frac{1}{r} - \frac{1}{r-2M} + \frac{2}{r}\right] \\ -\frac{1}{2}\frac{\partial ln(\omega n^2)}{\partial t}A &= \frac{A}{2\tau}\end{aligned}$$

Using $\frac{1}{n}\ \frac{r}{r-2M} = \frac{1}{1-\left(\frac{2M}{r}\right)^2} \approx 1$, we find

$$\begin{aligned}
\text{l.h.s. of eqn (6.35)} &= -\frac{A}{nr} \\
\text{r.h.s. of eqn (6.35)} &= -\frac{A}{nr}\left[1+\frac{\frac{-M}{r}}{1-\frac{2M}{r}}\right] \approx -\frac{A}{nr}
\end{aligned}$$

[A simpler way of deriving the l.h.s. as suggested by Kip is: in part (c) we already showed that along the ray $\{t - r_*, \theta, \phi\}$ are constants, i.e. $\frac{d}{dt}$ acting on them gives zero. As a result, the l.h.s. of the propagation equation for A is just $\frac{dA}{dt} = -\ \frac{A}{r}\frac{dr}{dt} = -\ \frac{A}{r}V_g = -\ \frac{A}{nr}$.]

Thus the propagation law (6.35) is satisfied to leading order.

B.

Ex. 6.8 Geometric Optics for the Schrodinger equation [by Kip Thorne/99]

(a) We write ψ as

$$\psi = (A + \hbar B + ...)e^{iS/\hbar}$$

Note that $\hbar$ is playing the role of the two-lengthscale expansion parameter ϵ [compare the above equation with eq. (6.31)]. Then

$$\begin{aligned}
&-\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = \left(-A\frac{\partial S}{\partial t} + i\hbar\frac{\partial A}{\partial t} - \hbar B\frac{\partial S}{\partial t}\right)e^{iS/\hbar} + O\left(\hbar^2\right) \\
&\frac{1}{2m}\left(\frac{\hbar}{i}\nabla\right)^2\psi = \left[\frac{A}{2m}\left(\nabla S\right)^2 - \frac{i\hbar}{m}\nabla A\cdot\nabla S - \frac{i\hbar}{2m}A\nabla^2 S + \frac{\hbar B}{2m}\left(\nabla S\right)^2\right]e^{iS/\hbar} + O\left(\hbar^2\right) \\
&V\psi = (VA + \hbar VB)e^{iS/\hbar}
\end{aligned}$$

Plugging the above results into the Schrodinger equation and collecting the leading order, $O(\hbar^0)$, terms, we get the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\nabla S\right)^2 + V = 0$$

which is the dispersion relation, because using $S \equiv \hbar\phi, \omega \equiv -\frac{\partial\phi}{\partial t}, \mathbf{k} \equiv \nabla\phi$, we get

$$\omega = \Omega(\mathbf{k}, \mathbf{x}, t) = \frac{\hbar\mathbf{k}^2}{2m} + \frac{V}{\hbar}$$

(b)The equations of motion can be derived from Hamilton's equation for this $\Omega(\mathbf{k}, \mathbf{x}, t)$. Alternatively, it can be derived as follows:

Taking the gradient of the Hamilton-Jacobi equation gives

$$
\begin{aligned}
&\frac{\partial \nabla S}{\partial t} + \frac{1}{2m}\nabla\left[(\nabla S)^2\right] + \nabla V = 0 \\
&\text{using } \mathbf{p} = \nabla S \\
&\Rightarrow \frac{\partial \mathbf{p}}{\partial t} + \frac{1}{2m}\nabla \mathbf{p}^2 + \nabla V = 0 \\
&\Rightarrow \frac{\partial \mathbf{p}}{\partial t} + \frac{1}{2m}\left[2\mathbf{p}\times(\nabla\times\mathbf{p}) + 2(\mathbf{p}\cdot\nabla)\mathbf{p}\right] + \nabla V = 0 \\
&\text{noting } \nabla\times\mathbf{p} = \nabla\times\nabla S = 0 \\
&\Rightarrow \frac{\partial \mathbf{p}}{\partial t} + \left(\frac{\mathbf{p}}{m}\cdot\nabla\right)\mathbf{p} = -\nabla V \qquad (1)
\end{aligned}
$$

Now we define the time derivative along the ray as

$$
\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{V}_g\cdot\nabla = \frac{\partial}{\partial t} + \frac{\mathbf{p}}{m}\cdot\nabla
$$

where we've used that fact that $\mathbf{V}_g \equiv \nabla_{\mathbf{k}}\Omega = \frac{\hbar\mathbf{k}}{m} = \frac{\mathbf{p}}{m}$ (recall $\mathbf{k} \equiv \nabla\phi = \frac{1}{\hbar}\nabla S = \frac{1}{\hbar}\mathbf{p}$).

Using the above expression for $\frac{d}{dt}$, we immediately find

$$
\frac{d\mathbf{x}}{dt} = \mathbf{V}_g = \frac{\mathbf{p}}{m}
$$

Also, we see that eqn (1) is just

$$
\frac{d\mathbf{p}}{dt} = -\nabla V
$$

(c) Collecting $O(\hbar^1)$ terms in the Schrodinger equation and noticing that by virtue of the Hamilton-Jacobi equation terms containing B cancel, one finds

$$
\frac{\partial A}{\partial t} + \frac{1}{m}\nabla A\cdot\nabla S + \frac{1}{2m}A\nabla^2 S = 0 \qquad (2)
$$

which is the propagation equation for the wave amplitude A.

By $[eqn.(2)\cdot A^* + c.c.]$ we get,

$$
\begin{aligned}
&\frac{\partial |A|^2}{\partial t} + \nabla |A|^2\cdot\frac{\mathbf{p}}{m} + |A|^2\,\nabla\cdot\frac{\mathbf{p}}{m} = 0 \\
&\text{using } \frac{d}{dt} = \frac{\partial}{\partial t} + \left(\frac{\mathbf{p}}{m}\cdot\nabla\right) \\
&\Rightarrow \frac{d|A|^2}{dt} + |A|^2\,\frac{\nabla\cdot\mathbf{p}}{m} = 0
\end{aligned}
$$

We can also write the above equation as

$$
\frac{\partial |A|^2}{\partial t} + \nabla\cdot\left(|A|^2\,\frac{\mathbf{p}}{m}\right) = 0
$$

This equation is nothing but the familiar probability conservation equation in quantum mechanics, with $|A|^2 = |\psi|^2$ being the probability density and $|A|^2 \frac{\mathbf{p}}{m} = |\psi|^2 \frac{\mathbf{p}}{m}$ being the probability flux.

C.

1. Ex. 6.10 Matrix Optics for a Simple Refracting Telescope [by Xinkai Wu/00]

(a) Denote the focal length of the first lens as f_1, that of the second lens as f_2, and the distance between them as S. The whole system being axisymmetric, we can assume that the rays are in the $x-z$ plane. Let the transverse position of the ray right on the front surface of the first lens be x' and its slope be $\dot{x}'$; the transverse position of the ray on the back surface of the second lens be x and its slope be $\dot{x}$.

The transfer matrix of the whole system is as follows (a thin converging lens followed by a straight section, then followed again by a thin converging lens):

$$\begin{aligned} J &= \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1-\frac{S}{f_1} & S \\ \frac{S}{f_1 f_2} - \frac{1}{f_1} - \frac{1}{f_2} & 1-\frac{S}{f_2} \end{pmatrix} \end{aligned}$$

This gives

$$\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = J \begin{pmatrix} x' \\ \dot{x}' \end{pmatrix} = \begin{pmatrix} \left(1-\frac{S}{f_1}\right) x' + S\dot{x}' \\ \left(\frac{S}{f_1 f_2} - \frac{1}{f_1} - \frac{1}{f_2}\right) x' + \left(1-\frac{S}{f_2}\right)\dot{x}' \end{pmatrix}$$

To convert parallel rays into parallel rays, $\dot{x}$ must be independent of x', i.e. J_{21} must vanish:

$$\begin{aligned} J_{21} &= \frac{S}{f_1 f_2} - \frac{1}{f_1} - \frac{1}{f_2} = 0 \\ &\Rightarrow S = f_1 + f_2 \end{aligned}$$

which is the familiar result one learns in optics.

(b) When $S = f_1 + f_2$ is satisfied,

$$-M\theta = \dot{x} = \left(1 - \frac{S}{f_2}\right)\dot{x}' = \left(1 - \frac{f_1+f_2}{f_2}\right)\dot{x}' = -\frac{f_1}{f_2}\dot{x}' = -\frac{f_1}{f_2}\theta$$

Thus $M = \frac{f_1}{f_2}$.

2. Ex. 6.11 Rays bouncing between two mirrors [by unknown author/unknown year :)]

(a) The transfer matrix between $\mathbf{x}_{k+1}$ and $\mathbf{x}_k$ is given by

$$\begin{aligned} J &= \begin{pmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1-\frac{2d}{R} & 2d\left(1-\frac{d}{R}\right) \\ -\frac{4}{R}\left(1-\frac{d}{R}\right) & 1-\frac{6d}{R}+\frac{4d^2}{R^2} \end{pmatrix} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{aligned}$$

note that $detJ = 1$.

Now let's find the recursive relations:

$\mathbf{x}_{k+2}$ and $\mathbf{x}_k$ are related by J^2, and we find

$$\mathbf{x}_{k+2} = (A^2 + BC)\mathbf{x}_k + B(A+D)\dot{\mathbf{x}}_k$$

Also, using J, we find

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\dot{\mathbf{x}}_k$$

Combining the above two equation to eliminate $\dot{\mathbf{x}}_k$, we find

$$\begin{aligned} &\mathbf{x}_{k+2} - 2b\mathbf{x}_{k+1} + \mathbf{x}_k = 0 \\ &\text{where } b = 1 - \frac{4d}{R} + \frac{2d^2}{R^2} \end{aligned}$$

This difference equation can be written as

$$(\mathbf{x}_{k+2} - \mathbf{x}_{k+1}) - (\mathbf{x}_{k+1} - \mathbf{x}_k) = -2(1-b)\mathbf{x}_{k+1}$$

which is obviously a difference-equation analogue of the simple-harmonic-oscillator equation with $2(1-b)$ being the "spring constant".

(b) Plugging the trial solution $\mathbf{x}_k = \mathbf{A}coskl + \mathbf{B}sinkl$ into the difference equation, one finds

$$\begin{aligned} &\mathbf{A}ReS + \mathbf{B}ImI = 0 \\ &\text{with } S = e^{ikl}\left(e^{2il} - 2be^{il} + 1\right) \end{aligned}$$

For $\mathbf{A}$ and $\mathbf{B}$ to be independent constant vectors, we must have $S = 0$, which gives

$$b = cosl, \text{i.e. } l = cos^{-1}b$$

Thus, the difference equation has the general solution

$$\mathbf{x}_k = \mathbf{A}cos(kcos^{-1}b) + \mathbf{B}sin(kcos^{-1}b)$$

(c) Let $t \equiv \frac{d}{R}$, then $b = 1 - 4t + 2t^2 = 2(t-1)^2 - 1$, which is a parabola. Easily seen, if $0 < t < 2$ (i.e. $0 < d < 2R$), then $-1 < b < 1$, thus $cos^{-1}b$ is real, and the solution is oscillatory. When $t > 2$ (i.e. $d > 2R$), $b > 1$, then $cos^{-1}b$ becomes imaginary, and our solution exhibits exponential divergence.

(d) Let's take $\mathbf{A} = \mathbf{e}_x, \mathbf{B} = \mathbf{e}_y$. Easily seen this choice gives the desired circular Harriet delay line pattern, with a angular step $cos^{-1}b$. So

$$\theta = cos^{-1}b \Rightarrow t = d/R = 1 \pm \sqrt{\frac{cos\theta + 1}{2}} = 1 \pm cos\frac{\theta}{2}$$

D.

1.Ex. 6.6 Propagation of Sound Waves in a Wind [by Shuyun Qi]

(a) Let R be the ground's frame and R' be the rest frame of a thin layer of air at height z, which moves at the velocity $\mathbf{u} = Sz\mathbf{e}_x$ w.r.t. frame R. Coordinate transformation between frames R and R' gives $\mathbf{x}' = \mathbf{x} - \mathbf{u}t$. In frame R', $\omega' = ck'$ is the dispersion relation. The phase of the wave is frame-independent:

$$\begin{aligned} \phi &= -\omega' t + \mathbf{k}' \cdot \mathbf{x}' = -\omega' t + \mathbf{k}' \cdot (\mathbf{x} - \mathbf{u}t) \\ &= -(\omega' + \mathbf{k}' \cdot \mathbf{u})t + \mathbf{k}' \cdot \mathbf{x} \\ &= -\omega t + \mathbf{k} \cdot \mathbf{x} \end{aligned}$$

Thus

$$\begin{aligned} &\mathbf{k} = \mathbf{k}' \\ &\omega = \omega' + \mathbf{k}' \cdot \mathbf{u} = \omega' + \mathbf{k} \cdot \mathbf{u} \\ &\quad = c\,|\mathbf{k}'| + \mathbf{k} \cdot \mathbf{u} = c\,|\mathbf{k}| + Szk_x = \Omega(\mathbf{k}, \mathbf{x}, t) \end{aligned}$$

which is the dispersion relation as seen in the ground's frame.

(b) $\frac{dk_x}{dt} = -\frac{\partial\Omega}{\partial x} = 0$ so k_x is conserved along a ray. Using the dispersion relation found in part (a), we readily get

$$\begin{aligned} &\frac{\omega}{k_x} - Sz = \frac{k}{k_x}c \\ &\text{i.e. } \frac{\omega}{k_x} - u_x(z) = \frac{k}{k_x}c \end{aligned}$$

When $\left|\frac{\omega}{k_x} - u_x(z)\right| < c$, we have $k < |k_x|$. Since $k = \sqrt{k_x^2 + k_z^2}$, this means that k_z is an imaginary number, i.e. the wave will exponentially decay in the z direction. So we see that sound waves will not propagate when $\left|\frac{\omega}{k_x} - u_x(z)\right| < c$.

(c) Consider the Hamilton-Jacobi equations:

$$\begin{aligned} &\frac{dk_x}{dt} = -\frac{\partial\Omega}{\partial x} = 0 \\ &\Rightarrow k_x \text{is constant along a ray path} \end{aligned}$$

and

$$\frac{dk_z}{dt} = -\frac{\partial\Omega}{\partial z} = -Sk_x$$
$$\Rightarrow k_z = -Sk_x t + \text{const}$$
denote as θ the initial angle between $\mathbf{k}$ and the x direction
$$k_z(t=0) = k_x tan\theta \Rightarrow k_z = -k_x(St - tan\theta)$$

and

$$\frac{dx}{dt} = \frac{\partial\Omega}{\partial k_x} = \frac{k_x}{k}c + Sz$$
$$\frac{dz}{dt} = \frac{\partial\Omega}{\partial k_z} = \frac{k_z}{k}c$$
$$x(t=0) = 0; \; z(t=0) = 0$$

Let $\xi \equiv St - tan\theta$, then $\xi(t=0) = -tan\theta$ and $k_z = -k_x\xi$. We have

$$\frac{dz}{d\xi} = \frac{1}{S}\frac{dz}{dt} = \frac{c}{S}\left(\frac{-\xi}{\sqrt{1+\xi^2}}\right)\frac{k_x}{|k_x|}$$
$$\Rightarrow z(\xi) = \frac{c}{S}\left(|sec\theta| - \sqrt{1+\xi^2}\right)\frac{k_x}{|k_x|}$$

Since $\frac{dz}{dt}(t=0) > 0$, we get $k_z(t=0) > 0 \Rightarrow -k_x\xi(t=0) > 0 \Rightarrow k_x tan\theta > 0$, namely, k_x and $tan\theta$(thus $sec\theta$) have the same sign, $\frac{k_x}{|k_x|} = \frac{sec\theta}{|sec\theta|}$.

Thus

$$z(\xi) = \frac{c}{S}\left(sec\theta - \frac{sec\theta}{|sec\theta|}\sqrt{1+\xi^2}\right)$$

Also we have

$$\frac{dx}{d\xi} = \frac{1}{S}\frac{dx}{dt} = \frac{c}{S}\left[\frac{sec\theta}{|sec\theta|}\left(\frac{1}{\sqrt{1+\xi^2}} - \sqrt{1+\xi^2}\right) + sec\theta\right]$$
$$\Rightarrow x(\xi) = \frac{c}{S}\left\{\frac{sec\theta}{|sec\theta|}\frac{1}{2}\left[ln(\xi + \sqrt{1+\xi^2}) - \xi\sqrt{1+\xi^2}\right] + sec\theta\;\xi \right.$$
$$\left. - \frac{sec\theta}{|sec\theta|}\frac{1}{2}ln(|sec\theta| - tan\theta) + \frac{1}{2}tan\theta\;\sec\theta\right\}$$

$z(\xi), x(\xi)$ as given above describes the ray in a parametric form.

When $\theta < \frac{\pi}{2}$, $z(\xi) = \frac{c}{S}(sec\theta - \sqrt{1+\xi^2})$, and $z \rightarrow 0$ when ξ is sufficiently large.

When $\theta > \frac{\pi}{2}$, $z(\xi) = \frac{c}{S}(sec\theta + \sqrt{1+\xi^2})$, $z(\xi) \rightarrow \frac{c}{S}\xi$ when $\xi \rightarrow \infty$; and $x(\xi) \rightarrow \frac{c}{S}\left[\frac{1}{2}\xi^2 + sec\theta\;\xi + const\right]$ when $\xi \rightarrow \infty$. Thus for large t, $x \approx \frac{Sz^2}{2c} + sec\theta\;z + const$, which is a parabola propagating to infinity. See Fig.1 for

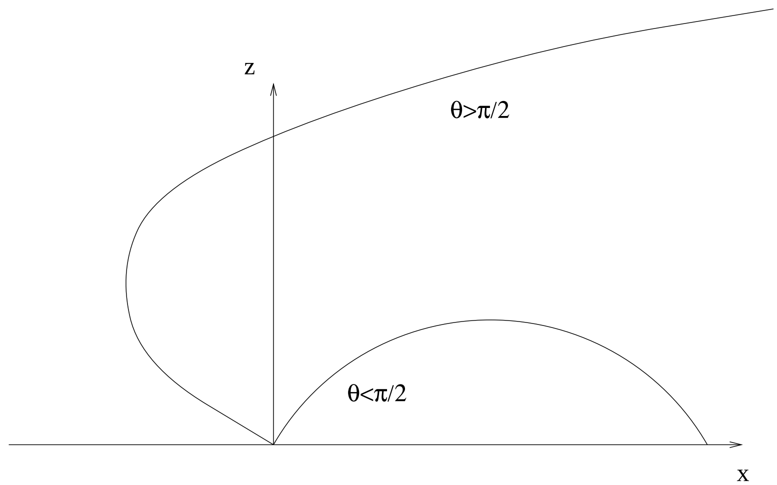

Figure 1: Ex 6.6 Sound wave in a wind

a sketch of the rays.

2. Ex. 6.13 Point-mass gravitational lens [by Xinkai Wu/02]
(a) Under the impulse approximation, the deflection is given by eqn (6.95)

$$\alpha = \frac{4GM}{bc^2} \quad b\text{: impact parameter}$$

The angle between the observer-lens line and the deflected ray if the ray is to meet the observer's eye is given by

$$\theta = \frac{b}{D}$$

Thus

$$\alpha = \theta \Rightarrow \frac{4GM}{bc^2} = \frac{b}{D} \Rightarrow b = \left(\frac{4GMD}{c^2}\right)^{1/2}$$

And we find the angular radius of the Einstein ring

$$\theta_E = \frac{b}{D} = \left(\frac{4GM}{Dc^2}\right)^{1/2}$$

(b) Now suppose the ray from the source before deflection has an angle β w.r.t. the observer-lens line, then

$$\beta + \theta = \alpha \Rightarrow b^2 + D\beta b = \frac{4GMD}{c^2}$$

in terms of θ, this is

$$\theta^2 + \beta\theta - \theta_E^2 = 0$$

Solving the above quadratic equation gives two solutions (taking the absolute values) corresponding to two images:

$$\theta_+ = \sqrt{\theta_E^2 + \left(\frac{\beta}{2}\right)^2} + \left(\frac{\beta}{2}\right) > \theta_E \quad \text{image outside Einstein ring}$$
$$\theta_- = \sqrt{\theta_E^2 + \left(\frac{\beta}{2}\right)^2} - \left(\frac{\beta}{2}\right) < \theta_E \quad \text{image inside Einstein ring}$$

(c) [this part by Roger Blandford/02]
In units in which $\theta_E = 1$,

$$\beta = \theta - 1/\theta$$

which implies

$$\theta_\pm = \beta/2 \pm \sqrt{1 + \beta^2/4} \tag{3}$$

nb also from Eq. [3],

$$\theta_+\theta_- = -1$$

Evaluate the magnification M from the Jacobian Eq. 6.92 using polar coordinates, $\beta, \phi \to \theta, \phi$

$$M = \frac{\theta}{\beta}\frac{d\theta}{d\beta} = (1 - \frac{1}{\theta^4})^{-1}$$

Hence

$$R \equiv \frac{M_+}{M_-} = \frac{1 - \frac{1}{\theta_-^4}}{1 - \frac{1}{\theta_+^4}} = \frac{1 - \frac{\theta_+^2}{\theta_-^2}}{1 - \frac{\theta_-^2}{\theta_+^2}} = -\left(\frac{\theta_+}{\theta_-}\right)^2 \tag{4}$$

using Eq. [4] We then obtain $\theta_\pm = \pm R^{\pm 1/4}$, inspecting the signs. Alternatively, evaluating directly,

$$R = \frac{\beta\sqrt{1 + \beta^2/4} + 1 + \beta^2/2}{\beta\sqrt{1 + \beta^2/4} - 1 - \beta^2/2}$$