Solution for Chapter 2

(compiled by Xinkai Wu, revised by Kip Thorne)

A.

Ex. 2.3 Observation of Cosmic Microwave Radiation from a Moving Earth [by Alexander Putilin]
a.

$$\begin{aligned} & I_\nu = \frac{h^4\nu^3}{c^2}\mathcal{N} \\ & \mathcal{N} = \frac{g_s}{h^3}\eta = \frac{2}{h^3}\eta\ (for\ photons) \\ & \Longrightarrow I_\nu = \frac{2h\nu^3}{c^2}\eta = \frac{(2h/c^2)\nu^3}{e^{h\nu/kT_0}-1} \\ & in\ its\ mean\ rest\ frame. \end{aligned}$$

let $x = h\nu/kT_0$,

$$I_\nu = \frac{2(kT_0)^3}{h^2c^2}\frac{x^3}{e^x-1} = (3.0\times10^{-15}\frac{erg}{cm^2})\frac{x^3}{e^x-1}$$

from Fig. 1, we see the intensity peak is at $x_m = 2.82$, which corresponds to $\nu_m = 1.6\times10^{11}s^{-1}, \lambda_m = 0.19cm$.
b. From chapter 1, we already know that the photon's energy as measured in the mean rest frame is $h\nu = -\vec{p}\cdot\vec{u}_0$, then (2.43) follows immediately.
c. Let $\mathbf{n}$ be the direction at which the receiver points, and $\mathbf{v}$ be the earth's velocity w.r.t. to the microwave background, then in the earth's frame, $\vec{u}_0 = (1/\sqrt{1-\mathbf{v}^2}, -\mathbf{v}/\sqrt{1-\mathbf{v}^2})$, $\vec{p} = (h\nu, -h\nu\mathbf{n})$. Plugging the above expressions into (2.43), we find(let θ be the angle between $\mathbf{v}$ and $\mathbf{n}$)

$$\begin{aligned} I_\nu = \frac{2h\nu^3}{c^2}\eta = \frac{2h}{c^2}\frac{\nu^3}{e^{h\nu/kT}-1} \\ with\ T = T_0\left(\frac{\sqrt{1-v^2}}{1-vcos\theta}\right) \end{aligned}$$

For small v, we can keep only terms linear in v and find $T \approx T_0(1+vcos\theta)$ which exhibits a dipolar anisotropy. And the maximal relative variation $\Delta T/T \approx (T(\theta=0)-T(\theta=\pi))/T_0 = 2v/c = 4\times10^{-3}$.

B.

Ex. 2.8 Vlasov Implies Conservation of Particles and of 4-Momentum [by Alexander Putilin]

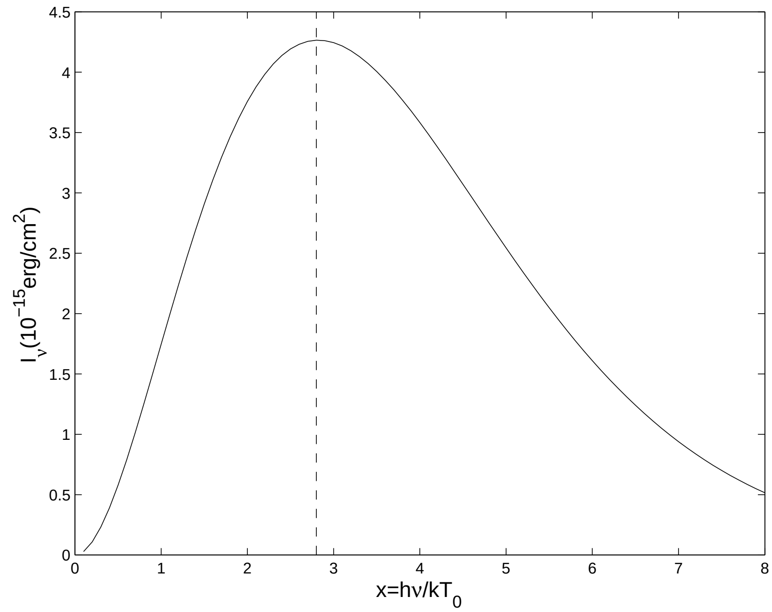

Figure 1: Ex. 2.3a

(a).

$$\frac{d\mathcal{N}}{d\zeta} = \frac{\partial \mathcal{N}}{\partial x^\mu}\frac{dx^\mu}{d\zeta} + \frac{\partial \mathcal{N}}{\partial p^j}\frac{dp^j}{d\zeta} = 0$$

$$note\ \frac{dp^j}{d\zeta} = 0\ for\ freely\ moving\ particles$$

$$also\ \frac{dx^\mu}{d\zeta} = p^\mu$$

thus the Vlasov eqn. can be written as

$$\mathcal{N}_{;\ \mu} p^\mu = \frac{\partial \mathcal{N}}{\partial x^\mu} p^\mu = 0$$

thus

$$S^\mu = \int \mathcal{N} p^\mu \frac{d\mathcal{V}_p}{p^0}$$

$$S^\mu{}_{;\ \mu} = \int \mathcal{N}_{;\ \mu} p^\mu \frac{d\mathcal{V}_p}{p^0} = 0$$

$$T^{\mu\nu} = \int \mathcal{N} p^\mu p^\nu \frac{d\mathcal{V}_p}{p^0}$$

$$T^{\mu\nu}{}_{;\ \nu} = \int \mathcal{N}_{;\ \nu} p^\mu p^\nu \frac{d\mathcal{V}_p}{p^0} = \int \mathcal{N}_{;\ \nu} p^\nu p^\mu \frac{d\mathcal{V}_p}{p^0} = 0$$

(b). In a Lorentz frame,

$$0 = S^{\alpha}{}_{;\ \alpha} = S^{\alpha}{}_{,\ \alpha} = S^{0}{}_{,\ 0} + S^{j}{}_{,\ j}$$
$$note\ S^0 = n,\ the\ number\ density\ of\ particles$$
$$\Longrightarrow \frac{\partial n}{\partial t} + \frac{\partial S^j}{\partial x^j} = \frac{\partial n}{\partial t} + \nabla \cdot \mathbf{S} = 0$$

C.

Ex. 2.7 Equation of State for Electron-Degenerate Hydrogen [by Alexander Putilin]

Mean occupation number of electron gas:

$$\eta = \frac{1}{e^{\frac{\tilde{E}-\tilde{\mu}_e}{kT}} + 1}, \quad \tilde{E}^2 = p^2 + m_e^2$$

Gas is degenerate if $\tilde{\mu}_e - m_e >> kT$. In this limit $\eta(\tilde{E})$ looks like Fig.2
The width of the "transition" region where $\eta(\tilde{E})$ goes from 0 to 1 is $\sim kT$, so in

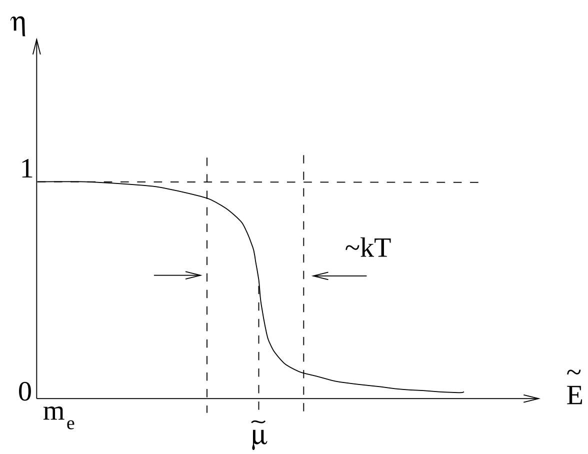

Figure 2: Ex. 2.7

the limit $\tilde{\mu}_e - m_e >> kT$ we can approximate $\eta(\tilde{E})$ by step function: $\eta(\tilde{E}) = 1$ for $\tilde{E} < \tilde{\mu}_e$; $\eta(\tilde{E}) = 0$ for $\tilde{E} > \tilde{\mu}_e$.
The number density n is given by

$$\begin{aligned} n &= \int_0^\infty 4\pi \mathcal{N} p^2 dp = 4\pi \frac{2}{h^3} \int_0^\infty \eta p^2 dp \\ &= \frac{8\pi}{h^3} \int_0^{p_F} p^2 dp = \frac{8\pi}{3h^3} p_F^3 \end{aligned}$$

where $p_F = \sqrt{\tilde{\mu}_e^2 - m_e^2}$.
The energy density $\rho = \rho_p + \rho_e$. Protons are nonrelativistic so $\rho_p = m_p n = 8\pi m_p p_F^3/3h^3$. while

$$\begin{aligned} \rho_e &= 4\pi \int_0^\infty \mathcal{N}\tilde{E}p^2 dp = \frac{8\pi}{h^3}\int_0^{p_F} \sqrt{p^2+m_e^2}\, p^2 dp \\ \rho_e &\approx m_e n,\ if\ p_F << m_e\ (non-relativistic\ case); \\ \rho_e &\approx \frac{2\pi}{h^3}p_F^4,\ if\ p_F >> m_e\ (ultra-relativistic\ case) \end{aligned}$$

In both cases $\rho_e << \rho_p$, provided that $p_F << m_p$, i.e. protons remain non-relativistic. Thus

$$\rho \approx \rho_p = \frac{8\pi m_p}{3h^3}p_F^3 = \frac{8\pi m_p}{3(h/m_e)^3}x^3, \quad x = \frac{p_F}{m_e}$$

Now turn to pressure. Electron's pressure

$$\begin{aligned} P_e &= \frac{4\pi}{3}\int_0^\infty \mathcal{N}\tilde{E}^{-1}p^4 dp = \frac{8\pi}{3h^3}\int_0^{p_F}\frac{p^4 dp}{\sqrt{p^2+m_e^2}} \\ &= \frac{8\pi m_e^4}{3h^3}\int_0^x \frac{z^4 dz}{\sqrt{z^2+1}}\ (let\ z = \frac{p}{m_e}) \\ &= \frac{\pi m_e^4}{h^3}\psi(x),\ where\ \psi(x) = \frac{8}{3}\int_0^x \frac{z^4 dz}{\sqrt{1+z^2}} \end{aligned}$$

Using Mathematica we find

$$\psi(x) = sinh^{-1}x - x\left(1 - \frac{2x^2}{3}\right)\sqrt{1+x^2}$$

$$for\ x << 1,\ \psi(x) \approx \frac{8}{15}x^5;\ for\ x >> 1,\ \psi(x) \approx \frac{2}{3}x^4$$

Proton pressure $P_p = nkT << P_e$ in both cases. Thus

$$P \approx P_e = \frac{\pi m_e^4}{h^3}\psi(x)$$

D.

Ex. 2.9 Solar Heating of the Earth: The Greenhouse Effect [by Alexander Putilin]

(a). The energy per unit time per unit frequency emmited by the surface element dA of the sun into the solid angle $d\Omega$ centered around unit vector $\hat{\mathbf{n}}$ is (see Fig. 3) $d\tilde{E}/dt = I_\nu dA cos\theta d\Omega d\nu$. And the total energy flux is thus

$$\begin{aligned} F = \int I_\nu cos\theta d\Omega d\nu &= \int_0^{\pi/2} 2\pi sin\theta d\theta \int_0^{+\infty} d\nu\ cos\theta \frac{2h}{c^2}\frac{\nu^3}{e^{h\nu/kT_\odot}-1} \\ &= \frac{2\pi k^4 T_\odot^4}{c^2 h^3}\int_0^{+\infty}\frac{x^3 dx}{e^x - 1}\quad (let\ x = h\nu/kT_\odot) \end{aligned}$$

The value of the above integral is $\pi^4/15$, thus we find

$$F = \sigma T_{\odot}^4, \quad where\ \sigma = \frac{2\pi^5}{15}\frac{k^4}{h^3c^2}$$

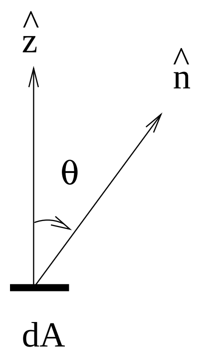

Figure 3: Ex. 2.9a

(b)(See Fig. 4). Similarly, the flux arriving at the earth is given by

$$F_e = \int I_\nu cos\theta d\Omega d\nu = \int_0^{\theta_0} 2\pi sin\theta cos\theta d\theta \int I_\nu d\nu = sin^2\theta_0 \sigma T_{\odot}^4$$

From Fig.4, we see $sin\theta_0 = R_{\odot}/r$. Thus

$$F_e = \sigma T_{\odot}^4 \left(\frac{R_{\odot}}{r}\right)^2$$

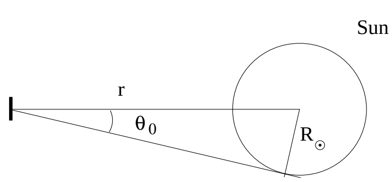

Figure 4: Ex. 2.9b

(c) (See Fig. 5) Radiated power $\left(d\tilde{E}/dt\right)_{radiated} = \sigma T_{\oplus}^4 4\pi R_{\oplus}^2$, while absorbed power $\left(d\tilde{E}/dt\right)_{absorbed} = F_e \int_{\theta=0}^{\theta=\pi/2} R_{\oplus}^2 cos\theta d\Omega = \pi F_e R_{\oplus}^2$. Then in thermal equilibrium, $\left(d\tilde{E}/dt\right)_{radiated} = \left(d\tilde{E}/dt\right)_{absorbed}$ immediately tells us

$$T_{\oplus} = T_{\odot}\left(\frac{R_{\odot}}{2r}\right)^{1/2} = 280K$$

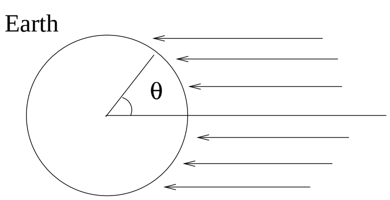

Figure 5: Ex. 2.9c

(d) If we take albedo A into account, $\left(d\tilde{E}/dt\right)_{absorbed} = (1-A)\pi R_\oplus^2 F_e$, while $\left(d\tilde{E}/dt\right)_{radiated}$ remains the same. Thus we get

$$T_\oplus = T_\odot \left(\frac{\sqrt{1-A}\; R_\odot}{2r}\right)^{1/2} = 255K$$

(e) Due to Greenhouse Effect, $\left(d\tilde{E}/dt\right)_{radiated} = 57\% \cdot 4\pi R_\oplus^2 \sigma T_\oplus^4$, and then $\left(d\tilde{E}/dt\right)_{radiated} = \left(d\tilde{E}/dt\right)_{absorbed}$ gives us $T_\oplus = 293K$.

Ex. 2.10 Olber's Paradox and Solar Furnace [by Alexander Putilin]
Place an observer at some spot on the earth and choose some arbitrary direction $\hat{\mathbf{n}}$. (See Fig.6)

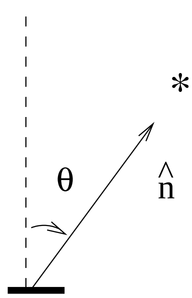

Figure 6: Olber's Paradox

Since the universe is assumed to be flat, it must be infinite in space and time so the observer will see some star in that direction.
Vlasov equation then gives $I_\nu(\hat{\mathbf{n}})/\nu^3 = I_\nu/\nu^3|_{at\ the\ star's\ surface}$. And since there's no gravitational and Doppler shifts in a flat stationary universe: $I_\nu(\hat{\mathbf{n}}) = I_\nu(at\ the\ star's\ surface)$.

The energy flux received by the observer is (see Ex. 2.9)

$$F = \int I_\nu cos\theta d\Omega d\nu = \int_0^{\pi/2} 2\pi sin\theta d\theta \int_0^{+\infty} d\nu \ cos\theta \frac{2h}{c^2} \frac{\nu^3}{e^{h\nu/kT_{star}} - 1}$$

$$= \sigma T^4, \ where \ T^4 = < T^4_{star} > = \sum_{i=1}^{N} \frac{1}{N} T_i^4$$

the summation is over all the visible stars and T_i is the temperature of the i-th star.
The hotter stars will dominate in this sum, so that $T \approx T_{hotter\ stars} \approx 10^4 K$.
When the earth come into thermal equilibrium $F = \sigma T_\oplus^4$, so its surface temperature will be $T_\oplus = T \approx 10^4 K$.
We are protected from being from fried because the universe is not stationary but rather is expanding (and has finite lifetime). The stars first formed when the universe was about 2 billion years old (at a redshift ~ 5). When we look out beyond that point, we see no more stars or galaxies. This means that only a small fraction of our sky is actually covered by stellar surfaces.
Now let's talk about solar furnace(see Fig. 7). We can use a lens of large

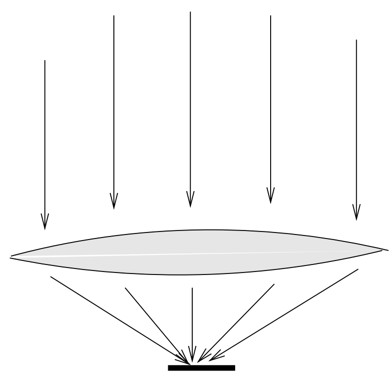

Figure 7: Solar Furnace

diameter D and small focal length $f << D$ to focus the sun's rays. At the spot where the rays are focused, the specific intensity I_ν is the same as at the surface of the sun:

$$I_\nu = \frac{2h}{c^2} \frac{\nu^3}{e^{h\nu/kT_\odot} - 1}$$

And the energy flux $F = \int I_\nu cos\theta d\Omega d\nu$, where the integration over the solid angle is almost over the whole upper hemisphere: $0 \leq \theta \leq \pi/2$. Thus we find $F = \sigma T_\odot^4$. So at equilibrium the temperature of the spot is $T = T_\odot$.
The effect of the lens is that it enlarges the image of the sun so that the image is spread over almost all the sky.

E.

Ex. 2.11 Diffusion Coefficient Computed in the "Collision-Time" Approximation [by Xinkai Wu]
a. let's use $d\mathbf{p}$ to denote $dp_x dp_y dp_z$

$$\int \left(\frac{d\mathcal{N}}{dt}\right)_{collision} d\mathbf{p} = \frac{1}{\hat{\tau}} \int (\mathcal{N}_0 - \mathcal{N}) d\mathbf{p}$$
$$\int \mathcal{N} d\mathbf{p} = n \ by \ definition$$
$$\int \mathcal{N}_0 d\mathbf{p} = n \int \frac{e^{-p^2/2mkT}}{(2\pi mkT)^{3/2}} d\mathbf{p} = n, as \ can \ be \ easily \ computed$$
$$thus \int \left(\frac{d\mathcal{N}}{dt}\right)_{collision} d\mathbf{p} = 0$$

b. Similar to eqn (2.116) of the text,

$$\left(\frac{d\mathcal{N}}{dt}\right)_{collision} = scattering - out \ term + scattering - in \ term$$

where the scattering-out term is given by $-\mathcal{N}_0/\hat{\tau}$(interpreting $1/\hat{\tau}$ as the scattering probability per unit time), and the scattering-in term is given by

$$\int \frac{d\sigma(\mathbf{n}' \to \mathbf{n})}{d\Omega} \mathcal{N}_0(t, \mathbf{x}, \mathbf{n}', p) d\Omega'$$

We use $\mathcal{N}_0$ in the above integral because we assume that when a particle gets scattered, its direction is randomized and its energy is thermalized at the scattering centers' temperature. Since $\mathcal{N}_0$ is independent of the direction of the momentum, we can take it out of the integral, and thus

$$scattering - in \ term = \mathcal{N}_0 \int \frac{d\sigma(\mathbf{n}' \to \mathbf{n})}{d\Omega} d\Omega' = \mathcal{N}_0/\hat{\tau}$$
$$\left(taking \frac{d\sigma(\mathbf{n}' \to \mathbf{n})}{d\Omega} = \frac{3}{16\pi\hat{\tau}} \left[1 + (\mathbf{n} \cdot \mathbf{n}')^2\right]\right)$$

thus we conclude that the collision term is $(\mathcal{N}_0 - \mathcal{N})/\hat{\tau}$.
c. The mean free path of the particle is $\lambda \sim \hat{\tau}\bar{v} \sim \hat{\tau}\sqrt{3kT/m}$. For the diffusion approximation to be reasonably accurate, we must have $\lambda << \mathcal{L}$, namely $\hat{\tau}\sqrt{3kT/m} << \mathcal{L}$.

d.(see Fig. 8) Similar to what is done in the text, let's take the density gradient to be along the z axis. Consider particles exchanged between two layers separated by a mean free path λ.
The flux to the right is $\sim \alpha n(0)\bar{v}$, and the flux to the left is $\sim \alpha n(\lambda)\bar{v}$, where α is a dimensionless constant of order 1/4(the 1/4 comes from the averaging

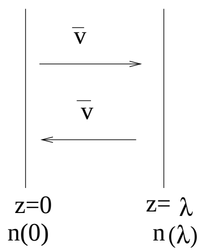

Figure 8: Ex. 2.11

$\frac{1}{2}\int_{\theta=0}^{\theta=\pi/2} cos\theta d\Omega / \int_{\theta=0}^{\theta=\pi/2} d\Omega)$.
The net flux to the right is: $S^z = \alpha n(0)\bar{v} - \alpha n(\lambda)\bar{v} = -\alpha\bar{v}\lambda\frac{dn}{dz}$. Comparing this with $\mathbf{S} = -D\nabla n$, we see $D \sim \alpha\bar{v}\lambda \sim \alpha\hat{\tau}\frac{3kT}{m} \sim \frac{kT}{m}\hat{\tau}$.
e. The law of particle conservation reads

$$\frac{\partial n}{\partial t} + \nabla \cdot \mathbf{S} = 0$$
$$we\ have\ \mathbf{S} = -D\nabla n$$
$$thus\ \frac{\partial n}{\partial t} + \nabla \cdot (-D\nabla n) = 0$$

f. $\frac{d\mathcal{N}}{dt} = \frac{\partial \mathcal{N}}{\partial t} + \frac{dx_j}{dt}\frac{\partial \mathcal{N}}{\partial x_j} + \frac{dp_j}{dt}\frac{\partial \mathcal{N}}{\partial p_j}$. For the "fiducial particle", $\frac{dx_j}{dt} = \frac{p_j}{m}, \frac{dp_j}{dt} = 0$. Combining this with the Boltzmann eqn. $\frac{d\mathcal{N}}{dt} = \left(\frac{d\mathcal{N}}{dt}\right)_{collision} = \frac{\mathcal{N}_0 - \mathcal{N}}{\hat{\tau}}$, we get eqn. (2.136).
g. Plugging $\mathcal{N} = \mathcal{N}_0 + \mathcal{N}_1$ into (2.136), and noting that we are considering steady state solution, we get

$$\frac{p_j}{m}\frac{\partial \mathcal{N}_0}{\partial x_j} + \frac{p_j}{m}\frac{\partial \mathcal{N}_1}{\partial x_j} = \frac{-\mathcal{N}_1}{\hat{\tau}}$$
$$neglecting\ the\ 2nd\ term\ on\ the\ l.h.s.\ gives$$
$$\mathcal{N}_1 = -\hat{\tau}\frac{p_j}{m}\frac{\partial \mathcal{N}_0}{\partial x_j}$$

Using the form of $\mathcal{N}_0$ given in eqn. (2.131) and noting that its $\mathbf{x}$ dependence comes soly from that of n, we get (2.137).
As a check, $\frac{\mathcal{N}_1}{\mathcal{N}_0} \sim \frac{(p_j\hat{\tau}/m)(\partial n/\partial x_j)}{n} \sim \frac{v_j\hat{\tau}}{\mathcal{L}} \sim \frac{\lambda}{\mathcal{L}} << 1$ (see part c).
h. The particle flux is $S_j = \int \mathcal{N} p_j \frac{d\mathcal{V}_p}{m}$. Only $\mathcal{N}_1$ contributes to S_j because $\mathcal{N}_0$ is isotropic in $\mathbf{p}$. Using the expression for $\mathcal{N}_1$ worked out in part g, we find

$$S_j = -D_{ji}\frac{\partial n}{\partial x_i}$$
$$where\ D_{ji} = \frac{\hat{\tau}}{m^2}\int p_j p_i \frac{e^{-p^2/2mkT}}{(2\pi mkT)^{3/2}} d\mathcal{V}_p = D\delta_{ji}$$
$$where\ D = \frac{1}{3}\frac{\hat{\tau}}{m^2}\int p^2 \frac{e^{-p^2/2mkT}}{(2\pi mkT)^{3/2}} d\mathcal{V}_p = \frac{kT}{m}\hat{\tau}$$

thus we have shown $\mathcal{S} = -D\nabla n$, with $D = \frac{kT}{m}\hat{\tau}$.

Ex. 2.12 Neutron Diffusion in a Nuclear Reactor [by Xinkai Wu]
Denote the distribution function by $\mathcal{N}(E,t)$. (As will become clear in a moment, it helps to think of $\mathcal{N}(E,t)$ as $n(x,t)$, where $n(x,t)$ is some density function in the coordinate space.)
Use $n_s, n_a, \sigma_s, \sigma_a$ to denote the density of scattering centers(i.e. moderator atoms), absorbing centers(i.e. ^{238}U atoms), the scattering cross section, and the absorption cross section, respectively.
A neutron with speed $\sqrt{2E/m}$ has a probability of getting scattered per unit time $\sqrt{2E/m}\, n_s\sigma_s$, and a probability of getting scattered per unit time $\sqrt{2E/m}\, n_a\sigma_a$.
Now we must find the energy decrement during each scattering. We can first go to the center-of-mass frame, get the final velocities of the particles, then transform back to the lab frame, and average over the 4π solid angle using the fact that the scattering cross section is isotropic in the center-of-mass frame. Chapter VI of Glasstone and Edlund does this for us and the result is:

$$dE = -\xi E,\ with\ \xi = 1 + \frac{(A-1)^2}{2A} ln\left(\frac{A-1}{A+1}\right)$$
$$where\ A\ is\ the\ atomic\ number\ of\ the\ moderator\ atom.$$

Thus we find that the rate of slowing down, i.e. the neutron's "velocity" in energy-space, is given by

$$\frac{dE}{dt} = -\sqrt{\frac{2E}{m}}\, n_s\sigma_s\xi E$$

Note the "flux" in energy-space is given by $(\mathcal{N}(E,t)dE/dt)$.
Thus we find the following "number conservation law" for $\mathcal{N}(E,t)$:

$$\frac{\partial\mathcal{N}}{\partial t} + \frac{\partial}{\partial E}\left(\mathcal{N}\frac{dE}{dt}\right) = -\mathcal{N}(E,t)\sqrt{\frac{2E}{m}} n_a\sigma_a$$

The term on the r.h.s. is a "sink" term corresponding to the absorption by ^{238}U.
We consider steady state thus the first term on the l.h.s. vanishes.
In general σ_s, σ_a both depend on E, and in the following we'll make this E-dependence explicit.
We are interested in the critical energy region between $E_1 = 7ev$ and $E_2 = 6eV$ where σ_a is non-zero.
Define the ratio between the "flux" at energy E and that at energy E_1 as:

$$\eta \equiv \frac{\mathcal{N}(E)dE/dt}{(\mathcal{N}(E)dE/dt)|_{E_1}} = \frac{\mathcal{N}(E)E^{3/2}\sigma_s(E)}{\mathcal{N}(E_1)E_1^{3/2}\sigma_s(E_1)}$$

Divide both sides of the "number conservation law" by the constant $(\mathcal{N}(E)dE/dt)|_{E_1}$.
The equation becomes

$$\frac{\partial\eta}{\partial E} = \frac{\eta}{\xi E}\frac{n_a\sigma_a(E)}{n_s\sigma_s(E)}$$

In the critical energy region, $\sigma_a(E)$ is approximately a constant $\sigma_{abs} \sim 2000\ barns$, also $\sigma_s(E)$ is approximately a constant σ_{scat}. Thus we can integrate the above differential equation over the critical region and get

$$ln(\eta(E_2)) = \frac{1}{\xi}\frac{n_a\sigma_{abs}}{n_s\sigma_{scat}} ln\frac{6}{7}$$

Thus $\eta(E_2) > 1/2$ requires

$$\frac{n_a}{n_s} < \frac{\xi ln2}{ln(7/6)}\frac{\sigma_{scat}}{\sigma_{abs}}$$

When we use carbon as moderator, $\xi = 0.158$, and $\sigma_{scat} = 4.8\ barns$(see table 3.79 of Glasstone and Edlund). And one must have

$$\frac{n_a}{n_s} < 1.7 \times 10^{-3}$$