Solution for Chapter 1

(compiled by Xinkai Wu, revised by Kip Thorne)

A.

1.1 Geometrized Units [by Alexei Dvoretskii]

$$(a)\quad t_P = \sqrt{\frac{G\hbar}{c^5}};\quad t_P = 5.36\times10^{-44}s;\quad t_P = 1.61\times10^{-35}m$$

$$(b)\quad m\frac{d\mathbf{v}}{dt} = e(\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B})$$

$$(c)\quad \mathbf{p} = \frac{\hbar\omega}{c}\mathbf{n}$$

How tall I am: $5.9\times10^{-9}s$; How old I am: $2.5\times10^{19}cm$

1.5 Numerics of Component Manipulations [by Xinkai Wu]

$$\begin{aligned}
&\mathbf{T}(\vec{A},\vec{A}) = T^{\alpha\beta}A_\alpha A_\beta = -9;\\
&\textit{denote } \mathbf{T}(\vec{A},_) \textit{ as } \vec{B},\ \textit{then } B^\beta = T^{\alpha\beta}A_\alpha \Longrightarrow\\
&B^0 = 1,\ B^1 = -4,\ B^2 = B^3 = 0;\\
&\textit{denote } \vec{A}\otimes\mathbf{T} \textit{ as } \mathbf{S},\ \textit{then } S^{\alpha\beta\gamma} = A^\alpha T^{\beta\gamma} \Longrightarrow\\
&S^{000} = 3,\ S^{001} = S^{010} = 2,\ S^{011} = -1,\\
&S^{100} = 6,\ S^{101} = S^{110} = 4,\ S^{111} = -2,\ \textit{all other components vanish}
\end{aligned}$$

1.6 Meaning of Slot Naming Index Notation [by Alexei Dvoretskii]

$$\begin{aligned}
&A_\alpha B^{\beta\gamma} \textit{ means } \vec{A}\otimes\mathbf{B}\\
&A_\alpha B^{\beta\alpha} \textit{ means } \mathbf{B}(_,\vec{A})\\
&S_{\alpha\beta\gamma} = T_{\gamma\beta\alpha} \textit{ means } \mathbf{S} = \mathbf{T}_{\textit{with slots 1 and 3 interchanged}}\\
&A_\alpha B^\alpha = g_{\mu\nu}A^\mu B^\nu \textit{ means } \vec{A}\cdot\vec{B} = \mathbf{g}(\vec{A},\vec{B})
\end{aligned}$$

1.15 Vectorial Identities for th Cross Product and Curl [by Alexei Dvoretskii]

$$\begin{aligned}
a.\quad & (\nabla\times(\nabla\times\mathbf{A}))_i = \epsilon_{ijk}(\nabla\times\mathbf{A})_{k;j} = \epsilon_{ijk}\epsilon_{klm}A_{m;lj}\\
= \quad & (\delta^i_l\delta^j_m - \delta^i_m\delta^j_l)A_{m;lj} = A_{j;ij} - A_{i;jj} = A_{j;ji} - A_{i;jj}\\
= \quad & (\nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A})_i \Longrightarrow \nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}
\end{aligned}$$

In the above equations all indices (slots) that follow the semicolon are gradient indices.

b. $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = \epsilon_{ijk}A_jB_k\epsilon_{ilm}C_lD_m = \epsilon_{jki}\epsilon_{lmi}A_jB_kC_lD_m$

$= (\delta^j_l\delta^k_m - \delta^j_m\delta^k_l)A_jB_kC_lD_m = A_jB_kC_jD_k - A_jB_kC_kD_j$

$= (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$

c. *using the identity demonstrated at the beginning of the problem*

$\mathbf{E}\times(\mathbf{F}\times\mathbf{G}) = (\mathbf{E}\cdot\mathbf{G})\mathbf{F} - (\mathbf{E}\cdot\mathbf{F})\mathbf{G}$

we easily get

$(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = [(\mathbf{A}\times\mathbf{B})\cdot\mathbf{D}]\,\mathbf{C} - [(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}]\,\mathbf{D}$

B.

1.7 Frame-Independent Expressions for Energy, Momentum, and Velocity [by Alexei Dvoretskii]

a. The energy E measured by the observer is the time-component p^0 of the particle's 4-momentum $\vec{p}$ in the observer's frame. In that frame, $\vec{p} = (p^0, \mathbf{p})$, $\vec{U} = (1, \mathbf{0})$, and thus $-\vec{p}\cdot\vec{U} = p^0 = E$

b. $\vec{p}^2 = (m\vec{u})^2 = m^2\vec{u}^2$, using $\vec{u}^2 = -1$ one gets the desired result.

c. In part a we showed (in the observer's frame) $\vec{p}\cdot\vec{U} = -p^0$; and by definition $\vec{p}\cdot\vec{p} = -(p^0)^2 + \mathbf{p}^2$. Thus we have $|(\vec{p}\cdot\vec{U})^2 + \vec{p}\cdot\vec{p}|^{1/2} = |(p^0)^2 - (p^0)^2 + \mathbf{p}^2|^{1/2} = |\mathbf{p}|$

d. $\mathbf{p} = m\gamma\mathbf{v}$, $E = m\gamma$, where $\gamma = 1/\sqrt{1-\mathbf{v}^2}$, thus we have $|\mathbf{v}| = |\mathbf{p}|/E$

e. From the previous parts we already know $\vec{p}\cdot\vec{U} = -p^0$, and $\mathbf{v} = \mathbf{p}/E = \mathbf{p}/p^0$. So we have

$$\frac{\vec{p} + (\vec{p}\cdot\vec{U})\vec{U}}{-\vec{p}\cdot\vec{U}} = \frac{(p^0, \mathbf{p}) - p^0(1,0)}{p^0} = \frac{(0,\mathbf{p})}{p^0} = (0,\mathbf{v}) = \vec{v}$$

1.18 3+1 Split of Charge-Current 4-Vector [by Xinkai Wu]

In the rest frame of an observer with 4-velocity $\vec{w}$, the charge-current 4-vector $\vec{J} = (\rho_{\vec{w}}, \mathbf{j})$, $\vec{w} = (1, \mathbf{0})$; and the 4-vector $\vec{j}_{\vec{w}} = (0, \mathbf{j})$. As can be easily verified, in this frame $\rho_{\vec{w}} = -\vec{J}\cdot\vec{w}$, and $\vec{j}_{\vec{w}} = \vec{J} + (\vec{J}\cdot\vec{w})\vec{w}$. Inverting these two expressions gives $\vec{J} = \vec{j}_{\vec{w}} + \rho_{\vec{w}}\vec{w}$. These relations are written in a frame-independent way, thus valid in any Lorentz frame.

C.

1.8 Doppler Shift Derived without Lorentz Transformations [by Alexei Dvoretskii]

The case of photon:

In frame F, $\vec{U} = (\gamma, \gamma\mathbf{v})$ with $\gamma = 1/\sqrt{1-\mathbf{v}^2}$, and $\vec{p} = (E_F, E_F\mathbf{n})$. Then using Eq. (1.69), we find the photon energy as measured by the emitting atom to be $E = -\vec{p}\cdot\vec{U} = E_F\gamma(1-\mathbf{v}\cdot\mathbf{n})$, i.e. $E_F/E = 1/[\gamma(1-\mathbf{v}\cdot\mathbf{n})]$.
The case of a particle with finite rest mass m:
Now $\vec{U}$ is same as in the photon case, but $\vec{p} = (E_F, |\mathbf{p}|\mathbf{n})$, where $|\mathbf{p}| = \sqrt{E_F^2 - m^2}$. And we find $E = -\vec{p}\cdot\vec{U} = \gamma(E_F - \sqrt{E_F^2 - m^2}\ \mathbf{v}\cdot\mathbf{n})$.

1.16 Reconstruction of **F** [by Alexei Dvoretskii]
Just like in the derivation of (1.107), we only need to show (1.108) holds in the rest frame of the observer $\vec{w}$(since it's written in a frame-independent way, it's true in any Lorentz frame if it's true in the observer's rest frame). In this frame, $w^0 = 1, w^j = 0$, and $E^0_{\vec{w}} = 0, E^j_{\vec{w}} = E_j, B^0_{\vec{w}} = 0, B^j_{\vec{w}} = B_j$. Both sides of (1.108) are manifestly antisymmetric in (α, β), thus we only need to check the $(0j)$ and (ij) components.
$F^{0j} = E_j$, while the r.h.s. of (1.108) is given by $w^0 E^j_{\vec{w}} - w^j E^0_{\vec{w}} + \epsilon^{0j}{}_{\gamma\delta} w^\gamma B^\delta_{\vec{w}}$, using the component forms of $\vec{w}$ and $\vec{E}_{\vec{w}}, \vec{B}_{\vec{w}}$ given above one easily finds $r.h.s. = E_j$.
$F^{ij} = \epsilon_{ijk}B_k$, while the r.h.s. of (1.108) is given by $w^i E^j_{\vec{w}} - w^j E^i_{\vec{w}} + \epsilon^{ij}{}_{\gamma\delta} w^\gamma B^\delta_{\vec{w}}$. Again, using the component forms of $\vec{w}$ and $\vec{E}_{\vec{w}}, \vec{B}_{\vec{w}}$ one easily finds $r.h.s. = \epsilon_{ijk}B_k$.

D.

1.11 Spacetime Diagrams [by Alexei Dvoretskii]
The spacetime diagrams are Fig. 1 through Fig. 6. In these figures, we use t', x' to denote $\bar{t}, \bar{x}$, and $\theta = tan^{-1}\beta$.
a. (See Fig. 1) Events A and B are simultaneous in $\bar{F}$. However because of the slope a $\bar{t} = const$ line has in frame F, A will occur before B in frame F(A is the event that's "farther back").
b. (See Fig. 2) Events A and B occur at the same spatial location in $\bar{F}$ but not in F.
c. (See Fig. 3) If P_1 and P_2 have a timelike separation, then P_2 lies inside the light cone and $\theta < 45°$. Hence in a boosted frame with $\beta = tan\theta < 1$ the two events will occur at the same spacial location. In $\bar{F}$, $\sqrt{-\Delta s^2} = \Delta\tau = \Delta\bar{t}$.
d. (See Fig. 4) Analogously P_2 will lie outside of the light cone and hence the angle θ(between $\overrightarrow{P_1P_2}$ and the x-axis) is less than 45°. By boosting into $\bar{F}$ with $tan\theta = \beta < 1$ we see that $\overrightarrow{P_1P_2}$ is parallel to the $\bar{x}$ axis, i.e. in $\bar{F}$ these two events are simultaneous. And we have $\sqrt{\Delta S^2} = |\Delta\bar{\mathbf{x}}|$.
e. (See Fig. 5. In the figure, the hyperbola is given by $t^2 - x^2 = \bar{t}^2$.) Let's consider how much time will elapse as measured by observers in F and $\bar{F}$ between O and P. $(\Delta\bar{t})^2 = (\Delta t)^2 - (\Delta t)^2 tan^2\theta = (\Delta t)^2(1-\beta^2)$, and thus $\Delta\bar{t} = \Delta t/\gamma$, i.e. time is slowed in a boosted frame.
f. (See Fig. 6. In the figure, the hyperbola is given by $x^2 - t^2 = \bar{x}^2$.) By analogous reasoning, $(\Delta\bar{x})^2 = (\Delta x)^2 - (\Delta x)^2 tan^2\theta = (\Delta x)^2(1-\beta^2)$, thus $\Delta\bar{x} = \Delta x/\gamma$,i.e. objects in a boosted frame are contracted along the boost.

since there are no boosts along y and z, the length along those axes is unchanged.

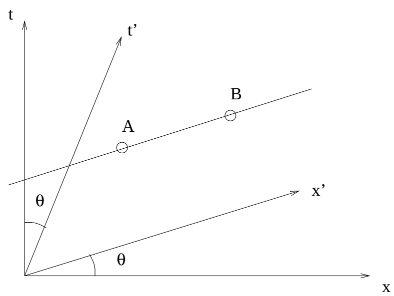

Figure 1: 1.11a

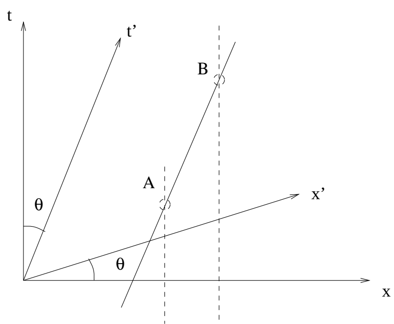

Figure 2: 1.11b

1.13 Twins Paradox [by Xinkai Wu]
a. Since $\vec{u}^2 = -1$, we have $0 = d(\vec{u}\cdot\vec{u})/d\tau = 2\vec{u}\cdot d\vec{u}/d\tau = 2\vec{u}\cdot\vec{a}$. In the observer's rest frame $\vec{u} = (1, \mathbf{0})$, thus $0 = \vec{u}\cdot\vec{a} \Longrightarrow \vec{a} = (0, \mathbf{a})$. So we get $\vec{a}\cdot\vec{a} = \mathbf{a}^2$, namely, $|\mathbf{a}| = \sqrt{\vec{a}\cdot\vec{a}}$.
b.
Denote $x^0,\ x^1$ coordinates in Methuselah's ref. frame as $t,\ x$, and the proper time of Florence as τ. We have

$$\frac{dt}{d\tau} = u^0,\ \frac{dx}{d\tau} = u^1$$
$$\frac{du^0}{d\tau} = a^0,\ \frac{du^1}{d\tau} = a^1$$

Using what we learned from part a, we have

$$0 = \vec{a}\cdot\vec{u} = -a^0u^0 + a^1u^1,\ g^2 = \vec{a}\cdot\vec{a} = -(a^0)^2 + (a^1)^2$$

which tells us that for $\tau \in [0, T_{Florence}/4] \cup [3T_{Florence}/4, T_{Florence}]$, $a^0 = gu^1, a^1 = gu^0$, and for $\tau \in [T_{Florence}/4, 3T_{Florence}/4]$, $a^0 = -gu^1, a^1 = -gu^0$.

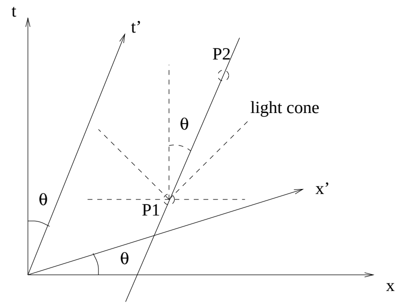

Figure 3: 1.11c

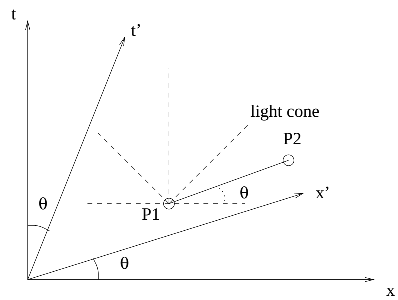

Figure 4: 1.11d

Then it's easy to integrate the acceleration twice and find $t(\tau), x(\tau)$. The answer is(we only give the $\tau \in [3T_{Florence}/4, T_{Florence}]$ part here b/c this is all we need)

$$\begin{aligned}
&for\ \tau \in [3T_{Florence}/4, T_{Florence}] \\
&t(\tau) = \frac{1}{g} sinh\left[g(\tau - T_{Florence})\right] + \frac{4}{g} sinh(\frac{1}{4} g T_{Florence}) \\
&x(\tau) = \frac{1}{g} cosh\left[g(\tau - T_{Florence})\right] - \frac{1}{g}
\end{aligned}$$

Thus we get(restoring c)

$$T_{Methuselah} = t(\tau = T_{Florence}) = \frac{4c}{g} sinh(\frac{g T_{Florence}}{4c})$$

Note that as $T_{Florence}$ increases, $T_{Methuselah}$ grows exponentially. A few numerical values are given below:

$$\begin{aligned}
&T_{Florence} = 10\ years\ gives\ T_{Methuselah} = 25\ years \\
&T_{Florence} = 50\ years\ gives\ T_{Methuselah} = 7.6 \times 10^5\ years \\
&T_{Florence} = 80\ years\ gives\ T_{Methuselah} = 1.7 \times 10^9\ years
\end{aligned}$$

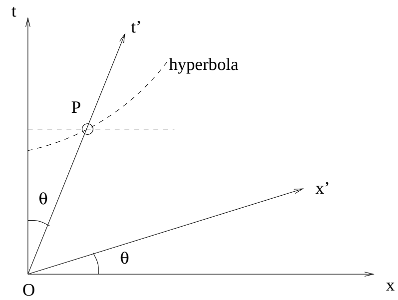

Figure 5: 1.11e

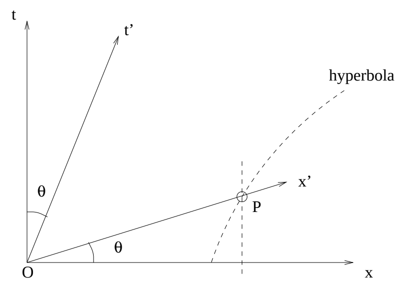

Figure 6: 1.11f

1.14 Around the World on TWA [by Xinkai Wu]

The 1972 Science papers of Hafele and Keating explain the experiment well. Let's summarize it as follows.

We analyze this problem in the non-rotating inertial frame whose origin coincides with the earth's center. Denote the proper time as measured by the eastward clock, the westward clock, and the clock in the ground laboratory as τ_e, τ_w, τ_g, respectively. For a clock moving with a speed $\mathbf{v}$ in this frame, to leading order of relativistic corrections, its proper time is related to the coordinate time by $d\tau = [1 - g(R-h)/c^2 - \mathbf{v}^2/2c^2]dt$, where R is the earth's radius, h is the clock's altitude, and g the surface value of the acceleration of gravity. The 2nd term in this expression is a general relativistic effect while the 3rd terms is a special relativistic one.

For the clock in the ground lab, $\mathbf{v}_g = \Omega R cos\lambda_g \mathbf{e}_\phi$, with Ω being the angular velocity of the earth's rotation, and λ_g the clock's latitude. For the eastward clock, $\mathbf{v}_e = (\Omega R cos\lambda_e + \nu cos\theta_e)\mathbf{e}_\phi + \nu sin\theta_e \mathbf{e}_\theta$, with λ_e being the eastward clock's latitude, ν being its speed w.r.t. the ground, and θ_e being the angle between its velocity and the eastward direction. For the westward clock, $\mathbf{v}_w = (\Omega R cos\lambda_w + \nu cos\theta_w)\mathbf{e}_\phi + \nu sin\theta_w \mathbf{e}_\theta$, with λ_w being the westward clock's latitude, and θ_w being the angle between the its velocity and the eastward direction. Note that $cos\theta_e > 0$ while $cos\theta_w < 0$

Using the above facts and eliminating dt, we find

$$d\tau_e = \left[1 + \frac{gh_e}{c^2} - \frac{\Omega^2 R^2(cos^2\lambda_e - cos^2\lambda_g) + \nu^2}{2c^2} - \frac{\Omega R\nu cos\lambda_e cos\theta_e}{c^2}\right] d\tau_g$$

and $d\tau_w$ given by the same formula with the subscript e replaced by w.
Integrating the above expressions gives the relation between τ_e and τ_g, τ_w and τ_g. In the real experiment $\nu, h_e, \lambda_e, \theta_e$(and h_w, λ_w, θ_w) changes with time so one must perform the integral numerically. For pedagogical purpose, here we consider the simplified case, where $\lambda_g = \lambda_e = \lambda_w = 0,\ \theta_e = 0, \theta_w = \pi$, and h_e, h_w are constants. We find

$$\tau_e - \tau_g = \frac{1}{c^2}\left[gh_e - \frac{\nu^2}{2} - \Omega R\nu\right]\frac{2\pi R}{\nu}$$
$$\tau_w - \tau_g = \frac{1}{c^2}\left[gh_w - \frac{\nu^2}{2} + \Omega R\nu\right]\frac{2\pi R}{\nu}$$

Take $\nu = 893km/hour$(based on the fact that it took about 45 hours to fly around the earth), $h_e = h_w = h = 10km$, we find

$$\frac{gh}{c^2}\frac{2\pi R}{\nu} = 178ns,\ \frac{\nu^2}{2c^2}\frac{2\pi R}{\nu} = 55ns,\ \frac{\Omega R\nu}{c^2}\frac{2\pi R}{\nu} = 208ns$$

(so we see that general relativistic effect is comparable to the special relativistic effect).
So we get: $\tau_e - \tau_g = -85ns,\ \tau_w - \tau_g = 331ns$.(the experimental data gives $\tau_e - \tau_g = -59 \pm 10ns,\ \tau_w - \tau_g = 273 \pm 7ns$).
One remark: the defference between the aging of the two flying clocks is given by

$$\tau_w - \tau_e = 2\frac{\Omega R}{c^2}2\pi R = 416ns$$

(the experimental result is $332 \pm 17ns$). Note that the velocity-independent general relativistic effect(and also all ν-dependence) cancels out in $\tau_w - \tau_e$.

E.

1.21 Clobal Conservation of 4-Momentum in a Lorentz Frame [by Alexei Dvoretskii]
The parallelepiped has eight faces, two perpendicular to each of the axes.

$$\begin{aligned}\int_{\partial\nu} T^{0\beta}d\Sigma_\beta = \quad & \Delta x\Delta y\Delta z(T^{00}(t+\Delta t) - T^{00}(t)) \\ & +\Delta x\Delta y\Delta t(T^{0z}(z+\Delta z) - T^{0z}(z)) \\ & +\Delta x\Delta z\Delta t(T^{0y}(y+\Delta y) - T^{0y}(y)) \\ & +\Delta y\Delta z\Delta t(T^{0x}(x+\Delta x) - T^{0x}(x))\end{aligned}$$

The first term gives the change of the energy in a 3d volume $\Delta x \Delta y \Delta z$ in time Δt. The other three terms give the flow of energy out of the 3d volume through the faces in time Δt.
The conservation law states that if the energy contained in a 3d volume increased/decreased then it flowed into/out of the volume. It's not created of destroyed in the volume itself, i.e. it's conserved.

1.22 Stress-Energy Tensor for a Perfect Fluid [by Alexei Dvoretskii]
The stress-energy tensor should be a symmetric tensor made from $\vec{u}, \mathbf{g}, \rho, \, and \, P$, so it must be of the form: $T^{\alpha\beta} = Au^\alpha u^\beta + Bg^{\alpha\beta}$, where A, B are scalars to be determined. In the local rest frame, $T^{jk} = P\delta^{jk}$ tells us $B = P$; and then $T^{00} = \rho$ tells us $A = \rho + B = \rho + P$; note $T^{0j} = 0$ is satisfied automatically. Thus we've derived the stress-energy tensor given in (1.142).