



Introduction to  
Nuclear and  
Particle Physics

Second Edition



# Introduction to Nuclear and Particle Physics

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*University of Rochester*

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*To  
Our Teachers  
and  
Our Students*



# Preface

This book is based on a one-semester course on Nuclear and Particle Physics that we have taught to undergraduate juniors and seniors at the University of Rochester. Naturally, the previous experience and background of our students determined to a large extent the level at which we presented the material. This ranged from a very qualitative and hand-waving exposition to one class that consisted of a mix of about six engineering and math majors, to relatively formal and quantitative developments for classes that were composed of about ten to fifteen well-prepared physics majors. It will not come as a great surprise that, independent of the degree of sophistication of our students, they were invariably fascinated by the subject matter, which provided great wonderment and stimulation to them. In class, we strove to stress the general underlying ideas of nuclear and particle physics, and we hope that in transforming our lecture notes into this more formal text, we have not committed the common sin of sacrificing physical content and beauty for difficulty and rigor.

It is quite remarkable how much has changed since we first wrote this book in 1989. The field of heavy-ion collisions has blossomed, the top quark and the  $\tau$  neutrino were discovered, a very small direct contribution to  $CP$  violation has been confirmed in  $K^0$  decays, large  $CP$  violation was found in interactions of neutral  $B$  mesons, the Standard Model has gained complete acceptance, and many exciting ideas have been proposed for possibilities for physics beyond the scale of the Standard Model. Furthermore, the confirmation of a finite mass for neutrinos has revealed the first chink in the armor, and a clear need for expansion of the Standard Model. The developments in the related field of cosmology have, if anything, been even more dramatic. We were tempted to include some of these in this second edition of our book, but fearing that this might expand it beyond its current scope

and sensible length, we decided not to pursue that option. Nevertheless, we have updated the original material, clarified several previous discussions, and added problems to help test the understanding of the material.

## Apologies

This book is intended primarily for use in a senior undergraduate course, and particularly for students who have had previous contact with quantum mechanics. In fact, more than just slight contact is required in order to appreciate many of the subtleties we have infused into the manuscript. A one-semester course in Quantum Mechanics should be of great help in navigating through the fantastic world of nuclear and particle phenomena. Although, in principle, our book is self-contained, there are parts of several chapters that will be daunting. For example, the sections on Relativistic Variables and Quantum Treatment of Rutherford Scattering in Chapter 1, some of the more formal material in Chapters 10, 11, 13, and 14, and the section on Time Development and Analysis of the  $K^0 - \bar{K}^0$  System in Chapter 12, are all especially demanding. Although the treatment of the mass matrix for the kaon system may be considered too advanced, and not essential for the overall development of the material in the book, we believe that the other sections are quite important. (Also, we felt that mathematically advanced students would appreciate some of the more challenging excursions.) Nevertheless, if deemed necessary, the formal concepts in these harder sections can be de-emphasized in favor of their phenomenological content.

Having chosen a somewhat historical development for particle physics, we had difficulty in infusing the quark structure of hadrons early into our logical development. We felt that this early introduction was important for familiarizing students with the systematics of hadrons and their constituents. To achieve this goal, we introduced the properties of quarks in the Problems section of Chapter 9, well before the discussion of their relevance in the Standard Model in Chapter 13. Although this might not be the best approach, it should nevertheless provide students, through problems, with the valuable experience of interpreting hadrons in terms of their quark content, and in reducing the possible confusion and frustration caused by keeping track of the many different hadrons.

## Units and Tables of Nuclear and Particle Properties

We use the cgs system of units throughout the text, except that energy, mass, and momentum are specified in terms of eV. This often requires the use of  $\hbar c$  to convert from cgs to the mixed system. Whenever possible, we have shown explicitly in the text how such change in units is made. Periodically, when we depart from our normal convention, as we do for the case of magnetic moments, we warn the reader of this change, and again offer examples or problems to ease the transition between different conventions.

We have found that the best source of information on properties of nuclei and particles, as well as on fundamental constants, is the all-inclusive *CRC Handbook of Chemistry and Physics* (CRC Press, Inc.) Because every library has copies of this work, we have not provided such detailed information in our manuscript, and urge students to consult the CRC tables when need arises. We have, nevertheless, included some useful physical constants in an appendix to this book.

## Other References

The subjects of nuclear and particle physics share a common heritage. The theoretical origins of the two fields and their reliance on quantum mechanics, as well as the evolution of their experimental techniques, provide much overlap in content. It is therefore sensible to present these two areas of physics, especially at the undergraduate level, in a unified manner. And, in fact, there are several excellent texts that have recently been published, or extensively revised, that provide the kind of combined exposition that we have presented. The books *Subatomic Physics* by Hans Frauenfelder and Ernest Henley (Prentice-Hall, Inc.), *Particles and Nuclei* by B. Povh, et al (Springer-Verlag), and *Nuclear and Particle Physics* by W. S. C. Williams (Oxford University Press) are particularly worthy of noting, because they offer a panoramic view of nuclear and particle physics of the kind that we have attempted to give in our book. We believe that the emphasis in all three of these works is sufficiently different and original to make them all complementary and of value to students learning these two exciting fields of physics.

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A. Das and T. Ferbel  
University of Rochester  
June, 2003

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## Chapter 1

# Rutherford Scattering

### 1.1 Introductory Remarks

Matter has distinct levels of structure. For example, atoms, once considered the ultimate building blocks, are themselves composed of nuclei and electrons. The nucleus, in turn, consists of protons and neutrons, which we now believe are made of quarks and gluons. Gaining an understanding of the fundamental structure of matter has not been an easy achievement, primarily because the dimensions of the constituents are so small. For example, the typical size of an atom is about  $10^{-8}$ cm, the average nucleus is about  $10^{-12}$ cm in diameter, neutrons and protons have radii of about  $10^{-13}$ cm, while electrons and quarks are believed to be without structure down to distances of at least  $10^{-16}$ cm (namely, they behave as particles of  $\lesssim 10^{-16}$ cm in size).

The study of the structure of matter presents formidable challenges both experimentally and theoretically, simply because we are dealing with the sub-microscopic domain, where much of our classical intuition regarding the behavior of objects fails us. Experimental investigations of atomic spectra provided our first insights into atomic structure. These studies ultimately led to the birth of quantum mechanics, which beautifully explained, both qualitatively and quantitatively, not only the observed spectra and the structure of the atom, but also clarified the nature of chemical bonding, and a host of phenomena in condensed matter. The remarkable success of quantum theory in explaining atomic phenomena was mainly due to two reasons. First, the interaction responsible for holding the atom together is the long-ranged electromagnetic force, whose properties were well understood in the classical domain, and whose principles carried over quite readily to the quantum regime. Second, the strength of the electromagnetic

coupling is weak enough (recall that the dimensionless coupling constant is represented by the fine structure constant,  $\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}$ ) so that the properties of even complex atomic systems can be estimated reliably using approximations based on perturbative quantum mechanical calculations. Peering beyond the atom into the nuclear domain, however, the situation changes drastically. The force that holds the nucleus together – the nuclear force as we will call it – is obviously very strong since it holds the positively charged protons together inside a small nucleus, despite the presence of the Coulomb force that acts to repel them. Furthermore, the nuclear force is short-ranged, and therefore, unlike the electromagnetic force, more difficult to probe. (We know that the nuclear force is short-ranged because its effect can hardly be noticed outside of the nucleus.) There is no classical equivalent for such a force and, therefore, without any intuition to guide us, we are at a clear disadvantage in trying to unravel the structure of the nucleus.

It is because of the lack of classical analogies that experiments play such important roles in deciphering the fundamental structure of subatomic matter. Experiments provide information on properties of nuclei and on their constituents, at the very smallest length scales; these data are then used to construct theoretical models of nuclei and of the nuclear force. Of course, the kinds of experiments that can be performed in this domain present interesting challenges in their own right, and we will discuss some of the techniques used in the field in Chapter 7. In general, much of the experimental information, both in nuclear and particle physics, is derived from scattering measurements – similar, in principle, to those that Ernest Rutherford and his collaborators performed in discovering the nucleus. In such experiments, beams of energetic particles are directed into a fixed target, or, alternately, two beams of energetic particles are made to collide. In either case, the results of collisions in such scattering experiments provide invaluable, and often the only attainable, information about subatomic systems. Since the basic principles in most of these experiments are quite similar, we will next sketch the ideas behind the pioneering work of Rutherford and his colleagues that was carried out at the University of Manchester, England, around 1910 and which provided the foundation for nuclear and particle physics.

## 1.2 Rutherford Scattering

The series of measurements performed by Hans Geiger and Ernest Marsden under Rutherford's direction at Manchester provide a classic example of a "fixed target" experiment. The target was a thin metal foil of relatively large atomic number, while the projectiles consisted of a collimated beam of low energy  $\alpha$ -particles, which, as we will see in the next chapter, are nothing more than the nuclei of helium atoms. The basic outcome of these experiments was that most of the  $\alpha$ -particles went straight through the foil with very little angular deviation. Occasionally, however, the deflections were quite large. A detailed analysis of these observations revealed the structure of the target, which ultimately led to the nuclear model of the atom.

To fully appreciate the beauty of these experiments, it is essential to analyze the results in their proper historical context. Prior to this work, the only popular model of the atom was due to Joseph Thomson, who visualized the electrically neutral atom as a "plum pudding" where negatively charged electrons were embedded, like raisins, within a uniform distribution of positive charge. If this model were correct, one would expect only small deviations in the  $\alpha$ -particles' trajectories (primarily due to scattering from the electrons), unlike what was found by Geiger and Marsden. To see this, let us do a few simple kinematic calculations. Because the velocities of the  $\alpha$ -particles in these experiments were well below  $0.1c$  (where  $c$  refers to the speed of light), we will ignore relativistic effects.

Let us assume that an  $\alpha$ -particle with mass  $m_\alpha$  and initial velocity  $\vec{v}_0$  collides head-on with a target particle of mass  $m_t$ , which is initially at rest (see Fig. 1.1). After the collision, both particles move with respective velocities  $\vec{v}_\alpha$  and  $\vec{v}_t$ . Assuming that the collision is elastic (namely, that no kinetic energy is converted or lost in the process), momentum and energy conservation yield the following relations.

*Momentum conservation:*

$$\begin{aligned} m_\alpha \vec{v}_0 &= m_\alpha \vec{v}_\alpha + m_t \vec{v}_t, \\ \text{or } \vec{v}_0 &= \vec{v}_\alpha + \frac{m_t}{m_\alpha} \vec{v}_t. \end{aligned} \tag{1.1}$$



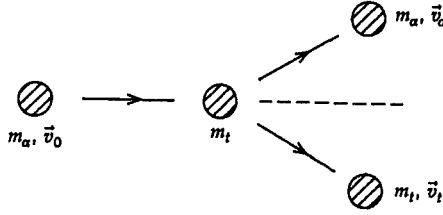


Fig. 1.1 Collision of a particle of mass  $m_\alpha$  and velocity  $\vec{v}_0$  with a target particle of mass  $m_t$ .

*Energy conservation:*

$$\begin{aligned} \frac{1}{2} m_\alpha v_0^2 &= \frac{1}{2} m_\alpha v_\alpha^2 + \frac{1}{2} m_t v_t^2, \\ \text{or } v_0^2 &= v_\alpha^2 + \frac{m_t}{m_\alpha} v_t^2, \end{aligned} \quad (1.2)$$

where we have labeled  $(\vec{v}_i)^2 = \vec{v}_i \cdot \vec{v}_i$  as  $v_i^2$ , for  $i = 0, \alpha$  and  $t$ . Squaring the relation in Eq. (1.1) and comparing with Eq. (1.2), we obtain

$$\begin{aligned} v_0^2 &= \left( v_\alpha^2 + \left( \frac{m_t}{m_\alpha} \right)^2 v_t^2 + 2 \frac{m_t}{m_\alpha} \vec{v}_\alpha \cdot \vec{v}_t \right) = v_\alpha^2 + \frac{m_t}{m_\alpha} v_t^2, \\ \text{or } v_t^2 \left( 1 - \frac{m_t}{m_\alpha} \right) &= 2 \vec{v}_\alpha \cdot \vec{v}_t. \end{aligned} \quad (1.3)$$

It is clear from this analysis that, if  $m_t \ll m_\alpha$ , then the left hand side of Eq. (1.3) is positive and, consequently, from the right hand side we conclude that the motion of the  $\alpha$ -particle and the target must be essentially along the incident direction. In other words, in such a case, one would expect only small deviations in the trajectory of the  $\alpha$ -particle. On the other hand, if  $m_t \gg m_\alpha$ , then the left hand side of Eq. (1.3) is negative, which implies large angles between the trajectories of the  $\alpha$ -particle and the recoiling nucleus, or large-angle scattering. To get a feeling for the magnitude of the numbers, let us recall that the masses of the electron and the  $\alpha$ -particle have the following approximate values

$$\begin{aligned} m_e &\approx 0.5 \text{ MeV}/c^2, \\ m_\alpha &\approx 4 \times 10^3 \text{ MeV}/c^2. \end{aligned} \quad (1.4)$$

Therefore, if we identify

$$m_t = m_e,$$

then,

$$\frac{m_t}{m_\alpha} \approx 10^{-4}. \quad (1.5)$$

Now, from Eq. (1.3) it follows that  $v_e = v_t \lesssim 2v_\alpha$ , and then Eq. (1.2) yields  $v_\alpha \approx v_0$ . Therefore,  $m_e v_e = m_\alpha \frac{m_e}{m_\alpha} v_e \lesssim 2 \times 10^{-4} m_\alpha v_\alpha \approx 2 \times 10^{-4} m_\alpha v_0$ , and the magnitude of the momentum transfer to the electron target is therefore  $\lesssim 10^{-4}$  of the incident momentum. Consequently, the change in the momentum of the  $\alpha$ -particle is quite small and, in the framework of the “plum pudding” model of the atom, we would expect only slight deviations in the  $\alpha$ -trajectory after scattering from atomic electrons; thus, the outcome of the experiments, namely the occasional scatters through large angles, would pose a serious puzzle. On the other hand, if we accept the nuclear model, wherein the atom has a positively charged core (the nucleus) containing most of the mass of the atom, and electrons moving around it, then the experimental observations would follow quite naturally. For example, setting the mass of the target to that of the gold nucleus

$$m_t = m_{\text{Au}} \approx 2 \times 10^5 \text{ MeV}/c^2, \quad (1.6)$$

yields

$$\frac{m_t}{m_\alpha} \approx 50. \quad (1.7)$$

A simple analysis of Eq. (1.3) gives  $v_t \leq \frac{2m_\alpha v_\alpha}{m_t}$ , and from Eq. (1.2) we again obtain that  $v_\alpha \approx v_0$ . Therefore,  $m_t v_t \leq 2m_\alpha v_\alpha \approx 2m_\alpha v_0$ . This means that the nucleus can carry away up to twice the incident momentum, which implies that the  $\alpha$ -particle can recoil backwards with a momentum essentially equal and opposite to its initial value. Such large momentum

transfers to the nucleus can, therefore, provide large scattering angles. Consequently, in the Rutherford picture, we would expect those  $\alpha$ -particles that scatter off the atomic electrons in gold to have only small-angle deflections in their trajectories, while the  $\alpha$ -particles that occasionally scatter off the massive nuclear centers to suffer large angular deviations.

The analysis of the scattering process, however, is not this straightforward, and this is simply because we have completely ignored the forces involved in the problem.<sup>1</sup> We know that a particle with charge  $Ze$  produces a Coulomb potential of the form

$$U(\vec{r}) = \frac{Ze}{r}. \quad (1.8)$$

We also know that two electrically charged particles separated by a distance  $r = |\vec{r}|$  experience a Coulomb force giving rise to a potential energy

$$V(r) = \frac{ZZ'e^2}{r}. \quad (1.9)$$

Here  $Ze$  and  $Z'e$  are the charges of the two particles. An important point to note about the Coulomb force is that it is conservative and central. A force is said to be conservative if it can be related to the potential energy through a gradient, namely

$$\vec{F}(\vec{r}) = -\vec{\nabla}V(\vec{r}), \quad (1.10)$$

and it is defined to be central if

$$V(\vec{r}) = V(|\vec{r}|) = V(r). \quad (1.11)$$

In other words, the potential energy associated with a central force depends only on the distance between the particles and not on their angular coordinates. Because the description of scattering in a central potential is no more complicated than that in a Coulomb potential, we will first discuss the general case.

Let us consider the classical scattering of a particle from a fixed center. We will assume that the particle is incident along the  $z$ -axis with an initial

<sup>1</sup>We have also tacitly assumed, in the context of the Thomson model, that contributions to large-angle scattering from the diffuse positively charged nuclear matter can be ignored. This is, in fact, the case, as discussed by Thomson in his historic paper.

velocity  $\vec{v}_0$ . (It is worth noting that, outside the foil, the incident and the outgoing trajectories are essentially straight lines, and that all the deflection occurs at close distances of the order of atomic dimensions, where the interaction is most intense.) If we assume that the potential (force) falls off at infinity, then conservation of energy would imply that the total energy equals the initial energy

$$E = \frac{1}{2} m v_0^2 = \text{constant} > 0. \quad (1.12)$$

Equivalently, we can relate the incident velocity to the total energy

$$v_0 = \sqrt{\frac{2E}{m}}. \quad (1.13)$$

Let us describe the motion of the particle using spherical coordinates with the fixed center as the origin (see Fig. 1.2). If  $r$  denotes the radial coordinate of the incident particle, and  $\chi$  the angle with respect to the  $z$ -axis, then the potential (being central) would be independent of  $\chi$ . Consequently, the angular momentum will be a constant during the entire motion. (That is, since  $\vec{r}$  and  $\vec{F}$  are collinear, the torque  $\vec{r} \times \vec{F}$  vanishes, and the angular momentum  $\vec{r} \times m\vec{v}$  cannot change.) For the incident particle, the angular momentum is clearly perpendicular to the plane of motion and has a magnitude  $\ell = m v_0 b$ , where  $b$  is known as the impact parameter. The impact parameter represents the transverse distance that the incident particle would fly by the source if there was no force acting. Using Eq. (1.13), we can obtain the following relation

$$\begin{aligned} \ell &= m \sqrt{\frac{2E}{m}} b = b \sqrt{2mE}, \\ \text{or } \frac{1}{b^2} &= \frac{2mE}{\ell^2}. \end{aligned} \quad (1.14)$$

From its definition, the angular momentum can also be related to the angular frequency,  $\dot{\chi}$ , as follows

$$\ell = |\vec{r} \times m\vec{v}| = |m\vec{r} \times \left( \frac{dr}{dt} \hat{r} + r \frac{d\chi}{dt} \hat{\chi} \right)| = m r^2 \frac{d\chi}{dt} \equiv m r^2 \dot{\chi}, \quad (1.15)$$

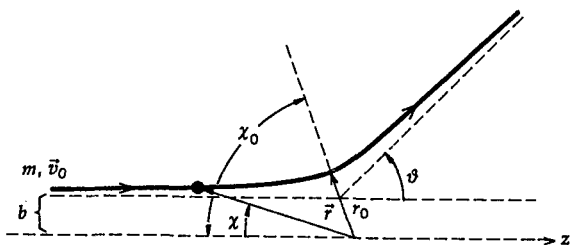


Fig. 1.2 The scattering of a particle of mass  $m$ , with initial (asymptotic) velocity  $\vec{v}_0$ , from a center of force at the origin.

where, as usual, we have defined a unit vector  $\hat{\chi}$  perpendicular to  $\vec{r} = r\hat{r}$ , with  $\vec{v}(\vec{r}) = \dot{r}\hat{r} + r\dot{\chi}\hat{\chi}$  expressed in terms of a radial and an angular component of the velocity, and the dot above a variable stands for differentiation with respect to time. Equation (1.15) can be rewritten as

$$\frac{d\chi}{dt} = \frac{\ell}{mr^2}. \quad (1.16)$$

The energy is identical at every point of the trajectory, and can be written as

$$\begin{aligned} E &= \frac{1}{2} m \left( \frac{dr}{dt} \right)^2 + \frac{1}{2} mr^2 \left( \frac{d\chi}{dt} \right)^2 + V(r) \\ &= \frac{1}{2} m \left( \frac{dr}{dt} \right)^2 + \frac{1}{2} mr^2 \left( \frac{\ell}{mr^2} \right)^2 + V(r), \end{aligned}$$

$$\text{or } \frac{1}{2} m \left( \frac{dr}{dt} \right)^2 = E - \frac{\ell^2}{2mr^2} - V(r),$$

$$\text{or } \frac{dr}{dt} = - \left[ \frac{2}{m} \left( E - V(r) - \frac{\ell^2}{2mr^2} \right) \right]^{\frac{1}{2}}. \quad (1.17)$$

The term  $\frac{\ell^2}{2mr^2}$  is referred to as the centrifugal barrier, which for  $\ell \neq 0$  can be considered as a repulsive contribution to an overall effective potential  $V_{\text{eff}}(r) = V(r) + \frac{\ell^2}{2mr^2}$ . Both positive and negative roots are allowed in Eq. (1.17), but we have chosen the negative root because the radial coordinate decreases with time until the point of closest approach, and that is the time

domain we will be examining.<sup>2</sup> Rearranging the factors in Eq. (1.17) and using Eq. (1.15), we obtain

$$\begin{aligned} \frac{dr}{dt} &= - \left[ \frac{2}{m} \frac{\ell^2}{2mr^2} \left\{ \frac{2mEr^2}{\ell^2} \left( 1 - \frac{V(r)}{E} \right) - 1 \right\} \right]^{\frac{1}{2}} \\ &= - \frac{\ell}{mr} \left[ \frac{r^2}{b^2} \left( 1 - \frac{V(r)}{E} \right) - 1 \right]^{\frac{1}{2}} \\ &= - \frac{\ell}{mrb} \left[ r^2 \left( 1 - \frac{V(r)}{E} \right) - b^2 \right]^{\frac{1}{2}}. \end{aligned} \quad (1.18)$$

From Eqs. (1.16) and (1.18), we now obtain

$$\begin{aligned} d\chi &= \frac{\ell}{mr^2} dt = \frac{\ell}{mr^2} \frac{dt}{dr} dr \\ &= - \frac{\ell}{mr^2} \frac{dr}{\frac{\ell}{mrb} \left[ r^2 \left( 1 - \frac{V(r)}{E} \right) - b^2 \right]^{\frac{1}{2}}}, \\ \text{or } d\chi &= - \frac{bdr}{r \left[ r^2 \left( 1 - \frac{V(r)}{E} \right) - b^2 \right]^{\frac{1}{2}}}. \end{aligned} \quad (1.19)$$

Integrating this between the initial point and the point of closest approach, we obtain

---

<sup>2</sup>The motion is completely symmetric about the point of closest approach ( $r = r_0$ ), and consequently the positive and negative roots provide identical information. In fact, if the  $\alpha$ -particle approached the target with the velocity  $v_0$  along the exiting trajectory in Fig. 1.2, it would then emerge on the entering trajectory, with the same asymptotic velocity. A simple way to see that this is true is to imagine the collision as observed from both above and below the plane of scattering shown in Fig. 1.2. Viewed from these two perspectives, the motion in Fig. 1.2 appears as the mirror image of the reversed trajectory. This symmetry is a consequence of time-reversal invariance of the equations of motion, a concept that will be discussed in Chapter 11.

$$\int_0^{\chi_0} d\chi = - \int_{\infty}^{r_0} \frac{bdr}{r \left[ r^2 \left( 1 - \frac{V(r)}{E} \right) - b^2 \right]^{\frac{1}{2}}},$$

$$\text{or } \chi_0 = b \int_{r_0}^{\infty} \frac{dr}{r \left[ r^2 \left( 1 - \frac{V(r)}{E} \right) - b^2 \right]^{\frac{1}{2}}}. \quad (1.20)$$

The point of closest approach is determined by noting that, as the particle approaches from infinity, its velocity decreases continuously (assuming the repulsive potential for the case of an  $\alpha$ -particle approaching a nucleus), until the point of closest approach, where the radial velocity ( $\frac{dr}{dt}$ ) vanishes and subsequently changes sign. That is, beyond this point, the velocity of the particle increases again. Therefore, at the distance of closest approach, when  $r = r_0$ , both the radial and the absolute velocities attain a minimum, and we have

$$\left. \frac{dr}{dt} \right|_{r=r_0} = 0,$$

which, from Eqs. (1.17) and (1.18), means that

$$E - V(r_0) - \frac{\ell^2}{2mr_0^2} = 0,$$

$$\text{or } r_0^2 \left( 1 - \frac{V(r_0)}{E} \right) - b^2 = 0. \quad (1.21)$$

Thus, given a specific form of the potential, we can determine  $r_0$ , and therefore  $\chi_0$ , as a function of the impact parameter  $b$ .<sup>3</sup> Defining the scattering angle  $\theta$  as the change in the asymptotic angles of the trajectory, we get

$$\theta = \pi - 2\chi_0 = \pi - 2b \int_{r_0}^{\infty} \frac{dr}{r \left[ r^2 \left( 1 - \frac{V(r)}{E} \right) - b^2 \right]^{\frac{1}{2}}}. \quad (1.22)$$

---

<sup>3</sup>We note that, in general, with  $\ell \neq 0$  and  $E > 0$ , that is, for  $b \neq 0$ ,  $\frac{dx}{dt}$  is maximum at  $r = r_0$  (see Eq. (1.16)). Also, for  $\ell \neq 0$ , even for an attractive Coulomb potential, there will be a finite result for  $r_0$  as determined from Eq. (1.21). This is because the centrifugal barrier for  $\ell \neq 0$  acts as a repulsive potential that dominates over Coulomb attraction at small distances.

Consequently, given an impact parameter  $b$ , and a fixed energy  $E$ , the scattering angle of a particle in a potential can, at least in principle, be completely determined.

As an application of the general result, let us now return to the scattering of a charged particle from a repulsive Coulomb potential, for which the potential energy is given by Eq. (1.9)

$$V(r) = \frac{ZZ'e^2}{r}, \quad (1.23)$$

where  $Z'e$  represents the charge of the incident particle and  $Ze$  the charge of the scattering center. (The scattering of an  $\alpha$ -particle from a nucleus would then correspond to  $Z' = 2$ , with  $Ze$  representing the nuclear charge.) The distance of closest approach can be obtained from Eq. (1.21)

$$r_0^2 - \frac{ZZ'e^2}{E} r_0 - b^2 = 0,$$

$$\text{or } r_0 = \frac{\frac{ZZ'e^2}{E} \pm \sqrt{\left(\frac{ZZ'e^2}{E}\right)^2 + 4b^2}}{2}. \quad (1.24)$$

Since the radial coordinate can by definition only be positive, we conclude that

$$r_0 = \frac{ZZ'e^2}{2E} \left( 1 + \sqrt{1 + \frac{4b^2 E^2}{(ZZ'e^2)^2}} \right). \quad (1.25)$$

Consequently, from Eq. (1.22), we obtain

$$\theta = \pi - 2b \int_{r_0}^{\infty} \frac{dr}{r \left[ r^2 \left( 1 - \frac{ZZ'e^2}{r} \right) - b^2 \right]^{\frac{1}{2}}}. \quad (1.26)$$

Let us define a new variable

$$x = \frac{1}{r}, \quad (1.27)$$

which gives



$$x_0 = \frac{1}{r_0} = \frac{2E}{ZZ'e^2} \left( 1 + \sqrt{1 + \frac{4b^2 E^2}{(ZZ'e^2)^2}} \right)^{-1}. \quad (1.28)$$

From Eq. (1.27), we obtain

$$dx = -\frac{dr}{r^2}, \quad \text{or} \quad dr = -\frac{dx}{x^2},$$

and, in terms of this new variable, we can write

$$\begin{aligned} \theta &= \pi - 2b \int_{x_0}^0 \left( -\frac{dx}{x^2} \right) \frac{x}{\left[ \frac{1}{x^2} - \frac{ZZ'e^2}{x} - b^2 \right]^{\frac{1}{2}}} \\ &= \pi + 2b \int_{x_0}^0 \frac{dx}{\left( 1 - \frac{ZZ'e^2}{E} x - b^2 x^2 \right)^{\frac{1}{2}}}. \end{aligned} \quad (1.29)$$

Now, using the following result from the integral tables

$$\int \frac{dx}{\sqrt{\alpha + \beta x + \gamma x^2}} = \frac{1}{\sqrt{-\gamma}} \cos^{-1} \left( -\frac{\beta + 2\gamma x}{\sqrt{\beta^2 - 4\alpha\gamma}} \right), \quad (1.30)$$

we obtain

$$\begin{aligned} \theta &= \pi + 2b \times \frac{1}{b} \cos^{-1} \left( \frac{\frac{ZZ'e^2}{E} + 2b^2 x}{\sqrt{\left( \frac{ZZ'e^2}{E} \right)^2 + 4b^2}} \right) \Bigg|_{x_0}^0 \\ &= \pi + 2 \cos^{-1} \left( \frac{1 + \frac{2b^2 E}{ZZ'e^2} x}{\sqrt{1 + \frac{4b^2 E^2}{(ZZ'e^2)^2}}} \right) \Bigg|_{x_0}^0 \\ &= \pi + 2 \cos^{-1} \left( \frac{1}{\sqrt{1 + \frac{4b^2 E^2}{(ZZ'e^2)^2}}} \right) - 2 \cos^{-1}(1) \\ &= \pi + 2 \cos^{-1} \left( \frac{1}{\sqrt{1 + \frac{4b^2 E^2}{(ZZ'e^2)^2}}} \right). \end{aligned} \quad (1.31)$$

Equivalently, we can write

$$\frac{1}{\sqrt{1 + \frac{4b^2 E^2}{(ZZ'e^2)^2}}} = \cos\left(\frac{\theta}{2} - \frac{\pi}{2}\right),$$

$$\text{or } \frac{1}{1 + \frac{4b^2 E^2}{(ZZ'e^2)^2}} = \cos^2\left(\frac{\theta}{2} - \frac{\pi}{2}\right) = \sin^2 \frac{\theta}{2} = \frac{1}{\operatorname{cosec}^2 \frac{\theta}{2}},$$

$$\text{or } \frac{2bE}{ZZ'e^2} = \cot \frac{\theta}{2},$$

$$\text{or } b = \frac{ZZ'e^2}{2E} \cot \frac{\theta}{2}. \quad (1.32)$$

This relates the scattering angle, which is a measurable quantity, to the impact parameter which cannot be observed directly. Note that, for fixed  $b$ ,  $E$  and  $Z'$ , the scattering angle is larger for a larger value of  $Z$ . This is consistent with our intuition in that the Coulomb potential is stronger for larger  $Z$ , and leads to a larger deflection. Similarly, for a fixed  $b$ ,  $Z$  and  $Z'$ , the scattering angle is larger when  $E$  is smaller. Qualitatively, we can understand this as follows. When the particle has low energy, its velocity is smaller and, therefore, it spends more time in the potential and suffers a greater amount of scattering. Finally, for fixed  $Z$ ,  $Z'$  and  $E$ , the scattering angle is larger for smaller  $b$ . Namely, when the impact parameter is small, the particle feels the force more strongly and hence the deflection is larger. Equation (1.32) therefore incorporates all the qualitative features that we expect of scattering in the Coulomb field.

### 1.3 Scattering Cross Section

As we have seen, the scattering of a particle in a potential is completely determined once we know the impact parameter and the energy of the particle; and, for a fixed incident energy, the deflection is therefore defined by just the impact parameter. To perform an experiment, we prepare an incident flux of beam particles of known energy, and measure the number of particles scattered out of the beam at different  $\theta$ . Because this number is determined entirely by the impact parameters involved in the collisions,

such measurements reflect these impact parameters and thereby the range of the interaction and the effective size of the scattering center.

Let  $N_0$  denote the number of particles incident on the target foil per unit area per unit time. Because we assume the target density in the foil to be low, this flux will always be uniform over the thickness of the target material. Any incident particle with impact parameter between  $b$  and  $b + db$  relative to any scattering center will undergo an angular deflection between  $\theta$  and  $\theta + d\theta$ , and will scatter into a solid angle  $d\Omega$ . (The larger the impact parameter, the smaller is the scattering angle.) The number of such particles scattered per unit time is  $2\pi N_0 b db$ , since  $2\pi b db$  is the relevant area of the circular ring around each scattering center through which any particle must pass in order to be emitted into the solid angle between  $\theta$  and  $\theta + d\theta$ . It may seem puzzling that we do not have to be concerned with the fact that we have many target particles in our foil, and that any single beam particle, in principle, comes within some impact parameter of all of them! This would clearly provide a great complication to our analysis. But we are assuming that our foil is exceedingly thin, so that multiple collisions of one beam particle are negligible; and we also have the Rutherford atomic model in mind, which means that the separation between nuclei is vast relative to their size. Normally, very large impact parameters provide very little scattering, so it is the trajectory that comes nearest to any single nuclear center that matters most. (And, of course, the effect of electrons, because of their small mass, is also quite negligible.) When the thickness or density of the medium cannot be ignored, other interesting phenomena, involving coherence and interference between scattering centers, come into play. Cherenkov radiation and the density effect in ionization have origin in such ramifications (see Chapters 6 and 7).

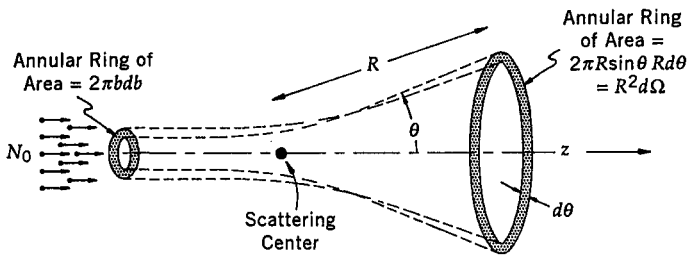


Fig. 1.3 Incident particles within the area  $2\pi b db$  of any scattering center are emitted into the annular ring of area  $R^2 d\Omega$  at angle  $\theta$ .

For a central potential, at any impact parameter  $b$ , we can think of the scattering center as presenting an effective transverse cross-sectional area  $\Delta\sigma = 2\pi b db$  for scattering of an incident particle by an angle  $\theta$  into  $d\Omega$ . Because the specific relationship between  $b$  and  $\theta$  (e.g., Eq. (1.32)) depends rather explicitly on the nature of the force, for example, whether it has  $r^{-2}$  behavior, or whether it is central, and so forth, in general,  $\Delta\sigma$  can depend on both  $\theta$  and  $\phi$ , so that we can write

$$\Delta\sigma(\theta, \phi) = b db d\phi = -\frac{d\sigma}{d\Omega}(\theta, \phi)d\Omega = -\frac{d\sigma}{d\Omega}(\theta, \phi) \sin\theta d\theta d\phi, \quad (1.33)$$

which defines the differential cross section  $\frac{d\sigma}{d\Omega}$ , and where the negative sign reflects the fact that  $\theta$  decreases as  $b$  increases. When there is no azimuthal dependence in the scattering – e.g., when the interaction has spherical symmetry – we can integrate over  $\phi$  (as we already did implicitly in our discussion of the annular rings of area  $2\pi b db$ ) and write

$$\begin{aligned} \Delta\sigma(\theta) &= -\frac{d\sigma}{d\Omega}(\theta)2\pi \sin\theta d\theta = 2\pi b db \\ \text{or } \frac{d\sigma}{d\Omega}(\theta) &= -\frac{b}{\sin\theta} \frac{db}{d\theta}. \end{aligned} \quad (1.34)$$

We wish to note that, since the Coulomb potential is central (depends only on distance and not on angle), we have assumed azimuthal symmetry in the scattering. This means that all positions along the annular ring of radius  $b$  are equivalent and that the differential cross section is only a function of the angle  $\theta$  and not  $\phi$ . It follows, therefore, that measuring the yield as a function of  $\theta$ , or the differential cross section, is equivalent to measuring the entire effect of the scattering.

In subatomic experiments, the unit, normally used to measure cross-sectional area is the barn, which is defined as  $10^{-24}$  cm<sup>2</sup>. This is a very small quantity – but then we have to remember that the typical size of a nucleus is about  $10^{-12}$  cm and, therefore, the cross-sectional area for a medium-size nucleus (if we assume it to be a sphere) would be of the order of a barn. This is, consequently, a relatively natural unit for such measurements. The units of solid angle are steradians, and  $4\pi$  sr corresponds to a sum over all solid angles around a point – that is, all  $\theta$  and  $\phi$ . We can also define a total scattering cross section by integrating the differential cross section over all angles

$$\sigma_{\text{TOT}} = \int d\Omega \frac{d\sigma}{d\Omega}(\theta, \phi) = 2\pi \int_0^\pi d\theta \sin\theta \frac{d\sigma}{d\Omega}(\theta), \quad (1.35)$$

where, in the last step, we have again assumed azimuthal symmetry. (If there is any  $\phi$ -dependence observed in the scattering, the last step in Eq. (1.35) would not hold.) The total cross section represents, in some sense, the effective size that the source of the potential presents for scattering at all possible impact parameters.

Let us now calculate the cross section for Rutherford scattering. We know from Eq. (1.32) that

$$b = \frac{ZZ'e^2}{2E} \cot \frac{\theta}{2}.$$

It follows, therefore, that

$$\frac{db}{d\theta} = -\frac{1}{2} \frac{ZZ'e^2}{2E} \operatorname{cosec}^2 \frac{\theta}{2}. \quad (1.36)$$

The negative sign in Eq. (1.36) again reflects the fact that, as  $b$  increases,  $\theta$  decreases, and that there is less deflection for larger impact parameter. Substituting this back into the definition of the scattering cross section, we obtain

$$\frac{d\sigma}{d\Omega}(\theta) = -\frac{b}{\sin\theta} \frac{db}{d\theta} = \left(\frac{ZZ'e^2}{4E}\right)^2 \operatorname{cosec}^4 \frac{\theta}{2} = \left(\frac{ZZ'e^2}{4E}\right)^2 \frac{1}{\sin^4 \frac{\theta}{2}}. \quad (1.37)$$

If we now integrate this relation over  $\theta$  (note from Eq. (1.34) that, because there is no azimuthal dependence,  $d\Omega = 2\pi \sin\theta d\theta$ ), we obtain the total cross section

$$\begin{aligned} \sigma_{\text{TOT}} &= \int \frac{d\sigma}{d\Omega}(\theta) d\Omega = 2\pi \int_0^\pi d\theta \sin\theta \frac{d\sigma}{d\Omega}(\theta) \\ &= 8\pi \left(\frac{ZZ'e^2}{4E}\right)^2 \int_0^1 d\left(\sin \frac{\theta}{2}\right) \frac{1}{\sin^3 \frac{\theta}{2}} \rightarrow \infty. \end{aligned} \quad (1.38)$$

This divergence may seem troublesome, but it is consistent with our earlier discussion. Namely, the total cross section reflects the largest values of impact parameter a particle can have and still undergo scattering. In the case

of the Coulomb potential, the long-ranged force extends to infinity and, consequently, a particle very far away from the center will still experience the Coulomb force, albeit only very slightly, and this is the origin of the divergence. Because the Coulomb force drops off rapidly with distance, and does not lead to any appreciable scattering beyond some finite value of the impact parameter, it is therefore appropriate to cut off the angular integration at some finite  $\theta = \theta_0 > 0^0$ , corresponding to some realistic cutoff for the impact parameter. This cutoff provides a finite  $\sigma_{\text{TOT}}$  for observable scattering angles (that is, for  $\theta > \theta_0$ ), which can be compared with experimental measurements. Finally, we should point out that our results cannot be valid for impact parameters much larger than the innermost electron levels in atoms because such electrons will shield and thereby reduce the effective nuclear charge.

#### 1.4 Measuring Cross Sections

Let us now see how we would go about performing a measurement in order to extract a cross section. Macroscopically, we have a beam of  $\alpha$ -particles (Geiger and Marsden used a collimated source of  $\alpha$ -particles from a sample of radioactive radon), a thin foil, and some scintillating material for detecting the scattered particles. This was, initially, a thin coat of ZnS phosphor deposited on a glass screen, and viewed by eye through a telescope. The telescope was able to rotate in one plane and thereby trace out the counting rate as a function of  $\theta$  (but not  $\phi$ ). Schematically, the apparatus can be represented as in Fig. 1.4.

Now, if we have our flux of  $N_0$   $\alpha$ -particles per unit area per second impinging on the thin foil, then some of these will pass through essentially undeflected, while others will be scattered through an angle between  $\theta - d\theta$  and  $\theta$ , corresponding to impact parameters between  $b + db$  and  $b$ . Here  $d\theta$  can be regarded as the angle subtended by the aperture of the telescope. In fact, the telescope views a small area of the screen given approximately by  $Rd\theta \cdot R \sin \theta d\phi = R^2 d\Omega$ , where  $R$  is the distance from the foil to the point of observation on the screen. The scattered particles that appear in this part of the screen are those that pass through and emerge from the part of the annular ring of impact radius  $b$ , width  $db$ , and arc length  $bd\phi$ . Had Geiger and Marsden constructed a circle of telescopes about the beam center to view the  $\alpha$ -particles emitted for all values of  $\phi$  for any particular fixed angle  $\theta$ , they would have certainly increased their event rate by  $\frac{2\pi}{d\phi}$ ,

but at the substantial cost of greatly complicating their experiment.

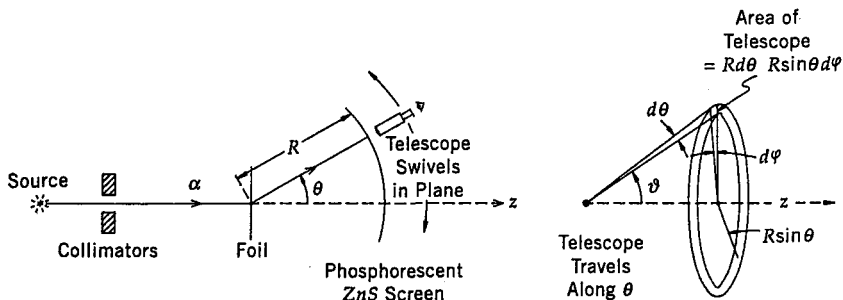


Fig. 1.4 Sketch of the macroscopic geometry of Rutherford scattering.

Now, the fraction of the incident particles that approach our nuclear targets within the small areas  $\Delta\sigma = bd\phi db$  at impact parameters  $b$ , is, in fact, the same fraction  $-\frac{dn}{N_0}$  that will be removed from the beam and scattered into the area  $R^2 d\Omega$  that is being viewed at angles  $(\theta, \phi)$  of the screen. This fraction is identical to the ratio of the sum of all the small  $bd\phi db$  areas for the  $N$  nuclear centers within the foil, divided by the entire area ( $S$ ) of the foil, or, stated in another way, it is the probability for the incident particles to enter within the  $N$  little areas, divided by the total probability of hitting the foil

$$-\frac{dn}{N_0} = \frac{Nbd\phi db}{S} = \frac{N}{S} \Delta\sigma(\theta, \phi). \quad (1.39)$$

For a foil of thickness  $t$ , density  $\rho$ , atomic weight  $A$ ,  $N = \left(\frac{\rho t S}{A}\right) A_0$ , where  $A_0$  is Avogadro's number of atoms per mole. Thus, for the number of  $\alpha$ -particles scattered per unit time into the detector at angles  $(\theta, \phi)$ , we can write

$$\begin{aligned} dn &= \frac{N_0 \rho t}{A} A_0 \frac{d\sigma}{d\Omega}(\theta, \phi) d\Omega, \\ \text{or } \frac{dn}{d\Omega} &= \frac{N_0 \rho t A_0}{A} \frac{d\sigma}{d\Omega}(\theta, \phi). \end{aligned} \quad (1.40)$$

For any given detector situated at angles  $(\theta, \phi)$  relative to the beam axis, and subtending a solid angle  $d\Omega$  (which is determined by the transverse

dimensions of the detector – its area/ $R^2$ ), we will observe  $dn$  counts per second

$$dn = N_0 \frac{N}{S} \frac{d\sigma}{d\Omega}(\theta, \phi) d\Omega. \quad (1.41)$$

This is a general expression, valid for any scattering process, independent of the existence of a theory that might be able to provide a specific formula for  $\frac{d\sigma}{d\Omega}$ . Thus, the counts observed in any experiment will be proportional to the number of incident beam particles, the number of scattering centers per unit area of target material, to the solid angle subtended by the detector, and to an effective cross section that each scattering center presents for bringing about the process of interest. (We are still assuming that corrections due to multiple collisions are small, that is, we are dealing with thin targets.) Geiger and Marsden performed very detailed measurements of  $dn$  as a function of  $\theta$ , using different target material (of relatively large  $Z$ ), different  $\alpha$ -particle sources of different energy, different thicknesses of foil, and found their data to be in complete agreement with Rutherford's prediction, as given in Eq. (1.37). That is, knowing  $N_0$ ,  $\frac{N}{S}$  and  $d\Omega$ , they measured  $dn$ , and extracted a form for the differential cross section that agreed beautifully with Rutherford's prediction, and thereby verified the presence of nuclei within atoms. It should be recognized that, although Geiger and Marsden's measurements provided clear evidence for the existence of a nuclear center, these experiments shed very little light on the nature of the nuclear force. The low-energy  $\alpha$ -particles never penetrated into the nucleus of the atom because of the repulsion from the nuclear Coulomb barrier.

### 1.5 Laboratory Frame and the Center-of-Mass Frame

So far, we have discussed collisions of a particle with a fixed center. In reality, however, the target also moves (recoils) as a result of the scattering. In some experiments we may be interested in colliding two beams of particles of comparable energy with each other. Although such situations may appear to be extremely complicated at first glance, when the potential is central, the problem can be reduced to the one we have just studied; this can be achieved through the separation of the motion of the center of mass.

Let us assume that we have two particles with masses  $m_1$  and  $m_2$ , at coordinates  $\vec{r}_1$  and  $\vec{r}_2$ , interacting through a central potential. The



equations for the motion can be written as

$$\begin{aligned} m_1 \ddot{\vec{r}}_1 &= -\vec{\nabla}_1 V(|\vec{r}_1 - \vec{r}_2|), \\ m_2 \ddot{\vec{r}}_2 &= -\vec{\nabla}_2 V(|\vec{r}_1 - \vec{r}_2|), \end{aligned} \quad (1.42)$$

where  $\vec{\nabla}$  is the gradient operator, which has the following form in spherical coordinates

$$\vec{\nabla}_i = \hat{r}_i \frac{\partial}{\partial r_i} + \frac{\hat{\theta}_i}{r_i} \frac{\partial}{\partial \theta_i} + \frac{\hat{\phi}_i}{r_i \sin \theta_i} \frac{\partial}{\partial \phi_i} \quad i = 1, 2. \quad (1.43)$$

Since the potential energy depends only on the relative separation of the two particles, let us define the variables:

$$\begin{aligned} \vec{r} &= \vec{r}_1 - \vec{r}_2, \\ \vec{R}_{\text{CM}} &= \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}, \end{aligned} \quad (1.44)$$

where  $\vec{r}$  denotes the coordinate of  $m_1$  relative to  $m_2$ , and  $\vec{R}_{\text{CM}}$  defines the coordinate of the center-of-mass of the system (see Fig. 1.5). From Eqs. (1.42) and (1.44) we can easily obtain the following:

$$\begin{aligned} \frac{m_1 m_2}{m_1 + m_2} \ddot{\vec{r}} &\equiv \mu \ddot{\vec{r}} = -\vec{\nabla} V(|\vec{r}|) = -\frac{\partial V(|\vec{r}|)}{\partial r} \hat{r}, \\ (m_1 + m_2) \ddot{\vec{R}}_{\text{CM}} &= M \ddot{\vec{R}}_{\text{CM}} = 0, \quad \text{or} \quad \dot{\vec{R}}_{\text{CM}} = \text{constant} \times \hat{R}, \end{aligned} \quad (1.45)$$

where we have used the fact that  $V(|\vec{r}|) = V(r)$  depends only on the radial coordinate  $r$ , and not on the angular variables associated with  $\vec{r}$ , and where we have defined

$$\begin{aligned} M &= m_1 + m_2 = \text{total mass of the system}, \\ \mu &= \frac{m_1 m_2}{m_1 + m_2} = \text{“reduced” mass of the system}. \end{aligned} \quad (1.46)$$

It is clear from the above analysis that, when the potential is central, the motion of two particles can be decoupled when rewritten in terms of a relative coordinate and the coordinate of the center of mass.

We also note from Eq. (1.45) that the motion of the center-of-mass is trivial in the sense that it corresponds to that of a free, non-accelerating,

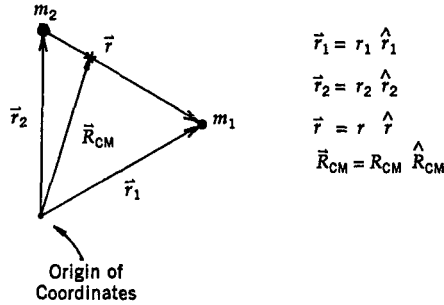


Fig. 1.5 Position of the center of mass and the definition of the relative coordinate for the two particles with masses  $m_1$  and  $m_2$ .

particle. In other words, the center-of-mass moves in the laboratory with a constant velocity ( $\vec{R}_{CM}$  is fixed) independent of the specific form of the potential. The dynamics is contained completely in the motion of a fictitious particle with the reduced mass  $\mu$  and coordinate  $\vec{r}$ . In the frame in which the center-of-mass is at rest, the complete dynamics, then, becomes equivalent to the motion of a single particle, with mass  $\mu$ , scattering from a fixed central potential, a situation that we have already analyzed in detail. A simplification that occurs in the center-of-mass frame is that the sum of the momenta of the interacting objects vanishes, which follows from Eq. (1.44) when  $\vec{R}_{CM}$  is set to zero. Because of this, it is more common to define the center-of-mass frame as the frame in which the total momentum vanishes, and we often refer to the center-of-mass frame equivalently as the center-of-momentum frame.

To understand how various quantities can be transformed between the laboratory frame and the center-of-mass frame, let us return to the scattering from a fixed target. We assume that the particle with mass  $m_2$  is initially at rest in the laboratory frame, and the particle of mass  $m_1$  is incident along the  $z$ -axis with a velocity  $v_1$ . Let the scattering angle of particle  $m_1$  in the laboratory frame be given by  $\theta_{\text{Lab}}$ , and its speed after scattering by  $v$ . For this case, the center-of-mass moves along the  $z$ -axis with a speed  $v_{\text{CM}}$

$$v_{\text{CM}} = \dot{R}_{\text{CM}} = \frac{m_1 v_1}{m_1 + m_2}. \quad (1.47)$$

In the center-of-mass, the two particles therefore move towards each other along the  $z$ -axis (see Fig. 1.6)

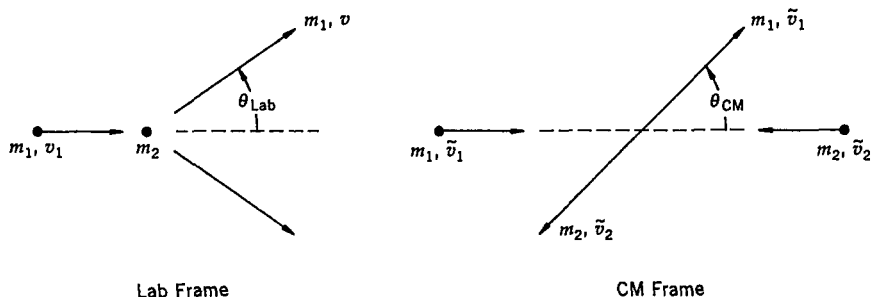


Fig. 1.6 Collision of  $m_1$  with  $m_2$ , as viewed in the Lab and in the center-of-mass frame.

$$\begin{aligned}\tilde{v}_1 &= v_1 - v_{\text{CM}} = \frac{m_2 v_1}{m_1 + m_2}, \\ \tilde{v}_2 &= v_{\text{CM}} = \frac{m_1 v_1}{m_1 + m_2}.\end{aligned}\tag{1.48}$$

where  $\tilde{v}_1$  and  $\tilde{v}_2$  are the speeds of beam and target particle, respectively, as viewed in the center-of-mass frame. Here we see explicitly that the momenta of the two particles in the center-of-mass are equal and opposite.

For elastic scattering, the magnitudes of the velocities of the particles do not change in the collision, but the angles at which they emerge depend on the dynamics. Let  $\theta_{\text{CM}}$  denote the scattering angle as measured in the center-of-mass frame. Note that because  $\theta_{\text{CM}}$  represents the change in the direction of the relative position vector ( $\vec{r}$ ) as a result of the collision, it must be identical to the scattering angle for the particle with reduced mass. To obtain a relation between  $\theta_{\text{Lab}}$  and  $\theta_{\text{CM}}$ , we note that the velocities in the laboratory frame and in the center-of-mass frame are related through the velocity of the center-of-mass. In particular, after scattering, the  $z$ -components of the velocities of particle of mass  $m_1$  are related through

$$v \cos \theta_{\text{Lab}} - v_{\text{CM}} = \tilde{v}_1 \cos \theta_{\text{CM}},$$

$$\text{or } v \cos \theta_{\text{Lab}} = \tilde{v}_1 \cos \theta_{\text{CM}} + v_{\text{CM}},\tag{1.49}$$

while, the components of the velocities perpendicular to the  $z$ -axis are related through

$$v \sin \theta_{\text{Lab}} = \tilde{v}_1 \sin \theta_{\text{CM}}. \quad (1.50)$$

From Eqs. (1.50) and (1.49), we obtain the non-relativistic result

$$\tan \theta_{\text{Lab}} = \frac{\sin \theta_{\text{CM}}}{\cos \theta_{\text{CM}} + \frac{v_{\text{CM}}}{\tilde{v}_1}} = \frac{\sin \theta_{\text{CM}}}{\cos \theta_{\text{CM}} + \zeta}, \quad (1.51)$$

where we have defined

$$\zeta = \frac{v_{\text{CM}}}{\tilde{v}_1} = \frac{m_1}{m_2}, \quad (1.52)$$

and where the last equality in Eq. (1.52) holds only for elastic scattering. For future use, we rewrite Eq. (1.51) in an alternative form

$$\cos \theta_{\text{Lab}} = \frac{\cos \theta_{\text{CM}} + \zeta}{(1 + 2\zeta \cos \theta_{\text{CM}} + \zeta^2)^{\frac{1}{2}}}, \quad (1.53)$$

where we have used the transformation of the final velocity to the center-of-mass frame for the case of elastic scattering.

Using the relationship between  $\theta_{\text{Lab}}$  and  $\theta_{\text{CM}}$ , we can also relate the differential cross sections in the two frames, arguing as follows. The particles that scatter through an angle  $\theta_{\text{Lab}}$  into the solid angle  $d\Omega_{\text{Lab}}$  in the laboratory frame are the same ones that scatter by  $\theta_{\text{CM}}$  into the corresponding solid angle  $d\Omega_{\text{CM}}$  in the center-of-mass frame. (That is, these are two equivalent ways of looking at the same process.) Because  $\phi$  is transverse to the boost direction between the two reference frames, it follows that  $d\phi_{\text{Lab}} = d\phi_{\text{CM}}$ . Thus, ignoring the azimuthal coordinate, we must have

$$\begin{aligned} \frac{d\sigma}{d\Omega_{\text{Lab}}} (\theta_{\text{Lab}}) \sin \theta_{\text{Lab}} d\theta_{\text{Lab}} &= \frac{d\sigma}{d\Omega_{\text{CM}}} (\theta_{\text{CM}}) \sin \theta_{\text{CM}} d\theta_{\text{CM}}, \\ \text{or } \frac{d\sigma}{d\Omega_{\text{Lab}}} (\theta_{\text{Lab}}) &= \frac{d\sigma}{d\Omega_{\text{CM}}} (\theta_{\text{CM}}) \frac{d(\cos \theta_{\text{CM}})}{d(\cos \theta_{\text{Lab}})}. \end{aligned} \quad (1.54)$$

The right hand side of Eq. (1.54) can be evaluated using Eq. (1.53), leading to

$$\frac{d\sigma}{d\Omega_{\text{Lab}}}(\theta_{\text{Lab}}) = \frac{d\sigma}{d\Omega_{\text{CM}}}(\theta_{\text{CM}}) \frac{(1 + 2\zeta \cos \theta_{\text{CM}} + \zeta^2)^{\frac{3}{2}}}{|1 + \zeta \cos \theta_{\text{CM}}|}. \quad (1.55)$$

## 1.6 Relativistic Variables

In the Appendix we review the basics of special relativity, and use those results here to discuss briefly the kinematics in terms of relativistic variables. In the scattering of any two particles with rest masses  $m_1$  and  $m_2$ , the velocity of the center-of-mass is obtained from the ratio of the total relativistic momentum and the total relativistic energy

$$\frac{\vec{v}_{\text{CM}}}{c} = \vec{\beta}_{\text{CM}} = \frac{(\vec{P}_1 + \vec{P}_2) c}{E_1 + E_2}. \quad (1.56)$$

If  $m_1$  refers to the mass of the projectile and  $m_2$  to that of a target particle, then using laboratory variables, we obtain (with the target initially at rest)

$$\vec{\beta}_{\text{CM}} = \frac{\vec{P}_1 c}{E_1 + m_2 c^2} = \frac{\vec{P}_1 c}{\sqrt{P_1^2 c^2 + m_1^2 c^4} + m_2 c^2}, \quad (1.57)$$

where our convention is to define  $|\vec{P}_i| = P_i$  for  $i = 1, 2$ . At very low energies, namely when  $m_1 c^2 \gg P_1 c$ , this reduces to our nonrelativistic expression of Eq. (1.47)

$$\vec{\beta}_{\text{CM}} = \frac{m_1 \vec{v}_1 c}{m_1 c^2 + m_2 c^2} = \frac{m_1 \vec{v}_1}{(m_1 + m_2) c}. \quad (1.58)$$

At very high energies, when  $m_1 c^2 \ll P_1 c$  and  $m_2 c^2 \ll P_1 c$ , we can write the following for the value of  $\beta_{\text{CM}}$

$$\beta_{\text{CM}} = |\vec{\beta}_{\text{CM}}| = \frac{1}{\sqrt{1 + \left(\frac{m_1 c^2}{P_1 c}\right)^2 + \frac{m_2 c^2}{P_1 c}}} \approx 1 - \frac{m_2 c}{P_1} - \frac{1}{2} \left(\frac{m_1 c}{P_1}\right)^2. \quad (1.59)$$

When  $m_1$  and  $m_2$  are comparable, Eq. (1.59) simplifies to  $\beta_{\text{CM}} \approx \left(1 - \frac{m_2 c}{P_1}\right)$ , and, for this case,  $\gamma_{\text{CM}}$  becomes

$$\begin{aligned}\gamma_{\text{CM}} &= (1 - \beta_{\text{CM}}^2)^{-\frac{1}{2}} \approx [(1 + \beta_{\text{CM}})(1 - \beta_{\text{CM}})]^{-\frac{1}{2}} \\ &\approx \left[ (2) \left( \frac{m_2 c}{P_1} \right) \right]^{-\frac{1}{2}} = \sqrt{\frac{P_1}{2m_2 c}}.\end{aligned}\quad (1.60)$$

In general, we can obtain an expression for  $\gamma_{\text{CM}}$  in the following way. We note from Eq. (1.57) that

$$\beta_{\text{CM}}^2 = \frac{P_1^2 c^2}{(E_1 + m_2 c^2)^2}, \quad (1.61)$$

so that

$$\begin{aligned}1 - \beta_{\text{CM}}^2 &= \frac{E_1^2 + 2E_1 m_2 c^2 + m_2^2 c^4 - P_1^2 c^2}{(E_1 + m_2 c^2)^2} \\ &= \frac{m_1^2 c^4 + m_2^2 c^4 + 2E_1 m_2 c^2}{(E_1 + m_2 c^2)^2},\end{aligned}\quad (1.62)$$

where we have substituted  $m_1^2 c^4$  for  $E_1^2 - P_1^2 c^2$ . It therefore follows that

$$\gamma_{\text{CM}} = (1 - \beta_{\text{CM}}^2)^{-\frac{1}{2}} = \frac{E_1 + m_2 c^2}{(m_1^2 c^4 + m_2^2 c^4 + 2E_1 m_2 c^2)^{\frac{1}{2}}}, \quad (1.63)$$

which, in the high-energy limit of  $E_1 \approx P_1 c \gg m_1 c^2$  and  $P_1 c \gg m_2 c^2$ , reduces to the result of Eq. (1.60).

The quantity in the denominator of Eq. (1.63), despite its appearance, is an invariant scalar. This can be deduced by evaluating the square of the following four-vector in the laboratory frame ( $\vec{P}_2 = 0$ )

$$\begin{aligned}s &= (E_1 + E_2)^2 - (\vec{P}_1 + \vec{P}_2)^2 c^2 \\ &= (E_1 + m_2 c^2)^2 - P_1^2 c^2 = E_1^2 + m_2^2 c^4 + 2E_1 m_2 c^2 - P_1^2 c^2 \\ &= m_1^2 c^4 + m_2^2 c^4 + 2E_1 m_2 c^2.\end{aligned}\quad (1.64)$$

Because  $s$  is a scalar, it has the same value when calculated in any reference frame. In particular, it has a simple meaning in the center-of-mass frame, where the two particles have equal and opposite momenta (i.e., the total momentum vanishes in the center-of-mass frame)

$$\begin{aligned}
 s &= m_1^2 c^4 + m_2^2 c^4 + 2E_1 m_2 c^2 = (E_{1\text{CM}} + E_{2\text{CM}})^2 - \left( \vec{P}_{1\text{CM}} + \vec{P}_{2\text{CM}} \right)^2 c^2 \\
 &= (E_{1\text{CM}} + E_{2\text{CM}})^2 = \left( E_{\text{CM}}^{\text{TOT}} \right)^2. \quad (1.65)
 \end{aligned}$$

Thus  $s$  is the square of the total energy available in the center-of-mass. Hence, for  $m_2$  initially at rest, we can write

$$\gamma_{\text{CM}} = \frac{E_1 + m_2 c^2}{E_{\text{CM}}^{\text{TOT}}} = \frac{E_{\text{Lab}}^{\text{TOT}}}{E_{\text{CM}}^{\text{TOT}}}. \quad (1.66)$$

The variable  $s$  is used frequently in describing high energy collisions, and  $E_{\text{CM}}^{\text{TOT}}$  is often referred to as  $\sqrt{s}$ . Clearly, from its structure in Eq. (1.65),  $\frac{\sqrt{s}}{c^2}$  can also be regarded as the rest mass or the invariant mass of the two colliding objects.

In discussing scattering, it is often convenient to define another invariant called  $t$ , the square of the four-momentum transfer in a collision. This variable is just the square of the difference in the energy-momentum four-vectors of the projectile before and after the scattering

$$t = \left( E_1^f - E_1^i \right)^2 - \left( \vec{P}_1^f - \vec{P}_1^i \right)^2 c^2. \quad (1.67)$$

Because momentum and energy are conserved separately in all collisions, we can express  $t$  just as well in terms of target variables

$$t = \left( E_2^f - E_2^i \right)^2 - \left( \vec{P}_2^f - \vec{P}_2^i \right)^2 c^2. \quad (1.68)$$

Furthermore, since, just as  $s$ ,  $t$  is also an invariant scalar, we can calculate its value in any reference frame. In particular, let us analyze this quantity in the center-of-mass frame. For simplicity, we will restrict ourselves to the case of elastic scattering, for which  $|\vec{P}_{\text{CM}}^i| = |\vec{P}_{\text{CM}}^f| = |\vec{P}_{\text{CM}}|$  and consequently  $E_{\text{CM}}^i = E_{\text{CM}}^f$  for the two particles in the center-of-mass frame. It follows from Eq. (1.67) that

$$\begin{aligned}
 t &= - \left( P_{1\text{CM}}^{f2} + P_{1\text{CM}}^{i2} - 2\vec{P}_{1\text{CM}}^f \cdot \vec{P}_{1\text{CM}}^i \right) c^2 \\
 &= -2P_{\text{CM}}^2 c^2 (1 - \cos \theta_{\text{CM}}). \quad (1.69)
 \end{aligned}$$

where we have set  $|\vec{P}_{1\text{CM}}^f| = |\vec{P}_{1\text{CM}}^i| = P_{\text{CM}}$ , and where  $\theta_{\text{CM}}$  denotes the scattering angle in the center-of-mass frame. Since  $-1 \leq \cos \theta_{\text{CM}} \leq 1$ , we conclude that for elastic scattering through any finite angle,  $t < 0$ . On the other hand, from its definition in Eq. (1.67), we can also think of  $t$  as the square of the mass of an exchanged particle (with energy  $E_1^f - E_1^i$  and momentum  $\vec{P}_1^f - \vec{P}_1^i$ ) that mediates the scattering. Consequently, we must conclude that if such an exchange process can be used to describe scattering, then the object being exchanged cannot be physical since it has an imaginary rest mass. This means that although this “virtual” object cannot be detected, if the picture is correct, its consequences can be calculated and observed. Diagrams of the kind shown in Fig. 1.7, were pioneered by Richard Feynman in the calculation of scattering amplitudes in quantum electrodynamics (QED) and are referred to as Feynman diagrams (graphs).

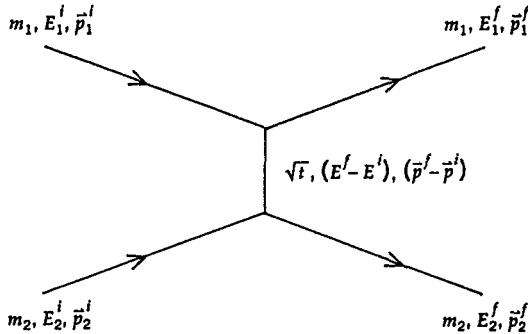


Fig. 1.7 Exchange of a mediating object of mass  $\sqrt{t}$  in the collision of masses  $m_1$  and  $m_2$ .

For convenience, let us define a variable  $q^2$  given by  $q^2 c^2 = -t$ . In the laboratory frame, we have that  $\vec{P}_{2\text{Lab}}^i = 0$  and, therefore, from Eq. (1.68) we obtain

$$\begin{aligned}
 q^2 c^2 &= - \left[ \left( E_{2\text{Lab}}^f - m_2 c^2 \right)^2 - \left( P_{2\text{Lab}}^f c \right)^2 \right] \\
 &= - \left[ \left( E_{2\text{Lab}}^f \right)^2 - \left( P_{2\text{Lab}}^f c \right)^2 - 2 E_{2\text{Lab}}^f m_2 c^2 + m_2^2 c^4 \right] \\
 &= - \left[ 2 m_2^2 c^4 - 2 E_{2\text{Lab}}^f m_2 c^2 \right] \\
 &= 2 m_2 c^2 \left( E_{2\text{Lab}}^f - m_2 c^2 \right) = 2 m_2 c^2 T_{2\text{Lab}}^f, \\
 \text{or } q^2 &= 2 m_2 T_{2\text{Lab}}^f, \tag{1.70}
 \end{aligned}$$



where, in the last step, we replaced  $E_{2\text{Lab}}^f$  by  $T_{2\text{Lab}}^f + m_2c^2$ . Thus, in the nonrelativistic limit, where  $T_{2\text{Lab}} \approx \frac{1}{2}m_2v_2^2$ ,  $q^2$  is just the square of the momentum transferred to the target, namely  $q^2 \approx (m_2v_2)^2$ . We consequently expect  $q^2$  to reflect the “hardness” of a collision, with small  $q^2$  being characterized by long-range ( $R \approx \frac{\hbar}{q}$ ) soft collisions. Referring back to Eq. (1.69), we see that small  $q^2$  must correspond to small  $\theta_{\text{CM}}$ . We also see that, for small  $\theta_{\text{CM}}$ ,  $q^2 \approx P_{\text{CM}}^2 \theta_{\text{CM}}^2 \approx p_T^2$ , or the square of the transverse momentum developed as a result of the collision.

We will leave it to the reader to show, with the help of Appendix A, that the relativistic equivalent of Eq. (1.51) is

$$\tan \theta_{\text{Lab}} = \frac{\tilde{\beta} \sin \theta_{\text{CM}}}{\gamma_{\text{CM}}(\tilde{\beta} \cos \theta_{\text{CM}} + \beta_{\text{CM}})}, \quad (1.71)$$

where  $\tilde{\beta}c$  is the velocity of the scattered particle in the center-of-mass frame. Note also that Eq. (1.71) reduces to Eq. (1.51) in the limit of low velocities.

Finally, let us rewrite the Rutherford cross section of Eq. (1.37) in terms of the momentum transferred between the two objects involved in the scattering process. From Eq. (1.69) (and the relation between  $t$  and  $q^2$ ), we can deduce that

$$dq^2 = -2P^2 d(\cos \theta) = \frac{P^2 d\Omega}{\pi} \quad (1.72)$$

where we have ignored the small difference between center-of-mass and Lab variables ( $P_{1\text{Lab}} \approx P_{1\text{CM}} = P = m_1v_0$ ). Specializing to the case of Rutherford scattering at low velocities and, for convenience, setting  $m = m_1 \ll m_2$ , and  $v = v_0$  in Eq. (1.37), we obtain

$$\begin{aligned} \frac{d\sigma}{\frac{\pi}{(mv)^2} dq^2} &= \frac{(ZZ'e^2)^2}{(2mv^2)^2} \frac{1}{\left(\frac{1-\cos \theta}{2}\right)^2}, \\ \text{or } \frac{d\sigma}{dq^2} &= \frac{4\pi(ZZ'e^2)^2}{v^2} \frac{1}{q^4}. \end{aligned} \quad (1.73)$$

The  $q^{-4}$  divergence of the cross section is characteristic of Coulomb scattering, and reflects the  $r^{-1}$  dependence of the potential. It is important to recognize that there is a distribution in  $q^2$ , with different events having different momentum transfers. The rapid fall-off with  $q^2$  implies that the typical value of momentum transfer is small. The dispersion in this mean

has important physical consequences that will be brought up in Chapter 6. Although the minimum value of  $q^2$  can be zero, this corresponds to no scattering; the maximum value (a rare occurrence, indeed!) is  $4P^2$ . Although Eq. (1.73) was obtained using nonrelativistic kinematics, it also holds, in fact, as  $v^2 \rightarrow c^2$  (see, however, our comments in the section "Sizes of Nuclei" in Chapter 2).

## 1.7 Quantum Treatment of Rutherford Scattering

We arrived at Eq. (1.73) through a rather circuitous classical route. We will now end this section by sketching how the Rutherford cross section can be calculated using quantum mechanics. This will be done through an application of Fermi's Golden Rule,<sup>4</sup> according to which the transition probability to continuum states per unit time in perturbation theory is given by

$$P = \frac{2\pi}{\hbar} |H_{fi}|^2 \rho(E_f) \quad (1.74)$$

where  $\rho(E_f)$  is the density of final states and  $H_{fi}$  denotes the matrix element of the perturbation Hamiltonian between the initial and the final states

$$H_{fi} = \langle f|H|i\rangle = \int d^3r \psi_f^*(\vec{r}) H(r) \psi_i(\vec{r}). \quad (1.75)$$

For the case of elastic Rutherford scattering, the wave functions are plane waves, corresponding to free-particle states approaching ( $i$ ) and leaving ( $f$ ) the scattering center, and the perturbation Hamiltonian is the Coulomb potential energy given in Eq. (1.23). For the incident and outgoing momenta  $\vec{p}$  and  $\vec{p}'$ , respectively, we can define the wave vectors  $\vec{k} = \frac{\vec{p}}{\hbar}$  and  $\vec{k}' = \frac{\vec{p}'}{\hbar}$ , and a momentum transfer that results from the scattering  $\vec{q} = \hbar(\vec{k}' - \vec{k})$ . Except for an overall normalization of the wave functions, our matrix element  $H_{fi}$  can now be written as

$$H_{fi} \approx \int_{\text{all space}} d^3r e^{i\vec{k}' \cdot \vec{r}} V(r) e^{-i\vec{k} \cdot \vec{r}} = \int_{\text{all space}} d^3r V(r) e^{i\vec{q} \cdot \vec{r}}. \quad (1.76)$$

---

<sup>4</sup>A discussion of this famous result for transitions between states can be found in standard texts on quantum mechanics.

The integral on the right is the Fourier transform of  $V(r)$ , and can be thought of as the potential energy in momentum space. Doing the integration,<sup>5</sup> we find that

$$V(\vec{q}) = \int_{\text{all space}} d^3r V(r) e^{i\vec{q}\cdot\vec{r}} = \frac{(ZZ'e^2)(4\pi\hbar^2)}{q^2}. \quad (1.77)$$

Evaluating the density of final states,<sup>6</sup> substituting into Eq. (1.74), and relating the transition probability to the scattering cross section, leads to the same expression as obtained in Eq. (1.73). Thus Rutherford's result, without any apparent reference to  $\hbar$ , is also in agreement with quantum mechanics (when effects of intrinsic spin are ignored).

## Problems

**1.1** Using Eq. (1.38) calculate the approximate total cross sections for Rutherford scattering of a 10 MeV  $\alpha$ -particle from a lead nucleus for impact parameters  $b$  less than  $10^{-12}$ ,  $10^{-10}$  and  $10^{-8}$  cm. How well do these agree with the values of  $\pi b^2$ ?

**1.2** Prove that Eq. (1.55) follows from the relations in Eqs. (1.53) and (1.54).

**1.3** Sketch  $\cos\theta_{\text{Lab}}$  as a function of  $\cos\theta_{\text{CM}}$  for the nonrelativistic elastic scattering of particles of unequal mass, for the cases when  $\zeta = 0.05$  and  $\zeta = 20$  in Eqs. (1.52) and (1.53).

**1.4** What would be the approximate counting rate observed in the Rutherford scattering of 10 MeV  $\alpha$ -particles off lead foil at an angle of  $\theta = \frac{\pi}{2}$  in the laboratory? Assume an incident flux of  $10^6$   $\alpha$ -particles per second on the foil, a foil 0.1 cm thick, and a detector of transverse area 1 cm  $\times$  1 cm placed 100 cm from the interaction point, and density of lead of 11.3 g/cm<sup>3</sup>. What would be the counting rate at  $\theta = 5^\circ$ ? By about how much

<sup>5</sup>The Fourier transform corresponds to a generalization of the Fourier decomposition of functions into series. Transforms of different functions can be found in mathematical tables and are useful for a variety of applications in physics. See, for example, L. Schiff, *Quantum Mechanics*, (New York, McGraw Hill, 1968); A. Das and A. C. Melissinos, *Quantum Mechanics*, (New York, Gordon & Breach, 1986); A. Das, *Lectures on Quantum Mechanics*, (New Delhi, Hindustan Book Agency 2003).

<sup>6</sup>See a discussion of this issue, and matters pertaining to this entire section, in A. Das and A. C. Melissinos, *Quantum Mechanics*, pp 199-204, A. Das, *Lectures on Quantum Mechanics*, (New Delhi, Hindustan Book Agency 2003).

would your answers change if the above angles were specified for the center-of-mass – be quantitative, but use approximations where necessary. (Why don't you have to know the area of the foil?)

**1.5** Sketch the cross section in the laboratory frame as a function of  $\cos \theta_{\text{Lab}}$  for the elastic scattering of equal-mass particles when  $\frac{d\sigma}{d\Omega_{\text{CM}}}$  is isotropic and equal to 100 mb/sr. What would be your result for  $\zeta = 0.05$  in Eq. (1.52)? (You may use approximations where necessary.)

**1.6** Certain radioactive nuclei emit  $\alpha$  particles. If the kinetic energy of these  $\alpha$  particles is 4 MeV, what is their velocity if you assume them to be nonrelativistic? How large an error do you make in neglecting special relativity in the calculation of  $v$ ? What is the closest that such an  $\alpha$  particle can get to the center of a Au nucleus?

**1.7** An electron of momentum 0.511 MeV/ $c$  is observed in the laboratory. What are its  $\beta = \frac{v}{c}$ ,  $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$ , kinetic energy, and total energy?

**1.8** What are the approximate values of the kinetic energy for the recoiling lead nucleus and the momentum transfers (in eV units) at the cutoffs specified in Problem 1.1?

**1.9** Taking the ultrarelativistic limit of Eq. (1.71), find an approximate expression for  $\theta_{\text{Lab}}$  at  $\theta_{\text{CM}} = \frac{\pi}{2}$ , and evaluate  $\theta_{\text{Lab}}$  for  $\gamma_{\text{CM}} = 10$  and  $\gamma_{\text{CM}} = 100$ . Does the approximation hold best for particles with small or large mass values?

**1.10** What is the minimum impact parameter needed to deflect 7.7 MeV  $\alpha$ -particles from gold nuclei by at least  $1^\circ$ ? What about by at least  $30^\circ$ ? What is the ratio of probabilities for deflections of  $\theta > 1^\circ$  relative to  $\theta > 30^\circ$ ? (See the *CRC Handbook* for the density of gold.)

**1.11** Consider a collimated source of 8 MeV  $\alpha$ -particles that provides  $10^4$   $\alpha$ /sec that impinge on a 0.1 mm gold foil. What counting rate would you expect in a detector that subtends an annular cone of  $\Delta\theta = 0.05$  rad, at a scattering angle of  $\theta = 90^\circ$ ? Compare this to the rate at  $\theta = 5^\circ$ . Is there a problem? Is it serious (see Problem 1.12). (*Hint*: You can use the small-angle approximation where appropriate, and find the density of gold in the *CRC Handbook*.)

**1.12** Consider the expression Eq. (1.41) for Rutherford Scattering of  $\alpha$ -particles from gold nuclei. Integrate this over all angles to obtain  $n$ . In principle,  $n$  cannot exceed  $N_0$ , the number of incident particles. Why?

What cut-off value for  $\theta$  would be required in the integral, that is, some  $\theta = \theta_0 > 0$ , to assure that  $n$  does not exceed  $N_0$  in Problem 1.4? (*Hint:* After integrating, use the small-angle approximation to simplify the calculation.) Using the Heisenberg uncertainty principle  $\Delta p_x \Delta x \approx \hbar$ , where  $\Delta x$  is some transverse distance corresponding to a change in transverse momentum of  $\Delta p_x = p_{in} \theta_0 \approx \sqrt{2mE} \theta_0$ , calculate the distances  $\Delta x$  to which you have to restrict the description of the scattering. Are these distances sufficiently restrictive? Explain!

### **Suggested Readings**

Geiger, H. and E. Marsden, *Philos. Mag.* **25**, 604 (1913).

Rutherford, E., *Philos. Mag.* **21** 669, (1911).

Thomson, J. J., *Cambridge Lit. Phil. Soc.* **15**, 465 (1910).

## Chapter 2

# Nuclear Phenomenology

### 2.1 Introductory Remarks

The original Rutherford-scattering experiments demonstrated that each atom had a positively charged central core that we call its nucleus. However, even the original experiments of Geiger and Marsden showed deviations from the Rutherford formula at  $\alpha$ -particle energies above 25 MeV, and especially for scattering from nuclei of low- $Z$ . Also, in the late 1920s, James Chadwick noticed serious discrepancies between expectations from Coulomb scattering and the elastic scattering of  $\alpha$ -particles on helium. The observed differences could not be attributed to expected quantum effects, first calculated by Neville Mott. All this indicated very clearly that there was more than just the Coulomb force involved in nuclear scattering.

Prior to the discovery of the neutron by Chadwick in 1932, it was thought that the nucleus contained protons and electrons, but it is now recognized that the nucleus consists of protons and neutrons – collectively known as *nucleons*. Most of what we know about nuclei and the nuclear force has been obtained through decades of painstaking experimentation. In what follows, we will merely summarize the main features of the physics of the nucleus, and only occasionally present some of the crucial experimental underpinnings that led to the elucidation of nuclear phenomena.

### 2.2 Properties of Nuclei

#### 2.2.1 *Labeling of Nuclei*

The nucleus of any atom  $X$ , can be labeled uniquely by its electric charge or atomic number  $Z$ , and its total number of nucleons  $A$ , and is conventionally

represented as  ${}^A X^Z$ . Alternatively, it can be specified by the number of protons ( $Z$ ) and the number of neutrons ( $N = A - Z$ ). Because the whole atom is electrically neutral, the nucleus must be surrounded by a cloud of  $Z$  electrons. A great many nuclei, with different  $Z$  and  $A$  values have been found in nature or produced in the laboratory. Nuclei with the same number of protons but different number of neutrons are known as *isotopes*; thus  ${}^A X^Z$  and  ${}^{A'} X^Z$  are isotopes of nucleus  $X$ , and all such atoms have similar chemical properties. Nuclei that have the same total number of nucleons but different number of protons are called *isobars*; thus  ${}^A X^Z$  and  ${}^A Y^{Z'}$  are isobars. Just as an atom can be found in its ground state as well as in an excited state, so also can a nucleus be excited to higher levels, and such states are referred to as a *resonances* or isomers of the ground state.

### 2.2.2 Masses of Nuclei

As we already mentioned, a nucleus,  ${}^A X^Z$ , contains  $Z$  protons and  $(A - Z)$  neutrons. Thus, naively, we would expect the mass of the nucleus to be

$$M(A, Z) = Zm_p + (A - Z)m_n, \quad (2.1)$$

where  $m_p$  and  $m_n$  denote, respectively, the mass of the proton and the neutron, with

$$\begin{aligned} m_p &\approx 938 \cdot 27 \text{ MeV}/c^2, \\ m_n &\approx 939 \cdot 56 \text{ MeV}/c^2. \end{aligned} \quad (2.2)$$

However, the measured values of nuclear masses reveal that the mass of a nucleus is smaller than the sum of the masses of its constituents.<sup>1</sup> Namely,

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<sup>1</sup>As an aside about masses, we should point out that isotope charts usually give masses of neutral atoms and not of nuclei. To get the nuclear mass one must subtract the electron masses ( $Zm_e$ ) from the atomic weights (ignoring the small differences in electron bindings). Unfortunately, chemists and physicists use different mass scales. Chemists assign 16.0 atomic mass units (amu) to the "natural" isotopic mixture of oxygen found on earth, while physicists assign 16.0 amu to the atom of  ${}^{16}\text{O}^8$ . One amu is the mass in grams of one fictitious atom that has an atomic weight of 1.0000 gm. Thus 1 amu =  $(A_0^{-1}) \text{ gm} = 1.6606 \times 10^{-24} \text{ gm}$ . (The latest value for  $A_0$  is  $(6.022098 \pm 0.000006) \times 10^{23} \text{ mole}^{-1}$ .) There is also the unified mass unit "u", defined as  $\frac{1}{12}$  of the mass of the  ${}^{12}\text{C}$  atom. We will use  $m_p = 1.00728 \text{ amu} = 938.27 \text{ MeV}/c^2 = 1.6726 \times 10^{-24} \text{ g}$ , and  $m_n = m_p + 1.29332 \text{ MeV}/c^2$ .

$$M(A, Z) < Zm_p + (A - Z)m_n. \quad (2.3)$$

This explains why an isolated nucleus cannot just fall apart into its constituents, because that would violate the principle of conservation of energy. The mass deficit, defined as

$$\Delta M(A, Z) = M(A, Z) - Zm_p - (A - Z)m_n, \quad (2.4)$$

is negative, and can be thought of as being proportional to the nuclear binding energy (B.E.); the absolute value of  $\Delta M$  is related to the minimum energy required to break up the nucleus into its components. Thus a negative B.E. will assure that the nucleus holds together, and the more negative is the value of  $\Delta M$ , the more stable is the nucleus. The mass deficit and the B.E. are related simply through  $c^2$

$$\text{B.E.} = \Delta M(A, Z)c^2, \quad (2.5)$$

where  $c$  is the speed of light. Thus,  $-\Delta Mc^2$  or  $-\text{B.E.}$  is the amount of energy required to release all the nucleons from their captivity within the nucleus. It is also useful to define a binding energy per nucleon, or average energy needed to release a nucleon from a nucleus, as

$$\begin{aligned} \frac{B}{A} &= \frac{-\text{B.E.}}{A} = \frac{-\Delta M(A, Z)c^2}{A} \\ &= \frac{(Zm_p + (A - Z)m_n - M(A, Z))c^2}{A}. \end{aligned} \quad (2.6)$$

This quantity has been measured for a wide range of stable nuclei (see Fig. 2.1) and, except for some fine structure that we will discuss later, shows some remarkable features.

For low-mass nuclei ( $A \lesssim 20$ ),  $\frac{B}{A}$  oscillates somewhat and increases rapidly with  $A$ , and then saturates, reaching a peak value of about 9 MeV per nucleon near  $A = 60$ ; for larger  $A$ ,  $\frac{B}{A}$  drops very slowly. An approximate average value of  $\frac{B}{A}$  for a wide range of nuclei can therefore be taken as about 8 MeV per nucleon. As we will see, these characteristics have important implications for the nature of the nuclear force and the structure of the nucleus. One immediate deduction is that if we deposit about 8 MeV of kinetic energy inside the nucleus, and transfer it all to one nucleon, then we



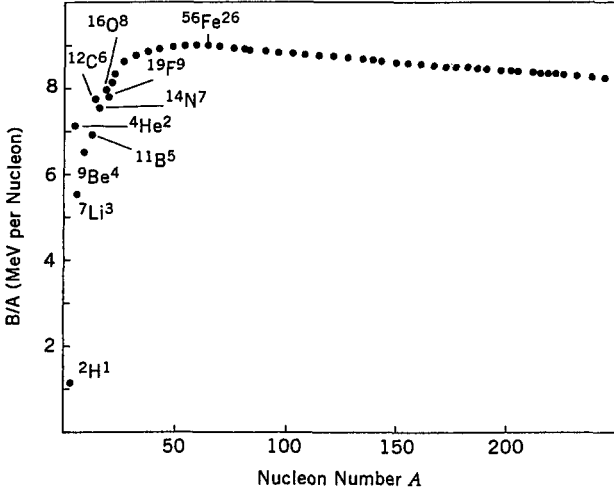


Fig. 2.1 The binding energy per nucleon for the most stable nuclei.

can, in principle, free that nucleon from the binding of the strong force, and it can escape the nucleus and materialize. To appreciate the significance of this observation, let us recall that all quantum-mechanical objects display wave behavior. In fact, for any particle with momentum  $p$ , we have an associated wavelength (according to de Broglie's hypothesis)

$$\lambda = \frac{\hbar}{p}, \quad (2.7)$$

where  $\hbar$  and  $\lambda$  are respectively Planck's constant and the wavelength  $\lambda$  divided by  $2\pi$  (referred to as the *reduced* wavelength). (The de Broglie bound-state requirement corresponds to  $2\pi r = n\lambda$ , and  $\lambda$  consequently reflects a typical radial size.) Now, let us assume that we transfer about 8 MeV of kinetic energy to a nucleon within a nucleus. Being quite massive ( $m \approx 940 \text{ MeV}/c^2$ ), the nucleon will be essentially non-relativistic. Calculating its wavelength from non-relativistic kinematics, we obtain

$$\begin{aligned} \lambda &= \frac{\hbar}{p} = \frac{\hbar}{\sqrt{2mT}} = \frac{\hbar c}{\sqrt{2mc^2T}} \\ &\approx \frac{197 \text{ MeV}\cdot\text{fm}}{\sqrt{2 \times 940 \times 8 \text{ MeV}}} \approx \frac{197}{120} \text{ fm} \approx 1.6 \text{ fm}, \end{aligned}$$

$$\text{or } \lambda \approx 1.6 \times 10^{-13} \text{cm}, \quad (2.8)$$

where 1 fm is a femto-meter ( $10^{-15}$ m) or one fermi (after Enrico Fermi). This wavelength is within the typical range of nuclear dimensions, and it is therefore reasonable to expect to localize nucleons of such energies within the nucleus. Consequently, a nucleon with about 8 MeV of kinetic energy (or 120 MeV/c momentum) could either be absorbed into or emitted from a nucleus. On the other hand, if electrons were present inside a nucleus, a kinetic energy of about 8 MeV would make them relativistic, and in this case  $pc \approx T \approx 8$  MeV would yield a far larger de Broglie wavelength

$$\begin{aligned} \lambda &= \frac{\hbar}{p} \approx \frac{\hbar c}{T} \approx \frac{\hbar c}{8 \text{ MeV}} \approx \frac{197 \text{ MeV-fm}}{8 \text{ MeV}} \\ &\approx 25 \text{ fm} \approx 2.5 \times 10^{-12} \text{cm}. \end{aligned} \quad (2.9)$$

With a de Broglie wavelength substantially larger than any nuclear radius, it would be unnatural to imagine an electron of  $\approx 8$  MeV energy residing inside a nucleus. Well, then what about an electron with momentum of 120 MeV/c? That kind of electron could, in principle, fit into a nucleus, but it would have 120 MeV of energy, and would therefore not be consistent with the energy scales of  $\approx 8$  MeV characterizing nuclear binding. This is, of course, a rather heuristic argument against the presence of electrons within nuclei, but more direct experimental observations also support this deduction. (We will return later to other implications of  $\frac{B}{A}$ .)

### 2.2.3 Sizes of Nuclei

The size of a subatomic object must be defined rather carefully. For a quantum mechanical system, the size normally refers to the expectation value of the coordinate operator in an appropriate state. For an atom, this would correspond to the average coordinate of the outermost electron. This can usually be calculated, at least perturbatively. In the nuclear domain, there is no simple expression for the force, and we therefore have to rely on interpretation of experiments to determine size.

There are several ways to go about this. First, for the low-energy Rutherford-scattering experiment, when the impact parameter is zero, namely when the projectile collides head-on with the scattering center, the distance of closest approach is a minimum (see Eq. (1.25)), given by

$$r_0^{\min} = \frac{ZZ'e^2}{E}. \quad (2.10)$$

Such particles will, of course, be scattered backwards ( $\theta = \pi$ ), and this distance of closest approach provides an upper bound on the size of the nucleus. The assumption is that low energy  $\alpha$ -particles cannot overcome the repulsive Coulomb barrier of the nucleus, and therefore cannot penetrate into the nucleus. Such low energy measurements yield relatively poor upper limits, typically,

$$R_{\text{Au}} \lesssim 3.2 \times 10^{-12} \text{cm}, \quad R_{\text{Ag}} \lesssim 2 \times 10^{-12} \text{cm}. \quad (2.11)$$

An alternative way to measure the sizes of nuclei is to scatter very high energy charged particles such as electrons off nuclei. For head-on collisions (i.e., when the impact parameter vanishes) we see from Eq. (2.10) that as  $E$  increases

$$r_0^{\min} \longrightarrow 0. \quad (2.12)$$

That is, higher-energy particles probe deeper into the nucleus. Because electrons interact mainly through the electromagnetic force, and are not sensitive to the nuclear force, they are influenced primarily by the electric-charge structure of the nucleus. In other words, using electron scattering, we can deduce the distribution of charge (the “form factor”) in a nucleus, and the radius of the charge distribution can be defined as an effective size of the nucleus. At relativistic energies, the magnetic moment of the electron also contributes to the scattering cross section. Neville Mott was first to formulate Rutherford scattering in the quantum domain, and to include such spin effects. Systematic studies of the scattering of high-energy electrons, initiated by Robert Hofstadter and his colleagues during the late 1950s, revealed the effects of spin and the extended nature of the nuclear charge distribution, including that of the proton.

For any given spatial charge distribution  $\rho(\vec{r})$  normalized to unity, we can define a form factor of the target in terms of its Fourier transform  $F(\vec{q})$  in momentum transfer, as given in Eq. (1.77)

$$F(\vec{q}) = \int_{\text{all space}} d^3r \rho(\vec{r}) e^{i\vec{q}\cdot\vec{r}}. \quad (2.13)$$

In general, this form factor modifies the cross section for elastic scattering of electrons from a point-like center, as follows

$$\frac{d\sigma}{dq^2} = |F(\vec{q})|^2 \left( \frac{d\sigma}{dq^2} \right)_{Mott}, \quad (2.14)$$

where the subscripted differential cross section is the Mott cross section for the scattering of point particles, which for high-energy scattering of electrons on a massive nuclear target can be related to the Rutherford formula as

$$\left( \frac{d\sigma}{d\Omega} \right)_{Mott} = 4 \cos^2 \frac{\theta}{2} \left( \frac{d\sigma}{d\Omega} \right)_{Rutherford} \quad (2.15)$$

Thus, deviations from the distribution expected for point-scattering provide a measure of size (and structure) of the objects involved in the collision. Because electrons are thought to be point particles, the observed distribution, therefore, reflects the size of the nuclear target.

There is yet another way of studying sizes of nuclei by taking advantage of the strong force. In particular, the relatively weak Coulomb interaction can be neglected in the elastic scattering of sufficiently energetic strongly-interacting particles (such as  $\pi$  mesons, protons, etc.) from nuclear targets. Such projectiles interact quite readily with nuclei, and are thereby “absorbed” out of the beam – very similar to the way light gets removed by an absorbing disc. The result of the absorption is a diffraction pattern – again, similar to that observed in the scattering of light from a slit or grating. The size of the nucleus, which acts in many ways as an absorbing disc, can therefore be inferred from the diffraction pattern.

All these phenomenological investigations have provided a remarkably simple relation for the radial size of the nucleus as a function of its nucleon number  $A$

$$\begin{aligned} R &= r_0 A^{\frac{1}{3}} \\ &\approx 1.2 \times 10^{-13} A^{\frac{1}{3}} \text{ cm} = 1.2 A^{\frac{1}{3}} \text{ fm}. \end{aligned} \quad (2.16)$$

From the preceding we can conclude that nuclei have enormous mass densities of  $\approx 10^{14} \text{ gm/cm}^3$ , and that nucleons are tightly packed inside the nucleus.

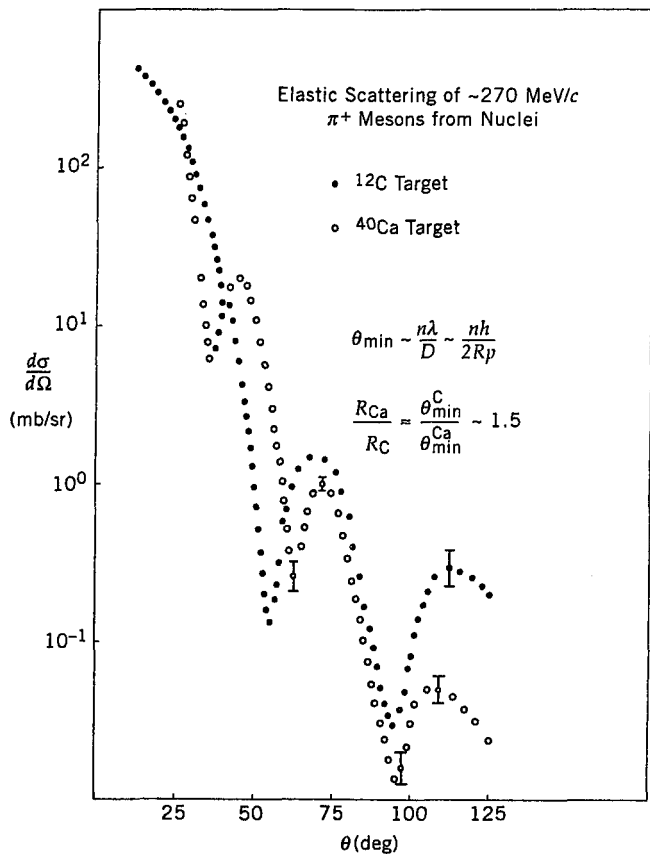


Fig. 2.2 Differential elastic cross sections for scattering of  $\pi^+$  mesons of momentum of  $\approx 270$  MeV/c from carbon and calcium targets. The unit mb is  $10^{-3}$  of a barn (see p. 15). Using the de Broglie wavelength for the  $\pi^+$ , and the optical analogy for the first minimum in the scattering, yields nuclear radii close to values expected from Eq. (2.16); also, the ratio of the angles at the minima for the two targets scale as the radii of these nuclei. (Data are based on C. H. Q. Ingram in *Meson-Nuclear Physics - 1979*, AIP Conference Proc. No. 54.)

### 2.2.4 Nuclear Spins and Dipole Moments

Both the proton and the neutron have spin angular momentum of  $\frac{1}{2}\hbar$ . Furthermore, just as electrons in an atom can have orbital angular momentum, so also can nucleons inside a nucleus. We know from quantum mechanics that orbital angular momentum can take on only integral values. The total angular momentum of the constituents – namely, the vector sum of the or-

bital and intrinsic spin angular momenta – defines the spin of the nucleus. Thus, it is not surprising that nuclei with even atomic number have integral nuclear spin whereas nuclei with odd atomic number have half-integral nuclear spin. However, what is surprising is that all nuclei with an even number of protons and an even number of neutrons (even-even nuclei) have zero nuclear spin. It is equally surprising that large nuclei have very small nuclear spins in their ground states. These facts lend credence to the hypothesis that spins of nucleons inside a nucleus are very strongly paired so as to cancel their overall effect.

Every charged particle has a magnetic dipole moment associated with its spin, given by

$$\vec{\mu} = g \frac{e}{2mc} \vec{S}, \quad (2.17)$$

where  $e$ ,  $m$  and  $\vec{S}$  are the charge, mass and the intrinsic spin of the charged particle. The constant  $g$  is known as the Landé factor, which for a point particle, such as the electron, is expected to have the value  $g = 2$ . (In fact, small deviations at the level of  $10^{-3}$  have been observed for the “point-like” electron, but this agrees with expectation from field-theoretical calculations based on quantum electrodynamics, or QED.) When  $g \neq 2$ , the particle is said to possess an anomalous magnetic moment, which is usually ascribed to the particle having a substructure. For the electron (with  $|S_z| = \frac{1}{2} \hbar$ ), the dipole moment  $\mu_e \approx \mu_B$ , where  $\mu_B$  is the Bohr magneton, defined as

$$\mu_B = \frac{e\hbar}{2m_e c} = 5.79 \times 10^{-11} \text{ MeV/T}, \quad (2.18)$$

where a magnetic field of 1 tesla (T) corresponds to  $10^4$  gauss (G). The magnetic dipole moment for nucleons is measured in terms of the nuclear magneton, defined using the proton mass

$$\mu_N = \frac{e\hbar}{2m_p c}. \quad (2.19)$$

From the ratio of  $\frac{m_p}{m_e}$ , we deduce that the Bohr magneton is about 2000 times larger than the nuclear magneton.

The magnetic moments of the proton and the neutron are

$$\begin{aligned}\mu_p &\approx 2.79 \mu_N, \\ \mu_n &\approx -1.91 \mu_N.\end{aligned}\tag{2.20}$$

Consequently, both nucleons have large anomalous contributions to their moments. This provides indirect evidence that these particles have additional structure. In fact, since the neutron is electrically neutral, its sizable magnetic moment is particularly dramatic, and points to the fact that the neutron must have an extended charge distribution. The measurement of magnetic dipole moments for different nuclei has yielded the surprising result that all their values lie between  $-3 \mu_N$  and  $10 \mu_N$ . This again is evidence for strong pairing inside the nucleus. Furthermore, this also shows that electrons cannot be present inside nuclei because it would then be particularly hard to explain the small values of nuclear moments, since even one electron would produce a moment a thousand times that observed for nuclei.

### 2.2.5 Stability of Nuclei

When we examine the characteristics of stable nuclei, we find that for  $A \lesssim 40$  the number of protons equals the number of neutrons ( $N = Z$ ). But beyond  $A = 40$ , stable nuclei have  $N \approx 1.7Z$ ; namely, neutrons far outnumber protons (see Fig. 2.3). This can be understood from the fact that, in larger nuclei, the charge density, and therefore the destabilizing effect of Coulomb repulsion, is smaller when there is a neutron excess.

Furthermore, a survey of the stable nuclei (see Table 2.1) reveals that even-even nuclei are the ones most abundant in nature. This again lends support to the strong-pairing hypothesis, namely that pairing of nucleons leads to nuclear stability.

**Table 2.1** Number of stable nuclei in nature.

$N$	$Z$	Number of Stable Nuclei
Even	Even	156
Even	Odd	48
Odd	Even	50
Odd	Odd	5

### 2.2.6 Instability of Nuclei

In 1896, through sheer accident, Henri Becquerel discovered natural radioactivity. He was studying fluorescent properties of uranium salts by exposing the material to the sun and then photographing the emission spectrum. Because the weather was cloudy, he stored the compound as well as some photographic plates inside a desk drawer. When he subsequently developed the plates, he noticed that they were overexposed, and surmised that the uranium compound must have emitted penetrating radiation of a variety quite different from fluorescence. This was the first observation of natural nuclear radioactivity, and subsequent studies have revealed that such spontaneous emission is a common phenomenon, especially for large nuclei.

Nuclear radioactivity involves the emission of essentially three kinds of radiation:  $\alpha$ -radiation,  $\beta$ -radiation and  $\gamma$ -radiation. Each of these emanations has distinct properties that can be characterized in the following way. Consider a radioactive source located at the bottom of a narrow and deep cavity within a piece of lead. Because lead easily absorbs nuclear radiation, the cavity will therefore function as a source of a well collimated beam of radiation (see Fig. 2.4). If a magnetic field is applied perpendicular to the plane of the paper in Fig. 2.4, the beam will bend if it contains any charged components. The direction of bending will depend on the sign of

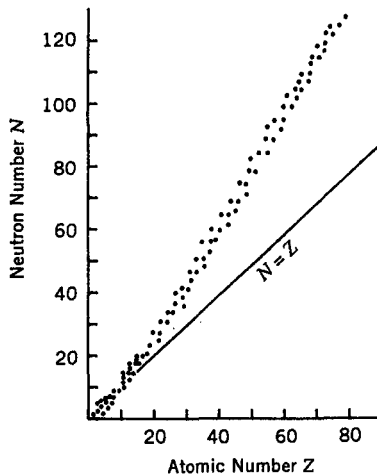


Fig. 2.3 Neutron number as a function of atomic number for a representative sample of most stable nuclei.



the electric charge, and the amount of bending will be determined by the momentum of the particles. Such simple experiments demonstrated that  $\alpha$ -rays have positive charge, and because most of them arrived close to the same spot on the screen, they indicated that the  $\alpha$ -particles in the beam were essentially mono-energetic, with typical velocities of about  $0.1c$ . Furthermore, the range of  $\alpha$ -particles was found to be relatively short. (We will discuss in Chapter 7 how such measurements can be carried out.) In contrast, the most common forms of  $\beta$ -rays were found to bend in a direction opposite to that of  $\alpha$ -rays, indicating that  $\beta$ -radiation consisted of negatively charged particles. The  $\beta$ -particles were observed to be well dispersed along the screen, which meant that, unlike the  $\alpha$ -particles, the  $\beta$ -particles had a continuous spectrum of velocities, which were as high as  $0.99c$ . Other measurements revealed that  $\beta$ -particles had longer ranges and were less ionizing than  $\alpha$ -particles. (We will discuss ionization in more detail in Chapter 6.) It took about 3 mm of lead to stop typical  $\beta$ -particles, while a piece of paper sufficed to stop  $\alpha$ -particles. Finally, a third form of emission, namely  $\gamma$ -radiation, was observed to arrive undeflected at the center of the screen, suggesting that these objects had no charge. In fact,  $\gamma$ -rays behaved in all respects like electromagnetic radiation, and it was therefore concluded that they were photons that traveled with the speed of light. Measurements revealed that  $\gamma$ -rays had much longer ranges and produced even less ionization per unit path than  $\beta$ -rays. It took typically several cm of lead to completely stop (absorb)  $\gamma$ -rays.

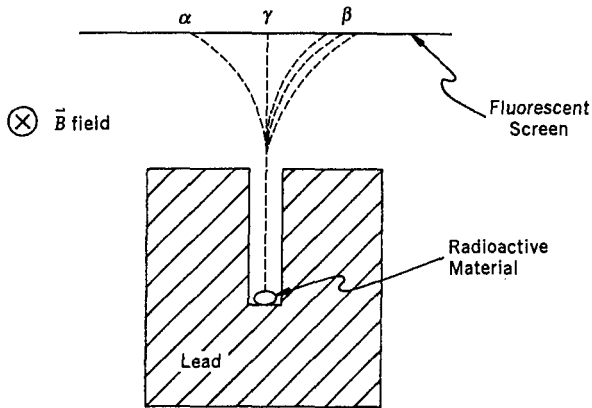


Fig. 2.4 Separation of  $\alpha$ ,  $\beta$  and  $\gamma$ -rays in a magnetic field.

It is, of course, also possible to deflect charged particles using an elec-

tric field. In fact, by applying an electric field in the plane of the paper in Fig. 2.4, perpendicular to both  $\vec{B}$  and to the beam axis, and adjusting the magnitudes of the electric and the magnetic fields, the deflection of any charged particle can be varied, and even completely canceled. For a given electric and magnetic field, the deflection is a function of the charge and mass of the radioactive emission, and such measurements of deflections revealed that  $\alpha$ -particles carried two units of positive charge and four units of atomic mass. In other words,  $\alpha$ -particles were merely the very stable nuclei of helium atoms, namely  ${}^4\text{He}^2$ . Similarly, through such measurements,  $\beta$ -particles were identified as electrons. Thus, the most common forms of natural nuclear radiation, namely  $\alpha$ -rays,  $\beta$ -rays and  $\gamma$ -rays correspond, respectively, to the spontaneous emission of helium nuclei, electrons and energetic photons by heavy nuclei. It should be recognized, however, that any nuclear fragment can also be regarded as a form of radiation. (More quantitative aspects of  $\alpha$ ,  $\beta$  and  $\gamma$  emission will be treated in Chapter 4.)

### 2.3 Nature of the Nuclear Force

In addition to determining the properties of nuclei, scattering experiments also provide more global information on the character of the nuclear force, as we will summarize below.

First of all, it is clear that the nuclear force has no classical analog. The gravitational attraction between nucleons is far too weak to bind them together. And the nuclear force cannot have an electromagnetic origin, since the nucleus of the deuteron contains only one proton and one neutron, and the neutron, being charge-neutral, has only very weak electromagnetic interactions (due to its magnetic dipole moment). In fact, the electromagnetic interaction (namely Coulomb repulsion) primarily destabilizes the nucleus.

It is also clear that the nuclear force must be extremely short-ranged. For example, the structure of the atom is explained exceedingly well just by the electromagnetic interaction. Consequently, the range of the nuclear force cannot be much greater than the size of the nucleus, simply because, otherwise, it would affect the excellent agreement between theory and experimental observations in atomic physics. This argument would suggest that the range of the nuclear force is limited to about  $10^{-13}\text{cm} - 10^{-12}\text{cm}$ , which corresponds to the approximate size of nuclei.

Other important evidence for the short-ranged nature of the nuclear force comes from the fact that the binding energy per nucleon is a constant,

essentially independent of the size of the nucleus. In fact, if the nuclear force had a long range like the Coulomb force, then given  $A$  nucleons, there would be  $\frac{1}{2}A(A - 1)$  pairwise interactions between them (that is, the total number of independent combinations for  $A$  nucleons, taken two at a time). Correspondingly, the binding energy, which basically reflects the total potential energy of all possible interactions among the nucleons, would grow with the number of nucleons as

$$B \propto A(A - 1). \quad (2.21)$$

Thus, for large values of  $A$ , we would have

$$\frac{B}{A} \propto A. \quad (2.22)$$

In other words, if the force between any two nucleons were independent of the presence of other nucleons, the binding energy per nucleon would grow linearly with  $A$ . This is, in fact, what happens for the Coulomb force, and it is primarily because a long-ranged force does not saturate, in the sense that any single particle can interact with as many other particles as are available. The net effect of this kind of force is that the binding becomes ever tighter as the number of interacting objects increases, and, as a result, the size of the interaction region remains fairly constant. This is the situation for the case of atomic binding, where atoms with a large number of electrons have sizes comparable to those with few electrons.

For the case of nuclei, however, we see from Fig. 2.1 that the binding energy per nucleon is essentially constant, and therefore we conclude that the nuclear force must saturate. Namely, any given nucleon can interact with only a finite number of nucleons in its neighborhood. Adding more nucleons to a nucleus therefore only increases the size of the nucleus but not the binding energy per nucleon. As we have seen before in Eq. (2.16), the size of a nucleus grows slowly with atomic number in a way so as to keep the nuclear density essentially fixed. These observations again lend support to the fact that the nuclear force is short-ranged.

In general, to keep the nucleons within a nucleus, the nuclear force must be attractive. However, experiments in which high energy particles were scattered off nuclei have revealed that the nuclear force has a repulsive core. Namely, we find that below a certain length scale, the nuclear force changes from attractive to repulsive. (The presence of the repulsive core

is best attributed to a quark substructure of the nucleon.) Conceptually, this result is appealing because, if the nuclear force were attractive at all distances, then the nucleus would collapse in on itself. Pictorially, we can represent the behavior of the nuclear force through a square-well potential that an incident nucleon can sense as it moves toward the nuclear center (see Fig. 2.5).

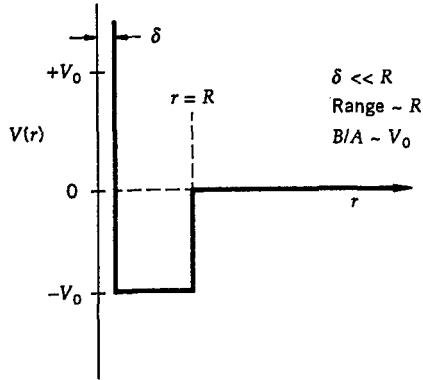


Fig. 2.5 Approximate description of the nuclear potential as a function of distance to the center. The repulsive core is sensed only at small distances ( $\delta \ll R$ ).

Because low energy particles cannot probe short-distance behavior in the nucleus, to an excellent approximation the repulsive core can be ignored in problems pertaining to low-energy nuclear structure, and the nuclear force can be represented adequately through just a square-well potential.

We should point out that we do not expect the nuclear density nor the nuclear force to cut off suddenly at some  $r = R$ , and so the square well is meant to represent only the general effects of the nuclear force. It is more appropriate for incident neutrons than, for example, for protons, or for other incident nuclei, which, in addition, are subject to the repulsive Coulomb potential due to the positive charge of the nucleus (see Fig. 2.6). In the presence of Coulomb repulsion, an incident proton of total energy  $E_0$  senses the Coulomb barrier as it approaches the nucleus. Classically, the proton cannot get closer than  $r = r_0^{\min}$ , because, for  $R < r < r_0^{\min}$ ,  $V(r)$  would exceed  $E_0$ , and the kinetic energy would have to be negative, which is not physically possible. However, ignoring the repulsive core for  $r \lesssim \delta$  in Fig. 2.5, a neutron of same energy could penetrate into the nuclear center.

It was once the hope that low-energy scattering experiments could be

used to obtain the exact shape of the nuclear potential, but it turns out that the results of the scattering are not very sensitive to the details of the shape, but primarily to the range and the height of the potential. The square well is one of several forms of potentials that can provide a good phenomenological description of the nuclear force.

The fact that the nuclear force can be described through a potential energy function of the kind shown in Fig 2.5, suggests on the basis of quantum theory that nuclear systems can have discrete energy levels and corresponding bound states similar to the type found in atomic systems. The presence of such nuclear quantum states, and the transitions between them, have been confirmed in a variety of ways. They can be inferred from scattering experiments and through studies of the energies observed for emitted nuclear radiation. The modeling of ground levels and excited nuclear states formed one of the early testing grounds for quantum mechanics. Some of the experimental evidence for nuclear levels, and several successful nuclear models, will be described in the following chapters.

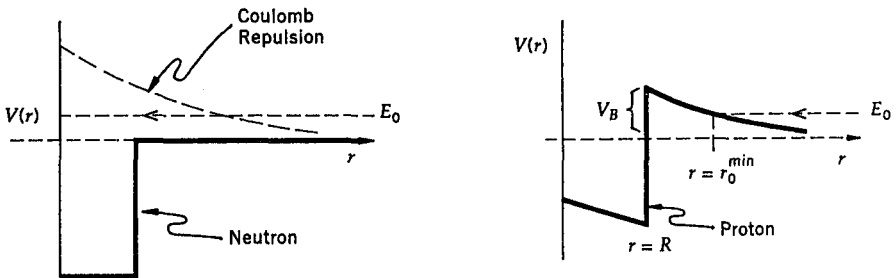


Fig. 2.6 The potential energy of a proton and a neutron incident on a nuclear target. (Recall that, as a charged particle enters the nucleus, it sees less of the total nuclear charge, and the character of the classical potential changes from  $\frac{1}{r}$  to  $(3R^2 - r^2)$ , and consequently the potential remains finite at  $r = 0$ .)

Studies of mirror nuclei,<sup>2</sup> and the scattering of protons and neutrons, demonstrate the interesting fact that, once we correct for known Coulomb effects, the force between two neutrons is the same as the force between two protons, which also coincides with the force between a proton and a neutron. This property of the nuclear force is referred to as *charge independence*. Namely, the strong nuclear force between two particles is independent of the

<sup>2</sup>Mirror nuclei are isobars that have proton and neutron numbers interchanged as in  ${}^A X^Z$  and  ${}^A Y^{A-Z}$  (e.g.,  ${}^{15}\text{O}^8$  and  ${}^{15}\text{N}^7$ ). Such pairs of nuclei have the same number of  $n$ - $p$  interactions, but differ in their number of  $p$ - $p$  and  $n$ - $n$  interactions.

electric charge carried by these particles. This is a remarkable result and, as we will see in Chapter 9, it leads to the concept of strong isotopic-spin symmetry. Very briefly, this symmetry implies that, just as the “spin-up” and the “spin-down” states of an electron correspond to two different spin states of the same particle, so the proton and the neutron correspond to two states of the same particle called the nucleon. If we could turn off the Coulomb field, the neutron and the proton would be indistinguishable in their nuclear interactions. This is analogous to the indistinguishability of spin up and spin down states in the absence of a magnetic field. We will discuss this symmetry in more detail in Chapter 9.

Let us examine the question of range of the nuclear force from a somewhat different perspective, and note that the electromagnetic force between two interacting charged particles can be understood as a result of an exchange of a photon between them. Photon propagation is described by the Maxwell Equations, which correspond to propagation at the speed of light (see Chapter 13). Consequently, we presume that the photon is massless. Furthermore, the Coulomb force is represented by the potential

$$V(r) \propto \frac{1}{r}, \quad (2.23)$$

which, of course, shows explicitly that it is a long-ranged force.

For the case when the exchanged particle is massive, Hideki Yukawa showed in 1934 that the corresponding potential takes the form

$$V(r) \propto \frac{e^{-\frac{mc}{\hbar}r}}{r}, \quad (2.24)$$

where  $m$  is the mass of the particle mediating the interaction.

In the limit that  $m$  vanishes, we recover the Coulomb potential of Eq. (2.23), and conclude again that the range of the Coulomb force is infinite, which is consistent with experiment. From the form of the Yukawa potential, the range for the interaction is given by some characteristic value of  $r$ , which also corresponds to the Compton wavelength of the object of mass  $m$

$$\lambda = \frac{\hbar}{mc}. \quad (2.25)$$

Therefore, once we know the mass of the exchanged particle, we can predict the range of the force. Conversely, if we know the range of the force, we

can also predict the mass of the particle being exchanged. For the case of the nuclear force, a simple calculation shows that

$$m = \frac{\hbar}{\lambda c},$$

$$\text{or } mc^2 = \frac{\hbar c}{\lambda} \approx \frac{197 \text{ MeV}\cdot\text{fm}}{1.2 \times 10^{-13} \text{ cm}} \approx 164 \text{ MeV}. \quad (2.26)$$

But this is approximately the mass of the well known  $\pi$  meson (pion). There are, in fact, three pions, with masses

$$\begin{aligned} m_{\pi^+} = m_{\pi^-} &= 139.6 \text{ MeV}/c^2, \\ m_{\pi^0} &= 135 \text{ MeV}/c^2. \end{aligned} \quad (2.27)$$

This suggests that pions might be the mediators of the nuclear force. We will return later to a discussion of pions, other mesons, and their place in the development of the full story of charge independence of the strong force.

## Problems

**2.1** Calculate the approximate density of nuclear matter in  $\text{gm}/\text{cm}^3$ . What would be the mass of a neutron star that had the diameter of an orange?

**2.2** Calculate the difference between the binding energy of a nucleus of  $^{12}\text{C}$  and the sum of the binding energies of three  $^4\text{He}$  nuclei ( $\alpha$ -particles). Assuming that  $^{12}\text{C}$  is composed of three  $\alpha$ -particles in a triangular structure, with three effective “ $\alpha$ -bonds” between them, what would be the binding energy per  $\alpha$ -bond? (See *CRC Handbook for Chemistry and Physics* for mass values.)

**2.3** Calculate the binding energy of the last neutron in  $^4\text{He}$  and the last proton in  $^{16}\text{O}$ . How do these compare with  $\frac{B}{A}$  for these nuclei? What does this tell you about the stability of  $^4\text{He}$  relative to  $^3\text{He}$ , and of  $^{16}\text{O}$  relative to  $^{15}\text{N}$ ? [Hint: the binding energy of the last neutron needed to form a nucleus  $(A, Z)$  is given by  $[M(A-1, Z) + m_n - M(A, Z)]c^2$ . An analogous expression holds for the last proton.]

**2.4** Starting with cgs quantities, calculate the value of  $\mu_B = \frac{e\hbar}{2m_e c}$ , and convert it to  $\text{MeV}/\text{T}$  units. (Hint: you can relate forces and magnetic fields through the Lorentz force  $\vec{F} = \frac{q\vec{v} \times \vec{B}}{c}$ .)

**2.5** Assume that the spin of a proton can be represented by a positive pion moving at a speed  $c$  in a circular orbit of radius  $10^{-13}$  cm about a neutral center. Calculate the current and the magnetic moment associated with this motion. Compare this with the known magnetic moment of the proton. (Hint: recall that using cgs units you can write a magnetic moment  $\vec{\mu} = \left(\frac{I}{c}\right) \vec{A}$ , where  $I$  is the current flowing around the area  $A$ .)

**2.6** We argued previously that the  $\pi^+$  mesons in Fig. 2.2 scattered not from individual nucleons, but rather (*coherently*) from the entire nuclei. In fact, the first minima ( $n = 1$ ) corresponded to  $\theta \approx \frac{n\hbar}{2Rp}$ , with  $R$  being consistent with  $1.2A^{\frac{1}{3}}$ . At higher energies, when larger momenta can be transferred to nuclei, it is possible to dislodge a single proton or neutron from the nucleus. When this happens, the  $\pi^+$  mesons can be termed to scatter elastically from quasi “free” nucleons. How would this affect the diffraction pattern in Fig. 2.2? What about if you could scatter from very small point-like constituents within nucleons? (Would the fact that a  $\pi^+$  is not a point particle affect your answer?)

**2.7** Normally, in optics, one looks at the diffraction pattern as a function of angle  $\theta$ . In this case, the value of  $\theta$  at the first minimum changes with wavelength or momentum. Can you see any advantage to using a variable such as  $q^2 \approx p_T^2 \approx (p\theta)^2$  to examine diffraction patterns at different scattering energies? Sketch how the pattern might look for scattering of  $\pi^+$  mesons of different energies from nuclear targets. Now, as energy increases, and larger  $q^2$  become possible, what would be the effect of having nucleon substructure within the nucleus? What about point substructure within the nucleon? (Does your answer depend on whether the  $\pi^+$  has such substructure?)

**2.8** What are the frequencies that correspond to typical splitting of lines for nuclear magnetic moments in magnetic fields of  $\approx 5$  tesla?

**2.9** Show that when non-relativistic neutrons of kinetic energy  $E_0$  collide head-on with stationary nuclei of mass number  $A$ , the smallest energy that elastically-scattered neutrons can have is given approximately by

$$E_{\min} = E_0 \left( \frac{A-1}{A+1} \right)^2.$$

What will be the approximate energies of the neutrons after one, two, and any number  $j$  of such consecutive collisions, if the target nucleus is hydro-



gen, carbon, and iron?

**2.10** Using the results of Problem 2.9, calculate the number of collisions needed to reduce the energy of a 2 MeV neutron to 0.1 MeV through elastic collisions between the neutron and carbon nuclei.

**2.11** For  $q^2 \ll 1$ , the exponential in the elastic form factor of Eq. (2.13) can be approximated as  $1 + i\vec{k} \cdot \vec{r} - \frac{1}{2}(\vec{k} \cdot \vec{r})^2$ , where  $\vec{k} = \frac{1}{\hbar}\vec{q}$ . Calculate  $|F(q)|^2$  in terms of a root-mean-square radius of the charge distribution  $R = \sqrt{\langle r^2 \rangle}$ , for  $\rho(r)$  described by (a) a uniform distribution of charge within  $r = R$ , and (b) a Gaussian  $\rho(r) = \frac{1}{R} \sqrt{\frac{2}{\pi}} e^{-\frac{2r^2}{R^2}}$ , and show that in both cases  $|F(q)|^2$  falls off approximately exponentially with  $q^2$ . (*Hint:* Use symmetry arguments to eliminate the  $\vec{k} \cdot \vec{r}$  term by recognizing that  $\vec{k} \cdot \vec{r} = k_x x + k_y y + k_z z$ . Also, note that for a spherically symmetric  $\rho(r)$ ,  $\langle x^2 \rangle = \langle y^2 \rangle = \langle z^2 \rangle = \frac{1}{3} \langle r^2 \rangle$ , and  $\langle r^2 \rangle = \int 4\pi r^2 dr r^2 \rho(r)$ .)

### Suggested Readings

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## Chapter 3

# Nuclear Models

### 3.1 Introductory Remarks

A variety of early experiments demonstrated that the character of the nuclear force differed markedly from any previously encountered in classical physics. However, a quantitative description of the nuclear force has turned out to be elusive. As we learned from atomic physics, where the correct level structure was found only after the classical Coulomb interaction between the nucleus and the electrons was extended to the atomic domain through quantum mechanics, knowing the properties of a force is only the first step in developing a theory of structure. Although neutrons and protons were known to be the nuclear constituents, the absence of a fundamental understanding of the nuclear force made it difficult to determine the structure of the nucleus. It is not surprising therefore that, instead of a theory, phenomenological models of the nucleus were constructed to accommodate the many remarkable experimental findings. In the following, we describe only a few such models. We should also keep in mind that, unlike the case of atomic physics, most of these nuclear models were proposed to explain only limited aspects of the data, which is precisely what they do.

### 3.2 Liquid Drop Model

The liquid drop model of the nucleus was one of the earliest phenomenological successes constructed to account for the binding energy of a nucleus. As we have already discussed, experiments revealed that nuclei were essentially spherical objects, with sizes that could be characterized by radii proportional to  $A^{1/3}$ , which suggested that nuclear densities were almost independent of nucleon number. This leads quite naturally to a model that

envisions the nucleus as an incompressible liquid droplet, with nucleons playing the role analogous to molecules in a drop of normal liquid. In this picture, known as the *liquid drop model*, the individual quantum properties of nucleons are completely ignored.

As in the case of a liquid drop, the nucleus is imagined as composed of a stable central core of nucleons for which the nuclear force is completely saturated, and a surface layer of nucleons that is not bound as tightly (forces not saturated). This weaker binding at the surface decreases the effective binding energy per nucleon ( $\frac{B}{A}$ ), and provides a “surface tension”, or an attraction of the surface nucleons towards the center (see Fig. 3.1). If, as experiments suggest, a constant binding energy (B.E.) per nucleon can be attributed to the saturation of the nuclear force, then on the basis of these considerations we can write a general form for the binding energy of a nucleus as follows

$$\text{B.E.} = -a_1 A + a_2 A^{\frac{2}{3}}, \quad (3.1)$$

where the first term represents a volume energy for the case of uniform saturated binding (remember that volume  $\propto R^3 \propto A$ ), and the second term corrects for any over-estimation due to the surface tension. It is clear that the correction to the binding energy per nucleon in Eq. (3.1) is higher for lighter nuclei because these have a larger surface-to-volume ratio of nucleons. That is, small nuclei have relatively more nucleons on the surface than in the core. This can explain why the binding energy per nucleon is smaller for lighter nuclei.

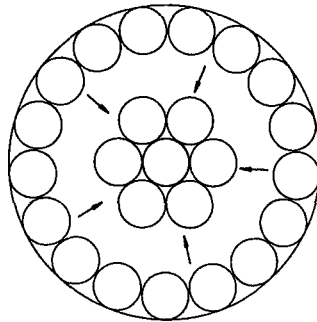


Fig. 3.1 Surface layer and core of nucleus in the liquid drop model.

In this same model, the small decrease in binding energy per nucleon for very heavy nuclei can be understood as due to Coulomb repulsion. Namely, if the nucleus has  $Z$  protons, then the electrostatic (Coulomb) energy of these protons, which has a destabilizing effect, has the form  $\propto \frac{Z^2}{R}$ . Thus, adding such a positive term to reduce the binding strength, we can write

$$\text{B.E.} = -a_1 A + a_2 A^{\frac{2}{3}} + a_3 \frac{Z^2}{A^{\frac{1}{3}}}. \quad (3.2)$$

The three terms in Eq. (3.2) arise from purely classical considerations. Unfortunately, they do not accommodate the fact that lighter nuclei with an equal number of protons and neutrons are particularly stable. In other words, Eq. (3.2) does not lead to stronger binding and greater stability (that is, more negative B.E.) for light nuclei that have  $N = Z$ . Similarly, Eq. (3.2) does not provide the natural abundance of even-even nuclei nor the paucity of odd-odd nuclei. Such observations can be understood mainly as arising from quantum effects (spin, statistics, etc.). Within the framework of the liquid drop model, they can be included by generalizing the empirical formula for binding energy to contain additional phenomenological terms

$$\text{B.E.} = -a_1 A + a_2 A^{\frac{2}{3}} + a_3 \frac{Z^2}{A^{\frac{1}{3}}} + a_4 \frac{(N - Z)^2}{A} \pm a_5 A^{-\frac{3}{4}}, \quad (3.3)$$

where all the coefficients  $a_1, a_2, a_3, a_4, a_5$  are assumed to be positive. Note that the fourth term implies that, unless  $N = Z$ , the binding energy will contain a positive contribution that will destabilize the nucleus. For small  $Z$ , where destabilization from the  $a_3$  term is not very important, the  $a_4$  term reflects the stability of  $N = Z$  nuclei. In the last term, the positive sign is chosen for odd-odd nuclei, implying that such nuclei are relatively unstable. On the other hand, for even-even nuclei, the sign is taken as negative, implying greater stability and, therefore, abundance of such nuclei in nature. For odd- $A$  nuclei, the value of  $a_5$  is chosen to be zero, primarily because the binding energy for such nuclei can be described quite well without the last term in Eq. (3.3).

The arbitrary coefficients can be determined by fitting the empirical formula to experimentally observed binding energies for a wide range of nuclei. The following set of values provides a rather good fit

$$\begin{aligned}
 a_1 \approx 15.6 \text{ MeV}, \quad a_2 \approx 16.8 \text{ MeV}, \quad a_3 \approx 0.72 \text{ MeV}, \\
 a_4 \approx 23.3 \text{ MeV}, \quad a_5 \approx 34 \text{ MeV}. \quad (3.4)
 \end{aligned}$$

Given the phenomenological formula for the binding energy, we can also write an equivalent empirical relation for masses of nuclei as follows (see Eqs. (2.4) and (2.5))

$$\begin{aligned}
 M(A, Z) &= (A - Z)m_n + Zm_p + \frac{\text{B.E.}}{c^2} \\
 &= (A - Z)m_n + Zm_p - \frac{a_1}{c^2} A \\
 &\quad + \frac{a_2}{c^2} A^{\frac{2}{3}} + \frac{a_3}{c^2} \frac{Z^2}{A^{\frac{1}{3}}} + \frac{a_4}{c^2} \frac{(A - 2Z)^2}{A} \pm \frac{a_5}{c^2} A^{-\frac{3}{4}}. \quad (3.5)
 \end{aligned}$$

This expression, known as the Bethe-Weizsäcker semi-empirical mass formula, can be used to predict stability and masses of unknown nuclei of arbitrary  $A$  and  $Z$ . It also plays a crucial role in a quantitative understanding of the theory of fission, as we will see in Chapter 5.

### 3.3 The Fermi-Gas Model

The *Fermi-gas model* was one of the earliest attempts to incorporate quantum mechanical effects into the discussion of nuclear structure. It assumes that a nucleus can be regarded as a gas of free protons and neutrons confined to a very small region of space, namely to the nuclear volume. Under such conditions, the nucleons would be expected to populate discrete (quantized) energy levels within the nucleus. We can think of the protons and neutrons as moving inside a spherically symmetric well, whose range is given by the radius of the nucleus, and whose depth can be adjusted to obtain the correct binding energy. Because protons carry electric charge, as discussed in Chapter 2, they sense a potential that differs from that sensed by neutrons. The observed energy levels for neutrons and protons will therefore differ somewhat, depending on the specific range and depth of the individual potentials. We will see in Chapter 9 that all elementary particles can be classified as either bosons or fermions, and that protons

and neutrons being fermions obey Fermi-Dirac statistics. According to the Pauli exclusion principle, this implies that any given energy level can be filled by at most two “identical” nucleons (i.e., in the sense of same energy and charge) of opposite spin projection.

Since the lowest levels in a well have strongest binding, to achieve greatest stability for the ground state, we expect the energy levels to fill from the bottom up. The highest level that is completely filled defines what is referred to as the *fermi level*, of energy  $E_F$ . If there is no fermion beyond the fermi level, the binding energy of the last nucleon is given simply by  $E_F$ . Otherwise, the energy of the fermion in the next level reflects the binding energy of the last nucleon.

If the depths of the wells for neutrons and protons were the same, then, in heavier nuclei, where the number of neutrons exceeds the number of protons, the fermi level for neutrons would lie higher than for protons. If this were the case, then the binding energy of the last nucleon would be charge dependent, namely different for protons and neutrons. This is inconsistent with experiment, and leads us to conclude that, to have fermi levels of same energy for neutrons and protons, protons must move in shallower potential wells (see Fig. 3.2). In fact, if this were not the case, all such nuclei would be unstable, and neutrons would drop down to lower proton levels through  $\beta^-$  emission ( $\beta^-$  decay is discussed in Chapter 4).

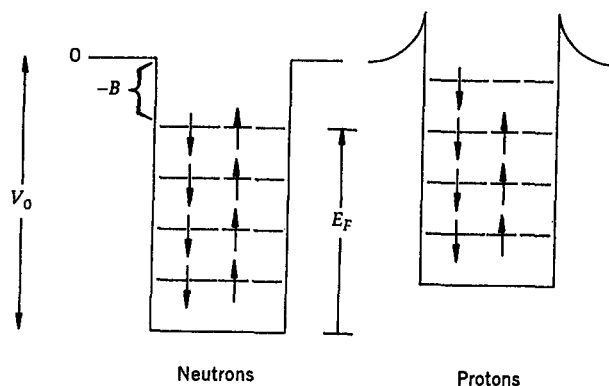


Fig. 3.2 Ground-state energy levels for neutrons and protons in nuclei.

Let us next relate the energy of the fermi level to the number of fermions. We define the momentum associated with the fermi level through

$$E_F = \frac{p_F^2}{2m}, \quad (3.6)$$

where  $m$  is the mass of a nucleon. Ignoring the presence of fermions beyond the fermi level, we can write the volume for states in momentum space as

$$V_{p_F} = \frac{4\pi}{3} p_F^3. \quad (3.7)$$

If  $V$  denotes the physical nuclear volume, then the total volume for states in what we may call “phase space” is given by the product

$$\begin{aligned} V_{\text{TOT}} &= V \times V_{p_F} = \frac{4\pi}{3} r_0^3 A \times \frac{4\pi}{3} p_F^3 \\ &= \left(\frac{4\pi}{3}\right)^2 A (r_0 p_F)^3, \end{aligned} \quad (3.8)$$

which is proportional to the total number of quantum states of the system. We know from the Heisenberg uncertainty principle that, for any quantum state, the same components of momentum and position obey the inequality

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}. \quad (3.9)$$

This relation can be used to provide a restriction on the minimum volume that can be associated with any physical state of the system, which can be shown to be

$$V_{\text{state}} = (2\pi\hbar)^3 = h^3. \quad (3.10)$$

It follows, therefore, that the number of fermions that can fill states up to and including the fermi level is

$$n_F = 2 \frac{V_{\text{TOT}}}{(2\pi\hbar)^3} = \frac{2}{(2\pi\hbar)^3} \left(\frac{4\pi}{3}\right)^2 A (r_0 p_F)^3 = \frac{4}{9\pi} A \left(\frac{r_0 p_F}{\hbar}\right)^3, \quad (3.11)$$

where the factor of 2 arises because each state can be occupied by two fermions with opposite spins.

For simplicity, let us now consider a nucleus with  $N = Z = \frac{A}{2}$ , and assume that all the states up to and including the fermi level are filled. In this case we have

$$N = Z = \frac{A}{2} = \frac{4}{9\pi} A \left( \frac{r_0 p_F}{\hbar} \right)^3,$$

$$\text{or } p_F = \frac{\hbar}{r_0} \left( \frac{9\pi}{8} \right)^{\frac{1}{3}} \quad (3.12)$$

In other words, the fermi momentum for this case is a constant, independent of the nucleon number. It follows that

$$E_F = \frac{p_F^2}{2m} = \frac{1}{2m} \left( \frac{\hbar}{r_0} \right)^2 \left( \frac{9\pi}{8} \right)^{\frac{2}{3}} \approx \frac{2.32}{2mc^2} \left( \frac{\hbar c}{r_0} \right)^2$$

$$\approx \frac{2.32}{2 \times 940} \left( \frac{197}{1.2} \right)^2 \text{ MeV} \approx 33 \text{ MeV}. \quad (3.13)$$

Taking the average binding energy per nucleon of about  $-8$  MeV to represent the binding of the last nucleon, it follows from our simple approximation that the depth of the potential well is about 40 MeV, namely,

$$V_0 = E_F + B \approx 40 \text{ MeV}. \quad (3.14)$$

This result is consistent with the value of  $V_0$  obtained through other considerations. The Fermi-gas model has been used to study excited states of complex nuclei, which can be accessed by “raising the temperature” of the nucleon gas (i.e., by adding kinetic energy to the nucleus). The model can also be shown to account in a natural way for the presence of the  $a_4$  term in the Bethe-Weizsäcker mass formula of Eq. (3.5).

### 3.4 Shell Model

The shell model of the nucleus is based on its analog in atomic physics, namely the orbital structure of electrons in complex atoms. The model can account for many crucial nuclear properties, and we will therefore review



several features of atomic structure before discussing the application to the nuclear domain

As we know, the binding of electrons to a nucleus in a complex atom is attributed to the central Coulomb potential. Electron orbits and energy levels for such a quantum system can be obtained by solving the appropriate Schrödinger equation. In general, the solutions are quite complicated because they involve the Coulomb field of the nucleus as well as that of the other electrons, and cannot be obtained in closed analytic form. Nevertheless, certain characteristic features of the motion of an electron in a hydrogen atom have general relevance, and we will discuss these first. For example, the orbits and atomic energy levels that electrons can occupy are labeled by a principal quantum number  $n$  (this determines the eigenvalue of the energy in the case of hydrogen), which can assume only integral values

$$n = 1, 2, 3, \dots \quad (3.15)$$

In addition, for any given value of the principal quantum number, there are energy-degenerate levels with orbital angular momentum given by

$$\ell = 0, 1, 2, \dots, (n - 1). \quad (3.16)$$

For any given orbital angular momentum, there are  $(2\ell + 1)$  sub-states ( $m_\ell$ ) with different projections of the orbital angular momentum along any chosen axis

$$m_\ell = -\ell, -\ell + 1, \dots, 0, 1, \dots, \ell - 1, \ell. \quad (3.17)$$

Due to the rotational symmetry of the Coulomb potential, all such sub-states are degenerate in energy. Furthermore, since electrons have an intrinsic spin angular momentum of  $\frac{\hbar}{2}$ , each of the above states can be occupied by an electron with spin “up” or “down”, corresponding to the spin-projection quantum number

$$m_s = \pm \frac{1}{2}, \quad (3.18)$$

and, again, the energy corresponding to either of these spin configurations will be the same.

Thus, any energy eigenstate in a hydrogen atom is labeled by four quantum numbers, namely  $(n, \ell, m_\ell, m_s)$ . For a given value of  $n$ , it follows that the number of such degenerate energy states is given by

$$\begin{aligned}
 n_d &= 2 \sum_{\ell=0}^{n-1} (2\ell + 1) \\
 &= 2 \left( 2 \sum_{\ell=0}^{n-1} \ell + n \right) \\
 &= 2 \left( 2 \times \frac{1}{2} n(n-1) + n \right) \\
 &= 2(n^2 - n + n) = 2n^2.
 \end{aligned} \tag{3.19}$$

However, all of these states are degenerate only if there is no preferred direction in space that can break the rotational symmetry of the Coulomb interaction. That is, when there is a preferred direction, for example, defined by some magnetic field, then the energy of the system can also depend on the  $m_\ell$  and  $m_s$  quantum numbers. Consequently, an interaction term such as  $-\vec{\mu} \cdot \vec{B}$  added to the Coulomb potential can split the degenerate energy levels. Interactions such as spin-orbit coupling (see Fig. 3.3), between the spin magnetic moment of the electron ( $\vec{\mu} \propto \vec{S}$ ) and the magnetic field ( $\vec{B} \propto \vec{L}$ ) due to the motion of the nucleus (as observed in the electron's rest frame), can change the energies of levels and thereby remove some of the degeneracies. In particular, spin-orbit interactions in atoms lead to a fine structure in the energy levels that has been well-studied. Because the effects of such interactions are usually quite small, they are often neglected in elementary discussions of atomic physics; however, as we shall see, they provide a key element in determining the nature of nuclear structure.

Consequently, ignoring fine structure, we can view the hydrogen atom as consisting of allowed electron orbits corresponding to shells of a given value of  $n$ , with each shell containing degenerate sub-shells specified by the value of the orbital angular momentum. Going beyond hydrogen, and introducing the electron-electron Coulomb interactions, leads to a splitting in any energy level  $n$  according to the  $\ell$ -value of the state. The larger the  $\ell$ , the more aspherical is the orbit, which produces less average binding, and a greater shift up in energy. The degeneracy in  $m_\ell$  and  $m_s$  is not affected greatly, even in more complex atoms. Any shell can still accommodate only

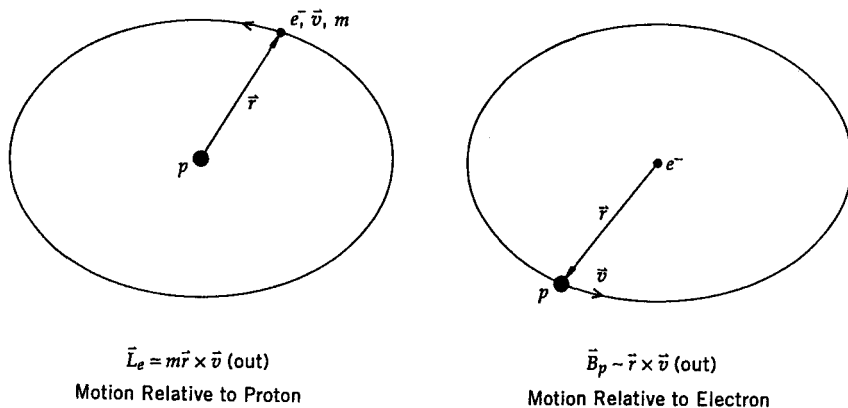


Fig. 3.3 Spin-orbit coupling of the electron and proton in a hydrogen atom. Classically, the orbital motion of an electron is equivalent to a magnetic field due to the circulating proton. Thus a  $\vec{\mu}_e \cdot \vec{B}_p$  term is equivalent to an  $\vec{L} \cdot \vec{S}$  operator for the electron.

$2n^2$  electrons, in consistency with the Pauli principle. It also follows that if a shell or a sub-shell is filled, we have

$$\begin{aligned} \sum m_s &= 0, \\ \sum m_\ell &= 0. \end{aligned} \tag{3.20}$$

In other words, there is a strong pairing effect for closed shells, and from the antisymmetry of the fermionic wave function (see Chapter 9) it can be shown that we get in general

$$\begin{aligned} \vec{L} &= 0 = \vec{S}, \\ \vec{J} &= \vec{L} + \vec{S} = 0. \end{aligned} \tag{3.21}$$

For any atom containing a closed shell or a closed sub-shell, all electrons are paired off and consequently no valence electrons are available. As a result, such atoms will be chemically inert. In fact, if we examine the inert elements, we find that they have just such structure. For example, both electrons in the He ( $Z = 2$ ) atom fill up the shell corresponding to  $n = 1$ . Similarly, Ne ( $Z = 10$ ) has closed shells corresponding to  $n = 1$  and  $n = 2$ . Ar ( $Z = 18$ ) has closed shells corresponding to  $n = 1, 2$ , and closed sub-shells corresponding to  $n = 3, \ell = 0, 1$ . The electrons in Kr ( $Z = 36$ ) fill up

shells corresponding to  $n = 1, 2, 3$ , as well as the sub-shells corresponding to  $n = 4, \ell = 0, 1$ . Finally, Xe ( $Z = 54$ ) has closed shells corresponding to  $n = 1, 2, 3$ , and closed sub-shells corresponding to  $n = 4, \ell = 0, 1, 2$ , as well as  $n = 5, \ell = 0, 1$ . (The energies of the oblong  $n = 4, \ell = 3$  levels lie above the more spherical  $n = 5, \ell = 0, 1$  levels; the latter therefore get filled first.) These inert elements are exceedingly stable. In fact, their ionization energies are particularly large, as is consistent with their greater stability. The above atomic numbers, namely,

$$Z = 2, 10, 18, 36, 54, \quad (3.22)$$

are called the *magic* numbers of atomic physics and correspond to closed-shell structures.

In nuclei, there is also evidence for magic numbers. In fact, although the binding energy per nucleon varies smoothly on a broad scale, a close examination shows peaks corresponding to specific values of nucleon numbers:

$$\begin{aligned} N &= 2, 8, 20, 28, 50, 82, 126, \\ Z &= 2, 8, 20, 28, 50, 82. \end{aligned} \quad (3.23)$$

Nuclei with either proton or neutron number corresponding to any of these magic values appear to be particularly stable, and are referred to as *magic nuclei*. Nuclei where both the proton and the neutron numbers are magic (e.g.,  ${}^4\text{He}^2$ ,  ${}^{16}\text{O}^8$ ,  ${}^{208}\text{Pb}^{82}$ ) are known as *doubly magic*, and have even greater stability.

In addition to stronger binding of magic nuclei, other interesting features also suggest that nuclei possess shell structure. For example, magic nuclei have many more stable isotopes and isotones than their neighbors do. (Isotones are nuclei with the same number of neutrons but a different number of protons.) Thus Sn ( $Z = 50$ ) has ten stable isotopes, whereas In ( $Z = 49$ ) and Sb ( $Z = 51$ ) have only two each. Similarly, for  $N = 20$ , there are five stable isotones, whereas  $N = 19$  has none, and  $N = 21$  has only one,  ${}^{40}\text{K}^{19}$ , which is not very stable (has a mean life of about  $10^9$  years.) Also, we know that a departure from a spherical charge distribution inside a nucleus can give rise to an electric quadrupole moment. Such moments are known to vanish for magic nuclei, whereas neighboring nuclei display large values. Again, this is reminiscent of behavior expected from

shell structure. Similarly, neutron-capture cross sections – measured by scattering neutrons from nuclei of different neutron number – show a sharp drop for magic nuclei relative to their neighbors. This again suggests a shell structure for neutrons within nuclei.

Although there are many suggestive indications for shell structure in nuclei, in trying to set up and solve an appropriate Schrödinger equation, we face two essential differences from the case of atoms. First, there is no apparent central core that can provide the binding potential. Consequently, for the nuclear analog, we must picture the nucleons as moving in some effective mean potential within the nucleus. Second, whereas the well-understood Coulomb potential provides binding in atoms, the exact form of the nuclear potential is unknown. Nevertheless, since we are interested in obtaining shell structure, it is not unreasonable to first assume that the mean potential in which the nucleons move is central. The Schrödinger equation for a central potential  $V(r)$  has the form

$$\left( -\frac{\hbar^2}{2m} \vec{\nabla}^2 + V(r) \right) \psi(\vec{r}) = E\psi(\vec{r}),$$

$$\text{or} \quad \left( \vec{\nabla}^2 + \frac{2m}{\hbar^2} (E - V(r)) \right) \psi(\vec{r}) = 0, \quad (3.24)$$

where  $E$  is the energy eigenvalue. Because we assume that the potential is spherically symmetric, the energy eigenstates will also be eigenstates of the angular momentum operator. (In other words, the system has rotational invariance, as a result of which its angular momentum will be conserved. The angular momentum operator will therefore commute with the Hamiltonian of the system, and will have simultaneous eigenstates.) The energy eigenstates can therefore be labeled by the angular momentum quantum numbers. Under these circumstances it is convenient to use spherical coordinates, in which case we can write

$$\vec{\nabla}^2 = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{1}{\hbar^2 r^2} \vec{L}^2, \quad (3.25)$$

where  $\vec{L}^2$  is the angular momentum operator in coordinate space, whose eigenstates are the spherical harmonics  $Y_{\ell, m_\ell}(\theta, \phi)$ , satisfying

$$\begin{aligned}
\vec{L}^2 Y_{\ell, m_\ell}(\theta, \phi) &= -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] Y_{\ell, m_\ell}(\theta, \phi) \\
&= \hbar^2 \ell(\ell + 1) Y_{\ell, m_\ell}(\theta, \phi), \\
L_z Y_{\ell, m_\ell}(\theta, \phi) &= -i\hbar \frac{\partial}{\partial \phi} Y_{\ell, m_\ell}(\theta, \phi) = \hbar m_\ell Y_{\ell, m_\ell}(\theta, \phi). \tag{3.26}
\end{aligned}$$

Now, writing the Schrödinger wave function in separable form, namely<sup>1</sup>

$$\psi_{n\ell m_\ell}(\vec{r}) = \frac{u_{n\ell}(r)}{r} Y_{\ell, m_\ell}(\theta, \phi), \tag{3.27}$$

where  $n$ ,  $\ell$  and  $m_\ell$  are, respectively, the radial, orbital and projection quantum numbers, and substituting Eq. (3.27) back into Eq. (3.24), we obtain the radial equation

$$\left( \frac{d^2}{dr^2} + \frac{2m}{\hbar^2} \left( E_{n\ell} - V(r) - \frac{\hbar^2 \ell(\ell + 1)}{2mr^2} \right) \right) u_{n\ell}(r) = 0. \tag{3.28}$$

The radial equation has the form of a one-dimensional Schrödinger equation, but with two differences. First, for  $\ell \neq 0$ , there is an additional potential term due to a centrifugal barrier resulting from the orbital motion. Second, the boundary condition for the radial wave function  $u_{n\ell}(r)$  is that it must vanish both at infinite separation as well as at the origin. (This is essential for having a normalizable wave function.) The radial quantum number  $n$  defines directly the number of nodes in the radial solution, and also determines the energies of the states. (Compare this with the hydrogen atom, where the number of nodes in the radial solution is given by  $n - \ell - 1$ .) In the general case  $n$  and  $\ell$  are therefore not correlated, and can take on any integral values.

<sup>1</sup>The symmetry of a wave function, that is, its response to some particular transformation, has important consequences. We will discuss these issues in greater detail in Chapters 10 and 11, when we get to particle physics. Here we only wish to point out that, under inversion of coordinates, namely  $\vec{r} \rightarrow -\vec{r}$ , the length  $r$  does not change,  $\theta \rightarrow \pi - \theta$ , and  $\phi \rightarrow \pi + \phi$ . The net effect of this transformation is that the  $Y_{\ell, m_\ell}(\theta, \phi)$ , and therefore the total wave function, picks up a phase of  $(-1)^\ell$ . This defines the “parity” of a state. Thus when  $\ell$  is even, there is no change in sign of the wave function and the parity of the state is termed even. When the sign changes (for odd  $\ell$ ), the parity of the level is termed odd. Atomic and nuclear states have unique parity – they are either even or odd, but not mixtures of the two. (See the Appendix B for a discussion of the properties of the  $Y_{\ell, m_\ell}(\theta, \phi)$ .)

It is not possible to extract any additional information about the energy levels of a nucleus without assuming some specific form for the potential. Two simple potentials that are used commonly to solve Eq. (3.28) are the infinite square well and the harmonic oscillator. Although these potentials yield exact solutions for the system, they are not realistic because, among other things, they do not provide the possibility of barrier penetration through quantum tunneling. A more realistic potential, such as a finite square well, can yield only numerical solutions and is therefore not very useful for gaining overall insights. Fortunately, the qualitative features of the solutions are not very sensitive to the specific form used for the potential, and so in what follows we will restrict ourselves to the simpler potentials.

### 3.4.1 Infinite Square Well

This potential is defined by

$$V(r) = \begin{cases} \infty & r \geq R, \\ 0 & \text{otherwise,} \end{cases} \quad (3.29)$$

where  $R$  denotes the nuclear radius. The radial equation for  $R \geq r \geq 0$  takes the form

$$\left( \frac{d^2}{dr^2} + \frac{2m}{\hbar^2} \left( E_{n\ell} - \frac{\hbar^2 \ell(\ell+1)}{2mr^2} \right) \right) u_{n\ell}(r) = 0. \quad (3.30)$$

The solutions that are regular at the origin are given by the oscillatory “spherical Bessel” functions (see Appendix C), namely

$$u_{n\ell}(r) = j_\ell(k_{n\ell}r), \quad (3.31)$$

where

$$k_{n\ell} = \sqrt{\frac{2mE_{n\ell}}{\hbar^2}}. \quad (3.32)$$

Since the height of the well is infinite, nucleons cannot escape, and consequently the radial wave function must vanish at the boundary. In other words, we must have

$$u_{n\ell}(R) = j_\ell(k_{n\ell}R) = 0, \quad \ell = 0, 1, 2, 3, \dots, \\ \text{and } n = 1, 2, 3, \dots, \text{ for any } \ell. \quad (3.33)$$

This boundary condition leads to the quantization of energy levels. In fact, the energy eigenvalue corresponding to any  $k_{n\ell}$  is given by the  $n$ th zero of the  $\ell$ th spherical Bessel function. Since the zeros of Bessel functions are all distinct (nondegenerate), it follows that in the present case there is no degeneracy in energy corresponding to different combinations of  $n$  and  $\ell$  values. Rotational invariance, however, still provides a  $(2\ell + 1)$  degeneracy in energy levels that corresponds to different  $m_\ell$  values for a given  $\ell$ . Also, because nucleons have a spin angular momentum of  $\frac{\hbar}{2}$ , as usual, each state can accommodate two neutrons or two protons, in consistency with the Pauli principle. Thus, we conclude that, for the case of an infinite square well, each shell can contain  $2(2\ell + 1)$  protons or neutrons. It now follows that, for  $n = 1$ , closed shells can occur for any of the following proton or neutron bold-faced numbers

$$\mathbf{2}, \mathbf{2 + 6 = 8}, \mathbf{8 + 10 = 18}, \mathbf{18 + 14 = 32}, \mathbf{32 + 18 = 50}, \dots \quad (3.34)$$

It is heartening to see that we can obtain several of the known magic numbers. But, unfortunately, this simple analysis does not yield the desired magic numbers 20, 82, and 126. (We should add that we were somewhat careless in presenting the above results, in that we ignored all but the  $n = 1$  solutions. The specific order in which energy levels are filled depends on the exact values of the zeros of the different Bessel functions. Taking other  $n$  values into account does not greatly affect our overall conclusions, namely, that the infinite square well potential does not reproduce all the nuclear magic numbers.)

### 3.4.2 Harmonic Oscillator

The radial equation for the three-dimensional harmonic-oscillator potential

$$V(r) = \frac{1}{2} m\omega^2 r^2, \quad (3.35)$$

takes the form



$$\left( \frac{d^2}{dr^2} + \frac{2m}{\hbar^2} \left( E_{n\ell} - \frac{1}{2} m\omega^2 r^2 - \frac{\hbar^2 \ell(\ell+1)}{2mr^2} \right) \right) u_{n\ell}(r) = 0. \quad (3.36)$$

The solutions are related to the associated Laguerre polynomials, as follows

$$u_{n\ell}(r) \propto e^{-\frac{m\omega r^2}{2\hbar}} r^{\ell+1} L_{\frac{\ell+1}{2}}^{\frac{n+\ell-1}{2}} \left( \sqrt{\frac{m\omega}{\hbar}} r \right), \quad (3.37)$$

and the energy eigenvalues of the bound states are given by

$$E_{n\ell} = \hbar\omega \left( 2n + \ell - \frac{1}{2} \right), \quad n = 1, 2, 3, \dots,$$

$$\text{and } \ell = 0, 1, 2, \dots, \text{ for any } n. \quad (3.38)$$

By defining a quantum number  $\Lambda$ , this can be rewritten in the more familiar form of the analysis based on Cartesian coordinates, namely, with

$$\Lambda = 2n + \ell - 2, \quad (3.39)$$

we have

$$E_{n\ell} = \hbar\omega \left( \Lambda + \frac{3}{2} \right) \quad \Lambda = 0, 1, 2, \dots, \quad (3.40)$$

where the ground state  $\Lambda = 0$  has the characteristic non-vanishing zero-point energy.

As in the case of the infinite square well, rotational invariance implies a  $(2\ell + 1)$ -fold degeneracy for every value of  $\ell$ , corresponding to different  $m_\ell$  values. However, there is more degeneracy in the energy eigenvalues corresponding to different  $\ell$  and  $n$  combinations that yield the same  $\Lambda$ . In fact, we note from Eq. (3.39) that when  $\Lambda$  is an even integer, then all the states with the following  $(\ell, n)$  values

$$(\ell, n) = \left( 0, \frac{\Lambda+2}{2} \right), \left( 2, \frac{\Lambda}{2} \right), \left( 4, \frac{\Lambda-2}{2} \right), \dots, (\Lambda, 1), \quad (3.41)$$

will be degenerate in energy. Similarly, if  $\Lambda$  is an odd integer, the states with  $(\ell, n)$  values

$$(\ell, n) = \left(1, \frac{\Lambda+1}{2}\right), \left(3, \frac{\Lambda-1}{2}\right), \left(5, \frac{\Lambda-3}{2}\right), \dots, (\Lambda, 1), \quad (3.42)$$

will have the same energy. Thus, the total number of degenerate states corresponding to some even value of  $\Lambda$  is

$$\begin{aligned} n_\Lambda &= \sum_{\ell=0,2,4,\dots}^{\Lambda} 2(2\ell+1) \\ &= \sum_{k=0}^{\frac{\Lambda}{2}} 2(4k+1) \\ &= 2 \left( 4 \times \frac{1}{2} \frac{\Lambda}{2} \left( \frac{\Lambda}{2} + 1 \right) + \left( \frac{\Lambda}{2} + 1 \right) \right) \\ &= 2 \left( \frac{\Lambda}{2} + 1 \right) (\Lambda + 1) = (\Lambda + 1)(\Lambda + 2). \end{aligned} \quad (3.43)$$

Similarly, for a given odd value of  $\Lambda$ , the total number of degenerate states is

$$\begin{aligned} n_\Lambda &= \sum_{\ell=1,3,5,\dots}^{\Lambda} 2(2\ell+1) \\ &= \sum_{k=0}^{\frac{\Lambda-1}{2}} 2(2(2k+1)+1) \\ &= 2 \sum_{k=0}^{\frac{\Lambda-1}{2}} 4(4k+3) \\ &= 2 \left( 4 \times \frac{1}{2} \frac{\Lambda-1}{2} \left( \frac{\Lambda-1}{2} + 1 \right) + 3 \left( \frac{\Lambda-1}{2} + 1 \right) \right) \\ &= 2 \left( \frac{\Lambda+1}{2} \right) (\Lambda - 1 + 3) = (\Lambda + 1)(\Lambda + 2). \end{aligned} \quad (3.44)$$

Thus, we see that, for any value of  $\Lambda$ , the total degeneracy of states is given by

$$n_{\Lambda} = (\Lambda + 1)(\Lambda + 2). \quad (3.45)$$

It now follows that, for the three-dimensional harmonic oscillator potential, closed shells can occur for proton or neutron numbers of 2, 8, 20, 40, 70, etc. Once again, this model predicts some of the magic numbers, but not all of them.

### 3.4.3 Spin-Orbit Potential

It was fairly clear by the 1940s that a central potential could not reproduce all the magic numbers. The crucial breakthrough came in 1949 when Maria Goeppert Mayer and Hans Jensen suggested – once again following the lead from atomic physics – that inside the nucleus, in addition to the central potential, there is a strong spin-orbit interaction, and therefore the total potential sensed by a nucleon has the form

$$V_{\text{TOT}} = V(r) - f(r)\vec{L} \cdot \vec{S}, \quad (3.46)$$

where  $\vec{L}$  and  $\vec{S}$  are the orbital and the spin angular momentum operators for a nucleon, and  $f(r)$  is an arbitrary function of the radial coordinates. In atomic physics, a spin-orbit interaction splits the two degenerate  $j = \ell \pm \frac{1}{2}$  energy levels and produces a fine structure. The spin-orbit interaction in Eq. (3.46) has precisely the same form as in atomic physics, except for the presence of the function  $f(r)$ . Also, the sign of this interaction must be chosen to be consistent with the data, so that the state with  $j = \ell + \frac{1}{2}$  can have a lower energy than the state with  $j = \ell - \frac{1}{2}$ , which is opposite to what happens in atoms.

Now, the total angular momentum operator is given by

$$\vec{J} = \vec{L} + \vec{S}, \quad (3.47)$$

and therefore

$$\begin{aligned} \vec{J}^2 &= \vec{L}^2 + \vec{S}^2 + 2\vec{L} \cdot \vec{S}, \\ \text{or } \vec{L} \cdot \vec{S} &= \frac{1}{2} (\vec{J}^2 - \vec{L}^2 - \vec{S}^2) \end{aligned} \quad (3.48)$$

where we have used the fact that orbital and spin angular momentum operators commute, and therefore their order in a product does not matter. Thus, in a state with definite  $\ell$ ,  $s$ , and  $j$  values (that is, a quantum state can be labeled either by the eigenvalues  $\ell$ ,  $m_\ell$ ,  $s$ ,  $m_s$  or  $\ell$ ,  $s$ ,  $j$ ,  $m_j$ , and it is the second basis that is appropriate for our calculation), we have

$$\begin{aligned}
 \langle \vec{L} \cdot \vec{S} \rangle &= \langle \frac{1}{2} (\vec{J}^2 - \vec{L}^2 - \vec{S}^2) \rangle \\
 &= \frac{\hbar^2}{2} [j(j+1) - \ell(\ell+1) - s(s+1)] \\
 &= \frac{\hbar^2}{2} \left[ j(j+1) - \ell(\ell+1) - \frac{3}{4} \right] \\
 &= \begin{cases} \frac{\hbar^2}{2} \ell & \text{for } j = \ell + \frac{1}{2}, \\ -\frac{\hbar^2}{2} (\ell + 1) & \text{for } j = \ell - \frac{1}{2}, \end{cases} \quad (3.49)
 \end{aligned}$$

where we have substituted  $s = \frac{1}{2}$  for the spin of a nucleon.

The shifts in the energies from their degenerate central values can be written as

$$\begin{aligned}
 \Delta E_{n\ell} \left( j = \ell + \frac{1}{2} \right) &= -\frac{\hbar^2 \ell}{2} \int d^3r |\psi_{n\ell}(\vec{r})|^2 f(r), \\
 \Delta E_{n\ell} \left( j = \ell - \frac{1}{2} \right) &= \frac{\hbar^2 (\ell + 1)}{2} \int d^3r |\psi_{n\ell}(\vec{r})|^2 f(r), \quad (3.50)
 \end{aligned}$$

so that the total splitting between the two levels becomes

$$\begin{aligned}
 \Delta &= \Delta E_{n\ell} \left( j = \ell - \frac{1}{2} \right) - \Delta E_{n\ell} \left( j = \ell + \frac{1}{2} \right) \\
 &= \hbar^2 \left( \ell + \frac{1}{2} \right) \int d^3r |\psi_{n\ell}(\vec{r})|^2 f(r). \quad (3.51)
 \end{aligned}$$

We see that the splitting due to the spin-orbit interaction is larger for higher values of orbital angular momentum, and can consequently produce level crossing. Namely, for large  $\ell$ , the splitting of any two neighboring degenerate levels can shift the  $j = \ell - \frac{1}{2}$  state of the initially lower level to lie above the  $j = \ell + \frac{1}{2}$  state of the previously higher level. Thus, as shown in Fig. 3.4, for an appropriately chosen  $f(r)$ , the energy levels for

a finite square well can split upon the addition of a spin-orbit interaction. And we can therefore conclude that by including a spin-orbit interaction we can reproduce all the desired magic numbers, and thereby accommodate a shell-like structure in nuclei. The energy level diagram of Fig. 3.4 is labeled according to the spectroscopic notation of atomic physics, namely as  $(nL_j)$ . The multiplicity of any final level is given, as usual, by  $(2j + 1)$ . We have not shown the levels beyond  $1G_{7/2}$ ; these are  $2D_{5/2}$ ,  $2D_{3/2}$ ,  $3S_{1/2}$ ,  $1H_{11/2}$ , and so forth.

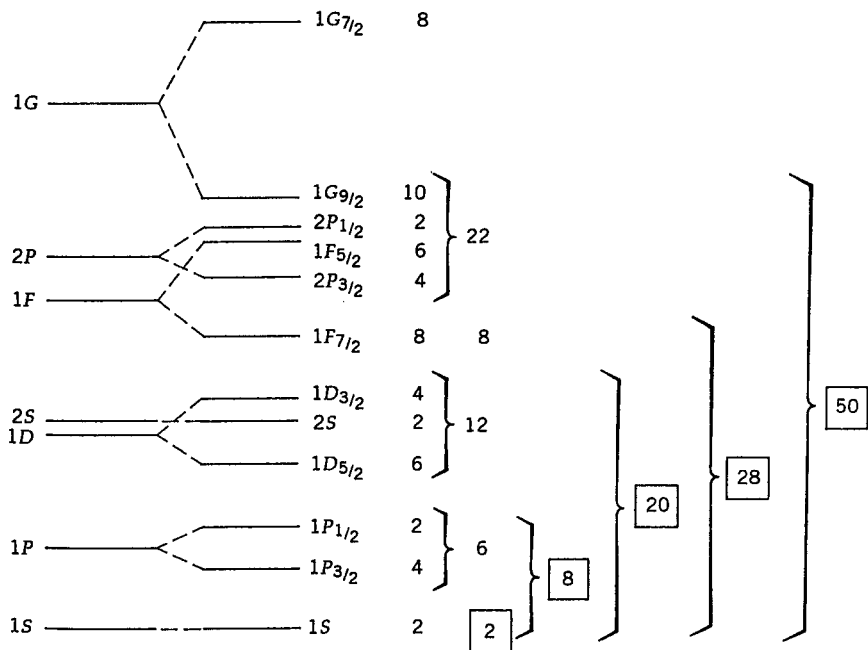


Fig. 3.4 Energy levels in a single-particle shell model. The boxed integers correspond to the magic nuclear numbers.

It is worth pointing out that in our discussion of the energy spectrum, we have treated protons and neutrons on an equal footing. It is clear, however, that the effect of the Coulomb potential must shift the energy levels for protons to somewhat higher values. Upon applying such corrections, it is found that the qualitative features of the spectrum remain essentially unchanged.

### 3.4.4 Predictions of the Shell Model

The shell model accounts for a wide variety of properties of complex nuclei. For example, it provides the correct spin-parity assignment for the ground states of a large number of odd- $A$  nuclei. According to the model, the proton and the neutron levels fill up independently and, due to the Pauli exclusion principle, only two neutrons or two protons can occupy any given level (with their intrinsic spins anti-parallel). If we assume that nucleons pair off in every filled level, yielding zero total angular momentum, then the last unpaired nucleon must determine the spin-parity of the ground state. An immediate consequence of this picture is that the ground states of all even-even nuclei must have zero spin, which is experimentally correct. The single-particle shell model cannot predict the ground-state spins of odd-odd nuclei because there is no a priori constraint on how unpaired protons and neutrons should couple.

Let us next examine the spin-parity assignments of several odd- $A$  nuclei in greater detail. Consider the isobars  $^{13}\text{C}^6$  and  $^{13}\text{N}^7$ . (Note that these are, in fact, mirror nuclei.) The six protons in  $^{12}\text{C}$  and the six neutrons in  $^{13}\text{N}$  should be completely paired off, while the remaining seven nucleons in both cases should fill the following shells

$$\left(1S_{\frac{1}{2}}\right)^2 \left(1P_{\frac{3}{2}}\right)^4 \left(1P_{\frac{1}{2}}\right)^1. \quad (3.52)$$

Thus, the last unpaired nucleon – a neutron for  $^{13}\text{C}^6$  and a proton for  $^{13}\text{N}^7$  – has total angular momentum  $j = \frac{1}{2}$  and orbital angular momentum  $\ell = 1$ . (Recall, from our previous comments in the footnote pertaining to Eq. (3.27), that  $\ell = 1$  corresponds to a state of odd parity.) Hence, according to the shell model, the spin-parity of the ground state for these nuclei is expected to be  $\left(\frac{1}{2}\right)^-$ , which is, in fact, the observed value. Similarly, for the isobars  $^{17}\text{O}^8$  and  $^{17}\text{F}^9$ , the nine neutrons for  $^{17}\text{O}^8$  and nine protons for  $^{17}\text{F}^9$  will fill the following levels

$$\left(1S_{\frac{1}{2}}\right)^2 \left(1P_{\frac{3}{2}}\right)^4 \left(1P_{\frac{1}{2}}\right)^2 \left(1D_{\frac{5}{2}}\right)^1. \quad (3.53)$$

The total angular momentum of the last unpaired nucleon in the  $\ell = 2$  state is  $\frac{5}{2}$ . Thus, the spin-parity of these nuclei is expected to be  $\left(\frac{5}{2}\right)^+$ , which is again consistent with experiment.

For  $^{33}\text{S}^{16}$ , the measured value of the ground state spin-parity is  $\left(\frac{3}{2}\right)^+$ .

According to the shell model, the seventeen neutrons will fill up the levels as follows

$$\left(1S_{\frac{1}{2}}\right)^2 \left(1P_{\frac{3}{2}}\right)^4 \left(1P_{\frac{1}{2}}\right)^2 \left(1D_{\frac{5}{2}}\right)^6 \left(2S_{\frac{1}{2}}\right)^2 \left(1D_{\frac{3}{2}}\right)^1, \quad (3.54)$$

once again leading to a prediction consistent with experiment. However, certain spin-parity assignments of the shell model do not agree with observation. For example, the neutrons in  $^{47}\text{Ti}^{22}$  would be expected to fill the levels as

$$\left(1S_{\frac{1}{2}}\right)^2 \left(1P_{\frac{3}{2}}\right)^4 \left(1P_{\frac{1}{2}}\right)^2 \left(1D_{\frac{5}{2}}\right)^6 \left(2S_{\frac{1}{2}}\right)^2 \left(1D_{\frac{3}{2}}\right)^4 \left(1F_{\frac{7}{2}}\right)^5, \quad (3.55)$$

leading to a ground state spin-parity of  $\left(\frac{7}{2}\right)^-$ , whereas the experimental value is  $\left(\frac{5}{2}\right)^-$ . Such discrepancies can be remedied by slightly modifying the assumptions of the single-particle shell model to allow pairing between all "valence" nucleons, namely between any nucleons that occupy unfilled levels.

The shell model can also be used to calculate magnetic moments of nuclei. As measurements show, the proton and the neutron have intrinsic dipole moments of  $2.79 \mu_N$  and  $-1.91 \mu_N$ , respectively. Thus, we expect the intrinsic magnetic moment of any unpaired nucleon to contribute to the total magnetic moment of the nucleus. In addition, since protons are charged, the orbital motion of any unpaired proton can also contribute to the magnetic moment of the nucleus. For the deuteron, for example, if we assume that the proton and the neutron are in  $1S_{\frac{1}{2}}$  states, then, without orbital angular momentum for the proton ( $\ell = 0$ ), we expect the magnetic moment of the deuteron to be the sum of the intrinsic dipole moments of the proton and the neutron

$$\mu_d = 2.79 \mu_N - 1.91 \mu_N = 0.88 \mu_N. \quad (3.56)$$

The observed magnetic moment of the deuteron is  $0.86 \mu_N$  - in good agreement with expectation. The nucleus of tritium ( $^3\text{H}^1$ ) has two neutrons and one proton, all in the  $1S_{\frac{1}{2}}$  state. Since the neutrons are paired, they should not contribute to the magnetic moment. The unpaired proton, having  $\ell = 0$ , will have no contribution from its orbital motion. Consequently, the total magnetic moment of  $^3\text{H}^1$  should be the same as that of the un-

paired proton, namely  $2.79 \mu_N$ , which is in good agreement with measured value of  $2.98 \mu_N$ . For  ${}^3\text{He}^2$ , the unpaired nucleon is a neutron in a  $1S_{\frac{1}{2}}$  state. Consequently, the total magnetic moment should be the same as that of the neutron, which is  $-1.91 \mu_N$ , again, close to the observed value of  $-2.13 \mu_N$ .  ${}^4\text{He}^2$  ( $\alpha$ -particle) has a closed shell structure (in fact, it is doubly magic), and the shell model would therefore predict no spin and no magnetic moment, which is indeed experimentally correct. In  ${}^{10}\text{B}^5$ , the five protons and the five neutrons have the same level structure, namely,

$$\left(1S_{\frac{1}{2}}\right)^2 \left(1P_{\frac{3}{2}}\right)^3. \quad (3.57)$$

Thus, there is one unpaired proton and one unpaired neutron. The unpaired proton will be in an  $\ell = 1$  state, and therefore the orbital motion will contribute  $\mu = \frac{e\hbar}{2m_N c} \ell = \mu_N$  to the total magnetic moment, which will yield a value

$$2.79 \mu_N - 1.91 \mu_N + \mu_N = 1.88 \mu_N. \quad (3.58)$$

This compares quite well with the measured value of  $1.80 \mu_N$ .

We see therefore that the shell model, in addition to providing the known magic numbers, also describes other important properties of light nuclei. For heavy nuclei, however, there is marked difference between the predictions of the shell model and the measured quantities.

### 3.5 Collective Model

For heavy nuclei, many predictions of the single-particle shell model do not agree quantitatively with experiment. The discrepancies are particularly severe for magnetic dipole moments. Also, the shell model predicts vanishingly small quadrupole moments for closed shells, and quadrupole moments of opposite sign for neighboring nuclei with atomic numbers  $Z \pm 1$ . Although this agrees qualitatively with experiment, the measured values of quadrupole moments are very different from the predictions. In fact, some heavy nuclei appear to have large permanent electric quadrupole moments, suggesting a nonsphericity in the shape of these nuclei. This is certainly not consistent with the assumptions of the shell model, where rotational



symmetry plays a crucial role.<sup>2</sup>

In a revival of the liquid drop model, Aage Bohr noted that many properties of heavy nuclei could be attributed to a surface motion of the nuclear liquid drop. Furthermore, James Rainwater showed that excellent agreement between the expected and measured values of magnetic dipole and electric quadrupole moments could be obtained under the assumption that the liquid drop had an aspherical shape. These successes presented somewhat of a dilemma because the liquid drop model and the single-particle shell model had fundamentally opposite viewpoints about the nature of nuclear structure. Individual particle characteristics, such as intrinsic spin and orbital angular momentum, play no role in a liquid drop picture, where collective motion that involves the entire nucleus has prime importance. On the other hand, individual nucleon properties, especially of the valence nucleons, are crucial to the success of the independent-particle shell model. The shell model had yielded too many important nuclear features to be abandoned outright, and a reconciliation between the two extreme views was needed.

The reconciliation was brought about by Aage Bohr, Ben Mottelson and James Rainwater who proposed a collective model for the nucleus that provided many features that were not present in either the shell or the liquid drop model. In what follows, we describe this model only qualitatively. Its basic assumption is that a nucleus consists of a hard core of nucleons in the filled shells, and outer valence nucleons that behave like the surface molecules in a liquid drop. The surface motion (rotation) of the valence nucleons introduces a nonsphericity in the central core, which in turn affects the quantum states of the valence nucleons. In other words, one can think of the surface motion as a perturbation that causes the quantum states of the valence nucleons to change from the unperturbed states of the shell model. This adjustment accounts for the difference in predictions for dipole and quadrupole moments from those given by the shell model.

Physically, one can view the collective model as a shell model with a potential that is not spherically symmetric. Spherically symmetric nuclei are, of course, insensitive to rotations, and consequently rotational motion cannot produce additional (rotational) energy levels in such nuclei. Aspherical nuclei, on the other hand, can have additional energy levels because of the presence of rotational and vibrational degrees of freedom. These types of

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<sup>2</sup>Finite quadrupole moments of charge distributions arise when the second moments  $\langle x^2 \rangle$ ,  $\langle y^2 \rangle$  and  $\langle z^2 \rangle$  differ from each other, namely when the distribution of charge is not spherical.

effects modify the predictions of the simple shell model. In particular, large nonsphericity in nuclei can provide large permanent dipole and quadrupole moments. Mathematically, these ideas can be incorporated as follows. For simplicity, we assume the nucleus to be an ellipsoid defined by the form

$$ax^2 + by^2 + \frac{z^2}{ab} = R^2, \quad (3.59)$$

where  $a$  and  $b$  are parameters related to the deformation from a spherical shape of radius  $R$ . The mean potential for nuclear motion can then be chosen as

$$V(x, y, z) = \begin{cases} 0 & \text{for } ax^2 + by^2 + \frac{z^2}{ab} \leq R^2, \\ \infty & \text{otherwise.} \end{cases} \quad (3.60)$$

Needless to say, more realistic calculations in the collective model provide even better descriptions of nuclear properties, but they also become far more complicated.

One of the important predictions of the collective model is the existence of rotational and vibrational levels in a nucleus. These levels can be derived much the same way as is done for the case of molecules. Thus, we can choose the Hamiltonian for rotations to be

$$H = \frac{\vec{L}^2}{2I}, \quad (3.61)$$

with eigenvalues  $\frac{\ell(\ell+1)}{2I} \hbar^2$ , where the effective moment of inertia  $I$  is a function of the nuclear shape. If there is rotation about an axis perpendicular to the symmetry axis of the ellipsoid, it can then be shown that the angular momentum of the rotational levels can only be even. Thus, we see that rotational and vibrational levels in a nucleus are predicted with specific values of angular momentum and parity. Such excitations have indeed been found through the observation of photon quadrupole transitions ( $\Delta\ell = 2$ ) between levels.

Finally, the collective model accommodates quite naturally the decrease, with increasing  $A$ , of the spacing between the first excited state and the ground level in even-even nuclei, as well as the fact that the spacing is largest for nuclei with closed shells. The first follows simply because the moment of inertia grows with  $A$ , which decreases the energy eigenvalue

of the first excited rotational state. The latter is due to the fact that a nucleus with a closed shell should not have a rotational level because such a nucleus would tend to be spherical. On the other hand, such a nucleus can have vibrational excitations. However, vibrational excitations involve the entire core and not just the surface. The core being much more massive, implies that the energy level for vibration will lie far higher, and the spacing between the ground state and the first excited state will be much greater.

### 3.6 Superdeformed Nuclei

Throughout our discussion of nuclear phenomena we have emphasized that nuclei tend to have relatively small intrinsic spins. We can imagine that under certain circumstances nuclei could be greatly deformed and yet not fission (see Chapter 5). In fact, particularly stable superdeformed nuclei have been predicted to exist for values of  $A$  between 150 and 190. Such nuclei were expected to be spheroidal in character, with semi-major and semi-minor axes differing by about a factor of two. During the late 1980s, a series of experiments was carried out on the scattering of heavy ions on heavy ions. When such collisions take place, superdeformed nuclei are produced with remarkably large angular momenta of about  $60 \hbar$ . These nuclei de-excite through a series of (quadrupole) emissions of  $\sim 50$  keV  $\gamma$ -rays down to lower levels that have more symmetric nuclear shapes. However, because the observed level spacings (photon energies) remain essentially fixed, this poses a problem from the point of view of the collective model, where we would expect the moment of inertia to decrease with a decrease in deformation. In fact, different nuclei appear to have essentially identical emissions as they “spin down”. This is an even greater puzzle, because of the known effects of nucleon pairing on binding energy and level spacing. This is currently a very active area of study in nuclear physics, that may yet offer additional surprises.

## Problems

**3.1** The Bethe-Weizsäcker formula of Eq. (3.5) provides an excellent representation of the mass systematics of nuclei. Show explicitly that, for fixed  $A$ ,  $M(A, Z)$  has a minimum value. Is there evidence for the “valley of stability” observed in Fig. 2.3? What is the stablest nucleus with  $A = 16$ ? What about  $A = 208$ ? (You can differentiate Eq. (3.5), or simply plot  $M$

as a function of  $Z$ .)

**3.2** Using Eq. (3.3) compute the total binding energy and the value of  $\frac{B}{A}$  for  ${}^8\text{Be}^4$ ,  ${}^{12}\text{C}^6$ ,  ${}^{56}\text{Fe}^{26}$  and  ${}^{208}\text{Pb}^{82}$ . How do these values compare with experiment? (See *CRC Handbook of Chemistry and Physics* for data.)

**3.3** You might conclude from Problem 3.2 that  ${}^8\text{Be}^4$  is stable. This is, in fact, not the case. Can you provide a model to explain this result? (Hint: see Problem 2.2.)

**3.4** Calculate the binding energy of the last neutron in  ${}^{15}\text{N}^7$  and of the last proton in  ${}^{15}\text{O}^8$ , and contrast with the last neutron in  ${}^{16}\text{N}^7$  and in  ${}^{16}\text{O}^8$ .

**3.5** What would you expect for the spin and parity of the ground states of  ${}^{23}\text{Na}^{11}$ ,  ${}^{35}\text{Cl}^{17}$  and  ${}^{41}\text{Ca}^{20}$  on the basis of the single-particle shell model? Do these predictions agree with experimental values? What about the magnetic moments of these nuclei? (See *CRC Handbook* for data.)

**3.6** Consider a somewhat more sophisticated model for the anomalous contribution to the magnetic moment of a nucleon. Assume that the proton can be regarded as a fixed neutral center with a  $\pi^+$  meson circling about in an  $\ell = 1$  orbit. Similarly, take a neutron as an effective proton center with a  $\pi^-$  meson in an  $\ell = 1$  orbit around it. Using  $m_\pi = 140 \text{ MeV}/c^2$ , calculate  $\mu = \left(\frac{e\hbar}{2m_\pi c}\right)\ell$ , and compare results with those of Problem 2.5.

**3.7** The ground state of  ${}^{137}\text{Ba}^{56}$  has spin-parity  $\frac{3}{2}^+$ . That is, its spin is  $\frac{3}{2}$  and parity  $+$ . The first two excited states have spin parity  $\frac{1}{2}^+$  and  $\frac{11}{2}^-$ . According to the shell model, what assignments would be expected for these excited states? (Hint: The surprise has to do with “pairing energy”.)

### Suggested Reading

Frauenfelder, H., and E. M. Henley, *Subatomic Physics*, Prentice-Hall (1991).

Krane, K. S., *Introductory Nuclear Physics*, Wiley (1987).

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Williams, W. S. C., *Nuclear and Particle Physics*, Oxford Univ. Press (1997).



## Chapter 4

# Nuclear Radiation

### 4.1 Introductory Remarks

In previous chapters we indicated that many nuclei are unstable and often emit  $\alpha$ ,  $\beta$  or  $\gamma$  particles. We will now discuss several more quantitative aspects of nuclear radioactivity and its historical impact on our understanding of nuclear structure and nuclear transmutation.

### 4.2 Alpha Decay

As we have seen before,  $\alpha$ -decay represents the disintegration of a parent nucleus to a daughter through the emission of the nucleus of a helium atom, and the transition can be characterized as



As we will see in Chapter 5,  $\alpha$ -decay can be regarded as the spontaneous fission of the parent nucleus into two daughter nuclei with highly asymmetric masses. If we assume that the parent nucleus is initially at rest, then conservation of energy requires

$$M_P c^2 = M_D c^2 + T_D + M_\alpha c^2 + T_\alpha, \quad (4.2)$$

where  $M_P$ ,  $M_D$  and  $M_\alpha$  are the masses of the parent, daughter and the  $\alpha$ -particle, respectively. Similarly,  $T_D$  and  $T_\alpha$  represent the kinetic energies of the daughter and of the  $\alpha$ -particle. Equation (4.2) can also be rewritten as

$$T_D + T_\alpha = (M_P - M_D - M_\alpha) c^2 = \Delta M c^2. \quad (4.3)$$

Although the right hand side of Eq. (4.3) involves nuclear masses, we can, in fact, use atomic masses in the expression since the masses of the electrons cancel. Thus, we can write

$$T_D + T_\alpha = (M(A, Z) - M(A - 4, Z - 2) - M(4, 2)) c^2 \equiv Q, \quad (4.4)$$

where we have defined the disintegration energy or  $Q$ -value as the difference in the rest masses of the initial and final states. It is clear that  $Q$  also equals the sum of the kinetic energies of the final state particles. For non-relativistic particles, the kinetic energies can be written as

$$\begin{aligned} T_D &= \frac{1}{2} M_D v_D^2, \\ T_\alpha &= \frac{1}{2} M_\alpha v_\alpha^2, \end{aligned} \quad (4.5)$$

with  $v_D$  and  $v_\alpha$  representing the magnitude of the velocities of the daughter and of the  $\alpha$ -particle.

Since the parent nucleus decays from rest, the daughter nucleus and the  $\alpha$ -particle must necessarily move in opposite directions to conserve momentum, satisfying

$$M_D v_D = M_\alpha v_\alpha,$$

$$\text{or } v_D = \frac{M_\alpha}{M_D} v_\alpha. \quad (4.6)$$

When the mass of the daughter nucleus is much greater than that of the  $\alpha$ -particle, then  $v_D \ll v_\alpha$ , and consequently the kinetic energy of the daughter nucleus is far smaller than that of the  $\alpha$ -particle.

Let us eliminate  $v_D$  and write expressions for  $T_D$  and  $T_\alpha$  in terms of the  $Q$ -value

$$\begin{aligned}
T_D + T_\alpha &= \frac{1}{2} M_D v_D^2 + \frac{1}{2} M_\alpha v_\alpha^2 \\
&= \frac{1}{2} M_D \left( \frac{M_\alpha}{M_D} v_\alpha \right)^2 + \frac{1}{2} M_\alpha v_\alpha^2 \\
&= \frac{1}{2} M_\alpha v_\alpha^2 \left( \frac{M_\alpha}{M_D} + 1 \right), \\
\text{or } T_D + T_\alpha &= T_\alpha \frac{M_\alpha + M_D}{M_D}. \tag{4.7}
\end{aligned}$$

Using Eq. (4.4), this can be rewritten as

$$T_\alpha = \frac{M_D}{M_\alpha + M_D} Q = \frac{1}{1 + \frac{M_\alpha}{M_D}} Q. \tag{4.8}$$

The kinetic energy of the emitted  $\alpha$ -particle cannot be negative, that is,  $T_\alpha \geq 0$ . Consequently, for  $\alpha$ -decay to occur, we must have an exothermic process

$$\Delta M \geq 0, \quad Q \geq 0. \tag{4.9}$$

For massive nuclei, which is our main interest, most of the energy is carried off by the  $\alpha$ -particle. The kinetic energy of the daughter nucleus is obtained from Eqs. (4.4) and (4.8)

$$T_D = Q - T_\alpha = \frac{M_\alpha}{M_\alpha + M_D} Q = \frac{M_\alpha}{M_D} T_\alpha \ll T_\alpha. \tag{4.10}$$

If we use the approximation  $\frac{M_\alpha}{M_D} \simeq \frac{4}{A-4}$ , we can then write

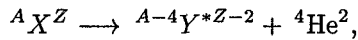
$$\begin{aligned}
T_\alpha &\approx \frac{A-4}{A} Q, \\
T_D &\approx \frac{4}{A} Q, \tag{4.11}
\end{aligned}$$

which can be used to estimate the energy released in the decay.

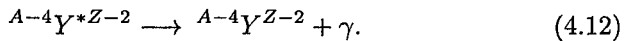
We note from Eq. (4.8) that the kinetic energy (and therefore the magnitude of the velocity) of the  $\alpha$ -particle in the decay is unique, which is consistent with our earlier discussion. This is a direct consequence of the



fact that the process is a two body decay of a parent initially at rest. Careful measurements, however, have revealed a fine splitting in the energies of  $\alpha$ -particles emitted from any radioactive material, corresponding to possibly different  $Q$  values. The most energetic  $\alpha$ -particles are observed to be produced alone, but less energetic  $\alpha$ -decays are always accompanied by the emission of photons. This suggests the presence of energy levels and of an underlying quantum structure of discrete states in nuclei. If this is correct, then a parent nucleus can transform to the ground state of the daughter nucleus by emitting an  $\alpha$ -particle with energy corresponding to the entire  $Q$  value, or it can decay to an excited state of the daughter nucleus, in which case the effective  $Q$  value is lower. And, as in the case of atomic transitions, the daughter nucleus can subsequently de-excite to its ground state by emitting a photon. Hence, the decay chain would involve



with



The difference in the two  $Q$  values would then correspond to the energy of the emitted photon. For example, the spectrum of observed  $\alpha$ -particle energies in the decay of  ${}^{228}\text{Th}$  to  ${}^{224}\text{Ra}$  can be associated schematically with the level structure shown in Fig. 4.1.

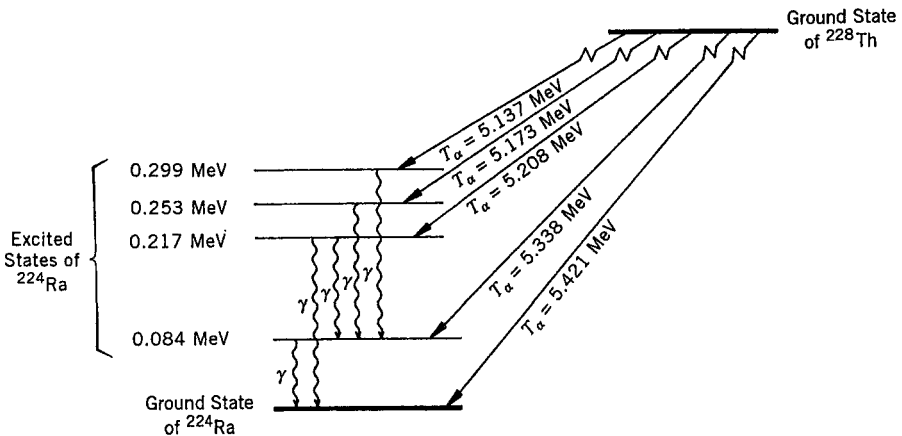
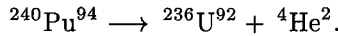


Fig. 4.1  $\alpha$ -particle transitions observed in the decay of  ${}^{228}\text{Th}$ .

The underlying level structure shown in Fig. 4.1 can be determined by measuring the kinetic energies of the different  $\alpha$ -particles observed in these decays, which in turn yield the  $Q$  values for the transitions through Eq. (4.8). Based on the assumption of discrete nuclear levels, the difference in the  $Q$  values will then yield the expected energies of the emitted photons. The measured energies of such accompanying (coincident) photons have, in fact, confirmed the overall picture and therefore the existence of discrete nuclear levels.

### Example 1

Consider the  $\alpha$ -decay of  $^{240}\text{Pu}^{94}$



The emitted  $\alpha$ -particles are observed to have energies of 5.17 MeV and 5.12 MeV. Substituting these two values into the first relation in Eq. (4.11)

$$Q \approx \frac{A}{A-4} T_\alpha,$$

we obtain the two  $Q$  values

$$Q_1 \approx \frac{240}{236} \times 5.17 \text{ MeV} \approx 1.017 \times 5.17 \text{ MeV} \approx 5.26 \text{ MeV},$$

$$Q_2 \approx \frac{240}{236} \times 5.12 \text{ MeV} \approx 1.017 \times 5.12 \text{ MeV} \approx 5.21 \text{ MeV}.$$

Thus, when  $^{240}\text{Pu}$  decays with disintegration energy  $Q_2 \approx 5.21 \text{ MeV}$ , the daughter nucleus  $^{236}\text{U}^{92}$  is left in an excited state and transforms to the ground state by emitting a photon of energy

$$Q_1 - Q_2 \approx 5.26 \text{ MeV} - 5.21 \text{ MeV} = 0.05 \text{ MeV}.$$

This is, indeed, consistent with the observed energy of 0.045 MeV for the photon. Thus, we can conclude from such studies of  $\alpha$ -decays that there are discrete energy levels in nuclei, very much like those found in atoms, and that the spacing between nuclear levels is about 100 keV, whereas the corresponding spacing in atomic levels is of the order of 1 eV.

### 4.3 Barrier Penetration

The  $\alpha$ -particles emitted in nuclear decay have typical energies of about 5 MeV. When such low-energy particles are scattered from a heavy nucleus they cannot penetrate the Coulomb barrier and get sufficiently close to the nucleus to interact through the strong force. The height of the Coulomb barrier for  $A \approx 200$  is about 20–25 MeV, and a 5 MeV  $\alpha$ -particle therefore cannot overcome this barrier to get absorbed into the center. On the other hand, a low-energy  $\alpha$ -particle that is bound in a nuclear potential well sees that same barrier, and yet is able to escape. How this could happen constituted a great puzzle, until it was recognized that the emission of  $\alpha$ -particles was a quantum-mechanical phenomenon.

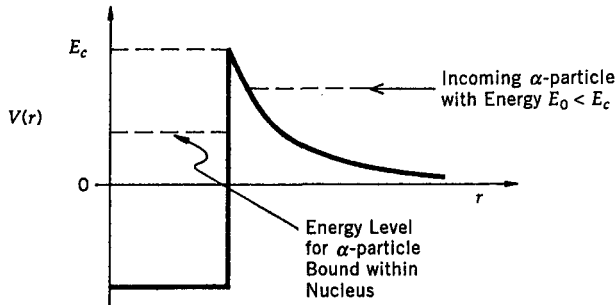
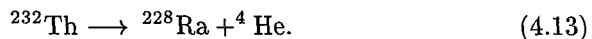


Fig. 4.2 Potential energy function for an  $\alpha$ -particle interacting with a nucleus.

The first quantitative understanding of  $\alpha$ -decay came in 1929 from the work of George Gamow and of Ronald Gurney and Edward Condon. Assuming that the  $\alpha$ -particle and the daughter nucleus exist within the parent nucleus prior to its dissociation, we can treat the problem as an  $\alpha$ -particle moving in the potential of the daughter nucleus, with the Coulomb potential preventing their separation (see Fig. 4.2). For concreteness, consider the decay



The kinetic energy of the emitted  $\alpha$ -particle is observed to be  $E = 4.05$  MeV, and the lifetime of  ${}^{232}\text{Th}$  is  $\tau = 1.39 \times 10^{10}$  years. The radius of the thorium nucleus obtained from the formula  $R = 1.2 \times 10^{-13} A^{\frac{1}{3}}$  cm is  $\approx 7.4 \times 10^{-13}$  cm.

The  $\alpha$ -particle must penetrate the Coulomb barrier in order for the decay to take place. The calculation of barrier penetration for a three-dimensional Coulomb potential is rather complicated. However, since we are interested only in order-of-magnitude estimates, we will ignore the angular dependence of the Schrödinger equation and consider the potential as effectively one-dimensional. Furthermore, we will replace the Coulomb potential by a square barrier of equal area, which approximates the effect of the Coulomb repulsion, and is computationally much simpler (see Fig. 4.3). As long as  $V_0$  is chosen so that it is larger than  $E$ , then the transmission through the barrier is sensitive primarily to the product of  $\sqrt{V_0 - E}$  and  $a$ , and not to the precise value of  $V_0$ . For  $Z \approx 90$ , we can choose

$$\begin{aligned} V_0 &= 14 \text{ MeV}, \\ 2a &= 33 \text{ fm} = 33 \times 10^{-13} \text{ cm}. \end{aligned} \quad (4.14)$$

A straightforward quantum-mechanical treatment of the transmission through the square barrier shown in Fig. 4.3, yields the following transmission coefficient

$$T = \frac{\frac{4k_1 k}{(k_1 + k)^2}}{1 + \left[ 1 + \left( \frac{\kappa^2 - k_1 k}{\kappa(k_1 + k)} \right)^2 \right] \sinh^2 2\kappa a}, \quad (4.15)$$

with

$$\begin{aligned} k_1 &= \left[ \frac{2M_\alpha}{\hbar^2} (E + U_0) \right]^{\frac{1}{2}}, \\ k &= \left[ \frac{2M_\alpha}{\hbar^2} E \right]^{\frac{1}{2}}, \\ \kappa &= \left[ \frac{2M_\alpha}{\hbar^2} (V_0 - E) \right]^{\frac{1}{2}}, \end{aligned} \quad (4.16)$$

where  $M_\alpha$  is the rest mass and  $E$  is the kinetic energy of the emitted  $\alpha$  particle (outside of the barrier). For  $M_\alpha c^2 \approx 4000 \text{ MeV}$ ,  $E = 4.05 \text{ MeV}$ ,  $V_0 = 14 \text{ MeV}$  and  $U_0 \approx 40 \text{ MeV}$  (the calculation is not very sensitive to the depth of the nuclear potential), we have

$$\begin{aligned}
 \kappa &= \frac{1}{\hbar c} [2M_\alpha c^2 (V_0 - E)]^{\frac{1}{2}} \\
 &\approx \frac{1}{197 \text{ MeV-fm}} [2 \times 4000 \text{ MeV} (14 - 4) \text{ MeV}]^{\frac{1}{2}} \sim 1.4 \text{ fm}^{-1}, \\
 k &\approx 0.9 \text{ fm}^{-1}, \\
 k_1 &\approx 3.0 \text{ fm}^{-1}.
 \end{aligned} \tag{4.17}$$

Now,  $2\kappa a \approx 33 \text{ fm} \times 1.4 \text{ fm}^{-1} \approx 46$ , which means that  $2\kappa a \gg 1$ , and allows us to write

$$\sinh^2 2\kappa a \approx \left( \frac{e^{2\kappa a}}{2} \right)^2 = \frac{1}{4} e^{4\kappa a} \approx \frac{1}{4} e^{92} \gg 1. \tag{4.18}$$

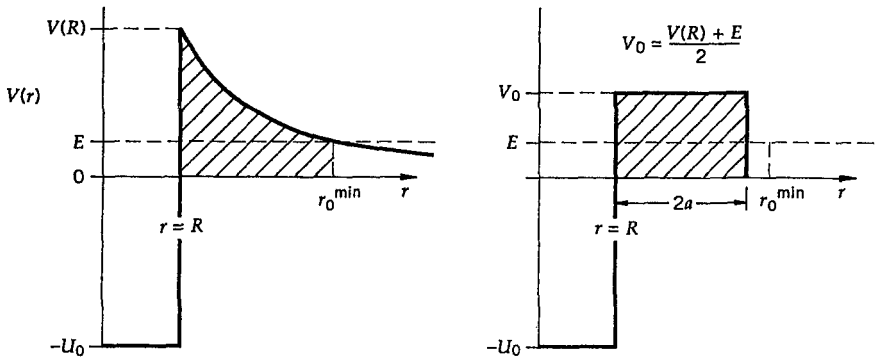


Fig. 4.3 Potential energy for scattering of 4 MeV  $\alpha$ -particle from  $^{228}\text{Ra}$ , and the equivalent one-dimensional square-well potential.

We see that the transmission coefficient  $T$  is determined essentially by this exponent, and is not very sensitive to the choice of  $k_1$  and  $k$ . Because we are interested only in estimating  $T$ , we can therefore simplify Eq. (4.15) by taking the limit of large  $k_1$  (i.e.,  $k_1^2 \gg \kappa^2$  and  $k_1^2 \gg k^2$ ). In this limit, the transmission coefficient of Eq. (4.15) becomes

$$\begin{aligned}
T &\approx \frac{4\kappa^2}{\kappa^2 + k^2} \frac{k}{k_1} (\sinh^2 2\kappa a)^{-1} \\
&\approx \frac{4(V_0 - E)}{V_0} \left( \frac{E}{E + U_0} \right)^{\frac{1}{2}} \left[ 4e^{-\frac{4a}{\hbar}} [2M_\alpha(V_0 - E)]^{\frac{1}{2}} \right] \\
&\approx \frac{4(10)}{14} \left( \frac{4}{44} \right)^{\frac{1}{2}} (4e^{-92}) \\
&\approx 3.5 \times e^{-92} \approx 4 \times 10^{-40}.
\end{aligned} \tag{4.19}$$

Thus, the  $\alpha$ -particle has an exceedingly small probability for penetrating the barrier. This explains why low energy  $\alpha$ -particles cannot be absorbed by heavy nuclei. However, for an  $\alpha$ -particle bound in a nucleus, the situation is quite different. The kinetic energy of the  $\alpha$ -particle within the well is

$$T_\alpha \approx U_0 + E \approx 44 \text{ MeV}, \tag{4.20}$$

and the corresponding velocity is

$$\begin{aligned}
v_\alpha &= \sqrt{\frac{2T_\alpha}{M_\alpha}} = c \sqrt{\frac{2T_\alpha}{M_\alpha c^2}} \\
&\approx c \sqrt{\frac{2 \times 44 \text{ MeV}}{4000 \text{ MeV}}} \approx 0.15 c.
\end{aligned} \tag{4.21}$$

Being confined to a small region of  $\approx 10^{-12}$  cm, the  $\alpha$ -particle will bounce against the barrier with a frequency given approximately by

$$\begin{aligned}
\frac{v_\alpha}{R} &\approx \frac{0.15 \times 3 \times 10^{10} \text{ cm/sec}}{7.4 \times 10^{-13} \text{ cm}} \\
&\approx 6.0 \times 10^{21} / \text{sec}.
\end{aligned} \tag{4.22}$$

Every time the  $\alpha$ -particle hits the barrier, the probability of escape is given by Eq. (4.19). We conclude therefore, that the probability for the  $\alpha$ -particle to escape per second is simply

$$\begin{aligned}
 P(\alpha\text{-emission}) &\approx \frac{v_\alpha}{R} T \approx 6.0 \times 10^{21} / \text{sec} \times 4 \times 10^{-40} \\
 &\approx 2.4 \times 10^{-18} / \text{sec}.
 \end{aligned}
 \tag{4.23}$$

This is what is called the decay constant (denoted by  $\lambda$ ), and is the probability of decay per unit time. The mean lifetime for the decay process (to be discussed in the next chapter) is the inverse of the decay constant

$$\begin{aligned}
 \tau &= \frac{1}{P(\alpha\text{-emission})} \\
 &\approx \frac{1}{2.4 \times 10^{-18} / \text{sec}} \\
 &\approx 0.4 \times 10^{18} \text{ sec} \\
 &\approx 1.3 \times 10^{10} \text{ yrs.}
 \end{aligned}
 \tag{4.24}$$

This lifetime is remarkably close to the observed value.

We have presented an oversimplified calculation of  $\alpha$ -decay. The quantitative result for the coefficient, therefore, cannot be trusted in detail. Nevertheless, in general, for  $V_0 \gg E$ , the decay constant can be represented as

$$P(\alpha\text{-emission}) \propto E^{\frac{1}{2}} e^{-\frac{4a}{\hbar}} [2M_\alpha(V_0 - E)]^{\frac{1}{2}}.
 \tag{4.25}$$

This shows that the probability for decay is quite sensitive to the mass and the energy of the  $\alpha$ -particle. In particular, it shows why spontaneous fission through barrier penetration into heavier daughter nuclei (of large  $M$ ), a subject that we will discuss more fully in the next chapter, is a slow process. It also connects the decay constant and the lifetime for a process with the energy of the  $\alpha$ -particle. We note that  $P(\alpha\text{-emission})$  is proportional to  $E^{\frac{1}{2}}$ , and, as a result, the larger the  $E$ , the shorter is the lifetime. That is, for large  $E$ , the decay is fast, which is consistent with naive expectation. We also note from Eq. (4.25) that, for  $V_0 \gg E$ , and  $(V_0 - E)^{\frac{1}{2}}$  varying slowly with  $E$ , we can write approximately

$$\log P(\alpha\text{-emission}) \propto (\log E + \text{constant}).
 \tag{4.26}$$

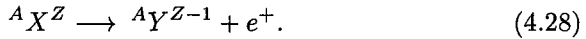
This result provides a quantitative relationship between the decay constant and the energy of the decaying particle, and is known as the Geiger-Nuttall rule. This relation was discovered in data prior to the development of the theoretical formulation.

#### 4.4 Beta Decay

A nucleus with an over abundance of neutrons (i.e., with a value of  $\frac{N}{Z}$  greater than that for stable nuclei) can transform to a more stable nucleus by emitting an electron. This kind of process is known as  $\beta$ -decay, and the transformation can be denoted by



From electric-charge conservation, it follows that the proton number of the daughter nucleus in such decays increases by one unit. However, the nucleon number remains unchanged. There are two other processes that are also referred to as  $\beta$ -decays. In one case, a proton-rich nucleus emits a positron (positrons are antiparticles of electrons, and have the same mass as electrons but positive electric charge), and thereby reduces the nuclear charge by one unit. In this case, the process can be represented by



In addition, a proton-rich nucleus can also reduce its nuclear charge by one unit by absorbing an atomic electron. This process is referred to as electron capture, and can be represented as



The electron is normally captured from an inner  $K$ -shell of an atom. As a result, the outer electrons of the atom cascade down to fill the lower atomic levels, and one or more X-rays are usually emitted. In all three of these processes, the nuclear transformation can be characterized by  $\Delta A = 0$  and  $|\Delta Z| = 1$ .

Because only the electron and the recoiling daughter nucleus were observed in  $\beta$ -decay, the process was initially assumed to be a two body



disintegration, very much like  $\alpha$ -decay. Thus, for the decay in Eq. (4.27), where the parent nucleus is at rest, conservation of energy requires

$$E_X = E_Y + E_{e^-} = E_Y + T_{e^-} + m_e c^2,$$

$$\begin{aligned} \text{or } T_{e^-} &= (E_X - E_Y - m_e c^2) = (M_X - M_Y - m_e)c^2 - T_Y \\ &= Q - T_Y \approx Q. \end{aligned} \tag{4.30}$$

In other words, for a two-body process, just as in  $\alpha$ -decay, the lighter emitted particle, (the electron), would be expected to carry away most of the released energy, which would have a unique value given by Eq. (4.30). However, as we have already discussed in Chapter 2, these electrons are emitted with a continuous spectrum of energies. In fact, the observed differential distribution in the number of emitted electrons as a function of their energy has the shape given in Fig. 4.4, and, within experimental accuracy, has an endpoint (the maximum energy of any emitted electron) given by the value in Eq. (4.30). That is, the electrons have a spectrum of energies, with most values lying well below that predicted by energy conservation in two-body decays. When this was first observed, it appeared to threaten the survival of one of the most cherished conservation laws in physics, namely energy conservation! In addition, a consideration of the change in angular momentum in  $\beta$ -decay processes reveals that angular momentum could not be conserved if the decays produced only two particles in the final state. Examining the decay in Eq. (4.27), we note that the number of nucleons does not change in the transition. However, an electron, which is a fermion, is emitted in the process. The electron, as well as each of the nucleons, have spin angular momentum of  $\frac{\hbar}{2}$ . Consequently, independent of any possible change in the value of the orbital angular momentum, which must always have integral value, it is clear that angular momentum cannot be conserved in this kind of a process.

For a while, it seemed that the principles of conservation of momentum, energy and angular momentum might not apply in  $\beta$ -decay. This would have implied, through Emmy Noether's theorem (discussed in Chapter 10), that the universe is not isotropic, and that there is an absolute coordinate system and an absolute time scale, all of which would have severely impacted physical behavior. Physics, as we know it, would have had to be abandoned. To extricate science from this abyss, Wolfgang Pauli proposed

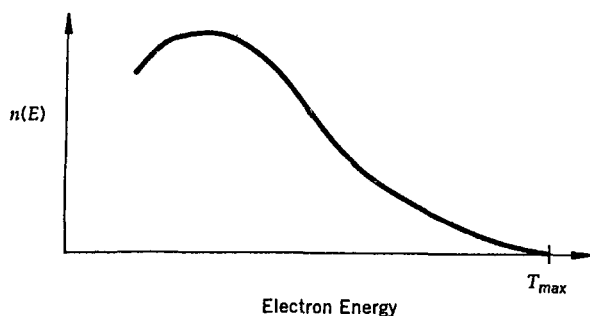


Fig. 4.4 The energy spectrum of electrons emitted in  $\beta$ -decay.

that an additional particle, one that was difficult to detect, was emitted in  $\beta$ -decay. Conservation of electric charge required this particle to be electrically neutral, just like the neutron and the photon. Ostensibly, this would explain why it was so hard to detect this particle. We know now that this neutral particle, the *neutrino*, does not interact readily with matter, and this is the main reason why it is so difficult to observe. Because the maximum energies for electrons emitted in  $\beta$ -decay corresponded to the disintegration energy of the nucleus, it meant that this new particle had to be essentially massless. Furthermore, if the postulated neutrino were to restore the conservation of angular momentum, then it would have to be a fermion with spin angular momentum  $\frac{\hbar}{2}$ . In some ways, such a particle would resemble a neutron, except that it would be much lighter, and Fermi therefore coined for it the name neutrino (diminutive for neutron), and it is denoted by the Greek letter  $\nu$ .

Every elementary particle appears to have an antiparticle, and the neutrino is no exception; its antiparticle is known as the *antineutrino* ( $\bar{\nu}$ ). Since both the neutrino and the antineutrino are electrically neutral, an interesting question is what specific property distinguishes them from each other. The neutron and the antineutron are also neutral, but they have magnetic dipole moments of opposite sign, and, as we will discuss in Chapters 9 and 11, opposite nucleon or “baryon” numbers that distinguish them. However, the neutrino is an essentially massless point particle, without structure, and has neither nucleon number nor a magnetic dipole moment. (Until the discovery of neutrino *mixing* or *oscillations* in the late 1990s, all neutrinos were thought to have been massless. We will touch on this development in Chapter 12.) Experiments on  $\beta$  decay indicate that neutrinos that accompany positrons (“ $\nu_{e+}$ ”) are left handed, whereas the ones that accompany

electrons (“ $\nu_{e-}$ ”) are right-handed, where by left-handed we mean that the particle has its spin pointing opposite to its momentum, and by right-handed that it has its spin pointing in the direction of its line of flight. (This convention is opposite to the definitions of left-handed and right-handed polarizations used in optics.) If we define  $e^-$  as the particle and  $e^+$  as its antiparticle, then it is tempting to call the  $\nu_{e-}$  the antineutrino ( $\bar{\nu}$ ) and the  $\nu_{e+}$  the neutrino ( $\nu_e$ ). (This assignment will be justified shortly.) Consequently, the handedness is one of the distinguishing characteristics between a neutrino and its antiparticle (antineutrino), and has far-reaching consequences, as we will see later. Using our new nomenclature, we can rewrite our three  $\beta$ -decay processes as

$$\begin{aligned} {}^A X^Z &\longrightarrow {}^A Y^{Z+1} + e^- + \bar{\nu}, \\ {}^A X^Z &\longrightarrow {}^A Y^{Z-1} + e^+ + \nu, \\ {}^A X^Z + e^- &\longrightarrow {}^A Y^{Z-1} + \nu. \end{aligned} \quad (4.31)$$

If the parent nucleus decays from rest, then conservation of energy for electron emission will yield

$$\begin{aligned} M_P c^2 &= T_D + M_D c^2 + T_{e^-} + m_e c^2 + T_{\bar{\nu}} + m_{\nu} c^2, \\ \text{or } T_D + T_{e^-} + T_{\bar{\nu}} &= (M_P - M_D - m_e - m_{\nu}) c^2 \\ &= \Delta M c^2 = Q, \end{aligned} \quad (4.32)$$

where  $M_P$ ,  $M_D$ ,  $m_e$  and  $m_{\nu}$  are, respectively, the masses of the parent nucleus, the daughter nucleus, the electron and the antineutrino. Similarly,  $T_D$ ,  $T_{e^-}$  and  $T_{\bar{\nu}}$  represent the kinetic energies of the decay daughter nucleus, the electron and antineutrino. We see from Eq. (4.32) that electron emission can take place only if the disintegration energy  $Q$  is positive, that is, when the mass of the parent nucleus is greater than the sum of the masses of the decay products. In fact, neglecting small differences in atomic binding energies, we conclude that electron emission will take place if

$$\begin{aligned} Q &= (M(A, Z) - M(A, Z + 1) - m_{\nu}) c^2 \\ &\approx (M(A, Z) - M(A, Z + 1)) c^2 \geq 0, \end{aligned} \quad (4.33)$$

where  $M(A, Z)$  represents the atomic weight, including the atomic electrons, and we have neglected the small mass of the neutrino. Furthermore, because the daughter nucleus is much heavier than either the electron or the antineutrino, the small recoil energy of the daughter can be ignored, and for any  $\beta$ -decay we can write

$$T_{e^-} + T_{\bar{\nu}} \approx Q. \quad (4.34)$$

It is now clear that, with a  $\bar{\nu}$  in the final state, the energy of the electron is no longer unique. In fact, any continuous value  $0 \leq T_{e^-} \leq Q$  is kinematically allowed, and the maximum electron energy, corresponding to  $T_{\bar{\nu}} = 0$ , is given by the endpoint value of Eq. (4.32)

$$(T_{e^-})_{\max} = Q. \quad (4.35)$$

Pauli's postulate therefore accommodates the continuous energy spectrum in  $\beta$ -decay, and simultaneously restores all the accepted conservation laws.

For completeness, let us note that the disintegration energy for positron emission is given by

$$\begin{aligned} Q &= (M_P - M_D - m_e - m_\nu)c^2 \\ &= (M(A, Z) - M(A, Z - 1) - 2m_e - m_\nu)c^2 \\ &\approx (M(A, Z) - M(A, Z - 1) - 2m_e)c^2, \end{aligned} \quad (4.36)$$

where, again, all the  $M(A, Z)$  in the last line of Eq. (4.36) refer to full atomic weights, and  $Q$  must be positive for the decay to occur. Similarly, electron capture can take place only if

$$\begin{aligned} Q &= (M_P + m_e - M_D - m_\nu)c^2 \\ &= (M(A, Z) - M(A, Z - 1) - m_\nu)c^2 \\ &\approx (M(A, Z) - M(A, Z - 1))c^2 \geq 0. \end{aligned} \quad (4.37)$$

As stated before, all of these relations neglect the  $\approx$  eV differences in binding energies of electrons in atoms.

Just as a proton or a neutron is defined to be a nucleon with nucleon or baryon number +1, so is an electron defined to be a *lepton* with a lepton number +1. A positron, being the antiparticle of an electron, has lepton number -1, just as an antiproton or an antineutron has nucleon number -1. We will see in Chapter 9 that both lepton number and nucleon number

appear to be conserved in all interactions; we can therefore conclude from the three processes in Eq. (4.31) that neutrinos must also be leptons with lepton number +1, while the lepton number for antineutrinos is -1.

#### 4.4.1 Lepton Number

Three charged leptons appear to exist in nature, all with their own associated neutrinos, namely  $(e^-, \nu_e)$ ,  $(\mu^-, \nu_\mu)$  and  $(\tau^-, \nu_\tau)$ . The muon and the  $\tau$  lepton have properties similar to that of the electron, but are far more massive. The three types of neutrinos are also known to be distinct from one another. For example, when neutrinos produced in a decay such as  $\pi^+ \rightarrow \mu^+ + \nu_\mu$  are allowed to interact with matter, they never produce charged leptons other than  $\mu^-$ . That is,

$$\begin{aligned}\nu_\mu + {}^A X^Z &\rightarrow {}^A Y^{Z+1} + \mu^-, \\ \nu_\mu + {}^A X^Z &\not\rightarrow {}^A Y^{Z+1} + e^-, \\ \nu_\mu + {}^A X^Z &\not\rightarrow {}^A Y^{Z+1} + \tau^-, \end{aligned} \quad (4.38)$$

and  $\nu_e$  interacting with matter produce electrons

$$\begin{aligned}\nu_e + {}^A X^Z &\rightarrow {}^A Y^{Z+1} + e^-, \\ \nu_e + {}^A X^Z &\not\rightarrow {}^A Y^{Z+1} + \mu^-, \\ \nu_e + {}^A X^Z &\not\rightarrow {}^A Y^{Z+1} + \tau^-. \end{aligned} \quad (4.39)$$

Similarly, although not studied as extensively,  $\nu_\tau$  produces  $\tau^-$ , and not  $e^-$  or  $\mu^-$ . This family structure for leptons and their antiparticles plays a major role in constructing theories of fundamental interactions.

#### 4.4.2 Neutrino Mass

The issue of whether neutrinos have mass has important ramifications. As is clear from Eqs. (4.33) and (4.35), the mass of the neutrino can be determined from the end point of the  $\beta$ -spectrum. If  $m_\nu = 0$ , then the end point of the spectrum is tangential to the abscissa, whereas if  $m_\nu \neq 0$ , then the end point is tangential to the ordinate (see Fig. 4.5). Thus, the shape of the  $\beta$ -spectrum near the end point can be used to extract the mass of the neutrino. However, in practice, the shape at the end point is very sensitive

to measurement resolution. There are also other methods of determining the masses of neutrinos, and at present the most stringent direct limit on the mass of the electron neutrino is  $m_{\nu_e} < 2 \text{ eV}/c^2$ .

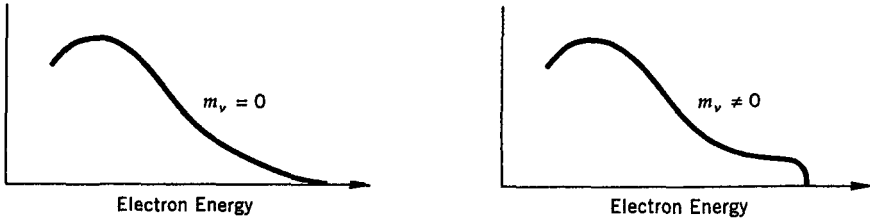
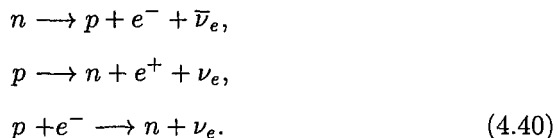


Fig. 4.5 Dependence of the end point of the  $\beta$  spectrum on the mass of the neutrino.

A small but finite neutrino mass is interesting from a cosmological perspective because a massive neutrino can contribute to the mass of the universe as *dark matter*. A finite mass for the neutrino leads naturally to concepts such as *mixing between different neutrino states*, and to the possibility of conversion of one species to another, much in the spirit of beat phenomena and energy exchange observed for weakly-coupled oscillators. The possibility of neutrino mixing provides one of the experimental methods for establishing a non-vanishing neutrino mass. For example, starting out with  $\nu_\mu$  we can look for the evolution of  $e^-$  production in matter as a function of distance traveled by the neutrino. In fact, measurements of the neutrino flux from the sun and from interactions of cosmic rays, performed by Ray Davis and Masatoshi Koshiba and their collaborators, imply that there is a finite probability for different types of neutrinos to transform into each other, which requires that neutrinos have finite mass. (These kinds of issues will be discussed in greater detail in Chapter 12.)

#### 4.4.3 The Weak Interaction

The  $\beta$ -decay processes of Eq. (4.31) can be written equivalently as



Because a neutron is more massive than a proton, a free neutron can de-

cay as in Eq. (4.40). However, by the same token, since the proton is lighter than a neutron, a proton cannot  $\beta$ -decay in free space. That is, a proton can undergo  $\beta$ -decay, but only inside a nucleus. Free neutrons, on the other hand, do decay in the laboratory with lifetimes of about 900 sec. This lifetime is much longer than the time scales involved in nuclear and electromagnetic reactions. (We note that the typical time scale for a nuclear reaction is about  $10^{-23}$  sec, while the corresponding time scale for an electromagnetic process is about  $10^{-16}$  sec.) Thus, we conclude that although  $\beta$ -decay is a nuclear phenomenon, it does not involve the strong nuclear force. (Its origin also cannot be electromagnetic.) This result led Fermi to postulate the existence of a new force that is responsible for  $\beta$ -decay. It is called the *weak force*, and is short-ranged, since it is effective only within the nuclear domain. The weakness of the strength of this force is responsible for the long lifetimes observed in  $\beta$ -decays. In terms of relative strengths, the nuclear, electromagnetic, weak and the gravitational interactions can be characterized by the ratios  $1 : 10^{-2} : 10^{-5} : 10^{-39}$ . As in the case of electromagnetism, the weak coupling strength of this force also allows us to calculate any of its effects through perturbative techniques.

As we have noted before, nuclei do not contain electrons. Consequently, electrons produced in  $\beta$ -decay cannot originate from within the nucleus. Rather, they must be produced at the time of the decay. This is quite analogous to the situation in atomic transitions, where photons do not exist within atoms, but are produced during the transitions. Just as a transition in an atom can be understood as being induced, for example, by a dipole interaction, and can be calculated using perturbation theory, in a similar way,  $\beta$ -decay can be understood as being induced by the weak force of the weak-interaction Hamiltonian. The transition probability per unit time, or the “width”, for the process can also be calculated from perturbation theory using Fermi’s Golden Rule (discussed in Chapter 1)

$$P = \frac{2\pi}{\hbar} |H_{fi}|^2 \rho(E_f), \quad (4.41)$$

where  $\rho(E_f)$  is the density of states for the decay products, and  $H_{fi}$  denotes the matrix element of the weak-interaction Hamiltonian,  $H_{\text{wk}}$ , taken between the initial and the final states

$$H_{fi} = \langle f | H_{\text{wk}} | i \rangle = \int d^3x \psi_f^*(x) H_{\text{wk}} \psi_i(x). \quad (4.42)$$

From Eq. (4.40) we see that the weak-interaction Hamiltonian must connect four fermionic states, otherwise the matrix element in Eq. (4.42) would not describe  $\beta$ -decay. The Hamiltonian for the theory of  $\beta$ -decay as proposed by Fermi – also known as the *four-fermion interaction* or *current-current interaction* – was relativistic, and based on the properties of the Dirac equation for fermions. Over the years, experimental studies have greatly restricted the structure of the four-fermion theory to a form that is in excellent agreement with all experimental measurements of low energy  $\beta$ -decay processes.

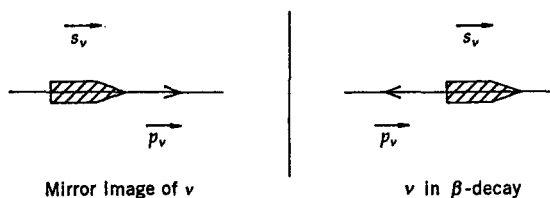


Fig. 4.6 Sketch of the momentum and spin vector of a neutrino and its mirror (inverted) image.

One of the features of the modern form of the theory is that there are only left-handed neutrinos and right-handed antineutrinos, which accommodates quite naturally the large violation of parity observed in weak interactions. A system is parity-invariant if it does not change under reflection of spatial coordinates, that is, if it is indistinguishable from its mirror image. For the left-handed neutrinos emitted in  $\beta$ -decay, however, the mirror reflected process, that is, the mirror image, will involve right handed neutrinos. This follows because under reflection  $\vec{r} \rightarrow -\vec{r}$ ,  $\vec{p} \rightarrow -\vec{p}$ , and  $\vec{s}$  must transform as  $\vec{L} = \vec{r} \times \vec{p} \rightarrow (-\vec{r}) \times (-\vec{p}) = \vec{L}$ . (Handedness can be defined as  $\frac{\vec{p} \cdot \vec{s}}{|\vec{p}||\vec{s}|}$ .) Therefore, as shown in Fig. 4.6, the direction of motion changes under reflection, but spin, being an angular momentum, does not, leading to a change in the handedness. (We urge the reader to look at a rotating screw and its image in a mirror, to become convinced that the sense of rotation is preserved under reflection.) Thus, we see that the process of  $\beta$ -decay is distinguishable from its mirror image. However, since to all intents right-handed neutrinos (and left-handed antineutrinos) do not appear in nature, the parity transformed process, in effect, does not exist, and consequently parity must be violated in weak interactions. This agrees completely with experiment, and these issues will be discussed again in later chapters.



## 4.5 Gamma Decay

As we have already seen, when a heavy nucleus disintegrates by emitting an  $\alpha$ -particle or a  $\beta$ -particle, the daughter nucleus may be left in an excited state. If the excited nucleus does not break apart or emit another particle, it can de-excite to the ground state by emitting a high energy photon or gamma ( $\gamma$ ) ray. As we saw in Example 1, the characteristic spacing of nuclear energy levels is about 50 keV, and typical energies of nuclear  $\gamma$ -rays can therefore range from a fraction to several MeV. Because this kind of de-excitation is electromagnetic, we expect lifetimes for such processes to be about  $10^{-16}$  sec.<sup>1</sup> As in atomic transitions, the photon carries away at least one unit of angular momentum (the photon, being described by the vector electromagnetic field, has spin angular momentum of  $\hbar$ ), and the process conserves parity.

The study of the emission and absorption of nuclear  $\gamma$ -rays, forms an essential part of the development of nuclear spectroscopy. The subject has a direct parallel in the study of atomic spectroscopy, however, there are important differences. Consider, for example, a system initially in a state of energy  $E_i$  making a transition to a state with energy  $E_f$  through the absorption or emission of a photon of frequency  $\nu$ . In such processes, we can define what are known as resonant or recoilless transitions, for which

$$h\nu = \mp (E_i - E_f), \quad (4.43)$$

where “-” corresponds to absorption and “+” to emission. Thus, in principle, measuring  $\nu$  determines the level spacings. However, in absorbing or emitting a photon, any system must, in fact, recoil to conserve momentum. If  $M$  denotes the mass of the final-state object and  $v$  the magnitude of its recoil velocity, it then follows from conservation of momentum that

$$\frac{h\nu}{c} = Mv. \quad (4.44)$$

Consequently, conservation of energy yields to a modified version of Eq. (4.43)

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<sup>1</sup>When we speak about “typical” lifetimes for different interactions, it should be recognized that these can vary substantially for any specific process because of differences in the phase space and transition operators under different circumstances. Thus, for example, electromagnetic reactions have “typical” lifetimes ranging between  $\approx 10^{-19}$  sec in certain particle decays to  $\approx 10^{-8}$  sec for atomic dipole transitions.

$$\begin{aligned}
 E_i - E_f &= \mp h\nu + \frac{1}{2} Mv^2, \\
 &= \mp h\nu + \frac{1}{2M} \left( \frac{h\nu}{c} \right)^2, \\
 \text{or } h\nu &= \mp \left( E_i - E_f - \frac{h^2\nu^2}{2Mc^2} \right) = \mp (E_i - E_f - \Delta E_R), \quad (4.45)
 \end{aligned}$$

where  $\Delta E_R$  denotes the kinetic energy of the recoil.

Now, every unstable energy level has a “natural” width  $\delta E = \Gamma$  and a lifetime  $\tau$ , which can be related through the uncertainty principle:

$$\tau\Gamma \approx \hbar$$

$$\text{or } \Gamma \approx \frac{\hbar}{\tau} \approx \text{uncertainty in } (E_i - E_f). \quad (4.46)$$

In other words, the exact value of an energy level is uncertain, and cannot be defined in any given transition to better than  $\approx \Gamma$ . Consequently, if the kinetic energy of the recoil is such that  $\Delta E_R \ll \Gamma$ , then Eq. (4.45) is essentially equivalent to Eq. (4.43), and resonant absorption can take place. On the other hand, if  $\Delta E_R \gg \Gamma$ , it is then impossible to excite the system to a higher level through resonant absorption within the bounds (or “umbrella”) provided by the uncertainty relation.

To appreciate this more fully, consider an atom with  $A = 50$ . The typical spacing of atomic levels is of the order of 1 eV, and we will therefore consider absorption of a photon of energy of  $h\nu = 1$  eV. For the atom, we have  $Mc^2 \approx 50 \times 10^3 \text{ MeV} = 5 \times 10^{10} \text{ eV}$ , and, consequently,

$$\Delta E_R = \frac{(h\nu)^2}{2Mc^2} \approx \frac{1(\text{eV})^2}{2 \times 5 \times 10^{10} \text{ eV}} = 10^{-11} \text{ eV}. \quad (4.47)$$

Because typical lifetimes associated with excited atomic levels are about  $10^{-8}$  sec, we see that

$$\begin{aligned}
 \Gamma &\approx \frac{\hbar}{\tau} \approx \frac{6.6 \times 10^{-22}}{10^{-8} \text{ sec}} \text{ MeV-sec} \\
 &= 6.6 \times 10^{-14} \text{ MeV} = 6.6 \times 10^{-8} \text{ eV}. \quad (4.48)
 \end{aligned}$$

Consequently,  $\Delta E_R \ll \Gamma$ , and, for atomic transitions, resonant absorption can therefore take place.

In contrast, typical nuclear spacings have  $h\nu \gtrsim 100 \text{ keV} = 10^5 \text{ eV}$ . If we consider again a nucleus with  $A = 50$ , we still have  $Mc^2 \approx 5 \times 10^{10} \text{ eV}$ , but now, with the higher photon energy, the nuclear recoil energy is given by

$$\Delta E_R = \frac{(h\nu)^2}{2Mc^2} \approx \frac{(10^5 \text{ eV})^2}{10^{11} \text{ eV}} = 10^{-1} \text{ eV}. \quad (4.49)$$

If we assume a typical lifetime of about  $10^{-12} \text{ sec}$  for a nuclear level, then

$$\begin{aligned} \Gamma &\approx \frac{\hbar}{\tau} \approx \frac{6.6 \times 10^{-22} \text{ MeV-sec}}{10^{-12} \text{ sec}} \\ &= 6.6 \times 10^{-10} \text{ MeV} = 6.6 \times 10^{-4} \text{ eV}. \end{aligned} \quad (4.50)$$

It is clear, therefore, that for such nuclear transitions  $\Delta E_R \gg \Gamma$ , and resonant absorption cannot occur.

In fact, for resonant absorption to take place in nuclei, the recoil energy must somehow be reduced, and this is done beautifully through what is known as the Mössbauer effect (named after its discoverer Rudolf Mössbauer). The basic idea rests on the fact that, the heavier the recoiling system, the smaller is the recoil energy (see Eq. (4.49)). An enormous increase in the mass of the recoil can be achieved by freezing the nucleus into a rigid crystal lattice, which, of course, has a much larger mass than a single nucleus. As a result, the mass of the recoiling system becomes the mass of the macroscopic crystal, thereby increasing the effective mass of the recoil by many orders of magnitude, and consequently making the recoil energy  $\Delta E_R$  negligible relative to  $\Gamma$ . Because of this feature, the Mössbauer technique can provide exceedingly precise estimates of widths of levels. For example, level widths in iron have been measured to an accuracy of about  $10^{-7} \text{ eV}$ , which leads to an accuracy of about 1 part in  $10^{12}$  in level spacing. The technique is therefore extremely useful in determining hyperfine splittings of nuclear energy levels.

## Problems

**4.1** Calculate the  $Q$  values for the following  $\alpha$ -decays between ground-state levels of the nuclei: (a)  $^{208}\text{Po} \rightarrow ^{204}\text{Pb} + \alpha$  and (b)  $^{230}\text{Th} \rightarrow ^{226}\text{Ra} + \alpha$ .

What are the kinetic energies of the  $\alpha$ -particles and of the nuclei in the final state if the decays proceed from rest?

**4.2** Estimate the relative contribution of the centrifugal barrier and the Coulomb barrier in the scattering of a 4 MeV  $\alpha$ -particle from  $^{238}\text{U}$ . In particular, consider impact parameters of  $b = 1$  fm and  $b = 7$  fm. What are the orbital quantum numbers in such collisions. (Hint:  $|\vec{L}| \sim |\vec{r} \times \vec{p}| \sim \hbar kb \sim \hbar \ell$ .)

**4.3** Free neutrons decay into protons, electrons and antineutrinos, with a mean life of 889 sec. If the neutron-proton mass difference is taken as 1.3 MeV/ $c^2$ , calculate to at least 10% accuracy the maximum kinetic energies that electrons and protons can have. What would be the maximum energy that the antineutrinos can have? (Assume decay from rest and that the antineutrino is massless.)

**4.4** If the stable isotope of sodium is  $^{23}\text{Na}$ , what kind of radioactivity would you expect from (a)  $^{22}\text{Na}$  and (b)  $^{24}\text{Na}$ ?

**4.5** Specify any additional particles needed in the following weak reactions to assure the conservation of lepton number: (a)  $\mu^- \rightarrow e^- + ?$  (b)  $\tau^+ \rightarrow e^+ + ?$  (c)  $e^- + {}^A X^Z \rightarrow ?$  (d)  $\nu_\mu + n \rightarrow ?$  (e)  ${}^A X^Z \rightarrow {}^A Y^{Z-1} + ?$  (f)  $\bar{\nu}_e + p \rightarrow ?$

**4.6** Calculate the typical kinetic energy expected of an  $\alpha$ -particle confined within a nucleus if its emitted energy is 10 MeV. What is the momentum of such an  $\alpha$ -particle inside the nucleus and after it is emitted. Is the wavelength of such an  $\alpha$ -particle acceptable for it to be contained within a nucleus of  $^{12}\text{C}$ ? What about  $^{238}\text{U}$ ?

**4.7** When you examine the dependence of  $Z$  on  $N$  for stable nuclei, you find that  $\beta^+$  emitters lie above the region of stability (have proton excess) and  $\beta^-$  emitters lie below that region (have neutron excess). For example,  $^8\text{B}$  emits  $\beta^+$ , while  $^{12}\text{B}$  emits  $\beta^-$ . Stable nuclei are those that do not seem to have sufficient mass for either emission to take place, that is, they are the nuclei with greatest binding or smallest mass. As discussed in Problem 3.1, this suggests that stable nuclei should correspond to a “valley” in the  $M$ - $Z$  space, that is, specified by  $\frac{\partial M}{\partial Z} = 0$ . Using the semi-empirical mass formula for  $M$ , show that the relationship between  $Z$  and  $A$  for this valley of stability is  $Z \approx \frac{A}{(2+0.008 A^{2/3})}$ . Several nuclei with  $Z$  beyond 110 were discovered in the late 1990s. Is it possible that there could be more “islands” of stability for  $Z > 120$ ? Consider, for specifics, the possibility

of binding of  $Z = 125$ ,  $Z = 126$ , and  $Z = 164$ . Even more massive nuclei have been hypothesized with  $Z > 200$ . These would have rather exotic bubble-like or toroidal structure. Why would such structures be expected to be more stable than spherical nuclei?

### **Suggested Readings**

Frauenfelder, H., and E. M. Henley, *Subatomic Physics*, Prentice-Hall (1991).

Krane, K. S., *Introductory Nuclear Physics*, Wiley (1987).

Povh, B., et al., *Particles and Nuclei*, Springer Verlag (2002).

Williams, W. S. C., *Nuclear and Particle Physics*, Oxford Univ. Press (1997).

## Chapter 5

# Applications of Nuclear Physics

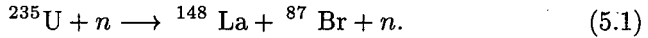
### 5.1 Introductory Remarks

Studies of properties of nuclei and of the nuclear force have contributed significantly to the formulation of the fundamental laws of nature. The understanding of physical laws has in the past led to applications that have benefited mankind. For example, the principles of electromagnetism led to the commercialization of electricity, which has proven indispensable in our daily life. Similarly, the explanation of atomic phenomena has given us the laser, the transistor and a host of amazing devices. Needless to say, many applications have also arisen from our understanding of nuclear physics. However, because these developments have been put to both constructive as well as destructive use, they have often led to controversy. In this chapter, we will describe only a few of these applications and the principles behind them.

### 5.2 Nuclear Fission

Neutrons, being electrically neutral, do not sense the direct Coulomb force. As a consequence, unlike protons that are repulsed by the nuclear charge, low energy neutrons can get quite close to the nuclei and interact with them through the attractive nuclear potential to form bound states. In the early days of nuclear physics, the capture of low energy neutrons within nuclei was promoted as a technique for producing new nuclei of higher  $A$  values. In experiments designed to make transuranic elements through neutron capture, it was often observed that scattering of low energy thermal neutrons (at room temperature  $T \approx 300$  K,  $kT \approx \frac{1}{40}$  eV) from odd- $A$  nuclei such as  $^{235}\text{U}$  did not produce heavier nuclei, but instead the parent nucleus

fragmented into two smaller-mass daughter nuclei. Such fragmentation of a heavy nucleus into two medium-size nuclei and any other remnants is known as *nuclear fission*. Certain heavy nuclei can also undergo spontaneous fission with only minimal external perturbation. A typical example of induced fission of an odd- $A$  nucleus is given by the absorption of thermal neutrons by  $^{235}\text{U}$



On the other hand, the scattering of thermal neutrons from even- $A$  nuclei such as  $^{238}\text{U}$  does not produce fragmentation. Nevertheless, fission can take place in such nuclei when the neutrons have kinetic energies of the order of 2 MeV.

Fission, therefore, appears to be an inherent characteristic of large nuclei, and it has come to play an important role in our lives because the process can release a large amount of energy. An estimate of the energy released in the fission of a heavy nucleus can be obtained from the graph of the binding energy per nucleon (see Fig. 2.1). The binding energy per nucleon is smaller for very large- $A$  values than for medium- $A$  nuclei, where it attains a maximum. The process of fission therefore involves the breakup of a comparatively lightly bound heavy nucleus into two tightly bound medium- $A$  nuclei, and as a result this must lead to a release of energy. Thus, if we use  $-7.5$  MeV as the approximate binding energy per nucleon for  $^{235}\text{U}$  and about  $-8.4$  MeV for the fission products (recall that  $\frac{B}{A}$  is defined as the negative of the binding energy per nucleon), we then obtain an energy release of about 0.9 MeV per nucleon in a typical fission. Consequently, the total energy released per fission of one  $^{235}\text{U}$  nucleus, and shared among the end products, can be estimated to be

$$235 \times 0.9 \text{ MeV} = 211.5 \text{ MeV} \approx 200 \text{ MeV}. \quad (5.2)$$

This is, indeed, a lot of kinetic energy, and consequently the harnessing of nuclear fission can, in principle, provide a substantial source of power.

### 5.2.1 Basic Theory of Fission

The phenomenon of nuclear fission can be understood both qualitatively and quantitatively on the basis of the liquid drop model. Qualitatively, the model assumes nuclei to be spherical, and this is indeed consistent

with much of the data. However, for very large nuclei, a spherical shape need not necessarily be stable. Furthermore, an external perturbation, such as an incident neutron, can create surface waves that can lead to a change in the shape of a liquid drop. The liquid drop can, for example, elongate as a result of the perturbation. If the produced deformation is sufficiently large, Coulomb repulsion between the elongated portions of the drop can produce a two-lobe structure that can push the lobes further apart, causing a complete split or fission of the initial nuclear drop into two droplets. On the other hand, if the initial deformation is not very large, then the deformed liquid drop can form an excited state of the compound nucleus (consisting of the incident neutron and the parent nucleus of nucleon number  $A$ ), which can eventually de-excite to a lower energy state of a nucleus with nucleon number  $(A + 1)$  through the emission of a photon. This second scenario is commonly referred to as the *radiative capture* of a neutron. These processes are represented pictorially in Fig. 5.1.

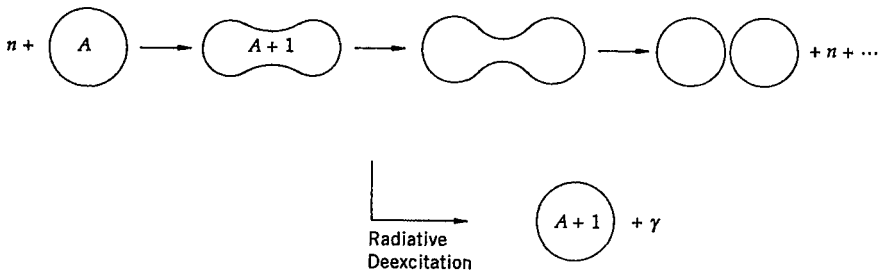


Fig. 5.1 Neutron absorption leading either to fission or to radiative capture.

The liquid drop model also provides an excellent quantitative description of nuclear fission. As we have already seen, the model provides a natural and successful parameterization of the binding energy of nuclei. The empirical formula for the binding energy (see Eq. (3.3)) has three classical terms that depend explicitly on the shape of the drop, namely the volume energy, the surface energy and the Coulomb energy. We can therefore perform a simple classical calculation to analyze the stability of a liquid drop under any external perturbation. Thus, if we assume that a spherical liquid drop of radius  $R$  deforms very slightly under some external perturbation to an ellipsoid of the same volume (recall that nuclear matter behaves like an incompressible liquid), with semi-major and semi-minor axes  $a$  and  $b$ , respectively, we can write  $a$  and  $b$  in terms of a small parameter of deformation  $\epsilon$ , as (see Fig. 5.2)



$$\begin{aligned}
 a &= R(1 + \epsilon), \\
 b &= \frac{R}{(1 + \epsilon)^{1/2}}.
 \end{aligned}
 \tag{5.3}$$

(See Problem 5.13 for the connection between the parameter  $\epsilon$  and the usual eccentricity of an ellipsoid.) This choice of parametrization guarantees that the volume of the liquid drop remains unchanged

$$V = \frac{4}{3} \pi R^3 = \frac{4}{3} \pi a b^2.
 \tag{5.4}$$

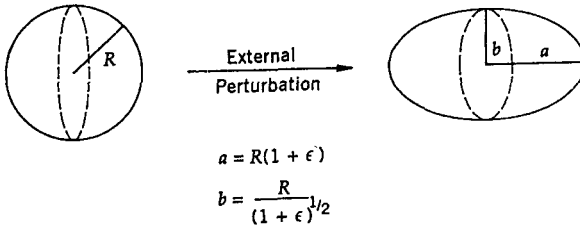


Fig. 5.2 Deformation of a sphere into an ellipsoid of same volume.

Since the volume is identical for the sphere and for the ellipsoid, the volume energy will be the same for both the original and the deformed liquid drops. However, the surface energy and the Coulomb energy will differ for the two cases. In fact, it can be shown (by comparing the surface area of an ellipsoid to that of a sphere) that the surface energy for the ellipsoid assumes the form

$$a_2 A^{2/3} \longrightarrow a_2 A^{2/3} \left( 1 + \frac{2}{5} \epsilon^2 \right),
 \tag{5.5}$$

while the Coulomb energy changes to

$$a_3 \frac{Z^2}{A^{1/3}} \longrightarrow a_3 \frac{Z^2}{A^{1/3}} \left( 1 - \frac{1}{5} \epsilon^2 \right).
 \tag{5.6}$$

The above deformation increases the surface energy while decreasing the Coulomb term. The stability of the droplet therefore depends on how these

two terms compete with each other. The total change in binding energy due to the deformation can now be written as

$$\begin{aligned}\Delta &= \text{B.E. (ellipsoid)} - \text{B.E. (sphere)} \\ &= \frac{2}{5} \epsilon^2 a_2 A^{\frac{2}{3}} - \frac{1}{5} \epsilon^2 a_3 \frac{Z^2}{A^{\frac{1}{3}}} \\ &= \frac{1}{5} \epsilon^2 A^{\frac{2}{3}} \left( 2a_2 - a_3 \frac{Z^2}{A} \right).\end{aligned}\quad (5.7)$$

Clearly, if this energy difference is positive, then the spherical drop will be more tightly bound and consequently stable under a small external perturbation. In fact, from the values of  $a_2$  and  $a_3$  given in Eq. (3.4), namely  $a_2 \approx 16.8$  MeV and  $a_3 \approx 0.72$  MeV, we find that  $\Delta > 0$  when

$$\begin{aligned}2a_2 - a_3 \frac{Z^2}{A} &> 0, \\ \text{or } \frac{Z^2}{A} &< 47.\end{aligned}\quad (5.8)$$

This simple classical analysis shows therefore that a spherical nucleus is stable under infinitesimal perturbations only if  $Z^2 < 47A$ . There are, of course, quantum-mechanical corrections that have to be considered, but they do not affect the qualitative features of the result, namely that spherical nuclei with  $Z^2 > 47A$  are expected to be highly unstable and subject to spontaneous fission. Because large nuclei have  $Z < \frac{1}{2}A$  (see Fig. 2.3), it follows, in fact, that they all satisfy  $Z^2 < 47A$ , and a spherical shape therefore provides them maximal binding. However, even for  $Z^2 < 47A$ , the binding energy of two daughter nuclei can be smaller than the binding energy of a spherical parent nucleus, in which case, the spherical parent can fission, and thereby transform to a state of lower energy.

Let us consider the simple example of a parent nucleus fragmenting into two identical daughter nuclei. (We are therefore assuming that both the  $A$  and  $Z$  values for the parent nucleus are even.) Neglecting the quantum-mechanical terms, namely the terms with  $a_4$  and  $a_5$  in Eq. (3.2), we can calculate the difference in the binding energy of the initial nucleus and the fission products when they are far apart from each other. Since the volume energy cancels out, we obtain

$$\begin{aligned}
\Delta(\text{B.E.}) &= \text{B.E.}(A, Z) - 2 \text{B.E.}\left(\frac{A}{2}, \frac{Z}{2}\right) \\
&= a_2 A^{\frac{2}{3}} \left(1 - 2 \left(\frac{1}{2}\right)^{\frac{2}{3}}\right) + a_3 \frac{Z^2}{A^{\frac{1}{3}}} \left(1 - 2 \frac{(\frac{1}{2})^2}{(\frac{1}{2})^{\frac{1}{3}}}\right) \\
&= a_2 A^{\frac{2}{3}} (1 - 2^{\frac{1}{3}}) + a_3 \frac{Z^2}{A^{\frac{1}{3}}} (1 - 2^{-\frac{2}{3}}). \tag{5.9}
\end{aligned}$$

Using the values of  $a_2$  and  $a_3$  from Eq. (3.4), we obtain

$$\begin{aligned}
\Delta(\text{B.E.}) &\approx A^{\frac{2}{3}} \left(-0.27 a_2 + 0.38 a_3 \frac{Z^2}{A}\right) \\
&= A^{\frac{2}{3}} \left(-0.27 \times 16.8 \text{ MeV} + 0.38 \times 0.72 \text{ MeV} \frac{Z^2}{A}\right) \\
&\approx 0.27 A^{\frac{2}{3}} \left(-16.5 + \frac{Z^2}{A}\right) \text{ MeV}. \tag{5.10}
\end{aligned}$$

This calculation shows therefore that for  $Z^2 > 16.5 A$ , we will have  $\Delta(\text{B.E.}) > 0$ , which corresponds to the condition that the two daughter nuclei will be more tightly bound than the parent nucleus. It follows therefore that for  $16.5 A < Z^2 < 47 A$ , when the spherical shape of the parent nucleus is stable under small perturbations, it is, nevertheless, energetically favorable for the parent to fragment into two lighter nuclei.

Our previous discussion can now be incorporated more quantitatively into a graph of the potential energy of the two fission fragments as a function of their separation distance (see Fig. 5.3). When the two daughter nuclei are far apart, their potential energy relative to the parent is given by Eq. (5.10). For  $A \approx 240$  and  $Z \approx 92$ , this corresponds to  $\approx 200$  MeV for two smaller nuclei of comparable size. As the fragments are brought closer together, they sense the repulsive Coulomb potential, which increases as the separation between them decreases. For  $r = r_0$ , approximately when the daughter nuclei start touching, the Coulomb potential is at its maximum, and is of the order of  $\approx 250$  MeV. (This reduces by  $\approx 10$ – $15\%$  for daughter nuclei with asymmetric  $Z$ -values of  $\approx 2:1$ .) For  $r < r_0$ , the two nuclei begin to fuse into a single deformed nucleus and, as we have discussed, there are two possibilities for the evolution of the system. (Note that when  $r < r_0$ , the value of  $r$  provides an effective measure of the elongation of the

deformed nucleus and is therefore proportional to the deformation parameter  $\epsilon$  given in Eq. (5.3).) First, when  $Z^2 > 47 A$ , the spherical shape is unstable, and that the energy decreases quadratically with deformation (see Eq. (5.7)). This corresponds to Branch-I in the potential energy. In this situation, at the slightest perturbation, the spherical parent nucleus evolves into two separate nuclei because continued separation of the original object is energetically favorable for all  $r$  values. The nucleus will therefore “roll downhill” very rapidly, and spontaneously fission. When  $Z^2 < 47 A$ , the spherical parent nucleus corresponds to a stable bound state whose energy increases quadratically with deformation. This is represented by Branch-II of the potential-energy graph. In this case, classically, the parent nucleus will be at the bottom of the potential well, but, due to quantum corrections, the ground state acquires a zero-point energy given by some  $E_0$ . If  $E_c$  denotes the peak of the Coulomb barrier, then this is the classical amount of energy a nucleus must have in order to undergo fission. In other words, a nucleus must acquire an amount of energy  $E_c - E_0$  in order to split apart. This is known as the *activation energy*, and its value is typically between 6–8 MeV for nuclei with  $A \approx 240$ . For Branch-II, the parent nucleus can also fission through quantum mechanical tunneling through the barrier. However, the probability for this, as mentioned in our treatment of barrier penetration in Chapter 4, is exceedingly small since the fragment masses are large and, correspondingly, the lifetime for such a process will be quite long. Because the evolution of nuclei into two daughter nuclei on Branch-I is always energetically favorable, such fissions will be very fast.

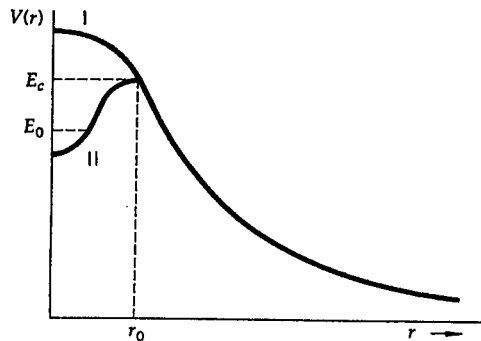


Fig. 5.3 Potential energy for the interaction of two medium-size nuclei as a function of their separation distance. The nuclei just touch at  $r = r_0$ , and coalesce at  $r = 0$ .

This elementary theory of fission, based on the liquid drop model, is due

to Niels Bohr and John Wheeler, and although classical in its conception, it leads to a surprisingly good understanding of both natural and induced fission. It explains, in particular, why thermal neutrons induce fission in  $^{235}\text{U}$ , whereas only higher-energy neutrons can produce fission of  $^{238}\text{U}$ . This difference can be argued in two ways. First, from a qualitative viewpoint, because  $^{235}\text{U}$  is an odd-even nucleus and  $^{238}\text{U}$  is even-even, it follows that the ground state of  $^{235}\text{U}$  will lie higher (less tightly bound) in the potential well of its fragments than that of  $^{238}\text{U}$ . Hence, to induce fission, a smaller perturbation will be needed for  $^{235}\text{U}$  than for  $^{238}\text{U}$ . More quantitatively, we can estimate the activation energy that is required for the fissioning of  $^{236}\text{U}$  and  $^{239}\text{U}$ ; this can be calculated as about 5 MeV for the former and over 6 MeV for the latter. Now, in capturing another neutron,  $^{235}\text{U}$  becomes an even-even compound nucleus, and the process therefore changes an odd-even nucleus to a more tightly bound even-even nucleus; consequently, this kind of transformation releases energy (the binding energy of the last neutron is  $-6.5$  MeV), which is sufficient for providing the activation energy needed for the compound nucleus to fission into its fragments. The kinetic energy of the incident neutron is irrelevant in this process, and hence even thermal neutrons can induce fission in  $^{235}\text{U}$ . In contrast, the capture of a neutron by  $^{238}\text{U}$ , changes it from an even-even to an odd-even nucleus. In other words, neutron capture in this case changes a tightly bound nucleus to a less tightly bound one, which is a less exothermic process (the binding energy of the last neutron in  $^{239}\text{U}$  is  $-4.8$  MeV, short of the more than 6 MeV required for fission). It is for this reason that higher energy neutrons, of energy greater than 1.2 MeV, are needed to provide the additional activation energy required for  $^{238}\text{U}$  to fission. We also wish to note, that although the pairing term (i.e., the last term in Eq. (3.2)) is negative for even-even nuclei and zero for odd-even nuclei, and reflects the qualitative behavior of the two systems, this term alone does not account for the entire difference observed in the neutron induced fission of  $^{235}\text{U}$  and  $^{238}\text{U}$ .

In the preceding example, we assumed that the fission fragments have equal mass, which would appear to be most natural. In general, however, the fission fragments have quite asymmetrical mass distributions (this reduces the effective size of the Coulomb barrier). In fact, masses of daughter nuclei tend to cluster around nucleon numbers of  $A \approx 95$  and  $A \approx 140$ . Thus far there is no fundamental understanding of this particular clustering. Just after fission, daughter nuclei are usually left in excited states and they decay to ground states through neutron emission or *evaporation*. Thus, neutrons are often produced along with the larger fission products.

In general, the original fission products, being neutron rich, do not lie on the stability line in the  $N$ - $Z$  plane but above it, and eventually decay to the stability line through  $\beta^-$  emission.

### 5.2.2 Chain Reaction

It is clear from the above discussion that each nuclear fission produces a large amount of energy. This in itself would not be very interesting, because what is needed for useful applications is a steady supply of energy. What makes fission attractive as a possible source of commercial power is the fact that neutrons are often produced along with the daughter nuclei. For example, in  $^{235}\text{U}$ , an average of 2.5 neutrons are produced per nuclear fission. Since such neutrons can induce additional fission, they can, in principle, sustain a continuous process, and thereby provide a useful output of energy.

Let us define the ratio of neutrons produced in the successive stages of fission by

$$k = \frac{\text{Number of neutrons produced in the } (n + 1) \text{ stage of fission}}{\text{Number of neutrons produced in the } n \text{ stage of fission}}. \quad (5.11)$$

If this ratio is less than unity, i.e., if  $k < 1$ , the process is called *sub-critical*. It is clear that in this case the fissioning of some sample of material cannot continue indefinitely, and eventually the reaction stops. This condition is therefore not very useful for generating power. If  $k = 1$ , namely, the number of neutrons inducing fission remains constant at every stage, then the process is called *critical*. In this case, a continued reaction rate is possible. This is the most desirable condition for providing a constant supply of power in a nuclear reactor. When  $k > 1$ , then more and more neutrons are produced at every stage of fission, causing a runaway chain reaction. This scenario is called *supercritical*, and it causes the output energy to grow rapidly, leading to an uncontrollable explosion. Needless to say, this kind of condition finds application in the design of nuclear weapons.

In a controlled environment, such as a nuclear reactor, the chain reaction can be put to practical use for generating power. Very briefly, a nuclear reactor consists of several components, the most important of which is the core (see Fig. 5.4). The core contains the fissile material, or fuel elements, the control rods and the moderator. Natural uranium can be used as the fuel in a reactor. However, because natural uranium is a mixture of  $^{235}\text{U}$

and  $^{238}\text{U}$ , and  $^{235}\text{U}$  has a shorter lifetime (about  $7 \times 10^8$  yr) than  $^{238}\text{U}$  (about  $5 \times 10^9$  yr), the natural mixture contains only a small fraction of  $^{235}\text{U}$  (the ratio of  $^{235}\text{U}$  to  $^{238}\text{U}$  in natural uranium is about 1:138). As a result, most of the thermal neutrons incident on such a sample will be captured radiatively by  $^{238}\text{U}$  nuclei, and will not induce fission. It is for this reason that enriched uranium, which is essentially pure  $^{235}\text{U}$ , is used as fuel in nuclear reactors.

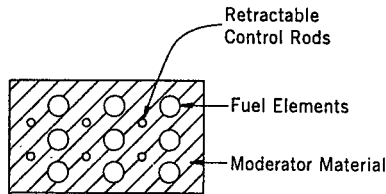


Fig. 5.4 Sketch of the elements of the core of a reactor.

The control rods in a reactor are often made of cadmium, which has a high absorption cross section for neutrons. Therefore, by retracting or inserting the control rods, the number of neutrons available for inducing fission can be regulated. This mechanism is the key element in maintaining a constant  $k$ -value and, therefore a constant power output. The fuel elements are usually surrounded by a moderator, whose main function is to slow down any fast neutrons that may be produced in the course of fission, so that they will have a larger probability of being absorbed and thereby induce more fission (higher energy neutrons have smaller absorption cross sections). It is advantageous to have moderator material that is inexpensive, and with a negligible cross section for absorption of neutrons. Heavy water ( $\text{D}_2\text{O}$ ), for example, is preferred as a moderator over normal water ( $\text{H}_2\text{O}$ ) because the cross section for neutron capture by protons in normal water (to make deuterons) is much larger than that for capture by deuterons in heavy water (to make tritium nuclei).

In a power plant (see Fig. 5.5), the reactor core is immersed in a coolant (often water), which removes the heat energy produced in the core, and keeps the core at sufficiently-low temperature to prevent a meltdown. (The heat is generated from energy deposited by the fission remnants as they ionize the material in the core.) The entire set up is surrounded by heavy shielding needed to minimize any leakage of radiation. When the nuclear reactor starts operation, the value of  $k$  is set slightly higher than unity,

and is maintained at that value until the desired power output is achieved, after which point the  $k$ -value is lowered to unity. As we have just noted, the coolant removes the heat energy produced in the fission process, and this can then be used to boil water and produce steam. The steam, in turn, can run turbines that generate electricity. This is, of course, only a very basic outline of the design and functioning of a nuclear power reactor. In practice, the design and construction is far more complicated, especially because of the many safety features needed to avoid accidents.

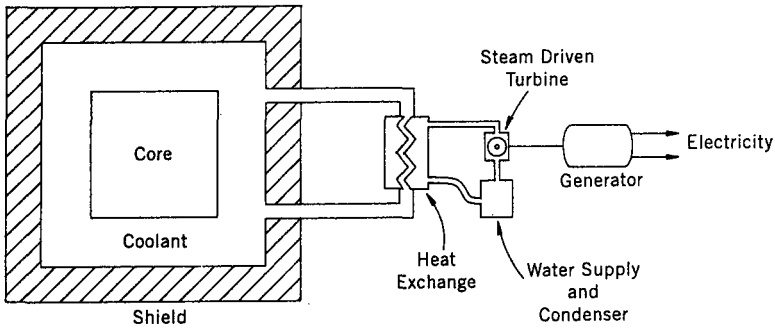


Fig. 5.5 Sketch of the main elements in a nuclear power plant.

Finally, it is interesting to calculate the maximum energy expected from a nuclear reactor. As we have seen, the fission of one  $^{235}\text{U}$  nucleus yields  $\approx 200$  MeV or  $3.2 \times 10^{-11}$  joules. Now, one gram of any element contains  $\frac{A_0}{A}$  atoms, where  $A_0$  is Avogadro's number, and one gram of  $^{235}\text{U}$  contains therefore about  $6 \times \frac{10^{23}}{235} \approx 3 \times 10^{21}$  atoms. It therefore follows that the complete fission of one gram of  $^{235}\text{U}$  can yield a total energy of

$$\begin{aligned}
 &\approx 3 \times 10^{21} \times 3.2 \times 10^{-11} \text{ joules} \\
 &\approx 10^{11} \text{ joules} \\
 &\approx 1 \text{ MWD (MegaWatt-Day)}.
 \end{aligned}
 \tag{5.12}$$

That is, the fission of one gram of  $^{235}\text{U}$  can produce one MW of power for an entire day. To compare this yield to the energy expected from one gram of coal, we recall that burning one ton of coal yields a thermal energy of 0.36 MWD. Thus, ignoring relative efficiency of converting the energy to electric power, a gram of completely fissioned  $^{235}\text{U}$  yields about  $\approx 3 \times 10^6$  more energy than a gram of coal.



### 5.3 Nuclear Fusion

The data for the binding energy per nucleon has an interesting structure in that it shows a maximum for medium-size nuclei with  $A \approx 60$ . As we have seen, the subsequent slow decrease in this value for increasing  $A$  is what makes nuclear fission possible. For lighter nuclei, the binding energy per nucleon falls much more sharply, indicating that, with the exception of magic nuclei, lighter nuclei are less tightly bound than medium-size nuclei. We can therefore imagine using a process just opposite of fission that can serve as another source for generating energy. Namely, if we combine (fuse) two light nuclei into a relatively heavy and tightly bound nucleus, this can then lead to a release of energy due to the difference in binding energies of the initial and the final states. This process is known as nuclear *fusion*, and the energy released per nucleon is comparable to that in fission. However, since lighter nuclei contain fewer nucleons, the total energy released per fusion is smaller. On the other hand, since there is an abundance of light and stable nuclei in nature, fusion provides an attractive alternative for generating power. Fusion is, in fact, the mechanism responsible for energy generation in the interior of the sun and of other stars.

In principle, fusion can take place when two light nuclei are brought close enough to each other so that they can overlap and fuse, and thereby release energy. However, for this to happen, the Coulomb barrier between the two nuclei has to be overcome. The value of the repulsive Coulomb energy is a maximum when the two nuclei are just touching, and has the form

$$V_{\text{Coulomb}} = \frac{ZZ'e^2}{R + R'}, \quad (5.13)$$

where  $Z$  and  $Z'$  are the atomic numbers of the two nuclei, and  $R$  and  $R'$  are their respective radii. Recalling Eq. (2.16), we can rewrite this as

$$\begin{aligned} V_{\text{Coulomb}} &= \frac{e^2}{\hbar c} \frac{\hbar c Z Z'}{1.2[A^{\frac{1}{3}} + (A')^{\frac{1}{3}}] \text{fm}} \\ &= \frac{1}{137} \frac{197 \text{ MeV-fm}}{1.2 \text{ fm}} \frac{Z Z'}{A^{\frac{1}{3}} + (A')^{\frac{1}{3}}} \\ &\approx \frac{Z Z'}{A^{\frac{1}{3}} + (A')^{\frac{1}{3}}} \text{ MeV} \approx \frac{1}{8} A^{\frac{5}{3}} \text{ MeV}, \end{aligned} \quad (5.14)$$

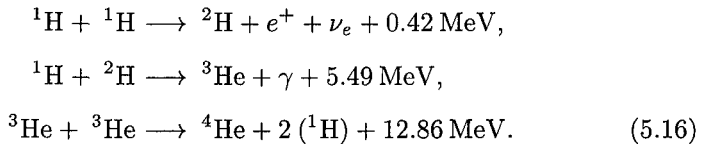
where  $A$  and  $A'$  are the nucleon numbers of the two light nuclei, and the final expression is obtained by setting  $A \approx A' \approx 2Z \approx 2Z'$ . Thus, the Coulomb barrier between two nuclei of  $A \approx 8$  is about 4 MeV. Consequently, for fusion to take place, we must provide kinetic energies of the order of a few MeV to overcome the Coulomb barrier (clearly, the exact value depends on the specific nuclear masses and charges).

It would therefore appear that a natural way to achieve fusion is by colliding two energetic beams of light nuclei. In such a process, however, most of the nuclei get scattered elastically and, as a result, this turns out to be an inefficient way of inducing fusion. An alternative method is to heat up the relevant nuclei to high temperatures to provide them with sufficient kinetic energy to overcome the Coulomb barrier. To estimate these temperatures, let us assume that each of the nuclei needs a kinetic energy of about 2 MeV (that is, the Coulomb barrier is roughly 4 MeV). Recalling that room temperature (300 K) corresponds to  $\frac{1}{40}$  eV, we obtain that 2 MeV corresponds to

$$\frac{2 \times 10^6 \text{ eV}}{\left(\frac{1}{40}\right) \text{ eV}} \times 300 \text{ K} \approx 10^{10} \text{ K.} \quad (5.15)$$

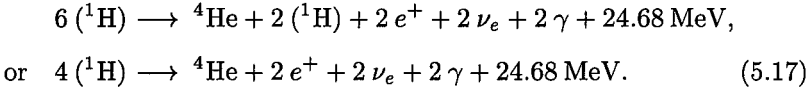
Although this is beyond the typical scale of  $\approx 10^7$  K temperatures found inside the sun and the stars, the Maxwellian tail of the spectrum in the sun goes out far enough to provide the required excursions in energy, and explains why fusion can take place in stellar interiors. There is a variety of fusion reactions that can take place inside stars, and we will describe only two of the “burning cycles”.

Our sun has a mass of about  $10^{30}$  kg, which consists primarily of about  $10^{56}$  hydrogen atoms. Consequently, we expect that the main source of energy in the sun is derived from the burning of hydrogen. This happens through the proton-proton cycle, as suggested initially by Hans Bethe



The large amount of kinetic energy released in the last step is due to the fact that the  ${}^4\text{He}$  nucleus is doubly magic, and is bound extremely tightly. The final kinetic energies are shared among the end-products of the reactions,

and can be deposited in the stellar material. Thus, in effect, a proton-proton cycle burns four hydrogen atoms to obtain

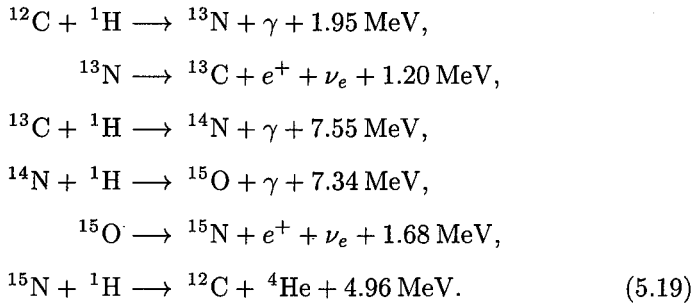


The atoms inside the sun are in a highly ionized plasma state, and consequently the positrons ( $e^+$ ) emitted in this cycle can annihilate with the prevalent electrons and contribute to an increase in the total release of energy. Similarly, the produced photons can interact with stellar matter, and deposit their energy. From the fact that the age of the universe is about  $10^{10}$  yr, and from the power output of the sun, we can estimate that the sun will continue to burn for about another  $10^9$  yr before it runs out of fuel for fusion.

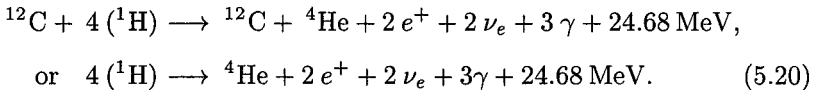
Another fusion cycle that plays a fundamental role within stars is the carbon or CNO cycle. The helium produced in the proton-proton cycle can produce carbon nuclei through the reaction



Subsequently, the carbon nucleus absorbs a hydrogen nucleus leading to

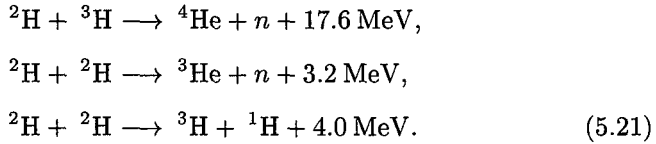


Thus, in the full carbon cycle, we effectively get



The types of burning cycles that take place in different stars determine their specific evolution.

Finally, we mention that there is a considerably large effort world-wide to achieve controlled thermonuclear fusion. In fact, the following reactions have already been observed under laboratory conditions



The main difficulty in producing fusion on a large scale lies in the problem of containing the fuel material sufficiently long at the high temperatures needed to penetrate the Coulomb barrier. At present there are two popular methods vying for this goal. One is magnetic confinement, wherein a hot plasma of  ${}^2\text{H}$  and  ${}^3\text{H}$  circulates and fuses within a region of confining electromagnetic fields. The other is inertial confinement, wherein electromagnetic energy (laser light or beams of heavily ionizing ions) is injected into a small region that contains the fuel material. However, much work remains to be done before fusion can be put to practical use.

#### 5.4 Radioactive Decay

As we have seen, unstable nuclei can often transmute into other nuclei through the emission of  $\alpha$ ,  $\beta$  or  $\gamma$  particles. Any such spontaneous transition from one state to another is known as a *radioactive decay*, and in this chapter we will describe some general properties of such processes.

As we mentioned before, radioactive decay can be described as a statistical process. Namely, if we have a large number of radioactive nuclei, we cannot say specifically which nucleus will decay at any given time. But there is a unique constant probability of decay associated with each nucleus. Thus, if  $N$  denotes the number of radioactive nuclei of any specified type, at a given time, and  $\lambda$  is the constant probability for decay per unit time (that is, the *decay constant*), then the change in the number of nuclei during an infinitesimal time interval  $dt$  is defined by

$$dN = N(t + dt) - N(t) = -N(t)\lambda dt. \tag{5.22}$$

The negative sign in the above equation represents, as usual, the fact that the number of nuclei decreases as a result of decay. If we assume  $N_0$  to be the initial number of nuclei at  $t = 0$ , then the number of nuclei  $N(t)$  at any later time can be obtained from Eq. (5.22) as

$$\begin{aligned} \frac{dN}{N} &= -\lambda dt, \\ \text{or } \int_{N_0}^N \frac{dN}{N} &= -\lambda \int_0^t dt, \\ \text{or } \ln \frac{N(t)}{N_0} &= -\lambda t, \\ \text{or } N(t) &= N_0 e^{-\lambda t}. \end{aligned} \tag{5.23}$$

In other words, for a radioactively decaying system, the number of nuclei that survive decreases exponentially, and vanishes only at infinite times. This is the characteristic law for all such statistical decay processes.

There are several time scales that can be associated with a radioactive system. We can denote by  $t_{\frac{1}{2}}$  the time interval during which half of the nuclei in the sample decay. It then follows that

$$N\left(t_{\frac{1}{2}}\right) = \frac{N_0}{2} = N_0 e^{-\lambda t_{\frac{1}{2}}},$$

$$\text{or } \lambda t_{\frac{1}{2}} = \ln 2,$$

and

$$t_{\frac{1}{2}} = \frac{\ln 2}{\lambda} = \frac{0.693}{\lambda}. \tag{5.24}$$

If the decay constant is known (or can be calculated), then the *half-life*  $t_{\frac{1}{2}}$  can be obtained and compared directly with measured values. Another useful time scale for describing decays is the average or *mean life* of a radioactive material. This can be calculated using Eq. (5.23)

$$\begin{aligned}
 \langle t \rangle = \tau &= \frac{\int_0^\infty t N(t) dt}{\int_0^\infty N(t) dt} \\
 &= \frac{N_0 \int_0^\infty t e^{-\lambda t} dt}{N_0 \int_0^\infty e^{-\lambda t} dt} \\
 &= \frac{\lambda^{-2}}{\lambda^{-1}} = \frac{1}{\lambda},
 \end{aligned} \tag{5.25}$$

where the definite integrals can be performed directly or found in tables (they are related to "Gamma" functions). Consequently, as we mentioned in the previous chapter in connection with barrier penetration, the mean life of the sample is the inverse of the decay constant. Furthermore,  $\tau$  is related to the half-life through the multiplicative constant  $\ln 2 = 0.693$ .

We noted before that Eq. (5.23) implied that it would take an infinite amount of time for the total sample to disintegrate. Nevertheless, after several half-lives, the number of decaying nuclei is often too small to be detected. The number of disintegrations per unit time, or the *activity*, of a material is defined as follows

$$A(t) = \left| \frac{dN}{dt} \right| = \lambda N(t) = \lambda N_0 e^{-\lambda t}. \tag{5.26}$$

The activity is clearly a function of time and, in fact, also falls off exponentially with time. For example, for  $^{226}\text{Ra}$ , whose half-life is 1620 years, we have  $t_{\frac{1}{2}} = 1620 \text{ yr} = 1.62 \times 10^3 \text{ yr} \approx 1.62 \times 10^3 \times 3.1 \times 10^7 \text{ sec} \approx 5 \times 10^{10} \text{ sec}$ . Consequently,

$$\lambda = \frac{0.693}{t_{\frac{1}{2}}} \approx \frac{0.693}{5 \times 10^{10} \text{ sec}} \approx 1.4 \times 10^{-11} / \text{sec}. \tag{5.27}$$

If the radioactive sample at  $t = 0$  consists of one gram of  $^{226}\text{Ra}$ , then the original number of radioactive nuclei in the sample is

$$N_0 \approx \frac{6 \times 10^{23}}{226} \approx 2.7 \times 10^{21}, \tag{5.28}$$

and the activity of the sample at  $t = 0$  is consequently

$$\begin{aligned}
 \mathcal{A}(t = 0) &= \lambda N_0 \\
 &\approx 1.4 \times 10^{-11} \times 2.7 \times 10^{21} / \text{sec} \\
 &\approx 3.7 \times 10^{10} \text{ disintegrations/sec.} \qquad (5.29)
 \end{aligned}$$

This initial activity falls off exponentially in time with the same decay constant as given in Eq. (5.27).

The natural activity of  $^{226}\text{Ra}$  has been used to define a unit of radioactivity. Thus, any sample with  $3.7 \times 10^{10}$  disintegrations per second is said to have a radioactivity of 1 curie (Ci), named after Pierre Curie. Typical laboratory samples usually have far smaller radioactivities of the order of a millicurie = 1 m Ci =  $10^{-3}$  Ci =  $3.7 \times 10^7$  disintegrations/sec, or a microcurie = 1  $\mu$  Ci =  $10^{-6}$  Ci =  $3.7 \times 10^4$  disintegrations/sec. A more rational unit of activity is known as the rutherford (rd), defined as  $10^6$  disintegrations/sec. An activity of 1 micro-rutherford ( $\mu\text{rd}$ ) in a material corresponds therefore to 1 disintegration per second, and is referred to as one becquerel (Bq).

### Example 1

Let us suppose that we have a small sample of radioactive substance that has a mean life  $\tau$  of  $10^3$  seconds. At some time  $t = 0$  we observe  $10^6$  disintegrations per second. At a later time  $t$ , we would expect from Eq. (5.26) that the activity would be

$$\mathcal{A}(t) = \mathcal{A}(0)e^{-\lambda t}.$$

Thus if we want the number of disintegrations expected in any 10 sec interval centered on  $t$ , this would be

$$\begin{aligned}
 \Delta N(t) &= \int_{t-5}^{t+5} dt \mathcal{A}(t) = -\frac{1}{\lambda} \mathcal{A}(0) e^{-\lambda t} \Big|_{t-5}^{t+5} \\
 &= \tau \mathcal{A}(0) e^{-\lambda t} \Big|_{t-5}^{t+5} = \tau \mathcal{A}(0) \left( e^{-\lambda(t-5)} - e^{-\lambda(t+5)} \right).
 \end{aligned}$$

Let us suppose that we wish to know  $\Delta N(t)$  at  $t = 1000$  sec. Then, for a 10 sec interval centered on  $t = 1000$ , we would predict

$$\begin{aligned}
\Delta N(1000) &= \tau \mathcal{A}(0) \left[ e^{-\frac{995}{1000}} - e^{-\frac{1005}{1000}} \right] \\
&= \tau \mathcal{A}(0) e^{-1} \left( e^{\frac{5}{1000}} - e^{-\frac{5}{1000}} \right) \\
&\approx \tau \mathcal{A}(0) \frac{1}{e} \left( \left( 1 + \frac{5}{1000} + \dots \right) - \left( 1 - \frac{5}{1000} + \dots \right) \right) \\
&\approx \tau \mathcal{A}(0) \frac{10}{1000e} = \frac{10^3 \times 10^6 \times 10}{10^3 \times 2.7} \approx 4 \times 10^6 \text{ counts.}
\end{aligned}$$

In fact, the general expression for an arbitrary  $\Delta t$  would be

$$\Delta N(t) = \tau \mathcal{A}(0) e^{-\frac{t}{\tau}} \left( e^{\frac{\Delta t}{2\tau}} - e^{-\frac{\Delta t}{2\tau}} \right),$$

which for  $\Delta t \ll \tau$  reduces to

$$\Delta N(t) \approx \tau \mathcal{A}(0) \frac{\Delta t}{\tau} e^{-\frac{t}{\tau}} = \mathcal{A}(0) \Delta t e^{-\frac{t}{\tau}}.$$

Clearly, the expected number of disintegrations for our chosen time interval will drop with time. There is, of course, no a priori way of determining specifically which of our nuclei will disintegrate; we know only the expected average number of disintegrations. In statistical processes, where the probability of any occurrence ( $p$ ) is small, but there is a large sample of events ( $N$ ) that can contribute to the process, Poisson statistics can be used to describe the system. For Poisson statistics, when the expected mean is  $\Delta N = pN$ , then the error or standard deviation on the mean can be shown to be just  $\sqrt{pN} = \sqrt{\Delta N}$ . (Note that for our chosen interval of  $\Delta t = 10$  sec, the probability  $p = \lambda \Delta t = 10^{-2} \ll 1$ , and therefore Poisson statistics are appropriate.)

Going back to our specific example, where  $\Delta N$  is  $4 \times 10^6$ , we must now interpret the predicted result as follows. We can state that, in any given experiment performed to count  $\Delta N$ , we will rarely observe the exact expected mean number of  $\Delta N$  counts. What we will see is that, in about 68% of such experiments (assuming a Gaussian approximation for the error), the observed counting rate will fall between  $\Delta N - \sqrt{\Delta N}$  and  $\Delta N + \sqrt{\Delta N}$ . Thus, if we expect  $\Delta N = 4 \times 10^6$  counts, then  $\frac{\sqrt{\Delta N}}{\Delta N}$  is only  $5 \times 10^{-4}$ , so the fluctuations about the mean will be at the level of  $\approx 0.05\%$ . However,



if we wish to look at the counting rate at a somewhat later time of  $t = 10^4$  sec, then  $\Delta N(t = 10^4)$  will be far smaller

$$\Delta N(10^4) \approx 10^6 \times 10 \times e^{-10} \approx 450,$$

and  $\sqrt{\Delta N}$  will be  $\approx 21$ , and the relative deviations from the expected value will therefore be larger and easier to observe.

#### 5.4.1 *Radioactive Equilibrium*

As we have indicated, upon decay, a radioactive parent nucleus produces what is called a daughter nucleus. The daughter nucleus can either be stable or radioactive. If it is radioactive, then it decays into a granddaughter nucleus and so on. Thus, each radioactive parent nucleus initiates a series of decays, with each decay-product having its own characteristic decay constant and, therefore, a different half-life. In general, the mean life of the parent nucleus is much longer than that of any other member of the decay chain, and this will be important for the observations that follow.

Consider a radioactive sample of material where the parent nucleus has a very long life time, and therefore the number of parent nuclei barely changes during some small time interval. Let us suppose that the daughter, granddaughter, etc., decay comparatively fast. After a certain lapse in time, a situation may develop where the number of nuclei of any member of the decay chain stops changing. In such a case, one says that radioactive equilibrium has set in. To see when this can occur, let us denote by  $N_1, N_2, N_3, \dots$  the number of nuclei of species 1, 2, 3,  $\dots$  in the series, at some specified time, and by  $\lambda_1, \lambda_2, \lambda_3, \dots$ , respectively, the decay constants for these members of the decay chain. The equations governing the time-evolution of the populations  $N_1, N_2, N_3, \dots$  can be deduced from the contributions to the change in any species, as follows. The daughter nuclei are produced at a rate of  $\lambda_1 N_1$  due to the decay of the parent nuclei (see Eq. (5.22)), and they in turn decay at a rate of  $\lambda_2 N_2$ . The difference between the two gives the net rate of change of the daughter nuclei. For any nucleus in the chain, there will be a similar increase in population from the feed-down and a decrease from decay, except for the parent nucleus, for which there is no feed-down possible. Thus, for the change in the number of parent, daughter, granddaughter nuclei, etc., in a time interval  $\Delta t$ , we can write

$$\begin{aligned}
 \Delta N_1 &= -\lambda_1 N_1 \Delta t, \\
 \Delta N_2 &= \lambda_1 N_1 \Delta t - \lambda_2 N_2 \Delta t, \\
 \Delta N_3 &= \lambda_2 N_2 \Delta t - \lambda_3 N_3 \Delta t, \\
 &\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots.
 \end{aligned}
 \tag{5.30}$$

Dividing Eq. (5.30) by  $\Delta t$ , and taking the limit of infinitesimal time intervals, we can rewrite relations (5.30) in terms of our species numbers  $N_1, N_2, N_3, \dots$ , as follows

$$\begin{aligned}
 \frac{dN_1}{dt} &= -\lambda_1 N_1, \\
 \frac{dN_2}{dt} &= \lambda_1 N_1 - \lambda_2 N_2, \\
 \frac{dN_3}{dt} &= \lambda_2 N_2 - \lambda_3 N_3, \\
 &\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots.
 \end{aligned}
 \tag{5.31}$$

We say that a *secular equilibrium* is reached when

$$\frac{dN_1}{dt} = \frac{dN_2}{dt} = \frac{dN_3}{dt} = \dots = 0.
 \tag{5.32}$$

Note that by assumption,  $\tau_1$  is very large, and the change in  $N_1$  is therefore very small ( $\frac{dN_1}{dt} \approx 0$ ). Clearly, Eq. (5.32) holds only when

$$\lambda_1 N_1 = \lambda_2 N_2 = \lambda_3 N_3 = \dots,
 \tag{5.33}$$

or, equivalently, when

$$\frac{N_1}{\tau_1} = \frac{N_2}{\tau_2} = \frac{N_3}{\tau_3} = \dots.
 \tag{5.34}$$

Consequently, under these conditions, the daughter, the granddaughter, etc., will all be in equilibrium with each other, as well as with the parent nucleus (i.e., their numbers will effectively not change with time).

### 5.4.2 Natural Radioactivity and Radioactive Dating

There are about 60 radioactive nuclei found in nature. This is much smaller than the order of  $\approx 1000$  radioactive isotopes that have been produced artificially in laboratories. If, at the time of the formation of our planet, all isotopes were almost equally abundant, then their absence in nature can be used to estimate the age of the Solar System. In fact, our Solar System is believed to be about 10 billion years old ( $10^{10}$  yr), it is therefore not surprising that during this time most of the radioactive nuclei with shorter lifetimes have completely decayed away.

The naturally occurring radioactive nuclei have atomic numbers mostly between  $Z = 81$  and  $Z = 92$ , and are characterized by substantial neutron excess. Nevertheless, the presence of a large number of protons in these nuclei leads to strong Coulomb repulsion and instability. Such nuclei can decay by successive emission of one or more  $\alpha$  particles (two protons and two neutrons). The resulting daughter nuclei, will therefore have an even larger neutron to proton ratio and will tend to decay through the emission of  $\beta^-$  particles. The granddaughters may still be unstable and decay again through the emission of more  $\alpha$ -particles. This chain of  $\alpha$  and  $\beta$  decays will continue until the nucleus reaches the  $N-Z$  stability band (Fig. 2.3). Because an  $\alpha$ -particle has four nucleons, the alternate  $\alpha$  and  $\beta$  decays will define a radioactive nuclear series with atomic mass numbers that differ by four nucleon units. This leads naturally to the four known series of heavy  $\alpha$ -emitters, whose daughter remnants differ progressively by four nucleons in their values of  $A$

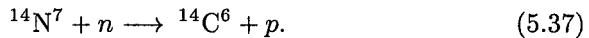
$$\begin{array}{ll}
 A = 4n & \text{Thorium series,} \\
 A = 4n + 1 & \text{Neptunium series,} \\
 A = 4n + 2 & \text{Uranium-Radium series,} \\
 A = 4n + 3 & \text{Uranium-Actinium series,}
 \end{array} \tag{5.35}$$

where  $n$  is an integer. Each of the series is labeled using the historical name of its parent nucleus, which is the longest-lived nuclide in the decay chain. (The parent of the “actinium” series is, in fact,  $^{235}\text{U}$ .) From the measured values of the mean lives of the parents

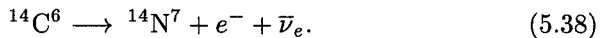
$$\begin{aligned}
\tau(\text{thorium } ^{232}\text{Th}^{90}) &\approx 1.39 \times 10^{10} \text{ yr}, \\
\tau(\text{neptunium } ^{237}\text{Np}^{93}) &\approx 2.2 \times 10^6 \text{ yr}, \\
\tau(\text{uranium } ^{238}\text{U}^{92}) &\approx 4.5 \times 10^9 \text{ yr}, \\
\tau(\text{"actinium" } ^{235}\text{U}^{92}) &\approx 7.15 \times 10^8 \text{ yr},
\end{aligned}
\tag{5.36}$$

and from the fact that the age of the universe is about  $10^{10}$  yr, we should not expect to find any of the radioactive isotopes of the Neptunium series on earth. In fact, we have natural evidence only for the parents of the other three series. It is also curious that isotopes of lead define the stable ends for each of these three series, namely,  $^{208}\text{Pb}^{82}$ ,  $^{206}\text{Pb}^{82}$  and  $^{207}\text{Pb}^{82}$ , corresponding to the Thorium, Uranium and the Actinium series, respectively. In addition to the heavier nuclei, there exist a few medium-size nuclei in nature, such as  $^{40}\text{K}^{19}$  ( $t_{\frac{1}{2}} \approx 1.3 \times 10^9$  yr) and  $^{115}\text{In}^{49}$  ( $t_{\frac{1}{2}} \approx 5 \times 10^{14}$  yr), that are also radioactive.

One of the important applications of radioactivity is in determining the age of organic material that may be thousands of years old. The method is based on the following simple observation. Our atmosphere contains many gases, including  $^{14}\text{N}$  and  $^{12}\text{C}$ . Furthermore, the atmosphere is constantly being bombarded with high energy cosmic rays, consisting of protons, heavier nuclei, photons, and other particles. These cosmic rays interact with nuclei in the atmosphere, and produce particles of lower energy. Any slow neutrons produced in these collisions can be absorbed by  $^{14}\text{N}$  to produce an isotope of carbon that is radioactive



$^{14}\text{C}$  decays with a half-life of 5730 years through  $\beta^-$  emission



At any particular time therefore, our atmosphere contains a large amount of  $^{12}\text{C}$  and a very small amount of  $^{14}\text{C}$ , both of which can form carbon dioxide ( $\text{CO}_2$ ) molecules. Living organisms, such as plants, consume  $\text{CO}_2$  from the atmosphere and consequently contain both of these carbon isotopes. The intake of  $\text{CO}_2$  stops with the death of the organism. Subsequently,  $^{14}\text{C}$ , being radioactive, continues to decay, whereas the amount of  $^{12}\text{C}$  remains

unchanged. As a result, the relative concentration of the two isotopes in any fossil changes with time. By measuring directly the relative amounts of  $^{14}\text{C}$  and  $^{12}\text{C}$  in a fossil, and comparing this result with that in a corresponding living organism, we can estimate the age of the fossil. Alternatively, we can compare the activity of  $^{14}\text{C}$  in a fossil with that in a living organism and thereby also deduce the fossil's age. The second method is known as *radioactive dating*, or  $^{14}\text{C}$  *dating*, and finds great use in archaeological and anthropological studies. The idea of carbon dating was suggested initially by Walter Libby.

### Example 2

As an example, consider a piece of wood, weighing 50 g, which has an activity of 320 disintegrations/minute from  $^{14}\text{C}$ . The corresponding activity in a living plant is 12 disintegrations/minute/gm, and we wish to determine the age of the wood. (The half-life of  $^{14}\text{C}$  is  $t_{\frac{1}{2}} = 5730$  yr, and  $\lambda = \frac{0.693}{t_{\frac{1}{2}}}$ .) We are given that the initial and current activities are

$$\begin{aligned} \mathcal{A}(t = 0) &= 12/\text{min}/\text{gm}, \\ \mathcal{A}(t) &= \frac{320}{50}/\text{min}/\text{gm}. \end{aligned}$$

From the definition of activity, we can relate the activities at our two times as follows

$$\mathcal{A}(t) = \left| \frac{dN}{dt} \right| = \lambda N(t) = \lambda N_0 e^{-\lambda t} = \mathcal{A}(t = 0) e^{-\lambda t}.$$

Therefore, we obtain

$$\begin{aligned} \lambda t &= \ln \frac{\mathcal{A}(t = 0)}{\mathcal{A}(t)}, \\ \text{or } t &= \frac{1}{\lambda} \ln \left( \frac{12 \times 50}{320} \right) \approx \frac{5730 \text{ yr}}{0.693} \times 0.626 \\ &\approx 5170 \text{ years.} \end{aligned}$$

In other words, the piece of wood is about 5170 years old. Recently, carbon dating techniques have greatly improved through the use of nuclear mass

spectrometers, which can measure directly very small differences in the concentrations of  $^{14}\text{C}$  and  $^{12}\text{C}$  in any material. Using sample sizes of about 1 mg (as opposed to 1 gram in the older counting method), measurements with sensitivity of  $\approx 10^{-14}$  in the  $^{14}\text{C}/^{12}\text{C}$  ratio have been achieved. In our example we have ignored variations in concentrations of  $^{14}\text{C}$  stemming from any time dependence in the flux of cosmic rays, or from other, more recent, sources such as atmospheric nuclear testing. Such effects can be detected and must be taken into account in radioactive carbon dating.

## Problems

**5.1** To study neutron absorption cross sections at very low energies, one must often slow down (moderate) energetic ( $\approx 1$  MeV) neutrons that are produced in reactors. Show that paraffin would be a better moderator than aluminum, by specifically calculating the maximum energy that a 1 MeV neutron can transfer in a collision with a proton (within paraffin) as opposed to that with an Al nucleus.

**5.2** Calculate the energy released when 1 gm of  $^{235}\text{U}$  fissions into  $^{148}\text{La}$  and  $^{87}\text{Br}$ . Compare this to the energy released in fusing deuterium and tritium nuclei in 1 gm of tritiated water with 1 gm of deuterated water (i.e.,  $\text{T}_2\text{O}$  and  $\text{D}_2\text{O}$ ).

**5.3** The counting rate for a radioactive source is measured for one minute intervals every hour, and the resulting counts are: 107, 84, 65, 50, 36, 48, 33, 25, ... Plot the counting rate versus time, and from the graph roughly estimate the mean life and the half-life. Recalling that the expected error on  $N$  counts is  $\sqrt{N}$ , do the data points seem reasonable? (*Hint*: use "semi-log" paper to plot  $\log N$  vs  $t$ .)

**5.4** A relic from an Egyptian tomb contains 1 gm of carbon with a measured activity of  $4 \times 10^{-12}$  Ci. If the ratio of  $^{14}\text{C}/^{12}\text{C}$  nuclei in a live tree is  $1.3 \times 10^{-12}$ , how old is the relic? Assume the half-life of  $^{14}\text{C}$  is 5730 yr.

**5.5** If the lifetime of the proton is  $10^{33}$  yr, how many proton decays would you expect per year in a mass of  $10^3$  metric tons of water? What would be the approximate number expected in the year 2050?

**5.6** Calculate the surface energies and Coulomb energies for the following nuclei

$$^{228}\text{Th}, \ ^{234}\text{U}, \ ^{236}\text{U}, \ ^{240}\text{Pu}, \ ^{243}\text{Pu}.$$

Based on your calculations which nuclei would you expect to fission most easily?

**5.7** If the efficiency for conversion of heat to electricity is only 5%, calculate the rate of consumption of  $^{235}\text{U}$  fuel in a nuclear reactor operating at a power level of 500 MW of electricity.

**5.8** In the fission of  $^{235}\text{U}$ , the mass ratio of the two produced fission fragments is 1.5. What is the ratio of the velocities of these fragments?

**5.9** How much energy is liberated when 1 gram of hydrogen atoms is converted into helium atoms through fusion? Compare this with the energy liberated in the fission of 1 gm of  $^{235}\text{U}$ .

**5.10** The half life of radioactive cobalt-60 is 5.26 yr.

a) Calculate its mean life and disintegration constant.

b) What is the activity of 1 gm of  $^{60}\text{Co}$ ? Express this in curies and in rutherfords.

c) What is the mass of a 10-Ci sample of cobalt-60?

**5.11** Suppose that atoms of type 1 decay to type 2, which, in turn, decay to stable atoms of type 3. The decay constants of 1 and 2 are  $\lambda_1$  and  $\lambda_2$ , respectively. Assume that at  $t = 0$ ,  $N_1 = N_0$  and  $N_2 = N_3 = 0$ . What are the values for  $N_1(t)$ ,  $N_2(t)$  and  $N_3(t)$  at any later time  $t$ ?

**5.12** The activity of a certain material decreases by a factor of 8 in a time interval of 30 days. What is its half life, mean life and disintegration constant?

**5.13** For a prolate spheroid (ellipsoid) with eccentricity  $x$ , the semi-major axis  $a$  and semi-minor axis  $b$  in Fig. 5.2 are related through  $b = \sqrt{1 - x^2} a$ . If the volume and surface area of the nuclear ellipsoid are given, respectively, as  $\frac{4}{3}\pi ab^2$  and  $2\pi b \left( b + \frac{a \sin^{-1} x}{x} \right)$ , defining  $\epsilon = \frac{1}{3}x^2$ , show that Eq. (5.5) holds for small values of  $x$ . (*Hint*: Assume that the volume does not change under distortion; expand functions of  $x$ , and keep all terms up to order  $x^5$ .) Using this result, roughly, how would you argue that Eq. (5.6) has the right dependence?

**5.14** Secular equilibrium can also be defined through the requirement that

$$\frac{d}{dt} \left( \frac{N_2}{N_1} \right) = \frac{d}{dt} \left( \frac{N_3}{N_2} \right) = \frac{d}{dt} \left( \frac{N_4}{N_3} \right) \dots = 0.$$

Assuming  $\lambda_1 \ll \lambda_2, \lambda_3, \lambda_4 \dots$ , show explicitly that you retrieve the first three relations in Eq. (5.33). What happens for the final state of the decay chain? Is this sensible?

### Suggested Readings

Bevington, P. R., *Data Reduction and Analysis for the Physical Sciences*, McGraw-Hill (1969).

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## Chapter 6

# Energy Deposition in Media

### 6.1 Introductory Remarks

Physics is an experimental science and experiments provide the foundation for our understanding of nature and of physical laws. As we have argued repeatedly, nowhere has the need for experiments been greater than in the development of nuclear and particle physics. In these sub-atomic domains, scattering of particles from each other provides the primary source of information. The experiments are often quite challenging in their own right, and the experimental techniques can be as fascinating as the underlying structure they are meant to study. In this and in chapter the following two chapters, we will discuss some of the principles and devices that form the basis of experimentation in nuclear and particle physics. Most modern experiments rely on the application of a variety of exceedingly sophisticated electronic and computer tools. These tools provide the means for automatically preselecting interactions of greatest interest and of handling of enormous volumes of scientific data. We will not cover these important areas of experimentation, but will rather restrict ourselves to the more general ideas encountered in the acceleration of probe particles to high energies and in the detection of particles produced in sub-atomic collisions. We begin with the principles underlying the detection of different kinds of particles, and defer the description of detectors and accelerators to following chapters.

In order to be detected, an object must leave some trace of its presence. That is, it must deposit energy in its wake. Ideally, detectors should help us observe particles without affecting them in any measurable way, but, as we will see later, this is not always possible. Independent of the sizes or shapes of particle detectors, their operation is usually based on the electromagnetic interactions of particles with matter. Energetic charged particles, for exam-

ple, can ionize atoms, and thereby release electrons that can subsequently be accelerated to produce small detectable currents. Most electrically neutral particles can also interact with matter and transfer some or all of their energies to the charged nuclei or to the atomic electrons of the medium, which in turn can yield detectable electric signals. Particles such as neutrinos, which have no electromagnetic interactions, and therefore have very low probabilities for colliding in matter (that is, have small cross sections), are therefore especially difficult to detect. We will now discuss some of the more straightforward ways in which particles can deposit their energies in matter.

## 6.2 Charged Particles

When a charged particle moves through a medium, it interacts primarily with the atomic electrons within that medium. If the particle has sufficient kinetic energy, it can deposit that energy in the medium by ionizing the atoms in its path or by exciting atoms or molecules to higher states; the excited systems can subsequently drop down to their ground levels through photon emission. When the charged particle is massive, its interactions with the atomic electrons (Rutherford-like scattering) will not greatly affect its trajectory (see the discussion in Chapter 1). A particle can also suffer more catastrophic nuclear collisions, but these have smaller cross sections, and are therefore relatively rare. Consequently, most of the energy that a particle deposits in the medium can be attributed to its collisions with atomic electrons.

A convenient variable that describes the ionization properties of any medium is the stopping power  $S(T)$ , which is defined as the amount of kinetic energy lost by any incident object per unit length of path traversed in the medium (this is often termed ionization-energy loss, or simply *energy loss*):

$$S(T) = -\frac{dT}{dx} = n_{\text{ion}} \bar{I}, \quad (6.1)$$

where  $T$  is the kinetic energy of the particle,  $n_{\text{ion}}$  is the number of electron-ion pairs formed per unit path length, and  $\bar{I}$  denotes the average energy needed to ionize an atom in the medium. (For large atomic numbers,  $\bar{I}$  can be approximated as  $10Z$  in eV units.) The negative sign in Eq. (6.1) simply reflects the fact that a particle's energy decreases as it moves along (that is,

the change in kinetic energy between  $x$  and  $x + dx$ ,  $dT = T(x + dx) - T(x)$ , is negative). For any given medium, the stopping power is, in general, a function of the energy of the incident particle, and it must, of course, also depend on the particle's electric charge. We will see later that the dependence on energy becomes very weak for relativistic particles.

Because the stopping power involves only electromagnetic interactions, it can be calculated quite reliably. Hans Bethe and Felix Bloch derived the following expression for relativistic particles

$$S(T) = \frac{4\pi Q^2 e^2 n Z}{m\beta^2 c^2} \left[ \ln \left( \frac{2mc^2 \gamma^2 \beta^2}{\bar{I}} \right) - \beta^2 \right], \quad (6.2)$$

where  $m$  is the rest mass of the electron,  $\beta = \frac{v}{c}$  is the particle's velocity relative to the speed of light in vacuum,  $\gamma$  is the particle's Lorentz factor  $(1 - \beta^2)^{-\frac{1}{2}}$ ,  $Q = ze$  is its charge,  $Z$  is the atomic number of the medium, and  $n$  is the number of atoms per unit volume (equal to  $\frac{\rho A_0}{A}$ , as given in Eq. (1.40)).

In natural  $\alpha$ -decay of nuclei, the emitted  $\alpha$ -particles have kinetic energies of the order of a few MeV, and because of their large mass ( $\approx 4000$  MeV/ $c^2$ ) the relativistic corrections in Eq. (6.2) can be ignored, which simplifies  $S(T)$  to

$$S(T) = \frac{4\pi Q^2 e^2 n Z}{m\beta^2 c^2} \ln \left[ \frac{2m\beta^2 c^2}{\bar{I}} \right]. \quad (6.3)$$

However, for energetic particles produced in accelerator experiments, or for electrons from nuclear  $\beta$ -emission, the relativistic corrections are usually substantial, and Eq. (6.2) must be used. (In fact, for electrons, there are additional small correction terms.) The above expressions for  $S(T)$  have been confirmed for different kinds of media and various types of particles, over a wide range of energies.

In light of the arguments presented in Chapter 1, it may seem puzzling that energy loss due to scattering from atomic electrons dominates over that for scattering from nuclei. The reason for this is that large angular deviations in elastic scattering correspond to large changes in the direction of momenta, which need not be accompanied by significant energy loss. For example, for the case of the elastic scattering of  $\alpha$ -particles in the nuclear Coulomb field, there is a significant change in the direction of momentum of the  $\alpha$  particle, but very little transfer of energy to the massive nucleus. On

the other hand, scattering from the weakly-bound atomic electrons (and ionization) represents an inelastic process that requires energy transfer. To be more specific, a momentum transfer of 0.1 MeV/ $c$  to an electron target would require an energy transfer of about 10 keV, while the same momentum transfer to a gold nucleus would correspond to less than 0.1 eV of energy transfer. Consequently, the dependence of energy loss in Eq. (6.2) on the inverse of the mass of the target, supports our previous contention that, ignoring strong nuclear collisions, small-angle scattering from atomic electrons is the dominant mechanism of energy deposition for massive charged particles traversing matter.

Because of the  $\beta^{-2}$  dependence in Eq. (6.2), at low particle velocities, the ionization loss is quite sensitive to particle energy. In fact, this dependence on  $v^{-2}$  suggests that particles of different rest mass ( $M$ ) but same momentum ( $p$ ) can be distinguished because of their different rates of energy loss. Although  $S(T)$  has no explicit dependence on particle mass, for any fixed momentum, the effect of mass comes in through

$$S(T) \propto \frac{1}{v^2} = \frac{M^2 \gamma^2}{p^2}.$$

Consequently, at low velocities ( $\gamma \approx 1$ ), particles of same momentum but different mass will display significantly different energy loss.

Independent of particle mass, the stopping power decreases with increasing particle velocity, and  $S(T)$  displays a rather shallow minimum when  $\gamma\beta \approx 3$  (that is, the minimum occurs at higher momenta for more massive particles). This minimum in Eq. (6.2) is due to the convolution of the decrease in  $S(T)$  caused by the  $\beta^{-2}$  dependence ( $\beta$  saturates at  $\beta \approx 1$  at high energies), and the rise caused by the  $\ln\gamma^2$  term that is due to relativistic effects. When the stopping power is displayed as a function of  $\gamma\beta$  or  $\frac{p}{Mc}$ ,  $S(T)$  is almost independent of  $M$ , and we can therefore say that  $S(T)$  “scales as”  $\gamma\beta$  or  $\frac{p}{Mc}$  (see Fig. 6.1).

The relativistic  $\ln\gamma^2$  rise in  $S(T)$  for  $\gamma\beta > 3$  ( $v > 0.96c$ ) eventually plateaus (saturates) because of the presence of long-range inter-atomic screening effects (ignored in the Bethe-Bloch calculation). The total increase in ionization is rarely greater than 50% beyond the value measured for a “minimum-ionizing” particle, namely a particle that has  $v \approx 0.96c$ . The relativistic rise is best observed in gaseous media, and is only a several percent effect for dense materials. Nevertheless, this can be used to distinguish different particle types through their small differences in energy loss

in gaseous detectors for energies corresponding to  $\gamma\beta > 3$ .

At very high energies, after the saturation of the relativistic rise, ionization loss becomes an energy-independent constant rate, and it is therefore not possible to distinguish particle-types purely on the basis of ionization. Except in gaseous media, the stopping power at high energies can be approximated quite adequately by the value when  $\gamma\beta \approx 3$  (see next section). We should also point out that, at very low energies, the stopping power in Eq. (6.2) becomes unphysical (negative), reflecting the fact that ionization loss is very small when the velocity of the particle is small. In this regime, the details of the atomic-structure of the medium become important, and the incident particle can even capture electrons from the medium to form atomic systems of its own.

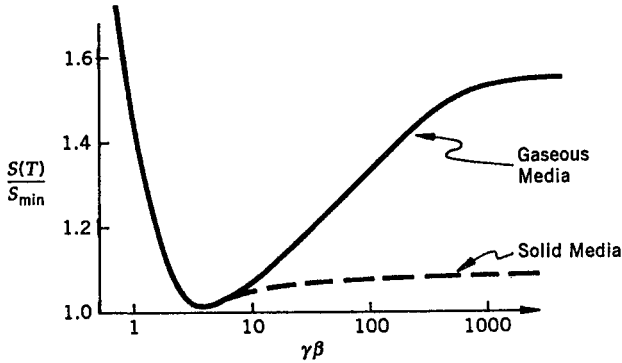


Fig. 6.1 Stopping power relative to its minimum value as a function of  $\frac{p}{Mc}$  (or  $\gamma\beta$ ).

Once we know the stopping power, we can calculate the expected range  $R$  of any particle in the medium, that is, the distance it will travel before it runs out of kinetic energy and comes to a halt

$$R = \int_0^R dx = \int_T^0 \frac{dx}{dT} dT = \int_0^T \frac{dT}{S(T)}. \quad (6.4)$$

At low energies, two particles of same kinetic energy but different mass can have substantially different ranges. For example, an electron with a kinetic energy of 5 MeV has a range that is several hundred times that of an  $\alpha$ -particle of the same kinetic energy. At high energies, where the range becomes essentially proportional to energy, the difference in path lengths

for particles of same kinetic energy becomes less pronounced.

### 6.2.1 Units of Energy Loss and Range

The units of  $S(T)$  in Eq. (6.2) are ergs/cm in the cgs system. The more common way to specify energy loss is in MeV/cm, or in terms of an equivalent thickness of gm/cm<sup>2</sup>, that is, in MeV/(gm/cm<sup>2</sup>) of material. Similarly, the range is commonly expressed in cm or in gm/cm<sup>2</sup>, where the two units are related simply through the density of the medium. When  $\gamma\beta \approx 3$ , the minimum value of  $S(T)$  for a particle with  $z = 1$  can be evaluated approximately from Eq. (6.2) as follows

$$\begin{aligned} S_{\min} &\approx \frac{4\pi e^4 A_0 \left(\frac{\rho Z}{A}\right)}{mc^2 \beta^2} \ln \left( \frac{2mc^2 \gamma^2 \beta^2}{\bar{I}} \right) \\ &\approx \frac{(12)(4.8 \times 10^{-10} \text{ esu})^4 (6 \times 10^{23} \text{ atoms/mole}) \left(\frac{\rho Z}{A}\right)}{(9.1 \times 10^{-28} \text{ gm})(3 \times 10^{10} \text{ cm/sec})^2 \left(\frac{9}{10}\right)} \\ &\quad \times \ln \left( \frac{2 \times 0.5 \times 10^6 \text{ eV} \times 9}{10 Z \text{ eV}} \right) \\ &\approx 5.2 \times 10^{-7} (13.7 - \ln Z) \rho \frac{Z}{A} \text{ erg/cm.} \end{aligned}$$

The  $\ln Z$  term is relatively small ( $< 4.5$ ), and it varies slowly with  $Z$ . Let us therefore use  $\langle Z \rangle \approx 20$  to get an approximate result

$$\begin{aligned} S_{\min} &\approx 5.6 \times 10^{-6} \rho \frac{Z}{A} \text{ erg/cm} \times 6.3 \times 10^5 \text{ MeV/erg} \\ &\approx 3.5 \rho \frac{Z}{A} \text{ MeV/cm,} \\ \text{or } S_{\min} &\approx 3.5 \frac{Z}{A} \text{ MeV/(gm/cm}^2\text{)}. \end{aligned} \tag{6.5}$$

As we mentioned before, Eq. (6.5) can also be used as a high-energy approximation for ionization loss in most media.

**Example 1**

The range of a 5 MeV  $\alpha$ -particle moving through air is given (in cm) approximately by  $R = 0.318 T^{\frac{3}{2}}$ , where  $T$  is in MeV units. If the stopping power of aluminum relative to air is 1600, calculate the range of an  $\alpha$ -particle in aluminum in cm and in the equivalent thickness in  $\text{gm}/\text{cm}^2$ .

The range in air is just  $0.318 \times 5^{\frac{3}{2}} \approx 3.56$  cm. Consequently, the range in aluminum foil is  $\frac{3.56}{1600}$  cm =  $2.225 \times 10^{-3}$  cm. Now, using the density of  $2.7 \text{ gm}/\text{cm}^3$  for aluminum, this yields an equivalent thickness of material of  $(2.225 \times 10^{-3} \text{ cm}) \times (2.7 \text{ gm}/\text{cm}^3) \approx 6.1 \times 10^{-3} \text{ gm}/\text{cm}^2$ , or  $6.1 \text{ mgm}/\text{cm}^2$ .

**Example 2**

Using the empirical formula for an electron's range-energy relation at low-energy, namely,  $R(\text{gm}/\text{cm}^2) = 0.53 T(\text{MeV}) - 0.16$ , calculate the energy of an electron that has a range in aluminum of  $2.5 \text{ gm}/\text{cm}^2$ .

The energy in MeV would be

$$T = \frac{1}{0.53} (R + 0.16) = \frac{1}{0.53} (2.5 + 0.16) \approx 5.0 \text{ MeV}.$$

Comparison of this with Example 1 shows that a 5 MeV electron has a range that is about 400 times longer than that of an  $\alpha$ -particle of the same kinetic energy.

### 6.2.2 Straggling, Multiple Scattering, and Statistical Processes

In our examples concerning the ranges of particles in matter, we calculated values expected on the basis of phenomenological expressions. On the average, these predictions are quite accurate, but substantial variations are observed from one event to another. The magnitude of the dispersion in individual ranges about the mean depends on the mass of the particle. Thus, the ranges of  $\alpha$ -particles of same energy have relatively little dispersion (or straggling) compared to that found for electrons stopping in matter.

The fundamental reason for such variations can be attributed to the inherent statistical nature of scattering processes. The energy that is transferred from an incident particle to target particles is not just a fixed and unique quantity, but rather has a range of values that are distributed according to some functional form. Thus, for example, for Rutherford scat-



tering, the distribution function is given by Eq. (1.73). Once a function of this kind is known, we can calculate a mean value and a dispersion about the mean for any chosen variable, such as, for example, the kinetic energy transferred to the target. Any finite dispersion about the mean implies the presence of variations in the process from one interaction to another. (We have already witnessed the presence of similar fluctuations in our discussion of natural radioactivity.) The range of a particle in matter is determined by the sum over a series of independent collisions with atomic electrons in the medium. It should therefore not be surprising that fluctuations in energy transfer in individual collisions can lead to variations in ranges of particles of same initial energy.

Another important effect that has statistical origin involves the angular deviation experienced by particles in their Rutherford scattering off atomic electrons in the medium. The consecutive collisions add up in a random fashion and provide some net deflection of any incident particle from its original line of flight. This “multiple-Coulomb scattering” also increases the path length that any particle follows as it traverses a given thickness of material. Because multiple scattering is a random process, the mean angular deviation for an ensemble of many particles passing through some thickness  $L$  of material must average to zero. However, the root-mean square (rms), or standard deviation  $\theta_{\text{rms}} = \sqrt{\langle \theta^2 \rangle}$  in the angle due to this “random walk” is finite and equals approximately

$$\theta_{\text{rms}} \approx \frac{20 \text{ MeV}}{\beta pc} z \sqrt{\frac{L}{X_0}}, \quad (6.6)$$

where  $z$  is the charge of the incident particle (in units of  $e$ ) of momentum  $p$  (in  $\text{MeV}/c$ ) and velocity  $\beta c$ , and  $X_0$  is the radiation length of the medium (see next section).

### Example 3

Calculate the mean kinetic energy transferred to a target at rest in the laboratory ( $\langle T \rangle$ ), and the dispersion about that mean ( $\Delta T$ ), for a process that can be characterized by the cross section

$$\frac{d\sigma}{dq^2} = e^{-8R^2 q^2}. \quad (6.7)$$

This is, in fact, the approximate form of the dependence for scattering of nucleons at small  $q^2$  from a nucleus of radius  $R$  in fm, and  $q^2$  is in  $(\text{GeV}/c)^2$  units (*cf* Problem 2.11).

The kinetic energy transferred to the target can be obtained from Eq. (1.70) to be

$$T = \frac{q^2}{2M}.$$

Hence, for the mean and for the second moment of  $T$ , we obtain

$$\begin{aligned} \langle T \rangle &= \frac{\int_0^\infty dq^2 \left( \frac{q^2}{2M} \right) e^{-8R^2 q^2}}{\int_0^\infty dq^2 e^{-8R^2 q^2}} = \frac{1}{16MR^2}, \\ \langle T^2 \rangle &= \frac{\int_0^\infty dq^2 \left( \frac{q^2}{2M} \right)^2 e^{-8R^2 q^2}}{\int_0^\infty dq^2 e^{-8R^2 q^2}} = \frac{1}{128M^2 R^4}, \end{aligned} \quad (6.8)$$

where, to evaluate the above integrals, we used the standard result

$$\int_0^\infty dx x^n e^{-ax} = \frac{n!}{a^{n+1}}. \quad (6.9)$$

Consequently, the dispersion in  $T$ , defined by the square root of the variance, reduces to

$$\begin{aligned} \Delta T &= \left[ \langle (T - \langle T \rangle)^2 \rangle \right]^{\frac{1}{2}} \\ &= [\langle T^2 \rangle - \langle T \rangle^2]^{\frac{1}{2}} = \frac{1}{16MR^2}. \end{aligned} \quad (6.10)$$

Thus, for this simple exponential dependence on  $q^2$ , the dispersion, or root-mean-square (rms) variation in  $\langle T \rangle$  from scattering to scattering is just equal to the mean value of  $T$ . Because  $M$  is almost equal to the atomic weight  $A$  in GeV units, and  $R \approx 1.2 A^{\frac{1}{3}}$ , we can write

$$\Delta T = T_{\text{rms}} = \langle T \rangle \approx (20A^{\frac{5}{3}})^{-1} \text{ GeV}. \quad (6.11)$$

From this example, we can see once again the very strong dependence of the kinetic energy transfer on target mass. For protons interacting with

protons,  $\langle T \rangle \approx 0.05$  GeV and momentum transfers are typically  $\approx 0.3$  GeV/c, but for proton-lead collisions  $\langle T \rangle \approx 7$  keV!, and momentum transfers are  $\approx 0.05$  GeV/c. (These results, including our exponential formula for  $q^2$ , hold only for elastic reactions when nuclei do not break apart in the course of the collision.)

### 6.2.3 Energy Loss Through Bremsstrahlung

Although Eq. (6.2) was derived for the case of Rutherford scattering of massive projectile particles, it also holds surprisingly well for incident electrons. The scattering of electrons in matter is more complicated because electrons have small mass, and consequently relativistic corrections become important for kinetic energies as low as several hundred keV. In addition, electron projectiles can transfer substantial fractions of their energies to the atomic electrons with which they collide, thereby producing what are referred to as  $\delta$ -rays, or knock-on electrons, which cannot be distinguished from the incident (i.e., scattered) electrons. This indistinguishability requires more delicate quantum-mechanical treatment of the scattering cross section. Despite these complications, Eq. (6.2) still provides an adequate approximation to the ionization loss by electrons for energies in excess of about 1 MeV. (The relativistic rise for electrons is somewhat smaller than it is for massive particles.)

However, unlike massive particles, electrons usually suffer large accelerations as a result of their interactions with atomic electric fields (and especially with the intense nuclear Coulomb fields). These accelerations can then lead to radiation of electromagnetic waves. Such emission of photons, or *bremsstrahlung* as it is termed, is an important mechanism for energy loss, especially for ultra-relativistic electrons. (Bremsstrahlung can also become significant for more massive particles, but only beyond  $10^{12}$  eV, or TeV energy scales.) Thus for the total energy loss by electrons traversing matter we can write schematically,

$$\left(-\frac{dT}{dx}\right)_{\text{tot}} = \left(-\frac{dT}{dx}\right)_{\text{ion}} + \left(-\frac{dT}{dx}\right)_{\text{brem}} \quad (6.12)$$

The ratio of the bremsstrahlung to ionization loss for high-energy electrons can be shown to be approximately equal to

$$\frac{\left(\frac{dT}{dx}\right)_{\text{brem}}}{\left(\frac{dT}{dx}\right)_{\text{ion}}} \approx \frac{TZ}{1200mc^2}, \quad (6.13)$$

where  $Z$  is the atomic number of the medium,  $m$  is the rest mass of the electron (projectile), and  $T$  is its kinetic energy in MeV. At high energies, the ionization loss is constant (saturated by the density effect), and given approximately by Eq. (6.5), and radiation dominates the total energy loss in Eq. (6.12). (This is illustrated in Figure 6.2.) According to Eq. (6.13), the radiated energy at high energies is proportional to the energy of the electron, and for this regime it is useful to define the radiation length ( $X_0$ ), which is the distance that an electron travels before its energy drops to  $\frac{1}{e}$  of its original value. From Eqs. (6.5) and (6.13) we obtain

$$\left(\frac{dT}{dx}\right)_{\text{brem}} = -\frac{T}{X_0}, \quad \text{with} \quad X_0 \approx 170 \frac{A}{Z^2} \text{ (in gm/cm}^2\text{)}. \quad (6.14)$$

Dividing  $X_0$  by the density of the medium, or multiplying the right-hand side of Eq. (6.14) by that density, converts the units to energy loss per cm (see Eq. (6.5)). At high energies ( $\gamma\beta > 3$ ) it is also useful to write an approximate expression for ionization loss in terms of the radiation length. Defining a critical energy ( $T_c$ ) as the energy at which energy loss due to collisions (ionization) is the same as that due to bremsstrahlung, we can write

$$\left(\frac{dT}{dx}\right)_{\text{brem}} = \left(\frac{dT}{dx}\right)_{\text{ion}} = -\frac{T_c}{X_0}, \quad (6.15)$$

where, from Eqs. (6.5) and (6.14), we get  $T_c \approx \frac{600}{Z}$  (in MeV).

Except for smallest  $Z$  values, the above expressions provide quite satisfactory approximations for calculating ionization-energy loss for any high-energy particle of unit charge, and the radiation loss for high-energy electrons. Substituting  $X_0$  from Eq. (6.14), into Eq. (6.15), and setting  $\frac{A}{Z} = 2.2$  for  $Z \approx 20$ , we get that  $\left(\frac{dT}{dx}\right)_{\text{ion}} \approx -1.6 \text{ MeV}/(\text{gm/cm}^2)$ . This high-energy approximation holds to  $\approx 30\%$  accuracy for every medium, except hydrogen.

An important consequence of Eq. (6.14) is that, because of bremsstrahlung, high energy electrons lose their kinetic energy exponentially with the distance traveled in matter. That is, integrating Eq. (6.14)

between an initial kinetic energy  $T_0$  and some later value  $T$ , provides the relation

$$T = T_0 e^{-\frac{x}{X_0}}. \quad (6.16)$$

Thus, energetic electrons radiate most of their energy within several radiation lengths of material. This characteristic behavior is particularly important in the design of electron detectors. More massive ultra-relativistic charged particles that do not radiate lose their energy through nuclear (strong) collisions or just through ionization loss.<sup>1</sup>

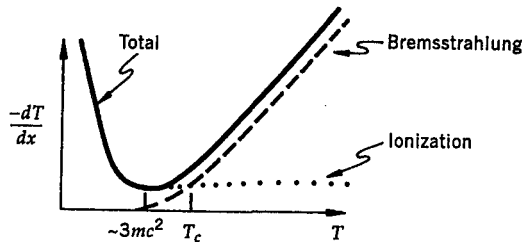


Fig. 6.2 Energy loss in matter as a function of electron energy.

#### Example 4

As an example of the effects of multiple scattering, let us calculate the typical angular deviation of a 5 MeV proton traversing 1 cm of argon gas at atmospheric pressure and 0°C, and compare this to the case of an electron of same kinetic energy.

The radiation length of gaseous argon at the stated conditions is approximately 105 m. The proton is non-relativistic, and its momentum can therefore be approximated as

$$p = \sqrt{2MT} \approx \sqrt{2 \times 1000 \text{ MeV}/c^2 \times 5 \text{ MeV}} \approx 100 \text{ MeV}/c.$$

<sup>1</sup>Muons, as we will see, are massive charged particles that do not have strong interactions, and therefore can neither radiate nor deposit their energies through large transfers of momentum to nuclei. Consequently, muons have ranges that are essentially proportional to their incident energies. Shielding radiation-sensitive equipment and personnel from excessive exposure to high-energy muons is therefore an issue of substantial concern at high-energy laboratories.

The proton's velocity can be calculated from

$$v = \sqrt{\frac{2T}{M}} \approx \sqrt{\frac{2 \times 5 \text{ MeV}}{1000 \text{ MeV}/c^2}} \approx 0.1 c.$$

The electron, on the other hand, is quite relativistic, and its momentum can therefore be taken as

$$p = \frac{E}{c} = \frac{T + mc^2}{c} \approx 5.5 \text{ MeV}/c.$$

The electron's velocity is essentially equal to  $c$ . Thus from Eq. (6.6), for the proton we obtain

$$\theta_{\text{rms}}^p \approx \frac{20}{0.1 \times 100} \sqrt{\frac{0.01}{105}} \approx 0.02 \text{ rad} = 20 \text{ mrad},$$

and for the electron we get

$$\theta_{\text{rms}}^e \approx \frac{20}{1 \times 5.5} \sqrt{\frac{0.01}{105}} \approx 40 \text{ mrad}.$$

Consequently, as expected on the basis of their small mass, electrons are scattered much farther away from their initial directions than the massive protons. And because low energy electrons also have far longer ranges, they also exhibit far greater dispersion or straggling than more massive particles of same kinetic energy.

### 6.3 Interactions of Photons with Matter

Because photons are electrically neutral, they do not experience the Coulomb force the way charged particles do. We might therefore conclude, incorrectly, that they cannot ionize atoms. In fact, photons are the carriers of electromagnetic force and can interact with matter in a variety of ways that lead to ionization of atoms and to energy deposition in a medium, as discussed below.

We can describe the attenuation of light (photons, X-rays or  $\gamma$ -rays) in a medium in terms of an effective *absorption coefficient*  $\mu$ , which reflects the total cross section for interaction. In general,  $\mu$  will depend on the

energy or frequency of the incident light. If  $I(x)$  represents the intensity of photons at any point  $x$  in the medium, then the change in intensity  $dI$  in an infinitesimal thickness of material  $dx$  can be written in terms of  $\mu$  as

$$dI = I(x + dx) - I(x) = -\mu I(x) dx, \quad (6.17)$$

where, as usual, the negative sign indicates that the intensity decreases with traversed distance. Integrating the above expression from some initial value  $I_0$  at  $x = 0$  to the final intensity  $I(x)$  at the point  $x$ , we obtain

$$\begin{aligned} \frac{dI}{I} &= -\mu dx, \\ \text{or } \int_{I_0}^I \frac{dI}{I} &= -\mu \int_0^x dx, \\ \text{or } I(x) &= I_0 e^{-\mu x}. \end{aligned} \quad (6.18)$$

As in the case of other statistical processes, such as radioactive decay, we can define a half-thickness,  $x_{\frac{1}{2}}$ , as the thickness of material that photons must traverse in order for their intensity to fall to half of the original value. This can be related to  $\mu$ , as follows. From Eq. (6.18), we can write

$$I(x_{\frac{1}{2}}) = \frac{I_0}{2} = I_0 e^{-\mu x_{\frac{1}{2}}},$$

which implies that

$$\mu x_{\frac{1}{2}} = \ln 2,$$

$$\text{or } x_{\frac{1}{2}} = \frac{\ln 2}{\mu} = \frac{0.693}{\mu}. \quad (6.19)$$

If  $x_{\frac{1}{2}}$  is expressed in cm, then  $\mu$  must have units of  $\text{cm}^{-1}$ , and when  $x_{\frac{1}{2}}$  is given in terms of  $\text{gm}/\text{cm}^2$ , then  $\mu$  has units of  $\text{cm}^2/\text{gm}$ . The value of  $\mu^{-1}$  is just the *mean free path* for absorption, or the average distance through which a beam of photons will propagate before their number drops to  $\frac{1}{e}$  of the initial value.

We will now turn to a brief discussion of the specific processes that contribute to absorption of photons in any medium.

### 6.3.1 Photoelectric Effect

In this process, a low-energy photon is absorbed by a bound electron, which is subsequently emitted with kinetic energy  $T_e$  (see Fig. 6.3). If we call the energy needed to free the atomic electron  $I_B$  (this is the negative of the binding energy), and the frequency of the photon  $\nu$ , then energy conservation requires that the Einstein relation holds, namely,

$$E_\gamma = h\nu = I_B + T_e,$$

$$\text{or } T_e = h\nu - I_B, \quad (6.20)$$

where  $I_B$  sets the scale for the appropriate photon energies that are required for the process to take place. The photoelectric effect has a large cross section in the range of X-ray energies (keV), and, ignoring the absolute normalization, scales approximately as

$$\begin{aligned} \sigma &\approx \frac{Z^5}{(h\nu)^{\frac{7}{2}}} \quad \text{for } E_\gamma < m_e c^2, \\ \sigma &\approx \frac{Z^5}{h\nu} \quad \text{for } E_\gamma > m_e c^2. \end{aligned} \quad (6.21)$$

Thus the process is particularly important in high- $Z$  atoms, and is not very significant above the 1 MeV range of photon energies. When the emitted electron originates from an inner shell of the atom, one of the outer electrons drops down to fill the lower (more stable) empty level, and the emitted electron is consequently accompanied by an X-ray photon produced in the subsequent atomic transition.

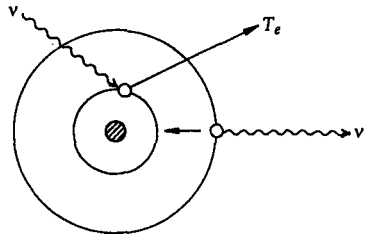


Fig. 6.3 Pictorial representation of the photoelectric effect.



### 6.3.2 Compton Scattering

Compton scattering can be thought of as equivalent to a photoelectric effect on a free electron. In conventional language, one can think of the process as involving the collision of two classical particles – the photon, with energy  $E = h\nu$  and momentum  $p = \frac{E}{c}$ , and an electron at rest. Alternatively, the process can be viewed as follows. The electron absorbs an incident photon, and forms an electron-like system that has an unphysical mass (see Problem 6.8); this virtual system (that is, “existing” only for very brief times as determined through the uncertainty relation  $\tau \approx \frac{\hbar}{\Delta mc^2}$ , where  $\Delta E = \Delta mc^2$  is the uncertainty in the system’s energy) then de-excites into a physical electron and to a photon of shifted frequency (or energy), as shown in Fig. 6.4.

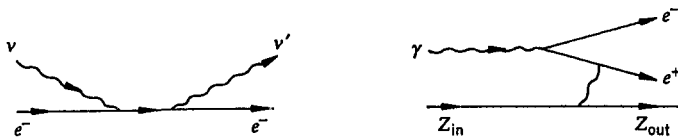


Fig. 6.4 Pictorial representations of Compton scattering (left) and of pair production (right).

The kinematics for the scattering assumes that the target electron is free. This means that the results are not expected to hold for incident photons of very low energy (much below 100 keV), where effects of atomic binding can be important. Treating the photon as a particle of energy  $h\nu$  and momentum  $\frac{h\nu}{c}$  (zero rest mass), and using fully relativistic momentum-energy expressions for the electron, it is straightforward to show that the kinematic relation between the frequency of the incident and the scattered photon ( $\nu'$ ), at a photon scattering angle  $\theta$ , is given by

$$\nu' = \frac{\nu}{1 + \frac{h\nu}{mc^2} (1 - \cos \theta)}, \quad (6.22)$$

where  $m$  is the rest mass of the electron. From the above expression, we see that, for any finite scattering angle, the energy of the scattered photon is smaller than that of the incident one. The incident photon must therefore transfer some of its energy to the electron, which consequently has a recoil energy that depends on the scattering angle.

Relying on special relativity, the quantization of light (that is the par-

ticle properties of photons), and quantum theory, the Compton reaction served as one of the early major confirmations of the veracity of the new ideas of 20th century physics. Again, ignoring absolute normalization, the cross section for Compton scattering appears to scale as

$$\sigma \approx \frac{Z}{h\nu}, \quad (6.23)$$

where  $Z$  is the atomic number of the medium. Compton scattering dominates energy deposition in the 0.1 to 10 MeV range of photon energies.

### 6.3.3 *Pair Production*

When a photon has sufficient energy, it can be absorbed in matter and produce a pair of oppositely charged particles. Such conversions can only take place when no known conservation laws are violated in the process. In addition to charge and momentum-energy conservation, other quantum numbers may restrict the possible final states. The best known conversion process, commonly referred to as pair production, involves the creation of a positron-electron ( $e^+e^-$ ) pair through the disappearance of a photon.

However, a massless photon cannot be converted into a pair of massive particles without violating momentum-energy conservation. This is best seen heuristically as follows. Let us suppose that the photon has a very small rest mass (far smaller than the mass of an electron). Now, in the photon's rest frame, the energy is its rest mass, namely close to zero, while, for the final state, the minimum energy is given by the sum of the rest masses of the two particles, which by assumption is relatively large. It follows therefore that a process such as pair production can only be observed in a medium in which, for example, a recoiling nucleus can absorb any momentum (but very little energy!) required to assure momentum-energy conservation. Since the mass of the positron equals that of the electron, the threshold for  $e^+e^-$  pair production is essentially  $h\nu \approx 2mc^2 = 2 \times 0.511$  MeV  $\approx 1.022$  MeV (see Problem 6.9).

The pair production cross section scales essentially as  $Z^2$ , where  $Z$  is the atomic number of the medium. It rises rapidly from threshold, and dominates all energy-loss mechanisms for photon energies  $\gtrsim 10$  MeV. At very high energies ( $> 100$  MeV), the  $e^+e^-$  pair cross section saturates, and can be characterized by a constant mean free path for conversion (or by a constant absorption coefficient) that essentially equals the electron

radiation length of the medium,

$$X_{\text{pair}} = (\mu_{\text{pair}})^{-1} \approx \frac{9}{7} X_0. \quad (6.24)$$

A natural question to ask is what happens to the positrons that are created in the conversion of photons in matter? Because positrons are the antiparticles of electrons, after production, they traverse matter, much as electrons do, and deposit their energies through ionization or through bremsstrahlung. Once a positron loses most of its kinetic energy, however, it captures an electron to form a hydrogen-like atom, referred to as positronium, where the proton is replaced by a positron. Unlike hydrogen, positronium atoms are unstable, and decay (annihilate) with lifetimes of about  $10^{-10}$  sec to form two photons

$$e^+ + e^- \longrightarrow \gamma + \gamma. \quad (6.25)$$

The process of annihilation produces photons of equal energy, back-to-back in the laboratory. To conserve momentum-energy, each photon carries away exactly 0.511 MeV. Thus pair annihilation provides a very clean signal for detecting positrons, as well as for calibrating the low-energy response of detectors.

The three processes that we have just discussed provide independent contributions to the absorption of photons in any medium. We can therefore write the total absorption coefficient as the sum of the three separate coefficients,

$$\mu = \mu_{\text{pe}} + \mu_{\text{Comp}} + \mu_{\text{pair}}. \quad (6.26)$$

The independent contributions as well as their sum, are shown as a function of photon energy in Fig. 6.5.

Finally, referring back to our discussion of Rutherford scattering in Chapter 1, we can relate any absorption coefficient to the scattering cross section as follows. We argue that an object scattered out of the beam produces a drop in the beam intensity or an equivalent increase in the counting rate for scattering. According to Eq. (1.39), the fraction of the incident beam that is scattered, or lost, is proportional to the cross section per nuclear scatterer ( $\sigma$ ) and to the thickness of target material ( $dx$ )

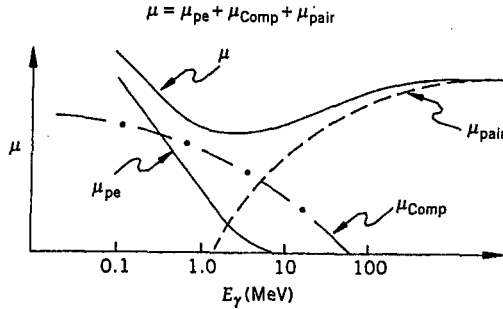


Fig. 6.5 Sketch of photon absorption coefficients as a function of energy for relatively high-Z material.

$$\frac{dn}{N_0} = \frac{A_0}{A} \rho \sigma dx. \tag{6.27}$$

This is just the negative of  $\frac{dI}{I}$ , or the fraction of the beam that is attenuated or absorbed, as given in Eq. (6.18). Equating the two terms, we obtain the following relationship between any cross section and an absorption coefficient

$$\mu = \rho \frac{A_0}{A} \sigma = n\sigma, \tag{6.28}$$

where  $\mu$  has units of  $\text{cm}^{-1}$ , when  $n$  is the number of scattering centers per  $\text{cm}^3$ , and  $\sigma$  is expressed in  $\text{cm}^2$ . When  $\mu$  is given in units of  $\text{cm}^2/\text{gm}$ , then  $n$  corresponds to the number of atoms per gm of material.

**Example 5**

The total absorption coefficient for 5 MeV photons in lead is about  $0.04 \text{ cm}^2/\text{gm}$ . If the density of lead is taken as  $11.3 \text{ gm}/\text{cm}^3$ , what is the half-thickness of lead for these  $\gamma$ -rays? What thickness of lead would be required to reduce the intensity of such photons to 0.06 of the initial value?

The absorption coefficient in  $\text{cm}^{-1}$  can be calculated as  $\mu = 0.04 \text{ cm}^2/\text{gm} \times 11.3 \text{ gm}/\text{cm}^3 = 0.45 \text{ cm}^{-1}$ . Hence,  $x_{\frac{1}{2}} = \frac{0.693}{\mu} = \frac{0.693}{0.45 \text{ cm}^{-1}} \approx 1.53 \text{ cm}$ , which is the thickness of lead that will reduce the photon intensity to half of its original value. To find the thickness needed to reduce the intensity to 0.06 of the initial value, we use Eq. (6.18)

$$\frac{I}{I_0} = e^{-\mu x},$$

$$\text{or } 0.06 = e^{-\mu x},$$

$$\text{or } \ln(0.06) = -\mu x,$$

$$\text{or } x = -\frac{\ln(0.06)}{\mu} = -\frac{\ln(0.06)}{0.45 \text{ cm}^{-1}} \approx 6.2 \text{ cm}.$$

If we characterize the range of 5 MeV photons in matter by the inverse of the absorption coefficient (that is, by the absorption length), we then obtain a typical penetration length or range in lead of  $\frac{1}{(0.45 \text{ cm}^{-1})} \approx 2.2 \text{ cm}$ , or equivalently  $2.2 \text{ cm} \times 11.3 \text{ gm/cm}^3 \approx 25 \text{ gm/cm}^2$ . Comparing this with the ranges of electrons and  $\alpha$ -particles of similar energy, it is clear that  $\gamma$ -rays are far more penetrating at such low energies.

### Example 6

What is the cross section that corresponds to an absorption coefficient of  $0.45 \text{ cm}^{-1}$  for photons in lead?

The relationship we need is given in Eq. (6.27), namely,

$$\sigma = \frac{\mu}{n} = \frac{A}{A_0} \frac{\mu}{\rho}.$$

Using  $A_0 = 6.02 \times 10^{23}$ ,  $A = 207.2 \text{ gm}$ , and  $\rho = 11.3 \text{ gm/cm}^3$ , we obtain the cross section

$$\sigma = \left( \frac{207.2 \text{ gm}}{6.02 \times 10^{23}} \right) \left( \frac{0.45 \text{ cm}^{-1}}{11.3 \text{ gm/cm}^3} \right) \approx 1.37 \times 10^{-23} \text{ cm}^2 = 13.7 \text{ b}.$$

### Example 7

The radiation length of lead at high energy is 5.6 mm. What is the value of the absorption coefficient, and what is the cross section for  $e^+e^-$  pair production on a lead target?

From Eq. (6.24), we can calculate the absorption coefficient for pair production as  $\mu \approx \frac{7}{9X_0} \approx 1.39 \text{ cm}^{-1}$ , and, following Example 6, we can calculate the cross section to be  $\sigma \approx 42.3 \text{ b}$ . This can be compared with the total inelastic nuclear (strong) cross section for nucleon interactions with lead at high energies, which is about 1.6 b. Consequently, we conclude that the mean free path for nuclear collisions in lead is about 15 cm, as opposed to the 0.6 cm radiation length that characterizes electromagnetic interactions. At high energies, far less material is therefore required to stop photons or electrons than particles that interact primarily through the strong force.

## 6.4 Interactions of Neutrons

As we have already mentioned, neutrons are in most respects very similar to protons. They are the constituents of nuclei, and have essentially the same mass, same nucleon number and spin as protons. They are, however, electrically neutral, and consequently, just like photons, cannot interact directly through the Coulomb force. (Although neutrons have small magnetic dipole moments, these do not provide substantial interactions in media.)

Neutrons do not sense the nuclear Coulomb force, and as a result even slow neutrons can be scattered or captured by the strong nuclear force. When low-energy neutrons interact inelastically, they can leave nuclei in excited states that can subsequently decay to ground levels through the emission of photons or other particles. Such emitted  $\gamma$ -rays or other particles can then be detected through their characteristic interactions with matter. Elastically scattered neutrons can transfer some of their kinetic energy to nuclear centers, which in recoiling can also provide signals (e.g., ionization) that can be used to reveal the presence of neutrons. In the elastic scattering of neutrons from nuclei, just as for the case of ionization loss, it is more difficult to transfer a sizable part of a neutron's kinetic energy to a nucleus if the nuclear mass is large (recall from Eq. (1.70) that, for any momentum transfer  $q$ , the transfer of kinetic energy goes as  $\frac{q^2}{2M}$  with nuclear mass  $M$ ). As we have already mentioned, this is the reason that hydrogen-rich paraffin is often used as a moderator to slow down energetic

neutrons.

When neutrons are produced in collisions, they can be quite penetrating, especially if their energies are in the range of several MeV, and there are no hydrogen nuclei available for absorbing their kinetic energies. The neutron shine, or “albedo”, at accelerators and reactors is often a major source of background to experiments, and can only be reduced through use of appropriate moderators and materials that have large neutron-absorption cross sections (e.g., boron, which captures low energy neutrons through  $^{10}\text{B} + n \rightarrow ^7\text{Li} + \alpha$ ).

## 6.5 Interaction of Hadrons at High Energies

All particles that interact through the strong nuclear force are known collectively as *hadrons*. Neutrons, protons,  $\pi$  mesons and K mesons, are the most common hadrons. We will discuss the intrinsic properties of such particles in Chapter 11, but will describe here the overall character of their interactions.

Protons are, of course, the nuclei of hydrogen atoms, and are therefore the easiest hadrons to accelerate and to use as particle beams (see Chapter 8). When proton beams interact with other protons, or with larger target nuclei, they can produce  $\pi$  mesons, K mesons, neutrons and other hadrons. At low beam energies (below  $\approx 2$  GeV), the interactions between pions and nucleons, kaons and nucleons, and between two nucleons, differ quite markedly. At such low energies, the collision cross sections between any two hadrons change rapidly (and often oscillate) with energy. All this is because certain hadronic systems resonate at specific energies while others do not. Beyond 5 GeV, the total cross sections for hadron-hadron interactions change (drop) only slightly with increasing energy. They reach minimum values, typically 20-40 mb ( $\approx \pi R^2$ ), at  $\approx 70$ -100 GeV, and then increase logarithmically with increasing beam energy.

Hadronic collisions, in the main, involve very small momentum transfers, small production angles, and interaction distances of the order of  $\approx 1$  fm. Central collisions, involving large momentum transfers are quite rare, but very interesting from the point of view of developing an understanding of the structure of hadrons. Typical momentum transfers in hadronic reactions are of the order of  $q^2 \approx 0.1$  (GeV/c) $^2$ . The mean multiplicity, or the average number of particles (usually pions) produced in a typical hadronic collision, grows logarithmically with incident energy, from  $\approx 3$  particles at 5 GeV

to  $\approx 12$  at 500 GeV, with great fluctuations around the mean occurring from one event to another. Thus when high-energy hadrons interact with matter, they break apart the nuclei, produce mesons and other hadrons, that can, in turn, interact again and deposit energy in the medium. This is essentially independent of electric charge of the hadron, and therefore, ignoring small differences due to Coulomb scattering, high-energy neutron and proton interactions in matter are almost indistinguishable. Any energy deposited by the primary or the secondary particles in matter can then be used to estimate the energy of the incident hadron (see our discussion of calorimeters in Chapter 7).

## Problems

**6.1** What is the minimum thickness of aluminum in cm that is needed to stop a 3 MeV  $\alpha$  particle? What about the thickness needed to stop a 3 MeV electron? (Use the approximate range-energy relationship provided in Examples 1 and 2.)

**6.2** About how much steel in cm is required to stop a 500 GeV muon if the muon deposits energy only via ionization loss? (Use Eq. (6.5) to calculate your result.) Would you need a comparable amount of material to stop 500 GeV electrons? What about 500 GeV protons?

**6.3** Multiple-scattering error often limits the ability to measure the direction of motion of a charged particle. To what accuracy can the incident angle of a 500 GeV muon be measured after the particle traverses one meter of iron?

**6.4** Typically, what fraction of a beam of 100 GeV photons will be transmitted through a 2 cm thick lead absorber?

**6.5** The capture cross section for thermal neutrons on  $^{27}\text{Al}$  is 233 mb. On average, how far can a beam of such neutrons penetrate a slab of aluminum ( $\rho = 2.7 \text{ gm/cm}^3$ ) before half of the beam is absorbed. (See relation (6.27).)

**6.6** Protons and  $\alpha$ -particles of 20 MeV pass through 0.001 cm of aluminum foil. How much energy do such particles deposit within the foil?

**6.7** Compare the stopping power of electrons, protons and  $\alpha$ -particles in copper, for particle velocities of  $0.5c$ .

**6.8** Calculate the mass (i.e.,  $\sqrt{s}$ ) of the virtual electron in Fig. 6.4 for an incident photon of wavelength of  $1.25 \times 10^{-10} \text{ cm}$ . What is the approximate



lifetime of such an object? Repeat your calculation for a wavelength of  $1.25 \times 10^{-12}$  cm.

**6.9** Consider the collision of a photon with a target of mass  $M$  that is initially at rest in the laboratory. Show that the minimum laboratory energy that a photon must have to produce an  $e^+e^-$  pair is  $E_\gamma = 2m_e c^2 (1 + \frac{m_e}{M})$ . (*Hint:* Equate the expression for  $s$  given in Eqs. (1.64) and (1.65).) Thus the threshold for pair production is essentially  $2m_e c^2$ .

**6.10** What is the mean free path for nuclear collisions of 10 GeV protons in liquid hydrogen if the proton-proton total cross section is 40 mb? (Assume a liquid hydrogen density of  $0.07$  gm/cm<sup>3</sup>.)

**6.11** Prove the kinematic relation given in Eq. (6.22).

### Suggested Readings

Fernow, R. C., *Introduction to Experimental Particle Physics*, Cambridge Univ. Press (1986).

Kleinknecht, K., *Detectors for Particle Radiation*, Cambridge Univ. Press (1998).

Knoll, G. F., *Radiation Detection and Measurement*, Wiley (1989).

Leo, W. R., *Techniques for Nuclear and Particle Physics Experiments*, Springer-Verlag (1994).

## Chapter 7

# Particle Detection

### 7.1 Introductory Remarks

The investigation of nuclear and particle collisions or disintegrations relies upon detectors for measuring the products of such interactions. Although subatomic particles are certainly too small to be observed through purely visual means, we can use the mechanisms for energy deposition we described in the previous chapter to detect such particles. Although we will describe only the very simplest prototype detectors, the principles underlying their performance are similar to those used in even the most sophisticated devices.

### 7.2 Ionization Detectors

*Ionization detectors* are devices designed to measure the ionization produced when an incident particle traverses some medium. If the number of detected electrons and positive ions is to reflect the energy deposited in the material, then any produced electron-ion pairs must be kept from immediately recombining into atoms. This can be done by applying a sufficiently high electric field across the medium. This field will separate the charges, start their drift towards their respective electrodes, and thereby keep them from recombining.

The basic ionization detector consists of a chamber that is filled with a suitable medium that can be easily ionized. The chamber has a cathode and an anode that are held at some large relative voltage, and the device is characterized by a capacitance ( $C$ ) that is determined by the geometry of the electrodes. The operating medium should be chemically stable (or inert) so that the moving ionization electrons are not easily captured by

the molecules of that medium. The medium should not be very sensitive to radiation damage so that its response to incident particles does not change markedly with use. In addition, the medium should have a low value of ionization potential ( $\bar{I}$ ) in order to maximize the amount of ionization produced per energy deposited by any incident particle.

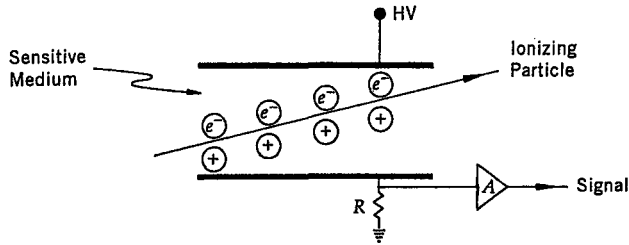


Fig. 7.1 Basic elements of an ionization detector.

As we have mentioned, when a charged particle traverses a sensitive region of a detector, it ionizes the medium and produces electron-ion pairs. These start drifting immediately along the electric field lines: electrons to the anode and the positive ions to the cathode. As the charges drift, they induce signals on the electrodes, which provide small currents that flow through some resistor  $R$  (see Fig. 7.1). This, in turn, produces a voltage drop that can be sensed with an amplifier  $A$ . The amplifier signal can be analyzed to obtain a pulse height that can be related to the amount of produced ionization. The amount of produced ionization depends primarily on the density and atomic structure of the ionizable medium and, of course, on the charge of and energy deposited by the incident particle. However, the amount of ionization that is detected is determined by many technical factors, foremost among which is the nature and size of the applied electric field, or the applied voltage (see Fig. 7.2).

When the voltage difference between electrodes is small, the electrons and ions can recombine soon after they are produced, and only a small fraction of the produced electrons and ions reach their respective electrodes. This provides an output signal that corresponds to fewer electron-ion pairs than are produced in the medium. The range of operating voltage where this occurs is referred to as the *recombination region*. As the voltage difference is increased beyond the point where dissociated electron-ion pairs can recombine, we obtain a signal that reflects the total amount of produced ionization. This operating range is called the *ionization region*. Increas-

ing the voltage further, provides the initially freed electrons with sufficient acceleration and energy so that they can ionize additional atoms of the medium. This increased ionization is often referred to as *signal amplification* or *multiplication*. The output signal in this operating region of voltages is larger than, but proportional to, the initially produced ionization, and, for this reason, this operating range is referred to as the *proportional range*. (It should be noted that proportional does not necessarily imply that the signal increases linearly with voltage.) Increasing the voltage even further, yields an avalanche of electron-ion pairs. In this mode, referred to as the *Geiger-operating region*, the energy of the original ionization electrons increases sufficiently rapidly so that they can excite or ionize more atoms, thereby providing more freed electrons or photons from de-excitation of atoms. This, in turn, produces more electron-ion pairs, and eventually leads to a discharge, that is, to a highly amplified output signal whose size is independent of the amount of original ionization. Finally, increasing the voltage beyond the *Geiger range*, leads to a breakdown that generates a continuous discharge of the medium, with the chamber no longer being sensitive to any incident ionization. Depending on the circumstances, most detectors are operated as ionization, proportional, or Geiger counters, depending on the circumstances, and most detectors use gas as the operating medium.

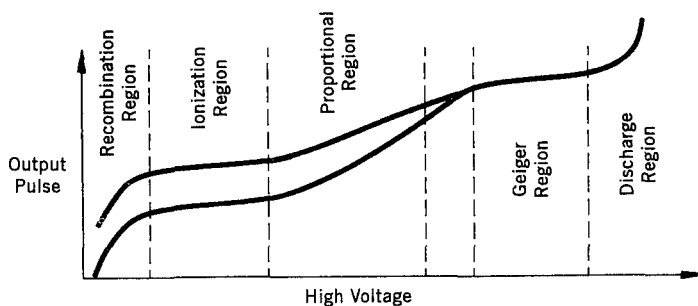


Fig. 7.2 Signal response to ionization loss as a function of imposed voltage for heavily ionizing (top curve) and minimum ionizing particles (lower curve). In the Geiger region, the output does not depend on the voltage, nor on the amount of deposited energy or initial ionization.

### 7.2.1 Ionization Counters

*Ionization chambers*, or *ionization counters*, operate at relatively low volt-

ages, and therefore provide no amplification of the original signal. Consequently, the output pulses for single minimum-ionizing particles tend to be quite small and usually require special low-noise amplifiers for attaining efficient operating performance. For heavily ionizing nuclear fragments, however, or for a flux of many particles, the fully integrated signals can be substantial and easy to detect. Ionization chambers are not very sensitive to voltage variations, and provide very linear output response for a wide range of input signals. Because there is no inherent amplification of signal, or discharge in the operating medium, these types of counters do not require much time to recover from large currents, and can therefore be used in environments with high interaction rates. In addition, because there is no amplification, they provide excellent energy resolution, which is limited primarily by electronic noise and by the inherent fluctuations in the production of the initial ionization. Liquid-argon ionization chambers have been used with great success as “sampling” detectors in high-energy calorimetric measurements of energy deposition (to be discussed later). Solid-state devices, pioneered in nuclear physics, are now used commonly as ionization counters in high energy experiments. Gaseous ionization chambers are useful for monitoring high levels of radiation; they were also used in the past to measure, for example, ranges of  $\alpha$ -particles from radioactive decays of nuclei.

Let us illustrate how an ionization counter can be employed to determine the range of 5.25 MeV  $\alpha$ -particles emitted in  $^{210}\text{Po}$  decay. The chamber can be chosen to be a precisely manufactured round-bottom flask with a radius of about 6 cm. The inside wall can be silver coated to serve as one of the two electrodes. A small sample of Po (about  $10\ \mu\text{Ci}$ ) can be suspended from a grounded insulated wire at the center of the flask. An operating gas can be admitted, under pressure if need be, and the flask sealed. (Alternatively, the chamber can be simply a parallel-plate device that has the  $\alpha$ -particle source embedded on one of the surfaces.) A potential can be applied to the silvered surface, and the current monitored through an amplifier, as sketched in Fig. 7.1. If the ionizing medium is air, then we expect to produce a small but detectable current. The value of  $\bar{I}$  for air is about 30 eV; thus, the number of electron-ion pairs that will be produced by one  $\alpha$ -particle will be about

$$n = \frac{5.25 \times 10^6 \text{ eV}}{30 \text{ eV}} = 1.75 \times 10^5. \quad (7.1)$$

The activity of the source is

$$\begin{aligned} \mathcal{A} &= 10 \mu\text{Ci} = 10 \times (10^{-6} \times 3.7 \times 10^{10}) \alpha\text{-particles/sec} \\ &= 3.7 \times 10^5 \alpha\text{-particles/sec.} \end{aligned} \quad (7.2)$$

Hence, the number of charged pairs produced per second is

$$N = n\mathcal{A} = (1.75 \times 10^5) \times (3.7 \times 10^5/\text{sec}) \approx 6.5 \times 10^{10}/\text{sec}. \quad (7.3)$$

Consequently, if both the positive and negative charges are collected, this will provide a current

$$\begin{aligned} J &= Ne = 6.5 \times 10^{10}/\text{sec} \times 1.6 \times 10^{-19} \text{ C} \\ &= 1.04 \times 10^{-8} \text{ C/sec} = 1.04 \times 10^{-8} \text{ A.} \end{aligned} \quad (7.4)$$

Currents of this size can be measured in a straightforward manner. (For the parallel-plate geometry, the current would be only  $\approx 5$  nA, because only half of the  $\alpha$ -particles would be emitted into the sensitive region.)

The measurement of range proceeds as follows. The current is monitored as a function of decreasing gas (air) pressure. As long as the density of the air is high enough to stop the  $\alpha$ -particles, the observed current, reflecting the total ionization produced by the  $\alpha$ -particles, remains constant. When the pressure drops below that critical value, the  $\alpha$ -particles do not lose all their kinetic energy within the gas volume, and therefore produce fewer electron-ion pairs in the sensitive region. As a result, the current drops, and keeps decreasing as the pressure is decreased further. For an  $\alpha$ -particle range of 6 cm in air, at a temperature of 25°C, the critical pressure  $P_{\text{crit}}$  is found to be 51 cm of Hg. Thus, at any other temperature and pressure, the range  $R$  can be calculated assuming the scaling of the simple gas law. In particular, for standard conditions of  $T = 288 \text{ K}$  and  $P = 76 \text{ cm of Hg}$ , we obtain

$$R = R_{\text{crit}} \times \frac{P_{\text{crit}}}{P} \frac{T}{T_{\text{crit}}} = 6 \text{ cm} \times \frac{51 \text{ cm Hg}}{76 \text{ cm Hg}} \frac{288 \text{ K}}{298 \text{ K}} = 3.9 \text{ cm.} \quad (7.5)$$

Although, in principle, the observed current provides an absolute measure of the total energy deposited in the form of ionization, it is always

wise to calibrate ionization chambers with sources or signals of known energy. This is particularly important when counting rates are very high and individual pulses must be counted in rapid succession. Under such conditions, small concentrations of impurities in the detector medium (often at less than a part per million level) can cause loss of electron signal through attachment. That is, some of the electrons drifting towards the anode can be attracted to the contaminant (electronegative) molecules that can form negative ions, which drift far slower than electrons, and therefore do not contribute to the fast output pulse that is produced by the electrons.

### 7.2.2 *Proportional Counters*

*Gaseous proportional counters* usually operate in high electric fields of the order of  $10^4$  V/cm and achieve typical amplification factors of about  $10^5$ . Such fields can be obtained using thin (diameters of 10 - 50  $\mu\text{m}$ ) metallic wires as anode field electrodes in a cylindrical chamber geometry. Because the fields are most intense near the axial anode wires, this is where the multiplication of charge, that is, secondary ionization, also takes place. For a large variety of gases, the output signals, even for minimum-ionizing particles, are quite large. Also, these detectors can operate over a relatively wide range of high voltage settings. Although proportional chambers can be used for measuring absolute energy deposition (pulse heights), their reliance on the multiplication of ionization in the medium makes them quite sensitive to the dependence of the output signal on the magnitude of the operating voltage.

Georges Charpak and his collaborators developed a variant of the proportional counter in the multiwire proportional chamber (MWPC) that has found primary application as a position detector in high-energy physics experiments. The idea is illustrated in Figs. 7.3 and 7.4. The principle is to have a plane of anode wires positioned precisely, with typical wire-spacings of about 2 mm. Such planes can be sandwiched between two similar cathode planes (or, alternatively, thin stretched aluminum foil can be used for cathodes). Typically, a space of  $\lesssim 1$  cm is left between cathode and anode planes. These doublets are then enclosed in some superstructure – usually with thin mylar-sheet windows on the outside. Operating gas can then be admitted into the regions between electrodes. Several doublet planes, with different orientations of anode wires, can be sandwiched together. Charged particles passing through the gaseous medium produce ionization along their paths, which produces pulses on the anode wires that are closest to

the trajectories. The anode wires, each with its own amplifier, function essentially as independent proportional counters, and can therefore be used to localize the position of any charged particle to an accuracy of the order of the spacing between the anode wires.

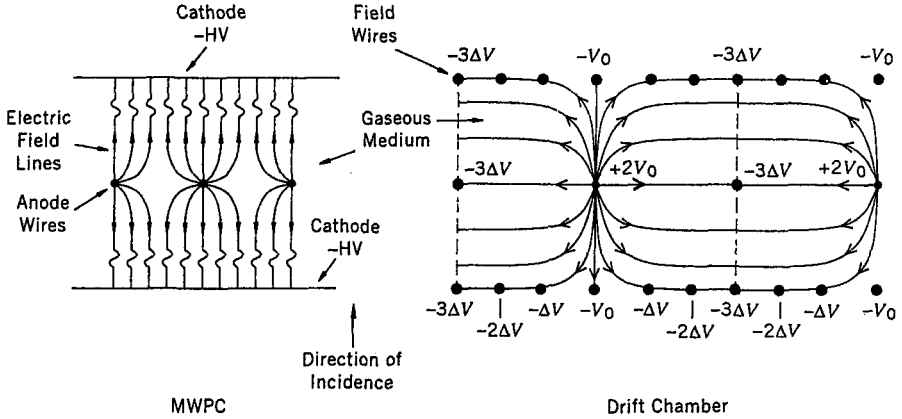


Fig. 7.3 Electric field structure in a multiwire proportional chamber and in a multiwire drift chamber.

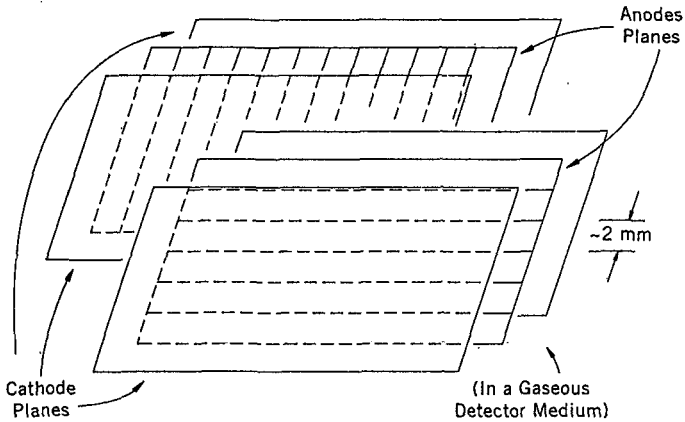


Fig. 7.4 Sketch of the geometry of multiwire proportional chamber planes.

A set of MWPC planes placed before and after a region that has an applied magnetic field can be used to obtain the change in the angle of a charged track as it passes through that region, which, in turn, provides the



momentum of the particle. The principle is shown in Fig. 7.5. Note that particles of different incident momentum will be dispersed, or fanned out, in  $\theta$ . The effect is very similar to the dispersion of white light through a prism. Thus, the position of a particle at the exit of the magnet is determined by its momentum. A system of the kind shown in Fig. 7.5 can therefore be used as a spectrometer to analyze the momentum distribution of particles in a beam.

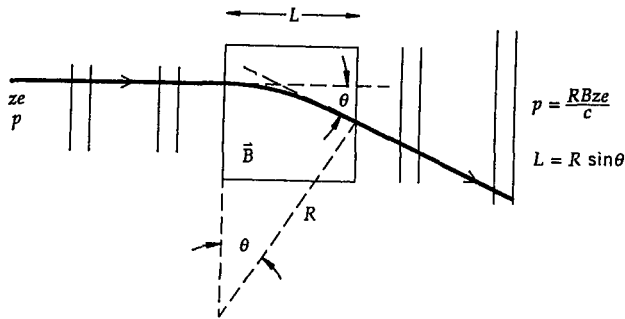


Fig. 7.5 Measuring the momentum of a charged particle using MWPCs, assuming the particle's trajectory traverses a region of constant magnetic field  $\vec{B}$  (normal to the plane). Reconstructing  $\theta$ , and knowing the length  $L$  and  $\vec{B}$ , provides  $p$ .

By modifying the electric field structure in a planar multiwire chamber, the position measuring accuracy can be improved substantially. The modified structure is shown in Fig. 7.3, where we have sketched the electric field lines in both a MWPC and in a *drift chamber*. The idea in the drift chamber is to provide a relatively constant electric field ( $E = -\frac{\Delta V}{\Delta x}$ ) in each cell in a direction transverse to normal incidence. This can be achieved by grading the potential uniformly along neighboring cathode field wires, and using additional "field-shaping" wires interspersed between the anode wires, as shown. The additional wires provide corrections needed to attain approximately constant electric fields between the anode wires. For most of a cell, the field is such that the ionization electrons drift along the electric field lines at a slow and constant velocity (about  $50 \mu\text{m}/\text{ns}$  or  $50 \text{mm}/\mu\text{s}$ ), until they arrive within a few diameters of the anode wire, where, in a short time, multiplication of signal takes place in the intense electric field. The time of arrival of the pulse at the wire, relative to some external fast signal, determines the distance of drift, and thereby yields a precise measure of the position of the incident particle. For, example, it is relatively straightforward to achieve precisions of  $\approx 200 \mu\text{m}$  in the measurement of position

for inter-anode spacings or drift distances of  $\approx 1$  cm.

### 7.2.3 Geiger-Müller Counters

A *Geiger-Müller counter*, or simply a *Geiger counter*, is an ionization detector that operates in the Geiger range of voltages, namely at a voltage high enough so that any produced ionization causes a gaseous discharge, independent of the initial energy deposited in the medium. To appreciate the advantage of a Geiger counter, let us consider the example of an electron, with a kinetic energy of 0.5 MeV, that loses all its energy within the counter. Suppose that the gaseous medium is helium, with an average ionization energy of 42 eV. The number of electron-ion pairs produced in the gas will be:

$$n = \frac{0.5 \times 10^6 \text{ eV}}{42 \text{ eV}} \approx 12,000. \quad (7.6)$$

If the detector operates as an ionization chamber, and has a capacitance of  $\approx 10^{-9} \text{ F} = 1 \text{ nF}$ , then the resulting voltage signal would correspond to

$$V = \frac{Q}{C} = \frac{ne}{C} = \frac{(12 \times 10^3)(1.6 \times 10^{-19} \text{ C})}{10^{-9} \text{ F}} \approx 2 \times 10^{-6} \text{ V}, \quad (7.7)$$

which is very small, indeed. On the other hand, if the detector operates in the Geiger mode, because of multiplication, the expected number of ion-pairs would be  $\approx 10^{10}$ , independent of electron energy. Consequently, the voltage pulse in this case would be a large and easily detectable  $\approx 1.6$  V.

The technical advantage of a Geiger counter is its simplicity of construction and its insensitivity to small voltage fluctuations. It is very useful for general measurement of nuclear radiation, but it has two important disadvantages. First, there is no information whatsoever on the nature of the ionization that caused the pulse. Second, because of the large avalanche induced by any ionization, a Geiger counter takes a long time (about 1 ms) to recover between successive pulses, and so it has a substantial dead-time, which means that it cannot be used for high counting rates.

## 7.3 Scintillation Detectors

The ionization produced by charged particles can excite atoms and molecules in the medium to higher energy levels. When these atoms and

molecules de-excite, they emit light that, in principle, can be detected and provide evidence for the traversal of the charged particle. *Scintillators* are kinds of materials that provide detectable photons in the visible part of the light spectrum, following the passage of a charged particle. There are primarily two types of scintillators in common use in nuclear and particle physics: organic or plastic scintillators and inorganic or crystalline scintillators. Although the physics of light emission is different in the two kinds of scintillators, and somewhat complicated, it is nevertheless well understood, but we will not discuss it here in any detail. Organic scintillators, such as anthracene or naphthalene, tend to emit ultraviolet light in their molecular de-excitation. Unfortunately, light of such frequency is rapidly attenuated, and consequently "wavelength shifter" material has to be admixed with the scintillator to permit detection of photons. That is, the initially produced light interacts with the wave-shifter material, which shifts the light to the visible part of the spectrum. Inorganic crystals, such as NaI or CsI, are usually doped with activators that can be excited by electron-hole pairs produced by charged particles in the crystal lattice; these dopants can then de-excite through photon emission.

Organic scintillators have fast decay times (typically  $\approx 10^{-8}$ sec), while inorganic crystals are usually far slower ( $\approx 10^{-6}$ sec), although some also have fast components in their response. Plastic scintillators are therefore more appropriate for use in high-flux environments. It takes substantially more energy to produce a detectable photon in a scintillator than an electron-ion pair through ionization (typically by a factor of 10), and because inorganic scintillators produce more light than organic scintillators, they are consequently better for applications at low energies.

In the pioneering days of nuclear studies, different phosphors were used routinely in experiments, and viewed by eye. The light produced in scintillators is usually very weak and normally cannot be seen in this manner. For scintillation light to be detected, the scintillator material must be transparent to its own radiation, that is, it cannot have a short attenuation length at the frequencies of interest. In addition, because of the low intensity of the emitted light, the photon signal must somehow be amplified in order to be counted. Signal amplification is achieved most commonly by using photomultiplier tubes (PMT) that view the scintillator either directly or through light guides.

A photomultiplier tube converts a weak photon signal to a detectable electric pulse. The device consists of several components (see Fig. 7.6). First, right after a thin entry window, is a photocathode, which is made

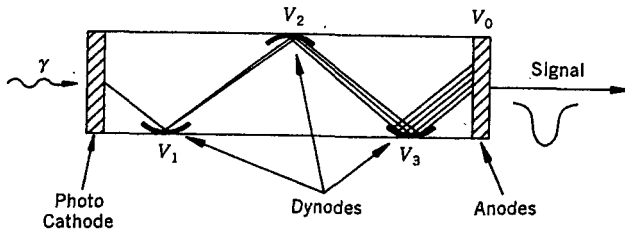


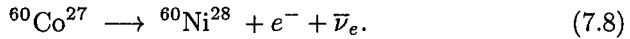
Fig. 7.6 Sketch of the main elements of a photomultiplier tube. The most common tubes have  $\approx 5$  cm diameters and are about  $\approx 20$  cm long.

of material in which the valence electrons are weakly bound and have a high cross section for converting photons to electrons via the photoelectric effect. As a result, any photon striking the photocathode will have a high probability for releasing an electron. (Cathode diameters are typically 2–12 cm, but far larger PMTs are also available.) Next, within the tube, there is a series (“stages”) of dynodes made of material of relatively low work function. These electrodes are operated at ever increasing potential ( $\Delta V \approx 100$ –200 V between dynodes), which can be provided, for example, through a regulated DC power supply and a resistor-divider chain. The dynodes accelerate the initial electrons to the next stage, and multiply them through secondary emission at each dynode. There are typically 6–14 dynode stages in PMTs, with a total gain, or electron amplification factor, in the range of  $\approx 10^4$ – $10^7$  (usually the multiplication factor is  $\approx 3$ –5 per dynode). The voltage is applied to the electrodes through pins embedded at the back end of the glass PMT; these pins are connected directly to the dynodes that are located within the vacuum region of the tube.

The quantum conversion efficiency of the photocathodes is typically  $\approx 0.25$  in the usual range of operating wavelengths ( $\approx 400$  nm). The output signal is usually taken from the anode or last dynode of the PMT. Except for statistical fluctuation, this signal is linearly proportional to the amount of light incident on the photocathode. Although there is some spread in the time of arrival of the signal due to different electron transit times (different paths and different velocities), this is typically only several nanoseconds. The output signals are therefore quite narrow, and, in conjunction with fast plastic scintillators, can be used very effectively for triggering any detector system on interesting events, and for timing intervals between successive signals.

A scintillator used in conjunction with a photomultiplier is consequently

an excellent detector of charged particles, and of any photons or neutrons that interact within the scintillator material. As an example, let us consider the  $\beta$ -decay of  $^{60}\text{Co}$



The  $^{60}\text{Ni}$  nucleus in this decay is, in fact, left in an excited state and decays to the ground level through two successive photon emissions: one of 1.17 MeV to the first excited level, which is followed by a 1.33 MeV photon to the ground level. Let us suppose that the  $^{60}\text{Co}$  sample is mounted on the front face of a NaI(Tl) crystal (thallium activated NaI scintillator), and a PMT is attached to the opposite side of the crystal. Because the anticipated signals are small, the crystal and PMT must be properly wrapped to prevent external light from leaking into the detector. Also, because crystals such as NaI are often hygroscopic, they must be well sealed to prevent deterioration through absorption of moisture. (Plastic scintillators do not have this disadvantage.)

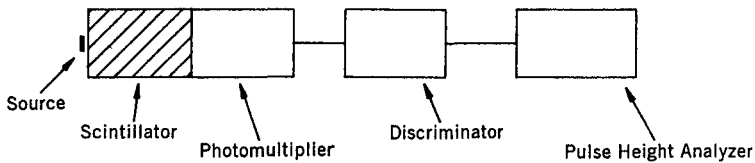


Fig. 7.7 Block diagram of apparatus needed for measuring the decay products in the decay of  $^{60}\text{Co}$ .

When photons from the de-excitation of  $^{60}\text{Ni}$  enter the scintillator, they can interact through the photoelectric effect, through Compton scattering or through  $e^+e^-$  pair production. Any photon that is converted to a photoelectron through the photoelectric effect generally deposits all of its energy within the scintillator in the form of ionization produced by the emitted electron. The intensity of the subsequently produced scintillation light is therefore proportional to the energy of the original photon. On the other hand, photons that undergo Compton scattering usually do not deposit all their energy within the scintillator, unless that scintillator block is exceedingly large. That is, although the scattered electrons often deposit their entire energy, the scattered photons tend to escape from the scintillator. (The radiation length of NaI is about 2.6 cm, while that of plastic scintillator is about 40 cm. It is therefore not surprising that, for detectors

several cm on a side, only a fraction of the energy of the incident photon is converted into ionization, and some leaves the detector.) Pair production is exceedingly unlikely for low-energy photons, but, when it occurs, the produced electron and positron deposit their kinetic energy in the scintillator, and eventually the positron annihilates with an atomic electron, yielding two 0.511 MeV photons.

Consequently, ignoring the low-energy electron in Eq. (7.8), the energy deposited in the NaI will, in general, have two kinds of contributions: First, the full energies of any of the photons that convert into photoelectrons, and, second, a continuous spectrum of energies deposited by the Compton-scattered recoil electrons. The scintillation light and the output from the PMT will therefore have signals equivalent to the deposition of 1.17 MeV, 1.33 MeV, and a continuum of energies below these peak values. (If enough 0.511 MeV photons are produced from the annihilation of  $e^+e^-$  pairs, they can provide photoelectrons and therefore a very useful calibration signal at 0.511 MeV.) The PMT output signals can be passed through a discriminator to eliminate any small signals from pulses produced through thermal electron emission from the cathode and dynode surfaces (“random noise”). After discrimination, the pulses can be digitized and displayed on a pulse-height analyzer (see Fig. 7.7). Because of fluctuations in ionization loss, differences in the efficiency of light collection, and fluctuations from electron multiplication, the 1.17 MeV and the 1.33 MeV energy signals will not be sharp, but will have a shape that reflects the experimental resolution of the detector system. The expected counting rate as a function of pulse height is sketched in Fig. 7.8. (The energy resolution for NaI(Tl) crystals in this range is about 10%.) We should point out that our simplified discussion has ignored the possibility of the simultaneous observation of the summed signals from the two emitted photons. We have left this common ramification to Problem 7.6.

## 7.4 Time of Flight

We have already remarked that a scintillation counter viewed with a PMT can provide excellent time resolution. In fact, with care, time resolutions of about  $10^{-10}$  sec (0.1 nsec) can be readily achieved. (This corresponds to a spatial resolution of  $\approx 3$  cm for particles traveling at the speed of light.) Thus arrays of scintillation counters can be used to measure the time of flight (TOF) of particles and thereby obtain their velocities. Such

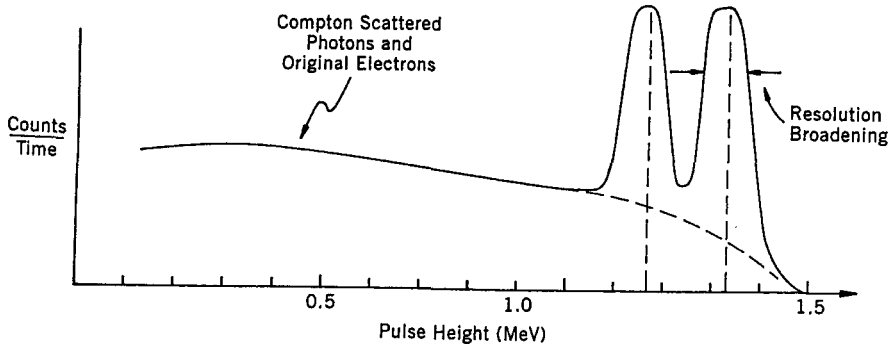


Fig. 7.8 Sketch of the counting rate as a function of pulse height expected for the products in the decay of  $^{60}\text{Co}$ .

TOF measurements have important applications in providing discrimination between particles of similar momentum but different mass that may be produced in a collision. For example, measuring the momentum ( $p$ ) of a charged particle in a magnetic field (see Fig. 7.5), as well as its time of flight ( $t$ ) for reaching some scintillation counter located at a distance  $L$  from the point of origin of the particle, determines the velocity and therefore the rest mass of the particle. Let us assume that the momentum measurement is very precise, and investigate the limitation of the TOF technique.

The TOF corresponds to just the distance traveled divided by the speed of the particle, namely,  $t = \frac{L}{v}$ . Consequently, the difference in the flight time of two particles of mass  $m_2$  and  $m_1$  will be

$$\Delta t = t_2 - t_1 = L \left( \frac{1}{v_2} - \frac{1}{v_1} \right) = \frac{L}{c} \left( \frac{1}{\beta_2} - \frac{1}{\beta_1} \right). \quad (7.9)$$

For our known momentum  $p$ , this can be rewritten as

$$\Delta t = \frac{L}{c} \left[ \frac{E_2}{pc} - \frac{E_1}{pc} \right] = \frac{L}{pc^2} \left[ (m_2^2 c^4 + p^2 c^2)^{\frac{1}{2}} - (m_1^2 c^4 + p^2 c^2)^{\frac{1}{2}} \right]. \quad (7.10)$$

In the non-relativistic limit, this reduces to the classical expression

$$\Delta t = \frac{L}{p} (m_2 - m_1) = \frac{L}{p} \Delta m, \quad (7.11)$$

which, for the most interesting (and most demanding) regime of  $m_2 \approx m_1 = m$  and  $v_2 \approx v_1 = v = \beta c$ , becomes

$$\Delta t = \frac{L}{\beta c} \frac{\Delta m}{m} = t \frac{\Delta m}{m}. \quad (7.12)$$

Now, using  $v = \frac{L}{t}$ , and  $dv = -\frac{L}{t^2} dt$ , we can write

$$\Delta v = v_2 - v_1 = -\frac{L}{t^2} \Delta t = -\frac{v^2}{L} \Delta t. \quad (7.13)$$

Consequently, at low energies ( $\beta \approx 0.1$ ), for a time resolution  $\Delta t \approx 2 \times 10^{-10}$  sec, and for a flight path of  $L \approx 10^2$  cm, we obtain a very respectable resolution in velocity

$$\begin{aligned} |\Delta v| &= \frac{v^2}{L} \Delta t \approx \frac{(0.1 \times 3 \times 10^{10})^2 \text{cm}^2/\text{sec}^2}{10^2 \text{cm}} \\ &\times 2 \times 10^{-10} \text{sec} \approx 2 \times 10^7 \text{cm/sec}. \end{aligned} \quad (7.14)$$

And because  $-\frac{\Delta v}{v} = \frac{\Delta t}{t} = \frac{\Delta m}{m}$ , this means that, using TOF, it is possible to discriminate between low-energy particles of same momentum but different mass to better than  $\approx 1\%$  level of accuracy. Clearly, the relative mass resolution deteriorates linearly with increasing momentum, and improves with increasing path length.

Now, for the relativistic limit, let us rewrite Eq. (7.10) as

$$\begin{aligned} \Delta t &= \frac{L}{c} \left[ \left( 1 + \frac{m_2^2 c^2}{p^2} \right)^{\frac{1}{2}} - \left( 1 + \frac{m_1^2 c^2}{p^2} \right)^{\frac{1}{2}} \right] \\ &\approx \frac{L}{c} \left[ 1 + \frac{m_2^2 c^2}{2p^2} - \left( 1 + \frac{m_1^2 c^2}{2p^2} \right) \right] \approx \frac{Lc}{2p^2} (m_2^2 - m_1^2), \end{aligned} \quad (7.15)$$

which, for  $m_1 \approx m_2$  and  $v_1 \approx v_2$ , becomes

$$\begin{aligned} \Delta t &= \frac{Lc}{2} \left( \frac{m_2^2}{p^2} - \frac{m_1^2}{p^2} \right) = \frac{Lc}{2} \left[ \frac{m_2^2}{m_2^2 \gamma_2^2 v_2^2} - \frac{m_1^2}{m_1^2 \gamma_1^2 v_1^2} \right] \\ &= \frac{Lc}{2} \left[ \frac{1 - \frac{v_2^2}{c^2}}{v_2^2} - \frac{1 - \frac{v_1^2}{c^2}}{v_1^2} \right] \approx \frac{L}{c} \frac{v_1 - v_2}{v} = -\frac{L}{c} \frac{\Delta v}{v}. \end{aligned} \quad (7.16)$$



Thus, when  $v \approx c$ , assuming the above conditions of  $\Delta t \approx 2 \times 10^{-10}$  sec and  $L \approx 10^2$  cm, we obtain for the resolution in velocity

$$|\Delta v| \approx \frac{c^2}{L} \Delta t \approx \frac{(3 \times 10^{10})^2 \text{cm}^2/\text{sec}^2}{10^2 \text{cm}} \times 2 \times 10^{-10} \text{sec} \approx 2 \times 10^9 \text{cm/sec}. \quad (7.17)$$

Although the resolution for particle velocity is still  $\approx 10\%$ , unfortunately, the relative mass resolution for  $v \approx c$  is no longer as good. In fact, from Eq. (7.16) we can deduce that

$$\Delta t = \frac{Lc}{2} \frac{(m_2 - m_1)(m_2 + m_1)}{p^2} \approx \frac{Lcm}{p^2} \Delta m = Lc \frac{m^2}{p^2} \frac{\Delta m}{m} \approx \frac{L}{c\gamma^2} \frac{\Delta m}{m},$$

so that

$$\frac{\Delta m}{m} = \frac{c\gamma^2}{L} \Delta t = \gamma^2 \frac{\Delta t}{t}. \quad (7.18)$$

Consequently, for momenta of  $\gtrsim 3$  GeV/ $c$ , and masses of  $\approx 1$  GeV/ $c^2$ , we have  $\gamma \gtrsim 3$ , and the resolution in mass discrimination is essentially lost. Our example is, of course, only true if the flight path cannot be increased greatly beyond  $\approx 100$  cm. For fixed-target experiments that involve highly relativistic particles, this can be a possible alternative, however, for most large collider experiments, increasing the flight path would also mean increasing the size of the overall detection system (see next chapter), which could be very expensive. It should also be recognized that the flight path cannot be increased without limit if the particles of interest decay with short lifetimes.

As we have implied previously, TOF can also be used to obtain momenta of low energy neutrons or of any photons that interact within our scintillation counter. In such cases, the initial collision time has to be obtained through other means, as, for example, from the interaction time as defined by a pulse produced by an incident beam particle. The difference in time between a signal in the scintillation counter and some initial "start" time can provide the TOF of any particle.

## 7.5 Cherenkov Detectors

When a charged particle moves with uniform velocity in vacuum, it does not emit radiation. However, if it enters a dielectric medium of index of refraction  $n > 1$ , with a speed greater than the speed of light in that medium (i.e.,  $v > \frac{c}{n}$  or  $\beta > \frac{1}{n}$ ), then it emits what is known as Cherenkov radiation (after Pavel Cherenkov, who first observed the effect in 1934). The direction of the emitted light can be calculated classically using Huygen's wave construction, and can be attributed to the emission of coherent radiation from the excitation of atoms and molecules in the path of the charged particle. The effect is completely analogous to the "shock" front produced by supersonic aircraft. The emitted light has a spectrum of frequencies, with the most interesting component being in the blue and ultraviolet band of wavelengths. The blue light can be detected with relatively standard photomultiplier tubes, while the ultraviolet light can be converted to electrons using photosensitive molecules that are mixed in with the operating gas in some ionization chamber (e.g., MWPC).

The angle of emission for Cherenkov light is given essentially by

$$\cos \theta_c = \frac{1}{\beta n}, \quad (7.19)$$

and the intensity of the produced radiation per unit length of radiator is proportional to  $\sin^2 \theta_c$ . Consequently, for  $\beta n > 1$ , light can be emitted, while for  $\beta n < 1$ ,  $\theta_c$  is complex and no light can be observed. The Cherenkov effect therefore provides a means for distinguishing two particles of same momentum but different mass. For example, protons, kaons and pions of 1 GeV/c momentum have  $\beta = 0.73$ , 0.89 and 0.99, respectively. Consequently, to observe Cherenkov light from these particles would require media of different refractive index. In particular, for protons to emit light, we would need a threshold  $n > 1.37$ , kaons would require  $n > 1.12$ , and pion  $n > 1.01$ . Now suppose that we arrange two Cherenkov counters in series, one filled with water ( $n = 1.33$ ) and the other filled with gas under pressure so that it has  $n = 1.05$ . If we pass a mixture of protons, kaons and pions through the two counters, the protons will not provide a signal in either detector, kaons will radiate Cherenkov light only in the water vessel, while pions will register signals in both counters. This can therefore provide a way of discriminating between particles that have different Cherenkov thresholds. When counters are used in this manner, they

are termed *threshold* counters. (Most of the large experiments searching for proton decay rely on Cherenkov light to identify the end products of the decay, e.g.,  $p \rightarrow e^+ \pi^0$ .)

We can also see from Eq. (7.19) that we can discriminate between particle types on the basis of the observed angle of emission of Cherenkov light. That is, for some fixed  $n$  value, the cone angle for light emitted by pions will be greater than that from kaons and protons. Cherenkov counters that are sensitive to different emission angles are known as *differential* counters.

Finally, more recent developments in this area have centered on the ultraviolet (UV) part of the emission spectrum ( $\approx 5$  eV). As we just mentioned, UV photons, through photo-ionization, can produce electrons that can be detected using MWPCs. At high energies, several UV photons can be emitted by a single charged particle. These photons will be distributed in a cone at angle  $\theta_c$  about the incident charged track. Consequently, in any ionization-sensitive device positioned transverse to the incident line of flight, electrons that are produced by these UV photons will be distributed in a ring pattern. Detectors that rely on this principle are known as *ring-imaging* Cherenkov counters (or RICH counters), and they are particularly useful in experiments in which many particles are produced in any given collision.

## 7.6 Semiconductor Detectors

The formation of an electron-hole pair in a semiconductor such as silicon or germanium requires an energy of only about 3 eV; consequently, when these crystals are used as solid-state ionization chambers, they can provide large signals for very little energy deposition in the medium. Solid-state devices can therefore be particularly advantageous for applications at low energies. They were, in fact, developed initially in nuclear physics for high-resolution measurements of energy, and for obtaining ranges and stopping power of nuclear fragments. More recently, silicon strip detectors and pixels have gained wide acceptance in both nuclear and particle physics for precision measurement of positions of charged particles.

Because the number of free charge carriers produced in semiconductors is so large, and both electrons and holes have high mobility, very thin wafers of crystal (about 200-300  $\mu\text{m}$ ) suffice for achieving good signals, even for minimum-ionizing particles. The performance of these detectors is quite linear in that the output signal is proportional to the ionization loss, pro-

vided that an imposed electric field within the medium is large enough to prevent recombination of the charge carriers. This can be achieved by using very pure semiconductors of high-resistivity, and operating these detectors as diodes with a reverse bias of about 100 V. The semiconductor wafer is sandwiched between very thin conducting electrodes (thickness of tens of  $\mu\text{gm}/\text{cm}^2$ ), which can be deposited in electrically separated stripes (or other patterns) on the surface of the wafer. Detectors  $5 \times 5 \text{ cm}^2$  in area are quite common; they often have 20-50  $\mu\text{m}$  stripes, and are used in series (just like planes of MWPCs) to determine charged-particle trajectories to position-accuracies of the order of several  $\mu\text{m}$  in the transverse direction. Such devices can be used to measure small impact parameters and thereby determine whether some charged particle originated from a primary collision or was the decay product of a particle that traveled a small distance from the original interaction, and then decayed.

Two silicon detectors positioned in series can be used to determine the kinetic energy and velocity of any low-energy particle or nuclear fragment, and therefore its rest mass. This determination is made by placing a very thin wafer in front of a thicker detector that can stop that particle. The velocity is deduced from the stopping power measured in the thin wafer, and the mass from the range or from the total kinetic energy loss in the thicker crystal (or array of thin wafers).

## 7.7 Calorimeters

Momenta of charged particles can be measured in a relatively straightforward fashion using magnetic spectrometers (see, for example, Fig. 7.5). In certain situations, however, magnetic measurement may not be viable. For example, precise magnetic measurements become difficult and expensive at very high energies because they require either large magnetic fields in extended regions of space, or very long lever arms for measuring small changes in the angular trajectories of particles passing through magnets, or both. Also, at times, specific design considerations may preclude the use of an analyzing magnet in an experiment. In addition, magnets cannot be used for measuring energies of neutral particles (e.g., neutrons or photons). Under circumstances when magnetic measurements are not of greatest value, we can turn to calorimetric detectors that rely purely on the measurement of total energy deposition in a medium. A *calorimeter* is a device that absorbs the full kinetic energy of a particle, and provides a signal that is

proportional to that deposited energy. One of the simplest calorimeters we can imagine is the device that we described earlier in this chapter for measuring the ranges of  $\alpha$  particles. Large calorimeters were developed during the early 1960s, especially for application in experiments dealing with high-energy cosmic rays, and they have become exceedingly important tools for measuring energies of particles produced at large accelerators.

We know that when high energy photons traverse matter they do not deposit energy until they convert into electron-positron pairs. The produced electrons and positrons deposit their energies, as usual, by ionizing atoms; however, when they are very energetic, they lose most of their energy through bremsstrahlung. When these bremsstrahlung photons have high energies, they can, in turn, convert to electron-positron pairs, which can radiate more photons, etc. This electromagnetic *shower* develops into a sea of low energy photons, electrons and positrons that eventually deposit all their remaining energies in the material.

Similarly, hadrons can deposit energy in matter through a series of successive interactions. However, because hadrons are relatively massive, and cannot radiate much of their energy through bremsstrahlung, they lose their energy mainly through multiple nuclear collisions. Thus, an incident hadron might produce several  $\pi$  mesons in its first nuclear collision; these pions will subsequently collide with downstream nuclei, producing more particles, etc., until the incident high energy is converted to many charged particles of low energy that eventually deposit that energy by ionizing atoms within the medium. Because in most materials (especially of  $Z > 10$ ) the mean free path for nuclear interactions is substantially greater than that for electromagnetic interactions (see Example 7 in Chapter 6), it takes far more thickness of material to generate a hadronic shower than an electromagnetic one. Consequently, calorimeters designed for measuring deposition of electromagnetic energy are physically far thinner than those designed for absorbing hadrons.

We should also mention that hadronic showers can have large fluctuations in their energy deposition. This is because such collisions often involve production of unstable particles that have neutrinos as decay products. Since neutrinos have exceedingly small interaction probabilities, they escape detection and thereby reduce the energy that a hadron deposits in the detector material. This occurs on a statistical basis, and consequently compromises the energy measurement of hadrons. Another important source of fluctuation in the energy deposited by hadrons is due to the production of neutral pions. These particles decay immediately into two photons

(lifetime of about  $10^{-16}$  sec), and the produced photons initiate their own showers. Since the energy deposited by photons is electromagnetic, it is therefore deposited relatively locally, and, as a result, the detected energy can be very sensitive to the detailed structure of the detector (see below). Because of such complications, the calorimetric energy resolution that can be obtained for hadronic showers is therefore expected to be worse than that for electromagnetically interacting particles.

We have discussed several mechanisms for energy deposition in matter, and have already described some of the most common ones that are often applied in the detection of particles. Similar principles apply in calorimetry. For example, relying on the production of ionization in a medium, calorimeters can function as ionization chambers (e.g., liquid-argon calorimeters); or, through the production of scintillation light, they can be scintillation-sensitive detectors (e.g., NaI); they can also rely entirely on the production of Cherenkov radiation (lead glass). In principle, a particle's energy can also be determined from the heating of a detector or from the deposited acoustic energy, and such techniques have, in fact, been explored in the past. Calorimeters can be constructed from homogeneous media (e.g., crystalline scintillators or lead glass), or they can be *sampling* detectors. Sampling calorimeters contain mainly uninstrumented absorber material that is interspersed with active sampling devices to sense the energy of a developing shower (see Fig. 7.9). Homogeneous detectors, which are usually sensitive to the full energy deposition, tend to have best resolution, but they are invariably very expensive. Sampling detectors, because they are subject to additional sampling fluctuations, usually have worse resolution, but are often easier and less expensive to construct. It is relatively straightforward to make large sampling calorimetric systems (with thousands of individual channels) with energy resolution in  $\frac{\Delta E}{E}$  of about  $\frac{0.2}{\sqrt{E}}$  for detection of electromagnetic components, and about  $\frac{1}{\sqrt{E}}$  for the hadronic components (where  $E$  is in GeV units), but anything beyond that is more challenging. (The improvement in relative resolution with increase in energy can be attributed to the expected decrease in sampling fluctuations.)

## 7.8 Layered Detection

Collisions at high energies involve the production of many and different kinds of particles. In any given event, there can be electrons, muons, neutrinos and a large number of pions. Some particles are stable, others short-

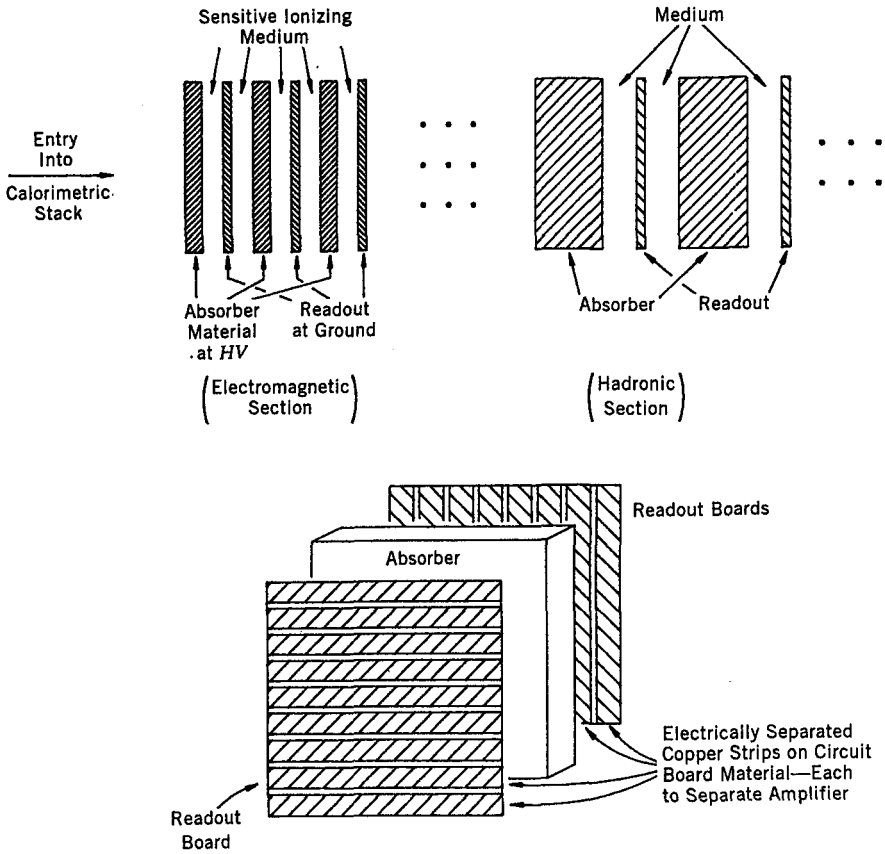


Fig. 7.9 A typical stack structure in a sampling calorimeter.

lived. To unravel the event structure, and to find out about the underlying physics of the collision, it is often required to measure all these species, and usually with substantial precision. Modern spectrometer systems, especially at colliders, which we discuss in the following chapter, are designed in a layered manner, with every layer having a unique function (see Fig. 7.10). For example, the region closest to the intersection point often has several layers of very thin silicon microstrip detectors to provide precise spatial information for trajectories of charged particles. (The reason for having "thin" material is to minimize multiple scattering and the conversion of photons into  $e^+e^-$  pairs in the silicon.) Such microstrip detectors also provide sensitivity to the characteristic decays of short-lived particles.

For a collider configuration, the next layer of instrumented detection might have, for example, several layers of drift chambers surrounding the silicon system. Such chambers are often located in an axial (i.e., solenoidal) magnetic field, which can be implemented through a thin superconducting coil that surrounds the chambers. (Again, the reason for “thin” is to minimize energy loss and multiple Coulomb scattering within the coil, which worsens the resolution.) The signals from the silicon detectors and from the drift chambers provide momentum information on any charged particles that emanate from the interaction point.

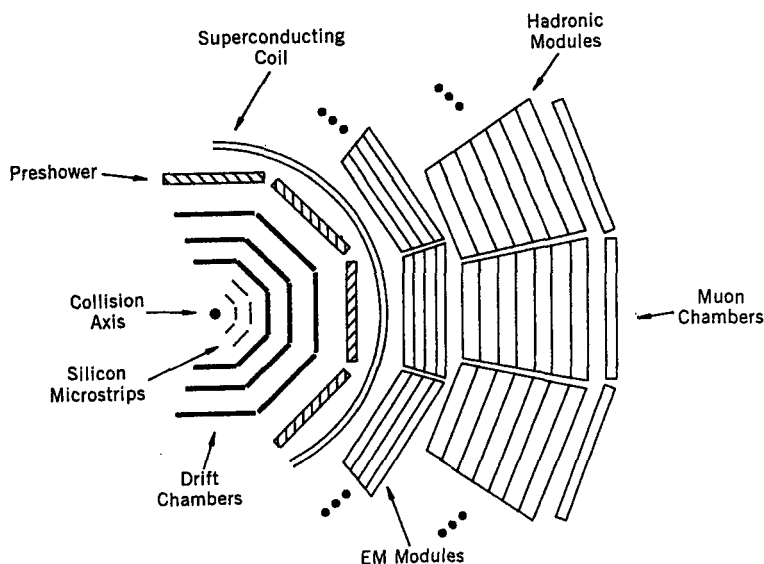


Fig. 7.10 Sketch of a “typical” layered detector for high energy physics experiments at colliders.

The next step in detection might involve segmented “preshower” counters, which usually consist of  $\approx 3$  radiation lengths of absorber followed by scintillation counters. (The scintillation counters can also provide information on TOF.) The inner tracking chambers usually have far less material, and consequently do not produce much photon conversion or bremsstrahlung of any original electrons. The preshower counters, on the other hand, can provide first evidence for the presence of electromagnetically interacting particles through the presence of large pulse heights from photons that convert into  $e^+e^-$  pairs or electrons that produce showers in the material in front of these counters. (The magnetic-field coil can function



as part of the radiator/absorber material when it precedes the preshower counters.)

With trajectories of charged particles determined, and some indication available for the presence of photons and electrons, the next step usually involves electromagnetic (EM) calorimetry. Typically, EM calorimeters are made of about 20 radiation lengths of high- $Z$  material, which usually corresponds to about one mean-free-path for hadronic interactions. Consequently, electrons and photons deposit essentially all their energies within the EM modules, and hadrons only start to interact there. Far thicker hadronic calorimeters, in which hadrons deposit most of their energy, follow the EM calorimeters. (The EM sections are about 30 cm thick, while the hadronic modules are usually about 150 cm thick, depending somewhat on the type of absorber and readout used.)

The particles that penetrate through the calorimetry are primarily neutrinos and high-energy muons (with energy greater than that given by the range within the intervening material). Muons that penetrate the calorimeters can be momentum analyzed again outside of the calorimeters, and their trajectories can be traced back and matched for consistency with the tracking information available from within the calorimeters. This leaves only neutrinos not accounted for. Their presence can be inferred from a lack of overall momentum balance in the event (especially balance in the transverse direction). In order to be sensitive to the presence of any missing energy that may be carried off by neutrinos, the detectors have to be designed to surround the intersection region over as much of the  $4\pi$  solid angle as is possible, and to provide a minimum of loss in coverage from the presence of structural elements, and the like. Needless to say, this poses very great technical challenges to the experimenter.

## Problems

**7.1** A radioactive source emits  $\alpha$ -particles with kinetic energies of 4 MeV. What must be the value of an applied magnetic field so that the radius of curvature of the orbit of the  $\alpha$ -particle is 10 cm? (Does your answer depend on the kind of medium into which the  $\alpha$ -particle is emitted?) Do the same calculation for electrons of same kinetic energy.

**7.2** The mass of a  $K^+$  is  $494 \text{ MeV}/c^2$  and that of a  $\pi^+$  is  $140 \text{ MeV}/c^2$ . If the rms time resolution of each of two scintillation counters that are 2 m apart is 0.2 nsec, calculate to better than 10% accuracy the momentum

at which the system will just be able to resolve a  $\pi^+$  from a  $K^+$  (by one standard deviation). (Hint: see Eq. (7.10).)

**7.3** What are the Cherenkov angles for electrons and pions of 1000 MeV/c for a radiator with  $n = 1.4$ ? What will be the ratio of the number of radiated photons for incident electrons and pions?

**7.4** About  $10^6$  electron-ion pairs are produced by a charged particle traversing a counter. If the typical ionization potential of the medium is  $\bar{I} = 30$  eV, in principle, how well can you measure the deposited energy using a Geiger counter, an ionization counter with a gain of unity, and a proportional counter with a gain of  $10^6$  that has gain variations of 5%?

**7.5** If you wish to measure the momentum of a 10 GeV/c singly-charged particle to 1% accuracy, in a 2 T field, using a 1 m long magnet, how well do you have to know the exit angle (see Fig. 7.5)? If you use MWPCs that have 2 mm inter-wire anode spacings to measure that angle, about how far do you have to separate two planes to achieve your goal? Now suppose that you use, instead, silicon microstrip detectors of 25  $\mu\text{m}$  spacing. What separation distance between two such planes could achieve the same goal?

**7.6** Sketch the pulse height spectrum that you would expect in the decay of  $^{60}\text{Co}$  in Eq. (7.8) when the two de-excitation photons are emitted simultaneously, namely within the time resolution of the detector.

### Suggested Readings

Fernow, R. C., *Introduction to Experimental Particle Physics*, Cambridge Univ. Press (1986).

Kleinknecht, K., *Detectors for Particle Radiation*, Cambridge Univ. Press (1998).

Knoll, G. F., *Radiation Detection and Measurement*, Wiley (1989).

Leo, W. R., *Techniques for Nuclear and Particle Physics Experiments*, Springer-Verlag (1994).



## Chapter 8

# Accelerators

### 8.1 Introductory Remarks

Accelerators are some of the most remarkable tools of modern science. They are precision instruments constructed on a gargantuan scale. They have to track and accelerate particles that traverse millions of km in just matters of seconds, and maneuver and constrain particle motion to accuracies of the order of  $1\ \mu\text{m}$ . They can provide sufficient numbers of energetic particles to vaporize macroscopic targets with single pulses of beam. Because of their immensity and their challenging complexity, and because of their symbolic reflection of the intellectual aspirations and creativity of mankind, modern accelerators have been likened by Robert R. Wilson to the great Gothic cathedrals of medieval Europe. Their impact on nuclear and particle physics, the fields for which they were initially developed, has, of course, been pivotal. They have served as the microscopes for probing nuclear and particle structure, and, in fact, were it not for the development of accelerators, the fields of nuclear and particle physics would still be in their infancies.

After the pioneering experiments of Rutherford and his colleagues revealed the presence of a nucleus within the atom, it became clear that higher-energy scattering experiments could provide an invaluable probe of the nucleus. For example, it was understood that with sufficient energy to penetrate the Coulomb barrier, projectiles could break apart nuclei and reveal their constituents. It was also recognized that the more energy a particle had, the more deeply it could probe within the nucleus. This is simply a consequence of the fact that, through the uncertainty principle, large momentum transfers correspond to small distances, and vice versa. The study of the short-distance behavior of nuclei and of elementary par-

ticles requires therefore the availability of high energy beams that can be used to impart large momentum transfers either to target particles or to other beams of particles.

Although high energy particles are available in the cosmic rays, their fluxes are quite low, and their energies, clearly, cannot be controlled. In fact, the excitement brought about by discoveries of new phenomena in experiments with cosmic rays only added impetus to the development of techniques for accelerating charged particles.

The increase in accelerator energies achieved over the past 75 years has been astounding. The first accelerators that were constructed (around 1930) provided beams of particles with energies of hundreds of keV, while the largest modern accelerators will soon have beam energies of almost  $10^8$  greater than that. And because of the advent of colliding-beam techniques, the effective increase in beam energy (that is, considering the energy available in the center-of-mass) has gone up by an even more spectacular factor of about  $10^{12}$ ! Such changes correspond to differences of about  $10^6$  in the sensitivity to distance scales that can be studied with the next-generation accelerators, which expect to provide sensitivity in the range of  $10^{-18}$  cm. Nowadays, besides being used in nuclear and particle physics, accelerators are in demand in a variety of applications, ranging from experiments in condensed matter physics, the electronics industry, biomedical and geophysical areas, to food processing and sewage treatment. Accelerator science is therefore no longer just an appendage of nuclear and particle physics, but is a separate intellectual discipline in its own right.

There is a variety of ways of accelerating charged particles, and the methods used for any specific application depend upon the kinds of probes that are required, their energies, the desired beam intensities, and, of course, any economic constraints. We will now sketch several of the key historical developments in particle acceleration during the past 75 years.

## 8.2 Electrostatic Accelerators

### 8.2.1 *Cockcroft-Walton Machines*

Cockcroft-Walton machines are the simplest types of accelerators. They are based on passing ions through sets of aligned electrodes that are operated at successively higher fixed potentials. Usually, voltage-doubling circuits are used for generating the high electric fields. A machine consists of an ion source (often hydrogen gas) located at one end, and a target at

the other, with the electrodes arranged in between. First, electrons can either be added to or stripped from the atoms of interest in order to produce ions, which are then passed through the series of accelerating regions. The kinetic energy gained by an ion of charge  $q$ , passing through the voltage difference  $V$ , is given simply by  $T = qV$ . John Cockcroft and Ernest Walton were the first to successfully apply these principles to particle acceleration, and used their device to disintegrate lithium nuclei using protons of about 400 keV. Cockcroft-Walton machines are limited to about 1 MeV energies because of the voltage breakdown and discharge that takes place much beyond voltages of 1 MV. Currently, Cockcroft-Walton accelerators are available commercially, and are often used as the first-step high-current injectors (of the order of 1 mA) in the multi-stage process of accelerating particles to high energies.

### 8.2.2 Van de Graaff Accelerator

The energy gained by a particle (ion) accelerated in a DC voltage machine is directly proportional to the applied voltage, consequently, clever construction of the high voltage source is of crucial importance. That is precisely what the Van de Graaff generator (named after Robert Van de Graaff) does. The basic principle relies on the fact that since the charge on any conductor resides on its outermost surface, if a conductor carrying charge touches another conductor that envelops it, then, irrespective of its potential, it will transfer all its charge to the outer conductor. This can be used to advantage to increase the charge on any conductor, and consequently to create a higher voltage.

In the Van de Graaff accelerator, charge is carried on a conveyor belt into a large metallic dome, where it is picked off, as shown in Fig. 8.1. The conveyor belt is made of insulating material, and goes over motor-driven rollers (R). A "sprayer" (S), connected to a discharge-voltage terminal, sprays positive ions to the conveyor belt (electrons go to  $P$ ). (Basically, the high voltage ionizes the gas, and the ions are collected on the conveyor belt.) The points where charges are sprayed or injected onto the belt are known as *corona points*. The conveyor belt takes the positive charges up to the dome, which is maintained at a positive voltage. The energy needed to do this work is provided by the motors. At the upper end of the conveyor there is a collector  $C$ , which collects the positive charges transferred to the dome. Typically, this technique can produce accelerating potentials of up to  $\approx 12$  MV. (A tandem generator is a modification of the Van de

Graaff generator, where negative ions that enter the accelerating tube from one side, are first accelerated to the positive HV terminal, where they are stripped and made positive, and then accelerated as positive ions down to ground potential on the other side of the terminal. This, effectively, doubles the acceleration energy to  $\lesssim 25$  MeV.)

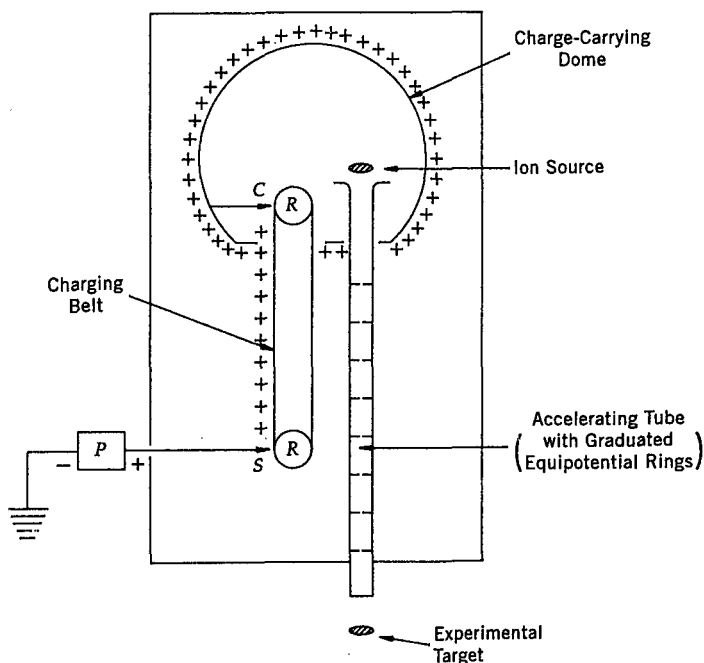


Fig. 8.1 The principle of a Van de Graaff accelerator.

The Van de Graaff has an evacuated tube through which ions from the ion source are accelerated to strike the target. This accelerating tube is constructed with equipotential metallic rings embedded within the insulated tube. The entire device operates within a pressurized chamber, containing some inert gas that does not breakdown easily (often  $\text{SF}_6$ ). Normally, the pressure of the gas inside the Van de Graaff is about 15 atm, and the limit for the highest energy in such a machine comes from the voltage at which there is electrical breakdown and discharge in the gas.

### 8.3 Resonance Accelerators

#### 8.3.1 Cyclotron

Fixed-voltage machines have an inherent limitation to their energy because of voltage breakdown and discharge. An alternative method, which uses the resonance principle, is more important for accelerating particles to higher energy.

The cyclotron (or cyclic accelerator), first built by Ernest Lawrence, is the simplest of the machines that use this principle (see Fig. 8.2). The accelerator is constructed out of two hollow evacuated *D*-shaped metal chambers (referred to as *Ds*), which are connected to an alternating high-voltage source. The entire system is placed inside a strong magnetic field perpendicular to the *Ds*. The principle of operation of the cyclotron is as follows.

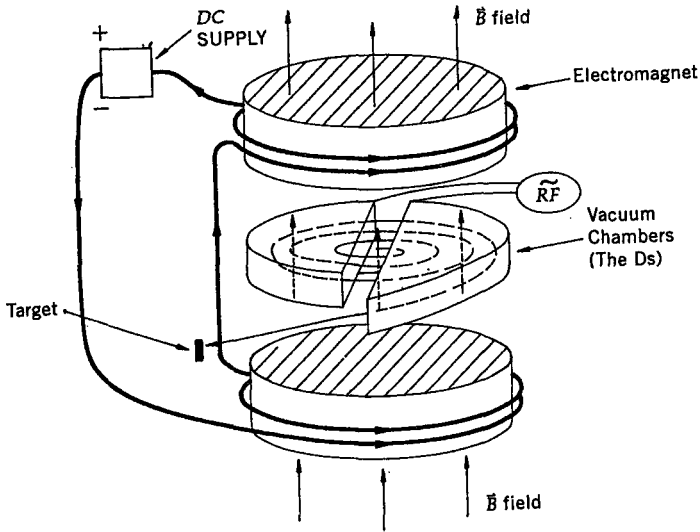


Fig. 8.2 Sketch of motion in a cyclotron.

Although the hollow *Ds* are connected to the source of high voltage, because of the shielding effect of the metallic chamber walls, there is no electric field within the *Ds*. Consequently, a strong alternating electric field exists only in the gap between the *Ds*. A source for producing ions is placed in the gap between the *Ds*, and, depending on the sign of the voltage at that moment, any ion in the gap is attracted towards one of the



Ds. However, the trajectory of the ions is circular because of the bending effect of the magnetic field. Once an ion is inside the D, it stops sensing the electric force, but continues in its circular motion because of the presence of the static magnetic field. But after a half circle, when the ion is about to emerge from the D, the direction of the voltage can be changed and the ion can be accelerated again before it enters the other D. Similarly, when it is about to exit from the second D, the applied voltage can again be reversed and the particle accelerated further. If the frequency of the alternating voltage source is just right, then the charged particle can be accelerated continuously and move in ever increasing radial orbits, until it is extracted to strike a target (for example, by suddenly turning off the  $B$  field).

For non-relativistic motion, the frequency appropriate for the alternating voltage can be calculated from the fact that the magnetic force provides the centripetal acceleration for a circular orbit. That is,

$$m \frac{v^2}{r} = q \frac{vB}{c},$$

$$\text{or } \frac{v}{r} = \frac{qB}{mc}. \quad (8.1)$$

Now, for circular motion at constant speed, the angular frequency  $\omega$  is related to the radius and circular velocity of the orbit as

$$\omega = \frac{v}{r}. \quad (8.2)$$

We can therefore express the frequency of the motion as

$$\nu = \frac{\omega}{2\pi} = \frac{qB}{2\pi mc} = \frac{1}{2\pi} \left( \frac{q}{m} \right) \frac{B}{c}. \quad (8.3)$$

Clearly, to keep the acceleration in phase with the particle motion requires that the frequency of the electric field be the same as  $\nu$ . This frequency is referred to as the *cyclotron resonance frequency*, and is the origin of the label "resonance accelerator" for this kind of machine. Equation (8.3) therefore provides a means for determining the frequency of the accelerating fields as a function of other parameters. The maximum energy that a charged particle has when it is extracted at a radius  $r = R$  is given by

$$\begin{aligned}
 T_{\max} &= \frac{1}{2} m v_{\max}^2 = \frac{1}{2} m \omega^2 R^2 \\
 &= \frac{1}{2} m \left( \frac{qB}{mc} \right)^2 R^2 = \frac{1}{2} \frac{(qBR)^2}{mc^2}
 \end{aligned} \tag{8.4}$$

Equation (8.4) relates the magnitude of the magnetic field and the size of the magnet that is needed to accelerate a particle to any given energy. In a typical cyclotron,  $B \lesssim 2$  T, the alternating voltage applied to the Ds is  $\approx 200$  kV, at a frequency of  $\approx 10$ – $20$  MHz. The maximum proton energy that can be attained in such cyclotrons is about 20 MeV (for Ds of  $R \approx 30$  cm), as will be shown shortly in an example.

As we increase the energy of charged particles, they become relativistic, and the frequency relation in Eq. (8.3) starts failing. Consequently, a fixed-frequency cyclotron cannot accelerate ions to relativistic energies. For electrons, relativistic effects set in at even lower energies, and consequently, such simple cyclotrons are not useful for accelerating electrons. Synchronous accelerators (discussed in the following section) are needed for attaining relativistic energies.

### Example 1

For a cyclotron operating at an extraction radius  $R = 0.4$  m, and a magnetic field of  $B = 1.5$  T =  $1.5 \times 10^4$  G, the frequency of the alternating source needed to accelerate protons, and the maximum energy gained by these protons, can be calculated from Eqs. (8.3) and (8.4) as

$$\begin{aligned}
 \nu &= \frac{qB}{2\pi m_p c} = \frac{1}{2\pi} \frac{4.8 \times 10^{-10} \text{ esu} \times c \times 1.5 \times 10^4 \text{ G}}{m_p c^2} \\
 &\approx \frac{4.8 \times 10^{-10} \text{ esu} \times 3 \times 10^{10} \text{ cm/sec} \times 1.5 \times 10^4 \text{ G}}{6.28 \times 10^3 \text{ MeV} \times (1.6 \times 10^{-6} \frac{\text{erg}}{\text{MeV}})} \\
 &\approx 22.8 \times 10^6 / \text{sec} = 22.8 \text{ MHz}, \\
 T_{\max} &= \frac{1}{2} \frac{(qBR)^2}{m_p c^2} = \frac{1}{2} \frac{(4.8 \times 10^{-10} \text{ esu} \times 1.5 \times 10^4 \text{ G} \times 40 \text{ cm})^2}{(1000 \text{ MeV}) \times (1.6 \times 10^{-6} \frac{\text{erg}}{\text{MeV}})} \\
 &\approx \frac{(3 \times 10^{-4})^2 \text{ erg}}{3.2 \times 10^{-3}} \approx 2.8 \times 10^{-5} \text{ erg} \approx 17 \text{ MeV}.
 \end{aligned}$$

Note that we have used appropriate cgs units in evaluating the above expressions, and consequently the results can also be assumed to be in cgs units. This implies that 1 esu-gauss is equivalent to 1 erg/cm, which is consistent with Problem 2.4.

### 8.3.2 *Linac or Linear Accelerator*

Linear accelerators, as the name implies, accelerate particles along linear trajectories rather than in circular orbits. These accelerators are also based on the resonance principle, and operate as follows. A series of metal tubes, called drift tubes, are located in a vacuum vessel and connected successively to alternate terminals of a radio frequency oscillator, as shown in Fig. 8.3. Let us suppose that at some time the fields are as shown in the figure. Positive ions from the source will then be accelerated by the electric field towards the first drift tube. If the alternator can change its direction before the ions pass through that tube, then they will be accelerated again on their way between the exit of the first and entry into the second tube, and so on. However, as the particles accelerate, their velocities increase, and consequently, if the drift tubes are all of the same length, the phase between the particle positions and the potentials at the next tube may not keep in step (that is, the next gap may not accelerate). To avoid this, the drift tubes are made longer along the path so that one radio-frequency (RF) alternator can accelerate the particles all the way to the end.

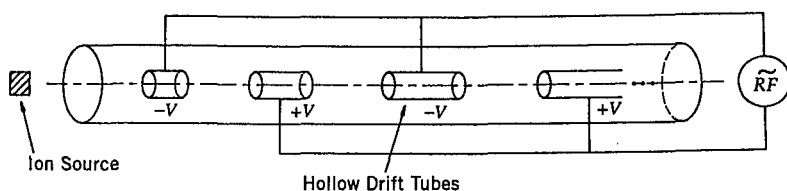


Fig. 8.3 A linear ion accelerator.

Because electrons become relativistic at relatively low energies, electron linear accelerators act on a slight variation of the principle just described. The electron source is usually a hot wire filament which, effectively, boils off electrons. These are accelerated through a positive potential grid and rapidly become relativistic. Bunches of these electrons are then passed through accelerating tubes that are fed with microwave power delivered by klystron amplifiers. Electrons radiate easily as they get accelerated (this is

referred to as *synchrotron radiation*), and therefore much power is needed to increase their energy. This power is supplied by the microwave fields that travel in step with electrons in specially shaped iris-loaded waveguides. The longest linear accelerator is the two-mile Stanford Linac (SLAC), and it accelerates electrons to energies of 50 GeV.

#### 8.4 Synchronous Accelerators

As we have discussed before, if we want to accelerate particles to very high energies, we must take relativity into account. For relativistic energies, the equation of motion for a particle of mass  $m$  and charge  $q$  in a magnetic field  $\vec{B}$  is

$$\frac{d\vec{p}}{dt} = q \frac{\vec{v} \times \vec{B}}{c}, \quad (8.5)$$

$$\text{or } m\gamma \frac{d\vec{v}}{dt} = m\gamma\vec{v} \times \vec{\omega} = q \frac{\vec{v} \times \vec{B}}{c}, \quad (8.6)$$

where, in the last step, we equated the centripetal force with the Lorentz force. Now, with  $|\vec{v}| \approx \text{constant} = c$ , the resonance relation follows from Eq. (8.6) (remember that both the magnetic field and the axis of circular bending are perpendicular to the direction of motion)

$$\omega = \frac{qB}{m\gamma c},$$

$$\text{or } \nu = \frac{\omega}{2\pi} = \frac{1}{2\pi} \left( \frac{q}{m} \right) \left( 1 - \frac{v^2}{c^2} \right)^{\frac{1}{2}} \frac{B}{c}. \quad (8.7)$$

Consequently, for this relation to hold during acceleration, either the alternating frequency has to decrease, or the magnetic field has to increase, or both must happen, as  $v \rightarrow c$ . Machines where the magnetic field is held constant but the frequency is varied are called *synchrocyclotrons*, and machines where the magnetic field is changed, irrespective of whether the frequency is changed, are known as *synchrotrons*. In electron synchrotrons,

the frequency is held constant and the magnetic field is varied, whereas in proton synchrotrons, both the frequency and magnetic field are varied.

With relativistic effects taken into account, Eq. (8.7) can be used to obtain the parameters for accelerating particles to any desired energy. Let us first rewrite Eq. (8.7) in terms of the momentum of the accelerated particle and the radius of the final orbit. For  $v \approx c$ , we can also express the frequency of the motion as

$$\nu = \frac{1}{2\pi} \frac{v}{R} \approx \frac{c}{2\pi R}. \quad (8.8)$$

Writing  $p = m\gamma v \approx m\gamma c$ , we can now obtain from Eq. (8.7) our usual relativistic relationship between  $p$ ,  $R$  and  $B$  (see also Fig. 7.5)

$$\frac{c}{2\pi R} = \frac{1}{2\pi} \left( \frac{q}{m} \right) \frac{1}{\gamma} \frac{B}{c},$$

$$\text{or } R = \frac{pc}{qB}. \quad (8.9)$$

It is convenient to write Eq. (8.9) in the mixed units of accelerator science, namely,

$$R \approx \frac{p}{0.3B}, \quad (8.9')$$

where  $p$  is in GeV/ $c$ ,  $B$  is in tesla and  $R$  in meters, and where we have assumed that  $q$  corresponds to the magnitude of the charge of a single electron.

Now we can see that, independent of the nature of the accelerating fields, any given momentum will be limited (often financially!) by the product of the radius of the final orbit and the largest magnetic field that can be supplied. At present, realistic bending magnets (dipoles) are limited to fields of strength  $\lesssim 2$  T for conventional electromagnets, and to  $\lesssim 10$  T for superconducting dipole magnets. Thus, for example, to accelerate protons to momenta of about 30 GeV/ $c$  requires an orbit radius of  $\approx 50$  m when using conventional magnets

$$R \approx \frac{p}{0.3B} \approx \frac{30}{(0.3)(2)} = 50 \text{ m}. \quad (8.10)$$

If the accelerator were a synchrocyclotron with a cyclotron type of electromagnet, ignoring the difficulties of construction, the cost of even the steel would be prohibitive. For the volume of the required magnet we can use  $\pi R^2 t$ , where  $t$  is the thickness of the poles ( $\approx 1$  meter each); using a density of  $8 \text{ gm/cm}^3$  for steel, this would yield a mass of  $\approx 2 \times 10^8$  lb, which, at current prices, would mean that the cost of the steel alone would exceed \$100M! A synchrocyclotron option is therefore impractical for energies much beyond several hundred MeV.

Synchrotrons in the GeV (or higher) range of energies have magnets positioned in a ring-like fashion (see Fig. 8.4). Particles with energies of hundreds of MeV are injected (often from a linac) into a narrow vacuum chamber that passes through all magnet apertures. At the start, the fields are set to low values that correspond to the momenta of the injected particles. The beam particles are constrained by the magnetic field to move in essentially circular orbits within the confines of the vacuum tube. To accelerate the particles to higher energy, RF power stations are placed in some convenient locations within the ring of magnets. Every time the particles pass through the RF cavities, they gain typically MeVs of energy from the electric fields. For this to work, the phases of the accelerating fields for different cavities must keep in step with each other as well as with the motion of the bunched beam. The magnetic field must also increase steadily to assure that, as the momenta of the particles increase, the radius of their orbit remains essentially fixed, in accord with Eq. (8.9).

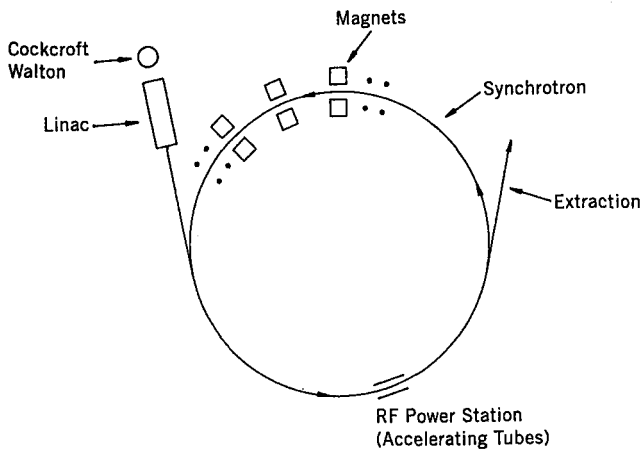


Fig. 8.4 Sketch of a synchrotron complex.

Most of the space along the ring is taken up by bending magnets. However, space is also needed for general servicing of the vacuum system, the powering and cooling of the magnets, for the RF stations, for the injection and extraction systems, and for other kinds of beam elements. A synchrotron therefore has many “straight” sections along the ring, where there is no bending. The accelerated particles consequently travel along circular arcs within the magnets, and in straight lines between them. In order to reach design energies, the beam particles must traverse the RF cavities and therefore the entire ring millions of times. A natural question is whether it is possible to maintain the beam within the small vacuum chamber for so many traversals. This is discussed in the next section.

## 8.5 Phase Stability

Before proceeding further with our description of different accelerators, we will introduce the principle of phase stability. Let us consider the operation of a synchrocyclotron, that is, a cyclotron with a fixed magnetic field and a variable accelerating electric RF field between the two Ds. Because there is always some finite spread in the momentum of particles in any beam, there is also a spread in the time of arrival of individual particles into the region of the RF field between the Ds. Referring to Fig. 8.5, let us label as synchronous, or “in time”, the particle that arrives at the time (or phase)  $\tau$  of the accelerating cycle. This particle will experience an electric field  $E_0$ , which will accelerate that particle while it is in the gap between the Ds. A particle that arrives earlier in the gap will therefore sense a somewhat larger field  $E_>$ . Thus the earlier particle will experience a larger accelerating force ( $qE$ ) while in the gap. This will increase the radius of its next orbit within the cyclotron D, and shift the particles’ next re-entry into the gap between Ds to a later time (towards that of the synchronous particle). On the other hand, a particle that arrives later than the synchronous one, will experience a smaller acceleration due to the smaller  $E_<$ . This will reduce the orbit radius in the D, and again shift the arrival time at the next acceleration stage towards that of the synchronous particle. On the next cycle, the synchronous particle will once again sense the same field  $E_0$ , while any particles arriving late will be accelerated less, and those arriving early will be accelerated more similarly for all following cycles. Particles that arrive at arbitrary times relative to  $\tau$  will experience random accelerations or even decelerations. Consequently, the self-correcting effect of the cyclical

field will lead to a grouping of all remaining particles into *bunches* centered on the time of the synchronous particle. This is the reason for the “RF structure” that one hears about in accelerators.

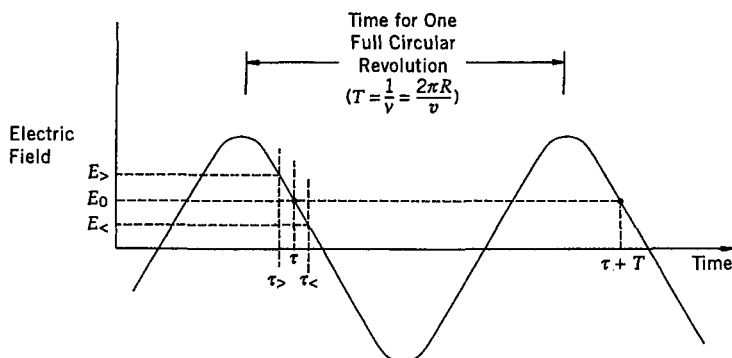


Fig. 8.5 Variation of electric field with time in the gap between the Ds of a synchrocyclotron.

A similar corrective effect occurs for particle motion in the vertical plane. As a consequence of Maxwell's equations, at large radii (that is near the edges), all dipole magnets have substantial fringe fields. This is shown in Fig. 8.6. If our beam particles are supposed to travel in the bending plane (horizontal circles in the figure), then, because of natural angular divergence in the beam, some particles will have trajectories that move them out of the median plane. These particles will experience a vertical restoring Lorentz force proportional to  $(\vec{v} \times \vec{B})$  that will tend to counter their divergence. Namely, it will deflect the particles back towards the median plane. The larger the divergence, the greater will be the vertical restoring or focusing force. There is no vertical correction for a particle moving in the median plane, but, for any finite divergence, there will always be a vertical component to the force.

For motion in a synchrotron, the fields in the apertures of individual dipole magnets also provide corrections to the transverse motion. Just as in the case of synchrocyclotrons, fringe fields along the edges of the magnets (see Fig. 8.6) focus the particles vertically towards the median plane. Because of the momentum dispersion of bending magnets, particles of higher momentum traverse the vacuum pipe at largest radii, and particles of smaller momentum move at smaller radii of the horizontal plane. And for velocities  $v \approx c$ , higher momentum particles therefore take longer to circle



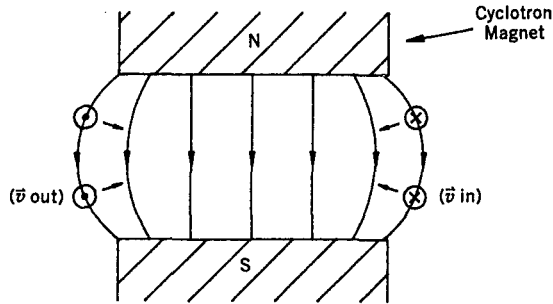


Fig. 8.6 Magnetic field in a synchrocyclotron and its effect on a charge circulating near the edge of the magnet.

the ring than smaller momentum particles, which, once again, provides the possibility for corrective acceleration at the locations of RF cavities, and leads to the kind of phase stability described in Fig. 8.5.

All these restoring forces induce very small oscillations in the motion about the mean trajectory for both the transverse direction (known as *betatron oscillations*) and for the longitudinal (energy or time) dimension (known as *synchrotron motion*), and make it possible to maintain particles in their orbits for long times. The entire concept, especially the corrective nature of the accelerating RF field, is referred to as the principle of *phase stability*; it was discovered independently by Edwin McMillan and Vladimir Veksler, and serves as the basis for stable operation of most modern high-energy accelerators.

In a proton synchrotron complex, the particles are usually accelerated first in a Cockcroft-Walton (to about 1 MeV), and then in a linac (to several hundred MeV), before they are injected into the synchrotron. As we described in the previous section, most synchrotrons have large radii, with a large number of magnets that are positioned in a ring along a circular acceleration path. The magnetic field is increased at a constant rate, usually from  $\approx$  several hundred gauss to the maximum value. This depends on the circumference of the accelerator and the fraction of the ring that is filled with magnets, as well as on whether the magnets are warm-temperature electromagnets or superconducting. The radio frequency of the accelerating fields is usually modulated between 0.3 MHz and 50 MHz, depending on the energy of the injected particles, the number of accelerating RF stations in the ring, the extraction (final) energy, etc. (A sketch of a typical accelerator complex was given in Fig. 8.4.)

Because protons do not radiate very much, most proton synchrotron

rings are filled essentially to capacity with the bending magnets required to maintain particles of maximum energy in their orbit. The size of the ring is therefore determined primarily by the desired magnetic field. Electrons, on the other hand, because of their small mass and centripetal acceleration in magnetic fields, emit substantial amounts of synchrotron radiation, which is inversely proportional to the radius of the orbit. (For example, 30 GeV electrons lose  $\approx 1.5$  GeV per turn in an orbit of  $R = 50$  meters! This amount of energy is difficult to supply through standard accelerating systems that provide  $\lesssim 10$  MeV/meter. The energy loss scales as  $\gamma^4$ , and is therefore insignificant for protons.) Consequently, electron synchrotrons usually have larger radii than proton synchrotrons of same energy; this is needed to reduce the amount of radiation and to provide adequate RF power for acceleration. The largest currently-operating proton synchrotron ring is the Tevatron at the Fermi National Accelerator Laboratory (Fermilab) in Batavia, IL. It has a circumference of about four miles, and contains superconducting magnets. It accelerates protons from  $\approx 150$  GeV to  $\approx 1000$  GeV (one TeV). The protons are accelerated first in a Cockroft-Walton, and then, as  $H^-$  ions, in a 400 MeV Linac. After the Linac, the beam is stripped of electrons, and the protons are accelerated to 8 GeV in the first "Booster" synchrotron. The beam from the Booster is passed to the Main Injector ring, and eventually, after acceleration to 150 GeV, the protons are injected into the Tevatron. The Booster and the Main Injector use conventional magnets.

## 8.6 Strong Focusing

The weak focusing provided by fringe fields of dipole magnets is insufficient for keeping large fluxes of high-energy particles within their orbits long enough to accelerate them to full energy. Fortunately, stronger focusing (larger gradients) can be attained through the use of quadrupole rather than dipole magnets. These serve essentially as lenses in optics, as is illustrated in Fig. 8.7. Imagine a positively-charged particle entering the field region along the axis of the magnet ( $x = y = 0$ ). We see that, in this case, the superposition of the magnetic field lines is such that there is no net deflection. Now suppose that the particle enters along  $x = 0$ , but  $y \neq 0$ ; here, for both positive and negative  $y$ -values, as the particle traverses the region of magnetic field it will be deflected towards the center of the magnet aperture (to smaller  $|y|$ ). The larger the  $|y|$  of the particle, the

stronger is the magnetic field, and the greater the deflection. Consequently, positively-charged particles entering these regions of the quadrupole field will be focused. For particles traversing the magnet along  $y = 0$ , but  $x \neq 0$ , the effect is opposite, namely, for both positive and negative  $x$ -values the particles will be deflected away from the center of the magnet, or be defocused. Because of the way the field in quadrupole magnets changes with position (that is, the fixed gradient, or constant change of field with position, is equal and opposite for the two orthogonal planes), the magnet will focus particles in one plane and defocus them in the other plane. It can be shown that by placing such magnets in alternating sequence along the beam, particles can, in fact, be focused in both planes. This is known as the principle of *alternate gradient focusing*, or, simply, *strong focusing*.

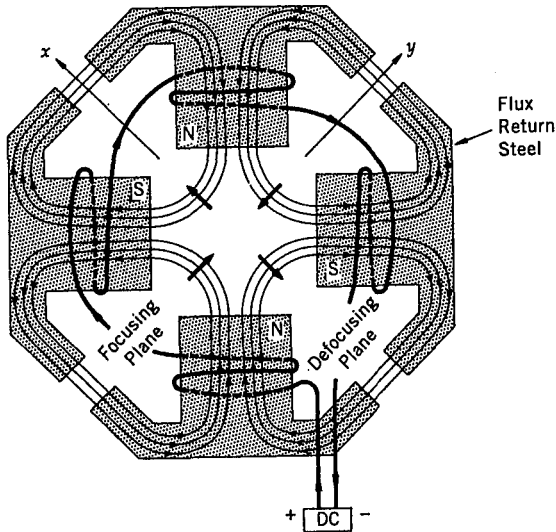


Fig. 8.7 Focusing/defocusing properties of a quadrupole magnet for positively charged particles entering the plane of the paper. The sense of the windings on the electromagnet is indicated by the arrows showing the direction of positive current flow. The path of the magnetic field lines through the flux-return iron is also indicated.

The principle of strong focusing, proposed independently by Ernest Courant, Stanley Livingston and Harland Snyder, and by Nicholas Christofilos in the early 1950s, was first applied in the construction of the 30 GeV proton accelerator, the Alternating Gradient Synchrotron (AGS), at Brookhaven National Laboratory in the late 1950s. The AGS design was based on the shaping of pole pieces (and therefore the fields) in dipole mag-

nets to provide large alternating gradients. All high energy synchrotrons now use quadrupole magnets along with dipole magnets to guide particles, as they circle the ring and keep gaining energy at the RF power stations. The use of such separate functions for dipoles (to bend and maintain particles in their orbits of fixed radius) and quadrupoles (to correct positions of particles within an orbit) was first developed for the accelerator at Fermilab.

Just as dipoles have weak focusing properties (moments corresponding to quadrupole terms in their field structure), so do quadrupoles have higher-order components in their fields. Correction coils (especially for sextupole effects, but also for even higher octupole terms) are needed in high-energy synchrotrons to assure that beams remain stable for the full acceleration cycle, and do not “blow up”, and leave their vacuum vessel. After the beams reach their final energy, they can be extracted to target stations, or made to collide with other beams of particles.

We will not discuss the variety of injection and extraction techniques, nor the manner in which beams are brought to external targets in experimental areas. The techniques employed for such purposes are based essentially on the same kinds of electromagnetic tools and principles we have already sketched, namely the use of dipoles, quadrupoles, RF cavities, etc. This is another important sub-branch of accelerator science that has wide applications in nuclear and particle physics, as well as in other disciplines.

## 8.7 Colliding Beams

As we know, the ultimate figure of merit in any high-energy scattering experiment is not just the laboratory energy of a colliding beam-particle, but rather the energy that is available for producing more particles, namely the energy in the center-of-mass of the collision. We have discussed some of these issues previously, but let us review several of the salient points. Let us assume that a particle of rest mass  $m$  and total energy  $E$  collides with a stationary particle of equal mass. The energy that is available in the center-of-mass of the collision is given by our expression for  $\sqrt{s}$  (Eq. (1.64))

$$E_{\text{CM}}^{\text{TOT}} = \sqrt{s} = \sqrt{2m^2c^4 + 2mc^2E}, \quad (8.11)$$

which for very high energies becomes

$$E_{\text{CM}}^{\text{TOT}} \approx \sqrt{2mc^2 E}. \quad (8.12)$$

This is the part of the incident energy that is available for converting energy into new particles; the rest of the incident energy cannot be used because it is required to maintain the motion of the center-of-mass, that is, to preserve conservation of momentum in the collision. We see therefore that in a collision of an accelerated particle with a fixed target, the energy in the center-of-mass increases only as the square-root of the accelerator energy. Consequently, to make massive objects such as the  $W$  and  $Z$  bosons with masses  $\approx 90 \text{ GeV}/c^2$ , requires enormous laboratory energies (just the threshold for single  $Z$  production in  $p$ - $p$  collisions is about 4 TeV).<sup>1</sup> Except for special purposes, colliding beam-particles with fixed targets would therefore appear to be a rather inefficient way to utilize the full energy of any machine. On the other hand, accelerating two separate beams of particles, and making them collide head-on, with the center-of-mass of the collision stationary in the laboratory, would make the entire energy of the beams available for producing new particles. This is the idea behind the development of colliding-beam accelerators.

There are different kinds of colliding-beam machines. The beams can have particles of same type, for example, heavy ions on heavy ions,  $p$  on  $p$  or  $e^-$  on  $e^-$ , or of opposite type (particles and their antiparticles), for example,  $\bar{p}$  on  $p$  or  $e^-$  on  $e^+$ , or of different type, for example,  $e^-$  on  $p$ . Both beams can have same or different energy. (Asymmetric energies are often used for technical reasons, that are either associated with the detection of short-lived particles, or when it is not possible to make beams of same momentum – as in the case of  $e^-$ - $p$  collisions, where electron energies are restricted to relatively low values because of the great synchrotron-radiation loss and the consequent financial implications of trying to reach the very top energies available for protons. Naturally, the center-of-mass in such asymmetric collisions is not stationary in the laboratory frame.) Also, colliders

<sup>1</sup>It could be argued that using more massive targets, such as lead nuclei or, better yet, blocks of lead would be advantageous. Unfortunately, this would not help increase the energy in the center-of-mass, except for interactions that are characterized by distances of the order of the size of lead nuclei ( $\approx 6 \text{ fm}$ ) or lead blocks (cm). That is, to produce  $Z$  and  $W$  objects, we have to have large momentum transfers, and therefore collisions at distances of the order of the Compton wavelength for these particles (about  $10^{-2} \text{ fm}$ ), and having larger targets has little relevance since the collisions of interest are between protons in the beam and in the target (or, in fact between constituents within these protons), and not between a proton and an extended object (see Problem 8.6).

can have a single ring or two independent rings of magnets. Clearly, independent systems must be used for all but antiparticle-on-particle colliders, since in the latter case the orbit of a particle moving along one direction of the ring can be maintained while moving the antiparticle in the same vacuum pipe but opposite direction. Whether there is a single ring or two rings of magnets, the two beams can be accelerated at the same time, and maneuvered into colliding orbits at selected intersection regions that contain detector systems. The beams pass through each other and interact at the collision regions until their intensity is greatly reduced (this often takes many hours – the reduction is due to beam-beam collisions as well as from interactions of the beams with any remnant molecules within the vacuum pipe), at which time they are removed safely from the collider, and the acceleration cycle restarted (the acceleration and filling part of the cycle is far shorter than the collision part).

The type of colliders that appear to be most appropriate for accelerating protons are the kind of ring synchrotrons that we have discussed in the previous sections. For electron-positron colliding accelerators, two options are possible. One is, again, the synchrotron variety, and the other is a newer concept that has been developed at SLAC, which involves *linear colliders*. Here, two accelerators are constructed to aim beams at each other. One accelerates electrons, and the other positrons. The full beams are then made to collide head on, in a single-shot fashion. Such linear colliders require high accelerating gradients of the order of  $\approx 100$  MeV/meter, large beam currents, and small transverse beam dimensions ( $\mu m$  in size) in order to produce enough collisions to make them competitive with ring machines, which have the advantage that they operate with stored beams that allow multiple passes of particles through the intersection regions.

The use of colliding beams, especially when it involves antiparticles, requires substantially more beam “gymnastics”. Antiparticles must first be produced, extracted, stored, accumulated and then accelerated in quantities to provide a sufficient number of interactions with the larger flux of opposing particles. This means that all particles must be stored for far longer times than in normal accelerators. The usual synchrotron focusing mechanisms are not sufficient, and must often be enhanced through “cooling” techniques that reduce the transverse momenta of particles in orbit, and prevent the beams from blowing up. The principle of stochastic cooling, developed by Simon Van der Meer, is used to assure that beam fluxes remain adequate. The idea is to sense the transverse position of a particle at some point in the ring, and send that information along a chord so that it arrives before

the particle does (this is possible because the particle must traverse the longer path along the arc of the ring). With the help of RF fields, this information is then used to correct the transverse position of the particle, and prevent its escape from orbit.

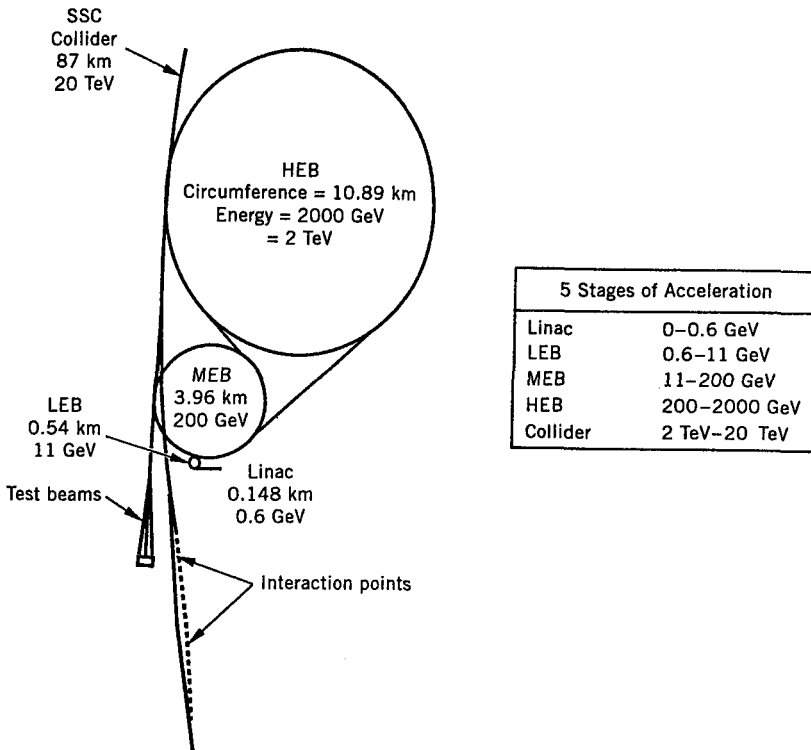


Fig. 8.8 The injector complex that was to be used for the Superconducting Supercollider (SSC). LEB, MEB and HEB refer, respectively, to the low-energy, medium-energy and high-energy boosters or synchrotrons.

The largest collider for proton collisions on protons is the Large Hadron Collider (LHC), which is expected to commence operations in  $\approx 2007$  at the CERN Laboratory, outside of Geneva, Switzerland. The LHC has one tunnel that is 27 km long, with two 7 GeV beams circulating in opposite directions in separate magnetic channels located in a common yoke and cryostat. The superconducting magnets will operate at almost 8.4 T, with currents of the order of 15,000 A (cable cooled to  $\approx 1.9$  K). The expected collision rate is about  $10^9$  interactions per second! This enormous luminos-

ity requires the development of novel detection and triggering techniques that can make effective use of the factor of  $\approx 100$  increase relative to previous collision rates. The LHC will also be able to collide protons with nuclei of Pb, and beams of Pb with Pb.

The canceled U.S. Superconducting Supercollider (SSC) was to have had two rings of magnets placed in an essentially circular pattern, each of which were to be about 90 km in circumference. Each ring was to contain about 4000 dipoles with superconducting coils (operating at about 7 T). In addition, there were to be about 1000 quadrupoles per ring, and other correction coils. The top energy was designed to be 20 TeV per beam. Figures 8.8 and 8.9 provide an idea of the scale of the SSC machine design.

## Problems

**8.1** Protons are accelerated in a cyclotron by an electric field with oscillating frequency of 8 MHz. If the diameter of the magnet is 1 m, calculate the value of magnetic field and the maximum energy that the protons can reach.

**8.2** To achieve an energy of 20 TeV, each of the SSC main rings was to contain about 4000 dipole magnets, each 16-meters long, with a field of 7 T. This means that over half of the  $\approx 60$  mile SSC tunnel was to be taken up by dipoles. If you were to build a single synchrotron for use in fixed-target collisions of equivalent energy in the center-of-mass ( $\sqrt{s} = 40$  TeV), and used a similar magnet design, how long would your tunnel have to be?

**8.3** If the capacitance of a Van de Graaff accelerator terminal is 250  $\mu\mu\text{F}$  (pF), and if it operates at a voltage of 4 MV, what is the total charge on the terminal? If the charging belt can carry a current of 0.2 mA, how long does it take to charge up the accelerator to 4 MV?

**8.4** Starting with cgs units, show that Relation (8.9') follows from Relation (8.9).

**8.5** Suggest a mechanism whereby an accelerated beam could be extracted from a circular accelerator, and directed onto an external target.

**8.6** Using Eq. (8.12), there is ostensibly sufficient energy in the center of mass in the collision of a 1 TeV  $\pi^0$  with a lead nucleus at rest to produce a Higgs boson ( $H^0$ ) of  $M_H \approx 120 \text{ GeV}/c^2$ . In principle, this can be done in a coherent collision, where the Pb nucleus remains intact. Does this make sense in light of Footnote 1? Assuming a nuclear form factor for Pb of  $\approx e^{-400q^2}$  (with  $q$  in GeV units), and considering the silly reaction  $\pi^0 + \text{Pb}$



$\rightarrow H^0 + Pb$ , what would be the approximate reduction in the probability for producing the Higgs at  $0^\circ$  as a result of the form factor? [*Hints*: To calculate the minimum value allowed for the quantity  $q^2 = (\vec{p}_\pi c - \vec{p}_H c)^2 - (E_\pi - E_H)^2$ , assume that the Higgs boson is relativistic, but approximate the terms to order  $(M_H c^2/E_H)^4$  in  $\beta$ . You should get that  $q^2 \approx q_{min}^2 \approx (M_H^2 c^4/2E_\pi)^2$ , when you ignore the small mass of the pion and set  $E_H = E_\pi$ .]

### Suggested Readings

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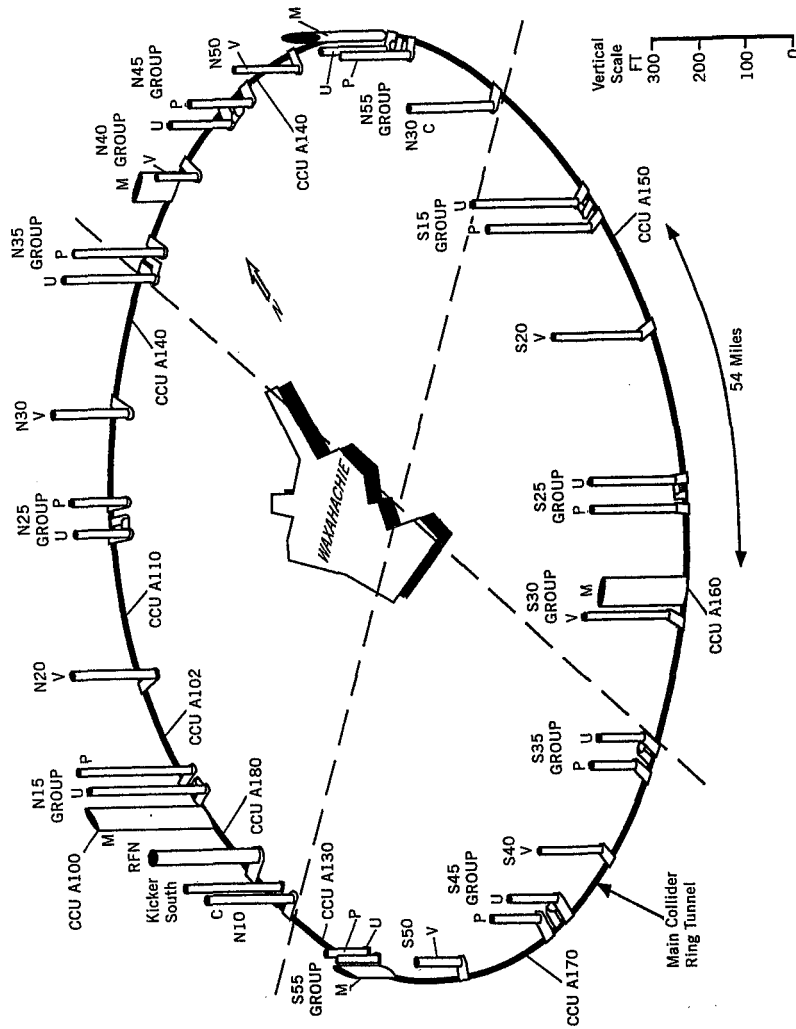


Fig. 8.9 Various access shafts and utility stations, and their locations below ground at the recently defunct SSC.



## Chapter 9

# Properties and Interactions of Elementary Particles

### 9.1 Introductory Remarks

After the discovery of the neutron in 1932, it was thought that the electron, the proton, and the neutron were the fundamental constituents of all matter. Subsequent experiments, involving cosmic rays as well as accelerator beams, revealed that there was a host of other particles that could be regarded as equally fundamental. We already mentioned the families of leptons and some of their properties in connection with nuclear  $\beta$ -decay. In addition, we also know of the existence of hadrons such as  $\pi$ -mesons,  $K$  mesons,  $\rho$  mesons, hyperons, and their many excited states. All these can be referred to collectively as “elementary” particles. Usually, an elementary particle is thought to be an object without any substructure, namely a point particle. However structure can be probed only up to any given scale that is limited by the available energy. Consequently, our definition of what is elementary or fundamental is always tentative, and must rely on experimental verification at ever higher energies. For example, to examine the structure of matter at length scales of  $\Delta r \lesssim 0.1$  fm, requires transverse-momentum transfers ( $\Delta p_T$ ) at least of the order

$$\Delta p_T \approx \frac{\hbar}{\Delta r} = \frac{\hbar c}{(\Delta r)c} \approx \frac{197 \text{ MeV}\cdot\text{fm}}{(0.1 \text{ fm})c} \approx 2000 \text{ MeV}/c. \quad (9.1)$$

In other words, to be sensitive to small length scales, the energy of the particles used as probes must be very high. Because of this need, the study of elementary particles has also come to be known as high-energy physics.

Whenever a higher-energy accelerator starts operating, we can probe deeper into the structure of matter and find that what was once considered elementary is not really so. This has, in fact, been the story of the proton,

the neutron, the  $\pi$ -mesons, the  $K$ -mesons, and so forth. Our current understanding of which particles should be considered as elementary is very different from that of only several decades ago. Nevertheless, we will start with the traditional (historical) perspective, and turn to the more modern view of elementary particles in a later chapter. We will begin at the one  $\text{GeV}/c^2$  mass scale, and discuss the properties of elementary particles and their interactions from a purely phenomenological point of view.

## 9.2 Forces

We are quite familiar with the classical electromagnetic and gravitational forces. We know that every particle, whether with or without rest mass, is subject to gravitational attraction. (The observed bending of light in a gravitational field shows conclusively that particles do not need to have rest mass but rather energy to experience the pull of gravity.) On the other hand, only particles that carry electric charge sense the Coulomb field directly. Both the Coulomb and the gravitational forces are long ranged. The photon is the carrier of Coulomb interaction, and from the fact that the electromagnetic force has infinite range, we can conclude that the photon must be massless. The carrier of the gravitational interaction is the conjectured graviton, which is also believed to be massless. From our discussion of nuclear phenomena, we learned that there are two more forces that have importance in the subatomic domain. There is the strong force, which, as we have seen, is responsible for the binding of nucleons inside a nucleus, and the weak force, which appears in processes such as  $\beta$  decay of nuclei. These forces have no classical analogs and, unlike the electromagnetic and the gravitational interactions, are exceedingly short ranged. Thus, it seems that we can point to four fundamental forces in nature

1. Gravitation,
2. Electromagnetism,
3. Weak Force,
4. Strong Force.

Because, in principle, all the forces can act at the same time, it could be asked how is it possible to determine which force contributes in any particular process? The answer is that the forces can be distinguished through the strengths of their interaction. We can estimate the relative magni-

tudes of these four forces in a heuristic way by considering their effective potentials. Although such potentials are fundamentally non-relativistic in concept, they provide a useful guide for rough comparison. Consider two protons separated by a distance  $r$ . The magnitudes of the Coulomb and of the gravitational potential energies for the two particles are

$$\begin{aligned} V_{\text{em}}(r) &= \frac{e^2}{r}, \\ V_{\text{grav}}(r) &= \frac{G_N m^2}{r}, \end{aligned} \quad (9.2)$$

where  $G_N$  is Newton's constant [ $6.7 \times 10^{-39} \hbar c (\text{GeV}/c^2)^{-2}$ ], and  $m$  is the mass of the proton. It is more instructive to write the potential energies in the Fourier transformed momentum space (see Eq. (1.77)) where, except for an overall normalization, they take the form

$$\begin{aligned} V_{\text{em}}(q) &= \frac{e^2}{q^2}, \\ V_{\text{grav}}(q) &= \frac{G_N m^2}{q^2}, \end{aligned} \quad (9.3)$$

where  $q$  refers to the magnitude of the momentum transfer that characterizes the interaction.

The absolute values of the potential energies for both interactions appear to decrease quadratically with momentum transfer, the ratio of  $V_{\text{em}}$  and  $V_{\text{grav}}$  is, in fact, independent of momentum scale, and we can evaluate this ratio as

$$\begin{aligned} \frac{V_{\text{em}}}{V_{\text{grav}}} &= \frac{e^2}{G_N m^2} = \left( \frac{e^2}{\hbar c} \right) \frac{1}{(mc^2)^2} \frac{\hbar c \times c^4}{G_N} \\ &\approx \left( \frac{1}{137} \right) \frac{1}{(1 \text{ GeV})^2} \frac{10^{39} \text{ GeV}^2}{6.7} \approx 10^{36}, \end{aligned} \quad (9.4)$$

where we have substituted  $1 \text{ GeV}/c^2$  for the mass of the proton, and used the value of  $\alpha = \frac{e^2}{\hbar c} = \frac{1}{137}$  for the electromagnetic fine-structure constant. Equation (9.4) shows that, for charged elementary particles, the gravitational force is inherently much weaker than the electromagnetic force.

Next, let us recall that since both the strong and the weak forces are short-ranged, they can be described phenomenologically by Yukawa potentials of the form

$$\begin{aligned} V_{\text{strong}} &= \frac{g_s^2}{r} e^{-\frac{m_\pi c^2 r}{\hbar c}}, \\ V_{\text{wk}} &= \frac{g_{\text{wk}}^2}{r} e^{-\frac{m_W c^2 r}{\hbar c}}, \end{aligned} \quad (9.5)$$

where  $g_s$  and  $g_{\text{wk}}$  represent the coupling constants (effective charges) for the strong and the weak interactions, and  $m_\pi$  and  $m_W$  represent the masses of the force-mediating (or exchanged) particles in the two cases. Once again, we can transform the above potentials to momentum space, and, except for an overall normalization constant, obtain

$$\begin{aligned} V_{\text{strong}} &= \frac{g_s^2}{q^2 + m_\pi^2 c^2}, \\ V_{\text{wk}} &= \frac{g_{\text{wk}}^2}{q^2 + m_W^2 c^2}. \end{aligned} \quad (9.6)$$

The values of the coupling constants can be estimated from experiment, and they are

$$\frac{g_s^2}{\hbar c} \approx 15, \quad \frac{g_{\text{wk}}^2}{\hbar c} \approx 0.004.$$

From our discussion in Chapter 2, we can think of the  $\pi$  meson ( $m_\pi \approx 140 \text{ MeV}/c^2$ ) as the mediator of the strong nuclear force. Also, from weak-interaction processes at low energies (e.g.,  $\beta$  decay), we can estimate that  $m_W \approx 80 \text{ GeV}/c^2$ . Consequently, we can compare the magnitude of the Coulomb potential energy to that for the strong and the weak interactions. However, there appears to be an explicit dependence on momentum scale in the ratio. Since we are considering the interaction of two protons, it is natural to choose the momentum scale to correspond to that of the proton mass. Thus, choosing  $q^2 c^2 = m^2 c^4 = (1 \text{ GeV})^2$ , we obtain

$$\begin{aligned}
\frac{V_{\text{strong}}}{V_{\text{em}}} &= \frac{g_s^2}{\hbar c} \frac{\hbar c}{e^2} \frac{q^2}{q^2 + m_\pi^2 c^2} = \frac{g_s^2}{\hbar c} \frac{\hbar c}{e^2} \frac{m^2 c^4}{m^2 c^4 + m_\pi^2 c^4} \\
&\approx 15 \times 137 \times 1 \approx 2 \times 10^3, \\
\frac{V_{\text{em}}}{V_{\text{wk}}} &= \frac{e^2}{\hbar c} \frac{\hbar c}{g_{\text{wk}}^2} \frac{m^2 c^4 + m_W^2 c^4}{m^2 c^4} \\
&\approx \frac{1}{137} \frac{1}{0.004} (80)^2 \approx 1.2 \times 10^4.
\end{aligned} \tag{9.7}$$

This shows once again that the strong force is stronger than the electromagnetic force, which in turn is stronger than the weak force, and that gravitation is the weakest of all the forces. For larger momentum scales of order  $\approx m_W$ , the weak and electromagnetic energies and strengths become more comparable, and suggest the interesting possibility for a unification of the two forces at very high energies. But because our phenomenological estimates are only qualitative, the ratios of the effective potentials as given in Eq. (9.7) should not be taken too literally.

The difference in the forces also manifests itself in the interaction time characterizing a particular process. Thus, for example, the typical time scale for a strong reaction is about  $10^{-24}$  sec, which is roughly the time it takes a light signal to traverse a proton's dimension, namely 1 fm. On the other hand, typical electromagnetic reactions of elementary particles occur in time intervals of the order of  $10^{-20}$ – $10^{-16}$  sec, whereas the typical time scales for weak decays are about  $10^{-13}$ – $10^{-6}$  sec.<sup>1</sup> In the GeV range of energies, the properties of the four fundamental forces are therefore quite different, and they can be used to classify the character of the elementary particles.

### 9.3 Elementary Particles

Before it was fully appreciated that quarks were the fundamental constituents of nuclear matter, all the known elementary particles were grouped into four classical categories that depended on the nature of their interac-

<sup>1</sup>Again, we wish to stress that these are only typical time scales. Specific transition rates have varying contributions from spin effects and factors corresponding to density of final states ("phase space") that can have large bearing on lifetimes. As we mentioned previously, the lifetime of the neutron, for example, is  $\approx 900$  sec, which is far from the norm for weak interactions.



tions. This is sketched in Table 9.1.

**Table 9.1** Different kinds of elementary particles, with their ranges in mass and electric charge (superscripted).

<i>Particle</i>	<i>Symbol</i>	<i>Range of Mass Values</i>
Photon	$\gamma$	$\lesssim 2 \times 10^{-16} \text{ eV}/c^2$
Leptons	$e^-, \mu^-, \tau^-, \nu_e, \nu_\mu, \nu_\tau$	$\lesssim 3 \text{ eV}/c^2 - 1.777 \text{ GeV}/c^2$
Mesons	$\pi^+, \pi^-, \pi^0, K^+, K^-, K^0,$ $\rho^+, \rho^-, \rho^0, \dots$	$135 \text{ MeV}/c^2 - \text{few GeV}/c^2$
Baryons	$p, n, \Lambda^0, \Sigma^+, \Sigma^-, \Sigma^0, \Delta^{++},$ $\Delta^0, N^{*0}, Y_1^{*+}, \Omega^-, \dots$	$938 \text{ MeV}/c^2 - \text{few GeV}/c^2$

All particles, including photons and neutrinos, participate in gravitational interactions. The photon can interact electromagnetically with any particle that carries electric charge. All charged leptons participate both in the weak and electromagnetic interactions, and neutral leptons, of course, have no direct electromagnetic coupling. (This is what made it so difficult to observe the neutrino in  $\beta$  decay.) Leptons do not sense the strong force. All hadrons (mesons and baryons) respond to the strong force and appear to participate in all the interactions. We will subsequently discuss the differences between mesons and baryons; their common characteristic, however, is that they appear to have substructure, and a size of the order of one femtometer.

All the particles in nature can be classified as either bosons or fermions, with the basic difference between them being the statistics that they obey. Bosons obey Bose-Einstein statistics whereas fermions satisfy Fermi-Dirac statistics. This is reflected in the structure of their wave functions. For example, the quantum mechanical wave function for a system of identical bosons is symmetric under the exchange of any pair of particles. That is,

$$\Psi_B(x_1, x_2, x_3, \dots, x_n) = \Psi_B(x_2, x_1, x_3, \dots, x_n), \quad (9.8)$$

where the  $x_i$  denote, collectively, space-time coordinates as well as internal quantum numbers of particle  $i$ . On the other hand, under similar assumptions, the quantum mechanical wave function for a system of identical fermions is antisymmetric under the exchange of any pair of particles, namely

$$\Psi_F(x_1, x_2, x_3, \dots, x_n) = -\Psi_F(x_2, x_1, x_3, \dots, x_n). \quad (9.9)$$

The Pauli exclusion principle is therefore automatically built into the antisymmetric fermionic wave function, thereby forbidding a pair of identical fermions to occupy the same quantum state. This follows because, for  $x_1 = x_2$ , the wave function in Eq. (9.9) would equal its negative value, and would therefore vanish.

It can be shown from fundamental principles that all bosons have integer values of spin angular momentum, while fermions have half integral spin values. In a subsequent section we will describe several ways to determine spins of elementary particles. From such studies, it has been learned that the photon and all mesons are bosons, whereas the leptons and all baryons are fermions. Also, as we have already indicated, every known particle has a corresponding antiparticle. The antiparticle has the same mass as the particle, but otherwise opposite quantum numbers. Thus, the positron ( $e^+$ ) is the antiparticle of the electron, and carries a negative lepton number and a positive charge. The antiproton ( $\bar{p}$ ) has one unit of negative charge and one unit of negative baryon number, in contrast to the proton which is positively charged and has a positive baryon or nucleon number. Certain particles cannot be distinguished from their own antiparticles. For example, the  $\pi^0$ , which has no electric charge, is its own antiparticle. It is clear that for a particle to be its own antiparticle, it must, at the very least, be electrically neutral. However, not all electrically neutral particles are their own antiparticles. The neutron has no electric charge, yet the antineutron is distinct because of its negative baryon number and the opposite sign of its magnetic moment. Similarly, the  $K^0$  meson, although charge neutral, has a distinct antiparticle. (It is still unknown whether the neutrino is distinct from its antiparticle.) Except where it is redundant, or where there is a special symbol, antiparticles are denoted by the same symbol as the particles, but with a bar over that symbol. Several examples are

$$\begin{aligned}\overline{e^-} &= e^+, \\ \overline{\pi^0} &= \pi^0, \\ \overline{\Sigma^-} &= \Sigma^+, \\ \overline{K^+} &= K^-. \end{aligned} \tag{9.10}$$

## 9.4 Quantum Numbers

As we emphasized in our treatment of nuclear phenomena, much of our physical intuition does not help in trying to understand the subatomic domain, and we have to rely on experiment as guide. The properties of the elementary particles and their interactions are even more mystifying, but there are many elementary particles, and many processes that can be studied. However, to derive any meaningful conclusions from observations, results must be organized in some coherent manner. Here our classical experience does help. Classically, we know that a process or a reaction can take place if it is allowed kinematically, and if it does not violate any recognized conservation law. Thus, for example, we are quite certain that a reaction that violates charge conservation will not take place. This certainty is based upon years of past studies and the development of a reliable theory for electromagnetic interactions. We believe that similar conservation principles hold in the subatomic domain, except that here we do not know all the relevant laws because we do not have a complete theoretical understanding of all the forces. Consequently, to formulate general principles, we must deduce from experiment the type of quantum numbers that are conserved and the conservation laws that are appropriate for each of the interactions of the elementary particles. One of the clearest results observed in reactions of elementary particles is that the number of fermions is always conserved (that is, if we count a fermionic antiparticle as a fermion, but with a negative fermion number), whereas the number of photons and mesons is not. This suggests that the conservation of fermion number is a fundamental feature of all interactions, as will be elaborated below.

### 9.4.1 *Baryon Number*

From differences in the magnitudes of observed transition rates, or from the absence (upper limits) of kinematically allowed processes, we can often infer the presence of possible conservation laws. As an example, consider the decay

$$p \rightarrow e^+ + \pi^0. \quad (9.11)$$

Since the proton is far more massive than the sum of the pion and positron masses, and since the above decay satisfies the conservation of electric charge, one might expect this process to take place. Nevertheless, proton decay is not observed. In fact, the upper limit on the probability for reaction (9.11) is a miniscule  $\lesssim 10^{-40}/\text{sec}$ . This suggests that there is some conservation principle that forbids the decay. In fact, we can account for this simply by asserting that baryons carry an additive and conserved quantum number (baryon or nucleon number) that equals  $B = 1$  for all baryons (and, of course,  $B = -1$  for antibaryons), but  $B = 0$  for photons, leptons, and mesons. Consequently, if baryon number is conserved in all physical processes, then the proton, being the lightest baryon, should not decay.

### 9.4.2 *Lepton Number*

Similarly, we can postulate a quantum number for leptons, namely assert that all leptons carry lepton number  $L = 1$ , whereas the photon and hadrons carry no lepton number. The introduction of a lepton quantum number is necessitated by many experimental observations. One example is the process

$$e^- + e^- \rightarrow \pi^- + \pi^-. \quad (9.12)$$

At high energies, this reaction is kinematically allowed, and it certainly satisfies charge conservation, but it is not observed. Naturally, lepton-number conservation would prevent this process from taking place.<sup>2</sup> In fact, reactions such as

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<sup>2</sup>Conservation of lepton number can also explain the absence of proton decay in the  $e^+\pi^0$  channel of Eq. (9.11).

$$\begin{aligned}\mu^- &\rightarrow e^- + \gamma, \\ \mu^- &\rightarrow e^- + e^+ + e^-, \end{aligned} \tag{9.13}$$

although kinematically allowed, have also never been observed. It is from such experimental findings that we arrive at the conclusion that there must be different kinds of lepton numbers within the family of leptons (see Table 9.2). Thus, the electron and its neutrino have an electron-lepton number  $L_e = 1$ , whereas the other leptons have  $L_e = 0$ . The muon and its neutrino have muon-lepton number  $L_\mu = 1$ , whereas the other leptons have  $L_\mu = 0$ , and similarly for the  $\tau$ -lepton and its neutrino. The net lepton number of any particle can therefore be expressed as the sum of the electron number, the muon number, and the  $\tau$ -lepton number.

**Table 9.2** Lepton numbers

	<i>Electron</i> Number $L_e$	<i>Muon</i> Number $L_\mu$	<i><math>\tau</math>-lepton</i> Number $L_\tau$	$L = L_e + L_\mu + L_\tau$
$e^-$	1	0	0	1
$\nu_e$	1	0	0	1
$\mu^-$	0	1	0	1
$\nu_\mu$	0	1	0	1
$\tau^-$	0	0	1	1
$\nu_\tau$	0	0	1	1

Leptons can therefore be split into three families, namely,  $(e^-, \nu_e)$ ,  $(\mu^-, \nu_\mu)$ ,  $(\tau^-, \nu_\tau)$ , with each family number conserved in all interactions. This would explain, for example, why the muon decays as

$$\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu. \tag{9.14}$$

It is worth noting that, although proton decay in Eq. (9.11) violates both baryon number and lepton number, the combination  $B - L$  is conserved in

the process. Which suggests that this interesting feature should be incorporated into any physical theory.

### 9.4.3 *Strangeness*

In early studies of cosmic-ray showers, it was found that certain particles, which have since been identified with  $K$  mesons and the  $\Sigma$  and  $\Lambda^0$  baryons, were produced strongly (that is, with large cross sections of the order of millibarns), but had lifetimes characteristic of weak interactions, namely  $\approx 10^{-10}$  sec. These particles were always produced in pairs, that is, a  $K$  in association with either a  $\Sigma$  or  $\Lambda^0$ . All this was certainly puzzling, and led to a suspicion that a new quantum number might be associated with such particles. When specific reactions, such as

$$\pi^- + p \rightarrow K^0 + \Lambda^0,$$

were studied with the  $\Lambda^0$  and  $K^0$  subsequently decaying as

$$\begin{aligned}\Lambda^0 &\rightarrow \pi^- + p, \\ K^0 &\rightarrow \pi^+ + \pi^-, \end{aligned} \tag{9.15}$$

it was observed that the  $\Lambda^0$  was always produced in association with a  $K^0$  and never with just a  $\pi^0$ . The  $\Lambda^0$  was also observed to be produced in association with a  $K^+$ , but not with a  $K^-$

$$\begin{aligned}\pi^- + p &\rightarrow K^+ + \pi^- + \Lambda^0, \\ \pi^- + p &\not\rightarrow K^- + \pi^+ + \Lambda^0, \\ \pi^- + p &\not\rightarrow \pi^- + \pi^+ + \Lambda^0. \end{aligned} \tag{9.16}$$

Similarly, for the reaction

$$\pi^+ + p \rightarrow \Sigma^+ + K^+,$$

with the  $\Sigma^+$  and  $K^+$  decaying subsequently as

$$\begin{aligned}\Sigma^+ &\rightarrow n + \pi^+, \\ K^+ &\rightarrow \pi^+ + \pi^0,\end{aligned}\tag{9.17}$$

it was observed that the  $\Sigma^+$  was always produced in association with a  $K^+$ , and never with just a  $\pi^+$ . Again,  $\Sigma^+$  baryons were also found to be produced in association with  $K^0$  mesons, but with an additional  $\pi^+$  required to conserve electric charge. Similarly,  $\Sigma^-$  baryons were produced in association with  $K^+$  mesons in  $\pi^-p$  collisions, but  $\Sigma^+K^-$  final states were not observed

$$\begin{aligned}\pi^+ + p &\rightarrow \Sigma^+ + \pi^+ + K^0, \\ \pi^- + p &\rightarrow \Sigma^- + K^+, \\ \pi^- + p &\not\rightarrow \Sigma^+ + K^-, \\ \pi^- + p &\not\rightarrow \Sigma^- + \pi^+.\end{aligned}\tag{9.18}$$

The production cross sections for reactions such as those given in Eqs. (9.15) and (9.17) for pion momenta of about 1 GeV/c were measured to be about 1 mb, whereas the total cross sections for  $\pi^\pm$  scattering on protons were known to be about 30 mb. Thus, it was clear that these production processes were strong. The subsequent decays of these particles were also studied, and revealed that the  $\Lambda^0$ , traveling at a speed of about  $0.1c$ , decayed after a flight path of about 0.3 cm. Consequently, the lifetime of this baryon was deduced to be

$$\tau_{\Lambda^0} \approx \frac{0.3 \text{ cm}}{3 \times 10^9 \text{ cm/sec}} = 10^{-10} \text{ sec.}$$

And similar lifetimes were observed for the other “strange” particles, leading to the conclusion that the decays involved weak interactions (see Figure 9.1)

The puzzle of *associated production* was clarified by Murray Gell-Mann and Abraham Pais, who proposed that these particles carried a new additive quantum number, which they called *strangeness*, which is conserved in strong production processes, but violated in weak decays. All the ordinary mesons and baryons (as well as the photon) were assumed to be non-strange

( $S = 0$ ). Thus, in any strong associated-production reaction with the initial state having no strangeness, the total strangeness of the particles in the final-state must also add up to zero. From the analysis of such reactions, it was deduced that the strangeness of the  $K^+$  and  $K^0$  must be opposite to that of the  $\Sigma^+$ ,  $\Sigma^0$ ,  $\Sigma^-$ , and  $\Lambda^0$ . In fact, if we arbitrarily choose

$$S(K^0) = 1, \quad (9.19)$$

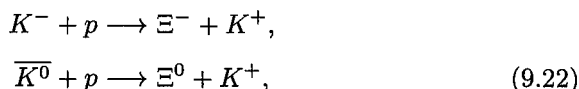
it follows that

$$S(K^+) = S(K^0) = 1, \quad (9.20)$$

and that

$$S(\Lambda^0) = S(\Sigma^+) = S(\Sigma^0) = S(\Sigma^-) = -1. \quad (9.21)$$

Similarly, from strong-production reactions such as



we deduce that the *cascade* particles  $\Xi^0$  and  $\Xi^-$ , can be assigned the strangeness number  $S = -2$ , if the  $\bar{K}^0$  and  $K^-$  have  $S = -1$ . The latter assignment is consistent with our identification of the  $K^-$  and  $\bar{K}^0$  as antiparticles of the  $K^+$  and  $K^0$ , respectively.

It is worth stressing that weak decays of hadrons do not conserve strangeness. Consequently, if we assume strangeness to be conserved only in strong and electromagnetic interactions, it then follows that we cannot assign unique strangeness quantum numbers to leptons.

#### 9.4.4 *Isospin*

The proton and the neutron are baryons with spin  $\frac{1}{2}$ , and are essentially degenerate in their mass. In fact, as we already indicated they are quite similar in their nuclear properties, except that the proton has a positive charge whereas the neutron is electrically neutral. Correspondingly, their electromagnetic interactions are quite different, and, as we discussed earlier, even their magnetic dipole moments have opposite sign.



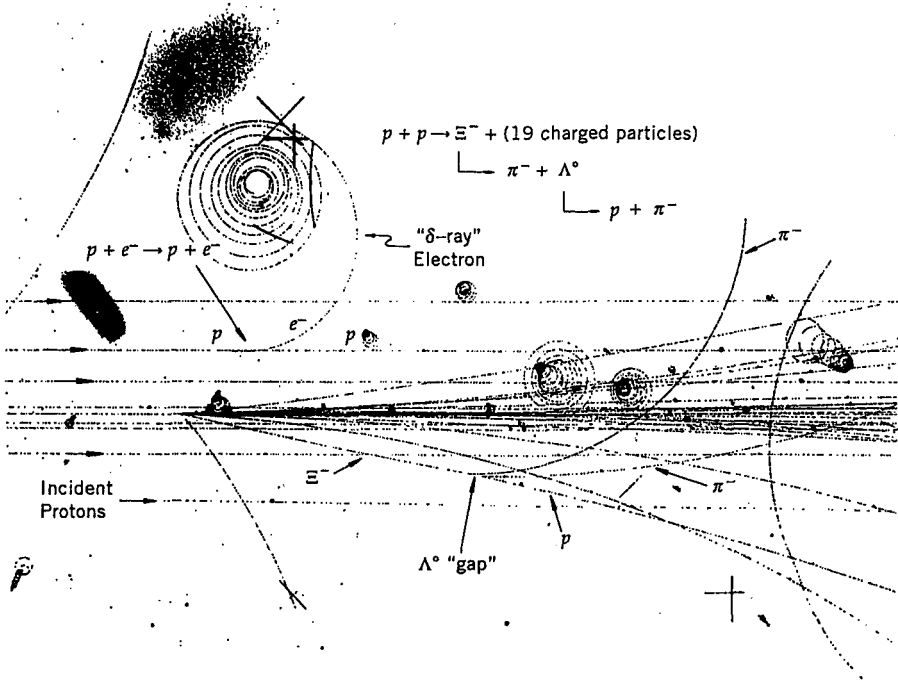


Fig. 9.1 A photograph of particle interactions and decays observed in a liquid-hydrogen bubble chamber at Fermilab. The tracks correspond to trajectories of charged particles that ionized the hydrogen atoms and thereby produced bubbles (local boiling) along their paths in the superheated liquid. A magnetic field in the chamber causes the curvature of the particle trajectories. The chamber is sensitive for about 1 msec, which is long compared to the lifetimes of some of the particles that can be observed to decay. The incident particles are 400 GeV protons. One of these interacts strongly with a target proton (nucleus of the hydrogen atom) and produces many other particles.

It has been known for a long time that the strong force does not depend on the charge of a particle. In fact, studies of mirror nuclei (e.g.,  ${}^3\text{H}$  and  ${}^3\text{He}$ ) have demonstrated that the strong binding force between  $p$ - $p$ ,  $p$ - $n$  and  $n$ - $n$  is essentially the same. Furthermore, scattering experiments have revealed that, if we correct for electromagnetic effects, the cross section for the scattering of two protons is the same as that for two neutrons. Thus, the strong interactions do not distinguish between a proton and a neutron. Consequently, if we imagine a world where only the strong force is present, and the weak and electromagnetic forces are turned off, then in such a world a proton would be indistinguishable from a neutron. (Our physical world is, of course, not like this. Nevertheless, because the strong force is so much

stronger than the other forces, we can pretend that our world is close to this, and that the presence of the other forces will then provide only small corrections to the simpler picture.) In such a world, we can think of the proton and the neutron as two orthogonal states of the same particle that we can call the nucleon, and write the states for the neutron and proton as

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (9.23)$$

This language is very similar to that used in discussing the “spin up” and the “spin down” states of a spin  $\frac{1}{2}$  particle, which are also indistinguishable in the absence of any interaction that breaks rotational symmetry (e.g., a magnetic field). The two spin states will be degenerate in energy until we apply an external magnetic field, which picks out a preferred direction in space, and removes the degeneracy of the two states. In much the same way, we can think of the proton and the neutron as being degenerate in mass because of some symmetry of the strong force (or of the strong Hamiltonian), and we call this symmetry the isotopic-spin or *isospin* symmetry. In reality, the presence of electromagnetic and weak forces breaks this symmetry, lifts the degeneracy in the masses, and allows us to distinguish between a neutron and a proton.

As indicated in Chapter 2, the three  $\pi$  mesons, namely  $\pi^+$ ,  $\pi^-$  and  $\pi^0$ , also have almost identical masses. And, just as in the case of the nucleons, the cross sections for scattering different pions on protons and neutrons are also found to be the same, once they are corrected for electromagnetic effects. Thus, it appears that the strong force does not distinguish between different kinds of  $\pi$  mesons. Therefore, in the absence of electromagnetic and weak forces, we can think of the three  $\pi$  mesons as corresponding to different states of one particle, the  $\pi$  meson, and we can represent the pion states as

$$\begin{aligned} \pi^+ &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \\ \pi^0 &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \\ \pi^- &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \end{aligned} \quad (9.24)$$

Of course, these three states are degenerate in mass in our hypothetical world. The analogy with spin now corresponds to the three spin projections of a  $J = 1$  particle that are degenerate in energy for a rotationally invariant Hamiltonian.

Similarly, the  $(K^+, K^0)$  doublet, the  $(\bar{K}^0, K^-)$  doublet and the  $(\Sigma^+, \Sigma^0, \Sigma^-)$  triplet, each correspond to states that can be considered as different manifestations of single particles, the  $K$ ,  $\bar{K}$  and  $\Sigma$ , respectively. In fact, this discussion can be extended to all the known hadrons, which can be classified into multiplets corresponding to some quantum number very much like the spin quantum number. We will refer to this quantum number as the strong isotopic spin or strong isospin, and its conservation suggests the invariance of the strong Hamiltonian under isospin transformations. These transformations correspond to rotations very much like those that occur for spin, but the rotations are in an internal Hilbert space and not in space-time. The isospin quantum number (or  $I$ -spin) is found to be conserved in strong interactions (it is a symmetry of the strong force). However,  $I$ -spin does not appear to be conserved in electromagnetic or weak processes.

Table 9.3 summarizes the strong isospin quantum numbers of different hadrons, as determined from scattering experiments. The assignment for the third-component, or projection of the isospin chosen in Table 9.3 is such that, in any given isospin multiplet, a particle with a larger positive charge has a higher value of the isospin projection. We have also denoted the projection as  $I_3$  instead of the conventional notation  $I_z$ , in order to emphasize that isospin is not a space-time symmetry. We cannot assign unique strong isospin quantum numbers to leptons or to the photon, because isospin transformations are a symmetry of only the strong-interaction Hamiltonian, and the photon and the leptons do not participate in strong reactions. As we will see in Chapter 13, there is another symmetry called the *weak isospin* symmetry, which is fundamental to the Standard Model, and involves leptons and quarks.

As we have indicated,  $I$ -spin is conserved in strong interactions, and this can be inferred from comparisons of different production and decay processes. We will leave these details to specific examples and to problems in Chapter 10.

**Table 9.3** Isotopic spin assignments of a representative group of relatively long-lived hadrons

<i>Hadron</i>	<i>Mass</i> (MeV/c <sup>2</sup> )	<i>I</i>	<i>I</i> <sub>3</sub>
<i>p</i>	938.3	1/2	1/2
<i>n</i>	939.6	1/2	-1/2
$\pi^+$	139.6	1	1
$\pi^0$	135.0	1	0
$\pi^-$	139.6	1	-1
$K^+$	494.6	1/2	1/2
$K^0$	497.7	1/2	-1/2
$\bar{K}^0$	497.7	1/2	1/2
$K^-$	494.6	1/2	-1/2
$\eta^0$	548.8	0	0
$\Lambda^0$	1115.6	0	0
$\Sigma^+$	1189.4	1	1
$\Sigma^0$	1192.6	1	0
$\Sigma^-$	1197.4	1	-1
$\Omega^-$	1672.4	0	0

### 9.5 Gell-Mann-Nishijima Relation

The assignment of the strangeness quantum number in Eq. (9.19), and the other choices we have made, may appear to be rather ad hoc. In fact, these were made originally with the phenomenological observation in mind that the electric charge of a hadron can be related to its other quantum numbers through the Gell-Mann-Nishijima relation

$$Q = I_3 + \frac{Y}{2} = I_3 + \frac{B + S}{2}, \quad (9.25)$$

where  $Y = B + S$  is known as the *strong hypercharge*. (We will see later, in the context of the Standard Model, that there is a different relation involving the *weak hypercharge*, which holds for all fundamental particles.) We summarize the quantum numbers of several typical long-lived hadrons in Table 9.4. These are all consistent with Relation (9.25).

**Table 9.4** Quantum numbers of a representative set of relatively long-lived hadrons

<i>Hadron</i>	$Q$	$I_3$	$B$	$S$	$Y = (B + S)$
$\pi^+$	1	1	0	0	0
$\pi^0$	0	0	0	0	0
$\pi^-$	-1	-1	0	0	0
$K^+$	1	1/2	0	1	1
$K^0$	0	-1/2	0	1	1
$\eta^0$	0	0	0	0	0
$p$	1	1/2	1	0	1
$n$	0	-1/2	1	0	1
$\Sigma^+$	1	1	1	-1	0
$\Lambda^0$	0	0	1	-1	0
$\Xi^-$	-1	-1/2	1	-2	-1
$\Omega^-$	-1	0	1	-3	-2

With the subsequent discovery of new particles with new *flavor* quantum numbers such as *charm* and *bottom*, in addition to strangeness, the Gell-Mann–Nishijima relation has been generalized to include these as well. In the expanded relation, the hypercharge is defined to be the sum of the baryon number, strangeness number, and all the new flavor quantum numbers. With this modification, the original relation, namely,

$$Q = I_3 + \frac{Y}{2}, \quad (9.26)$$

holds for all hadrons. Since charge and isospin are conserved in strong interactions, it follows that the generalized hypercharge is also conserved in such processes. In fact, each of the flavors is conserved independently in strong interactions.

## 9.6 Production and Decay of Resonances

We already mentioned in Chapters 2 and 4 about the existence of resonances or excitations of nuclear ground levels. Similarly, it has been found that there are also excited states of hadrons, and that such resonances have typical lifetimes of the order of  $10^{-23}$  sec. There are two ways to observe such short-lived particles. Let us first consider the  $\Delta(1232)$ , which is a  $\pi$ - $N$  state that has  $I = \frac{3}{2}$  (four different charge states). This was the first object of its kind to be found, and it was discovered by Enrico Fermi and his collaborators in the study of  $\pi$ - $N$  scattering as a function of energy. This direct way of searching for excited hadronic states is referred to as *formation* or *s-channel* studies. Using a pion beam, the probability of scattering from a nucleon target (i.e., the  $\pi$ - $N$  cross section  $\sigma_{\pi N}$ ) can be measured as a function of the momentum of the pion, or, equivalently, the invariant mass  $\frac{\sqrt{s}}{c^2}$  of the  $\pi$ - $N$  system, as given in Eq. (1.64). Figure 9.2 sketches what is observed at low energies for the elastic scattering of  $\pi^+$  on protons as a function of  $\sqrt{s}$ . The cross section rises from threshold (about 1080 MeV, corresponding to the sum of the masses of the  $\pi^+$  and the proton) and reaches a maximum at  $M_\Delta c^2 \approx 1230$  MeV; the peak has an observed full width at half maximum of  $\Gamma_\Delta c^2 \approx 100$  MeV. The excitation spectrum can be characterized, essentially, by a Lorentzian, or “Breit-Wigner” form (after Gregory Breit and Eugene Wigner). This peak can be interpreted as a resonance in the  $\pi$ - $N$  system, or an excited state of the nucleon. The intrinsic uncertainty in the mass associated with the observed width of the line shape (corrected for any small effects due to the experimental resolution) corresponds to a lifetime of the order of

$$\tau_\Delta \approx \frac{\hbar}{\Gamma_\Delta c^2} \approx \frac{6.6 \times 10^{-22} \text{ MeV-sec}}{100 \text{ MeV}} \approx 10^{-23} \text{ sec.} \quad (9.27)$$

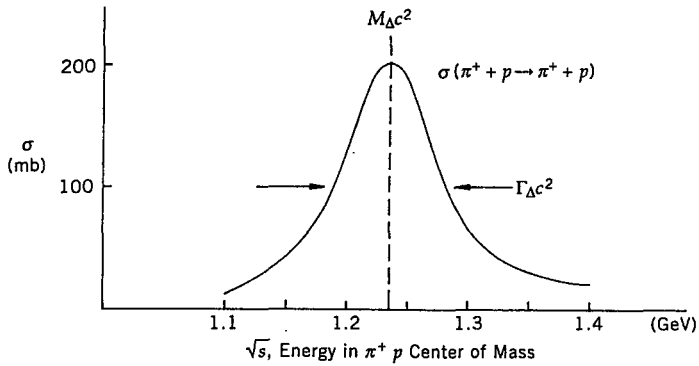
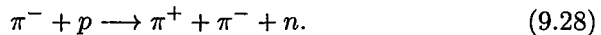
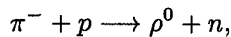


Fig. 9.2 Sketch of the pion-nucleon elastic scattering cross section at low energies.

Clearly, not all excited states of hadrons can be found in this fashion. For example, resonances among pions cannot be produced in formation experiments because the fluxes needed to observe such systems would be forbiddingly high. Objects such as the  $\rho$  meson were discovered in final states involving multiple-pion production. The following reaction, when studied at a fixed beam energy, is a rich source of  $\rho^0$  mesons



The way that the presence of a resonance can be detected in the final state is by plotting the invariant mass or the  $\frac{\sqrt{s_{\pi^+\pi^-}}}{c^2}$  of the  $\pi^+\pi^-$  system. If Reaction (9.28) proceeds through the intermediate step



followed by



then, after the  $\rho^0$  decays, the  $\pi^+$  and  $\pi^-$  will remain correlated. This is because energy-momentum conservation in the decay will assure that

$$\begin{aligned} E_\rho &= E_{\pi^+} + E_{\pi^-}, \\ \vec{p}_\rho &= \vec{p}_{\pi^+} + \vec{p}_{\pi^-}, \end{aligned} \quad (9.30)$$

and therefore the invariant mass of the two pions will maintain the mass of the  $\rho^0$ , namely,

$$M_\rho^2 c^4 = (E_\rho^2 - p_\rho^2 c^2) = (E_{\pi^+} + E_{\pi^-})^2 - (\vec{p}_{\pi^+} + \vec{p}_{\pi^-})^2 c^2 = s_{\pi\pi} \quad (9.31)$$

Consequently, when we plot the distribution for the effective or invariant mass of two pions for many events corresponding to Reaction (9.28), namely plot the number of events as a function of  $\frac{\sqrt{s_{\pi\pi}}}{c^2}$ , if there is a contribution from Reaction (9.29), we should then observe a peak in the distribution at  $\frac{\sqrt{s_{\pi\pi}}}{c^2} = M_\rho$ . A typical result for Reaction (9.28) is shown in Fig. 9.3, and displays a peak at  $M_\rho = 760 \text{ MeV}/c^2$ , with a width of  $\Gamma_\rho \approx 150 \text{ MeV}/c^2$ , which characterizes the strong resonant interaction of the two pions.

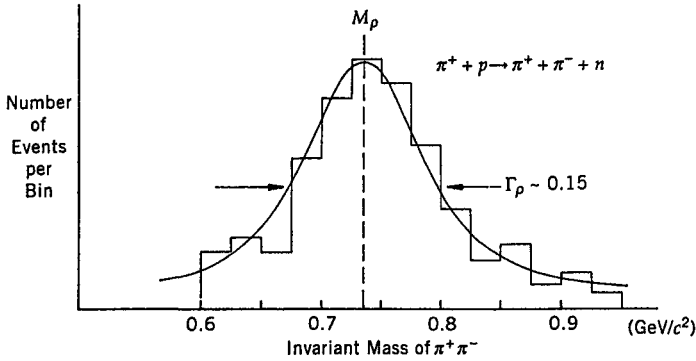


Fig. 9.3 Sketch of the invariant mass of  $\pi^+\pi^-$  pairs produced in Reaction (9.28).

The fact that resonances have Breit-Wigner form is an intrinsic consequence of the character of a decaying quantum state. The time dependence of the amplitude of any state with mean life  $\frac{\hbar}{\Gamma c^2}$ , and with a central value of mass of  $M = M_0$ , can be written in its own rest frame as (see the discussion on time development in Chapter 12)

$$\psi(t) \propto e^{-\frac{ic^2}{\hbar} (M_0 - i\frac{\Gamma}{2}) t}, \quad t > 0. \quad (9.32)$$

This assures exponential decay of a state with mean life  $\frac{\hbar}{\Gamma c^2}$ , namely,

$$|\psi(t)|^2 \propto e^{-\frac{\Gamma c^2 t}{\hbar}} \quad (9.33)$$



Taking the Fourier transform of Eq. (9.32) provides the amplitude in energy space (or in mass)

$$\psi(M) \propto \int_0^\infty dt \psi(t) e^{\frac{i}{\hbar} M c^2 t}. \quad (9.34)$$

This is simple to integrate, and, except for an overall normalization, yields

$$\psi(M) \propto \frac{1}{(M - M_0) + i\frac{\Gamma}{2}}, \quad (9.35)$$

which, upon squaring, yields the Lorentzian or Breit-Wigner form for a resonant effect at  $M = M_0$ ,

$$|\psi(M)|^2 \propto \frac{1}{(M - M_0)^2 + \frac{\Gamma^2}{4}}. \quad (9.36)$$

## 9.7 Determining Spins

The spins of some of the stable elementary particles can in principle be obtained, for example, through a Stern-Gerlach type of experiment. Thus, from the splitting of a beam of particles in a magnetic field, we can deduce that the electron as well as the proton have spin angular momentum  $\frac{1}{2}$ . The neutrino, as we have seen earlier, was postulated to have spin  $\frac{1}{2}$  in order to maintain angular momentum conservation in  $\beta$  decay. The spin of the photon can, of course, be determined from the classical properties of electromagnetic waves. Because the electromagnetic field is described by a vector potential, this implies that the photon is a vector particle with spin-1, and that its wave function is proportional to its polarization vector  $\vec{\epsilon}$ . Normally, a spin-1 state has three possible projections of the angular momentum, corresponding to  $s_z = 1, 0, -1$ . However, propagating electromagnetic waves are transverse, which means that the physical photon has no longitudinal degree of freedom. This is reflected in the fact that the electric ( $\vec{E}$ ) and magnetic ( $\vec{B}$ ) fields, and the polarization vector of the photon, are transverse to the direction of propagation ( $\hat{k} = \frac{\vec{k}}{|\vec{k}|}$ )

$$\vec{E} = \vec{\epsilon} E_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}, \quad \vec{B} = \hat{k} \times \vec{E}, \quad (9.37)$$

and obey the conditions

$$\vec{k} \cdot \vec{E} = 0, \quad \vec{k} \cdot \vec{B} = 0, \quad \vec{k} \cdot \vec{e} = 0. \quad (9.38)$$

These properties of the photon are related to its being massless, which is a feature that can be accommodated through the invariance of Maxwell's equations under what are traditionally referred to as "gauge" transformations of the electromagnetic potentials. (We shall discuss this in more detail in Chapter 13.)

The spin of the  $\pi^0$  meson can be deduced from the fact that it decays into two photons. In the rest frame of the  $\pi^0$ , the two photons must be emitted back-to-back with equal and opposite momenta (see Fig. 9.4). The final state in this decay consists of two identical bosons and, consequently, the final state wave function, which corresponds to the product of the wave functions for the two photons, must be symmetric under the exchange of the photons. As we have indicated, the photon wave function is proportional to its polarization vector. Letting  $\vec{k}$  denote the relative momentum of the two photons and  $\vec{e}_1$  and  $\vec{e}_2$  their polarization vectors, the only scalar or vector quantities that can be constructed from the vectors  $\vec{k}$ ,  $\vec{e}_1$  and  $\vec{e}_2$  that are linear in both  $\vec{e}_1$  and  $\vec{e}_2$ , and are symmetric under the exchange of the photon variables, are

$$\vec{k} \times (\vec{e}_1 \times \vec{e}_2), \quad \vec{k} \cdot (\vec{e}_1 \times \vec{e}_2), \quad \text{and} \quad \vec{e}_1 \cdot \vec{e}_2. \quad (9.39)$$

The first vanishes because the polarization vectors are transverse, that is,

$$\vec{k} \times (\vec{e}_1 \times \vec{e}_2) = (\vec{k} \cdot \vec{e}_2)\vec{e}_1 - (\vec{k} \cdot \vec{e}_1)\vec{e}_2 = 0.$$

Thus, the simplest combinations satisfying all symmetry properties are the scalar products in Eq. (9.39). Because the planes of polarization for the two photons are observed to be orthogonal, we conclude that the final state wave function must be proportional to the scalar product

$$\vec{k} \cdot (\vec{e}_1 \times \vec{e}_2). \quad (9.40)$$

For the decay of a  $\pi^0$  to take place (that is, for the transition amplitude not to vanish), the pion wave function must have a component corresponding to the final state wave function of the two photons. We therefore conclude

that the pion wave function must also be a scalar under rotations, that is, the pion must have spin zero. (We have ignored the possibility of the pion having a spin greater than  $J = 1$ .)

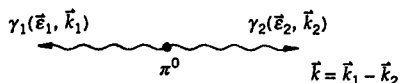


Fig. 9.4 The decay of a  $\pi^0$  into two photons, as viewed in its own rest frame.

Similarly, we can obtain the spin of the  $K^0$  meson from the decay process  $K^0 \rightarrow 2\pi^0$ . Once again, in the rest frame of the  $K^0$  meson, the two  $\pi$  mesons must have equal and opposite momentum (see Fig. 9.5). The final state consists of two spin-zero particles, and consequently the total angular momentum of the final state (and the spin of the  $K^0$ ) is the same as the relative orbital angular momentum of the two pions. Since the two  $\pi^0$  mesons are identical bosons, the final-state wave function must be symmetric under the exchange of the two particles. If we let  $\ell$  denote the orbital angular momentum of the final state, then the angular part of the wave function of the final state will be proportional to the spherical harmonics  $Y_{\ell,m}(\theta, \phi)$ . We have noted previously in Chapter 3 that, under the exchange of the two particles, these wave functions behave as<sup>3</sup>

$$Y_{\ell,m}(\theta, \phi) \longrightarrow (-1)^\ell Y_{\ell,m}(\theta, \phi). \quad (9.41)$$

It follows therefore that for the final state to be symmetric under the exchange of the two particles,  $\ell$  can take on only even values. Thus, we conclude that the spin of the  $K$  meson must be even, namely 0, 2, or 4, and so forth.

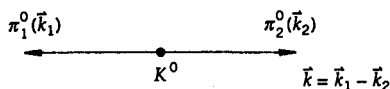


Fig. 9.5 The decay of a  $K^0$  into two  $\pi^0$  mesons, as viewed in the rest frame of the kaon.

We can also argue, as we did in the case of the  $\pi^0$ , that, since pions are spinless, in the rest frame of the  $K$  meson, the only vector available for describing the final state is the relative momentum of the two  $\pi^0$  mesons,

<sup>3</sup>Exchange of the two particles corresponds to  $\vec{r} \rightarrow -\vec{r}$ , which is the same as reflection or a parity transformation (see Chapter 11).

or  $\vec{k}$ . Under an exchange of pions, the  $\vec{k}$  vector changes sign, and we can conclude that the simplest wave function that can be constructed from this vector, and still satisfy all the symmetry properties of the final state, is a scalar (e.g., any function of  $\vec{k} \cdot \vec{k}$ ). Consequently, we deduce that the wave function for the  $K^0$  meson must be a scalar, and that the  $K^0$  is a spin zero particle. In fact, the decay characteristics of the  $K$  mesons exclude the higher spin assignments of  $J = 2, 4$ , etc. That is, the angular distribution of the  $\pi^0$  mesons in the  $K^0$  rest frame show no evidence for the presence of other than  $\ell = 0$  contributions to the decay.

Let us next discuss how the spins of some of the baryons can be obtained. For example, let us analyze the following collision of a high-energy  $\pi^-$  with a proton at rest



where the hyperon (strange baryon)  $Y_1^{*0}$  subsequently decays through



Let us use the beam direction, namely the direction of the incident pion in the laboratory, as the axis of quantization for angular momentum. The component of the incident pion's orbital angular momentum along its direction of motion will clearly vanish ( $\vec{L}_\pi \approx \vec{r} \times \vec{p}_\pi$ , which is perpendicular to  $\vec{p}_\pi$ ). Furthermore, the pion is spinless. For the initial state, the projection of the total angular momentum along the beam direction is therefore given by the projection of the intrinsic spin of the proton,  $s_z$ , namely,

$$j_z = s_z(p) = \pm \frac{1}{2} \hbar. \quad (9.44)$$

Now, let us restrict our study to the  $K^0$ s and  $Y_1^{*0}$ s that are produced only along the beam axis, namely, close to the forward-backward direction in the center-of-mass of Reaction (9.42). Again, along this axis, the relative orbital momentum of the  $K^0 - Y_1^{*0}$  system vanishes, and because the spin of the  $K^0$  is zero, from conservation of angular momentum we conclude that

$$s_z(Y_1^{*0}) = s_z(p) = \pm \frac{1}{2} \hbar. \quad (9.45)$$

This allows us to conclude that the spin of the  $Y_1^{*0}$  hyperon is  $\frac{1}{2}$ , or a higher half-integer value. To obtain the true spin, we would have to analyze the decay in Eq. (9.43) in substantial detail. Measuring the complexity of the angular decay distribution in the rest frame of the  $Y_1^{*0}$  would then determine its spin value, which is now known to be  $\frac{3}{2}$ .

Similar analyses of other high energy production processes, and the subsequent decays of the particles, have provided us with the values of spins of many hadrons. What has been found is that some of the particles can be related to each other and correlated into groups having similar quantum numbers.

## 9.8 Violation of Quantum Numbers

As we have seen, all quantum numbers appear to be conserved in strong processes, however, some are violated in electromagnetic and weak interactions, and we will discuss these with a few illustrative examples.

### 9.8.1 Weak Interactions

There are three kinds of weak processes in nature, which can be classified as follows: (a) hadronic decays, where only hadrons are present in the final state, (b) semi-leptonic processes, where both hadrons and leptons are present, and, finally, (c) leptonic processes, where only leptons are present. For example, the decays

$$\begin{aligned}\Lambda^0 &\rightarrow \pi^- + p, \\ n &\rightarrow p + e^- + \bar{\nu}_e, \\ \mu^- &\rightarrow e^- + \bar{\nu}_e + \nu_\mu,\end{aligned}\tag{9.46}$$

represent the three different kinds of weak processes. Since most of the strong quantum numbers are not defined for leptons, it is not meaningful to discuss their violation in leptonic processes. Furthermore, even in the case of the semi-leptonic processes, we can only speak about the conservation or the violation of quantum numbers between the initial and the final hadronic states. Keeping this in mind, let us now examine some typical reactions.

#### 9.8.1.1 Hadronic Weak Decays:

Consider the following decays of hadrons into other hadrons

$$\begin{array}{rcl}
 \Lambda^0 & \longrightarrow & \pi^- + p, \\
 I_3 = 0 & & I_3 = -1 \quad I_3 = \frac{1}{2} \\
 S = -1 & & S = 0 \quad S = 0 \\
 \\
 \Sigma^+ & \longrightarrow & p + \pi^0, \\
 I_3 = 1 & & I_3 = \frac{1}{2} \quad I_3 = 0 \\
 S = -1 & & S = 0 \quad S = 0 \\
 \\
 K^0 & \longrightarrow & \pi^+ + \pi^-, \\
 I_3 = -\frac{1}{2} & & I_3 = 1 \quad I_3 = -1 \\
 S = 1 & & S = 0 \quad S = 0 \\
 \\
 \Xi^- & \longrightarrow & \Lambda^0 + \pi^-. \\
 I_3 = -\frac{1}{2} & & I_3 = 0 \quad I_3 = -1 \\
 S = -2 & & S = -1 \quad S = 0
 \end{array} \tag{9.47}$$

We see that both isospin and strangeness are violated in these decays, and that a selection rule for such violations can be summarized by

$$|\Delta I_3| = \frac{1}{2}, \quad |\Delta S| = 1. \tag{9.48}$$

Also, we should add that, while both  $\Delta I = \frac{1}{2}$  and  $\Delta I = \frac{3}{2}$  transitions appear to contribute to these processes, the  $\Delta I = \frac{3}{2}$  contributions are found to be highly suppressed, and processes involving  $|\Delta S| = 2$  are exceptionally rare.

### 9.8.1.2 Semileptonic Processes:

Once again, we examine only a few examples to bring out the essential features of these decays, and emphasize that we consider only changes in the quantum numbers of initial and final-state hadrons.

$$\begin{aligned}
n &\longrightarrow p && + e^- + \bar{\nu}_e, \\
I_3 = -\frac{1}{2} & & I_3 = \frac{1}{2} \\
S = 0 & & S = 0 \\
\\
\pi^- &\longrightarrow \mu^- + \bar{\nu}_\mu, \\
I_3 = -1 \\
S = 0 \\
\\
\pi^+ &\longrightarrow \pi^0 && + e^+ + \nu_e, \\
I_3 = 1 & & I_3 = 0 \\
S = 0 & & S = 0 \\
\\
K^+ &\longrightarrow \mu^+ + \nu_\mu, \\
I_3 = \frac{1}{2} & & \\
S = 1 & & \tag{9.49} \\
\\
K^+ &\longrightarrow \pi^0 && + \mu^+ + \nu_\mu, \\
I_3 = \frac{1}{2} & & I_3 = 0 \\
S = 1 & & S = 0 \\
\\
\Lambda^0 &\longrightarrow p && + e^- + \bar{\nu}_e, \\
I_3 = 0 & & I_3 = \frac{1}{2} \\
S = -1 & & S = 0 \\
\\
\Sigma^- &\longrightarrow n && + e^- + \bar{\nu}_e. \\
I_3 = -1 & & I_3 = -\frac{1}{2} \\
S = -1 & & S = 0
\end{aligned}$$

Thus, we see that semi-leptonic decays can be classified into two types. The first kind has no change in the strangeness flavor of hadrons. These processes are known as *strangeness-preserving* decays, and are characterized, of course, by  $|\Delta S| = 0$ . In such processes we see that  $|\Delta I_3| = 1$ . Thus, the strangeness-conserving semi-leptonic processes satisfy

$$|\Delta S| = 0, \quad |\Delta I_3| = 1, \quad \Delta I = 1. \tag{9.50}$$

The second class of semi-leptonic decays do not conserve strangeness. Consequently, these decays are also known as *strangeness-changing* pro-

cesses, and for these we find

$$|\Delta S| = 1, \quad |\Delta I_3| = \frac{1}{2}, \quad \Delta I = \frac{1}{2} \quad \text{or} \quad \frac{3}{2}. \quad (9.51)$$

Again, it is observed that the processes with  $\Delta I = \frac{3}{2}$  are highly suppressed, as are  $|\Delta S| = 2$  transitions.

### 9.8.2 *Electromagnetic Processes*

Let us also consider several samples of electromagnetic decays. Again, since strong quantum numbers cannot always be defined for a photon, the meaningful quantity to analyze is the change in the quantum numbers of the hadrons

$$\begin{aligned} \pi^0 &\longrightarrow \gamma + \gamma, \\ I_3 = 0 & \\ S = 0 & \\ \\ \eta^0 &\longrightarrow \gamma + \gamma, \\ I_3 = 0 & \\ S = 0 & \\ \\ \Sigma^0 &\longrightarrow \Lambda^0 + \gamma. \\ I_3 = 0 & \quad I_3 = 0 \\ S = -1 & \quad S = -1 \end{aligned} \quad (9.52)$$

This shows that strangeness is conserved in electromagnetic processes, while isospin is not. In fact, these processes are characterized by

$$|\Delta S| = 0, \quad |\Delta I_3| = 0, \quad \Delta I = 1 \quad \text{and} \quad 0. \quad (9.53)$$

In the following chapters we will attempt to incorporate all these results into a theoretical framework, which is referred to as the Standard Model of particle physics.



## Problems

**9.1** What quantum numbers, if any, are violated in the following reactions? Are the interactions strong, weak, electromagnetic, or none of the above? (See the *CRC Handbook* for particle properties.)

- (a)  $\Omega^- \rightarrow \Xi^0 + \pi^-$ ,
- (b)  $\Sigma^+ \rightarrow \pi^+ + \pi^0$ ,
- (c)  $n \rightarrow p + \pi^-$ ,
- (d)  $\pi^0 \rightarrow \mu^+ + e^- + \bar{\nu}_e$ ,
- (e)  $K^0 \rightarrow K^+ + e^- + \bar{\nu}_e$ ,
- (f)  $\Lambda^0 \rightarrow p + e^-$ .

**9.2** What quantum numbers, if any, are violated in the following processes. Would the reaction be strong, electromagnetic, weak, or unusually suppressed? Explain. (See *CRC Handbook* for particle properties.)

- (a)  $\Lambda^0 \rightarrow p + e^- + \bar{\nu}_e$ ,
- (b)  $K^- + p \rightarrow K^+ + \Xi^-$ ,
- (c)  $K^+ + p \rightarrow K^+ + \Sigma^+ + \bar{K}^0$ ,
- (d)  $p + p \rightarrow K^+ + K^+ + n + n$ ,
- (e)  $\Sigma^+(1385) \rightarrow \Lambda^0 + \pi^+$ ,
- (f)  $\bar{p} + n \rightarrow \pi^- + \pi^0$ .

**9.3** A  $\pi^0$  meson with momentum 135 GeV/ $c$  decays into two photons. If the mean life of a  $\pi^0$  is  $8.5 \times 10^{-17}$  sec, calculate to 10% accuracy how far the high-energy  $\pi^0$  will travel prior to decay? What will be the approximate minimum value of the opening angle of its two decay photons in the laboratory?

**9.4** We will see in Chapter 13 that hadrons are composed of constituents known as quarks, and that mesons can be represented as quark–antiquark systems and baryons as three-quark systems. All quarks have baryon number  $\frac{1}{3}$ , and their other quantum numbers are listed in Table 9.5. Antiquarks have all their quantum numbers of opposite sign to the quarks. The isotopic spin of quarks can be inferred from the generalized Gell-Mann–Nishijima relation of Eq. (9.26). Free quarks are not observed in nature. The top quark is as free as a quark can get, but it decays so rapidly that it does not have sufficient time to form hadrons, reflecting the fact that its weak interactions are stronger than its strong interactions.

**Table 9.5** Properties of the quarks

Quark	Symbol	Rest	Electric	"Flavor" Quantum Numbers			
		Mass (GeV/c <sup>2</sup> )	Charge (e)	Strange	Charm	Bottom	Top
Up	$u$	$\lesssim 3 \times 10^{-3}$	$\frac{2}{3}$	0	0	0	0
Down	$d$	$\approx 7 \times 10^{-3}$	$-\frac{1}{3}$	0	0	0	0
Strange	$s$	$\approx 0.12$	$-\frac{1}{3}$	-1	0	0	0
Charm	$c$	$\approx 1.2$	$\frac{2}{3}$	0	1	0	0
Bottom	$b$	$\approx 4.2$	$-\frac{1}{3}$	0	0	-1	0
Top	$t$	$175 \pm 5$	$\frac{2}{3}$	0	0	-0	1

The quark system  $uds$  can exist in more than one isospin state. What is the value of  $I_3$  for this combination of quarks? What are the possible values of total  $I$ -spin for  $uds$  states? Can you identify them with any known particles? (See, e.g., *CRC Handbook*)

**9.5** What is the baryon number, hypercharge, and isotopic spin of the following quark systems: (a)  $u\bar{s}$ , (b)  $c\bar{d}$ , (c)  $\bar{u}\bar{u}\bar{d}$ , (d)  $ddc$ , (e)  $ubc$ , (f)  $s\bar{s}$ . Using the *CRC Handbook*, can you identify these states with any known particles?

**9.6** Consider the following decays

$$(a) N^+(1535) \longrightarrow p + \eta^0,$$

$$(b) \Sigma^+(1189) \longrightarrow p + \pi^0,$$

$$(c) \rho^0(770) \longrightarrow \pi^0 + \gamma.$$

From the properties listed in the *CRC* tables, discuss the interactions involved in each case. What quantum numbers are violated? What are the possible values of orbital angular momentum in the final states?

### Suggested Readings

Frauenfelder, H., and E. M. Henley, *Subatomic Physics*, Prentice-Hall (1991).

Griffiths, D., *Introduction to Elementary Particles*, Wiley (1989).

Perkins, D. H., *Introduction to High Energy Physics*, Cambridge Univ. Press (2000).

Williams, W. S. C., *Nuclear and Particle Physics*, Oxford Univ. Press (1997).

## Chapter 10

# Symmetries

### 10.1 Introductory Remarks

As we saw in the previous chapter, although several quantum numbers appear to be conserved in strong high-energy processes, some are violated in weak and electromagnetic interactions. This must reflect the inherent character of the underlying forces. Consequently, understanding the origin of conservation principles, and under what conditions they are violated, would appear to be an important element in the formulation of a quantitative description of particle interactions. We will therefore first address the question of how conservation laws arise in physical theories. As we will see shortly, the surprisingly simple answer is that whenever there is an underlying symmetry in a physical system, namely if our system is not affected by a change in some coordinate or other dynamical variable, then we can define a conserved “charge” (quantum number) associated with that symmetry. Conversely, if there is a conserved quantity associated with a physical system, then there exists an underlying invariance or symmetry principle responsible for its conservation. This observation, known as Noether’s Theorem (after Emmy Noether), gives rise to powerful restrictions on the structure of physical theories. We now turn to issues related to symmetries in physical systems.

### 10.2 Symmetries in the Lagrangian Formalism

In simple terms, any set of transformations that leaves the equations of motion of a system unchanged or invariant, defines what is known as a symmetry of that physical system. Symmetries can be discussed using either the Lagrangian or the Hamiltonian formalism, for both classical as

well as quantum theories. We begin the discussion using the Lagrangian framework, which is most appropriate for studying relativistic systems – our ultimate goal.

Consider first an isolated non-relativistic physical system consisting of two particles interacting through a potential that depends on their relative separation. The total kinetic and potential energies for this system are given by

$$\begin{aligned} T &= \frac{1}{2} m_1 \dot{\vec{r}}_1^2 + \frac{1}{2} m_2 \dot{\vec{r}}_2^2, \\ V &= V(\vec{r}_1 - \vec{r}_2), \end{aligned} \quad (10.1)$$

where  $m_1$  and  $m_2$  are the masses of the particles, and  $\vec{r}_1$  and  $\vec{r}_2$  their coordinates measured with respect to some specified origin. The equations of motion (Newton's equations), or the *dynamical equations*, for this system take the form

$$\begin{aligned} m_1 \ddot{\vec{r}}_1 &= -\vec{\nabla}_1 V(\vec{r}_1 - \vec{r}_2) = -\frac{\partial}{\partial \vec{r}_1} V(\vec{r}_1 - \vec{r}_2), \\ m_2 \ddot{\vec{r}}_2 &= -\vec{\nabla}_2 V(\vec{r}_1 - \vec{r}_2) = -\frac{\partial}{\partial \vec{r}_2} V(\vec{r}_1 - \vec{r}_2), \end{aligned} \quad (10.2)$$

where by  $\frac{\partial}{\partial \vec{r}_i} V(\vec{r}_1 - \vec{r}_2)$  we mean

$$\hat{x} \frac{\partial}{\partial x_i} V + \hat{y} \frac{\partial}{\partial y_i} V + \hat{z} \frac{\partial}{\partial z_i} V,$$

with  $i$  being 1 or 2, and  $\hat{x}$ ,  $\hat{y}$  and  $\hat{z}$  being the unit vectors along the  $x$ ,  $y$  and  $z$  axes of our fixed coordinate system.

Now, if we translate the origin of the coordinate system by a constant vector  $(-\vec{a})$ , namely if we transform the coordinates as

$$\begin{aligned} \vec{r}_1 &\longrightarrow \vec{r}'_1 = \vec{r}_1 + \vec{a}, \\ \vec{r}_2 &\longrightarrow \vec{r}'_2 = \vec{r}_2 + \vec{a}, \end{aligned} \quad (10.3)$$

then the dynamical equations for the system of particles in Eq. (10.2) do not change. This is simply a consequence of the fact that

$$V(\vec{r}_1 - \vec{r}_2) \longrightarrow V(\vec{r}_1 + \vec{a} - \vec{r}_2 - \vec{a}) = V(\vec{r}_1 - \vec{r}_2). \quad (10.4)$$

Thus, a translation of the origin of the coordinate system defines a symmetry of the two-particle system, and we say that the physical system is invariant under spatial translations, namely, our physical system is not sensitive to any particular choice for the origin of our coordinate system. The consequence of this symmetry is quite interesting. We note from the form of the potential that the total force acting on the system vanishes, that is,

$$\vec{F}_{\text{TOT}} = \vec{F}_1 + \vec{F}_2 = -\vec{\nabla}_1 V(\vec{r}_1 - \vec{r}_2) - \vec{\nabla}_2 V(\vec{r}_1 - \vec{r}_2) = 0. \quad (10.5)$$

(Equation (10.5) follows from the fact that  $\frac{\partial V}{\partial \vec{r}_1} = -\frac{\partial V}{\partial \vec{r}_2}$ .) Consequently, for the total momentum of the system we obtain

$$\frac{d\vec{P}_{\text{TOT}}}{dt} = \vec{F}_{\text{TOT}} = 0. \quad (10.6)$$

In other words, the total momentum associated with the system is conserved – that is, it is a constant of the motion, and independent of time.

This kind of result may seem to be purely accidental and applicable only for our simple two-body example, but, in fact, it can be shown that for any symmetry associated with a physical system, there exists a conserved quantity. To see this, let us rewrite the dynamical equations in Eq. (10.2) as

$$\begin{aligned} \frac{d}{dt} \frac{\partial T}{\partial \dot{\vec{r}}_1} &= -\frac{\partial V}{\partial \vec{r}_1}, \\ \frac{d}{dt} \frac{\partial T}{\partial \dot{\vec{r}}_2} &= -\frac{\partial V}{\partial \vec{r}_2}, \end{aligned} \quad (10.7)$$

where the kinetic energy is defined in Eq. (10.1). (Again, we are using our shorthand notation in Eq. (10.7), and each equation represents, in fact, three separate relations, namely,  $\frac{d}{dt} \frac{\partial T}{\partial \dot{q}} = -\frac{\partial V}{\partial q}$ , for  $q = x_i, y_i$  and  $z_i$ , and  $i = 1$  and  $2$ ). Furthermore, if we define the function

$$L = T - V, \quad (10.8)$$

then, recognizing that the coordinate and velocity of a particle are really independent variables, we can write the entire content of the dynamical equations in (10.2) or (10.7) schematically as

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\vec{r}}_i} - \frac{\partial L}{\partial \vec{r}_i} = 0, \quad i = 1, 2. \quad (10.9)$$

The quantity  $L(\vec{r}_i, \dot{\vec{r}}_i)$  is known as the Lagrangian of the system, and we note that by construction we have

$$\frac{\partial L}{\partial \dot{\vec{r}}_i} = \frac{\partial T}{\partial \dot{\vec{r}}_i} = m_i \dot{\vec{r}}_i = \vec{p}_i. \quad (10.10)$$

Consequently, the Hamiltonian,  $H = T + V = 2T - L$ , can now be obtained from the Lagrangian through the use of Eqs. (10.1) and (10.10), and written as

$$H = \sum_{i=1}^2 \vec{p}_i \cdot \dot{\vec{r}}_i - L(\vec{r}_i, \dot{\vec{r}}_i). \quad (10.11)$$

All the preceding considerations carry over quite naturally to more complicated systems, and the Lagrangian for a general system with  $n$ -degrees of freedom (namely,  $n$ -coordinates and  $n$ -velocities) can be represented as

$$L = L(q_i, \dot{q}_i), \quad i = 1, 2, \dots, n. \quad (10.12)$$

The momenta associated with, or *conjugate* to, the coordinates  $q_i$  can be defined as given in Eq. (10.10)

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad i = 1, 2, \dots, n, \quad (10.13)$$

and the general dynamical equations of motion can be written in line with Eq. (10.9) as

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0,$$

$$\text{or } \frac{dp_i}{dt} = \frac{\partial L}{\partial q_i}, \quad i = 1, 2, \dots, n. \quad (10.14)$$

Let us now suppose that the Lagrangian for a given physical system is independent of some particular coordinate  $q_m$ . It then follows that

$$\frac{\partial L}{\partial q_m} = 0, \quad \text{for the specified } m. \quad (10.15)$$

As a result, the dynamical equation for  $i = m$  in Eq. (10.14) will give

$$\frac{dp_m}{dt} = 0. \quad (10.16)$$

In other words, if the Lagrangian for a physical system does not depend explicitly on a given coordinate, then the corresponding conjugate momentum is conserved. Moreover, if a Lagrangian does not depend on some particular coordinate, it must be invariant under translations (redefinitions) of this coordinate, which brings to focus the connection between the invariance of a theory and a corresponding conserved quantity.

As example, we saw in Chapter 1, that if we rewrite the two body problem in terms of the relative coordinate  $\vec{r} = \vec{r}_1 - \vec{r}_2$  and the center-of-mass coordinate  $\vec{R}_{\text{CM}}$ , then the potential energy and therefore the Lagrangian will be independent of  $\vec{R}_{\text{CM}}$ . Consequently, the corresponding momentum  $\vec{P}_{\text{CM}}$ , which we have seen to correspond to the total momentum of the system, will be a constant – a result we already recognized in Eq. (10.6).

As a second simple demonstration of these ideas, let us consider the motion of a free rotor. In the absence of any force, the system has only kinetic energy and we can therefore write

$$L = T = \frac{1}{2} I \dot{\theta}^2, \quad (10.17)$$

where  $I$  denotes the moment of inertia of the rotor and  $\dot{\theta}$  its angular velocity. This Lagrangian is independent of the angular coordinate  $\theta$  of the rotor, and correspondingly we conclude (following our previous argument) that

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = I \dot{\theta} = \text{constant}. \quad (10.18)$$



Hence, the lack of an explicit  $\theta$ -dependence in the Lagrangian of a rotor gives rise to the rotational invariance of the system, and leads to a constant value of its angular momentum. As we have emphasized, such conclusions are, in fact, quite general, and we summarize in Table 10.1 several common transformations and the associated quantities that are conserved when physical systems are invariant under these transformations.

**Table 10.1** Invariance of a system under a transformation and the corresponding conserved quantity

<i>Transformation</i>	<i>Conserved Quantity for System</i>
Space translation	Momentum
Time translation	Energy
Spatial rotation	Angular momentum
Rotation in isotopic-spin space	Isotopic spin

The converse argument also holds, namely that for every conserved quantity of a physical system there exists an underlying invariance principle. All this is much easier to see using the Hamiltonian formalism, to which we now turn.

### 10.3 Symmetries in the Hamiltonian Formalism

The Hamiltonian formalism of classical mechanics goes over naturally to quantum mechanics and, therefore, the discussion of symmetries in the context of the Hamiltonian formalism can be quite illuminating. Let us recall that a Hamiltonian  $H(q_i, p_i)$  for a system with  $n$ -degrees of freedom is a function of  $n$ -coordinates and  $n$ -momenta. The equations of motion now comprise the first-order Hamilton relations, given by

$$\begin{aligned} \frac{dq_i}{dt} &= \dot{q}_i = \frac{\partial H}{\partial p_i}, \\ \frac{dp_i}{dt} &= \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, 2, \dots, n. \end{aligned} \quad (10.19)$$

Let us now introduce the bracket notation (*Poisson bracket*) for independent coordinates and momenta. In general, the Poisson bracket for any two functions of  $q_i$  and  $p_i$  can be defined in terms of their partial derivatives with respect to these variables as

$$\begin{aligned} \{F(q_i, p_i), G(q_i, p_i)\} &= \sum_{i=1}^n \left( \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right) \\ &= -\{G(q_i, p_i), F(q_i, p_i)\}. \end{aligned} \quad (10.20)$$

Thus, for the basic Poisson brackets of the coordinates and momenta (also known as the *canonical* Poisson brackets), we obtain

$$\begin{aligned} \{q_i, q_j\} &= 0, \\ \{p_i, p_j\} &= 0, \\ \{q_i, p_j\} &= -\{p_j, q_i\} = \delta_{ij}, \end{aligned} \quad (10.21)$$

where  $\delta_{ij}$  is the Kronecker “delta”, which equals unity for  $i = j$  and zero for  $i \neq j$ . (In the transition to quantum mechanics, Poisson brackets are replaced by commutators.) Using the brackets, we note that

$$\begin{aligned} \{q_i, H\} &= \sum_j \left( \frac{\partial q_i}{\partial q_j} \frac{\partial H}{\partial p_j} - \frac{\partial q_i}{\partial p_j} \frac{\partial H}{\partial q_j} \right) \\ &= \sum_j \delta_{ij} \frac{\partial H}{\partial p_j} = \frac{\partial H}{\partial p_i}, \end{aligned}$$

and

$$\begin{aligned} \{p_i, H\} &= \sum_j \left( \frac{\partial p_i}{\partial q_j} \frac{\partial H}{\partial p_j} - \frac{\partial p_i}{\partial p_j} \frac{\partial H}{\partial q_j} \right) \\ &= -\sum_j \delta_{ij} \frac{\partial H}{\partial q_j} = -\frac{\partial H}{\partial q_i}. \end{aligned} \quad (10.22)$$

Thus the dynamical equations in (10.19) can also be written as

$$\begin{aligned}\dot{q}_i &= \{q_i, H\}, \\ \dot{p}_i &= \{p_i, H\}.\end{aligned}\tag{10.23}$$

(To obtain the relations in Eq. (10.22) we used the fact that the  $q_i$  and  $p_i$  are independent variables, and therefore the partial derivatives  $\frac{\partial q_i}{\partial q_j}$  and  $\frac{\partial p_i}{\partial p_j}$  vanish for  $i \neq j$ , and all  $\frac{\partial q_i}{\partial p_j}$  and  $\frac{\partial p_i}{\partial q_j}$  vanish as well.) In fact, if any physical observable  $\omega(q_i, p_i)$  does not depend explicitly on time, then it follows from the chain rule of differentiation and Eq. (10.19) that its time evolution will be given by

$$\frac{d\omega(q_i, p_i)}{dt} = \{\omega(q_i, p_i), H\}.\tag{10.24}$$

### 10.3.1 Infinitesimal Translations

Let us next consider an infinitesimal translation of the coordinates of the form

$$\begin{aligned}q_i &\longrightarrow q'_i = q_i + \epsilon_i, \\ p_i &\longrightarrow p'_i = p_i,\end{aligned}\tag{10.25}$$

where  $\epsilon_i$  are infinitesimal constant (arbitrary) parameters defining the translation. Equivalently, we can write the infinitesimal changes in the dynamical variables as

$$\begin{aligned}\delta_\epsilon q_i &= q'_i - q_i = \epsilon_i, \\ \delta_\epsilon p_i &= p'_i - p_i = 0,\end{aligned}\tag{10.26}$$

where the subscript  $\epsilon$  on  $\delta$  signifies that our transformation involves only changes in coordinates with parameters  $\epsilon_i$ .

If we define a function  $g(q_i, p_i)$  as

$$g = \sum_j \epsilon_j p_j,\tag{10.27}$$

we then obtain

$$\begin{aligned}\frac{\partial g}{\partial q_i} &= \frac{\partial(\sum_j \epsilon_j p_j)}{\partial q_i} = 0, \\ \frac{\partial g}{\partial p_i} &= \frac{\partial(\sum_j \epsilon_j p_j)}{\partial p_i} = \sum_j \epsilon_j \delta_{ij} = \epsilon_i.\end{aligned}\quad (10.28)$$

Thus, from the definition of the Poisson brackets in Eq. (10.20), we can write

$$\begin{aligned}\{q_i, g\} &= \sum_j \left( \frac{\partial q_i}{\partial q_j} \frac{\partial g}{\partial p_j} - \frac{\partial q_i}{\partial p_j} \frac{\partial g}{\partial q_j} \right) \\ &= \sum_j \delta_{ij} \epsilon_j = \epsilon_i = \delta_\epsilon q_i, \\ \{p_i, g\} &= \sum_j \left( \frac{\partial p_i}{\partial q_j} \frac{\partial g}{\partial p_j} - \frac{\partial p_i}{\partial p_j} \frac{\partial g}{\partial q_j} \right) = 0 = \delta_\epsilon p_i,\end{aligned}\quad (10.29)$$

where we have used the results of Eq. (10.26) to relate the Poisson brackets to infinitesimal changes in the dynamical variables. Now, using Eq. (10.20) or Eq. (10.21), we can also verify that the original and the transformed variables satisfy the same Poisson-bracket relations, namely,

$$\begin{aligned}\{q'_i, q'_j\} &= 0 = \{p'_i, p'_j\}, \\ \{q'_i, p'_j\} &= \delta_{ij}.\end{aligned}\quad (10.30)$$

In other words, the infinitesimal translations in Eq. (10.25) preserve the canonical Poisson-bracket structure, and are correspondingly known as canonical transformations.

Because the Hamiltonian is a function of the coordinates and momenta, its change under the transformation of Eq. (10.25) can be calculated using the chain rule of differentiation, as follows

$$\begin{aligned}
\delta_\epsilon H &= \sum_i \left( \frac{\partial H}{\partial q_i} \delta_\epsilon q_i + \frac{\partial H}{\partial p_i} \delta_\epsilon p_i \right) \\
&= \sum_i \frac{\partial H}{\partial q_i} \epsilon_i = \sum_i \left( \frac{\partial H}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \\
&= \{H, g\},
\end{aligned} \tag{10.31}$$

where, in the middle step, we used the results of Eq. (10.28). Now, if the Hamiltonian does not change under our infinitesimal translation, that is, if

$$\delta_\epsilon H = \{H, g\} = 0, \tag{10.32}$$

then we can write

$$H(q'_i, p'_i) = H(q_i, p_i). \tag{10.33}$$

Furthermore, since the Poisson brackets between the  $q_i$  and the  $p_i$  do not change, it follows that the transformed dynamical equations coincide with the original equations of (10.23)

$$\begin{aligned}
\dot{q}'_i &= \{q'_i, H(q'_j, p'_j)\} = \{q_i, H(q_j, p_j)\}, \\
\dot{p}'_i &= \{p'_i, H(q'_j, p'_j)\} = \{p_i, H(q_j, p_j)\}.
\end{aligned} \tag{10.34}$$

That is, Eqs. (10.34) represent the same motion as Eqs. (10.23). In fact, this reflects the very general result that when  $H$  does not change under an infinitesimal transformation, that transformation defines a symmetry of the dynamical equations of the system, or simply a symmetry of the physical system. Thus, in our example of Eq. (10.32), translations are a symmetry of the system because the Hamiltonian does not change under translations.

We see, from Eqs. (10.29) and (10.31), that the changes in  $q_i$ ,  $p_i$  and  $H$  under the infinitesimal translation of Eq. (10.25) can be obtained from their Poisson brackets with  $g$ . In fact, from the procedure in Eq. (10.31), it follows that the change in any observable can be obtained from its Poisson bracket with  $g$ . Thus, we can think of  $g$  as generating the infinitesimal translation, and, as a result,  $g$  is termed the *generator* of the infinitesimal transformations. We see from Eqs. (10.24), (10.32), and (10.27) that an infinitesimal translation is a symmetry of a physical system when

$$\begin{aligned} \frac{dg}{dt} &= \{g, H\} = 0, \\ \text{or } \frac{dp_i}{dt} &= \{p_i, H\} = 0. \end{aligned} \quad (10.35)$$

In other words, if translations are a symmetry of a system, then the momenta are conserved, and, conversely, if the momenta are conserved, then translations are a symmetry of the physical system. This is, of course, the same result that we obtained in Eq. (10.16) using the Lagrangian formalism.

### 10.3.2 Infinitesimal Rotations

Let us next consider rotations in two dimensions, and, in particular, finite spatial rotations by an angle  $\theta$  about the  $z$  axis. The transverse coordinates in the two systems can be related through the set of transformations

$$\begin{aligned} x' &= x \cos \theta - y \sin \theta, \\ y' &= x \sin \theta + y \cos \theta. \end{aligned} \quad (10.36)$$

For infinitesimal  $\theta$ , we can replace  $\cos \theta$  by  $1 - \frac{\theta^2}{2}$ , and  $\sin \theta$  by  $\theta$ , and, to first order in  $\theta$ , we can write the transformations as

$$\begin{aligned} x' &= x - \theta y, \\ y' &= \theta x + y. \end{aligned} \quad (10.37)$$

Using matrix notation, this can be rewritten as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & -\theta \\ \theta & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \quad (10.38)$$

Defining  $\delta_\theta x$  and  $\delta_\theta y$  as the infinitesimal changes in the  $x$  and  $y$  coordinates, we can write

$$\delta_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' - x \\ y' - y \end{pmatrix} = \theta \begin{pmatrix} -y \\ x \end{pmatrix} = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \quad (10.39)$$

Now, in terms of the generalized coordinates and momenta, we can write an infinitesimal rotation about the  $z$ -axis as

$$\begin{aligned} q_1 &\longrightarrow q'_1 = q_1 - \epsilon q_2, \\ q_2 &\longrightarrow q'_2 = q_2 + \epsilon q_1, \\ p_1 &\longrightarrow p'_1 = p_1 - \epsilon p_2, \\ p_2 &\longrightarrow p'_2 = p_2 + \epsilon p_1, \end{aligned} \tag{10.40}$$

where  $q_1$  and  $q_2$  can be thought of as  $x$  and  $y$ , and  $p_1$  and  $p_2$  as  $p_x$  and  $p_y$ . Furthermore, identifying the infinitesimal angle of rotation with  $\epsilon$ , we can write

$$\begin{aligned} \delta_\epsilon q_1 &= q'_1 - q_1 = -\epsilon q_2, \\ \delta_\epsilon q_2 &= q'_2 - q_2 = \epsilon q_1, \\ \delta_\epsilon p_1 &= p'_1 - p_1 = -\epsilon p_2, \\ \delta_\epsilon p_2 &= p'_2 - p_2 = \epsilon p_1, \end{aligned} \tag{10.41}$$

and returning to our matrix notation, using column vectors for the coordinates and momenta, we get

$$\begin{aligned} \delta_\epsilon \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} &= \epsilon \begin{pmatrix} -q_2 \\ q_1 \end{pmatrix} = \begin{pmatrix} 0 & -\epsilon \\ \epsilon & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \\ \delta_\epsilon \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} &= \epsilon \begin{pmatrix} -p_2 \\ p_1 \end{pmatrix} = \begin{pmatrix} 0 & -\epsilon \\ \epsilon & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \end{aligned}$$

which is similar to the transformations in Eq. (10.38). If we define a function  $g(q_i, p_i)$  as proportional to the third or  $z$ -component of orbital angular momentum, namely, proportional to  $(\vec{r} \times \vec{p})_z$ , as

$$g = \epsilon(q_1 p_2 - q_2 p_1) = \epsilon \ell_z, \tag{10.42}$$

then we have

$$\begin{aligned}
\frac{\partial g}{\partial q_1} &= \epsilon p_2, & \frac{\partial g}{\partial q_2} &= -\epsilon p_1, \\
\frac{\partial g}{\partial p_1} &= -\epsilon q_2, & \frac{\partial g}{\partial p_2} &= \epsilon q_1.
\end{aligned}
\tag{10.43}$$

Using the definition for Poisson brackets, we therefore obtain

$$\begin{aligned}
\{q_1, g\} &= \frac{\partial g}{\partial p_1} = -\epsilon q_2 = \delta_\epsilon q_1, \\
\{q_2, g\} &= \frac{\partial g}{\partial p_2} = \epsilon q_1 = \delta_\epsilon q_2, \\
\{p_1, g\} &= -\frac{\partial g}{\partial q_1} = \epsilon p_2 = \delta_\epsilon p_1, \\
\{p_2, g\} &= -\frac{\partial g}{\partial q_2} = \epsilon p_1 = \delta_\epsilon p_2.
\end{aligned}
\tag{10.44}$$

Once again, it can be shown that the Poisson brackets do not change under the transformation in Eq. (10.40), and the change in the Hamiltonian can be obtained, as before

$$\begin{aligned}
\delta_\epsilon H &= \sum_{i=1}^2 \left( \frac{\partial H}{\partial q_i} \delta_\epsilon q_i + \frac{\partial H}{\partial p_i} \delta_\epsilon p_i \right) \\
&= \sum_{i=1}^2 \left( \frac{\partial H}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \\
&= \{H, g\} = -\{g, H\},
\end{aligned}
\tag{10.45}$$

where we have used the results of Eqs. (10.44) and (10.20) in the preceding derivation. Thus, arguing as we did in Eqs. (10.32)–(10.34), we see again that rotations are a symmetry of the dynamical equations when the Hamiltonian is invariant under such rotations, namely if

$$\delta_\epsilon H = -\{g, H\} = 0.
\tag{10.46}$$

Through Eq. (10.24), this implies that



$$\{g, H\} = \frac{dg}{dt} = \epsilon \frac{d\ell_z}{dt} = 0. \quad (10.47)$$

We see therefore that, if rotations about the  $z$ -axis define a symmetry of the system, then the  $z$ -component of the orbital angular momentum is conserved. Conversely, whenever the  $z$ -component of the orbital angular momentum is conserved, the physical system is invariant under rotations about the  $z$ -axis.

We can show similarly that for any general infinitesimal transformation we can define a generator of that transformation, and a physical system is invariant under that transformation if the corresponding generator is conserved; and, conversely, if the generator of an infinitesimal transformation is conserved, that transformation represents a symmetry of the system.

## 10.4 Symmetries in Quantum Mechanics

The transition from classical mechanics to quantum mechanics is best described within the framework of the Hamiltonian formalism. In quantum mechanics, classical observables are represented by Hermitian operators, and the Poisson brackets are replaced by appropriate commutation relations. The classical generators of infinitesimal transformations therefore become operators that define symmetry transformations for operators as well as for vectors in Hilbert space. In quantum theory, such symmetry transformations can be implemented in one of the two equivalent ways, namely, either by transforming the state vectors in the Hilbert space or by transforming the operators that act on them. This is quite similar to the two ways that a classical transformation can be implemented, namely, as a passive or an active transformation.

In quantum mechanics, any observable quantity corresponds to the expectation value of a Hermitian operator in a given quantum state, and its time evolution – if the operator does not depend explicitly on time – is given by Ehrenfest's theorem (compare with Eq. (10.24))

$$\frac{d}{dt}\langle Q \rangle = \frac{1}{i\hbar}\langle [Q, H] \rangle = \frac{1}{i\hbar}\langle (QH - HQ) \rangle, \quad (10.48)$$

where we have denoted the expectation value of an operator  $Q$  in a state  $|\psi\rangle$  as

$$\langle Q \rangle = \langle \psi | Q | \psi \rangle. \quad (10.49)$$

It is clear therefore that an observable quantity that does not depend explicitly on time will be conserved if and only if the corresponding quantum operator commutes with the Hamiltonian. That is, for any quantum state, we will obtain

$$\frac{d}{dt} \langle Q \rangle = 0,$$

if and only if

$$[Q, H] = 0. \quad (10.50)$$

This is the quantum analog of Eqs. (10.35) and (10.47), and we conclude that the infinitesimal transformations generated by an operator  $Q$  define a symmetry of the theory when Eq. (10.50) holds; and, as a consequence of the symmetry, the expectation value of  $Q$  in any quantum state is independent of time (is conserved). Conversely, when an observable or the expectation value of  $Q$  in any quantum state is conserved (is constant in time), then  $Q$  generates a symmetry of the underlying physical system.

In quantum mechanics, when two operators commute, they can be diagonalized simultaneously, that is, they can have a complete set of common eigenfunctions. Thus, when the Hamiltonian has an underlying symmetry defined by the generator  $Q$ , the energy eigenstates are also eigenfunctions of the operator  $Q$ , and can also be labelled by the quantum numbers corresponding to the eigenvalues of  $Q$ . Furthermore, these quantum numbers are conserved in any physical process where the interaction Hamiltonian for some transition (e.g., decay or reaction) is invariant under the symmetry transformation. However, for transitions in which interaction Hamiltonians are not invariant under symmetry transformations, the corresponding quantum numbers do not have to be conserved. This provides an understanding of why some quantum numbers are conserved whereas others are violated in different interactions, and points to an essential first step in constructing physical theories of fundamental interactions.

As an example of quantum symmetries, let us return to translations. For simplicity, let us restrict ourselves to one dimension, and consider an infinitesimal translation of the  $x$ -coordinate by a constant amount  $\epsilon$ . We

will implement the transformation on the state vectors and not on the operators, although the inverse is equally straightforward. Thus, for  $x \rightarrow x' = x + \epsilon$ , with  $\epsilon$  real, our wave function corresponding to a given state vector changes as<sup>1</sup>

$$\psi(x) \longrightarrow \psi(x - \epsilon) = \psi(x) - \epsilon \frac{d\psi(x)}{dx} + O(\epsilon^2). \quad (10.51)$$

Consequently, under this transformation, the expectation value of the Hamiltonian changes as follows

$$\begin{aligned} \langle H \rangle &= \int_{-\infty}^{\infty} dx \psi^*(x) H(x) \psi(x) \\ &\longrightarrow \langle H \rangle' = \int_{-\infty}^{\infty} dx \psi^*(x - \epsilon) H(x) \psi(x - \epsilon) \\ &= \int_{-\infty}^{\infty} dx \psi^*(x) H(x) \psi(x) - \epsilon \int_{-\infty}^{\infty} dx \frac{d\psi^*}{dx} H(x) \psi(x) \\ &\quad - \epsilon \int_{-\infty}^{\infty} dx \psi^*(x) H(x) \frac{d\psi(x)}{dx} + O(\epsilon^2). \end{aligned}$$

We can integrate the middle term by parts to write

$$\int_{-\infty}^{\infty} dx \frac{d\psi^*(x)}{dx} H(x) \psi(x) = \int_{-\infty}^{\infty} dx \left[ \frac{d}{dx} (\psi^* H \psi) \right] - \int_{-\infty}^{\infty} dx \psi^* \frac{d}{dx} (H \psi).$$

By assumption, the wave functions vanish at infinity, and therefore the first term on the right-hand side also vanishes, and we obtain

$$\begin{aligned} \langle H \rangle' &= \langle H \rangle - \epsilon \int_{-\infty}^{\infty} dx \psi^*(x) \left( H \frac{d}{dx} - \frac{d}{dx} H \right) \psi(x) + O(\epsilon^2), \\ \text{or } \langle H \rangle' &= \langle H \rangle - \frac{i\epsilon}{\hbar} \langle [H, p_x] \rangle + O(\epsilon^2), \end{aligned} \quad (10.52)$$

where, in the last step, we have identified the momentum operator with the spatial derivative

<sup>1</sup>Note that for the translation  $x \rightarrow x + \epsilon$ , the corresponding change in the wave function is  $\psi(x) \rightarrow \psi(x - \epsilon)$ . (See any standard book on Quantum Mechanics.)

$$p_x \longrightarrow -i\hbar \frac{d}{dx}. \quad (10.53)$$

Comparing this analysis to Eqs. (10.27) and (10.31) shows that, to first order in  $\epsilon$ , the quantum generator of infinitesimal space translations  $G$  can be identified with the momentum operator, namely,

$$g = \epsilon G = -\frac{i\epsilon}{\hbar} p_x, \quad (10.54)$$

and that the Hamiltonian will be invariant under translations of the  $x$ -coordinate if

$$[p_x, H] = 0. \quad (10.55)$$

Furthermore, if Eq. (10.55) holds, then, through Ehrenfest's theorem,  $\langle p_x \rangle$  will be conserved. Clearly, the Hamiltonian for a free particle of mass  $m$  in one dimension possesses this type of invariance

$$H_{\text{free particle}} = \frac{p_x^2}{2m}. \quad (10.56)$$

As we know, the energy eigenstates of the quantum mechanical free-particle Hamiltonian are plane waves, which are also the eigenstates of the momentum operator.

## 10.5 Continuous Symmetries

Broadly speaking, all symmetry transformations of a theory can be classified into two categories: those that depend on a continuous set of parameters and those that correspond to some kind of reflection. Accordingly, they are known, respectively, as *continuous* and *discrete* transformations. All the examples of symmetry transformations that we have considered thus far in this chapter can be identified with continuous transformations, since they depend on an arbitrary parameter of the transformation (e.g.,  $\epsilon$ ). In the next chapter we will turn to discrete transformations, but we will first proceed further with our development of continuous symmetries.

It is only meaningful to speak about infinitesimal transformations when the transformations are continuous. In fact, for continuous transformations,

the infinitesimal transformation has fundamental importance because any finite transformation can be described in terms of a series of successive infinitesimal transformations. This can be shown as follows. We note from Eq. (10.51) that the effect of an infinitesimal translation along the  $x$ -axis on a state  $|\psi\rangle$  is given by the operator

$$U_x(\epsilon) = 1 - \frac{i\epsilon}{\hbar} p_x, \quad (10.57)$$

acting on the state  $|\psi\rangle$ .

The operator corresponding to a finite translation along the  $x$ -axis, namely  $U_x(\alpha)$ , where  $\alpha$  is no longer infinitesimal, can be obtained as follows. First, let us consider  $N$  successive infinitesimal translations by an amount  $\epsilon$  along the  $x$ -axis. This corresponds to a total translation by an amount  $N\epsilon$ , and the operator representing such a transformation corresponds merely to the product of  $N$  infinitesimal translations applied in succession

$$\begin{aligned} U_x(N\epsilon) &= \left(1 - \frac{i\epsilon}{\hbar} p_x\right) \left(1 - \frac{i\epsilon}{\hbar} p_x\right) \\ &\dots \left(1 - \frac{i\epsilon}{\hbar} p_x\right) = \left(1 - \frac{i\epsilon}{\hbar} p_x\right)^N. \end{aligned} \quad (10.58)$$

Since  $\epsilon$  is infinitesimal,  $N\epsilon$  is also infinitesimal for any finite  $N$ . However, if  $N$  is very large, then the product can be finite. Thus, let us define  $\alpha$  as the parameter of finite translation which is identified with  $\alpha = N\epsilon$  in the limit  $\epsilon \rightarrow 0$  and  $N \rightarrow \infty$ . We can consequently regard a finite translation as a series of an infinitely large number of successive infinitesimal translations. It then follows from Eq. (10.58) that the operator corresponding to a finite translation is given by

$$\begin{aligned} U_x(\alpha) &= \lim_{\substack{N \rightarrow \infty \\ \epsilon \rightarrow 0 \\ N\epsilon = \alpha}} \left(1 - \frac{i\epsilon}{\hbar} p_x\right)^N \\ &= \lim_{\substack{N \rightarrow \infty \\ \epsilon \rightarrow 0 \\ N\epsilon = \alpha}} \left(1 - \frac{i\alpha}{N\hbar} p_x\right)^N = e^{-\frac{i}{\hbar} \alpha p_x}. \end{aligned} \quad (10.59)$$

We see therefore that the operator for finite transformations is obtained simply by exponentiating the generators of infinitesimal transformations.

(Clearly, similar expressions hold for finite translations along other axes.)

Symmetry transformations normally define what is known as a *group*. (See Appendix D for some basics on Group Theory.) Thus, for example, two successive translations can also be thought of as a single translation. Similarly, two successive rotations define a rotation. The rules for combining two transformations (otherwise also known as the group properties of the transformation) are completely determined by the commutation relations (or the *algebra*) of the generators of the transformation. Thus, for translations along the  $x$ -axis, we saw that the generators correspond to the commuting momentum operators that satisfy

$$[p_x, p_x] = 0. \quad (10.60)$$

In fact, all momentum operators (along different axes) commute with one another

$$[p_i, p_j] = 0, \quad i, j = x, y, \text{ or } z. \quad (10.61)$$

For obvious reasons, such an algebra is known as a *commutative* or Abelian (after Niels Abel) algebra. A consequence of Eqs. (10.59)–(10.61) is that

$$\begin{aligned} U_j(\alpha)U_k(\beta) &= e^{-\frac{i}{\hbar}\alpha p_j} e^{-\frac{i}{\hbar}\beta p_k} \\ &= e^{-\frac{i}{\hbar}\beta p_k} e^{-\frac{i}{\hbar}\alpha p_j} \\ &= U_k(\beta)U_j(\alpha), \quad k, j = x, y, z, \end{aligned} \quad (10.62)$$

and

$$\begin{aligned} U_x(\alpha)U_x(\beta) &= e^{-\frac{i}{\hbar}\alpha p_x} e^{-\frac{i}{\hbar}\beta p_x} \\ &= e^{-\frac{i}{\hbar}(\alpha + \beta)p_x} = U_x(\alpha + \beta) = U_x(\beta)U_x(\alpha). \end{aligned} \quad (10.63)$$

Namely, translations form what is referred to as a commutative or an Abelian group. The order of the two translations is not relevant. However, not all symmetry transformations have this property. As we know, infinitesimal rotations in quantum mechanics are generated by angular momentum operators (even classically, angular momenta generate rotations through Poisson-bracket relations)

$$\begin{aligned}
 L_1 &= x_2 p_3 - x_3 p_2, \\
 L_2 &= x_3 p_1 - x_1 p_3, \\
 L_3 &= x_1 p_2 - x_2 p_1,
 \end{aligned}
 \tag{10.64}$$

which satisfy the following quantum algebra (commutation relations)

$$[L_j, L_k] = \sum_{\ell} i\hbar \epsilon_{j k \ell} L_{\ell}, \quad j, k, \ell = 1, 2, 3,
 \tag{10.65}$$

where  $\epsilon_{j k \ell}$  is the totally anti-symmetric Levi-Civita tensor, which equals 1 if the  $j, k, \ell$  combination is cyclical,  $-1$  if it is not cyclical, and 0 if any two indices repeat. Equations (10.65) define the simplest *non-commutative* algebra (the generators do not commute), otherwise known as a non-Abelian algebra. A consequence of this non-commutative property is that the group of rotations behaves quite differently from translations. In particular, unlike translations along two different directions, rotations about two different axes do not commute. The order of rotations is important.

The group of spatial rotations in three dimensions [known as  $SO(3)$ ] has an algebraic structure very similar to that of the  $SU(2)$  group, which is a group relevant to certain internal symmetries, and is characterized by the properties of  $2 \times 2$  unitary matrices that have determinants equal to unity. As described below, the  $SU(2)$  group of transformations rotates state vectors in Hilbert space in a manner akin to spatial rotations.

States of a quantum mechanical system are defined by vectors in an abstract Hilbert space. And, just as normal vectors can be rotated in configuration (coordinate) space, so can the vectors corresponding to quantum mechanical states be rotated in an internal Hilbert space. Thus continuous symmetries for a quantum system can be associated either with space-time symmetry transformations or with internal symmetry transformations. The transformations in the internal Hilbert space do not affect the space-time coordinates, and consequently in all such transformations the space-time coordinates are kept fixed. Thus, if we consider a two-level system, where the two basic states are represented by the column vectors

$$\begin{pmatrix} \psi_1(x) \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ \psi_2(x) \end{pmatrix},$$

then a general rotation in the internal space of this two dimensional system can be represented as

$$\delta \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} = - \sum_{j=1}^3 i\epsilon_j \frac{\sigma_j}{2} \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}, \quad (10.66)$$

where the  $\sigma_j$  are the  $2 \times 2$  Pauli spin matrices, the infinitesimal generators of  $SU(2)$ , and defined as usual

$$I_j = \frac{\sigma_j}{2}, \quad j = 1, 2, 3,$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (10.67)$$

From the properties of the Pauli matrices, the  $I_j$  can be shown to satisfy the same algebra as satisfied by the angular momentum operators in Eq. (10.65).<sup>2</sup> In analogy with angular momentum, we can label the two states according to the eigenvalues of, say, the  $I_3$  operator. In fact, the states  $\begin{pmatrix} \psi_1(x) \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 \\ \psi_2(x) \end{pmatrix}$  are eigenstates of  $I_3$  with the eigenvalues  $\pm \frac{1}{2}$ . (Recall that, because the  $I_j$  do not commute, only one of them can be diagonal.) These two states will, of course, be degenerate in energy if such a rotation corresponds to a symmetry of the system. (Once again, this is very much like the situation with the spin up and the spin down states, which are degenerate in energy for a rotationally invariant system.) The strong isospin transformations that we discussed in the previous chapter correspond to such internal rotations, and the degeneracy of the proton and the neutron mass can be thought of as a consequence of invariance of the strong-interaction Hamiltonian under such a symmetry transformation. In general, if the Hamiltonian for any system is invariant under this kind of an internal rotation, then there will be a conserved quantum number.

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<sup>2</sup>Note that if we restrict the rotations to where  $\epsilon_1 = 0$ ,  $\epsilon_3 = 0$ , and  $\epsilon_2 = \epsilon$ , then these internal rotations take the form

$$\delta \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} = \frac{\epsilon}{2} \begin{pmatrix} -\psi_2(x) \\ \psi_1(x) \end{pmatrix} \quad (10.68)$$

which, when compared with Eq. (10.39), shows that the two components of the wave function can be thought of as coordinates in the internal space, and heightens the analogy with rotations in ordinary space.



### 10.5.1 Isotopic Spin

To bring the preceding formalism into focus, we will expand somewhat on the application of these ideas to isospin. If there is an isospin symmetry, then the implication is that our spin up proton ( $p$ ) with  $I_3 = \frac{1}{2}$  and our spin down neutron ( $n$ ) with  $I_3 = -\frac{1}{2}$  are indistinguishable. (We denote by  $I_3$  the quantum number associated with the projection of isospin, namely the eigenvalue of the  $I_3$ -operator.) We can consequently define a new neutron and proton state as some linear superposition of the  $|p\rangle$  and  $|n\rangle$  vectors. We note from Eq. (10.68) (compare with Eqs. (10.36) and (10.39)) that a finite rotation of our vectors in isospin space by an arbitrary angle  $\theta$  about the  $I_2$  axis leads to a set of transformed vectors  $|p'\rangle$  and  $|n'\rangle$

$$\begin{aligned} |p'\rangle &= \cos \frac{\theta}{2} |p\rangle - \sin \frac{\theta}{2} |n\rangle, \\ |n'\rangle &= \sin \frac{\theta}{2} |p\rangle + \cos \frac{\theta}{2} |n\rangle. \end{aligned} \quad (10.69)$$

Now, let us see what such an invariance implies about the nucleon-nucleon interaction. Our two-nucleon quantum states in the Hilbert space can be written in terms of the more fundamental states, which are either symmetric or antisymmetric under an exchange of particles. These correspond to the following four states

$$\begin{aligned} |\psi_1\rangle &= |pp\rangle, \quad |\psi_2\rangle = \frac{1}{\sqrt{2}} (|pn\rangle + |np\rangle), \\ |\psi_3\rangle &= |nn\rangle, \quad |\psi_4\rangle = \frac{1}{\sqrt{2}} (|pn\rangle - |np\rangle). \end{aligned} \quad (10.70)$$

Assuming, as in the case of normal spin, that  $I_3$  is an additive quantum number, we can identify the isospin projections of  $|\psi_1\rangle$  with  $I_3 = +1$ ,  $|\psi_2\rangle$ ,  $|\psi_4\rangle$  with  $I_3 = 0$ , and  $|\psi_3\rangle$  with  $I_3 = -1$ . Let us now see what is the impact of the isospin transformation, for example, on  $|\psi_1\rangle$  and on  $|\psi_4\rangle$ . Under the rotation given in Eq. (10.69), these states will transform schematically as

$$\begin{aligned}
|\psi'_1\rangle &= \left| \left( \cos \frac{\theta}{2} p - \sin \frac{\theta}{2} n \right) \left( \cos \frac{\theta}{2} p - \sin \frac{\theta}{2} n \right) \right\rangle \\
&= \cos^2 \frac{\theta}{2} |pp\rangle - \cos \frac{\theta}{2} \sin \frac{\theta}{2} (|pn\rangle + |np\rangle) + \sin^2 \frac{\theta}{2} |nn\rangle \\
&= \cos^2 \frac{\theta}{2} |\psi_1\rangle - \frac{1}{\sqrt{2}} \sin \theta |\psi_2\rangle + \sin^2 \frac{\theta}{2} |\psi_3\rangle, \tag{10.71}
\end{aligned}$$

$$\begin{aligned}
|\psi'_4\rangle &= \frac{1}{\sqrt{2}} \left( \left| \left( \cos \frac{\theta}{2} p - \sin \frac{\theta}{2} n \right) \left( \sin \frac{\theta}{2} p + \cos \frac{\theta}{2} n \right) \right\rangle \right. \\
&\quad \left. - \left| \left( \sin \frac{\theta}{2} p + \cos \frac{\theta}{2} n \right) \left( \cos \frac{\theta}{2} p - \sin \frac{\theta}{2} n \right) \right\rangle \right) \\
&= \frac{1}{\sqrt{2}} \left( \cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2} \right) (|pn\rangle - |np\rangle) = |\psi_4\rangle. \tag{10.72}
\end{aligned}$$

We see, therefore, that  $|\psi_4\rangle$  is totally insensitive to rotations in this space. It must consequently correspond to a scalar (or a “singlet”) combination, and represent the  $I = 0, I_3 = 0$  nucleon-nucleon system. We can also calculate the changes in the states  $|\psi_2\rangle$  and  $|\psi_3\rangle$  under the above rotation, and show that the remaining states and  $|\psi_1\rangle$  transform into one another under the isospin rotation, just as the three components of a vector do under a spatial rotation. If there is isospin invariance in the nucleon-nucleon strong interaction, then it follows that the three states  $|\psi_1\rangle, |\psi_2\rangle,$  and  $|\psi_3\rangle,$  corresponding to  $I_3 = 1, 0,$  and  $-1,$  respectively, are equivalent and cannot be distinguished from each other. Consequently, it appears that any two-nucleon system can be classified either as an  $I = 0$  singlet or an  $I = 1$  triplet in isotopic spin space. The singlet and the three triplet states are independent of each other, and the three substates of  $I = 1$  are indistinguishable if isospin is a symmetry of the system, that is, if the nucleon-nucleon strong interaction is not sensitive to the replacement of a neutron by a proton. Any breaking of the degeneracy of the  $I = 1$  states must arise from other interactions (e.g., electromagnetic contribution to the Hamiltonian, which can depend upon the electric charge of the system).

Similarly, we can form three-nucleon systems, that correspond to two doublets with  $I = \frac{1}{2},$  and an independent quartet of states with  $I = \frac{3}{2},$  etc. In this manner we can build multiplets in isotopic spin space in the same way as we combine angular momentum states.

One important application of isospin invariance lies in the calculation

of relative transition rates in decays or in interactions. As an example, let us examine how we might calculate the decay of the  $\Delta(1232)$  into a pion and a nucleon. As we mentioned in Chapter 9, the  $\Delta(1232)$  is a  $\pi$ - $N$  resonance that was discovered by Fermi and his colleagues in the scattering of  $\pi$  mesons from nucleons. There are four members of the  $\Delta(1232)$  family, which correspond to four charged states and to four  $I_3$  projections of an  $I = \frac{3}{2}$  isospin multiplet. To calculate the relative decay rates of the  $\Delta(1232)$  into a pion and a nucleon, we must recognize that, if there is isospin symmetry, then the total rate for  $\Delta^{++}(1232)$ ,  $\Delta^+(1232)$ ,  $\Delta^0(1232)$  and for  $\Delta^-(1232)$  must be identical because these members of the multiplet cannot be distinguished from each other on the basis of the strong interaction. In addition, under the transformation of Eq. (10.69),  $p$  and  $n$ , as well as  $\pi^+$ ,  $\pi^0$  and  $\pi^-$ , transform into one another, and cannot be told apart. We must assume therefore that the total rate for  $\Delta(1232)$  to decay with a neutron in the final state must equal that for a proton in the final state, and similarly for the three pions. We can therefore form a table in which we list all the possible initial  $\Delta(1232)$  members and all possible  $\pi$  and  $N$  combinations that do not violate charge conservation, and impose the above requirements of charge symmetry, or isospin invariance, for the strong decay. This is shown below in Table 10.2.

**Table 10.2** Transition rates for  $\Delta \rightarrow \pi N$ , assuming isospin symmetry in the decay.

<i>Charge State of <math>\Delta</math></i>	<i><math>I_3</math></i>	<i>Final State</i>	<i>Expected Rate</i>	<i>Solution</i>
$\Delta^{++}$	$\frac{3}{2}$	$p\pi^+$	1	1
$\Delta^+$	$\frac{1}{2}$	$p\pi^0$	$x$	$\frac{2}{3}$
		$n\pi^+$	$1 - x$	$\frac{1}{3}$
$\Delta^0$	$-\frac{1}{2}$	$p\pi^-$	$y$	$\frac{1}{3}$
		$n\pi^0$	$1 - y$	$\frac{2}{3}$
$\Delta^-$	$-\frac{3}{2}$	$n\pi^-$	1	1

By requiring that the sum of the  $p\pi^0$  and  $n\pi^+$  rates in  $\Delta^+$  decay, as well as the sum of the  $p\pi^-$  and  $n\pi^0$  rates in  $\Delta^0$  decay, add up to 1, we have

assured that all the  $\Delta$  members are equivalent in the sense that their total transition rates are the same, which we have for simplicity normalized to unity. Now, we impose the requirement that rates for decays involving a  $p$  or  $n$  in the final state are the same, namely,

$$1 + x + y = (1 - x) + (1 - y) + 1, \quad (10.73)$$

where  $x$ ,  $y$  are defined in Table 10.2. In addition, we stipulate that the final rates for decays involving a  $\pi^+$ ,  $\pi^0$  or  $\pi^-$  be identical

$$1 + (1 - x) = x + (1 - y) = y + 1. \quad (10.74)$$

We have more equations than unknowns, but there is a unique set of consistent solutions, which is given in Table 10.2. The result indicates that, for example, the  $\Delta^+(1232)$  will decay twice as often into  $p + \pi^0$  as into  $n + \pi^+$ , and that the  $\Delta^0(1232)$  will decay twice as often into  $n + \pi^0$  as into  $p + \pi^-$ , and so forth. These relative rates are purely a consequence of isospin symmetry. The fact that these transition rates agree with data, suggests that isotopic spin is a symmetry of the strong interaction, and that both  $I$  and  $I_3$  are conserved in strong processes. The solution we have just calculated could also have been obtained simply from tables of Clebsch-Gordan coefficients, which give the coupling of angular momenta. Our example, however, provides an instructive alternative procedure, one which has been emphasized by Robert Adair and Ilya M. Shmushkevich.

## 10.6 Local Symmetries

Whether they are space-time or internal, continuous symmetries can also be classified in two ways. First, the parameters of transformation can be constants, that is, universal parameters, implying that the transformation is the same at all space-time points, and this type of symmetry transformation is known as *global*. All the continuous transformations that we have considered thus far fall into this category, and, as we have seen, invariance of a theory under such transformations provides conserved charges (quantum numbers). In contrast, if the parameters of transformation depend on the space-time coordinates – namely, if the magnitude of the transformation is different from point to point – then the symmetry transformation is known as a *local* transformation. In this case, real forces must be introduced to

maintain the symmetry. As example, let us consider the time-independent Schrödinger equation

$$H\psi(\vec{r}) = \left( -\frac{\hbar^2}{2m} \vec{\nabla}^2 + V(\vec{r}) \right) \psi(\vec{r}) = E\psi(\vec{r}). \quad (10.75)$$

Clearly, if  $\psi(\vec{r})$  is a solution of this equation, then so is  $e^{i\alpha}\psi(\vec{r})$ , where  $\alpha$  is a constant parameter. In other words, any quantum mechanical wave function can be defined only up to a constant phase, and therefore a transformation involving a constant phase is a symmetry of any quantum mechanical system. This kind of transformation conserves probability density, and, in fact, the conservation of electric charge can be associated with just such a global phase transformation.

Consider next a local phase transformation

$$\psi(\vec{r}) \longrightarrow e^{i\alpha(\vec{r})}\psi(\vec{r}), \quad (10.76)$$

where the phase depends explicitly on the space coordinate, so that the wave function at every point in space has a different phase. (We wish to emphasize that we are not considering a change in space-time coordinates, but rather that the parameter of the phase transformation is different for different coordinate points.) Now, under the local phase transformation of Eq. (10.76), the gradient introduces an inhomogeneous term

$$\vec{\nabla} \left[ e^{i\alpha(\vec{r})}\psi(\vec{r}) \right] = e^{i\alpha(\vec{r})} \left[ i(\vec{\nabla}\alpha(\vec{r}))\psi(\vec{r}) + \vec{\nabla}\psi(\vec{r}) \right] \neq e^{i\alpha(\vec{r})}\vec{\nabla}\psi(\vec{r}). \quad (10.77)$$

Consequently, since the right-hand side of Eq. (10.75) remains homogeneous under the transformation of Eq. (10.76) while the left-hand side does not, we see that the Schrödinger equation cannot in general be invariant under a local phase transformation.

However, the transformation of Eq. (10.76) can be made a symmetry of the Schrödinger equation if an arbitrarily modified gradient operator is introduced as

$$\vec{\nabla} \longrightarrow \vec{\nabla} - i\vec{A}(\vec{r}). \quad (10.78)$$

Now, requiring the vector potential  $\vec{A}(\vec{r})$  to change under the transformation of Eq. (10.76) as

$$\vec{A}(\vec{r}) \longrightarrow \vec{A}(\vec{r}) + \vec{\nabla}\alpha(\vec{r}), \quad (10.79)$$

will cancel the inhomogeneous term in Eq. (10.77), and under the combined change we will have

$$\begin{aligned} (\vec{\nabla} - i\vec{A}(\vec{r}))\psi(\vec{r}) &\longrightarrow (\vec{\nabla} - i\vec{A}(\vec{r}) - i(\vec{\nabla}\alpha(\vec{r})))\left(e^{i\alpha(\vec{r})}\psi(\vec{r})\right) \\ &= e^{i\alpha(\vec{r})}(\vec{\nabla} - i\vec{A}(\vec{r}))\psi(\vec{r}), \end{aligned} \quad (10.80)$$

which means that the local phase transformation in Eq. (10.76) will be a symmetry of the modified time-independent Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}(\vec{\nabla} - i\vec{A}(\vec{r}))^2 + V(\vec{r})\right)\psi(\vec{r}) = E\psi(\vec{r}), \quad (10.81)$$

provided we require the added vector potential to transform as given in Eq. (10.79). We recognize Eq. (10.79) as a gauge transformation similar to that found in Maxwell's equations, and note that invariance under a local phase transformation requires the introduction of additional fields. These are known as *gauge fields* (in the present case,  $\vec{A}(\vec{r})$  can be interpreted as the electromagnetic vector potential), and lead to the introduction of definite physical forces. The symmetry group associated with the single-parameter phase transformations of Eq. (10.76) is Abelian (commuting symmetry), referred to as a  $U(1)$  group. (These ideas will be discussed further in Chapter 13.)

Although we have discussed all these results in the context of a simple local phase symmetry, the general conclusion, namely that additional fields must be introduced in order to get a local symmetry, holds as well for more complicated symmetries. This observation is vital for constructing modern physical theories. In particular, we can turn the argument around, and suggest that the distinct fundamental forces in nature arise from local invariances of physical theories, and that the associated gauge fields generate those forces. This idea is referred to as the *gauge principle*, and such theories as *gauge theories*, and together they provide our current understanding of the fundamental interactions.

## Problems

**10.1** Using isotopic spin decomposition for the decays of the  $\rho$  meson with  $I = 1$ :  $\rho^+ \rightarrow \pi^+\pi^0$ ,  $\rho^- \rightarrow \pi^-\pi^0$ ,  $\rho^0 \rightarrow \pi^+\pi^-$  and  $\rho^0 \rightarrow \pi^0\pi^0$ , prove that  $\rho^0 \rightarrow \pi^0\pi^0$  is forbidden on the basis of isospin invariance (that is, use the Adair–Shmushkevich analysis).

**10.2** Assuming invariance of strong interactions under rotations in isotopic-spin space and the usual isospin assignments for  $K$  and  $\pi$  mesons, what would you predict for the ratios of transition rates in the following decays:

(a) For an  $I = \frac{3}{2}$ ,  $K^*$  meson,

$$\frac{K^{*++} \rightarrow K^+\pi^+}{K^{*+} \rightarrow K^+\pi^0}, \quad \frac{K^{*+} \rightarrow K^+\pi^0}{K^{*+} \rightarrow K^0\pi^+}, \quad \frac{K^{*-} \rightarrow K^0\pi^-}{K^{*0} \rightarrow K^+\pi^-}.$$

(b) What would you expect for the above processes if the  $K^*$  meson had  $I = \frac{1}{2}$ ? (*Hint*: Consider the  $I_3$  of the final states.)

**10.3**  $N^*$  baryons are  $I = \frac{1}{2}$  excited states of the nucleon. On the basis of isospin invariance in strong interactions, compare the differences expected for  $N^*$  and  $\Delta$  decays into the  $\pi$ - $N$  systems discussed in Table 10.2.

**10.4** What are the possible values of isotopic spin for the following systems?

(a) A  $\pi^+$  meson and an antiproton, (b) two neutrons, (c) a  $\pi^+$  meson and a  $\Lambda^0$ , (d) a  $\pi^+$  and a  $\pi^0$  meson, (e) a  $u$  and a  $\bar{u}$  quark, (f) a  $c$ ,  $b$  and an  $s$  quark (for properties of quarks, see Table 9.5).

## Suggested Readings

Frauenfelder, H., and E. M. Henley, *Subatomic Physics*, Prentice-Hall (1991).

Goldstein, H., *Classical Mechanics*, Addison-Wesley (1980).

Griffiths, D., *Introduction to Elementary Particles*, Wiley (1987).

Perkins, D. H., *Introduction to High Energy Physics*, Cambridge Univ. Press (2000).

Sakurai, J. J., *Invariance Principles and Elementary Particles*, Princeton Univ. Press (1964).

Williams, W. S. C., *Nuclear and Particle Physics*, Oxford Univ. Press (1997).

Also, see standard texts on quantum mechanics, e.g., Das, A. and A. C. Melissinos, *Quantum Mechanics*, Gordon & Breach (1986).

## Chapter 11

# Discrete Transformations

### 11.1 Introductory Remarks

Any set of transformations – either involving space-time or some internal space – can be best understood when described in terms of a change in reference frame. Continuous as well as discrete transformations can be discussed within this kind of framework. The previous chapter dealt with continuous symmetries, and we now turn to discrete transformations.

### 11.2 Parity

As mentioned in previous chapters, parity, otherwise known as space inversion, is a transformation that takes us from a right handed coordinate frame to a left handed one, or vice versa. Under this transformation, which we denote by the symbol  $P$ , the space-time four-vector changes as follows:

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \xrightarrow{P} \begin{pmatrix} ct \\ -x \\ -y \\ -z \end{pmatrix}. \quad (11.1)$$

It is important to recognize that the parity operation is distinct from spatial rotations because a left handed coordinate system cannot be obtained from a right handed one through any combination of rotations. In fact, rotations define a set of continuous transformations, whereas the inversion of space coordinates does not. It is clear therefore that the quantum numbers corresponding to rotations and parity are distinct.



Classically, the components of position and momentum vectors change sign under inversion of coordinates, while their magnitudes are preserved

$$\begin{aligned}
 \vec{r} &\xrightarrow{P} -\vec{r}, \\
 \vec{p} = m\dot{\vec{r}} &\xrightarrow{P} -m\dot{\vec{r}} = -\vec{p}, \\
 r = (\vec{r} \cdot \vec{r})^{\frac{1}{2}} &\xrightarrow{P} [(-\vec{r}) \cdot (-\vec{r})]^{\frac{1}{2}} = (\vec{r} \cdot \vec{r})^{\frac{1}{2}} = r, \\
 p = (\vec{p} \cdot \vec{p})^{\frac{1}{2}} &\xrightarrow{P} [(-\vec{p}) \cdot (-\vec{p})]^{\frac{1}{2}} = (\vec{p} \cdot \vec{p})^{\frac{1}{2}} = p.
 \end{aligned} \tag{11.2}$$

This defines the behavior of normal scalar and vector quantities under space inversion. There are, however, scalar and vector quantities that do not transform under parity as shown in Eq. (11.2). Thus, for example, the orbital angular momentum, which changes like a vector under a rotation of coordinates, and which we therefore regard as a vector, behaves under space inversion as

$$\vec{L} = \vec{r} \times \vec{p} \xrightarrow{P} (-\vec{r}) \times (-\vec{p}) = \vec{r} \times \vec{p} = \vec{L}. \tag{11.3}$$

This is, in fact, just opposite of how a normal vector transforms. Such vectors are consequently called *pseudovectors* or *axial vectors*. Similarly, there exists a class of scalars, for example, the volume of a parallelepiped, that transform oppositely from normal scalars

$$\vec{a} \cdot (\vec{b} \times \vec{c}) \xrightarrow{P} (-\vec{a}) \cdot (-\vec{b} \times -\vec{c}) = -\vec{a} \cdot (\vec{b} \times \vec{c}). \tag{11.4}$$

Such quantities are known as *pseudoscalars*. Of course, any type of vector can be labeled by one index (namely, by its components). There are also more complex objects in physics that require more indices, and are known as *tensors*. The quadrupole moment, the (energy-momentum) stress tensor, and the relativistic electromagnetic field strength  $F_{\mu\nu}$ , are examples of second rank tensors (objects with two indices).

An important property of the parity operation is that two successive parity transformations leave the coordinate system unchanged, namely,

$$\vec{r} \xrightarrow{P} -\vec{r} \xrightarrow{P} \vec{r}. \tag{11.5}$$

If we think of  $P$  as representing the operator implementing a parity transformation, then from Eq. (11.5) we conclude that

$$P^2|\psi\rangle = +1|\psi\rangle. \quad (11.6)$$

The eigenvalues of the parity operator can therefore be only  $\pm 1$ . If we have a parity invariant theory, namely, a theory whose Hamiltonian  $H$  is invariant under inversion of coordinates, then, as discussed before,  $P$  commutes with  $H$ ,

$$[P, H] = 0. \quad (11.7)$$

When  $P$  and  $H$  commute, the eigenstates of the Hamiltonian are also eigenstates of  $P$ , with eigenvalues of either  $+1$  or  $-1$ . Because a wave function transforms under  $P$  as

$$\psi(\vec{r}) \xrightarrow{P} \psi(-\vec{r}), \quad (11.8)$$

this implies that the stationary states of any Hamiltonian invariant under a parity transformation have definite parity, and can be classified as either even or odd functions. As example, consider the one-dimensional harmonic oscillator, whose Hamiltonian is parity invariant

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 \xrightarrow{P} \frac{(-p)^2}{2m} + \frac{1}{2}m\omega^2(-x)^2 = H, \quad (11.9)$$

and, as we know, the energy eigenstates of the oscillator are Hermite polynomials, which are either even or odd functions of  $x$ , but never a mixture of odd and even functions.

Consider next a rotationally invariant system in three dimensions. As we noted in Chapter 10, the energy eigenstates in this case are also eigenstates of the angular momentum operator. The wave function for the system can be written as

$$\psi_{n\ell m}(\vec{r}) = R_{n\ell}(r)Y_{\ell m}(\theta, \phi), \quad (11.10)$$

where the  $Y_{\ell m}(\theta, \phi)$  are the spherical harmonics, discussed previously in Chapter 3. The parity transformation in spherical coordinates takes the form

$$\begin{aligned}
 r &\xrightarrow{P} r, \\
 \theta &\xrightarrow{P} \pi - \theta, \\
 \phi &\xrightarrow{P} \pi + \phi,
 \end{aligned}
 \tag{11.11}$$

and under this transformation, the spherical harmonics behave as

$$Y_{\ell m}(\theta, \phi) \xrightarrow{P} Y_{\ell m}(\pi - \theta, \pi + \phi) = (-1)^\ell Y_{\ell m}(\theta, \phi). \tag{11.12}$$

Consequently, parity transforms any wave function that is an eigenstate of orbital angular momentum as

$$\psi_{n\ell m}(\vec{r}) \xrightarrow{P} (-1)^\ell \psi_{n\ell m}(\vec{r}). \tag{11.13}$$

In general, a quantum mechanical wave function can have, in addition, an *intrinsic parity* or phase that is independent of its spatial transformation property of Eq. (11.13), and, correspondingly, a general quantum state that is described by eigenfunctions of orbital angular momentum will transform under parity as

$$\psi_{n\ell m}(\vec{r}) \xrightarrow{P} \eta_\psi (-1)^\ell \psi_{n\ell m}(\vec{r}), \tag{11.14}$$

where  $\eta_\psi$  is the intrinsic parity of the quantum state. We can think of the intrinsic parity as the phase analog of intrinsic spin, which when added to the orbital angular momentum yields the total angular momentum of a system. As a consequence of Eq. (11.6), the intrinsic parity satisfies the condition

$$\eta_\psi^2 = 1. \tag{11.15}$$

We can therefore define a total parity of any such quantum mechanical state as

$$\eta_{\text{TOT}} = \eta_\psi (-1)^\ell. \tag{11.16}$$

A detailed analysis of relativistic quantum theories reveals that bosons have the same intrinsic parities as their antiparticles, whereas the relative intrinsic parity of fermions and their antiparticles is odd (opposite).

The classical Newton's equation of motion for a point particle has the form

$$m \frac{d^2 \vec{r}}{dt^2} = \vec{F}. \quad (11.17)$$

If we assume the force  $\vec{F}$  to be either electromagnetic or gravitational, we can write

$$\vec{F} = \frac{C}{r^2} \hat{r}, \quad (11.18)$$

where  $C$  is a constant. Clearly, since under inversion of coordinates both the left-hand side of Eq. (11.17) and the right-hand side of Eq. (11.18) change sign, Newton's equation for electromagnetic or gravitational interactions is therefore invariant under space inversion. It can be shown in a similar fashion that Maxwell's equations are also invariant under a parity transformation.

### 11.2.1 Conservation of Parity

When parity is a good symmetry, then the intrinsic parities of different particles can be determined by analyzing different decay or production processes, as will be shown in the examples. It should be recognized, however, that it is not possible to determine an absolute parity of any system because, starting with some set of assignments, we can invert the parities of all states without observing a physical consequence of that change. This is similar, for example, to defining the absolute sign of electric charge or other quantum numbers. A convention is needed to define intrinsic parities of objects that differ in some fundamental way – either through their electric charge, strangeness, or other characteristics. The accepted convention is to choose the intrinsic parities of the proton, the neutron and the  $\Lambda$  hyperon as +1. The parities of other particles relative to these assignments can be obtained through the analysis of parity-conserving interactions involving such particles.

When parity is conserved, it then restricts the kind of decay processes that can take place. Let us consider, for example, particle  $A$  decaying in

its rest frame into particles  $B$  and  $C$

$$A \longrightarrow B + C. \quad (11.19)$$

If  $J$  denotes the spin of the decaying particle, then conservation of angular momentum requires that the total angular momentum of the final state also be  $J$ . In particular, if the two decay products are spinless, then their relative orbital angular momentum ( $\ell$ ) must equal the spin of  $A$ ,

$$\ell = J. \quad (11.20)$$

Conservation of parity in the decay then implies that

$$\eta_A = \eta_B \eta_C (-1)^\ell = \eta_B \eta_C (-1)^J. \quad (11.21)$$

If the decaying particle has spin-zero, then for the process in Eq. (11.19) to take place we must have

$$\eta_A = \eta_B \eta_C. \quad (11.22)$$

Hence, the allowed decays correspond to

$$\begin{aligned} 0^+ &\longrightarrow 0^+ + 0^+, \\ 0^+ &\longrightarrow 0^- + 0^-, \\ 0^- &\longrightarrow 0^+ + 0^-, \end{aligned} \quad (11.23)$$

with  $J^P = 0^+$  (or  $0^-$ ) representing the standard convention for labeling a spin and intrinsic parity of a particle. It also follows that certain decays are forbidden because they violate parity conservation, e.g.,

$$\begin{aligned} 0^+ &\not\rightarrow 0^+ + 0^-, \\ 0^- &\not\rightarrow 0^+ + 0^+, \\ 0^- &\not\rightarrow 0^- + 0^-. \end{aligned} \quad (11.24)$$

**Example 1: Parity of  $\pi^-$  Meson**

Consider the absorption of very low-energy (or, as is usually termed, “stopping”)  $\pi^-$  mesons on deuterium nuclei

$$\pi^- + d \longrightarrow n + n. \quad (11.25)$$

If  $\ell_i$  and  $\ell_f$  denote the orbital angular momenta in the initial and final states, respectively, then conservation of parity in the reaction would require

$$\eta_\pi \eta_d (-1)^{\ell_i} = \eta_n \eta_n (-1)^{\ell_f}, \quad (11.26)$$

where  $\eta_\pi$ ,  $\eta_d$ , and  $\eta_n$  represent intrinsic parities of the three particles. Because the intrinsic parity of the deuteron is  $+1$ , and  $\eta_n^2 = +1$ , it follows that

$$\eta_\pi = (-1)^{\ell_f - \ell_i} = (-1)^{\ell_f + \ell_i}. \quad (11.27)$$

The capture process is known to proceed from an  $\ell_i = 0$  state, and consequently, we get that

$$\eta_\pi = (-1)^{\ell_f}. \quad (11.28)$$

Now, with the spin of the deuteron being  $J_d = 1$ , this leaves the following possibilities for the state of the two neutrons

$$\begin{aligned} |\psi_{nn}^{(1)}\rangle &= |J = 1, s = 1, \ell_f = 0 \text{ or } 2\rangle, \\ |\psi_{nn}^{(2)}\rangle &= |J = 1, s = 1, \ell_f = 1\rangle, \\ |\psi_{nn}^{(3)}\rangle &= |J = 1, s = 0, \ell_f = 1\rangle, \end{aligned} \quad (11.29)$$

where the state with  $s = 0$  corresponds to the antisymmetric singlet spin state ( $\uparrow\downarrow - \downarrow\uparrow$ ), and the state with  $s = 1$  to the symmetric triplet spin state of two neutrons. Because the two neutrons are identical fermions, their overall wave function must be antisymmetric, which excludes all but  $|\psi_{nn}^{(2)}\rangle$  from consideration, and specifies that the pion is a pseudoscalar, or has an intrinsic parity of  $\eta_\pi = -1$ .

**Example 2: Parity of  $\Delta(1232)$** 

As we have discussed in Chapter 9, the  $\Delta(1232)$  is a  $\pi$ - $N$  resonance that decays strongly into a pion and a nucleon

$$\Delta(1232) \longrightarrow \pi + N. \quad (11.30)$$

The parity of the  $\Delta$  can therefore be written as

$$\eta_{\Delta} = \eta_{\pi}\eta_N(-1)^{\ell}, \quad (11.31)$$

where  $\ell$  is the relative orbital angular momentum in the final state. We have just seen that  $\eta_{\pi} = -1$  and  $\eta_N$  is defined as  $+1$ . Consequently, we have that the parity of the  $\Delta$  is given uniquely by the orbital wave found in the final state. (See, however, Problem 11.6.) From the observed angular distribution of the  $\pi$  and  $N$  in the rest frame of the  $\Delta$ , it is found that  $\ell = 1$ , and consequently the parity of the  $\Delta$  relative to the nucleon is  $\eta_{\Delta} = +1$ . (The spin of the  $\Delta$  is also known to be  $J = \frac{3}{2}$ .)

**11.2.2 Violation of Parity**

Until the late 1950s, it was believed that parity was a symmetry of all fundamental interactions. Namely, physics was believed to be the same whether described in a right handed coordinate system or in a left handed one. However, in the early 1950s, two weak decays were observed that were quite confounding, and were referred to as the “ $\tau - \theta$  puzzle” (this “ $\tau$ ” should not be confused with the more recently discovered  $\tau$  lepton)

$$\begin{aligned} \theta^+ &\longrightarrow \pi^+ + \pi^0, \\ \tau^+ &\longrightarrow \pi^+ + \pi^+ + \pi^-. \end{aligned} \quad (11.32)$$

These decays were very interesting because the  $\theta$  and  $\tau$  particles were observed to have essentially identical masses and lifetimes. (Subsequently, both were also found to have spin  $J = 0$ , a fact that we will use to simplify our argument.) Naively, it could be concluded that the  $\theta^+$  and  $\tau^+$  were one and the same particle. However, this presented a conflict with parity conservation. To see this, note that in the rest frame of the decaying particles, the total angular momentum of the initial state is zero (i.e.,

assuming both  $\theta^+$  and  $\tau^+$  have spin  $J = 0$ ). The final states in both processes involve only  $\pi$ -mesons, which also have spin zero. Thus, conservation of angular momentum requires that the relative orbital angular momentum in the  $\pi^+\pi^0$  final state must vanish ( $\ell_f = 0$ ). For the  $\pi^+\pi^+\pi^-$  final state, the situation is somewhat more complicated because there are two relative angular momenta in the final state (the relative angular momentum of the two  $\pi^+$  mesons and that of the  $\pi^-$  with respect to this system), both of which were found to be  $\ell = 0$ . Consequently, the intrinsic parities of the  $\theta^+$  and  $\tau^+$  had to be the same as the product of intrinsic parities of the  $\pi$ -mesons in their respective final states. But pions are pseudoscalars ( $\eta_{\pi^+} = \eta_{\pi^-} = \eta_{\pi^0} = -1$ ), and we must therefore conclude that

$$\begin{aligned}\eta_{\theta^+} &= \eta_{\pi^+}\eta_{\pi^0} = 1, \\ \eta_{\tau^+} &= \eta_{\pi^+}\eta_{\pi^+}\eta_{\pi^-} = -1.\end{aligned}\tag{11.33}$$

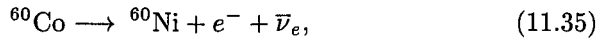
Consequently, if parity is conserved in these decays, then the  $\theta^+$  and  $\tau^+$  have opposite intrinsic parities and therefore cannot correspond to different decay modes of one object. Alternatively, we can assume that the  $\theta^+$  and  $\tau^+$  are the same particle, but, of course, only if parity is not conserved in their decays. Tsung-Dao Lee and Chen-Ning Yang, in fact, undertook a systematic study of all the experimentally known weak decays, and concluded that there was no evidence supporting conservation of parity in weak processes. They postulated that weak interactions violate parity, and suggested experiments to test their conjecture, which was clearly and rapidly confirmed to be correct. The decays in Eq. (11.32) are now understood as two weak-decay modes of the  $K^+$  meson, a process in which parity is not conserved<sup>1</sup>

$$\begin{aligned}K^+ &\longrightarrow \pi^+ + \pi^0, \\ K^+ &\longrightarrow \pi^+ + \pi^+ + \pi^-.\end{aligned}\tag{11.34}$$

The experiment that showed conclusively that parity is violated in weak interactions involved a study of the  $\beta$  decay of polarized  $^{60}\text{Co}$ . The experimental techniques were simple, yet quite powerful, and we will sketch out the procedures. Consider the decay

<sup>1</sup>The antiparticle of the  $K^+$ , namely the  $K^-$  meson, has decay channels analogous to those of Eqs. (11.32) and (11.34), these being  $K^- \rightarrow \pi^- + \pi^0$  and  $K^- \rightarrow \pi^- + \pi^+ + \pi^-$ .





which is equivalent to the  $\beta$  decay of a neutron. The experiment utilized a crystal of cobalt salt, whose nuclear spins were polarized by applying a strong external magnetic field. The temperature of the salt was lowered to about  $0.01\text{ K}$  in order to minimize thermal motion that leads to depolarization. The angular distribution of the emitted electrons ( $\theta_e$ ) was measured relative to the direction of the applied magnetic field, and the electrons were found to be emitted preferentially in a direction opposite to the field, and therefore opposite to the spin direction of the cobalt nuclei. That is, if  $\vec{s}$  denotes the spin of  ${}^{60}\text{Co}$  and  $\vec{p}$  the momentum of the emitted electron, then the experiment showed that the expectation value of  $\cos \theta_e$  was finite and negative

$$\langle \cos \theta_e \rangle = \left\langle \frac{\vec{s} \cdot \vec{p}}{|\vec{s}||\vec{p}|} \right\rangle = \langle \psi | \frac{\vec{s} \cdot \vec{p}}{|\vec{s}||\vec{p}|} | \psi \rangle < 0. \quad (11.36)$$

Because spin is an angular momentum, and therefore an axial vector, under a parity transformation, our observable  $\langle \cos \theta_e \rangle$  changes sign

$$\begin{aligned} \langle \cos \theta_e \rangle &= \left\langle \frac{\vec{s} \cdot \vec{p}}{|\vec{s}||\vec{p}|} \right\rangle \xrightarrow{P} \left\langle \frac{\vec{s} \cdot (-\vec{p})}{|\vec{s}||\vec{p}|} \right\rangle \\ &= - \left\langle \frac{\vec{s} \cdot \vec{p}}{|\vec{s}||\vec{p}|} \right\rangle = - \langle \cos \theta_e \rangle. \end{aligned} \quad (11.37)$$

Now, if a right handed and a left handed coordinate system are physically equivalent, then we must observe an identical value in the two frames, and consequently conclude from Eq. (11.37) that, if parity is conserved,  $\langle \cos \theta_e \rangle$  equals its negative value, and therefore vanishes, that is,

$$\langle \cos \theta_e \rangle \propto \langle \vec{s} \cdot \vec{p} \rangle = 0, \quad (11.38)$$

implying that the electrons must be emitted with equal probability for  $\cos \theta_e > 0$  and for  $\cos \theta_e < 0$ . The finite negative value observed for this quantity shows therefore that the two coordinate systems are not equivalent, and that parity is violated in weak interactions. In fact, since the original experiment by Chien-Shiung Wu, Ernest Ambler and collaborators, others have confirmed this result, and it appears that parity is violated maximally

in weak processes. The basic principle in these kinds of experiments is similar, namely, they all try to measure the expectation value of a quantity that should vanish if parity is conserved.

### 11.3 Time Reversal

In simple terms, time reversal corresponds to inverting the time axis, or the direction of the flow of time. In classical mechanics this transformation can be represented as

$$\begin{aligned} t &\xrightarrow{T} -t, \\ \vec{r} &\xrightarrow{T} \vec{r}, \\ \vec{p} = m\dot{\vec{r}} &\xrightarrow{T} -m\dot{\vec{r}} = -\vec{p}, \\ \vec{L} = \vec{r} \times \vec{p} &\xrightarrow{T} \vec{r} \times (-\vec{p}) = -\vec{L}. \end{aligned} \quad (11.39)$$

Newton's equations of motion (Eq. (11.17)), being second order in the time derivative, are invariant under time reversal for both electromagnetic and gravitational interactions. It can also be shown that Maxwell's equations are invariant under time inversion. However, not all macroscopic systems are time-reversal invariant. In fact, for macroscopic systems, statistical mechanics defines a unique direction for the flow of time as the one for which entropy (disorder) increases. Microscopic systems, on the other hand, appear to respect time reversal invariance. However, the implementation of time reversal symmetry into a theoretical formalism is not as straightforward as for the other symmetries.

Let us consider the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi. \quad (11.40)$$

Being a first-order equation in the time derivative, it cannot be invariant under the simplest time inversion

$$\psi(\vec{r}, t) \xrightarrow{T} \psi(\vec{r}, -t). \quad (11.41)$$

However, if we require the wave function to transform under time reversal as

$$\psi(\vec{r}, t) \xrightarrow{T} \psi^*(\vec{r}, -t), \quad (11.42)$$

then, assuming  $H$  is real (for a Hermitian operator), from the complex conjugate of Eq. (11.40), we get

$$-i\hbar \frac{\partial \psi^*(\vec{r}, t)}{\partial t} = H\psi^*(\vec{r}, t), \quad (11.43)$$

and now letting  $t \rightarrow -t$ , we obtain

$$i\hbar \frac{\partial \psi^*(\vec{r}, -t)}{\partial t} = H\psi^*(\vec{r}, -t). \quad (11.44)$$

Hence, the Schrödinger equation can be made invariant under time inversion, that is, both  $\psi$  and its time-reversed solution can obey the same equation, provided that time reversal for quantum mechanical wave functions is defined as in Eq. (11.42).

Consequently, the operator representing time reversal in quantum mechanics is quite unconventional in that it transforms a wave function to its complex conjugate. (Technically, such operators are called *antilinear*.) Because time dependent wave functions are necessarily complex, it follows that quantum mechanical wave functions cannot be eigenfunctions of the time reversal operator. Consequently, there is no simple quantum number that can be associated with time reversal invariance. Physically, however, invariance under time reversal implies that the transition amplitudes for the process  $i \rightarrow f$  and the time reversed one  $f \rightarrow i$  have the same magnitude, namely,

$$|M_{i \rightarrow f}| = |M_{f \rightarrow i}|, \quad (11.45)$$

where  $M_{i \rightarrow f}$  denotes the matrix element for the transition from an initial state  $|i\rangle$  to a final state  $|f\rangle$ . Equation (11.45) is referred to conventionally as the *principle of detailed balance*. It states that the quantum mechanical probability for a forward reaction to take place is the same as for the time reversed process. Nevertheless, the transition rates for the two processes can be quite different. From Fermi's Golden Rule, the rates are given by

$$\begin{aligned}
 W_{i \rightarrow f} &= \frac{2\pi}{\hbar} |M_{i \rightarrow f}|^2 \rho_f, \\
 W_{f \rightarrow i} &= \frac{2\pi}{\hbar} |M_{f \rightarrow i}|^2 \rho_i,
 \end{aligned}
 \tag{11.46}$$

where  $\rho_f$  and  $\rho_i$  represent the density of states for the end products in the two reactions. These can be quite different, depending on the masses of the particles involved, and, correspondingly, the rates can be different, even if detailed balance is valid. The principle of detailed balance has been verified for many processes, and has, in fact, been used to determine spins of particles by comparing the rates for the forward and backward reactions.

Time reversal invariance appears to be valid in almost all known fundamental processes, and the most spectacular test of this invariance principle for electromagnetic interactions comes from a search for an electric dipole moment of the neutron. As we have already discussed, although the neutron has no electric charge, it has a magnetic dipole moment, suggesting the presence of an extended charge distribution within the neutron. If the centers of the positive and the negative charge distributions do not coincide, then the neutron can also have an electric dipole moment. From simple dimensional arguments, we can estimate the magnitude of such a dipole moment to be

$$\mu_{el} \lesssim ed \approx e \times 10^{-13} \text{cm} \approx 10^{-13} e\text{-cm},
 \tag{11.47}$$

where we have used the fact that the typical size of the neutron and, therefore, the maximum separation of the charge centers is about  $d \approx 10^{-13} \text{cm}$ . Since the only possible preferred spatial direction for a neutron is its spin axis, it follows that if the neutron has a non-vanishing electric dipole moment, then it can point only along that axis, thereby yielding a finite value of  $\langle \vec{\mu}_{el} \cdot \vec{s} \rangle$ . The most sensitive searches for this effect provide an upper limit of

$$\mu_{el} \lesssim 10^{-25} e\text{-cm}.
 \tag{11.48}$$

This is, of course, consistent with the absence of an electric dipole moment, and is  $\approx 12$  orders of magnitude smaller than our naive limit from Eq. (11.47).

A finite value of  $\mu_{el}$ , and therefore a finite expectation value for  $\langle \vec{\mu}_{el} \cdot \vec{s} \rangle$ , would imply violation of  $T$ -invariance. This can be seen from the way the operator for the projection of the electric dipole moment along the spin direction transforms under time reversal

$$\vec{\mu}_{el} \cdot \vec{s} \xrightarrow{T} \vec{\mu}_{el} \cdot (-\vec{s}) = -\vec{\mu}_{el} \cdot \vec{s}, \quad (11.49)$$

where we used the fact that  $\vec{\mu}_{el}$  transforms as  $\sim e\vec{r}$ , which does not change under time reversal. The spin, on the other hand, being an angular momentum changes sign (see Eq. (11.39)). Consequently,

$$\langle \vec{\mu}_{el} \cdot \vec{s} \rangle \xrightarrow{T} -\langle \vec{\mu}_{el} \cdot \vec{s} \rangle, \quad (11.50)$$

and, if time reversal is a symmetry of the system, then this quantity must vanish. The result in Eq. (11.48) can therefore be regarded as an impressive upper limit on  $T$ -violation in electromagnetic interactions. There is, however, a question of interpretation of this result, because under the parity transformation we also obtain a change of sign

$$\langle \vec{\mu}_{el} \cdot \vec{s} \rangle \xrightarrow{P} \langle (-\vec{\mu}_{el}) \cdot \vec{s} \rangle = -\langle \vec{\mu}_{el} \cdot \vec{s} \rangle. \quad (11.51)$$

This means that a finite electric dipole moment could also arise as a consequence of parity violation. However, we know from other experiments that parity is conserved in electromagnetic interactions but violated in weak processes. Consequently, the presence of an electric dipole moment could arise from an interplay of electromagnetic and weak interactions. In fact, a small contribution to the electric dipole moment is expected from the weak interaction. Anything beyond that could be attributed to  $T$  violation in electromagnetic processes or to contributions from some new physical mechanisms. The upper limit on the value of the electric dipole moment of the neutron (which is a factor of about 100 larger than the upper limit on the electric dipole moment of the point-like electron) can therefore be interpreted as providing limits on  $T$  violation in electromagnetic and  $P$  violation in weak interactions.

## 11.4 Charge Conjugation

Both parity and time reversal are discrete space-time symmetry transformations, and it is natural to ask whether there are any discrete transformations in the internal Hilbert space of a quantum mechanical system. Charge conjugation is, in fact, this kind of transformation, under which the space-time coordinates are unchanged and the discrete transformation affects only the internal properties of the state.

Let us recall that the classification of the electron as particle and positron as antiparticle is arbitrary. In fact, the definition of positive and negative electric charge, positive and negative strangeness, the assignment of baryon number, etc., as we have stated before, are all a matter of convention. Once a choice is made, however, we can measure the quantum numbers of other particles relative to the defined assignments. The charge conjugation operation inverts all internal quantum numbers of states, and thereby relates particles to their antiparticles. Classically, one can represent charge conjugation as the following transformation on electric charge  $Q$

$$Q \xrightarrow{C} -Q. \quad (11.52)$$

Since electric charge is the source of electric and magnetic fields, it follows that under such a transformation

$$\begin{aligned} \vec{E} &\xrightarrow{C} -\vec{E}, \\ \vec{B} &\xrightarrow{C} -\vec{B}. \end{aligned} \quad (11.53)$$

(This is simply because both  $\vec{E}$  and  $\vec{B}$  are linear in electric charge.) It is straightforward to show that Maxwell's equations are invariant under such a transformation.

For a quantum mechanical state  $|\psi(Q, \vec{r}, t)\rangle$ , where  $Q$  represents all the internal quantum numbers such as electric charge, lepton number, baryon number, strangeness, etc., charge conjugation reverses all the charges,

$$|\psi(Q, \vec{r}, t)\rangle \xrightarrow{C} |\psi(-Q, \vec{r}, t)\rangle. \quad (11.54)$$

Consequently, a state can be an eigenstate of the charge conjugation operator  $C$  if, at the very least, it is electrically neutral. Thus, for example, the photon ( $\gamma$ ), the atom of positronium ( $e^- - e^+$ ), the  $\pi^0$  meson, etc., can

be eigenstates of  $C$ . However, not all charge-neutral states are eigenstates of  $C$  since they may carry other internal quantum numbers. For example, the following are, clearly, not eigenstates of  $C$

$$\begin{aligned} |n\rangle &\xrightarrow{C} |\bar{n}\rangle, \\ |\pi^- p\rangle &\xrightarrow{C} |\pi^+ \bar{p}\rangle, \\ |K^0\rangle &\xrightarrow{C} |\bar{K}^0\rangle. \end{aligned} \quad (11.55)$$

Because two consecutive charge conjugation transformations will leave a state unchanged, it follows that the eigenvalues of  $C$ , or the *charge parities* of an eigenstate, can be only  $\pm 1$ . Thus, for example, from Eq. (11.53) we conclude that the photon, the quantum of the electromagnetic field, must have a charge parity of  $-1$ ,

$$\eta_C(\gamma) = -1. \quad (11.56)$$

If charge conjugation is a symmetry of the theory, that is, if  $H$  and  $C$  commute

$$[C, H] = 0, \quad (11.57)$$

then the charge parity for any given process must be conserved. Because Maxwell's equations do not change under  $C$ , electromagnetic interactions should therefore be invariant under charge conjugation. Consequently, from the decay of the  $\pi^0$  into two photons

$$\pi^0 \longrightarrow \gamma + \gamma, \quad (11.58)$$

we conclude that the  $\pi^0$  must be even under  $C$ , if charge parity is to be conserved in the decay

$$\eta_C(\pi^0) = \eta_C(\gamma)\eta_C(\gamma) = (-1)^2 = +1. \quad (11.59)$$

Invariance under charge conjugation therefore leads to restrictions on the kinds of interactions or decays that can take place. For example, a  $\pi^0$  cannot decay to an odd number of photons because that would violate conservation of  $C$ -parity

$$\pi^0 \not\rightarrow n\gamma, \quad \text{for } n \text{ odd.} \quad (11.60)$$

And, in fact, the experimental upper limit on the branching ratio  $\frac{\pi^0 \rightarrow 3\gamma}{\pi^0 \rightarrow 2\gamma}$  is  $\approx 10^{-8}$ .

While charge conjugation is known to be a symmetry of electromagnetic and strong interactions, we can argue, as follows, that it must be violated in weak interactions. As we have emphasized, charge conjugation does not change space-time properties, and therefore the handedness of a quantum state is insensitive to such a transformation. Thus, under charge conjugation, we obtain

$$\begin{aligned} |\nu_L\rangle &\xrightarrow{C} |\bar{\nu}_L\rangle, \\ |\bar{\nu}_R\rangle &\xrightarrow{C} |\nu_R\rangle, \end{aligned} \quad (11.61)$$

where the subscripts  $L$  and  $R$  refer to left and right handed neutrinos (or antineutrinos), respectively. But we pointed out previously that there is no evidence for the existence of right handed neutrinos or left handed antineutrinos. Consequently, the charge conjugate process of  $\beta$ -decay cannot take place, and charge conjugation therefore cannot be a symmetry of such interactions. Nevertheless, although both  $P$  and  $C$  symmetry are violated in  $\beta$  decay, the combined transformation of  $CP$  appears to be a symmetry of such processes. One can see this heuristically as follows

$$\begin{aligned} |\nu_L\rangle &\xrightarrow{P} |\nu_R\rangle \xrightarrow{C} |\bar{\nu}_R\rangle, \\ |\bar{\nu}_R\rangle &\xrightarrow{P} |\bar{\nu}_L\rangle \xrightarrow{C} |\nu_L\rangle, \end{aligned} \quad (11.62)$$

that is, the combined operation of  $CP$  takes a physical state to another physical state, which is not what the  $C$  or  $P$  operations do individually. Nevertheless, the  $CP$  operation is not a symmetry of all weak interactions, as we will discover in the following chapter.

### 11.5 $CPT$ Theorem

We have seen that the discrete symmetries  $P$ ,  $T$  and  $C$  appear to be violated in some processes. However, it was shown independently by Georg Lüders, Wolfgang Pauli, and Julian Schwinger that the combined operation of  $CPT$



must be a symmetry of essentially any theory that is invariant under Lorentz transformations. That is, even if the individual transformations do not represent symmetries of any given theory, the product transformation will be a symmetry. This is known as the *CPT* theorem, and a consequence of *CPT* invariance leads to certain very interesting conclusions, which we summarize below.

- (1) Particles satisfy Bose-Einstein statistics when they have integer spin, and obey Fermi-Dirac statistics when they have half-integer spin. This has additional implications for relativistic theories, in that it requires an operator with integer spin to be quantized using commutation relations, and an operator with half-integer spin to be quantized using anti-commutation relations.
- (2) Particles and their antiparticles have identical masses and same total lifetimes.
- (3) All the internal quantum numbers of antiparticles are opposite to those of their partner particles.

The *CPT* theorem is consistent with all known observations, and *CPT* appears to be a true symmetry of all interactions.

## Problems

**11.1** The  $\rho^0(770)$  has  $J^P = 1^-$ , and it decays strongly into  $\pi^+\pi^-$  pairs. From symmetry and angular momentum considerations, explain why the decay  $\rho^0(770) \rightarrow \pi^0\pi^0$  is forbidden.

**11.2** What is the charge-conjugate reaction to  $K^- + p \rightarrow \overline{K^0} + n$ ? Can a  $K^-p$  system be an eigenstate of the charge conjugation operator? Similarly, discuss the reaction  $\overline{p} + p \rightarrow \pi^+ + \pi^-$ .

**11.3** If  $\rho^0$  mesons are produced in states with spin projection  $J_z = 0$  along their line of flight, what would you expect for the angular distribution of  $\rho^0 \rightarrow \pi^+ + \pi^-$  decay products in the  $\rho^0$  rest frame? (See Appendix B for the appropriate  $Y_{\ell,m}(\theta, \phi)$  functions.) What would be your answer if the initial  $\rho^0$  had spin projection  $J_z = +1$ ?

**11.4** The  $\Xi^-$  has  $J^P = \frac{1}{2}^+$ . It decays through weak interaction into a  $\Lambda^0$  and a  $\pi^-$  meson. If  $J_\Lambda^P = \frac{1}{2}^+$  and  $J_\pi^P = 0^-$ , what are the allowed relative orbital angular momenta for the  $\Lambda - \pi^-$  system?

**11.5** Which of the following decays are forbidden by *C*-invariance?

- (a)  $\omega^0 \rightarrow \pi^0 + \gamma$ ,
- (b)  $\eta' \rightarrow \rho^0 + \gamma$ ,
- (c)  $\pi^0 \rightarrow \gamma + \gamma + \gamma$ ,
- (d)  $J/\psi \rightarrow \bar{p} + p$ ,
- (e)  $\rho^0 \rightarrow \gamma + \gamma$ .

(Check the *CRC* tables to see if these decays take place.)

**11.6** Although the orbital wave for any strong  $\pi$ - $N$  state determines the parity of that state, different  $\ell$ -values do not necessarily yield different decay angular distributions. In particular, show that a  $J = \frac{1}{2}$ ,  $J_Z = +\frac{1}{2}$ ,  $\pi$ - $N$  resonance decays the same way whether it has  $\ell = 0$  or  $\ell = 1$ . Similarly, show that a  $J = \frac{3}{2}$ ,  $J_Z = +\frac{1}{2}$ ,  $\pi$ - $N$  system has the same decay angular distribution for  $\ell = 1$  as for  $\ell = 2$ . [*Hint*: Expand the wave function for the state in terms of the products of  $s = \frac{1}{2}$  spin-states and the appropriate  $Y_{\ell,m}(\theta, \phi)$ .]

### Suggested Readings

Frauenfelder, H., and E. M. Henley, *Subatomic Physics*, Prentice-Hall (1991).

Griffiths, D., *Introduction to Elementary Particles*, Wiley (1987).

Perkins, D. H., *Introduction to High Energy Physics*, Cambridge Univ. Press (2000).

Sakurai, J. J., *Invariance Principles and Elementary Particles*, Princeton Univ. Press (1964).

Williams, W. S. C., *Nuclear and Particle Physics*, Oxford Univ. Press (1997).

Also, see standard texts on quantum mechanics, e.g., Das, A. and A. C. Melissinos, *Quantum Mechanics*, Gordon & Breach (1986).



## Chapter 12

# Neutral Kaons, Oscillations, and $CP$ Violation

### 12.1 Introductory Remarks

As we saw in the previous chapter, weak interactions violate, separately, both the  $C$  and  $P$  symmetries. Nevertheless, it was thought until the early 1960s that the combined operation of  $CP$  might hold for all interactions. Because the  $CP$  transformation takes a physical particle state to a physical antiparticle state, as shown in the example of Eq. (11.62), invariance under  $CP$  is equivalent to having a particle-antiparticle symmetry in nature. However, the universe is known to be dominated by matter, with essentially no antimatter present, which is tantamount to saying that there is a definite particle-antiparticle asymmetry in the universe. This would suggest that  $CP$  may not be a symmetry of all the fundamental interactions, and, in fact, as we will describe below, there are processes in which the combined operation of  $CP$  is violated. In this chapter we discuss the violation of  $CP$  in weak interactions. It should be recognized that, if  $CPT$  is a symmetry of all physical systems, then a violation of  $CP$  automatically implies that  $T$  must also be violated.  $CP$  violation therefore implies that there are microscopic (subatomic) processes for which time has a unique direction of flow.

### 12.2 Neutral Kaons

We have already discussed the  $\tau$ - $\theta$  puzzle, where we concluded that the decays of the  $\theta^+(\theta^-)$  and the  $\tau^+(\tau^-)$  can be identified as two decay channels of the  $K^+(K^-)$ . There are analogous decays of the neutral kaon partners of the  $K^+$  and  $K^-$  that we will consider in this chapter. In particular, we will concentrate on the hadronic final states

$$\begin{aligned}
 \theta^0 &\longrightarrow \pi^0 + \pi^0, \\
 \theta^0 &\longrightarrow \pi^+ + \pi^-, \\
 \tau^0 &\longrightarrow \pi^0 + \pi^0 + \pi^0, \\
 \tau^0 &\longrightarrow \pi^+ + \pi^- + \pi^0.
 \end{aligned}
 \tag{12.1}$$

The first question that naturally comes to mind is how the  $\theta^0$  and  $\tau^0$  are related to the  $K^0$  and  $\overline{K}^0$ . To make this connection, we will first discuss the production and decay characteristics of neutral kaons.

Both  $K^0$  and  $\overline{K}^0$  mesons can be produced in strong-interaction processes such as

$$\begin{aligned}
 K^- + p &\longrightarrow \overline{K}^0 + n, \\
 K^+ + n &\longrightarrow K^0 + p, \\
 \pi^- + p &\longrightarrow \Lambda^0 + K^0.
 \end{aligned}
 \tag{12.2}$$

In these reactions, the kaons are produced in states of unique strangeness, namely  $S = +1$  for  $K^0$ , and  $S = -1$  for  $\overline{K}^0$ . As we know, the  $K^0$  can be identified as the  $I_3 = -\frac{1}{2}$  isospin partner of the  $K^+$ , and the  $\overline{K}^0$  as the  $I_3 = +\frac{1}{2}$  partner of the  $K^-$ . The  $\overline{K}^0$  is the antiparticle of the  $K^0$ , and the two can be distinguished due to their difference in strangeness. The neutral kaons produced in the above collisions are unstable, and decay through the weak interaction after traveling some distance  $\ell$  (in a time  $t_{\text{lab}}$ ) in the laboratory. The distance traveled prior to decay is related to the proper time through the velocity  $v$  of the kaon,

$$\ell = vt_{\text{lab}} = v\gamma t_{\text{proper}}, \quad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}, \tag{12.3}$$

and the mean proper time is just the lifetime  $\tau_{K^0}$  of the  $K^0$  in its own rest frame

$$\tau_{K^0} = \langle t_{\text{proper}} \rangle. \tag{12.4}$$

By measuring the velocity and the decay length ( $\ell$ ) of a  $K^0$ , we can determine its proper time, and from a sample of such events extract the lifetime.

Because the  $\overline{K}^0$  is the antiparticle of the  $K^0$ , it follows from the *CPT* theorem that the two particles must have identical masses and lifetimes.

The results of experiments that study  $\tau_{K^0}$  are sketched in Fig. 12.1, and are remarkable indeed. Instead of observing a single characteristic decay time (exponential drop off) that would be expected for any unique eigenstate of the free-particle Hamiltonian, the data indicate that there are two distinct lifetimes associated with both the  $K^0$  and the  $\overline{K}^0$ . This can only be understood if we assume that the  $K^0$  and the  $\overline{K}^0$  states consist of a superposition of two distinct states with different lifetimes: a short-lived one, originally labeled  $K_1^0$ , and a longer-lived one, labeled  $K_2^0$ . The events corresponding to  $K_1^0$  decays are of the  $\theta^0$  variety (namely, two-pion channels) while those corresponding to the  $K_2^0$  decays are of the  $\tau^0$  variety (that is, three-pion channels). The results found for  $K^0$  and  $\overline{K}^0$  decays are completely consistent with each other in the sense that the decay modes and lifetimes observed for the  $K_1^0$  and  $K_2^0$  components in both cases are the same

$$\begin{aligned}\tau_1 &\approx 0.9 \times 10^{-10} \text{ sec,} \\ \tau_2 &\approx 5 \times 10^{-8} \text{ sec.}\end{aligned}\tag{12.5}$$

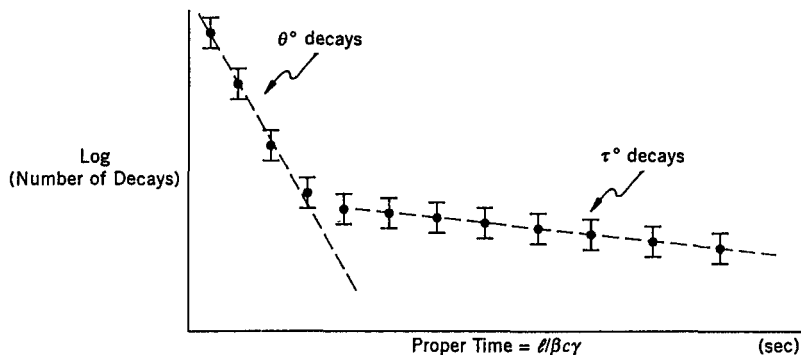


Fig. 12.1 Sketch of the distribution in proper time for  $K^0$  or for  $\overline{K}^0$  mesons, calculated from their velocities and distances traveled prior to decay.

The fact that the  $K^0$  and the  $\overline{K}^0$  can decay through common channels

$$\begin{aligned} K^0 &\longrightarrow \pi^0 + \pi^0, \\ \overline{K}^0 &\longrightarrow \pi^0 + \pi^0, \end{aligned} \quad (12.6)$$

suggests that these particles can mix through higher orders in the weak interaction. That is, although the  $K^0$  and  $\overline{K}^0$  are distinguishable because of their strangeness quantum number, and are represented by orthogonal states, they do not remain orthogonal as time evolves and weak interactions set in. That is, in the presence of weak interactions, the two kaons share the same decay channels. This is a consequence of the fact that weak interactions do not conserve strangeness. It is therefore possible to have transitions between a  $K^0$  and a  $\overline{K}^0$ , for example, through  $2\pi^0$  intermediate states (see Fig. 12.2)

$$K^0 \xrightarrow{H_{wk}} \pi^0 + \pi^0 \xrightarrow{H_{wk}} \overline{K}^0. \quad (12.7)$$

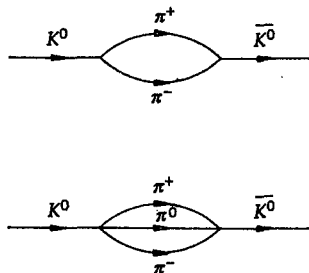


Fig. 12.2 Possible transformations of  $K^0$  to  $\overline{K}^0$ .

Thus, the  $K^0$  and  $\overline{K}^0$  particle states, although eigenstates of the strong-interaction Hamiltonian ( $H_{st}$ ), cannot be eigenstates of the weak interaction Hamiltonian ( $H_{wk}$ ). That is, schematically, we have for the strong interactions

$$\begin{aligned} \langle \overline{K}^0 | K^0 \rangle &= 0, \\ \langle \overline{K}^0 | H_{st} | K^0 \rangle &= 0, \end{aligned}$$

with  $H_{st}|K^0\rangle = m_{K^0}c^2|K^0\rangle$ ,  $H_{st}|\overline{K}^0\rangle = m_{\overline{K}^0}c^2|\overline{K}^0\rangle$ , and  $m_{K^0} = m_{\overline{K}^0} \approx 498 \text{ MeV}/c^2$ , and (where  $S$  is the operator measuring strangeness)

$$\begin{aligned}
 S|K^0\rangle &= +1|K^0\rangle, & S|\overline{K}^0\rangle &= -1|\overline{K}^0\rangle, \\
 I_3|K^0\rangle &= -\frac{1}{2}|K^0\rangle, & I_3|\overline{K}^0\rangle &= \frac{1}{2}|\overline{K}^0\rangle,
 \end{aligned}
 \tag{12.8}$$

while for the weak interactions we have

$$\langle \overline{K}^0 | H_{\text{wk}} | K^0 \rangle \neq 0.
 \tag{12.9}$$

Because the decay of  $K$ -mesons is a weak process, the observed  $K_1^0$  and  $K_2^0$  particles, with unique lifetimes, can be thought of as corresponding to the eigenstates of  $H_{\text{wk}}$ . Furthermore, because both  $K^0$  and the  $\overline{K}^0$  appear to be superpositions of  $K_1^0$  and  $K_2^0$  states, it follows that the  $K_1^0$  and  $K_2^0$  must also be superpositions of  $K^0$  and  $\overline{K}^0$ .

### 12.3 CP Eigenstates of Neutral Kaons

To determine which linear superpositions of states correspond to the eigenstates of the weak Hamiltonian, let us assume for simplicity that  $CP$  is a symmetry of weak interactions. Also, let us choose the phases for the  $K^0$  and  $\overline{K}^0$  states as follows

$$\begin{aligned}
 CP|K^0\rangle &= -C|K^0\rangle = -|\overline{K}^0\rangle, \\
 CP|\overline{K}^0\rangle &= -C|\overline{K}^0\rangle = -|K^0\rangle,
 \end{aligned}
 \tag{12.10}$$

where we have used the fact that  $K$ -mesons are pseudoscalars and consequently have odd intrinsic parities. Using Eq. (12.10), we can now define two linear orthonormal combinations of  $K^0$  and  $\overline{K}^0$  that will be eigenstates of the  $CP$  operator, namely,

$$\begin{aligned}
 |K_1^0\rangle &= \frac{1}{\sqrt{2}} \left( |K^0\rangle - |\overline{K}^0\rangle \right), \\
 |K_2^0\rangle &= \frac{1}{\sqrt{2}} \left( |K^0\rangle + |\overline{K}^0\rangle \right).
 \end{aligned}
 \tag{12.11}$$

Applying the  $CP$  operator to the  $K_1^0$  and  $K_2^0$  states, we can verify explicitly that



$$\begin{aligned}
CP|K_1^0\rangle &= \frac{1}{\sqrt{2}} \left( CP|K^0\rangle - CP|\bar{K}^0\rangle \right) \\
&= \frac{1}{\sqrt{2}} \left( -|\bar{K}^0\rangle + |K^0\rangle \right) = \frac{1}{\sqrt{2}} \left( |K^0\rangle - |\bar{K}^0\rangle \right) = |K_1^0\rangle, \\
CP|K_2^0\rangle &= \frac{1}{\sqrt{2}} \left( CP|K^0\rangle + CP|\bar{K}^0\rangle \right) \\
&= \frac{1}{\sqrt{2}} \left( -|\bar{K}^0\rangle - |K^0\rangle \right) \\
&= -\frac{1}{\sqrt{2}} \left( |K^0\rangle + |\bar{K}^0\rangle \right) = -|K_2^0\rangle.
\end{aligned} \tag{12.12}$$

Thus, the two states  $|K_1^0\rangle$  and  $|K_2^0\rangle$ , which do not carry unique strangeness, can be defined as eigenstates of  $CP$  with eigenvalues  $+1$  and  $-1$ , respectively. If  $CP$  is conserved in weak processes, we can then identify  $K_1^0$  and  $K_2^0$  with  $\theta^0$  and  $\tau^0$ , respectively. In fact, we see that in the rest frame of  $\theta^0$ , the two  $\pi^0$  mesons must have zero orbital angular momentum ( $\ell = 0$ ), and the final-state  $\pi^0\pi^0$  system is therefore an eigenstate of  $CP$  with eigenvalue  $+1$ . This is consistent with our identification of  $K_1^0$  with the  $\theta^0$  decay mode

$$\theta^0 = K_1^0 \longrightarrow \pi^0 + \pi^0. \tag{12.13}$$

Recalling that pions are pseudoscalar mesons, a similar analysis of the  $3\pi^0$  decay mode of the  $\tau^0$  shows that it is an eigenstate of  $CP$  with eigenvalue  $-1$ , which confirms the identification of  $K_2^0$  with the  $\tau^0$  decay

$$\tau^0 = K_2^0 \longrightarrow \pi^0 + \pi^0 + \pi^0 \tag{12.14}$$

Note that the momentum (and therefore the phase space or the density of states) available for the two-body decay in Eq. (12.13) is substantially larger than the phase space for the three body decay in Eq. (12.14). Consequently, if our analysis is valid, we predict that the rate for decay of the  $K_1^0$  will be much greater than that for  $K_2^0$ , leading to the expectation that the two particles will have different lifetimes –  $K_1^0$  being short-lived compared to  $K_2^0$ . This prediction of two lifetimes was, in fact, a principal result of an

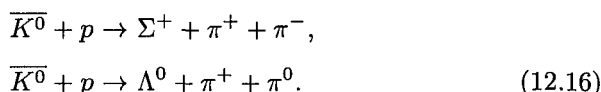
analysis of this problem by Murray Gell-Mann and Abraham Pais prior to the discovery of the  $K_2^0$ .

## 12.4 Strangeness Oscillation

We can invert the relations in Eq. (12.11) to obtain

$$\begin{aligned} |K^0\rangle &= \frac{1}{\sqrt{2}} (|K_1^0\rangle + |K_2^0\rangle), \\ |\overline{K}^0\rangle &= -\frac{1}{\sqrt{2}} (|K_1^0\rangle - |K_2^0\rangle). \end{aligned} \quad (12.15)$$

Thus the interactions involving the  $K^0$  and  $\overline{K}^0$  mesons can be understood as follows. In strong-production processes, such as given in Eq. (12.2), only the eigenstates of the strong Hamiltonian are produced, namely  $|K^0\rangle$  or  $|\overline{K}^0\rangle$ . However, as we see from Eq. (12.15), these states are superpositions of the  $|K_1^0\rangle$  and  $|K_2^0\rangle$ , which are eigenstates of the weak Hamiltonian. (The mass and lifetime eigenvalues of the  $K_1^0$  and  $K_2^0$  are discussed below.) At the time of production, the  $K^0$  and  $\overline{K}^0$  correspond to the superpositions of  $K_1^0$  and  $K_2^0$  given in Eq. (12.15). But as these specific mixtures of the  $|K_1^0\rangle$  and  $|K_2^0\rangle$  states propagate in vacuum, both the  $|K_1^0\rangle$  and  $|K_2^0\rangle$  components decay away. However, the state  $|K_1^0\rangle$  decays much faster than  $|K_2^0\rangle$ , and, after some time, the initial  $|K^0\rangle$  or  $|\overline{K}^0\rangle$  will therefore be composed primarily of  $|K_2^0\rangle$ . But Eq. (12.11) indicates that  $K_2^0$  has an equal admixture of  $K^0$  and  $\overline{K}^0$ , which means that, starting out either as a pure  $K^0$  or as a pure  $\overline{K}^0$  state, any neutral kaon will evolve into a state of mixed strangeness. This phenomenon is known as  $K^0 - \overline{K}^0$  or strangeness *oscillation*, and it can be observed as follows. For example, to detect the presence of  $\overline{K}^0$  resulting from the time evolution of initially pure  $K^0$ , we can examine the interactions of neutral kaons as a function of the distance from their point of production. At the point of origin, the  $K^0$  is a pure  $S = +1$  state, but, as the  $K_1^0$  component decays away, a  $\overline{K}^0$  component starts developing, and it can interact strongly with the medium (say, protons) to produce hyperons of  $S = -1$ , as follows



On the other hand, conservation of strangeness does not allow production of hyperons by  $K^0$  mesons, namely,

$$K^0 + p \not\rightarrow \Sigma^+ + \pi^+ + \pi^-,$$

$$K^0 + p \not\rightarrow \Lambda^0 + \pi^+ + \pi^0.$$

Consequently detecting hyperon production in the medium will signal the presence of  $\overline{K}^0$ . This is, in fact, exactly what is observed. Close to where the  $K^0$  are produced, there are no  $\overline{K}^0$ , and no secondary interactions of the kind given in Eq. (12.16). However, further downstream, there is evidence for the emergence of  $\overline{K}^0$ , as can be inferred from the observation of hyperon production.

As we will see shortly, the phenomenon of  $K^0 - \overline{K}^0$  oscillations can be used to measure the small mass difference between the  $K_1^0$  and  $K_2^0$ . The technique is similar to the one mentioned in Chapter 4 that was used to detect finite values of masses of neutrinos. Besides neutral kaons and neutrinos, neutral bottom-mesons ( $B^0$ ), composed of  $b\bar{d}$  or  $b\bar{s}$  quark systems (see Table 9.5) also display these interesting quantum effects. There have been searches for oscillations in neutral charm-mesons ( $D^0$  mesons, composed of  $c\bar{u}$  or  $b\bar{d}$  systems), and between neutrons and antineutrons, but no effects have been detected so far.

## 12.5 $K_1^0$ Regeneration

Another interesting process involving the  $K^0 - \overline{K}^0$  system is what is known as  $K_1^0$  regeneration. This was originally proposed as a possibility by Abraham Pais and Oreste Piccioni. The idea relies on the fact that the cross section for  $\overline{K}^0$  interactions with nucleons is different from (greater than) the cross section for  $K^0$  interactions with nucleons. (We might expect  $\sigma(\overline{K}^0 N) > \sigma(K^0 N)$  since strong  $\overline{K}^0 N$  collisions can produce everything that strong  $K^0 N$  collisions can yield, and, in addition, can produce hyperons, as indicated in Eq. (12.16).) Let us consider a beam of  $K^0$  that is allowed to evolve in vacuum (through  $K_1^0$  decay) into essentially a pure  $K_2^0$  beam. If we now let the  $K_2^0$  interact with some target material, then, because the absorption of the  $\overline{K}^0$  component is greater than that of the  $K^0$  component of the  $K_2^0$ , the admixture of  $K^0$  and  $\overline{K}^0$  in the  $K_2^0$  beam will change. For example, if all the  $\overline{K}^0$  is removed in strong interactions with the material, then we will be left with a transmitted beam that is pure  $K^0$ ,

and is consequently an equal mixture of  $K_1^0$  and  $K_2^0$ . Hence, starting with a  $K_2^0$  beam, we can regenerate  $K_1^0$  by passing the  $K_2^0$  through some absorbing medium. This interesting phenomenon has been observed in many experiments.

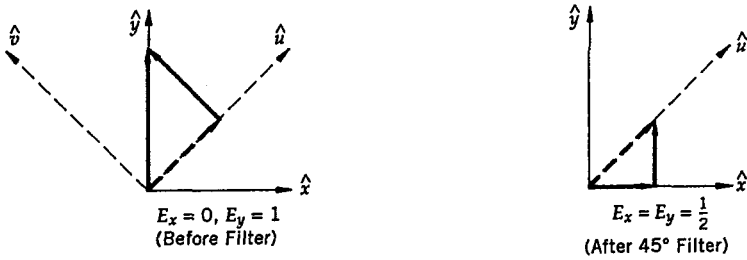


Fig. 12.3 Polarization vector of an electric field before and after a filter that transmits light along  $45^\circ$  ( $\hat{u}$ ).

Although  $K_1^0$  regeneration may appear to be somewhat exotic, it has, in fact, a simple analogy in optics in the absorption of linearly polarized light. Just as the  $K^0$  and  $\bar{K}^0$  can be expressed using the  $K_1^0$  and  $K_2^0$  basis vectors, and vice versa, so can light polarized along  $\hat{x}$  or  $\hat{y}$  be expressed in terms of polarization vectors  $\hat{u}$  and  $\hat{v}$  that are rotated by  $45^\circ$  relative to  $\hat{x}$  and  $\hat{y}$  (see Fig. 12.3), and vice versa. If we pass light polarized along  $\hat{y}$  through a filter that absorbs light along  $\hat{v}$  (that is, a filter rotated by  $45^\circ$  relative to  $\hat{y}$ ), then the partially transmitted light will be polarized along  $\hat{u}$ . This transmitted light can be decomposed along  $\hat{x}$  and  $\hat{y}$ , to yield an equal mixture of components along both directions. Thus, starting off with light that has the electric field polarized along  $\hat{y}$  (or with a  $K_2^0$  beam), we absorb away the component along  $\hat{v}$  (selectively deplete the  $\bar{K}^0$  fraction of  $K_2^0$ ), and thereby generate a component that is orthogonal to the incident polarization, namely along  $\hat{x}$  (regenerate the orthogonal  $K_1^0$  state).

### 12.6 Violation of CP Invariance

Since  $|K_2^0\rangle$  is a state with  $CP$  eigenvalue  $-1$ , then, if  $CP$  is conserved in weak interactions, we would conclude that such a state could not decay into two pions. That is, the following transitions could not take place

$$\begin{aligned}
 K_2^0 &\not\rightarrow \pi^0 + \pi^0, \\
 K_2^0 &\not\rightarrow \pi^+ + \pi^-.
 \end{aligned}
 \tag{12.17}$$

Nevertheless, an experiment performed in 1963 by James Christenson, James Cronin, Val Fitch and René Turlay revealed that the long-lived component of  $K^0$  did, in fact, decay into two pions. Because this implies that the long-lived and the short-lived components of  $K^0$  need not be the  $K_2^0$  and  $K_1^0$  eigenstates of  $CP$ , we will refer to them as  $K_L^0$  and  $K_S^0$  (for *K-long* and *K-short*). The branching rates of the  $K_L^0$  into  $\pi^+\pi^-$  and into  $\pi^0\pi^0$  are of the order of 0.1% of all  $K_L^0$  decays

$$\frac{K_L^0 \rightarrow \pi^+\pi^-}{K_L^0 \rightarrow \text{ALL}} \approx 2 \times 10^{-3}, \quad \frac{K_L^0 \rightarrow \pi^0\pi^0}{K_L^0 \rightarrow \text{ALL}} \approx 9 \times 10^{-4}.
 \tag{12.18}$$

From the lifetimes of the  $K_L^0$  and  $K_S^0$  in Eq. (12.5), it can be deduced that the decay rate of  $K_L^0 \rightarrow 2\pi$  is about  $4 \times 10^{-6}$  that of the decay rate of the short-lived  $K_S^0$  into two pions.

The experiment of Cronin, Fitch and their colleagues involved passing a pure  $\approx 1$  GeV/ $c$   $K^0$  beam through an evacuated tube that was about 15 meters long. The short-lived component in the beam had a decay length  $\langle \ell_{K_S^0} \rangle = \gamma\beta c\tau_{K_S^0} \approx 6$  cm, it was expected to decay away by the end of the tube, and the goal was to search for  $2\pi$  decays at the end, and thereby establish a more stringent upper limit on the  $2\pi$  mode of  $K_2^0$ . But, instead, the experiment observed  $2\pi$  decays, representing the first clear evidence for  $CP$  violation in particle interactions.

As we mentioned in the introduction, the violation of  $CP$  is important for gaining an understanding of the matter-antimatter asymmetry in the universe. However, the character of  $CP$  violation is somewhat unusual. In particular, parity violation in weak interactions is maximal, while  $CP$ , at least in  $K^0$  systems is violated only very slightly, and, for all practical purposes, it can be considered as essentially conserved. In fact,  $CP$  violation has been observed only in systems of  $K^0$ -mesons and neutral  $B^0$ -mesons (where the violation is large and rates are comparable to those for  $CP$  conserving decays). Thus, we can regard  $CP$  as a symmetry of almost all physical systems, and, because  $CP$  violation in  $K_L^0 \rightarrow 2\pi$  decay is very small, we therefore expect the  $K_L^0$  and  $K_S^0$  eigenstates of the physical particles to differ only slightly from  $CP$  eigenstates.

Unlike parity violation,  $CP$  violation was found to be more difficult to incorporate into theories, and, historically, this problem was viewed in two ways: One suggested that the interaction responsible for the  $K_L^0 \rightarrow 2\pi$  transition involved the standard weak Hamiltonian  $H_{wk}$ , and the other view was that it involved a new “extra-weak” interaction that appeared solely in systems such as  $K^0$  mesons. In the latter prescription,  $H_{wk}$  was assumed to be  $CP$  invariant, and the  $CP$  violation was attributed to the new extra-weak interaction. The second alternative, although distinctly possible, does not offer the most economical solution to the problem, as it invokes new interactions for systems that violate  $CP$  symmetry. In fact, most predictions of such phenomenologically-motivated “milli-weak” or “super-weak” theories have been found to be inconsistent with data for  $K^0$  and  $B^0$  decays, and will therefore not be elaborated any further.

As we have indicated, the transition  $\langle 2\pi | H_{wk} | K_L^0 \rangle$  clearly violates  $CP$  symmetry. But this can happen because the  $K_L^0$  and  $K_S^0$  eigenstates of the weak Hamiltonian are not eigenstates of the  $CP$  operator, in which case, the physical states are superpositions of  $CP$ -odd and  $CP$ -even components. This type of  $CP$  violation is commonly referred to as *indirect*, and arises in  $K_L^0 \rightarrow 2\pi$  decay solely because of the small inherent admixture of  $K_1^0$  in the  $K_L^0$  state. And, if this is the case,  $H_{wk}$  is even under the  $CP$  operation. But there is another possibility for a *direct* violation of  $CP$  through the  $H_{wk}$  Hamiltonian, and this can take place if  $\langle 2\pi | H_{wk} | K_2^0 \rangle \neq 0$ . Here,  $H_{wk}$  would have a component that is odd under  $CP$ . In fact, both the indirect and direct terms contribute to  $CP$  violation in the  $K_L^0 \rightarrow 2\pi$  transition, with the direct contribution being  $\approx 0.1\%$  of the strength of the indirect term.

To proceed further, let us define the two eigenstates of the weak Hamiltonian in terms of the eigenstates of  $H_{st}$ , i.e., in terms of  $|K^0\rangle$  and  $|\bar{K}^0\rangle$ , as follows (for a more formal discussion, see the next section)

$$\begin{aligned}
 |K_S^0\rangle &= \frac{1}{\sqrt{2(1+|\epsilon|^2)}} \left( (1+\epsilon)|K^0\rangle - (1-\epsilon)|\bar{K}^0\rangle \right) \\
 &= \frac{1}{\sqrt{2(1+|\epsilon|^2)}} \left[ (|K^0\rangle - |\bar{K}^0\rangle) + \epsilon(|K^0\rangle + |\bar{K}^0\rangle) \right] \\
 &= \frac{1}{\sqrt{(1+|\epsilon|^2)}} \left( |K_1^0\rangle + \epsilon |K_2^0\rangle \right), \tag{12.19}
 \end{aligned}$$

$$\begin{aligned}
|K_L^0\rangle &= \frac{1}{\sqrt{2(1+|\epsilon|^2)}} \left( (1+\epsilon)|K^0\rangle + (1-\epsilon)|\bar{K}^0\rangle \right) \\
&= \frac{1}{\sqrt{2(1+|\epsilon|^2)}} \left[ (|K^0\rangle + |\bar{K}^0\rangle) + \epsilon(|K^0\rangle - |\bar{K}^0\rangle) \right] \\
&= \frac{1}{\sqrt{(1+|\epsilon|^2)}} (|K_2^0\rangle + \epsilon|K_1^0\rangle), \tag{12.20}
\end{aligned}$$

where  $\epsilon$  is a very small, complex, parameter, representing the deviation of the  $K_L^0$  and  $K_S^0$  states from true  $CP$  eigenstates, and therefore reflecting the degree of indirect violation of  $CP$  in the system. Because we have constructed the physical short-lived and long-lived neutral  $K$ -mesons as explicit admixtures of the  $CP$  eigenstates, these new states therefore cannot be eigenstates of  $CP$ , as can be checked directly

$$\begin{aligned}
CP|K_S^0\rangle &= \frac{1}{\sqrt{(1+|\epsilon|^2)}} (CP|K_1^0\rangle + \epsilon CP|K_2^0\rangle) \\
&= \frac{1}{\sqrt{(1+|\epsilon|^2)}} (|K_1^0\rangle - \epsilon|K_2^0\rangle) \neq |K_S^0\rangle, \\
CP|K_L^0\rangle &= \frac{1}{\sqrt{(1+|\epsilon|^2)}} (CP|K_2^0\rangle + \epsilon CP|K_1^0\rangle) \\
&= \frac{1}{\sqrt{(1+|\epsilon|^2)}} (-|K_2^0\rangle + \epsilon|K_1^0\rangle) \neq -|K_L^0\rangle. \tag{12.21}
\end{aligned}$$

Moreover, these new physical states are not even orthogonal

$$\begin{aligned}
\langle K_L^0|K_S^0\rangle &= \frac{1}{1+|\epsilon|^2} (\langle K_2^0| + \epsilon^*\langle K_1^0|) (|K_1^0\rangle + \epsilon|K_2^0\rangle) \\
&= \frac{1}{1+|\epsilon|^2} (\epsilon\langle K_2^0|K_2^0\rangle + \epsilon^*\langle K_1^0|K_1^0\rangle) \\
&= \frac{\epsilon + \epsilon^*}{1+|\epsilon|^2} = \frac{2 \operatorname{Re} \epsilon}{1+|\epsilon|^2} = \langle K_S^0|K_L^0\rangle. \tag{12.22}
\end{aligned}$$

The lack of orthogonality of the two states is, in some sense, expected, since both have the same decay channels (such as the  $2\pi$  and  $3\pi$  modes), and this lack of orthogonality is, in fact, also a measure of the degree of  $CP$  violation. In other words, in this formulation, it is only the state  $|K_1^0\rangle$

that decays into two pions, and because  $|K_L^0\rangle$  contains a small admixture of  $|K_1^0\rangle$ , it follows that it too has a small probability for such decay.

It is worth re-emphasizing that the above indirect violation of  $CP$  in  $K_L^0 \rightarrow 2\pi$  decay arises purely from the small admixture of  $|K_1^0\rangle$  in the  $|K_L^0\rangle$  state. This kind of decay can be ascribed to  $\Delta S = 2$ ,  $\Delta I = \frac{1}{2}$  transitions. The recently observed far smaller direct  $CP$  component arises from the direct decay of  $K_2^0 \rightarrow 2\pi$ , which can only take place if  $H_{wk}$  has a  $CP$ -violating component (does not commute with  $CP$ ). This proceeds through a  $\Delta S = 1$ ,  $\Delta I = \frac{3}{2}$  transition, referred to in the Standard Model (see the following chapter) as a *penguin* contribution.

We can also express the ratios of the  $K_L^0$  and  $K_S^0$  transition amplitudes in terms of the following complex parameters:

$$\begin{aligned}\eta_{+-} &= \frac{K_L^0 \rightarrow \pi^+ + \pi^-}{K_S^0 \rightarrow \pi^+ + \pi^-}, \\ \eta_{00} &= \frac{K_L^0 \rightarrow \pi^0 + \pi^0}{K_S^0 \rightarrow \pi^0 + \pi^0}.\end{aligned}\tag{12.23}$$

While these weak decays can take place either through  $\Delta I = \frac{1}{2}$  or  $\frac{3}{2}$  transitions, as we mentioned previously, the amplitude for  $\Delta I = \frac{3}{2}$  is observed to be highly suppressed. Thus, for simplicity, if we assume that these decays proceed through  $\Delta I = \frac{1}{2}$ , this implies that the  $2\pi$  systems are in the  $I = 0$  state of isotopic spin. (This is equivalent to ignoring the direct term in  $CP$  violation.) From the definition of  $|K_S^0\rangle$  and  $|K_L^0\rangle$  in Eqs. (12.19) and (12.20), we can then conclude that

$$\eta_{+-} = \eta_{00} = \epsilon.\tag{12.24}$$

Thus, in this scenario of indirect violation of  $CP$ , we expect the ratios for the two decay modes in Eq. (12.23) to be the same. In fact, the measured values are consistent with this expectation, but only at the 1% level

$$\begin{aligned}|\eta_{+-}| &= (2.29 \pm 0.02) \times 10^{-3}, \\ \phi_{+-} &= (43 \pm 1)^\circ,\end{aligned}\tag{12.25}$$

$$\begin{aligned}|\eta_{00}| &= (2.27 \pm 0.02) \times 10^{-3}, \\ \phi_{00} &= (43 \pm 1)^\circ,\end{aligned}\tag{12.26}$$



where we have parameterized

$$\eta_{+-} = |\eta_{+-}|e^{i\phi_{+-}}, \quad (12.27)$$

and

$$\eta_{00} = |\eta_{00}|e^{i\phi_{00}}. \quad (12.28)$$

Including the  $\Delta I = \frac{3}{2}$  admixture, provides the small direct contribution expected from the Standard Model (see next chapter). In this more general scenario, the relationship between  $\eta_{00}$  and  $\eta_{+-}$  becomes modified

$$\frac{|\eta_{00}|^2}{|\eta_{+-}|^2} = 1 - 6 \operatorname{Re} \left( \frac{\epsilon'}{\epsilon} \right),$$

where  $\epsilon'$  is the parameter that represents the direct contribution. Based on very sensitive recent measurements of the above ratio of the ratios of decay rates specified in Eq. (12.23), the latest value of  $\operatorname{Re} \left( \frac{\epsilon'}{\epsilon} \right)$  is found to be  $\approx 1.6 \times 10^{-3}$ , which is inconsistent with the presence of just an indirect  $CP$ -violating term, but nevertheless still in agreement with expectations of the Standard Model.

## 12.7 Time Development and Analysis of the $K^0$ - $\overline{K}^0$ System

In the absence of weak interactions, the states  $|K^0\rangle$  and  $|\overline{K}^0\rangle$  are eigenstates of the strong Hamiltonian, and describe distinct particle and antiparticle states. These are stationary states in a 2-dimensional Hilbert space, and can be identified with the basis vectors

$$\begin{aligned} |K^0\rangle &\longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ |\overline{K}^0\rangle &\longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned} \quad (12.29)$$

Any normalized general state in this space can, of course, be written as a linear superposition of the two of the form

$$\begin{aligned}
 |\psi\rangle &= \frac{1}{(|a|^2 + |b|^2)^{\frac{1}{2}}} (a|K^0\rangle + b|\overline{K^0}\rangle) \\
 &\longrightarrow \frac{1}{(|a|^2 + |b|^2)^{\frac{1}{2}}} \begin{pmatrix} a \\ b \end{pmatrix}.
 \end{aligned}
 \tag{12.30}$$

In the presence of weak interactions, however, the states in Eq. (12.30) will not be stationary. In fact, as we have argued, they can decay through many available weak channels. Consequently, to describe the  $K^0$ - $\overline{K^0}$  system, requires an enlargement of our Hilbert space through the incorporation of the other final states. Alternatively, we can keep the analysis simple by restricting ourselves to a 2-dimensional Hilbert space, and incorporating the consequences of decay channels into some effective Hamiltonian. Because we are dealing with states that decay in time, that is whose probability is not conserved, the effective Hamiltonian will no longer be Hermitian (see, for example, Eq. (9.32)). Nevertheless, the time evolution of a general two-dimensional vector in this space will still be governed by the time-dependent Schrödinger equation

$$i\hbar \frac{\partial |\psi(t)\rangle}{\partial t} = H_{\text{eff}} |\psi(t)\rangle, \tag{12.31}$$

where  $H_{\text{eff}}$  is a  $2 \times 2$  complex (non-Hermitian) matrix operator, which can be written in general as

$$H_{\text{eff}} = M - \frac{i}{2} \Gamma, \tag{12.32}$$

with

$$\begin{aligned}
 M &= \frac{1}{2} (H_{\text{eff}} + H_{\text{eff}}^\dagger), \\
 \Gamma &= i (H_{\text{eff}} - H_{\text{eff}}^\dagger),
 \end{aligned}$$

so that

$$\begin{aligned}
 M^\dagger &= M, \quad \text{or} \quad M_{jk}^* = M_{kj}, \\
 \Gamma^\dagger &= \Gamma, \quad \text{or} \quad \Gamma_{jk}^* = \Gamma_{kj}, \quad j, k = 1, 2,
 \end{aligned}
 \tag{12.33}$$

where  $\Gamma$  and  $M$  are Hermitian  $2 \times 2$  matrices. Clearly, unless  $\Gamma$  vanishes,

$$H_{\text{eff}}^\dagger \neq H_{\text{eff}}. \tag{12.34}$$

In fact, as we have just emphasized, since  $H_{\text{eff}}$  has to incorporate decays, it cannot be Hermitian. We presume therefore that  $\Gamma$  is related to the lifetimes of the states.

For the time evolution of  $|\psi\rangle$  in Eq. (12.31) we can write

$$i\hbar \frac{\partial |\psi(t)\rangle}{\partial t} = H_{\text{eff}} |\psi(t)\rangle = \left( M - \frac{i}{2} \Gamma \right) |\psi(t)\rangle,$$

and

$$-i\hbar \frac{\partial \langle \psi(t) |}{\partial t} = \langle \psi(t) | H_{\text{eff}}^\dagger = \langle \psi(t) | \left( M + \frac{i}{2} \Gamma \right). \tag{12.35}$$

From these equations, it is straightforward to show that

$$\frac{\partial \langle \psi(t) | \psi(t) \rangle}{\partial t} = -\frac{1}{\hbar} \langle \psi(t) | \Gamma | \psi(t) \rangle. \tag{12.36}$$

Because decays reduce probability, we can conclude that the matrix  $\Gamma$ , in any state in this two-dimensional space, must satisfy the requirement

$$\langle \psi(t) | \Gamma | \psi(t) \rangle \geq 0. \tag{12.37}$$

In other words, the matrix  $\Gamma$  must have positive or vanishing eigenvalues, and, as anticipated, it represents the decay characteristics of the system. The eigenvalues of matrix  $M$ , which correspond to the real parts of the energy levels of the system, define the masses of the states in their own rest frames ( $\vec{p} = 0$ ). For this reason,  $M$  is known as the *mass matrix* of the system, while  $\Gamma$  is commonly referred to as the *decay matrix*.

In addition, if we represent  $H_{\text{eff}}$  by a general  $2 \times 2$  matrix of the form

$$H_{\text{eff}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{12.38}$$

then from Eq. (12.29) we obtain

$$\begin{aligned} \langle K^0 | H_{\text{eff}} | K^0 \rangle &= A, \\ \langle \bar{K}^0 | H_{\text{eff}} | \bar{K}^0 \rangle &= D. \end{aligned} \tag{12.39}$$

Invariance of  $H_{\text{eff}}$  under CPT then leads to the requirement that the mass of the  $K^0$  equal the mass of the  $\bar{K}^0$ , or

$$A = D. \tag{12.40}$$

We can therefore write the general form of  $H_{\text{eff}}$ , consistent with CPT invariance, as

$$H_{\text{eff}} = \begin{pmatrix} A & B \\ C & A \end{pmatrix}. \tag{12.41}$$

Let us next construct the eigenstates of  $H_{\text{eff}}$ , and take the following general parametrization for the two eigenstates

$$\begin{aligned} |K_S^0\rangle &= \frac{1}{(|p|^2 + |q|^2)^{\frac{1}{2}}} (p|K^0\rangle + q|\bar{K}^0\rangle) \\ &\rightarrow \frac{1}{(|p|^2 + |q|^2)^{\frac{1}{2}}} \begin{pmatrix} p \\ q \end{pmatrix}, \\ |K_L^0\rangle &= \frac{1}{(|r|^2 + |s|^2)^{\frac{1}{2}}} (r|K^0\rangle + s|\bar{K}^0\rangle) \\ &\rightarrow \frac{1}{(|r|^2 + |s|^2)^{\frac{1}{2}}} \begin{pmatrix} r \\ s \end{pmatrix}, \end{aligned} \tag{12.42}$$

where  $p, q, r, s$  are complex parameters that define the  $K_S^0$  and  $K_L^0$  eigenstates of  $H_{\text{eff}}$ , which in the particle's rest frame have eigenvalues  $m_S - \frac{i}{2}\gamma_S$  and  $m_L - \frac{i}{2}\gamma_L$ , respectively. That is,

$$\begin{aligned}
 H_{\text{eff}}|K_S^0\rangle &= \left(m_S - \frac{i}{2}\gamma_S\right)|K_S^0\rangle, \\
 H_{\text{eff}}|K_L^0\rangle &= \left(m_L - \frac{i}{2}\gamma_L\right)|K_L^0\rangle,
 \end{aligned}
 \tag{12.43}$$

where  $m_S$  and  $m_L$ , and  $\gamma_S$  and  $\gamma_L$ , correspond to the masses and widths of the two eigenstates, and we have set  $c = 1$ . In the basis of the  $K_L^0$ ,  $K_S^0$  eigenstates, the diagonal elements of  $H_{\text{eff}}$  are, of course, the two eigenvalues of Eq. (12.43). In this basis, the sum of the eigenvalues equals the trace ( $\text{Tr}$ ) of  $H_{\text{eff}}$ . However, because the trace of a matrix is the same in any basis,  $\text{Tr } H_{\text{eff}}$  always equals the sum of the two eigenvalues. Thus, from Eq. (12.41), we obtain

$$\begin{aligned}
 \text{Tr } H_{\text{eff}} &= 2A = \left(m_S - \frac{i}{2}\gamma_S\right) + \left(m_L - \frac{i}{2}\gamma_L\right), \\
 \text{or } A &= \frac{1}{2}(m_S + m_L) - \frac{i}{4}(\gamma_S + \gamma_L).
 \end{aligned}
 \tag{12.44}$$

Writing out the first equation in (12.43), we obtain

$$\begin{aligned}
 \begin{pmatrix} A & B \\ C & A \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} &= \left(m_S - \frac{i}{2}\gamma_S\right) \begin{pmatrix} p \\ q \end{pmatrix}, \\
 \text{or } \begin{pmatrix} A - m_S + \frac{i}{2}\gamma_S & B \\ C & A - m_S + \frac{i}{2}\gamma_S \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} &= 0.
 \end{aligned}
 \tag{12.45}$$

Equation (12.45) defines a set of coupled linear, homogeneous equations in the unknowns  $p$  and  $q$ , and a nontrivial solution in such a case exists only if the determinant of the coefficient matrix vanishes. In other words, for a nontrivial solution of Eq. (12.45) to exist, we must have

$$\det \begin{pmatrix} A - m_S + \frac{i}{2}\gamma_S & B \\ C & A - m_S + \frac{i}{2}\gamma_S \end{pmatrix} = 0,
 \tag{12.46}$$

$$\text{or } BC = \left( A - m_S + \frac{i}{2} \gamma_S \right)^2 = \left[ \frac{1}{2} (m_L - m_S) - \frac{i}{4} (\gamma_L - \gamma_S) \right]^2,$$

$$\text{or } \frac{1}{2} (m_L - m_S) - \frac{i}{4} (\gamma_L - \gamma_S) = \pm \sqrt{BC}. \quad (12.47)$$

Substituting this back into the first Eq. (12.45), provides the following solution for the coefficients  $p$  and  $q$

$$\frac{p}{q} = \pm \sqrt{\frac{B}{C}}. \quad (12.48)$$

Similarly, for the  $K_L^0$  eigenstate in Eq. (12.42), we obtain

$$\frac{r}{s} = \mp \sqrt{\frac{B}{C}} = -\frac{p}{q}. \quad (12.49)$$

Thus, if we choose  $r = p$  and  $s = -q$ , we can write

$$\begin{aligned} |K_S^0\rangle &= \frac{1}{(|p|^2 + |q|^2)^{\frac{1}{2}}} (p|K^0\rangle + q|\overline{K^0}\rangle), \\ |K_L^0\rangle &= \frac{1}{(|p|^2 + |q|^2)^{\frac{1}{2}}} (p|K^0\rangle - q|\overline{K^0}\rangle). \end{aligned} \quad (12.50)$$

And the choice made in Eqs. (12.19) and (12.20) therefore corresponds to

$$p = 1 + \epsilon, \quad q = -(1 - \epsilon). \quad (12.51)$$

Inverting the relations in Eq. (12.50), we can now write

$$\begin{aligned} |K^0\rangle &= \frac{(|p|^2 + |q|^2)^{\frac{1}{2}}}{2p} (|K_S^0\rangle + |K_L^0\rangle), \\ |\overline{K^0}\rangle &= \frac{(|p|^2 + |q|^2)^{\frac{1}{2}}}{2q} (|K_S^0\rangle - |K_L^0\rangle). \end{aligned} \quad (12.52)$$

Since  $|K_S^0\rangle$  and  $|K_L^0\rangle$  are eigenstates of  $H_{\text{eff}}$ , the solutions to Eqs. (12.31) and (12.43) are

$$\begin{aligned}
 |K_S^0(t)\rangle &= e^{-\frac{i}{\hbar}(m_S - \frac{1}{2}\gamma_S)t} |K_S^0\rangle, \\
 |K_L^0(t)\rangle &= e^{-\frac{i}{\hbar}(m_L - \frac{1}{2}\gamma_L)t} |K_L^0\rangle,
 \end{aligned}
 \tag{12.53}$$

where, again, we have set  $c = 1$  (cf. Eq. (9.32)). The states in Eq. (12.53) decay with lifetimes given by the squares of their amplitudes

$$\begin{aligned}
 \tau_S &= \frac{\hbar}{\gamma_S}, \\
 \tau_L &= \frac{\hbar}{\gamma_L},
 \end{aligned}
 \tag{12.54}$$

which correspond, of course, to the previously cited lifetimes of  $\tau_S \approx 0.9 \times 10^{-10}$  sec and  $\tau_L \approx 5 \times 10^{-8}$  sec. Also, as we stated before,  $m_L$  and  $m_S$  can be identified with the masses of the long-lived and the short-lived particles, respectively. It is worth emphasizing that, because the  $K_L^0$  and  $K_S^0$  are not each other's antiparticles (unlike the  $K^0$  and  $\bar{K}^0$ ), their lifetimes and masses do not have to be identical.

Let us now suppose that we start with an initially pure  $K^0$  beam. The evolution of such a beam can be obtained from Eqs. (12.52) and (12.53) as follows

$$\begin{aligned}
 |K^0(t)\rangle &= \frac{(|p|^2 + |q|^2)^{\frac{1}{2}}}{2p} (|K_S^0(t)\rangle + |K_L^0(t)\rangle) \\
 &= \frac{(|p|^2 + |q|^2)^{\frac{1}{2}}}{2p} \left[ e^{-\frac{i}{\hbar}(m_S - \frac{1}{2}\gamma_S)t} |K_S^0\rangle + e^{-\frac{i}{\hbar}(m_L - \frac{1}{2}\gamma_L)t} |K_L^0\rangle \right] \\
 &= \frac{(|p|^2 + |q|^2)^{\frac{1}{2}}}{2p} \left[ e^{-\frac{i}{\hbar}(m_S - \frac{1}{2}\gamma_S)t} \frac{1}{(|p|^2 + |q|^2)^{\frac{1}{2}}} (p|K^0\rangle + q|\bar{K}^0\rangle) \right. \\
 &\quad \left. + e^{-\frac{i}{\hbar}(m_L - \frac{1}{2}\gamma_L)t} \frac{1}{(|p|^2 + |q|^2)^{\frac{1}{2}}} (p|K^0\rangle - q|\bar{K}^0\rangle) \right] \\
 &= \frac{1}{2p} [p(e^{-\frac{i}{\hbar}(m_S - \frac{1}{2}\gamma_S)t} + e^{-\frac{i}{\hbar}(m_L - \frac{1}{2}\gamma_L)t}) |K^0\rangle \\
 &\quad + q(e^{-\frac{i}{\hbar}(m_S - \frac{1}{2}\gamma_S)t} - e^{-\frac{i}{\hbar}(m_L - \frac{1}{2}\gamma_L)t}) |\bar{K}^0\rangle].
 \end{aligned}
 \tag{12.55}$$

Thus, the probability of finding the state  $|K^0\rangle$  in the beam at a later time

$t$  is given by

$$\begin{aligned}
 P(K^0, t) &= |\langle K^0 | K^0(t) \rangle|^2 \\
 &= \frac{1}{4} |(e^{-\frac{i}{\hbar} (m_S - \frac{i}{2} \gamma_S)t} + e^{-\frac{i}{\hbar} (m_L - \frac{i}{2} \gamma_L)t})|^2 \\
 &= \frac{1}{4} \left( e^{-\frac{\gamma_S t}{\hbar}} + e^{-\frac{\gamma_L t}{\hbar}} + e^{-\frac{1}{2\hbar} (\gamma_S + \gamma_L)t} \times 2 \cos(m_L - m_S) \frac{t}{\hbar} \right) \\
 &= \frac{1}{4} e^{-\frac{t}{\tau_S}} + \frac{1}{4} e^{-\frac{t}{\tau_L}} + \frac{1}{2} e^{-\frac{1}{2} (\frac{1}{\tau_S} + \frac{1}{\tau_L})t} \cos \frac{\Delta m}{\hbar} t, \tag{12.56}
 \end{aligned}$$

where we have defined the mass difference

$$\Delta m = m_L - m_S. \tag{12.57}$$

Similarly, we can obtain the probability of finding the state  $|\bar{K}^0\rangle$  at a time  $t$  in the original  $|K^0\rangle$  beam as

$$\begin{aligned}
 P(\bar{K}^0, t) &= |\langle \bar{K}^0 | K^0(t) \rangle|^2 \\
 &= \left| \frac{q}{p} \right|^2 \left[ \frac{1}{4} e^{-\frac{t}{\tau_S}} + \frac{1}{4} e^{-\frac{t}{\tau_L}} \right. \\
 &\quad \left. - \frac{1}{2} e^{-\frac{1}{2} (\frac{1}{\tau_S} + \frac{1}{\tau_L})t} \cos \frac{\Delta m}{\hbar} t \right]. \tag{12.58}
 \end{aligned}$$

We see from Eqs. (12.57) and (12.58) that, if the two states  $|K_S^0\rangle$  and  $|K_L^0\rangle$  had identical masses, that is,  $\Delta m = 0$ , the beam intensities would then exhibit the sum of two exponential fall-offs, corresponding to the two characteristic lifetimes. However, what is observed in addition is an oscillatory behavior (strangeness oscillation), implying a finite mass difference for the two particles. This mass splitting can, in fact, be measured from the period of oscillation, and has the value

$$\Delta m = m_L - m_S \approx 3.5 \times 10^{-12} \text{ MeV}/c^2. \tag{12.59}$$

The mass splitting is indeed small (recall that  $m_{K^0} \approx 500 \text{ MeV}/c^2$ ), and, when converted to an upper limit on a possible  $K^0 - \bar{K}^0$  mass difference ( $m_{K^0} - m_{\bar{K}^0} < 10^{-18} m_{K^0}$ ), implies that  $CPT$  invariance holds to high accuracy for neutral kaons. It also suggests, as we will see in the next



chapter, that  $K^0-\overline{K}^0$  mixing is mainly a second-order effect in the weak Hamiltonian, generating  $\Delta S = 2$  through two  $\Delta S = 1$  transitions. This is the origin of the contribution from the indirect component of  $CP$  violation, whereas the contribution from the direct  $K_2^0 \rightarrow 2\pi$  decay arises from single  $\Delta S = 1$  transitions.

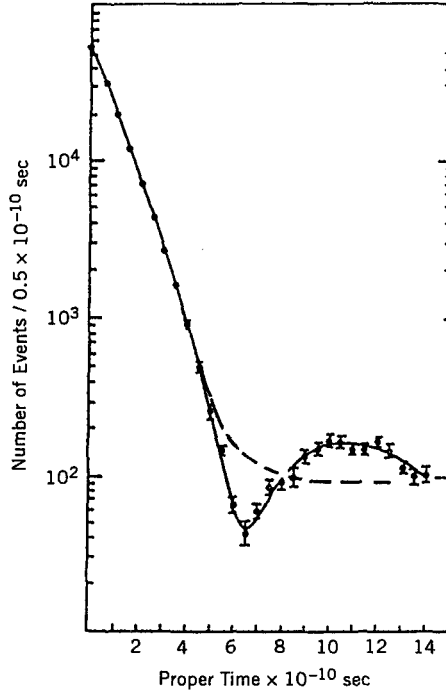


Fig. 12.4 Data for  $K_{L,S}^0 \rightarrow \pi^+\pi^-$  as a function of proper time, after passing a  $K_L^0$  beam through a carbon regenerator. The dashed curve shows the shape expected in the absence of  $K_L^0 - K_S^0$  interference. The solid curve shows a fit including interference, which is used to determine  $\phi_{+-}$ . [After W. C. Carithers et al., Phys. Rev. Lett. **34**, 1244 (1975).]

Starting off with Eq. (12.55), we can also calculate the probability of finding a  $|K_L^0\rangle$  or  $|K_S^0\rangle$  in the beam as a function of time. Because both  $|K_L^0\rangle$  and  $|K_S^0\rangle$  decay into  $\pi^+\pi^-$  pairs, by studying the number of  $\pi^+\pi^-$  (or  $\pi^0\pi^0$ ) decays as a function of proper time, we can observe quantum-mechanical interference in the two-pion decay modes of the  $|K_L^0\rangle$  and  $|K_S^0\rangle$ . That is, if we measure the square of the sum of the amplitudes

$$|(K_L^0 \rightarrow 2\pi) + (K_S^0 \rightarrow 2\pi)|^2, \quad (12.60)$$

such data can provide values of the relative phases  $\phi_{+-}$  and  $\phi_{00}$  from the interference term in Eq. (12.60). The result of this type of measurement is shown in Fig. 12.4.

## 12.8 Semileptonic $K^0$ Decays

Studies of  $K^0$  and  $\bar{K}^0$  produced in reactions such as given in Eq. (12.2), indicate that when a  $K^0$  decays semileptonically there is a positron in the final state, while for the decay of the  $\bar{K}^0$  there is an electron

$$\begin{aligned} K^0 &\longrightarrow \pi^- + e^+ + \nu_e, \\ \bar{K}^0 &\longrightarrow \pi^+ + e^- + \bar{\nu}_e. \end{aligned} \quad (12.61)$$

Under the  $CP$  operation, all particles, including neutrinos, transform to their physical antiparticles. Consequently, the above decays can, in principle, provide additional insight into  $CP$  violation in the kaon system.

One interesting possibility is to start off with a beam that is composed primarily of either  $K^0$  or  $\bar{K}^0$ , and use the fact that there is oscillation in strangeness as a function of time, to study the variation in the number of decays involving  $e^+$  (denoted by  $N^+$ ) and the number involving  $e^-$  (denoted by  $N^-$ ), as given in Eqs. (12.56) and (12.58). As the  $K_S^0$  component decays away, the original beam will turn completely into pure  $K_L^0$ . Now, if  $K_L^0$  is an eigenstate of  $CP$ , then it will have equal admixtures of  $K^0$  and  $\bar{K}^0$ , and therefore an identical number of  $e^+$  and  $e^-$  decays. If, however, the  $K_L^0$  is not an eigenstate of  $CP$ , namely if  $CP$  is violated in the neutral kaon system, then an asymmetry will eventually develop in the number of  $e^+$  and  $e^-$  decays. That is, as time goes by, and oscillations have died down (recall that  $\tau_S \ll \tau_L$ ), there will be an asymmetry observed in  $N^+$  and  $N^-$ , which depends on the relative strengths of the  $K^0$  and  $\bar{K}^0$  components in  $K_L^0$ . This strangeness asymmetry, which appears explicitly in our definition in Eq. (12.20), is observed to be  $\approx 3.3 \times 10^{-3}$  (corresponding to  $2 \operatorname{Re} \epsilon$ ), and also determines the value of  $|\frac{q}{p}|^2$  in Eq. (12.58). This result is shown in Fig. 12.5.

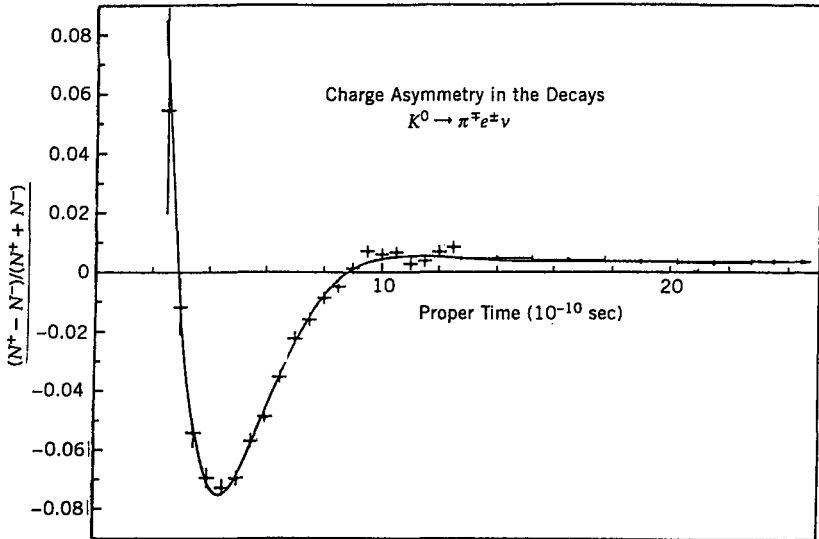


Fig. 12.5 The charge asymmetry observed for  $K^0 \rightarrow \pi^- e^+ \nu$  and  $\bar{K}^0 \rightarrow \pi^+ e^- \bar{\nu}$  as a function of proper time, when the starting beam is predominantly  $K^0$ . The observed interference effect is sensitive to the  $K_L^0 - K_S^0$  mass difference. For large values of proper time, the asymmetry represents a  $CP$  violating effect and determines the strangeness imbalance in  $K_L^0$ . [After S. Gjesdal et al., Phys. Lett. **52B**, 113 (1974).]

## Problems

**12.1** Ignoring  $CP$  violation, plot to  $\approx 10\%$  accuracy the probability of observing of  $\bar{K}^0$  as a function of time in a beam that is initially ( $t = 0$ ) pure  $K^0$ .

**12.2** Using the parameters  $\eta_{+-}$  and  $\phi_{+-}$  of Eq. (12.27), derive an expression for the rate of  $K^0 \rightarrow \pi^+ \pi^-$  decay as a function of time. Assume that you start with a pure  $K^0$  beam that develops according to Eq. (12.55). You may ignore the overall normalization of the decay rate.

## Suggested Readings

Frauenfelder, H., and E. M. Henley, *Subatomic Physics*, Prentice-Hall (1991).

Griffiths, D., *Introduction to Elementary Particles*, Wiley (1989).

Kabir, P. K., *The CP Puzzle*, Academic Press (1968).

Perkins, D. H., *Introduction to High Energy Physics*, Cambridge Univ. Press (2000).

Williams, W. S. C., *Nuclear and Particle Physics*, Oxford Univ. Press (1997).



## Chapter 13

# Formulation of the Standard Model

### 13.1 Introductory Remarks

In Chapter 9, we discussed the properties of only a few low-mass hadrons discovered prior to the mid 1970s. As the energies of accelerators increased, additional excited states of those particles, but with larger masses and higher spins, as well as particles with new flavors (see Table 9.5) were found. In fact, even by the mid 1960s, there was a whole host of particles to contend with, and it was questioned whether they could all be regarded as fundamental constituents of matter. As we argued previously, even the lightest baryons, namely the proton and the neutron, show indirect evidence of substructure. For example, the large anomalous magnetic moments observed for these particles, especially dramatic for the neutron, imply a complex internal distribution of currents. From the pattern of the observed spectrum of hadrons, Murray Gell-Mann and George Zweig suggested independently in 1964 that all such particles could be understood as composed of quark constituents. As shown in Table 9.5, these constituents had rather unusual properties, and were initially regarded as calculational tools rather than as true physical objects.

A series of measurements performed in the late 1960s at the Stanford Linear Accelerator Center (SLAC) on electron scattering from hydrogen and deuterium revealed that the data could be most easily understood if protons and neutrons were composed of point-like objects that had charges of  $-\frac{1}{3}e$  and  $+\frac{2}{3}e$ . These experiments, led by Jerome Friedman, Henry Kendall and Richard Taylor, corresponded to a modern parallel of the original work of the Rutherford group, where, instead of finding “point-like” nuclei within atoms, the presence of point-like *quarks* or *partons* was deduced from the characteristics of inelastically scattered electrons. In the

original experiments of the Rutherford group, nuclei were not probed very deeply, and therefore did not break apart in collisions with  $\alpha$ -particles. On the other hand, the scattering of electrons at SLAC involved sufficient momentum transfers to break apart the neutrons and protons.

It is perhaps worth expanding somewhat on the difference between elastic and inelastic scattering of electrons from nucleon targets. For elastic scattering at high energy, the form factor of Eq. (2.14), obtained from measurements at low energies, provides an adequate description of the differential cross section. However, inelastic scattering, where the proton does not stick together, offers the possibility of probing for substructure within the nucleon. In particular, the inelastic scattering of electrons at large  $q^2$  corresponds to interactions that take place at very small distances, and are therefore sensitive to the presence of point-like constituents within the nucleon. In fact, the form factor for inelastic scattering at large  $q^2$  becomes essentially independent of  $q^2$ , reflecting the presence of point-like objects within the nucleon. This is reminiscent of the large-angle contribution to the Rutherford scattering of low-energy  $\alpha$ -particles on the “point-like” nucleus of the atom. It was eventually clarified that the nucleon contained charged quark-partons as well as neutral *gluon*-partons, both described by their individual characteristic momentum distributions (the *parton distribution functions*).

By the early 1970s, it became quite apparent that hadrons were not fundamental point-like objects. In contrast, leptons still do not exhibit any evidence of structure, even at highest momentum transfers. It is natural therefore to regard leptons as elementary, but to regard hadrons as composed of more fundamental constituents. This line of thought – completely phenomenological in the beginning – merged the observations from electron scattering with those from particle spectroscopy and the quark model, and culminated in the present Standard Model. The Standard Model incorporates all the known fundamental particles, namely, the quarks, leptons and the gauge bosons, and it provides a theory describing three of the basic forces of nature – the strong, weak and electromagnetic interactions.

## 13.2 Quarks and Leptons

As we saw earlier, each charged lepton has its own neutrino, and there are three families (or flavors) of such leptons, namely,

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}, \quad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}. \quad (13.1)$$

In writing this, we have used the convention introduced previously in connection with strong isospin symmetry, namely, the higher member of a given multiplet carries a higher electric charge. The quark constituents of hadrons also come in three families (see Problem 9.4)

$$\begin{pmatrix} u \\ d \end{pmatrix}, \quad \begin{pmatrix} c \\ s \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} t \\ b \end{pmatrix}. \quad (13.2)$$

The charges and baryon content of the different quarks were given in Table 9.5. The baryon numbers are  $B = \frac{1}{3}$  for all the quarks, and the charges are

$$\begin{aligned} Q[u] = Q[c] = Q[t] &= +\frac{2}{3} e, \\ Q[d] = Q[s] = Q[b] &= -\frac{1}{3} e. \end{aligned} \quad (13.3)$$

Although the fractional nature of their electric charges was deduced indirectly from electron scattering for only the  $u$  and  $d$  quarks, phenomenologically, such charge assignments also provide a natural way for classifying the existing hadrons as bound states of quarks. Quarks also appear to have flavor quantum numbers, as given in Table 9.5. For example, because we defined the strangeness of the  $K^+$  as  $+1$ , we will see shortly that the strange quark will have to be assigned a strangeness of  $-1$ . The charm, top and the bottom quarks, correspondingly, carry their own flavor quantum numbers. Of course, each quark has its own antiquark, which has opposite electric charge and other internal quantum numbers such as strangeness and charm.

### 13.3 Quark Content of Mesons

The quarks, like the leptons, are point-like fermionic particles. In other words, they have spin angular momentum of  $\frac{1}{2}\hbar$ . This suggests that, since mesons have integer spin, then, if they are bound states of quarks, they can only consist of an even number of these particles. In fact, every known



meson can be described as a bound state of a quark and an antiquark. Thus, for example, a  $\pi^+$  meson, which has spin zero and electric charge  $+1$ , can be described as the bound state

$$\pi^+ = u\bar{d}. \quad (13.4)$$

It follows, therefore, that the  $\pi^-$  meson, which is the antiparticle of the  $\pi^+$ , can be described as the bound state

$$\pi^- = \bar{u}d. \quad (13.5)$$

The  $\pi^0$  meson, which is charge neutral, can, in principle, be described as a bound state of any quark and its antiquark. However, other considerations, such as the fact that all three  $\pi$ -mesons belong to a strong-isospin multiplet, and should therefore have the same internal structure, lead to a description of the  $\pi^0$  meson as

$$\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}). \quad (13.6)$$

The strange mesons can similarly be described as bound states of a quark and an antiquark, where one of the constituents is strange. Thus, we can identify the following systems

$$\begin{aligned} K^+ &= u\bar{s}, \\ K^- &= \bar{u}s, \\ K^0 &= d\bar{s}, \\ \bar{K}^0 &= \bar{d}s. \end{aligned} \quad (13.7)$$

It is quite easy to check that not only are the charge assignments right, but even the strangeness quantum numbers work out to be correct if we assign a strangeness quantum number  $S = -1$  to the  $s$ -quark. Because there are quarks with higher mass and new flavor quantum numbers, phenomenologically, on the basis of the quark model, we would also expect new kinds of mesons. Many such mesons have already been found. For example, the charge-neutral  $J/\psi$  meson, whose discovery in 1974 by independent groups headed by Samuel Ting and by Burton Richter suggested first evidence for

the existence of the charm quark, can be described as a bound state of charmonium (named in analogy with positronium)

$$J/\psi = c\bar{c}. \quad (13.8)$$

This is a “normal” meson, in the sense that the quantum numbers of charm add up to zero, but its properties (decays) cannot be explained using only the older  $u$ ,  $d$  and  $s$  quarks. There are, of course, mesons that contain *open* charm, such as

$$\begin{aligned} D^+ &= c\bar{d}, \\ D^- &= \bar{c}d, \\ D^0 &= c\bar{u}, \\ \overline{D^0} &= \bar{c}u. \end{aligned} \quad (13.9)$$

We can think of such mesons as the charm analogs of the  $K$  mesons, and the properties of these mesons have by now been studied in great detail. In analogy with the  $K^+$ , the  $D^+$  meson is defined to have charm flavor of +1, which then defines the charm quantum number for the  $c$ -quark to be +1. There are also mesons that carry both strangeness and charm quantum numbers, two of these are denoted as

$$\begin{aligned} D_s^+ &= c\bar{s}, \\ D_s^- &= \bar{c}s. \end{aligned} \quad (13.10)$$

Finally, there is also extensive evidence for hadrons in which one of the constituents is a bottom quark. For example, the  $B$  mesons, analogous to the  $K$  mesons, have structure of the form

$$\begin{aligned} B^+ &= u\bar{b}, \\ B^- &= \bar{u}b, \\ B_d^0 &= d\bar{b}, \\ \overline{B_d^0} &= \bar{d}b. \end{aligned} \quad (13.11)$$

The charge-neutral states involving  $b$  and  $s$  quarks are particularly interesting because, just like the  $K^0$ - $\bar{K}^0$  system, they exhibit  $CP$  violation in their decays

$$\begin{aligned} B_s^0 &= s\bar{b}, \\ \bar{B}_s^0 &= \bar{s}b. \end{aligned} \tag{13.12}$$

Recent experiments at two  $e^+e^-$  colliders, one at SLAC (“BaBar”) and one at the KEK accelerator at Tsukuba, Japan (“BELLE”), both referred to as “ $B$ -factories”, have studied these neutral  $B$  mesons in the clean environment of  $e^+e^-$  collisions, and have found clear evidence for large violation of  $CP$  symmetry in decays of  $B^0$  mesons produced in pairs

$$e^+ + e^- \longrightarrow B + \bar{B}. \tag{13.13}$$

Extensive studies of  $B$  decays are currently being pursued at both  $e^+e^-$  and at hadron colliders, to search for any discrepancies with expectations from the Standard Model.

### 13.4 Quark Content of Baryons

Just as mesons can be thought of as bound states of quarks and antiquarks, so can baryons be considered as constructed out of these constituents. But because baryons carry half-integral spin angular momenta (they are fermions), they can be formed from only an odd number of quarks. Properties of baryons are most consistent with being composed of only three quarks. Thus, we can think of the proton and the neutron as corresponding to the bound states

$$\begin{aligned} p &= uud, \\ n &= udd. \end{aligned} \tag{13.14}$$

Similarly, the hyperons, which carry a strangeness quantum number, can be described by

$$\begin{aligned}
 \Lambda^0 &= uds, \\
 \Sigma^+ &= uus, \\
 \Sigma^0 &= uds, \\
 \Sigma^- &= dds.
 \end{aligned}
 \tag{13.15}$$

Also, the cascade particles, which carry two units of strangeness, can be described as

$$\begin{aligned}
 \Xi^0 &= uss, \\
 \Xi^- &= dss.
 \end{aligned}
 \tag{13.16}$$

Since all baryons have baryon number of unity, it follows therefore that each quark must carry a baryon number of  $\frac{1}{3}$ . Furthermore, since a meson consists of a quark and an antiquark, and since an antiquark would have a baryon number  $-\frac{1}{3}$ , we conclude that mesons do not carry baryon number, which is consistent with our previous discussion.

### 13.5 Need for Color

Extending the quark model to all baryons, leads to a theoretical difficulty. We have already discussed the  $\Delta^{++}$  baryon, which is nonstrange, carries two units of positive charge, and has spin angular momentum of  $\frac{3}{2}$ . Thus, naively, we can conclude that the  $\Delta^{++}$  can be described by three up quarks

$$\Delta^{++} = uuu.
 \tag{13.17}$$

This substructure satisfies all the known quantum numbers, and, in the ground state (where there are no contributions from relative orbital waves), the three up quarks can have parallel spins to provide a resultant value of  $J = \frac{3}{2}$ . However, the wave function for this final state, representing three identical fermions, would therefore be symmetric under the exchange of any two quarks. This is, of course, incompatible with the Pauli principle, which requires a wave function containing identical fermions to be totally antisymmetric. It would appear, therefore, that the quark model cannot describe the  $\Delta^{++}$ . On the other hand, the model works so well for other

hadrons that it would seem unwise to give it up entirely. An interesting resolution can be attained if it is assumed that all quarks carry an additional internal quantum number, and that the final state in Eq. (13.17) is, in fact, antisymmetric in the space corresponding to this quantum number.

This additional degree of freedom is referred to as *color*, and it is believed that each of the quarks comes in three different colors. Namely, the quark multiplets take the form

$$\begin{pmatrix} u^a \\ d^a \end{pmatrix}, \quad \begin{pmatrix} c^a \\ s^a \end{pmatrix}, \quad \begin{pmatrix} t^a \\ b^a \end{pmatrix}, \quad a = \text{red, blue, green.} \quad (13.18)$$

At this point of our development, color can be regarded as merely a new quantum number needed for phenomenological reasons for understanding the substructure of hadrons. However, we will see shortly that, in fact, color is to the strong interaction what charge is to the electromagnetic force, namely the source of the respective fields.

Hadrons do not appear to carry any net color, and therefore correspond to bound states of quarks and antiquarks of zero total color quantum number, or, simply stated, hadrons are color-neutral bound states of quarks. Under the interchange of any two quarks, the color singlet wave function of three quarks changes sign, while that of a quark-antiquark color singlet does not. This hypothesis leads to an excellent description of all known baryons as bound states of three quarks, and of mesons as bound states of quark-antiquark pairs. In particular, it also explains the structure of the  $\Omega^-$  baryon which has a strangeness of  $-3$  and spin angular momentum of  $\frac{3}{2}$ , and corresponds to the ground state of three strange quarks

$$\Omega^- = sss. \quad (13.19)$$

We see once again how the symmetry property in color space plays a crucial role in assuring the overall antisymmetry of the fermionic wave function for this state.

The theoretical postulate of color seems rather ad hoc, especially since the observable hadrons do not carry a color quantum number. However, the existence of color can be established as follows. Consider the annihilation of an electron and positron, leading to the creation of a  $\mu^+\mu^-$  pair or a quark-antiquark pair. The reaction can be thought of as proceeding through the production of an intermediate virtual photon, as shown in Fig. 13.1. The

cross section for the production of hadrons in this process depends on the number of ways a photon can produce a quark-antiquark pair. This must therefore be proportional to the number of available quark colors. That is, the ratio of production cross sections

$$R = \frac{\sigma(e^-e^+ \rightarrow \text{hadrons})}{\sigma(e^-e^+ \rightarrow \mu^-\mu^+)}, \quad (13.20)$$

is proportional to the number of quark colors. And, indeed, this quantity is consistent with exactly three colors. Because the production of hadrons through the mechanism in Fig 13.1 depends, in addition, on the electric charges of the quarks, such data also confirm the fractional nature of electric charge carried by quarks.

In closing this section, we wish to point out that electron-positron annihilation at high energies is one of the cleanest ways to establish the presence of new quark flavors. For example, when the energy of the  $e^+e^-$  system exceeds the threshold for production of hadrons containing some new quark, the ratio in Eq. (13.20) must increase and display a step at that energy. Of course, in addition, beyond that threshold, new hadrons containing the new flavor can be observed in the final states of such collisions. This was found to be the case for charm and bottom quarks, where after the initial production of the analogs of positronium (the  $J/\psi$  for  $c\bar{c}$ , and the  $\Upsilon$  for  $b\bar{b}$ ), particles with open charm and open bottom flavor were observed to be produced at somewhat higher energies. However, this is not likely to repeat for the top quark, which, as we have indicated before, decays too rapidly after its production to be able to form hadrons.

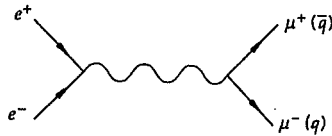


Fig. 13.1 The annihilation of  $e^+e^-$  through a virtual photon into  $\mu^+\mu^-$  or  $q\bar{q}$  pair.

## 13.6 Quark Model for Mesons

We will now apply the symmetry requirements of the strong interaction to  $q\bar{q}$  wave functions, and thereby deduce the quantum numbers that we would expect for the spectrum of charge-neutral meson states in a simple

non-relativistic quark model. Specifically, we will establish the restrictions on the spin ( $J$ ), parity ( $P$ ) and charge conjugation ( $C$ ) quantum numbers that apply to such systems. The  $q\bar{q}$  wave function is a product of separate wave functions, each of which has a unique symmetry under the exchange of the two particles

$$\Psi = \psi_{\text{space}} \psi_{\text{spin}} \psi_{\text{charge}}, \quad (13.21)$$

where  $\psi_{\text{space}}$  denotes the space-time part of the  $q\bar{q}$  wave function,  $\psi_{\text{spin}}$  represents the intrinsic spin, and  $\psi_{\text{charge}}$  the charge conjugation properties. We have ignored the part of the wave function that is associated with the color degree of freedom because we know that this will always have even symmetry for mesons.

The symmetry of  $\psi_{\text{space}}$  under the exchange of the  $q$  and  $\bar{q}$  is, as usual, determined by the spherical harmonics and the relative orbital angular momentum of the  $q$  and  $\bar{q}$ . If we call the exchange operation  $X$ , then, schematically, we have

$$X\psi_{\text{space}} \sim X Y_{\ell m}(\theta, \phi) = (-1)^\ell \psi_{\text{space}}. \quad (13.22)$$

Therefore, if  $\Psi$  is a state of definite parity, the spatial part of the wave function will be either symmetric or antisymmetric under exchange, depending on whether  $\ell$  is even or odd.

The effect of the exchange operation on  $\psi_{\text{spin}}$  will depend on whether the two quark spins are in a spin state  $s = 0$  or  $s = 1$ . Considering the states with  $s_z = 0$ , we obtain

$$\begin{aligned} s = 0: \quad X[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle] &= -[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle], \\ s = 1: \quad X[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle] &= +[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle]. \end{aligned} \quad (13.23)$$

Thus, we deduce that

$$X\psi_{\text{spin}} = (-1)^{s+1} \psi_{\text{spin}}. \quad (13.24)$$

Under the action of the exchange operator,  $q$  and  $\bar{q}$  become interchanged, and consequently, we can think of this as the operation of charge conjugation in the space of  $\psi_{\text{charge}}$ . To determine the charge conjugation properties of such a state, let us impose the Pauli principle on our two-fermion system,

namely, let us require that the overall wave function change sign under an interchange of  $q$  and  $\bar{q}$ . Note that we are using a generalized form of the Pauli principle, which treats  $q$  and  $\bar{q}$  as identical fermions corresponding to spin up and spin down states in the space of  $\psi_{\text{charge}}$ . Thus, we require

$$X\Psi = -\Psi. \quad (13.25)$$

Now, using the results of Eqs. (13.22), (13.24) and (13.25), we can write

$$\begin{aligned} X\Psi &= X\psi_{\text{space}}X\psi_{\text{spin}}X\psi_{\text{charge}} \\ &= (-1)^\ell\psi_{\text{space}}(-1)^{s+1}\psi_{\text{spin}}C\psi_{\text{charge}} = -\Psi. \end{aligned} \quad (13.26)$$

Consequently, for Eq. (13.26) to hold, we conclude that the meson state must be an eigenstate of charge conjugation with charge parity

$$\eta_C = (-1)^{\ell+s}. \quad (13.27)$$

Thus, for meson states that are eigenstates of charge conjugation, Eq. (13.27) establishes a relationship between the orbital wave, the intrinsic spin value, and the  $C$  quantum number of the  $q\bar{q}$  system.

The only relevant quantum number that is still missing in our discussion is the parity of the allowed states. The parity of  $\Psi$  is given by the product of the intrinsic parities of the constituents and the effect from inversion of spatial coordinates. As we discussed in Chapter 11, the relative intrinsic parity of a particle and an antiparticle with spin  $\frac{1}{2}$  is odd. Consequently, the total parity of our state  $\Psi$  is

$$P\Psi = -(-1)^\ell\Psi = (-1)^{\ell+1}\Psi,$$

or, the total parity quantum number is

$$\eta_P = (-1)^{\ell+1}. \quad (13.28)$$

Since the spins of mesons are obtained from the addition of the orbital and intrinsic angular momenta of the  $q\bar{q}$  pair

$$\vec{J} = \vec{L} + \vec{S}, \quad (13.29)$$



we now have all the ingredients for forming an allowed spectrum of mesons. Table 13.1 lists the possible lowest-lying states, all of which correspond to known mesons.

**Table 13.1** Lowest-lying meson states expected in the quark model.

$\ell$	$s$	$j$	$\eta_P$	$\eta_C$	Meson <sup>a</sup>
0	0	0	-	+	$\pi^0, \eta$
0	1	1	-	-	$\rho^0, \omega, \phi$
1	0	1	+	-	$b_1^0(1235)$
1	1	0	+	+	$a_0(1980), f_0(975)$
1	1	1	+	+	$a_1^0(1260), f_1(1285)$
1	1	2	+	+	$a_2^0(1320), f_2(1270)$

<sup>a</sup>For other properties of these mesons, see the *CRC Handbook*.

### 13.7 Valence and Sea Quarks in Hadrons

Regarding the substructure of hadrons, we have mentioned that the nucleon appears to contain both quarks ( $q$ ) and gluons ( $g$ ), and, in fact, we have also mentioned a momentum distribution for partons. The quark model of hadrons must be therefore be recognized as the analog of the description of valence electrons in an atom or the valence nucleons in a nucleus. Just as there are more constituents in the closed shells of atoms or nuclei, so there are also more “paired” quark-antiquark systems within hadrons. These quarks are referred to as the *sea quarks*, as opposed to the *valence quarks* that characterize hadronic quantum numbers. And, in fact, the additional contribution from the color-carrying gluons (see following two sections) corresponds to about half of the content of the nucleon.

Searches for other kinds of hadrons have been performed in order to identify new possible states of matter that would correspond to, for example, valence systems of  $q\bar{q}q\bar{q}$ , *hybrid* mesons composed of  $q\bar{q}g$ , or *glueballs*

made of  $gg$  systems. There is some evidence for the existence of such expected states, but it is as yet not compelling.

### 13.8 Weak Isospin and Color Symmetry

As we have seen, leptons and quarks come as doublets, or in pairs, and quarks, in addition, carry a color quantum number. The existence of such groupings, and the color degrees of freedom, suggest the presence of new underlying symmetries for this overall structure. From our discussion of spin and isospin, we can associate the doublet structure with a non-commuting (non-Abelian) symmetry group  $SU(2)$ . We will continue to refer to this underlying symmetry group as isospin, since it is an internal symmetry. Unlike strong isospin, which is used only to classify hadrons, the isospin in the present case also classifies leptons. Leptons, on the other hand, interact weakly and, therefore, this symmetry must be related to the weak interaction. Correspondingly, the isospin symmetry associated with the weak interactions of quarks and leptons is referred to as *weak isospin*. This symmetry is quite distinct from that of the strong isospin symmetry that we discussed previously. But, as with strong isospin, where the symmetry is discernible only when the electromagnetic interaction (electric charge) can be ignored, so is the essential character of the weak isospin symmetry also apparent only when the electromagnetic force is “turned off”. Under such circumstances, the up and down states of Eqs. (13.1) and (13.2) become equivalent and cannot be distinguished.

For the case of weak-isospin symmetry, we can define a weak hypercharge for each quark and lepton, based on a general form of the Gell-Mann–Nishijima relation of Eq. (9.26), namely,

$$Q = I_3 + \frac{Y}{2},$$

or  $Y = 2(Q - I_3),$  (13.30)

where  $Q$  is the charge of the particle, and  $I_3$  the projection of its weak isospin quantum number. Thus, for the  $(\nu, e^-)$  doublet we obtain

$$\begin{aligned}
 Y(\nu) &= 2 \left( 0 - \frac{1}{2} \right) = -1, \\
 Y(e^-) &= 2 \left( -1 + \frac{1}{2} \right) = -1.
 \end{aligned}
 \tag{13.31}$$

Similarly, for the  $(u, d)$  quark doublet, we have

$$\begin{aligned}
 Y(u) &= 2 \left( \frac{2}{3} - \frac{1}{2} \right) = 2 \times \frac{1}{6} = \frac{1}{3}, \\
 Y(d) &= 2 \left( -\frac{1}{3} + \frac{1}{2} \right) = 2 \times \frac{1}{6} = \frac{1}{3}.
 \end{aligned}
 \tag{13.32}$$

The weak hypercharge quantum number for other quark and lepton doublets can be obtained in the same manner. In fact, in the Standard Model, only left-handed particles have a doublet structure. The right-handed quarks and the right-handed charged leptons are all singlets with  $I = 0$ , and there are no right-handed neutrinos. As can be seen from Eq. (13.30), the weak hypercharge quantum number is the same for both members of any doublet, which is required if weak hypercharge is to be regarded as a  $U(1)$  symmetry of the type specified in Eq (10.76).

The color symmetry of quarks is also an internal symmetry. It can be shown that it is similar to isospin in that it involves rotations – however, the rotations are in an internal space of three dimensions – corresponding to the three distinct colors of the quarks. The relevant symmetry group is known as  $SU(3)$ . The interactions of quarks are assumed to be invariant under such  $SU(3)$  rotations in color space, leading to an equivalence of quarks of different color. (This is needed in order to have consistency with experimental observations.) Because the color quantum number is carried by quarks and not by leptons or photons, we expect this symmetry to be associated only with the strong interaction.

### 13.9 Gauge Bosons

As we have seen, the presence of a global symmetry can be used to classify particle states according to some quantum number (e.g., strong isospin), while the presence of a local symmetry requires the introduction of forces. Since weak isospin and color symmetry are associated with rather distinct

interactions, it is interesting to ask whether the corresponding physical forces – namely, the strong (color) and the weak forces – might arise purely from the requirement that these symmetries be local. Years of painstaking theoretical development, coupled with detailed experimental verification, has led to the conclusion that this is indeed very likely. It is the current understanding that the local symmetries underlying the electromagnetic, weak, and strong interactions have origin in the  $U_Y(1)$ ,  $SU_L(2)$ , and  $SU_{\text{color}}(3)$  symmetry groups, respectively. The group corresponding to the weak hypercharge symmetry,  $U_Y(1)$ , is a local Abelian symmetry group, while  $SU_L(2)$  and  $SU_{\text{color}}(3)$  are non-Abelian groups corresponding to the weak isospin and color symmetries.<sup>1</sup> From the Gell-Mann–Nishijima formula of Eq. (13.30), we see that electric charge is related to weak hypercharge and weak isospin, from which it follows that the electromagnetic  $U_Q(1)$  symmetry can be regarded as a particular combination of the weak isospin and weak hypercharge symmetries.

In Chapter 10, we showed in an example how local invariance necessarily leads to the introduction of gauge potentials, such as the vector potential in electromagnetic interactions. When these potentials are quantized, they provide the carriers of the force, otherwise known as gauge particles. Thus, the photon is the carrier of the electromagnetic interaction, or its gauge boson. All the gauge bosons have spin  $J = 1$ , and the number of gauge bosons associated with any symmetry reflects the nature of that symmetry group. There are three gauge bosons associated with the weak interactions, and they are known as the  $W^+$ ,  $W^-$ , and  $Z^0$  bosons. (These were discovered independently in 1983 by Carlo Rubbia and collaborators and Pierre Darriulat and collaborators at the antiproton-proton collider at the CERN Laboratory outside of Geneva, Switzerland.) For the strong interactions, there are eight gauge bosons, and all are referred to as gluons. (These are the same gluons we have been discussing in connection with the substructure of the nucleon.) The gluons, or the gauge bosons of color symmetry, are electrically neutral, but carry the color quantum number. This is in contrast to the photon, which is the carrier of the force between charged particles, but does not itself carry electric charge. This difference can be attributed to the Abelian nature of the  $U_Q(1)$  symmetry that describes the photon, and the non-Abelian nature of  $SU_{\text{color}}(3)$  that describes gluons.

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<sup>1</sup>Because the doublet structure of quarks and leptons involves only left-handed particles, the weak isospin symmetry group is also conventionally denoted by  $SU_L(2)$ . This kind of structure is essential for incorporating the properties of neutrinos and of parity violation in weak interactions.

Figure 13.2 displays several examples of how the different gauge bosons can induce transitions between different fermions, and, for the case of the color force, also between gluons.

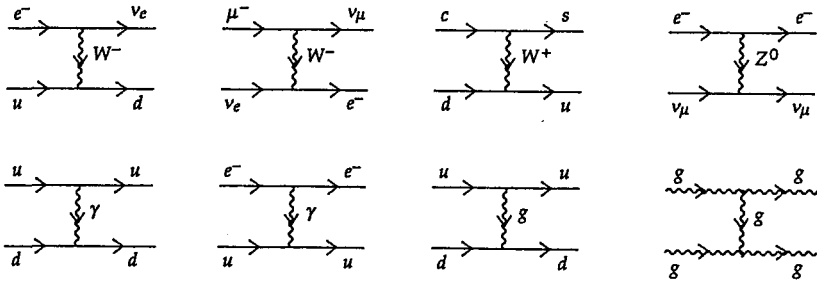


Fig. 13.2 Typical interactions among leptons and quarks and between gluons, mediated by different fundamental gauge bosons.

### 13.10 Dynamics of the Gauge Particles

In this section, we highlight some basic features of the dynamics of the gauge particles. For simplicity, we first consider Maxwell's equations, which describe the dynamics of the photon: the gauge particle associated with electromagnetic interactions. For the other interactions, the dynamical equations for the corresponding gauge bosons, although similar in form, are more complicated. However, the conclusions regarding the properties of such particles can be garnered from the simpler case. Consider Maxwell's equations in vacuum

$$\begin{aligned}
 \vec{\nabla} \cdot \vec{E} &= 0, \\
 \vec{\nabla} \cdot \vec{B} &= 0, \\
 \vec{\nabla} \times \vec{E} &= -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}, \\
 \vec{\nabla} \times \vec{B} &= \frac{1}{c} \frac{\partial \vec{E}}{\partial t}.
 \end{aligned} \tag{13.33}$$

From the structure of the second and the third equations, we see that we can define the electric and magnetic fields in terms of the usual scalar and vector potentials as

$$\begin{aligned}\vec{E} &= -\vec{\nabla}\phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}, \\ \vec{B} &= \vec{\nabla} \times \vec{A},\end{aligned}\tag{13.34}$$

where  $\vec{A}$  corresponds to the vector potential introduced in Eq. (10.78), and  $\phi$  is the scalar potential. The contents of Eqs. (13.33) can be expressed equally well in terms of these potentials. An important consequence of the definitions in Eq. (13.34) is that the electric and the magnetic fields are not sensitive to the following local changes or redefinitions of the gauge potentials

$$\begin{aligned}\delta\phi &= -\frac{1}{c} \frac{\partial \alpha(\vec{r}, t)}{\partial t}, \\ \delta\vec{A} &= \vec{\nabla}\alpha(\vec{r}, t).\end{aligned}\tag{13.35}$$

Namely, under the transformation given in Eq. (13.35), where  $\alpha(\vec{r}, t)$  is an arbitrary scalar function of space-time coordinates, the  $\vec{E}$  and  $\vec{B}$  fields do not change

$$\begin{aligned}\delta\vec{E} &= -\vec{\nabla}\delta\phi - \frac{1}{c} \frac{\partial \delta\vec{A}}{\partial t} \\ &= \vec{\nabla} \frac{1}{c} \frac{\partial \alpha}{\partial t} - \frac{1}{c} \frac{\partial}{\partial t} (\vec{\nabla}\alpha) = 0, \\ \delta\vec{B} &= (\vec{\nabla} \times \delta\vec{A}) = \vec{\nabla} \times (\vec{\nabla}\alpha) = 0.\end{aligned}\tag{13.36}$$

The transformations in Eq. (13.35) are, in fact, equivalent to the gauge transformation introduced in Eq. (10.79), where  $\alpha(\vec{r})$  was the static, position dependent, phase of the transformation. Because the  $\vec{E}$  and the  $\vec{B}$  fields do not change under the gauge transformations of Eq. (13.35), Maxwell's equations must therefore also be invariant under such redefinitions of the potentials. This invariance corresponds to the  $U_Q(1)$  symmetry of electromagnetic interactions.

A direct consequence of this gauge invariance of Maxwell's equations is that it provides solutions corresponding to transverse electromagnetic waves that propagate at the speed of light. To see this, we can take the curl of the third equation, and substitute for  $\vec{\nabla} \times \vec{B}$  using the fourth equation of Eq. (13.33), as follows

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{1}{c} \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B}) = -\frac{1}{c} \frac{\partial}{\partial t} \left( \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \right).$$

Expanding  $\vec{\nabla} \times (\vec{\nabla} \times \vec{E})$ , we get

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \vec{\nabla}^2 \vec{E} = -\frac{1}{c} \frac{\partial}{\partial t} \left( \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \right),$$

and setting  $\vec{\nabla} \cdot \vec{E} = 0$ , we obtain

$$\left( \vec{\nabla}^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{E} = 0. \quad (13.37)$$

Consequently, Eq. (13.37) follows from Eq. (13.33) and the feature that the electric field is transverse to the direction of propagation of the  $\vec{E}$  field. In fact, Eq. (13.37) describes a relativistic traveling wave propagating at the speed of light. Similarly, the other pair of Maxwell's equations yield

$$\left( \vec{\nabla}^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{B} = 0. \quad (13.38)$$

When these fields are quantized, they correspond to massless particles (photons) that reflect the long-range nature of the Coulomb interaction.

To see that masslessness of a gauge particle is a consequence of gauge invariance, let us write the equation for a traveling wave, but for a particle of mass  $m$  (this is known as the Proca Equation, after Alexandre Proca, with the corresponding equation for a spin-zero field being the Klein-Gordon equation)

$$\begin{aligned} \left( \vec{\nabla}^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{m^2 c^2}{\hbar^2} \right) \vec{E} &= 0, \\ \left( \vec{\nabla}^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{m^2 c^2}{\hbar^2} \right) \vec{B} &= 0. \end{aligned} \quad (13.39)$$

These equations follow from a set of Maxwell-like equations for massive vector fields ( $J = 1$ ) of the type

$$\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= -\frac{m^2 c^2}{\hbar^2} \phi, \\
\vec{\nabla} \cdot \vec{B} &= 0, \\
\vec{\nabla} \times \vec{E} &= -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}, \\
\vec{\nabla} \times \vec{B} &= \frac{1}{c} \frac{\partial \vec{E}}{\partial t} - \frac{m^2 c^2}{\hbar^2} \vec{A}.
\end{aligned} \tag{13.40}$$

The fact that Eq. (13.39) is a consequence of Eq. (13.40) can be checked as follows

$$\begin{aligned}
\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) &= -\frac{1}{c} \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B}) = -\frac{1}{c} \frac{\partial}{\partial t} \left( \frac{1}{c} \frac{\partial \vec{E}}{\partial t} - \frac{m^2 c^2}{\hbar^2} \vec{A} \right), \\
\text{or } \vec{\nabla} (\vec{\nabla} \cdot \vec{E}) - \vec{\nabla}^2 \vec{E} &= -\frac{1}{c} \frac{\partial}{\partial t} \left( \frac{1}{c} \frac{\partial \vec{E}}{\partial t} - \frac{m^2 c^2}{\hbar^2} \vec{A} \right), \\
\text{or } \vec{\nabla} \left( -\frac{m^2 c^2}{\hbar^2} \phi \right) - \vec{\nabla}^2 \vec{E} &= -\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} + \frac{m^2 c}{\hbar^2} \frac{\partial \vec{A}}{\partial t}, \\
\text{or } \left( \vec{\nabla}^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{E} + \frac{m^2 c^2}{\hbar^2} \left( \vec{\nabla} \phi + \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right) &= 0, \\
\text{or } \left( \vec{\nabla}^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{m^2 c^2}{\hbar^2} \right) \vec{E} &= 0,
\end{aligned} \tag{13.41}$$

where in the last step we used the definition of  $\vec{E}$  in Eq. (13.34). Similarly, the equation for the  $\vec{B}$ -field can be derived from the other pair of equations in (13.40). Thus, the modified Maxwell's equations given in Eq. (13.40) lead to massive traveling waves, which upon quantization yield massive particles. Unfortunately, unlike Maxwell's equations in (13.33), the set of equations in (13.40) depend explicitly on the gauge potentials and are therefore no longer invariant under the gauge transformation of Eq. (13.35), unless  $m = 0$ , which demonstrates the intimate connection between the



masslessness of gauge particles and gauge invariance, namely that gauge invariance holds only for massless gauge bosons.

The preceding analysis also points to the difficulty of generalizing the gauge principle to all forces. This is because, unlike the electromagnetic interaction, the strong and the weak forces are short ranged. If all of the forces had their origin in the existence of local symmetries, and had dynamics similar to those of Maxwell's equations, then it would follow that they should all describe long-range interactions, which is, of course, not the case. The resolution to the puzzle of how the weak and the strong forces can be short ranged, despite their apparent origin in a gauge principle, is quite interesting. The mechanisms responsible for the short-ranged character of the two forces are entirely different, and we discuss first how the weak forces can become short ranged.

### 13.11 Symmetry Breaking

The meaning of symmetries can be quite subtle. As we have already seen, the invariance of the dynamical equations of a system under a set of transformations defines a symmetry of the system, whose presence can be inferred from the properties of its Hamiltonian. But even if a set of dynamical equations is invariant under some set of transformations, the solutions (physical states) of the system need not possess that symmetry. Thus, as a heuristic example, let us consider magnetism, which can be thought of as arising from the interaction of spins ( $\vec{s}$ ) situated on a lattice, and which, for a ferromagnet, can be described by a Hamiltonian of the form

$$H = -\kappa \sum_i \vec{s}_i \cdot \vec{s}_{i+1}, \quad (13.42)$$

where  $\kappa$ , a positive quantity, denotes the strength of coupling between spins at nearest-neighbor sites. If all the spins are rotated by a constant angle, their inner product is not affected, and consequently such rotations correspond to a global symmetry of the Hamiltonian for a ferromagnet. On the other hand, we can recognize from the structure of the Hamiltonian that the ground state of the system – namely, the state with the lowest energy value – has all spins parallel. Thus, a typical configuration of the spins in the ground state can be represented as in Fig. 13.3. The ground state configuration therefore picks out, at random, a preferred direction in space,

and consequently breaks the rotational symmetry of the Hamiltonian.

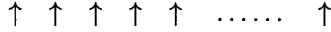


Fig. 13.3 Aligned spins in the ground state of a ferromagnet.

When the solution to a set of dynamical equations violates an inherent symmetry of these equations, we say that the symmetry of the system is broken *spontaneously*. In the case of the ground state of a ferromagnet, the spins have a long-range correlation, in that they all point along one specific direction. This kind of result appears to be a general feature of spontaneous breaking of symmetry, that is, when a symmetry is broken spontaneously, certain correlations become long-ranged, and can be thought to originate from the presence of zero-mass particles in the quantum mechanical theory. In other words, when there is spontaneous symmetry breaking, the spectrum of states of a relativistic quantum theory develops massless particles.

A more quantitative (albeit still heuristic) way to see the above result is to consider the two-dimensional classical Hamiltonian:

$$H = T + V = \frac{1}{2m} (p_x^2 + p_y^2) - \frac{1}{2} m\omega^2 (x^2 + y^2) + \frac{\lambda}{4} (x^2 + y^2)^2, \quad \lambda > 0. \quad (13.43)$$

Except for the negative rather than positive sign of the the second term, this is the Hamiltonian for a two-dimensional classical anharmonic oscillator. This Hamiltonian is invariant under rotations about the  $z$  axis, and such rotations are therefore a global symmetry of this system. The lowest energy solution, or the ground state of this system, clearly, requires the kinetic energy to vanish, since that is a positive quantity. Consequently, it follows that the minima for total energy must coincide with those of the potential. The extrema of the potential can be obtained by setting the derivatives with respect to  $x$  and  $y$  to zero

$$\begin{aligned} \frac{\partial V}{\partial x} &= x (-m\omega^2 + \lambda (x^2 + y^2)) = 0, \\ \frac{\partial V}{\partial y} &= y (-m\omega^2 + \lambda (x^2 + y^2)) = 0. \end{aligned} \quad (13.44)$$

Thus, the coordinates at the extrema of the potential satisfy the relation

$$x_{\min} = y_{\min} = 0, \quad (13.45)$$

$$\text{or } x_{\min}^2 + y_{\min}^2 = \frac{m\omega^2}{\lambda}. \quad (13.46)$$

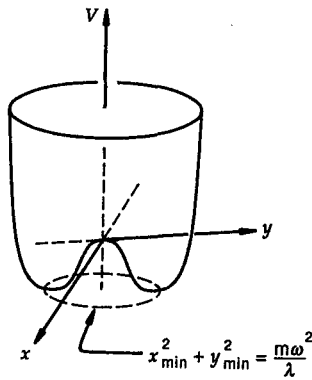


Fig. 13.4 A sketch of the potential in Eq. (13.43).

From the form of the potential (see Fig. 13.4), it can be seen that the point  $x_{\min} = y_{\min} = 0$  corresponds to a local maximum, which is a solution that is unstable under perturbation. The potential has the shape of a symmetric Mexican sombrero, with a continuous set of  $(x_{\min}, y_{\min})$  coordinates on the circle of Eq. (13.46) specifying the minimum value. If, for simplicity, we choose to define the minimum by

$$y_{\min} = 0, \quad (13.47)$$

$$x_{\min} = \sqrt{\frac{m\omega^2}{\lambda}},$$

then this choice has the effect of picking out a preferred direction in space, which breaks the rotational invariance of the system. Small motion (oscillations) around this minimum determines its stability, which can be investigated by expanding the potential about the coordinates in Eq. (13.47)

$$V(x_{\min} + x, y) = -\frac{1}{2} m\omega^2 \left( (x_{\min} + x)^2 + y^2 \right) + \frac{\lambda}{4} \left( (x_{\min} + x)^2 + y^2 \right)^2, \quad (13.48)$$

which, upon expansion to second order in  $x$  and  $y$ , and on substitution for  $x_{\min}$  from Eq. (13.47), yields

$$V(x_{\min} + x, y) = -\frac{m^2\omega^4}{4\lambda} + m\omega^2 x^2 + \text{higher orders}. \quad (13.49)$$

Consequently, the small oscillations along the  $x$ -axis are harmonic at an equivalent frequency of  $\omega_x = \sqrt{2} \omega$ , whereas the frequency of oscillation along the  $y$ -axis can be characterized by  $\omega_y = 0$ .

The small oscillations around the classical ground state also bring out the essential features of correlations in the quantum mechanical ground state, as follows. From Eq. (13.49), keeping terms only up to quadratic order, we see that for small oscillations the Hamiltonian takes the form

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + m\omega^2 x^2 - \frac{m^2\omega^2}{4\lambda}. \quad (13.50)$$

Consequently, the equation of motion along the  $y$ -axis satisfies

$$\begin{aligned} \dot{y}(t) &= \text{constant} = c, \\ \text{or } y(t) &= y(0) + ct. \end{aligned} \quad (13.51)$$

In quantum field theory, correlations are defined by the vacuum (or ground-state) expectation value of bilinear operator products involving different space-time points. Here, in analogy, we can write time correlations for  $y(t)$ , as follows<sup>2</sup>

$$\begin{aligned} \langle 0|y(t)y(0)|0\rangle &= \langle 0|y(0)y(0) + ct y(0)|0\rangle \\ &= \langle 0|y(0)y(0)|0\rangle, \end{aligned} \quad (13.52)$$

where we have used the fact that the second term vanishes when we integrate it over all space. That is, if the ground state has definite parity,

<sup>2</sup>Note that in quantum theory  $y(t)$  is a coordinate operator.

the expectation value of  $y(0)$  has to be zero for the integration in  $y$ . The expectation value in Eq. (13.52) is therefore independent of time, implying a long-time correlation. This is the simple analog to a long-distance correlation found for such systems in quantum field theories, and a result that is observed in the spin system of a ferromagnet.

We could have, of course, chosen any alternative solution to Eq. (13.46) such as

$$x_{\min} = y_{\min} = \sqrt{\frac{m\omega^2}{2\lambda}}. \quad (13.53)$$

In fact, for any solution of Eq. (13.46), it is easy to show that we can define normal modes of oscillation such that the frequency of oscillation for one of the modes is  $\sqrt{2}\omega$ , while the frequency for the orthogonal mode vanishes. Qualitatively, this can be discerned from the form of the potential: irrespective of which point is chosen as the minimum, the motion along the valley of the potential requires no expense of energy, and therefore corresponds to the mode with zero frequency. However, motion in the orthogonal direction requires energy, and therefore corresponds to a finite frequency.

Our result is a general feature of all theories that have spontaneous symmetry breaking, and, correspondingly, in such theories, the quantum mechanical system develops states of zero energy. For relativistic quantum mechanical systems, such states can be identified with massless particle states. In addition, the orthogonal mode with non-vanishing frequency corresponds to a massive particle. The massless particles, which, as we have emphasized, arise as a consequence of the spontaneous breaking of a global symmetry, are known as Nambu-Goldstone bosons (after Yoichiro Nambu and Jeffrey Goldstone). We should also mention that our simple example is to be regarded as purely illustrative of the basic features of spontaneous symmetry breaking, in that Nambu-Goldstone bosons appear only in relativistic field theories with two or more spatial dimensions. When the spontaneous symmetry breaking is of a local rather than of a global symmetry, the Nambu-Goldstone particles become transformed into the longitudinal modes of the gauge bosons. As a consequence, the resultant gauge bosons develop mass, or, equivalently, the corresponding "electric" and "magnetic" fields lose their purely transverse character.

The preceding discussion suggests a mechanism whereby the gauge bosons of the weak interaction can develop a mass and thereby give rise

to a short-ranged force. Conventionally, this is known as the Higgs mechanism (after Peter Higgs, but also discovered independently by Robert Brout and Francois Englert, and by Gerald Guralnik, Richard Hagen and Thomas Kibble), and for the weak interactions, the massive partner of the Nambu-Goldstone boson (corresponding to the mode with frequency  $\sqrt{2}\omega$ ) is a scalar particle referred to as the Higgs boson. The Higgs boson has yet to be observed, and it is not clear whether, if found, it will be a fundamental and structureless particle. Thus, in this kind of scenario, we expect the local weak-isospin symmetry to be broken spontaneously, and we therefore do not expect weak isospin to be a good (conserved) quantum number in weak interactions, which agrees with observation.<sup>3</sup> In fact, even the weak hypercharge symmetry is spontaneously broken. The breaking in weak isospin and weak hypercharge, however, compensate each other such that the particular combination in Eq. (13.30), corresponding to electric-charge symmetry, remains unbroken. Correspondingly, the photon remains massless, but the weak gauge bosons, namely the  $W^\pm$  and  $Z^0$ , become massive objects, with  $m_{W^\pm} \simeq 80.4 \text{ GeV}/c^2$  and  $m_{Z^0} \simeq 91.19 \text{ GeV}/c^2$ . They are assumed to be elementary particles that can decay into lepton-anti-lepton or quark-antiquark pairs, as can be surmised from Fig. 13.2, and as will be discussed in the next chapter.

Returning to our example of the ferromagnet, we note that, although the ground state spontaneously breaks rotational invariance because the aligned spins pick out a preferred spatial direction, when we heat up such a system, the thermal motion randomizes the spin orientations. Above some critical high temperature or energy, the spins indeed become randomly oriented, thereby restoring rotational invariance. This feature is also found in quantum mechanical field theories with spontaneous symmetry breaking. Namely, for theories displaying spontaneous breaking of symmetry,

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<sup>3</sup>This statement may confuse the intrepid reader, who by applying Eq. (13.30) to the  $W$  and  $Z$  bosons will deduce that they can be regarded as  $Y = 0$  objects. In fact, going further, the reader would conclude that all the transitions shown in Fig. 13.2, and the weak decays discussed in Chapter 9, conserve both weak hypercharge and weak isospin. It would, however, be incorrect to surmise that these quantum numbers are always conserved. We know that weak isospin must be a broken symmetry, because, for example, otherwise the masses of the members of the weak isospin doublets would be identical. In addition, the Higgs boson, in the context of the Standard Model, has  $I = \frac{1}{2}$ , but Higgs bosons interact with quarks and with  $W$  and  $Z$  bosons ( $H \rightarrow W^+ + W^-$ ,  $H \rightarrow Z + Z$ , etc.). This, clearly, cannot happen if weak isospin is conserved. Consequently, the breaking of weak isospin must affect the usual fermionic transitions, and violations of weak isospin in the more common processes are expected from higher-order contributions in the Higgs sector.

the symmetry is indeed restored above a certain temperature or energy. If we apply these ideas to weak interactions, we expect that, beyond some energy scale, weak-isospin symmetry is restored, and consequently the weak gauge bosons become massless, just like the photon. And, as was pointed out in connection with Eq. (9.7), because the strengths of the weak and the electromagnetic interactions become comparable at high momentum transfers, the two forces may indeed be unified at sufficiently high energy.

### 13.12 Chromodynamics (QCD) and Confinement

As we have just seen, the short-range nature of the weak interaction can be argued to arise from spontaneous breaking of local weak-isospin symmetry. However, the short-range nature of the strong nuclear force has a completely different origin. The dynamical theory of quarks and gluons that describes color interactions is known as Quantum Chromodynamics (QCD), and is a gauge theory of the non-Abelian color symmetry group  $SU(3)$ . This theory is very similar to Quantum Electrodynamics (QED), which describes the electromagnetic interactions of charges with photons. As we have mentioned, QED is a gauge theory of phase transformations corresponding to the commuting symmetry group  $U_Q(1)$ . Being a gauge theory of color symmetry, QCD also contains massless gauge bosons (namely gluons) that have properties similar to photons.

There are, however, essential differences between the two theories, which arise because of the different nature of the two symmetry groups. As we noted previously, the photon, which is the carrier of the force between charged particles, is itself charge neutral, and, as a result, the photon does not interact with itself. In contrast, the gluon, which is the mediator of color interactions, also carries color charge, and consequently has self-interactions. Another consequence of the non-Abelian nature of the color symmetry is related to how color-neutral states are formed. Consider, for example, a red-colored quark. We can obtain a color-neutral system by combining the red quark with an antired ( $\overline{\text{red}}$ ) antiquark. This is very much how electric charges add, namely,

$$\text{red} + \overline{\text{red}} = \text{color neutral.} \quad (13.54)$$

However, because three quarks with distinct colors can also yield a color-neutral baryon, there must therefore be an alternative way of obtaining a

color-neutral combination from three colored quarks, and this must be

$$\text{red} + \text{blue} + \text{green} = \text{color neutral}, \quad (13.55)$$

which is clearly different from the way electric charges add together.

This difference between color charge and electric charge has important physical consequences. For example, a classical test particle carrying positive electric charge polarizes a dielectric medium by creating pairs of oppositely charged particles (dipoles). Due to the nature of the Coulomb interaction, the negatively-charged parts of the dipoles are attracted towards the test particle, while the positively-charged parts are repelled (see Fig. 13.5). As a consequence, the charge of the test particle is shielded, and the effective charge seen at large distance is smaller than the true charge carried by the test particle. (Recall that the electric field in a dielectric medium is reduced relative to that in vacuum by the value of the dielectric constant of the medium.) In fact, the effective charge depends on the distance (or scale) at which we probe the test particle. The magnitude of the charge increases as we probe it at ever smaller distances, and only asymptotically (at largest momentum transfers) do we obtain the true point charge of the test particle. Since the distance probed is inversely proportional to the momentum transfer, it is stated conventionally that the effective electric charge, or the strength of the electromagnetic interaction, increases with momentum transfer, and, as we have just argued, this is purely a consequence of the screening of electric charge in a dielectric medium. Because of the presence of quantum fluctuations, a similar effect arises for charged particles in vacuum, the impact of which is that the fine structure constant  $\alpha = \frac{e^2}{\hbar c}$  increases, albeit only slightly, with momentum transfer. This has been confirmed in high-energy  $e^+e^-$  scattering, where  $\alpha(\sqrt{s} = m_Z c^2)$  is found to be  $\frac{1}{127.9}$ , or  $\approx 7\%$  larger than at low energy.

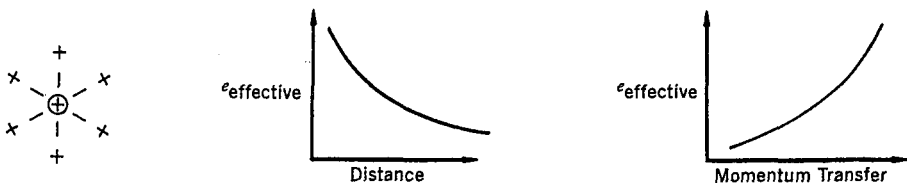


Fig. 13.5 Polarization of a dielectric medium around a positive electric charge, and the effective value of the charge as a function of probing distance and of momentum transfer.



In contrast, a test particle carrying color charge polarizes the medium in two ways. First, just as in the case of QED, it can create pairs of particles with opposite color charge. But it can also create three particles of distinct color, while still maintaining overall color neutrality. Consequently, for the color force, the effect of color charge on a polarized medium is more complex. A detailed analysis of QCD reveals that the color charge of a test particle is, in fact, *anti-screened*. In other words, far away from the test particle, the magnitude of the effective color charge is larger than that carried by the test particle. In fact, as we probe deeper, the magnitude of this charge decreases. Thus, the qualitative dependence of the color charge on probing distance, or on the probing momentum transfer, is exactly opposite of that for electromagnetic interactions (see Fig. 13.6). This implies that the strength of the strong interactions decreases with increasing momentum transfer, and vanishes asymptotically. Conventionally, this is referred to as *asymptotic freedom*, and refers to the fact that, at infinite energies, quarks behave as essentially free particles, because the effective strength of the coupling for interactions vanishes in this limit. (Asymptotic freedom of QCD was discovered independently by David Politzer, by David Gross and Frank Wilczek, and by Gerard 't Hooft.) This principle has the additional implication that, in very high energy collisions, hadrons consist of quarks that act as essentially free and independent particles. This limit of QCD for high-energy hadrons is known as the *parton model*, to which we alluded in the beginning of this chapter, and which agrees with many aspects of high-energy scattering.

The very fact that the strength of coupling in QCD decreases at high energies is extremely important, because it means that the effect of color interactions can be calculated perturbatively at small distances or large momentum transfers. Consequently, the predictions of QCD are expected to be particularly accurate at large momentum scales, and can be checked in experiments at high energies. At present, all such predictions are in excellent agreement with data (see next chapter).

At low energies, color interactions become stronger, thereby making perturbative calculations less reliable. But this property also points to the possibility that, as color couplings increase, quarks can form bound states, namely the colorless hadrons. In fact, as we have indicated, quarks alone cannot account for the properties of hadrons. As inferred from high-energy collisions, quarks carry only about one half of the momentum of the hadrons, the rest has to be attributed to the presence of other point-like constituents that appear to be electrically neutral, and have spin  $J = 1$ .

These constituents can be identified with the color gluons.

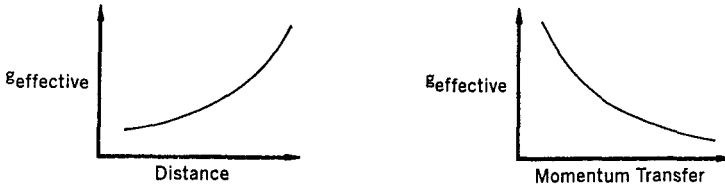


Fig. 13.6 The effective value of color charge as a function of probing distance and of momentum transfer.

There have been many attempts to understand the low-energy, non-perturbative, behavior of QCD. The present qualitative picture can be summarized best by a phenomenological linear potential between quarks and antiquarks of the form

$$V(r) \propto kr. \quad (13.56)$$

This kind of picture works particularly well for describing the interactions of the heavier quarks. Intuitively, we can think of the  $q\bar{q}$  system as being connected through a string. As a bound  $q\bar{q}$  pair is forced to separate, the potential between the two constituents increases. At some separation length, it becomes energetically more favorable for the  $q\bar{q}$  pair to split into two  $q\bar{q}$  pairs. Pictorially, we can describe the process as shown in Fig. 13.7. In other words, the strong color attraction increases with separation distance between the quarks, and therefore precludes the possibility of observing an isolated quark.<sup>4</sup> This effect, known as *confinement*, is, of course, consistent with observation. That is, all observed particles appear to be color neutral, and there has never been any evidence for the production of an isolated quark or gluon with color charge. When additional quarks are produced in high-energy collisions, they are always found in states whose total color adds up to zero (i.e., color neutral). As these quarks leave the region of their production, they *dress* themselves (become converted) into hadrons, and their presence can be inferred from a *jet* of particles that is formed from their initial energy. Similarly, gluons emitted in hadronic interactions also become dressed into hadrons and produce jets of particles as they leave the point of collision. While we presently believe in the confinement of quarks

<sup>4</sup>The splitting of a  $q\bar{q}$  pairs is similar to what happens when a bar magnet is cut into two. The net result is two separate bar magnets rather than a north and a south pole.

and gluons, a detailed proof of this requirement within the context of QCD is still lacking.

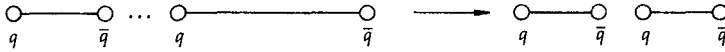


Fig. 13.7 Creation of a new  $q\bar{q}$  pair from the vacuum when the separation between the original  $q\bar{q}$  pair is increased.

In the context of the Standard Model, the strong nuclear force between hadrons can be thought of as a residual-color Van der Waals force, analogous to the Van der Waals force that describes the residual electromagnetic interactions of charge-neutral molecules. Namely, just as the Van der Waals force reflects the presence of charged atomic constituents that can interact through the Coulomb force, the strong nuclear force reflects the presence of far more strongly interacting color objects that are present within hadrons. The Van der Waals force falls off far more rapidly with distance than the Coulomb force, which suggests that a similar effect could be expected for the case of color, which would explain the origin of the short-range nature of the strong force between hadrons, both within as well as outside of nuclei.

### 13.13 Quark-Gluon Plasma

We have argued that quarks are confined within hadrons. However, increasing the temperature of our hadronic system, and thereby the random thermal motion of its constituents, could eventually lead to a complete disintegration of the hadron into free quarks and gluons, defining a new transformed kind of matter known as the *quark-gluon plasma* phase. This phase is quite similar to the plasma state of charged particles that exists inside the sun and the stars, where electrons and protons from ionized hydrogen atoms move about freely. The best theoretical evidence that a transition between the confined and the deconfined phase of quarks takes place as the temperature increases, comes from extensive computer simulations based on QCD. This kind of a quark-gluon plasma phase of matter was likely to have existed right after the Big Bang, when the temperature in the universe was very high. The phase is characterized by a large number of rapidly moving charged quarks that scatter and therefore radiate photons, leading to enhanced direct single-photon production. In addition, because of the high temperatures (or high energies), the production of quarks would not be limited to just low-mass flavors, but would also increase the produc-

tion of quarks with more exotic flavors, such as strangeness and charm. Experimental verification of such signals in high-energy interactions is an interesting area of research, and is being pursued in heavy-ion collisions at the Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory. These collider experiments study interactions of large- $A$  nuclei, each with energies of several hundred GeV per nucleon. The energy and matter densities in these experiments are expected to be large enough to observe the transformation of normal nuclei into free quark-gluon systems. The attempts to test such ideas are quite challenging. The expected properties of the quark-gluon plasma phase are not as yet completely formulated, and, furthermore, the experimental signatures for the presence of the quark-gluon plasma are also not entirely clear. In spite of this uncertainty, or perhaps because of it, it is exceedingly interesting to learn whether such states of matter can be produced in the laboratory.

## Problems

**13.1** Prove that Eq. (13.49) follows from Eq. (13.48).

**13.2** According to the quark model, wave functions of baryons are antisymmetric in color. Construct a wave function for the  $\Delta^{++}$  that is explicitly antisymmetric under the exchange of any two of its quark constituents in color space.

## Suggested Readings

Aitchison, I. J. R., and A. J. G. Hey, *Gauge Theories in Particle Physics: A Practical Introduction*, IOP (2003).

Frauenfelder, H., and E. M. Henley, *Subatomic Physics*, Prentice-Hall (1991).

Griffiths, D., *Introduction to Elementary Particles*, Wiley (1989).

Jackson, J. D., *Classical Electrodynamics*, Wiley (1999).

Kane, G., *Modern Elementary Particle Physics*, Addison-Wesley (1993).

Perkins, D. H., *Introduction to High Energy Physics*, Cambridge Univ. Press (2000).

Williams, W. S. C., *Nuclear and Particle Physics*, Oxford Univ. Press (1997).

Also, see standard texts on quantum mechanics, e.g., Das, A., and A. C. Melissinos, *Quantum Mechanics*, Gordon & Breach (1986).



## Chapter 14

# Standard Model and Confrontation with Data

### 14.1 Introductory Remarks

We have mentioned several times that the Standard Model appears to be in complete agreement with all measurements. In fact, with the exception of the surprising result that neutrinos possess finite mass, there have been no confirmed deviations between data and predictions of the Model. In this chapter, we provide comparisons with data, and expand somewhat on the phenomenological implications of the Standard Model.

### 14.2 Comparisons with Data

As example of the kind of agreement that has been observed between expectations from QCD and collisions studied at high energies, we show in Figs. 14.1 and 14.2, respectively, the data and theoretical predictions for production of  $W$  and  $Z$  bosons and for production of particle jets (quarks and gluons that form, or evolve into, color-neutral particles) in antiproton-proton collisions. The differential production cross section as a function of any variable, e.g., transverse momentum  $p_T$ , can be written schematically in QCD, in terms of the elastic scattering of a parton  $a$  from hadron  $A$  and a parton  $b$  from hadron  $B$ , as

$$\frac{d\sigma}{dp_T} = \int dx_b \frac{d\hat{\sigma}}{dp_T} f_A(x_a, \mu) f_B(x_b, \mu) dx_a, \quad (14.1)$$

where the term  $\frac{d\hat{\sigma}}{dp_T}$  refers to the point cross section for elastic scattering of the two partons, and can be calculated from fundamental principles of quantized field theory,  $x_i$  is the fraction of the momentum of hadron

$I$  carried by parton  $i$ ,  $f_I(x_i, \mu)$  represents the momentum distribution of parton  $i$  within hadron  $I$  at the scale  $q^2 = \mu^2$ . That is, just as for the case of  $\alpha$  in QED, and the strength of the color interaction in QCD, the parton distribution function  $f(x_i, \mu)$  also depends on (“runs with”) the momentum scale in any collision. Such dependence of parameters on  $q^2$  is usually referred to as the *scaling violation* of QCD. In fact, given some  $f(x_i, \mu_0)$  at  $q^2 = \mu_0^2$ , it is possible to calculate within QCD the resulting function  $f(x_i, \mu)$  at some other  $q^2 = \mu^2$ , using what are known as the DGLAP evolution equations (after Yuri Dokshitzer, Vladimir Gribov, Lev Lipatov, Guido Altarelli and Giorgio Parisi). The integrations in Eq. (14.1) have to be performed over all values of the  $x_i$ .

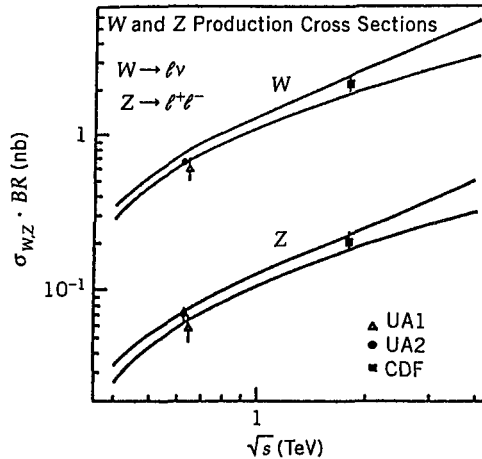


Fig. 14.1 The cross section for  $W$  and  $Z$  production in  $\bar{p}p$  collisions compared to theoretical predictions based on the Standard Model. (From A. G. Clark, *Techniques and Concepts of High Energy VI*, Plenum Press, T. Ferbel, ed. (1991).)

The primary uncertainty in the theory (displayed as the allowed regions between the two sets of smooth curves in Fig. 14.1) stems from the inability to predict the content and the momentum distributions of constituents that are bound within hadrons. This is an issue related to confinement and interactions of quarks and gluons at low momentum transfer, which cannot be calculated reliably in perturbation theory. However, as implied in the beginning of Chapter 13, parton distribution functions  $f(x_i, \mu_0)$  at some known scale  $q^2 = \mu_0^2$ , can be extracted from other reactions (e.g., from electron scattering off protons), and then applied to predict results for

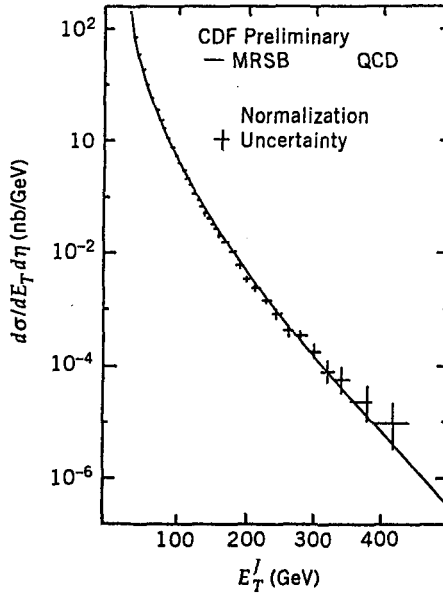


Fig. 14.2 Cross section and prediction from QCD for production of particle jets at large momentum transfers in  $\bar{p}p$  collisions at  $\sqrt{s} \sim 1.8$  TeV. (From A. G. Clark, loc cit.)

collisions of any partons located within separate hadrons. Thus, for the case of  $W$  production, the main contribution to the yield arises from the interaction of  $\bar{u}$  (or  $\bar{d}$ ) quarks in the antiproton that fuse with  $d$  (or  $u$ ) quarks within the proton to produce a  $W^-$  (or  $W^+$ ), and possible remnant jets of particles. The uncertainty on the gluon content of hadrons is larger than for quarks, because photons,  $W$  and  $Z$  bosons can interact directly with quarks, but only indirectly (at higher order in perturbation theory) with gluons, as can be inferred from Fig. 14.3.

For the case of jet production in hadron-hadron collisions, any parton in one of the interacting hadrons can scatter elastically off any parton in the other hadron, and then both partons can evolve into jets. Clearly, the scattered partons can appear at large angles relative to the collision axis, while the other (unscattered) constituents tend to evolve into color-neutral states at small angles along the collision axis. In fact, since momentum must be conserved in the direction transverse to the collision axis, we expect the scattered-parton jets to be emitted back-to-back. Two typical events of this kind are shown in Fig. 14.4. This kind of display is referred to as a *lego*



*plot.* The height of any entry is proportional to the energy observed in that region of coordinates. The axes correspond to the azimuth ( $\phi$ ) around the collision axis, and the polar angle  $\theta$  relative to the collision axis. The jets are observed at  $180^\circ$  relative to each other in azimuth, reflecting collisions between constituents contained within the hadrons.

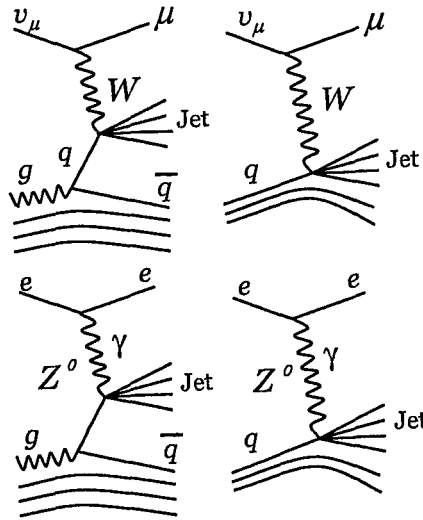


Fig. 14.3 Interactions of  $\nu_\mu$  and  $e$  with constituents of the nucleon through weak “charged currents” involving  $W$  exchange, and through electroweak “neutral currents” involving  $Z^0$  and  $\gamma$  exchange. To be sensed by the exchanged object, gluons must first dissociate into virtual  $q\bar{q}$  pairs. The quark struck by the virtual boson evolves into a jet of normal (colorless) particles. For this to happen, more color (“soft gluons”) must be exchanged between the interacting colored parton and the colored parton-remnants of the nucleon to assure that the total color in the final physical state remains the same as in the initial state. Nevertheless, the production of jets reflects the parton content of the nucleon.

### 14.3 Cabibbo Angle and the “GIM” Mechanism

We showed in Fig. 13.2 how  $W$  and  $Z$  bosons can produce transitions between members of the same weak isospin doublet. However, if  $W$  and  $Z$  bosons could not also provide transitions among particles belonging to different multiplets, it would clearly present a great puzzle concerning the origin of the  $|\Delta S| = 1$  strangeness-changing weak decays. The solution to this issue comes from our previous observation that strangeness is a quan-

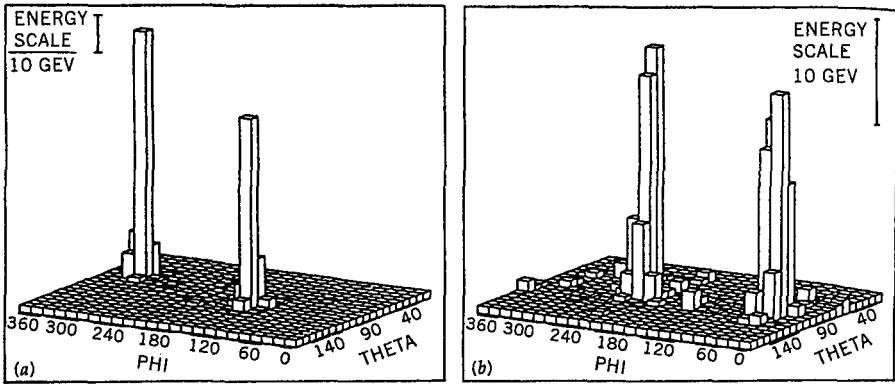


Fig. 14.4 Energy flowing in the direction transverse to the collision axis for production of particle jets in  $p\bar{p}$  collisions at  $\sqrt{s} \sim 600$  GeV. (From L. DiLella, *Techniques and Concepts of High Energy Physics IV*, Plenum Press, T. Ferbel, ed. (1987).)

tum number that is not conserved in weak interactions. Consequently, the eigenstates of the weak Hamiltonian are different from those of the strong Hamiltonian, and, in particular, do not have unique strangeness. In analogy with our analysis of the  $K^0-\bar{K}^0$  system, we can try to redefine the quark doublet eigenstates of the weak Hamiltonian as mixed states of the doublets of Eq. (13.2). Before the discovery of the charm quark, and based on the experimental results available at that time, Nicola Cabibbo showed that all data were consistent with altering the doublet corresponding to the first family of quarks as follows

$$\begin{pmatrix} u \\ d \end{pmatrix} \rightarrow \begin{pmatrix} u \\ d' \end{pmatrix}, \quad (14.2)$$

where the newly defined state  $d'$  is a mixture of  $d$  and  $s$  quarks

$$d' = \cos\theta_c d + \sin\theta_c s. \quad (14.3)$$

This kind of state clearly does not have a unique strangeness quantum number, and, if the weak gauge bosons can give rise to transitions within the  $u, d'$  multiplet, then they can, in fact, induce strangeness-changing processes. The angle  $\theta_c$  parameterizing the mixing between the  $d$  and  $s$  quarks in Eq. (14.3) is commonly called the Cabibbo angle, and its value determines the relative rates for processes such as

$$\begin{aligned}
 W^+ &\longrightarrow u\bar{s}, \\
 W^+ &\longrightarrow u\bar{d}, \\
 Z^0 &\longrightarrow u\bar{u}, \\
 Z^0 &\longrightarrow d\bar{s}.
 \end{aligned}
 \tag{14.4}$$

The Cabibbo angle can be determined experimentally through a comparison of  $\Delta S = 0$  and  $\Delta S = 1$  transitions, and has the value  $\sin \theta_c = 0.23$ . Figure 14.5 shows how the decay of a  $K^0$  into a  $\pi^+$  and a  $\pi^-$  can now be described in the Standard Model.

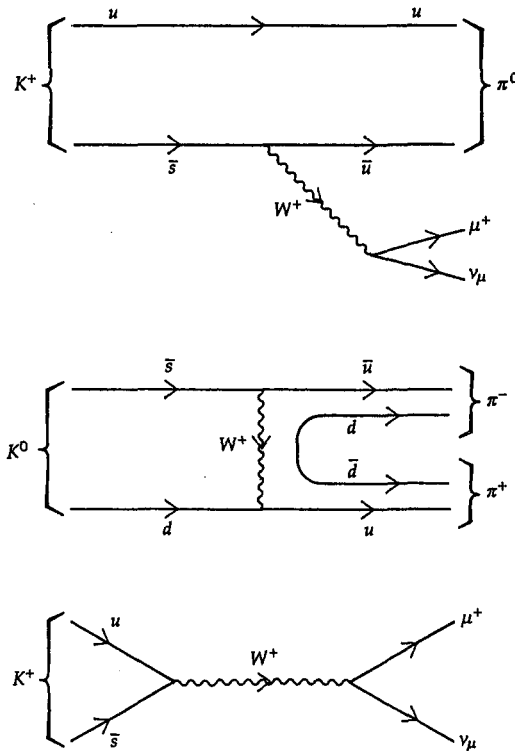


Fig. 14.5 Strangeness-changing transitions in the Standard Model. A  $\bar{d}d$  pair produced from the vacuum (in the middle graph) combines with the other quarks to produce a  $\pi^+$  and a  $\pi^-$  for the final state of the  $K^0$  decay.

While the Cabibbo hypothesis accommodated most of the  $W^\pm$  induced decays, certain strangeness-changing processes, particularly involving leptonic decay modes of the  $K^0$  remained puzzling. For example

$$\begin{aligned}\frac{1}{\hbar}\Gamma(K^+ \rightarrow \mu^+\nu_\mu) &\approx 0.5 \times 10^8 \text{ sec}^{-1}, \\ \frac{1}{\hbar}\Gamma(K_L^0 \rightarrow \mu^+\mu^-) &\approx 0.14 \text{ sec}^{-1},\end{aligned}\tag{14.5}$$

leading to

$$\frac{\Gamma(K_L^0 \rightarrow \mu^+\mu^-)}{\Gamma(K^+ \rightarrow \mu^+\nu_\mu)} \approx 3 \times 10^{-9}.\tag{14.6}$$

The smallness of this ratio could not be explained within the framework of the Cabibbo analysis. In fact, further investigation of this problem led Sheldon Glashow, John Iliopoulos and Luciano Maiani to propose the existence of a fourth quark (charm quark) in a doublet structure of the type given in Eq. (14.2), namely,

$$\begin{pmatrix} c \\ s' \end{pmatrix},\tag{14.7}$$

with

$$s' = -\sin\theta_c d + \cos\theta_c s.$$

This, indeed, leads to a resolution of all the leptonic decay modes of the strange mesons, and this proposal is commonly referred to as the ‘‘GIM mechanism’’.

The ideas of Cabibbo and GIM can be summarized by saying that for the two doublets

$$\begin{pmatrix} u \\ d' \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c \\ s' \end{pmatrix},$$

the weak eigenstates are related to the eigenstates of the strong Hamiltonian through the following unitary matrix

$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos \theta_c & \sin \theta_c \\ -\sin \theta_c & \cos \theta_c \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}. \quad (14.8)$$

#### 14.4 CKM Matrix

With the subsequent discovery of the  $b$  and  $t$  quarks, the Standard Model is now characterized by three doublets of quarks of the form

$$\begin{pmatrix} u \\ d' \end{pmatrix} \quad \begin{pmatrix} c \\ s' \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} t \\ b' \end{pmatrix}. \quad (14.9)$$

The relation between the three states  $d'$ ,  $s'$  and  $b'$  and the eigenstates  $d$ ,  $s$ , and  $b$  is somewhat more complicated and involves a  $3 \times 3$  matrix, which is known as the Cabibbo–Kobayashi–Maskawa unitary matrix

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}. \quad (14.10)$$

The elements of this matrix reflect the couplings of the  $W$  boson to all the possible quark pairs, e.g.,  $W \rightarrow t\bar{b}$ ,  $c\bar{b}$ ,  $c\bar{s}$ , etc. Clearly, the dominant transitions in  $W$  decay are to members of the same doublet, and therefore, to first order, the diagonal elements of the matrix are expected to be large (close to unity) and the off-diagonal elements small. In writing down the original  $3 \times 3$  matrix (prior to the discovery of the  $b$  and  $t$  quarks!), Makoto Kobayashi and Toshihide Maskawa noted that this kind of matrix representation introduces at least one complex phase into the Standard Model, which can then encompass the phenomenon of  $CP$  violation in transitions of neutral mesons, as discussed in Chapter 12.

With the generalization of Eq. (14.10), in which all quarks of same electric charge can mix, and the quark eigenstates of the weak Hamiltonian  $b'$ ,  $s'$ , and  $d'$  correspond to specific superpositions of the  $d$ ,  $s$ , and  $b$  quarks, the Standard Model can accommodate all measured particle decays and transitions. For example, the  $CP$  violating decays of the  $K^0$  mesons can be

calculated from the kind of processes indicated in Fig. 14.6 and Fig. 14.7, for the indirect and direct terms in the  $CP$  violating processes, respectively.

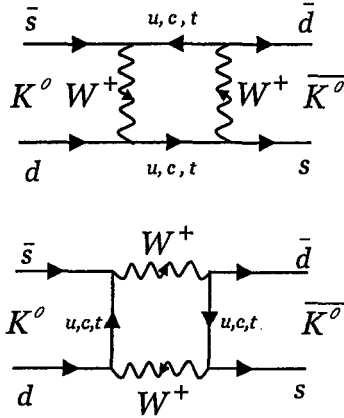


Fig. 14.6 The  $\Delta S = 2$  transition “box diagrams” via two consecutive weak processes that are responsible for  $K^0$ - $\bar{K}^0$  mixing and indirect  $CP$  violation in  $K^0$  decay.

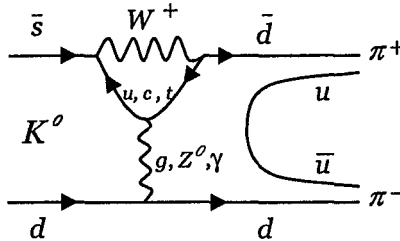


Fig. 14.7 The  $\Delta S = 1$  transition, or “penguin diagram”, that is responsible for direct  $CP$  violation in  $K^0$  decay.

### 14.5 Higgs Boson and $\sin^2 \theta_W$

We will remark only briefly on the connection between electrodynamics and the weak interaction. Sheldon Glashow and Abdus Salam and Steven Weinberg, independently formulated the Electroweak Model, which is one of the cornerstones of the Standard Model. Electroweak theory relates the strengths of the electromagnetic and weak interactions of the fundamental particles through the *weak mixing angle*,  $\theta_W$ , and through the masses of

the gauge bosons. In particular, we can write the following relations for the parameters

$$\sin^2 \theta_W = \frac{\pi\alpha}{\sqrt{2} G_F} \frac{1}{m_W^2} = 1 - \frac{m_W^2}{m_Z^2} = 0.23, \quad (14.11)$$

where  $\alpha$  is the fine structure “constant” and  $G_F$  is the weak (Fermi) coupling constant. The value of  $\theta_W$  has been measured in different scattering experiments, and the masses of the  $W$  and  $Z$  gauge bosons have been measured in  $\bar{p}p$  and in  $e^+e^-$  collisions. Except for one or two possibly serious, but not considered firm discrepancies, all current experimental data are consistent with Relation (14.11) and with all the other predictions of the electroweak theory of the Standard Model.

Finally, a word about the yet to be observed Higgs boson. In the Standard Model, the Higgs, through its interactions, is responsible for generating masses of the gauge bosons. It also provides a mechanism for generating masses of all fundamental fermions because of their coupling to the Higgs. In fact, the strength of the  $Hq\bar{q}$  coupling (e.g.,  $H \rightarrow q\bar{q}$ , either through a virtual or physical transition) reflects the mass of the quark. The stronger the coupling strength, the larger is the mass of  $q$ . The  $J = 0$  Higgs field permeates the entire universe, and the observed masses of all fundamental particles are characterized by their interactions with the Higgs field in vacuum.

At present, the best direct limit on the mass of the Higgs boson is from the experiments performed at LEP, which indicate that  $m_{H^0} > 114 \text{ GeV}/c^2$ . Because of the fact that the Higgs interacts with and gives mass to all the fundamental particles, the internal consistency of the Standard Model, through *radiative corrections* (virtual contributions of the type where a Higgs dissociates into virtual  $t\bar{t}$  pairs, which then recombine to reform the Higgs, or where a  $W$  breaks up into a virtual system of  $WH^0$ , which again recombine), also provides limits and a connection between  $m_t$ ,  $m_W$  and  $m_{H^0}$ . The latest correlation is exhibited in Fig. 14.8, and suggests that, in the context of the Standard Model, the constraints from the measured values of  $m_t$  and  $m_W$ , as well as from other electroweak measurements (not direct measurements of  $m_t$  and  $m_W$ ), require the Higgs mass to be below  $200 \text{ GeV}/c^2$ . The experiments currently underway at the Tevatron at Fermilab, and being planned at the LHC at CERN, are bound to provide a definitive answer to the existence of this universe-permeating Higgs field.

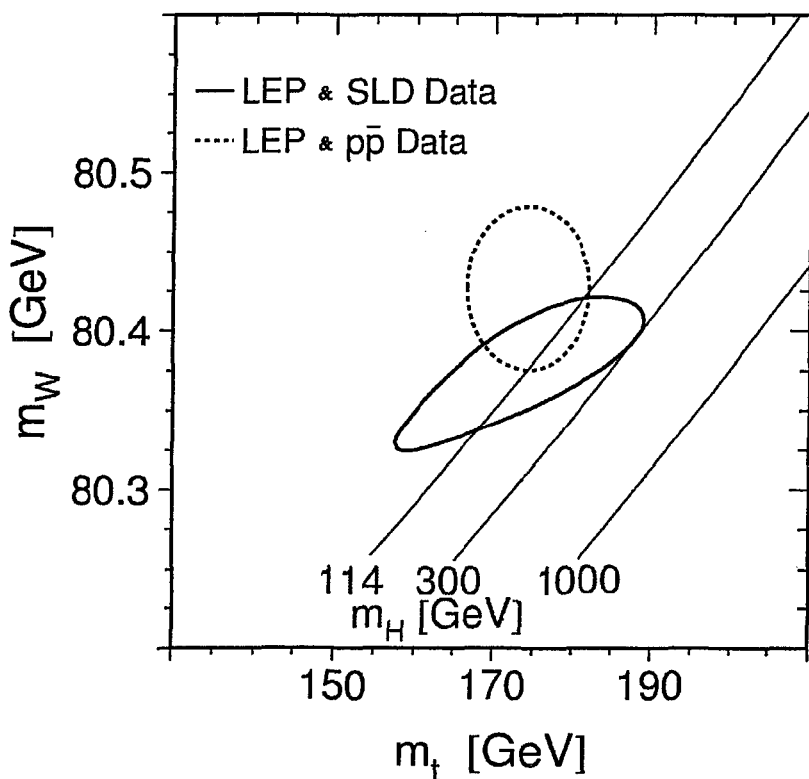


Fig. 14.8 The correlation between the known masses of the  $W$  boson and the top quark, and the unknown mass of the Higgs boson. The area within the dotted ellipse corresponds to the directly measured values of the masses of the top quark ( $m_t$ ) and of the  $W$  boson ( $m_W$ ), obtained at the Tevatron and at the CERN LEP  $e^+e^-$  collider. The best values are given by the center of the ellipse, with the boundary indicating one standard-deviation limits on the essentially independent measurements. The continuous contour corresponds to the same result, but now based on the indirect extraction of  $m_t$  and  $m_W$  from other electroweak measurements at the LEP and SLD (SLAC)  $e^+e^-$  experiments. Here the masses are correlated, but independent of the direct measurements. The lines correspond to values of the Higgs mass ( $m_H$ ) as a function of  $m_t$  and  $m_W$  in the Standard Model. The overlap of the lines with the two contours testify to the internal consistency of the Standard Model, and point to the small expected value of  $m_H$ . (Based on LEP Electroweak Working Group, <http://lepewwg.web.cern.ch>.)

## Problems

14.1 The mass of the top quark is larger than that of the  $W$  boson. It is consequently not surprising that the top quark decays into a  $W$  and a  $b$  quark ( $t \rightarrow W + b$ ). The expected width ( $\Gamma$ ) of the top quark in the



Standard Model is  $\approx 1.5$  GeV. (a) What can you say about the lifetime of the top quark? (b) If QCD color interactions can be characterized by the fly-by time of two hadrons (time needed to exchange gluons), what is the ratio of lifetime to interaction time for top quarks? (c) Because of the rapid fall-off in parton distributions  $f(x, \mu)$  with increasing  $x$ , the peak of the production cross section for  $t\bar{t}$  events in  $p\bar{p}$  collisions occurs essentially at threshold. What is the typical momentum of the  $b$  quark in  $p\bar{p}$  collisions that yield  $t\bar{t}$  events. (d) What are the typical  $x$  values of the colliding partons that can produce  $t\bar{t}$  events at the Tevatron ( $\sqrt{s} = 2$  TeV)? What about at the LHC ( $\sqrt{s} = 14$  TeV)? (*Hint*:  $\hat{s} = x_a x_b s$ , where  $\hat{s}$  is the value of the square of the energy in the rest frame of the partonic collision of  $a$  and  $b$ . Can you prove this?)

**14.2** In discussing weak decays proceeding through  $W$  or  $Z$  bosons, we have focused primarily on the fundamental transitions among quarks and leptons. However, such decays often involve hadrons that contain *spectator* quarks, in addition to the partons that participate in the weak interaction (see Fig. 14.3). For example, Fig. 14.5 shows a diagram for the decay of a  $K^0$  into a  $\pi^+\pi^-$  pair. Using similar *quark-line* diagrams, draw processes for the following decays: (a)  $K^+ \rightarrow \pi^+ + \pi^0$ , (b)  $n \rightarrow p + e^- + \bar{\nu}_e$ , (c)  $\pi^+ \rightarrow \mu^+ + \nu_\mu$ , (d)  $K^0 \rightarrow \pi^- + e^+ + \nu_e$ .

**14.3** Draw quark-line diagrams for the following reactions: (a)  $\pi^- + p \rightarrow \Lambda^0 + K^0$ , (b)  $\pi^+ + p \rightarrow \Sigma^+ + K^+$ , (c)  $\pi^+ + n \rightarrow \pi^0 + p$ , (d)  $p + p \rightarrow \Lambda^0 + K^+ + \pi^+$ , (e)  $\bar{p} + p \rightarrow K^+ + K^-$ .

**14.4** Draw quark-line diagrams for the following weak interactions, and include any required intermediate  $W$  or  $Z$  bosons: (a)  $\nu_e + n \rightarrow \nu_e + n$ , (b)  $\bar{\nu}_\mu + p \rightarrow \mu^+ + n$ , (c)  $\pi^- + p \rightarrow \Lambda^0 + \pi^0$ .

**14.5** One of the main reasons for the introduction of the GIM mechanism was the need to suppress *flavor-changing neutral currents*, in order to reduce the rate for  $K_L^0 \rightarrow \mu^+\mu^-$  to its observed small value. (a) Draw the quark-line diagram for this transition involving  $W$  bosons (via a higher-order box diagram), and the possible contribution from  $Z^0$  exchange. (b) Show that the  $Z^0$  contribution vanishes once the weak states of Eq. (14.8) are used to calculate that contribution. (*Hint*: Contrast the transition elements  $\langle d'\bar{d}'|Z^0\rangle$ ,  $\langle s'\bar{s}'|Z^0\rangle$ , and  $\langle d'\bar{s}'|Z^0\rangle$  by considering  $\langle d'\bar{d}'\rangle$ ,  $\langle s'\bar{s}'\rangle$ , and  $\langle d'\bar{s}'\rangle$ .)

**14.6** Consider the scattering of an electron from a proton (of mass  $m_p$ ), as shown in Fig. 14.3. Let  $W$  be the invariant mass of the entire recoiling hadronic system, and  $Q$  and  $P$  (except for multiplicative factors of  $c$ )

the four-momenta of the exchanged vector boson and target proton, respectively, and  $E$ ,  $E'$  and  $\theta$  the incident energy, the scattered energy and scattering angle of the electron in the laboratory (i.e., the rest frame of the proton). Defining  $Q^2$  as  $(\vec{k}' - \vec{k})^2 c^2 - \nu^2$ , where  $\vec{k}'$  and  $\vec{k}$  are the three-momenta of the scattered and incident electron, and  $\nu$  is the difference in electron energy, show that for very high energies, (a)  $Q^2 = 4EE' \sin^2 \frac{\theta}{2}$ , (b)  $W^2 = m_p^2 + \frac{2m_p\nu}{c^2} - \frac{Q^2}{c^4}$ . (c) What is the smallest value that  $W$  can assume? What type of scattering does that correspond to? (d) What is the largest  $Q^2$  possible? What does that correspond to? What is the mass of the vector boson in this case? (e) What is the largest possible value of  $W$ ?

**14.7** Now consider the scattering of Problem 14.6 in a frame in which the proton has an exceedingly large three-momentum, so that its mass  $m_p$  can be ignored, as can the transverse momenta of all its partons. Now, suppose that the collision involves a parton carrying a fraction  $x$  of the proton's four-momentum, and that it absorbs the exchanged "four-momentum"  $Q$ . (a) First, show that in the laboratory frame  $Q \cdot P = m_p \nu c^2$ . (b) Now, prove that, for very large  $Q^2$  (corresponding to deep-inelastic scattering), and in particular when  $Q^2 \gg x^2 m_p^2 c^4$ ,  $x = \frac{Q^2}{2m_p \nu c^2}$ . (c) Plot  $\frac{Q^2}{c^4}$  as a function of  $\frac{2m_p \nu}{c^2}$  for  $W = m_p$ ,  $W = \sqrt{5}m_p$  and  $W = 3m_p$ . (d) Indicate the regions in (c) that correspond to  $x < 1$ ,  $x < 0.5$  and  $x < 0.1$ . (e) Identify the approximate location of the point corresponding to  $E = 10$  GeV,  $E' = 1$  GeV and  $\theta = \frac{\pi}{3}$ , and the point corresponding to  $E = 10$  GeV,  $E' = 4$  GeV and  $\theta = \frac{\pi}{6}$ , on the plot in part (c).

### Suggested Readings

Aitchison, I. J. R., and A. J. G. Hey, *Gauge Theories in Particle Physics: A Practical Introduction*, IOP (2003).

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## Chapter 15

# Beyond the Standard Model

### 15.1 Introductory Remarks

The Standard Model of fundamental strong, weak and electromagnetic interactions is a gauge theory involving quarks and leptons based on the symmetry group  $SU_{\text{color}}(3) \times SU_L(2) \times U_Y(1)$ . As we have argued, the weak isospin and hypercharge symmetries, or the symmetry groups  $SU_L(2)$  and  $U_Y(1)$ , are spontaneously broken. As a result, the weak gauge bosons become massive, and the symmetry at low energy reduces to the gauge symmetry of electromagnetism and of color symmetry, namely  $SU_{\text{color}}(3) \times U_Q(1)$ . Ignoring the finer technical points, this is essentially the spirit of the Standard Model. The Standard Model leads to many interesting perturbative predictions, and all of them appear to hold true. In fact, the agreement between experiment and theory is quite remarkable (see previous chapter). Thus, it seems reasonable to conclude that the Standard Model leads to a correct description of fundamental interactions at low energies. However, the Standard Model has many parameters, e.g., masses of the leptons, quarks, gauge bosons, and of the Higgs, various coupling strengths, and the elements of the CKM matrix, with all values seemingly perplexing and *ad hoc*. Furthermore, the Standard Model does not incorporate gravity, which is another fundamental force that might be expected to unify with the other fundamental forces at some large mass scale.

The recent discovery that neutrinos carry mass implies, in analogy with the situation for quarks and neutral mesons, that neutrinos too can mix. In fact, the interesting possibility of neutrino mixing was suggested long before it was required by any available data (by Ziro Maki, Masami Nakagawa, Shoichi Sakata and Bruno Pontecorvo). Following the observation of neutrino oscillations at the Super-Kamiokande experiment, which was led

by Yoji Totsuka at the Kamioka Laboratory in Japan, a new series of experiments is about to be launched to measure the elements of the “MNKP” lepton-flavor mixing matrix

$$\begin{pmatrix} \nu'_e \\ \nu'_\mu \\ \nu'_\tau \end{pmatrix} = \begin{pmatrix} U_{e'e} & U_{e'\mu} & U_{e'\tau} \\ U_{\mu'e} & U_{\mu'\mu} & U_{\mu'\tau} \\ U_{\tau'e} & U_{\tau'\mu} & U_{\tau'\tau} \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}, \quad (15.1)$$

where the primed states refer to the mixed eigenstates of the weak Hamiltonian and the unprimed states are the states of definite lepton-flavor quantum number. The number of independent elements and the number of possible phases depends on the nature of the neutrino, namely whether it is a Dirac (with  $\nu$  and  $\bar{\nu}$  distinct) or a Majorana particle (with  $\nu$  and  $\bar{\nu}$  indistinguishable).

Although a finite neutrino mass does not call for the abandonment of the Standard Model, there are certain other puzzles, particularly in the context of the unification of all forces, that point to its inadequacy. One such puzzle, known as the *hierarchy* problem, can be sketched as follows. In field theory, the mass of any particle is determined from a sum over all of its interactions leading to its self-energy. For the Higgs boson, in particular, its interactions in the vacuum, give rise to radiative corrections to its mass (from virtual loops of the kind mentioned in the previous chapter). These corrections can be written in the form

$$\delta m_H^2 \approx g^2(\Lambda^2 + m_{EW}^2), \quad (15.2)$$

where  $\Lambda$  corresponds to a cutoff in energy, beyond which effects from any new forces become important (e.g., from quantum gravity),  $m_{EW}$  is the mass of any object contributing to a virtual loop relevant to the electroweak scale of  $\lesssim$  TeV, and  $g$  refers to the coupling of that object to the Higgs boson. Because the scale at which quantum gravity, the only other known force, becomes dominant is expected to be of the order of  $\Lambda \approx 10^{19}$  GeV (see later), the corrections to  $m_H$  would naturally be of that same order, as would therefore be the value of the mass of the Higgs. But we just argued in Chapter 14 that experiments exclude the possibility for  $m_H$  outside of the range of 114-200 GeV/ $c^2$ . This implies that there must exist new interactions at a scale well below that of quantum gravity (i.e., of order 1

TeV) that would lend stability to the mass of the Higgs, and not require a fortuitous and highly unlikely cancellation to 16 decimal places in the sum over all contributions of the kind given in Eq. (15.2).

Hence, although aside from the issue of neutrino mass, there does not seem to be great reason to look beyond the Standard Model, there are many theoretical motivations to do just that! In this chapter, we will describe various attempts to look beyond the Standard Model.

## 15.2 Grand Unification

Examining the properties of the quark and lepton multiplets (families), it appears that the electric charges of all fundamental particles can be regarded as quantized in units of  $\frac{1}{3}e$ . We recall that angular momentum is quantized in units of  $\frac{1}{2}\hbar$ . However, this quantization arises because the algebra of angular momentum is non-commutative. In other words, it is a general feature of non-commutative (non Abelian) symmetry groups that they give rise to conserved charges that have discrete, quantized, values. In contrast, the symmetry that gives rise to a conserved electric charge corresponds to a phase transformation that is described by the commuting  $U_Q(1)$  symmetry group. This symmetry group does not require the corresponding conserved charge to take on quantized values, and, consequently, within the framework of the Standard Model, the quantization of electric charge must be regarded as a great mystery. However, if for some reason, all these symmetries – namely, the  $U_Y(1)$ ,  $SU_L(2)$  and  $SU_{\text{color}}(3)$  groups – were part of a larger non-commutative symmetry group, then that could explain the origin of quantization of electric charge.

In addition, a certain phenomenological symmetry seems to exist between quarks and leptons, in that for every lepton family, there is a family of quarks with three different colors:

$$\begin{aligned} \begin{pmatrix} \nu_e \\ e^- \end{pmatrix} &\longleftrightarrow \begin{pmatrix} u^a \\ d^a \end{pmatrix}, \\ \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix} &\longleftrightarrow \begin{pmatrix} c^a \\ s^a \end{pmatrix}, \\ \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix} &\longleftrightarrow \begin{pmatrix} t^a \\ b^a \end{pmatrix}. \end{aligned} \tag{15.3}$$

where  $a$  labels the color degree of freedom.

A quark-lepton symmetry might be expected if quarks and leptons corresponded simply to different states of the same particle. For example, if each quark had four colors, with the fourth color reflecting the lepton quantum number, then this would lead naturally to an equal number of lepton and quark families.

The idea that quarks and leptons are different manifestations of one object, raises the very interesting possibility that the interactions of leptons and quarks, which appear to be rather different – leptons interacting weakly through the  $Z^0$  and  $W^\pm$ , and quarks interacting strongly through the color gluons – might also be related. That is, it would only make sense for these particles to be grouped together if the strong and the weak forces corresponded to different manifestations of a single fundamental force. This kind of simplification, namely that the three fundamental forces are different manifestations of one truly basic force, would lead to an elegance of the fundamental laws of nature that would be esthetically pleasing. This is the concept known as grand unification, and it can be examined in theories reaching beyond the Standard Model.

Because the strengths of coupling for the three forces are rather different, it is therefore not a priori clear in what way the separate forces can be regarded as manifestations of a single force. But our previous observations concerning the dependence of coupling constants on momentum scale (or distance) now become telling. We have argued that electric charge grows with momentum transfer, whereas the effective charge associated with non-Abelian symmetries, such as color, decreases with momentum. It is therefore conceivable that, at some large energy, the three coupling strengths could become equal, and the three kinds of interactions would therefore not be distinguishable, and could be described by a single force that would operate above the unifying energy scale. At low energies, that single force would simply separate into three of the four known fundamental forces of nature.

To understand how such a separation of forces can be achieved, we must recognize that, in order to incorporate quarks and leptons into a single family, we must also enlarge the overall symmetry group. (This prescription would also lead more naturally to the quantization of electric charge.) There are several symmetry groups, with different degrees of complexity, that can be used to implement the ideas of grand unification. Simplest is a symmetry group known as  $SU(5)$ , which is analogous to isospin, and corresponds to a rotation in an internal space of five dimensions. In this particular

model (suggested by Howard Georgi and Sheldon Glashow), it is assumed that the symmetry of the fundamental interactions beyond the unifying energy scale corresponds to a local symmetry based on this larger  $SU(5)$  group. Below the unification scale, however, the local symmetry breaks down spontaneously to the low-energy symmetry group of the Standard Model – namely  $SU_{\text{color}}(3) \times SU_L(2) \times U_Y(1)$  – which subsequently breaks down spontaneously to an even lower symmetry group of  $SU_{\text{color}}(3) \times U_Q(1)$  at the electroweak energy scale. This kind of mechanism can explain how a single force at very high energy can manifest itself as three separate forces at low energy. Extrapolating the couplings observed for the three interactions at low energy (from the electroweak scale) to higher energy, a detailed analysis and some theoretical prejudice suggest that the unification scale is close to  $10^{15}$  GeV.

With  $SU(5)$  as the symmetry group, each family of quarks and leptons can be incorporated consistently into a five-dimensional multiplet, and a ten-dimensional antisymmetric matrix of states. The five-dimensional multiplet consists of right-handed particles, whereas the ten-dimensional multiplet contains only left handed particles. Explicitly, these multiplets take the form

$$\begin{pmatrix} d \text{ red} \\ d \text{ blue} \\ d \text{ green} \\ e^+ \\ \bar{\nu}_e \end{pmatrix}_R, \quad \begin{pmatrix} 0 \ \bar{u} \text{ green} \ \bar{u} \text{ blue} \ u \text{ red} \ d \text{ red} \\ 0 \ \bar{u} \text{ red} \ u \text{ blue} \ d \text{ blue} \\ 0 \ u \text{ green} \ d \text{ green} \\ 0 \ e^+ \\ 0 \end{pmatrix}_L. \quad (15.4)$$

The particle representations for other unifying groups are more complex, but, generically, we can represent a multiplet of a unifying group as

$$\begin{pmatrix} q \\ \ell \end{pmatrix}, \quad (15.5)$$

with each multiplet necessarily containing both quarks and leptons.

As we discussed previously, the gauge bosons of the unifying group provide transitions among members of a given multiplet. Thus, from the structure of the multiplet in Eq. (15.5), we conclude that, in grand unified



theories, there will be transitions between quarks and leptons caused by some new massive gauge bosons (when the symmetry is broken). Of course, this implies that baryon and lepton numbers need not be conserved in such theories, and, as a consequence, the proton will decay. For the case of  $SU(5)$ , the process shown in Fig. 15.1 corresponds to a proton decaying into a  $\pi^0$  and  $e^+$

$$p \longrightarrow \pi^0 + e^+ \tag{15.6}$$

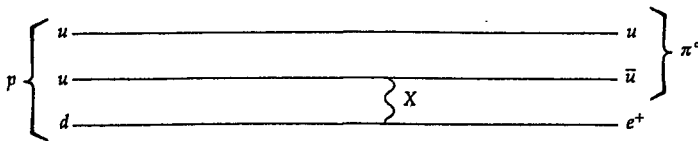


Fig. 15.1 Mechanism for proton decay involving the  $X$  gauge boson of  $SU(5)$ .

We considered this decay in Chapter 9 in the context of baryon-number conservation. The lifetime of the proton can, in fact, be calculated in grand unified models. From the observed stability of the universe, we would, of course, expect the proton to be quite long lived. In fact, geochemical experiments indicate that the lifetime of the proton  $\tau_p$  is

$$\tau_p > 1.6 \times 10^{25} \text{ years (independent of mode).} \tag{15.7}$$

The limits are far stronger ( $> 10^{31} - 10^{34}$  years) for specific decay modes of the kind given in Eq. (15.6), and disagree with the shorter lifetimes expected from simplest grand unified theory (GUT) based on the symmetry group  $SU(5)$ , and therefore rule out this possibility. There are, however, other models of grand unification (with more parameters and greater flexibility) that are still consistent with all current limits on  $\tau_p$ .

The idea of grand unification has also had considerable influence on cosmology – the study of the evolution of the universe. This application rests on the fact that when a symmetry is spontaneously broken, the system undergoes a phase transition. A way to see this is to consider again the example of the ferromagnet. At very high temperatures, thermal motion orients the spins in a random fashion, and therefore prevents the establishment of any order. Rotations are consequently a symmetry of the system. As the temperature drops, thermal motion diminishes, leading to a ground

state where spins are aligned, and the inherent rotational symmetry is therefore no longer apparent. Consequently, at a lower temperature, the system makes a transition to an ordered phase.

The above concepts can be applied to the evolution of our universe. In particular, immediately after the Big Bang, the temperature in the universe must have been exceedingly high, and, assuming the validity of grand unification, we expect that the larger symmetry of grand unified theories reflected the symmetry of the universe at that time. As the universe expanded and cooled to below the temperature corresponding to the unification scale, the symmetry of the system reduced to that of the Standard Model. In other words, there was a phase transition. Such transitions are normally exothermic, and energy is consequently released in the process. Recalling that energy is the source of the gravitational force, we would infer that such a phase transition would have influenced the dynamical evolution of the universe. In fact, using concepts of particle physics, it can be shown explicitly that such a phase transition would have led to an epoch where the universe expanded exponentially, and much faster than predicted by older cosmological models. This conjecture (made by Alan Guth) has the added benefit that it solves several other important problems in standard cosmology.

Grand unified theories may also provide a natural explanation for the baryon asymmetry of our universe. The argument is based on the observed ratio of the number of baryons to the number of photons, which has the value

$$\frac{n_B}{n_\gamma} \approx 4 \times 10^{-10}.$$

The photons (mostly from the  $3K$  background radiation – remnant of the Big Bang) have typical energies of about  $10^{-4}$  eV. This means that the visible energy in our universe is mainly in the form of matter (that is, *matter dominated*). As we have seen, grand unified theories can lead to baryon nonconservation through processes such as proton decay. Furthermore, if  $CP$  violation is built into such theories, they can then provide a baryon asymmetry and a prediction for the  $\frac{n_B}{n_\gamma}$  ratio, which comes out relatively close to the observed value. It is worth emphasizing that models based on GUTs are the only ones where the ratio  $\frac{n_B}{n_\gamma}$  is calculable, and can thereby provide a possible understanding of the origin of the baryon asymmetry in the universe.

However, in detail, the situation is not so clear. It appears that most of the matter in the universe ( $\approx 95\%$ ) is of unknown origin. About 25% of this is referred to as *dark matter*, and is not baryonic in character (neutrinos contribute only a small fraction of this). The main part of the content of the universe has a repulsive pressure-like quality, and is referred to as *dark energy*, which is supposedly responsible for the origin of the recently discovered *acceleration* in the expansion of the universe. The origin of the baryon asymmetry reflected in  $\frac{n_B}{n_\gamma}$  is still not fully understood, and will likely require input from physics beyond the Standard Model in order to be accommodated.

### 15.3 Supersymmetry (SUSY)

Thus far, our discussions of symmetry have been restricted to transformations that relate similar kinds of particles. For example, a rotation can take a spin-up electron to a spin-down electron state. An isospin rotation can take a proton state to a neutron state, or a  $\pi^+$  meson state to a  $\pi^0$  meson state, and so on. Thus, the conventional symmetry transformations rotate bosonic states to other bosonic states, and fermionic states to other fermionic states. A novel form of symmetry transformation would be one that would rotate a bosonic state to a fermionic state, and vice versa. If this were possible, it would imply that bosons and fermions could be merely different manifestations of the same state, and in some sense would correspond to an ultimate form of unification. For a long time, it was believed that this kind of symmetry transformation was not possible to implement in physical theories. Nevertheless, such transformations can be defined, and, in fact, there are theories that are invariant under such transformations. These transformations are known as *supersymmetry* (SUSY) transformations, and the corresponding theories invariant under such transformations are called supersymmetric theories.

To get a qualitative understanding of supersymmetry, let us consider a simple quantum mechanical example. For a bosonic harmonic oscillator in one dimension, the Hamiltonian can be written in terms of creation and annihilation operators as

$$H_B = \frac{\hbar\omega}{2} \left( a_B a_B^\dagger + a_B^\dagger a_B \right), \quad (15.8)$$

where  $a_B$  and  $a_B^\dagger$  lower and raise, respectively, the number of quanta in a

state, and they satisfy the commutation relations

$$\begin{aligned} [a_B, a_B] &= 0 = [a_B^\dagger, a_B^\dagger], \\ [a_B, a_B^\dagger] &= 1. \end{aligned} \quad (15.9)$$

The Hamiltonian of Eq. (15.8) can also be written in the more familiar form

$$H_B = \hbar\omega \left( a_B^\dagger a_B + \frac{1}{2} \right). \quad (15.10)$$

The energy spectrum of this Hamiltonian, and the associated quantum states with their corresponding energy eigenvalues, take the form

$$|n_B\rangle \longrightarrow E_{n_B} = \hbar\omega \left( n_B + \frac{1}{2} \right), \quad n_B = 0, 1, 2, \dots \quad (15.11)$$

We note, in particular, that the ground state energy of the system has the value

$$E_0 = \frac{\hbar\omega}{2}. \quad (15.12)$$

Quantum mechanical oscillators can also satisfy Fermi-Dirac statistics, and in this case, the Hamiltonian has the form

$$H_F = \frac{\hbar\omega}{2} \left( a_F^\dagger a_F - a_F a_F^\dagger \right). \quad (15.13)$$

Being fermionic operators,  $a_F$  and  $a_F^\dagger$  satisfy the anti-commutation relations

$$\begin{aligned} a_F^2 &= 0 = \left( a_F^\dagger \right)^2, \\ a_F a_F^\dagger + a_F^\dagger a_F &= 1. \end{aligned} \quad (15.14)$$

Using Eq. (15.14), we can also rewrite the Hamiltonian for the fermionic oscillator as

$$H_F = \hbar\omega \left( a_F^\dagger a_F - \frac{1}{2} \right). \quad (15.15)$$

This kind of system has only two energy eigenstates, and the corresponding energy eigenvalues are given by

$$|n_F\rangle \longrightarrow E_{n_F} = \hbar\omega \left( n_F - \frac{1}{2} \right), \quad n_F = 0, 1. \quad (15.16)$$

The simplicity of this spectrum is purely a consequence of Fermi-Dirac statistics, according to which any physical state can have either one fermionic quantum ( $n_F = 1$ ) or be the empty (bosonic) ground state containing no fermions ( $n_F = 0$ ).

If we now consider a mixed bosonic and fermionic oscillator of the same frequency, the Hamiltonian becomes

$$H = H_B + H_F = \hbar\omega \left( a_B^\dagger a_B + a_F^\dagger a_F \right). \quad (15.17)$$

Very roughly speaking, this Hamiltonian is invariant under an interchange of bosons and fermions

$$"a_B" \longleftrightarrow "a_F". \quad (15.18)$$

A way to see this from Eqs. (15.11) and (15.16) is to note that the energy spectrum of this system can be represented as

$$\begin{aligned} |n_B, n_F\rangle &\longrightarrow E_{n_B, n_F} = \hbar\omega(n_B + n_F), \\ n_F &= 0, 1 \text{ and } n_B = 0, 1, 2, \dots \end{aligned} \quad (15.19)$$

Thus, for any  $n_B \geq 1$ , the bosonic state  $|n_B, n_F = 0\rangle$  and the fermionic state  $|n_B - 1, n_F = 1\rangle$  are degenerate in energy, with eigenvalue

$$E = \hbar\omega n_B. \quad (15.20)$$

This degeneracy is a consequence of the invariance of the Hamiltonian in Eq. (15.17) under supersymmetry. Without going into details, we simply note

that generators of supersymmetry transformations exist for this example, and can be identified with

$$\begin{aligned} Q_F &= a_B^\dagger a_F, \\ Q_F^\dagger &= a_F^\dagger a_B, \end{aligned} \quad (15.21)$$

which can be shown to satisfy the anti-commutation relations

$$\left[ Q_F, Q_F^\dagger \right]_+ = Q_F Q_F^\dagger + Q_F^\dagger Q_F = \frac{H}{\hbar\omega}. \quad (15.22)$$

The operators  $Q_F$  and  $Q_F^\dagger$ , in analogy with raising and lowering operators of angular momentum, transform a bosonic state to a fermionic one of same energy, and vice versa. (It is worth emphasizing that the ground state for the supersymmetric oscillator, as can be seen from Eq. (15.19), has zero energy. This is, in fact, a general property of all supersymmetric theories, and has bearing on the nature of spontaneous symmetry breaking in such theories.)

Supersymmetry is not only a beautiful concept, but it also solves many technical difficulties, e.g., the hierarchy problem of unified theories. Without supersymmetry, it is exceedingly difficult to understand why the particles of the Standard Model are so light, when the scale of their unifying interaction is at least  $\approx 10^{15}$  GeV. The presence of supersymmetry can rather naturally prevent the Higgs and other fundamental particles from becoming far more massive, because the contributions of particles and their SUSY partners come in with opposite sign in Eq. (15.2), and therefore cancel exactly the  $g^2\Lambda^2$  contribution term-by-term.

There are many other reasons that have prompted the examination of supersymmetric grand unified theories. For the simplest supersymmetric GUTs, the calculated proton lifetime, again, turns out to be inconsistent with the current experimental lower limits on  $\tau_p$ , but there are other supersymmetric models where this is not the case, and such models are therefore viable. The main difficulty with accepting supersymmetric theories lies in the fact that the suggested doubling of the spectrum of fundamental particles in SUSY, that is, the required existence of bosonic partners for all fermions, and vice versa (in analogy with the two degeneracies of Eq. (15.19)) has yet to be verified in the laboratory. It is expected, however, that if supersymmetry is a true symmetry of nature, these new SUSY par-

ticles will be detected directly in experiments at the Tevatron or at the LHC.

Finally, a brief comment on the nature of spontaneous symmetry breaking at the electroweak scale, which remains an interesting and open question. We have noted that the spontaneous breakdown of a symmetry gives rise to fundamental massive particles, and that the massive Higgs boson of electroweak symmetry breaking has yet to be observed. There are two alternative scenarios for symmetry breaking. One involves a breaking that is induced by composite rather than fundamental bosons. “Technicolor” theories are those in which the symmetry is spontaneously broken by a composite state consisting of a fermion-antifermion pair (usually taken to be  $t\bar{t}$ ). These theories have an additional symmetry group known as the technicolor group, where new quarks, carrying new technicolor charges, form bound states that spontaneously break the low-energy symmetry of the Standard Model. At present, however, technicolor theories do not appear to reduce in a natural way to the  $SU_{\text{color}}(3) \times SU_L(2) \times U_Y(1)$  structure of the Standard Model. We will return to the other interesting alternative for spontaneous electroweak symmetry breaking in the final section of this chapter.

#### 15.4 Gravity, Supergravity and Superstrings

Grand unified theories, whether standard or the supersymmetric kind, are not complete because they leave out one of the four fundamental forces, namely gravitation. As we have seen, this force is very weak, and can be ignored for interactions involving the current sub-TeV energy scales. However, from the form of the gravitational potential energy

$$V_{\text{grav}}(r) = G_N \frac{m^2}{r}, \quad (15.23)$$

it is possible for this force to become appreciable at very small distances. In fact, for distances of the order of the Planck length of  $\approx 10^{-33}$  cm, or for equivalent energy scales of the order of  $10^{19}$  GeV, the effect of the gravitational interaction cannot be neglected. This can be seen heuristically in the following way. Considering two relativistic particles with energy  $E = pc$ , we can write the relation in Eq. (15.23) as

$$V_{\text{grav}} = \frac{G_N \left(\frac{E}{c^2}\right)^2}{r}. \quad (15.24)$$

Using the uncertainty principle, we can substitute

$$r \approx \frac{\hbar}{p} = \frac{\hbar c}{pc} = \frac{\hbar c}{E}, \quad (15.25)$$

and, therefore, express the potential energy as

$$V_{\text{grav}} \approx \frac{G_N}{\hbar c} \times E \times \left(\frac{E}{c^2}\right)^2. \quad (15.26)$$

From Eq. (15.26), we can deduce the energy scale at which the gravitational potential energy can no longer be neglected. This should correspond to the range when  $V \approx E$ , namely,

$$V_{\text{grav}} \approx \frac{G_N}{\hbar c} \times E \times \left(\frac{E}{c^2}\right)^2 \approx E,$$

$$\text{or} \quad \left(\frac{E}{c^2}\right)^2 \approx \frac{\hbar c}{G_N} \approx \frac{6}{6.7} \times 10^{39} \text{ (GeV}/c^2)^2,$$

$$\text{or} \quad E \approx 10^{19} \text{ GeV}. \quad (15.27)$$

Hence, at such energies, the effects of gravitation cannot be ignored. Since the unification scale is expected to be about  $10^{15}$  GeV, which is relatively close to the Planck scale, a consistent description of the fundamental interactions at these energies should therefore include gravity.

The primary reason why gravity is avoided in all such considerations is because Einstein's theory of gravity does not readily lend itself to quantization. In fact, if we naively quantize Einstein's theory, it yields divergent results for any calculated cross section. Divergences are not unheard-of features in quantum field theories, and, in fact, most relativistic quantum field theories contain divergences. However, in all these theories there is a systematic procedure for extracting meaningful physical quantities from the seemingly infinite results. This procedure, referred to as *renormalization*, fails for Einstein's gravity.



Supersymmetric theories, on the other hand, are known to have much better divergence behavior. Consequently, it is natural to consider supersymmetrizing Einstein's theory of gravity to see if this improves its divergence properties. (Supersymmetrizing Einstein's theory does not affect its desirable classical predictions, because any such modification to the theory proceeds through the addition of supersymmetric fermionic partners of gravity, which have no classical analog, and consequently do not contribute in the classical limit.) Supersymmetrized quantum gravity is found to have a local supersymmetry invariance, which is related to the fact that Einstein's gravity has no preferred reference frame. This supersymmetric gauge theory is known as *supergravity*, and it has indeed better divergence structure than ordinary quantum theories of gravity. Nevertheless, even the most sophisticated form of supergravity theory does not appear to be completely free of divergences. In addition, these theories also do not appear to reduce naturally to the Standard Model.

The divergences in relativistic quantum theories can be traced primarily to the presumed local nature of interactions. That is, to the assumption that all decays, emissions, or collisions, take place at specific space-time points, which implies that there is no uncertainty in their positions. This is essentially because we are dealing with point particles, and as a consequence such interactions have  $\Delta x = 0$ , and the uncertainty principle therefore requires infinite uncertainty in the conjugate momentum. This is the origin of divergences in ordinary quantum theories. A way to remedy this problem is to regard the fundamental constituents not as point particles, but rather as one dimensional objects of infinitesimal size (of the order of  $10^{-33}$  cm). In that case, the interaction vertices are no longer completely localized (see Fig. 15.2), uncertainties in momentum transfers become finite, and divergences disappear. Theories that describe particles as objects of infinitesimal extension are known as *string theories*, and they appear to incorporate gravity in a natural manner. This possibility offers the only currently known prescription for a fully quantized theory of gravity that does not suffer from problems of divergence.

Broadly speaking, there are two kinds of string theories – bosonic string theories and superstring (supersymmetric string) theories. They are quite elegant and incorporate many interesting symmetries. However, they can be formulated in a consistent manner only in 10 (for superstrings) or 26 (for bosonic strings) space-time dimensions. Of the two possibilities, superstrings appear to be more interesting for a variety of reasons. Furthermore, even within the framework of superstring theories, there are five consistent



Fig. 15.2 Difference between a point-like (left) and a string-like (right) interaction.

possible theories involving different gauge groups. If string theories are to describe the unique theory of the world, the possibility of five such theories presents a natural dilemma. In the past few years, however, a lot of work has been done showing that the string theories possess *duality* symmetries – of the kind that may be present in Maxwell’s theory (in the presence of Dirac monopoles), where the roles of electric and the magnetic fields can be interchanged. These duality symmetries have proven to be fundamental in many ways. First, they show that the five distinct superstring theories can be mapped into each other under a duality transformation, so that they need not be regarded as distinct theories. Second, they also suggest that there may be a more fundamental theory in 11 dimensions (conventionally called the *M-theory*), with the five superstring theories corresponding to different dimensional reductions of this fundamental theory. The existence of an eleven-dimensional fundamental theory is also esthetically satisfying in that its low energy limit can be identified with the eleven-dimensional supergravity theory (which is the largest possible supergravity theory). Duality symmetries also provide a way of learning the behavior of string theories in regions of strong coupling, based on calculations of their dual equivalents in regions of weak coupling. There have been many other exciting developments in this field that we cannot go into, but we simply point out that one of the main unresolved problems of string theories is the ambiguity in selecting the relevant physical sector of the theory upon reduction from ten to four space-time dimensions. That is, the dimensional reduction provides many possible ground states (vacuum states), and there is no guiding principle, at the moment, for selecting the correct physical one. Much more work remains to be done in this field before clean experimental tests can be formulated, nevertheless, the existence of supersymmetry is a key aspect and requirement of string theories.

Finally, we end our thumbnail sketch of physics beyond the Standard Model with a rather remarkable recent suggestion for eliminating (or, at the very least, minimizing) the hierarchy problem, in light of the possibility that

the universe has as many as ten dimensions. It is normally assumed that quantum gravity becomes important for particle interactions at energies close to the Planck scale. But this is based on the small value of  $G_N$  in our three-dimensional worldview. It was assumed in the past that, if extra dimensions did exist, they would have “curled up” in the first moments of the Big Bang, at the scale of the order of the Planck length, or string size. But, if these extra dimensions are, in fact, large compared to the Planck length, but not as small as the length characterizing the electroweak scale, it would then be possible to sense the impact of these “large” extra dimensions through the observation of enhanced effects of gravity in particle interactions. This possibility would require a modification of Newton’s Law at exceedingly small distances that can be probed only through interactions of point-like particles (see Problem 15.3). In the suggested scenario, the onset of quantum gravity would take place at distances of  $\approx 10^{-17}$  cm, or at an energy scale of  $\approx 1$  TeV. With quantum gravity operating this close to the electroweak scale, it would obviate the need for a huge value of  $\Lambda$  in Eq. (15.2), and allow the force of gravitation to be the possible cause of electroweak symmetry breaking. Although intoxicating in concept, this idea has not as yet gained any support from experiment. Again, it is hoped that the LHC might shed further light on this issue.

## Problems

**15.1** Show that Eq. (15.22) follows from the definitions given in Eq. (15.21).

**15.2** Using dimensional analysis, and the known value of  $G_N$ , show that you can write  $G_N = \frac{\hbar c}{M_P^2}$ , where  $M_P$  is the Planck mass or scale. What is the value of  $M_P$  in GeV units? Applying the uncertainty principle, you can define a Planck length and a Planck time, as in Eq. (15.25). What are these values in cm and sec, respectively?

**15.3** Ignoring, for the moment powers of  $\hbar c$ , Newton’s law for  $n$  extra dimensions can be written as

$$V_{\text{grav}}(r) \propto \frac{1}{M_S^{n+2}} \frac{m_1 m_2}{r^{n+1}},$$

where  $m_1$  and  $m_2$  are the interacting masses, and  $M_S$  corresponds to the effective Planck scale for  $n + 3$  spatial dimensions. Assuming that these  $n$

extra dimensions are compactified over equal radii  $R$ , then  $V(r)$  for  $r \gg R$ , that is, from the perspective of our 3-dimensional space, becomes

$$V_{\text{grav}}(r) \rightarrow \frac{1}{M_S^{n+2}} \frac{m_1 m_2}{R^n r}.$$

Now, using the fact that  $M_S^{n+2} R^n$  must equal  $M_P^2$ , calculate  $R$  in meters for  $n = 1, 2, 3$ , and  $\infty$ , with  $M_S$  set to the desired value of  $\approx 1 \text{ TeV}/c^2$ . From what you know of Newton's law, is it possible to have  $n = 1$ ? (*Hint:* Clearly, you cannot ignore  $\hbar c$  in calculating  $R$ ! Using the fact that  $(Mc) \times (R) \approx \hbar$ , and Problem 15.2, should enable you to get the right answers.)

### Suggested Readings

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## Appendix A

# Special Relativity

Essentially all of particle physics and many areas of nuclear physics deal with particles that travel at relativistic velocities, namely velocities that are close to the speed of light  $c$ . In this appendix we will therefore summarize some of the basic concepts and results of special relativity that are needed for interpreting relativistic processes.

Starting off with the assumption that the laws of physics do not depend on the relative motion of observers at rest in different inertial frames, and that the speed of light (in vacuum) is a constant of nature that is independent of the inertial frame, Albert Einstein showed that the space-time coordinates of an event observed in two such frames can be related through the Lorentz transformation. That is, for two inertial frames that move with a relative velocity  $v = v_z = \beta c$  with respect to each other, the relationship between the coordinates of any event in the two frames can be expressed as

$$\begin{aligned} ct' &= \gamma(ct - \beta z), \\ x' &= x, \\ y' &= y, \\ z' &= \gamma(z - \beta ct), \end{aligned} \tag{A.1}$$

where we have chosen to define the  $z$ -axis as the direction of relative motion of our two coordinate frames (with primed and unprimed coordinates), and  $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$ . The relations given in Eq. (A.1) can be written in the form of a matrix as

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \quad (A.2)$$

The inverse transformation involves just a change in the sign of  $v$  (and therefore of  $\beta$ ),

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}. \quad (A.2')$$

For a general Lorentz transformation, the matrix connecting the coordinates of the two reference frames is more complicated. But since we can always define the  $z$ -axis by the direction of relative motion, without affecting the physical situation, we will continue to use this simpler matrix for our transformations.

The four coordinates ( $x^0 = ct$ ,  $x^1 = x$ ,  $x^2 = y$ ,  $x^3 = z$ ), or  $(x^0, \vec{x})$  are referred to as the components of a space-time *four-vector*  $x$ . Now, just as the scalar or dot product of any two common "three-vectors"  $\vec{S}$  and  $\vec{R}$ , namely  $\vec{S} \cdot \vec{R}$ , is invariant (remains the same) under a rotation of coordinates, so is the following *contraction* of any two four-vectors  $x$  and  $y$  invariant under any Lorentz transformation that consists of rotations and *boosts*

$$x \cdot y = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3 = x^0 y^0 - \vec{x} \cdot \vec{y}. \quad (A.3)$$

Similarly, the momentum vector  $\vec{P}$  and the energy  $E$  of any particle, also define a four-vector  $p$ , commonly referred to as the energy-momentum four vector

$$p = \left( \frac{E}{c}, \vec{P} \right) = \left( \frac{E}{c}, P_x, P_y, P_z \right) = (p^0, p^1, p^2, p^3). \quad (A.4)$$

Although the individual components of such energy-momentum four-vectors are different in different inertial frames, they, can be related to

one another through the same Lorentz transformation that relates the coordinates, namely,

$$\begin{pmatrix} \frac{E'}{c} \\ P'_x \\ P'_y \\ P'_z \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \frac{E}{c} \\ P_x \\ P_y \\ P_z \end{pmatrix} \quad (\text{A.5})$$

Again, given any two energy-momentum four-vectors  $p = (p^0, \vec{P})$  and  $q = (q^0, \vec{Q})$ , the quantity  $p \cdot q = (p^0 q^0 - \vec{P} \cdot \vec{Q})$  is independent of Lorentz frame, that is, it is an invariant constant. In particular,  $p \cdot p$  is also an invariant, and for a particle with energy  $E$  and momentum  $\vec{P}$ , we note that

$$p \cdot p = \frac{E^2}{c^2} - |\vec{P}|^2 = \text{constant}. \quad (\text{A.6})$$

Since this quantity is independent of reference frame, we can, in particular, consider its value in the rest frame of the particle ( $\vec{P} = 0$ ), where we can relate it to the square of the rest energy, namely,

$$p \cdot p = \frac{E_{\text{rest}}^2}{c^2} = M^2 c^2, \quad (\text{A.6}')$$

where  $M$  is defined as the *rest mass* of the particle.

For any particle moving with a velocity  $\vec{v} = \vec{\beta}c$  relative to a stationary observer, the relativistic momentum and energy can be written as

$$\begin{aligned} \vec{P} &= M\gamma\vec{v} = M\gamma\vec{\beta}c, \\ E &= M\gamma c^2. \end{aligned} \quad (\text{A.7})$$

It follows from this that

$$\vec{\beta} = \frac{c\vec{P}}{E}, \quad \text{and} \quad \gamma = \frac{E}{Mc^2}. \quad (\text{A.8})$$

The total energy  $E$  can also be decomposed into the frame-independent rest energy,  $Mc^2$ , and the relativistic kinetic energy  $T$  as



$$E = T + Mc^2. \quad (\text{A.9})$$

Consequently, the kinetic energy of any particle can be expressed in terms of the momentum observed in any rest frame as

$$T = E - Mc^2 = \sqrt{(Mc^2)^2 + c^2|\vec{P}|^2} - Mc^2,$$

$$\text{or } c|\vec{P}| = \sqrt{T^2 + 2Mc^2T}. \quad (\text{A.10})$$

Because the sums and differences of any four-vectors are also four-vectors, it follows that the “square” of the sum or difference of any number of four-vectors is also a Lorentz invariant quantity. In particular, for any set of four-vectors  $q_i = (q_i^0, q_i^1, q_i^2, q_i^3)$ , if we define the sum of the  $q_i$  as a four vector  $q = \sum_i q_i$ , then the square of the four-vector  $q^2$  is an invariant quantity

$$q^2 = \left( \sum_i q_i^0 \right)^2 - \left( \sum_i q_i^1 \right)^2 - \left( \sum_i q_i^2 \right)^2 - \left( \sum_i q_i^3 \right)^2. \quad (\text{A.11})$$

For example, when the  $q_i$  represent the energy-momentum four-vectors of some group of particles, the quantity  $q^2$  is related to the square of the rest energy of the entire system. (When multiplied by  $c^2$ , this is the quantity we call  $s$  in Chapter 1.)

An unstable relativistic particle of energy  $E$  and momentum  $\vec{P}$ , that has a mean life  $\tau$  in its own rest frame, will have an observed mean life given by the appropriate Lorentz transformation for a time interval; that is, if a particle is produced at rest in the laboratory, its point of production will coincide with its point of decay ( $x_2 = x_1, y_2 = y_1, z_2 = z_1$ ) and its decay time ( $t_2 - t_1$ ) will be characterized by its mean life  $\tau$ . But if the particle has velocity  $v = \beta c = \frac{|\vec{P}|}{E}$  in the laboratory, its mean life will be dilated. The time interval in the laboratory is related to the one in the particle’s rest frame through the first of the relations in Eq. (A.1)

$$t'_2 - t'_1 = \gamma(t_2 - t_1) - \frac{\beta}{c} (z_2 - z_1). \quad (\text{A.12})$$

In the rest frame of the particle  $z_2 = z_1$ , and consequently the mean life observed in the laboratory ( $\tau' = t'_2 - t'_1$ ), is given by

$$\tau' = \gamma\tau. \tag{A.13}$$



## Appendix B

# Spherical Harmonics

The spherical harmonic functions  $Y_{\ell,m}(\theta, \phi)$  are eigenstates of both the square of the angular momentum operator  $L^2$ , as well as of  $L_z$ , the projection of  $\vec{L}$  on some specific axis  $z$  (see Eq. (3.26))

$$\begin{aligned}L^2 Y_{\ell,m}(\theta, \phi) &= \hbar^2 \ell(\ell + 1) Y_{\ell,m}(\theta, \phi), \\L_z Y_{\ell,m}(\theta, \phi) &= \hbar m Y_{\ell,m}(\theta, \phi).\end{aligned}\tag{B.1}$$

The  $Y_{\ell,m}(\theta, \phi)$  are products of periodic functions of  $\theta$  and of  $\phi$  that are often encountered in quantum mechanics and in other areas where we seek solutions to problems with spherical symmetry. The  $Y_{\ell,m}(\theta, \phi)$  can be written in terms of associated Legendre polynomials  $P_{\ell,m}(\cos \theta)$  and exponentials in  $\phi$  as

$$Y_{\ell,m}(\theta, \phi) = (-1)^{\frac{m+|m|}{2}} \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-|m|)!}{(\ell+|m|)!}} P_{\ell,m}(\cos \theta) e^{im\phi},\tag{B.2}$$

where the associated Legendre functions are given by

$$P_{\ell,m}(x) = \frac{(-1)^m}{2^\ell \ell!} (1-x^2)^{\frac{m}{2}} \frac{d^{\ell+m}}{dx^{\ell+m}} (x^2-1)^\ell,\tag{B.3}$$

with  $x = \cos \theta$ . The  $P_{\ell,m}(x)$  are defined such that the spherical harmonics obey the following normalization relation over the full solid angle

$$\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} Y_{\ell',m'}^*(\theta, \phi) Y_{\ell,m}(\theta, \phi) \sin \theta \, d\theta \, d\phi = \delta_{\ell'\ell} \delta_{m'm},\tag{B.4}$$

where the  $\delta_{nm}$  are the Kronecker symbols (see Eq. (10.21)). It follows from (B.2) that

$$Y_{\ell,m}^*(\theta, \phi) = (-1)^m Y_{\ell,-m}. \quad (B.5)$$

Some of the low-order spherical harmonics are

$$\begin{aligned} Y_{0,0}(\theta, \phi) &= \frac{1}{\sqrt{4\pi}}, \\ Y_{1,1}(\theta, \phi) &= -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi}, \\ Y_{1,0}(\theta, \phi) &= \sqrt{\frac{3}{4\pi}} \cos \theta, \\ Y_{1,-1}(\theta, \phi) &= \sqrt{\frac{3}{8\pi}} \sin \theta e^{-i\phi}, \\ Y_{2,2}(\theta, \phi) &= \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{2i\phi}, \\ Y_{2,1}(\theta, \phi) &= -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi}, \\ Y_{2,0}(\theta, \phi) &= \sqrt{\frac{5}{4\pi}} \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right). \end{aligned} \quad (B.6)$$

## Appendix C

# Spherical Bessel Functions

The spherical Bessel functions  $j_\ell(x)$  arise in solutions of the radial Schrödinger equation in spherical coordinates. These functions are related to the ordinary Bessel functions  $J_\ell(x)$  that are usually encountered in systems that possess cylindrical symmetry. The relation between the two type of functions are

$$j_\ell(x) = \sqrt{\frac{\pi}{2x}} J_{\ell+\frac{1}{2}}(x). \quad (C.1)$$

The more standard Bessel functions are given by the expansion

$$J_\ell(x) = \sum_{\lambda=0}^{\infty} \frac{(-1)^\lambda \left(\frac{x}{2}\right)^{\ell+2\lambda}}{\Gamma(\lambda+1)\Gamma(\lambda+\ell+1)}, \quad (C.2)$$

where  $\Gamma$  refers to the factorial function (“Gamma” function).

Using identities to relate  $\Gamma$  functions of different argument, it can be shown that the series obtained by substituting Eq. (C.2) into Eq. (C.1) can be identified with expansions of simple periodic functions. In particular, it follows that some of the lowest order spherical Bessel functions can be written as

$$\begin{aligned} j_0(x) &= \frac{\sin x}{x}, & j_1(x) &= \frac{\sin x}{x^2} - \frac{\cos x}{x}, \\ j_2(x) &= \left(\frac{3}{x^3} - \frac{1}{x}\right) \sin x - \frac{3 \cos x}{x^2}, \quad \text{etc.} \end{aligned} \quad (C.3)$$

All the  $j_\ell(x)$  are well behaved near  $x = 0$ . In fact, all but the  $\ell = 0$  function vanish at the origin, and  $j_0(0) = 1$ . The solutions of the radial

Schrödinger equation that are singular at the origin are known as the Neumann functions, but such functions are not normalizable and therefore do not correspond to physical solutions for bound quantum mechanical systems. They are, however, important in the study of problems that exclude the origin, e.g., in the study of scattering.

## Appendix D

# Basics of Group Theory

A group consists of a set of elements (objects, quantities) – finite or infinite in number – with a rule for combining the elements (*multiplication* rule) such that the set is *closed* under multiplication. Thus, if  $G$  represents a group, with  $(g_1, g_2, \dots, g_n)$  as its elements, then the combination of any two of its elements  $g_i$  and  $g_j$ , denoted by  $g_i \bullet g_j$ , also belongs to the group. (Mathematically,  $g \in G$  stands for the statement  $g$  belongs to  $G$ .) It should be understood that the combination rule for the elements (namely, the multiplication rule) does not necessarily have to be an ordinary product of the elements. It can also be any other operation such as addition.

The set of elements have to satisfy several other properties in order to define a group, and these are

- (1) The multiplication (combination) of the elements must be associative, namely,

$$g_1 \bullet (g_2 \bullet g_3) = (g_1 \bullet g_2) \bullet g_3 \in G. \quad (D.1)$$

- (2) There must be an identity element of the group, denoted as  $I$ , such that combining any element with the identity gives back the same element,

$$g \bullet I = g = I \bullet g. \quad (D.2)$$

- (3) For every element  $g \in G$ , there must exist a unique inverse element  $g^{-1} \in G$ , such that

$$g \bullet g^{-1} = I = g \bullet g^{-1}. \quad (D.3)$$



For a simple example of a group, let us assume that  $G$  consists of all the real numbers, both positive and negative. In this case, we can define

$$g_1 \bullet g_2 = g_1 + g_2. \quad (D.4)$$

With this combination formula, it is clear that the sum of any two real numbers is again a real number, and  $G$  is therefore closed under multiplication. We also recognize that ordinary addition is associative, and, for our example,

$$g_1 \bullet (g_2 \bullet g_3) = g_1 \bullet (g_2 + g_3) = g_1 + g_2 + g_3 = (g_1 \bullet g_2) \bullet g_3. \quad (D.5)$$

Furthermore, we can identify the identity element with the number zero, so that

$$g \bullet I = g + 0 = g = I \bullet g. \quad (D.6)$$

Finally, for any real number  $g$ , we can identify its inverse as  $g^{-1} = -g$ , such that

$$g \bullet g^{-1} = (g + g^{-1}) = (g - g) = 0 = I = g^{-1} \bullet g. \quad (D.7)$$

This shows that the set of all real numbers define a group, with ordinary addition representing the multiplication rule.

Let us next consider the set of all real phases, and denote them as

$$G = \{U(\alpha) = e^{i\alpha}, \quad \text{with } \alpha \text{ real and in the range } -\infty \leq \alpha \leq \infty\}. \quad (D.8)$$

The elements of this set are labeled by a continuous parameter,  $\alpha$ , and consequently this defines a continuous set. Note that if we choose ordinary multiplication to be the combination formula, then

$$U(\alpha) \bullet U(\beta) = U(\alpha)U(\beta) = e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)} = U(\alpha + \beta). \quad (D.9)$$

That is, we see that the combination is a phase and, therefore, belongs to the set. The set  $G$  is consequently closed under multiplication. The ordinary products are, of course, associative, and we have

$$\begin{aligned}
 U(\alpha) \bullet [U(\beta) \bullet U(\gamma)] &= U(\alpha)[U(\beta)U(\gamma)] = [U(\alpha)U(\beta)]U(\gamma) \\
 &= e^{i(\alpha+\beta+\gamma)} = U(\alpha + \beta + \gamma) \in G.
 \end{aligned}
 \tag{D.10}$$

For the identity element, we can choose the element with zero phase to correspond to  $I = 1$ , so that

$$U(\alpha) \bullet I = e^{i\alpha} \times 1 = e^{i\alpha} = U(\alpha) = I \bullet U(\alpha). \tag{D.11}$$

Furthermore, given a phase  $U(\alpha)$ , we can identify its inverse with  $U^{-1}(\alpha) = U(-\alpha)$ , such that

$$\begin{aligned}
 U(\alpha) \bullet U^{-1}(\alpha) &= U(\alpha)U^{-1}(\alpha) = U(\alpha)U(-\alpha) \\
 &= e^{i\alpha}e^{-i\alpha} = 1 = I = U^{-1}(\alpha) \bullet U(\alpha).
 \end{aligned}
 \tag{D.12}$$

Thus, the set of all real phases defines a group. For this case, the adjoint (or complex conjugate) element is also the inverse element, namely,

$$U^\dagger(\alpha) = e^{-i\alpha} = U(-\alpha) = U^{-1}(\alpha), \quad \text{for real } \alpha, \tag{D.13}$$

and such groups are referred to as unitary groups. Furthermore, because the group elements in this case are completely defined by a single parameter, the group is denoted as the group  $U(1)$ , or the unitary group in one dimension.

It is important to recognize that, in general, the combination rule for the elements of any group need not be commutative. That is,

$$g_1 \bullet g_2 \neq g_2 \bullet g_1. \tag{D.14}$$

In our simple example, however, we see that

$$U(\alpha) \bullet U(\beta) = U(\alpha)U(\beta) = U(\alpha + \beta) = U(\beta) \bullet U(\alpha). \tag{D.15}$$

Thus, we say that the group  $U(1)$  is commutative or Abelian.

In a similar manner, it can be shown that the set of all  $(2 \times 2)$  unitary matrices, with determinant (det) equaling unity, defines a group that has ordinary matrix multiplication representing the combination rule for the elements. This kind of group is known as the special (det = 1) unitary

group in two dimensions, and is denoted by  $SU(2)$ . An element of such a group can also be represented as a phase of the form

$$U(\vec{\alpha}) = e^{iT(\vec{\alpha})}, \quad (D.16)$$

where  $\vec{\alpha}$  denotes a vector parameter that labels the phase, and  $T(\vec{\alpha})$  corresponds to a  $2 \times 2$  matrix.<sup>1</sup> Note that for  $U(\vec{\alpha})$  to be unitary, we must have

$$U^\dagger(\vec{\alpha}) = U^{-1}(\vec{\alpha}),$$

$$\text{or } e^{-iT^\dagger(\vec{\alpha})} = e^{-iT(\vec{\alpha})}. \quad (D.17)$$

We conclude therefore that the matrices  $T(\vec{\alpha})$  must be Hermitian. Furthermore,  $\det U(\vec{\alpha})$  can equal unity only if the matrices  $T(\vec{\alpha})$  are traceless. This can be seen by noting that, for any matrix  $A$ , we can write, in general,

$$\det A = e^{\text{Tr } \ln A}, \quad (D.18)$$

where  $\text{Tr}$  refers to the trace of a matrix. Thus, requiring

$$\det U(\vec{\alpha}) = 1,$$

$$\text{or } e^{\text{Tr } \ln U(\vec{\alpha})} = 1,$$

$$\text{or } e^{i \text{Tr } T(\vec{\alpha})} = 1. \quad (D.19)$$

Because this must be true for any arbitrary vector  $\vec{\alpha}$ , we conclude that

$$\text{Tr } T(\vec{\alpha}) = 0. \quad (D.20)$$

It is well known that there are only three linearly independent, Hermitian, traceless  $2 \times 2$  matrices, namely the Pauli matrices

<sup>1</sup>The effect of the exponentiated matrix is equivalent to the normal series expansion of the operator:  $e^{iT(\vec{\alpha})} = I + iT(\vec{\alpha}) - \frac{1}{2!} T^2(\vec{\alpha}) - \frac{i}{3!} T^3(\vec{\alpha}) + \dots$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (D.21)$$

For a general element of  $SU(2)$  we can therefore write

$$U(\vec{\alpha}) = e^{iT(\vec{\alpha})} = e^{i \sum_{j=1}^3 \alpha_j T_j}, \quad (D.22)$$

where, conventionally, we identify

$$T_j = \frac{1}{2} \sigma_j. \quad (D.23)$$

The phases or elements of  $SU(2)$  are therefore labeled by three continuous parameters  $\alpha_1$ ,  $\alpha_2$ ,  $\alpha_3$ . Furthermore, we note that, because the matrix product is not commutative, for this case we have

$$U(\vec{\alpha})U(\vec{\beta}) = e^{i \sum_{j=1}^3 \alpha_j T_j} e^{i \sum_{k=1}^3 \beta_k T_k} \neq U(\vec{\beta})U(\vec{\alpha}). \quad (D.24)$$

The group  $SU(2)$  is, therefore, non-commutative or non-Abelian. However, the properties of the group can be determined completely, once we know the properties of the  $T_j$  matrices. These matrices satisfy the commutation relations

$$[T_j, T_k] = \left[ \frac{1}{2} \sigma_j, \frac{1}{2} \sigma_k \right] = i \epsilon_{jkl} \frac{1}{2} \sigma_l = i \epsilon_{jkl} T_l, \quad (D.25)$$

where  $\epsilon_{jkl}$  is the antisymmetric Levi-Civita symbol introduced in our discussion of continuous symmetries in Chapter 10, and this algebra is known as the Lie algebra for the group  $SU(2)$ .

Similarly, it can be shown that the set of all  $3 \times 3$  unitary matrices with  $\det=1$  constitute a group known as  $SU(3)$ . The set of all  $3 \times 3$  real orthogonal matrices with  $\det=1$  define a group known as  $SO(3)$ , and so on. The properties of these groups are fully determined once their Lie algebras are specified.



## Appendix E

# Table of Physical Constants

<i>Constant</i>	<i>Symbol</i>	<i>Value</i>
Avogadro's number	$A_0$	$6.0221420 \times 10^{23} \text{ mole}^{-1}$
Boltzmann's constant	$k$	$8.61734 \times 10^{-5} \text{ eV/K}$
Electron charge	$e$	$4.8032042 \times 10^{-10} \text{ esu}$
Mass of electron	$m_e$	$0.51099890 \text{ MeV}/c^2$ $9.1093819 \times 10^{-28} \text{ gm}$
Fermi's constant	$G_F(\hbar c)^3$	$1.16639 \times 10^{-5} \text{ GeV}^{-2}$
Fine structure constant	$\alpha = \frac{e^2}{\hbar c}$	$\frac{1}{137.0359998}$
Speed of light	$c$	$2.99792458 \times 10^{10} \text{ cm/sec (exact)}$
Newton's gravitational constant	$G_N$	$6.67 \times 10^{-8} \text{ cm}^3/\text{gm-sec}^2$ $6.71 \times 10^{-39} \hbar c (\text{GeV}/c^2)^{-2}$
("Reduced" Planck's constant) $\times c$	$\hbar c$	$197.326960 \text{ MeV-fm}$

Review of Particle Physics, K. Hagiwara et al, Phys. Rev. **D66**, 010001 (2002). See also the latest *CRC Handbook*. The physical constants are uncertain only in the last significant figure given in the table. The value of  $c$  is termed exact, and is used to define the meter as the distance traveled by light in  $\frac{1}{299792458}$  of a second.



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\*p is for problem and f is for figure.



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