IX. Compressible Flow

Compressible flow is the study of fluids flowing at speeds comparable to the local speed of sound. This occurs when fluid speeds are about 30% or more of the local acoustic velocity. Then, the fluid density no longer remains constant throughout the flow field. This typically does not occur with fluids but can easily occur in flowing gases.

Two important and distinctive effects that occur in compressible flows are (1) *choking* where the flow is limited by the sonic condition that occurs when the flow velocity becomes equal to the local acoustic velocity and (2) *shock waves* that introduce discontinuities in the fluid properties and are highly irreversible.

Since the density of the fluid is no longer constant in compressible flows, there are now four dependent variables to be determined throughout the flow field. These are pressure, temperature, density, and flow velocity. Two new variables, temperature and density, have been introduced and two additional equations are required for a complete solution. These are the *energy equation* and the fluid *equation of state*. These must be solved simultaneously with the *continuity* and *momentum* equations to determine all the flow field variables.

Equations of State and Ideal Gas Properties:

Two equations of state are used to analyze compressible flows: the *ideal gas* equation of state and the *isentropic flow* equation of state. The first of these describe gases at low pressure (relative to the gas critical pressure) and high temperature (relative to the gas critical temperature). The second applies to ideal gases experiencing isentropic (adiabatic and frictionless) flow.

The ideal gas equation of state is

$$\rho = \frac{P}{R\,T}$$

In this equation, R is the gas constant, and P and T are the absolute pressure and absolute temperature respectively. Air is the most commonly incurred compressible flow gas and its gas constant is R_{air} = 1717 ft^2/(s^2-^{o}R) = 287 m^2/(s^2-K).

Two additional useful ideal gas properties are the constant volume and constant pressure specific heats defined as

$$C_v = \frac{du}{dT} \quad and \quad C_p = \frac{dh}{dT}$$

where *u* is the specific internal energy and *h* is the specific enthalpy. These two properties are treated as constants when analyzing elemental compressible flows. Commonly used values of the specific heats of air are: c_v = 4293 ft^2/(s^2-°R) = 718 m^2/(s^2-K) and c_p = 6010 ft^2/(s^2-°R) = 1005 m^2/(s^2-K). Additional specific heat relationships are

$$R = C_p - C_v \quad and \quad k = \frac{C_p}{C_v}$$

The *specific heat ratio k* for air is 1.4.

When undergoing an isentropic process (constant entropy process), ideal gases obey the isentropic process equation of state:

$$\frac{P}{\rho^k} = \text{constant}$$

Combining this equation of state with the ideal gas equation of state and applying the result to two different locations in a compressible flow field yields

$$\frac{P_2}{P_1} = \left(\frac{T_2}{T_1}\right)^{k/(k-1)} = \left(\frac{\rho_2}{\rho_1}\right)^k$$

Note: The above equations may be applied to any ideal gas as it undergoes an isentropic process.

Acoustic Velocity and Mach Number

The *acoustic velocity* (speed of sound) is the speed at which an infinitesimally small pressure wave (sound wave) propagates through a fluid. In general, the acoustic velocity is given by

$$a^2 = \frac{\partial P}{\partial \rho}$$

The process experienced by the fluid as a sound wave passes through it is an isentropic process. The speed of sound in an ideal gas is then given by

$$a = \sqrt{k\,RT}$$

The *Mach number* is the ratio of the fluid velocity and speed of sound,

$$Ma = \frac{V}{a}$$

This number is the single most important parameter in understanding and analyzing compressible flows.

Mach Number Example:

An aircraft flies at a speed of 400 m/s. What is this aircraft's Mach number when flying at standard sea-level conditions (T = 289 K) and at standard 15,200 m (T = 217 K) atmosphere conditions?

At standard sea-level conditions, $a = \sqrt{k\,RT} = \sqrt{(1.4)(287)(289)} = 341\,m/s$ and at 15,200 m, $a = \sqrt{(1.4)(287)(217)} = 295\,m/s$. The aircraft Mach numbers are then

$$sea-level: \quad Ma = \frac{V}{a} = \frac{400}{341} = 1.17$$

$$15{,}200\ m: \quad Ma = \frac{V}{a} = \frac{400}{295} = 1.36$$

Note: Although the aircraft speed did not change, the Mach number did change because of the change in the local speed of sound.

Ideal Gas Steady Isentropic Flow

When the flow of an ideal gas is such that there is no heat transfer (i.e., adiabatic) or irreversible effects (e.g., friction, etc.), the flow is isentropic. The steady-flow energy equation applied between two points in the flow field becomes

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} = h_o = \text{constant}$$

where h_0 is called the *stagnation enthalpy* that remains constant throughout the flow field. Observe that the stagnation enthalpy is the enthalpy at any point in an isentropic flow field where the fluid velocity is zero or very nearly so.

The enthalpy of an ideal gas is given by $\text{h} = \text{C}_\text{p}\, T$ over reasonable ranges of temperature. When this is substituted into the adiabatic, steady-flow energy equation, we see that $\text{h}_\text{o} = \text{C}_\text{p}\, T_o = \text{constant}$ and

$$\frac{\text{T}_\text{o}}{\text{T}} = 1 + \frac{k-1}{2} Ma^2$$

Thus, the stagnation temperature T_o remains constant throughout an isentropic or adiabatic flow field and the relationship of the local temperature to the field stagnation temperature only depends upon the local Mach number.

Incorporation of the acoustic velocity equation and the ideal gas equations of state into the energy equation yields the following useful results for steady isentropic flow of ideal gases.

$$\frac{\text{T}_\text{o}}{\text{T}} = 1 + \frac{k-1}{2} Ma^2$$

$$\frac{a_o}{a} = \left(\frac{\text{T}_\text{o}}{\text{T}}\right)^{1/2} = \left(1 + \frac{k-1}{2} Ma^2\right)^{1/2}$$

$$\frac{P_o}{P} = \left(\frac{\text{T}_\text{o}}{\text{T}}\right)^{k/(k-1)} = \left(1 + \frac{k-1}{2} Ma^2\right)^{k/(k-1)}$$

$$\frac{\rho_o}{\rho} = \left(\frac{\text{T}_\text{o}}{\text{T}}\right)^{1/(k-1)} = \left(1 + \frac{k-1}{2} Ma^2\right)^{1/(k-1)}$$

The values of the ideal gas properties when the Mach number is 1 (i.e., sonic flow) are known as the *critical or sonic properties* and are given by

$$\frac{T_o}{T^*} = 1 + \frac{k-1}{2}$$

$$\frac{a_o}{a^*} = \left(\frac{T_o}{T^*}\right)^{1/2} = \left(1 + \frac{k-1}{2}\right)^{1/2}$$

$$\frac{P_o}{P^*} = \left(\frac{T_o}{T^*}\right)^{k/(k-1)} = \left(1 + \frac{k-1}{2}\right)^{k/(k-1)}$$

$$\frac{\rho_o}{\rho^*} = \left(\frac{T_o}{T^*}\right)^{1/(k-1)} = \left(1 + \frac{k-1}{2}\right)^{1/(k-1)}$$

given by

Isentropic Flow Example:

Air flowing through an adiabatic, frictionless duct is supplied from a large supply tank in which P = 500 kPa and T = 400 K. What are the Mach number Ma. the temperature T, density _, and fluid V at a location in this duct where the pressure is 430 kPa?

The pressure and temperature in the supply tank are the stagnation pressure and temperature since the velocity in this tank is practically zero. Then, the Mach number at this location is

$$Ma = \sqrt{\frac{2}{k-1}\left[\left(\frac{P_o}{P}\right)^{(k-1)/k} - 1\right]}$$

$$Ma = \sqrt{\frac{2}{0.4}\left[\left(\frac{500}{430}\right)^{0.4/1.4} - 1\right]}$$

$$Ma = 0.469$$

and the temperature is given by

$$T = \frac{T_o}{1 + \frac{k-1}{2} Ma^2}$$

$$T = \frac{400}{1 + 0.2(0.469)^2}$$

$$T = 383\,K$$

The ideal gas equation of state is used to determine the density,

$$\rho = \frac{P}{R\,T} = \frac{430{,}000}{(287)(383)} = 3.91\,kg/m^3$$

Using the definition of the Mach number and the acoustic velocity,

$$V = Ma\sqrt{k\,RT} = 0.469\sqrt{(1.4)(287)(383)} = 184\,m/s$$

Solving Compressible Flow Problems

Compressible flow problems come in a variety of forms, but the majority of them can be solved by

1. Use the appropriate equations and reference states (i.e., stagnation and sonic states) to determine the Mach number at all the flow field locations involved in the problem.
2. Determine which conditions are the same throughout the flow field (e.g., the stagnation properties are the same throughout an isentropic flow field).
3. Apply the appropriate equations and constant conditions to determine the necessary remaining properties in the flow field.
4. Apply additional relations (i.e., equation of state, acoustic velocity, etc.) to complete the solution of the problem.

Most compressible flow equations are expressed in terms of the Mach number. You can solve these equations explicitly by rearranging the equation, by using tables, or by programming them with spreadsheet or EES software.

Isentropic Flow with Area Changes

All flows must satisfy the continuity and momentum relations as well as the energy and state equations. Application of the continuity and momentum equations to a differential flow (see textbook for derivation) yields:

$$\frac{dV}{V} = \frac{1}{Ma^2 - 1}\frac{dA}{A}$$

This result reveals that when $Ma < 1$ (subsonic flow) velocity changes are the opposite of area changes. That is, increases in the fluid velocity require that the area decrease in the direction of the flow. For supersonic flow ($Ma > 1$), the area must increase in the direction of the flow to cause an increase in the velocity. Changes in the fluid velocity dV can only be finite in sonic flows ($Ma = 1$) when $dA = 0$. The effect of the geometry upon velocity, Mach number, and pressure is illustrated in Figure 1 below.

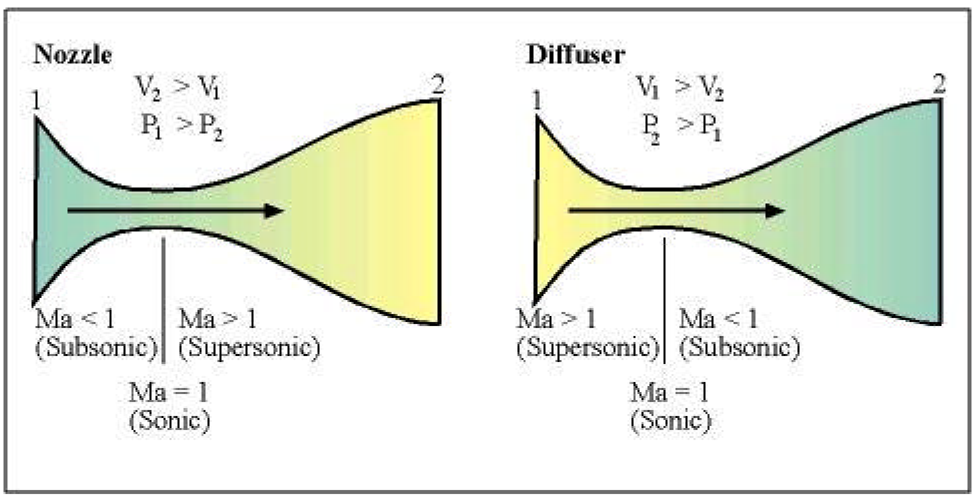


Figure 1

Combining the mass flow rate equation $\dot{m} = \rho A V = \text{constant}$ with the preceding isentropic flow equations yields

$$\frac{\rho^*}{\rho} = \left[\frac{2}{k+1}\left(1 + \frac{k-1}{2}Ma^2\right)\right]^{1/(k-1)}$$

$$\frac{V^*}{V} = \frac{1}{Ma}\left[\frac{2}{k+1}\left(1+\frac{k-1}{2}Ma^2\right)\right]^{1/2}$$

$$\frac{A}{A^*} = \frac{1}{Ma}\left[\frac{1+0.5(k-1)Ma^2}{0.5(k+1)}\right]^{(k+1)/[2(k-1)]}$$

where the sonic state (denoted with *) may or may not occur in the duct. If the sonic condition does occur in the duct, it will occur at the duct minimum or maximum area. If the sonic condition occurs, the flow is said to be choked since the mass flow rate $\dot{m} = \rho\, A\, V = \rho^*\, A^*\, V^*$ is the maximum mass flow rate the duct can accommodate without a modification of the duct geometry.

Review Example 9.4 of the textbook.

Normal Shock Waves

Under the appropriate conditions, very thin, highly irreversible discontinuities can occur in otherwise isentropic compressible flows. These discontinuities are known as *shock waves* which when they are perpendicular to the flow velocity vector are called *normal shock waves*. A normal shock wave in a one-dimensional flow channel is illustrated in Figure 2.

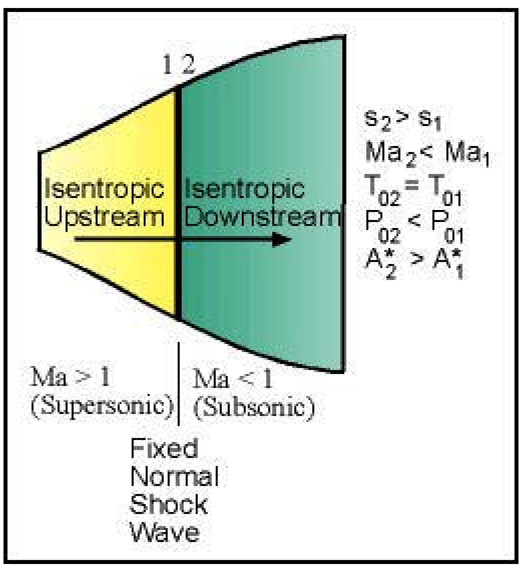


Figure 2

Application of the second law of thermodynamics to the thin, adiabatic normal shock wave reveals that normal shock waves can only cause a sharp rise in the gas pressure and must be supersonic upstream and subsonic downstream of the normal shock. *Rarefaction waves* that result in a decrease in pressure and increase in Mach number are impossible according to the second law.

Application of the conservation of mass, momentum, and energy equations along with the ideal gas equation of state to a thin, adiabatic control volume surrounding a normal shock wave yields the following results.

$$Ma_2^2 = \frac{(k-1)Ma_1^2 + 2}{2k\,Ma_1^2 - (k-1)}, \quad Ma_1 > 1$$

$$\frac{P_2}{P_1} = \frac{1 + k\,Ma_1^2}{1 + k\,Ma_2^2}$$

$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \frac{(k+1)\,Ma_1^2}{(k-1)\,Ma_1^2 + 2}$$

$$T_{o1} = T_{o2}$$

$$\frac{T_2}{T_1} = \left[2 + (k-1)Ma_1^2\right]\frac{2k\,Ma_1^2 - (k-1)}{(k+1)^2\,Ma_1^2}$$

$$\frac{P_{o2}}{P_{o1}} = \frac{\rho_{o2}}{\rho_{o1}} = \left[\frac{(k+1)\,Ma_1^2}{2 + (k-1)\,Ma_1^2}\right]^{k/(k-1)} \left[\frac{k+1}{2k\,Ma_1^2 - (k-1)}\right]^{1/(k-1)}$$

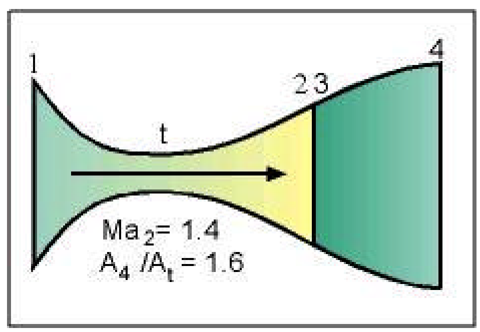
$$\frac{A_2^*}{A_1^*} = \frac{Ma_2}{Ma_1}\left[\frac{2 + (k-1)\,Ma_1^2}{2 + (k-1)\,Ma_2^2}\right]^{(k+1)/[2(k-1)]}$$

When using these equations to relate conditions upstream and downstream of a normal shock wave, keep the following points in mind:

1. Upstream Mach numbers are always supersonic while downstream Mach numbers are subsonic.
2. Stagnation pressures and densities decrease as one moves downstream across a normal shock wave while the stagnation temperature remains constant.
3. Pressures increase greatly while temperature and density increase moderately across a shock wave in the downstream direction.
4. The effective throat area increases across a normal shock wave in the downstream direction.
5. Shock waves are very irreversible causing the specific entropy downstream of the shock wave to be greater than the specific entropy upstream of the shock wave.

Moving normal shock waves such as those caused by explosions, spacecraft reentering the atmosphere, and others can be analyzed as stationary normal shock waves by using a frame of reference that moves at the speed of the shock wave in the direction of the shock wave.

Converging-Diverging Nozzle Example: **Also see Example 9.6 of textbook**



Air is supplied to the converging-diverging nozzle shown here from a large tank where P = 2 Mpa and T = 400 K. A normal shock wave in the diverging section of this nozzle forms at a point $P_{o1} = P_{o2} = 2\,MPa$ where the upstream Mach number is 1.4. The ratio of the nozzle exit area to the throat area is 1.6. Determine (a) the Mach number downstream of the shock wave, (b) the Mach number at the nozzle exit, (c) the pressure at the nozzle exit, and (d) the temperature at the nozzle exit.

This flow is isentropic from the supply tank (1) to just upstream of the normal shock (2) and also from just downstream of the shock (3) to the exit (4). Stagnation temperatures do not change in isentropic flows or across shock waves, $T_{o1} = T_{o2} = T_{o3} = T_{o4} = 400\,K$. Stagnation pressures do not change in isentropic flows, $P_{o1} = P_{o2} = 2\,MPa$ and $P_{o3} = P_{o4}$, but stagnation pressures change across shocks, $P_{o2} > P_{o3}$.

Based upon the Mach number at 2 and the isentropic relations,

$$\frac{A_2}{A_t} = \frac{A_3}{A_t} = \frac{A_2}{A_t^*} = \frac{1}{Ma_2}\frac{\left(1+0.2\,Ma_2^2\right)^3}{1.728} = 1.115$$

The normal shock relations can be used to work across the shock itself. The answer to (a) is then:

$$Ma_3 = \left[\frac{(k-1)Ma_2^2 + 2}{2\,k\,Ma_2^2 - (k-1)}\right]^{1/2} = \left[\frac{(0.4)(1.4)^2 + 2}{2\,(1.4)(1.4)^2 - 0.4}\right]^{1/2} = 0.740$$

Continuing to work across the shock,

$$P_{o4} = P_{o3} = P_{o2}\left[\frac{(k+1)Ma_2^2}{2 + (k-1)\,Ma_2^2}\right]^{k/(k-1)}\left[\frac{k+1}{2\,k\,Ma_2^2 - (k-1)}\right]^{1/(k-1)}$$

$$P_{o4} = P_{o3} = 2\left[\frac{(2.4)(0.74)^2}{2 + (0.4)(0.74)^2}\right]^{3.5}\left[\frac{2.4}{2\,(1.4)\,(0.74)^2 - 0.4}\right]^{2.5} = 1.92\,MPa$$

$$\frac{A_3^*}{A_2^*} = \frac{Ma_3}{Ma_2}\left[\frac{2 + (k-1)\,Ma_2^2}{2 + (k-1)\,Ma_3^2}\right]^{(k+1)/[2(k-1)]} = 1.044$$

Now, we know A_4/A_t, and the flow is again isentropic between states 3 and 4. Writing an expression for the area ratio between the exit and the throat, we have

$$\frac{A_4}{A_t} = 1.6 = \frac{A_4}{A_4^*}\frac{A_4^*}{A_3^*}\frac{A_3^*}{A_2^*}\frac{A_2^*}{A_t} = \frac{A_4}{A_4^*}(1)(1.044)(1.115)$$

Solving for $\frac{A_4}{A_4^*}$ we obtain $\qquad \frac{A_4}{A_4^*} = 1.374$

Using a previously developed equation for choked, isentropic flow, we can write

$$\frac{A_4}{A_4^*} = 1.374 = \frac{1}{Ma}\left[\frac{1 + 0.5(k-1)Ma^2}{0.5(k+1)}\right]^{(k+1)/[2(k-1)]}$$

or

$$1.374 = \frac{1}{Ma_4}\frac{\left(1 + 0.2Ma_4^2\right)^3}{1.728}$$

The solution of this equation gives answer (b) $Ma_4 = 0.483$.

Now that the Mach number at 4 is known, we can proceed to apply the isentropic relations to obtain answers (c) and (d).

$$P_4 = \frac{P_{o4}}{\left[1 + 0.5(k-1)Ma_4^2\right]^{k/(k-1)}} = \frac{1.92\,MPa}{\left[1 + 0.2(0.483)^2\right]^{3.5}} = 1.637\,MPa$$

$$T_4 = \frac{T_{o4}}{1 + 0.5(k-1)Ma_4^2} = \frac{400\,K}{1 + 0.2(0.483)^2} = 382\,K$$

Note: Observe how the sonic area downstream from the shock is not the same as upstream of the shock. Also, observe the use of the area ratios to determine the Mach number at the nozzle exit.

The following steps can be used to solve most one-dimensional compressible flow problems.

1. Clearly identify the flow conditions:e.g., isentropic flow, constant stagnation temperature, constant stagnation pressure, etc.
2. Use the flow condition relationships, tables, or software to determine the Mach number at major locations in the flow field.
3. Once the Mach number is known at the principal flow locations, one can proceed to use the flow relations, tables, or software to determine other flow properties such as fluid velocity, pressure, and temperature. This may require the reduction of property ratios to the product of several ratios, as was done with the area ratio in the above example to obtain the answer.

Operation of Converging-Diverging Nozzles

A converging-diverging nozzle like that shown in Figure 3 can operate in several different modes depending upon the ratio of the discharge and supply pressure P_d/P_s. These modes of operation are illustrated on the pressure ratio – axial position diagram of Figure 3.

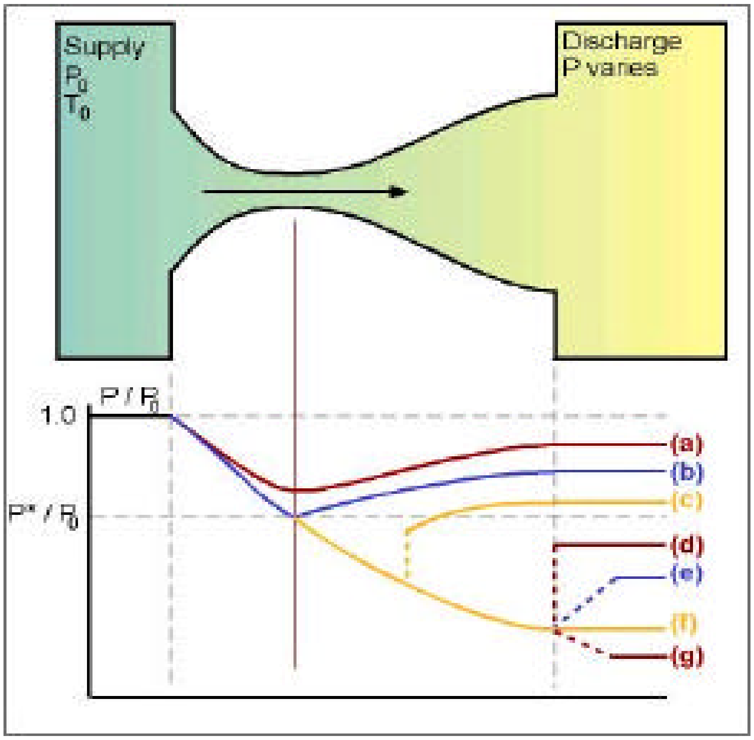


Figure 3

Mode (a) The flow is subsonic throughout the nozzle, supply, and discharge chambers. Without friction, this flow is also isentropic and the isentropic flow equations may be used throughout the nozzle.

Mode (b) The flow is still subsonic and isentropic throughout the nozzle and chambers. The Mach number at the nozzle throat is now unity. At the throat, the flow is sonic, the throat is choked, and the mass flow rate through the nozzle has reached its upper limit. Further reductions in the discharge tank pressure will not increase the mass flow rate any further.

Mode (c) A shock wave has now formed in the diverging section of the nozzle. The flow is subsonic before the throat, same as mode (b), the throat is choked, same as mode (b), and the flow is supersonic and accelerating between the throat and just upstream of the shock. The flow is isentropic between the supply tank and just upstream of the shock. The flow downstream of the shock is subsonic and decelerating. The flow is also isentropic downstream of the shock to the discharge tank. The flow is not isentropic across the shock. Isentropic flow methods can be applied upstream and downstream of the shock while normal shock methods are used to relate conditions upstream to those downstream of the shock.

Mode (d) The normal shock is now located at the nozzle exit. Isentropic flow now exists throughout the nozzle. The flow at the nozzle exit is subsonic and adjusts to flow conditions in the discharge tank, not the nozzle. Isentropic flow methods can be applied throughout the nozzle.

Mode (e) A series of two-dimensional shocks are established in the discharge tank downstream of the nozzle. These shocks serve to decelerate the flow. The flow is isentropic throughout the nozzle, same as mode (d).

Mode (f) The pressure in the discharge tank equals the pressure predicted by the supersonic solution of the nozzle isentropic flow equations. The pressure ratio is known as the supersonic design pressure ratio. Flow is isentropic everywhere in the nozzle, same as mode (d) and (e), and in the discharge tank.

Mode (g) A series of two-dimensional shocks are established in the discharge tank downstream of the nozzle. These shocks serve to decelerate the flow. The flow is isentropic throughout the nozzle, same as modes (d), (e), and (f).

Review Example 9.9 of the textbook.

Adiabatic, Constant Duct Area Compressible Flow with Friction

When compressible fluids flow through insulated, constant-area ducts, they are subject to Moody-like pipe-friction which can be described by an average Darcy-Weisbach friction factor $\bar{f}$. Application of the conservation of mass, momentum, and energy principles as well as the ideal gas equation of state yields the following set of working equations.

$$\frac{\bar{f}\,L^*}{D} = \frac{1 - Ma^2}{k\,Ma^2} + \frac{k+1}{2k}\ln\frac{(k+1)\,Ma^2}{2+(k-1)\,Ma^2}$$

$$\frac{P}{P^*} = \frac{1}{Ma}\left[\frac{(k+1)}{2+(k-1)\,Ma^2}\right]^{1/2}$$

$$\frac{\rho}{\rho^*} = \frac{V^*}{V} = \frac{1}{Ma}\left[\frac{2+(k-1)\,Ma^2}{k+1}\right]^{1/2}$$

$$\frac{T}{T^*} = \frac{a}{a^{*2}} = \frac{(k+1)}{2+(k-1)\,Ma^2}$$

$$\frac{P}{P_o^*} = \frac{\rho_o}{\rho_o^*} = \frac{1}{Ma}\left[\frac{2+(k-1)\,Ma^2}{k+1}\right]^{(k+1)/[2(k-1)]}$$

where the asterisk state is the sonic state at which the flow Mach number is one. This state is the same throughout the duct and may be used to relate conditions at one location in the duct to those at another location. The length of the duct enters these calculations by

$$\frac{\bar{f}\,\Delta L}{D} = \left(\frac{\bar{f}\,L^*}{D}\right)_1 - \left(\frac{\bar{f}\,L^*}{D}\right)_2$$

Thus, given the length ΔL of the duct and the Mach number at the duct entrance or exit, the Mach number at the other end (or location) of the duct can be determined.

Compressible Flow with Friction Example:

Air enters a 0.01-m-diameter duct ($\bar{f} = 0.05$) with Ma = 0.05. The pressure and temperature at the duct inlet are 1.5 MPa and 400 K. What are the (a) Mach number, (b) pressure, and (c) temperature in the duct 50 m from the entrance?

At the duct entrance, with $\bar{f} = 0.05$, D = 0.01 m, and Ma = 0.05, we obtain

$$\left(\frac{\bar{f}\,L^*}{D}\right)_1 = \left[\frac{1 - Ma^2}{k\,Ma^2} + \frac{k+1}{2k}\ln\frac{(k+1)\,Ma^2}{2+(k-1)\,Ma^2}\right]_1$$

$$\left(\frac{\bar{f}\,L^*}{D}\right)_1 = \left[\frac{1-0.05^2}{1.4(0.05)^2} + \frac{2.4}{2.8}\ln\frac{(2.4)\,0.05^2}{2+(0.4)\,0.05^2}\right]_1 = 280$$

Then, at the duct exit we obtain

$$\left(\frac{\bar{f}\,L^*}{D}\right)_2 = \left(\frac{\bar{f}\,L^*}{D}\right)_1 - \frac{\bar{f}\,\Delta L}{D} = 280 - \frac{(0.05)\,50}{0.01} = 30$$

We can not write for the duct exit that

$$\left(\frac{\bar{f}\,L^*}{D}\right)_2 = 30 = \left[\frac{1 - Ma^2}{k\,Ma^2} + \frac{k+1}{2k}\ln\frac{(k+1)\,Ma^2}{2+(k-1)\,Ma^2}\right]_2$$

or

$$30 = \frac{1 - Ma_2^2}{1.4\,Ma_2^2} + \frac{2.4}{2.8}\ln\frac{2.4\,Ma_2^2}{2+0.4\,Ma_2^2}$$

The solution of the second of these equations gives answer (a) Ma_2 = 0.145. Writing the following expression for pressure ratios yields for (b),

$$P_2 = P_1\frac{P_2}{P_2^*}\frac{P_2^*}{P_1^*}\frac{P_1^*}{P_1}$$

$$P_2 = (1.5)\frac{1}{Ma_2}\left[\frac{(k+1)}{2+(k-1)Ma_2^2}\right]^{1/2}(1)\frac{Ma_1}{1}\left[\frac{2+(k-1)Ma_1^2}{k+1}\right]^{1/2}$$

$$P_2 = (1.5)\frac{1}{0.145}\left[\frac{2.4}{2+(0.4)0.145^2}\right]^{1/2}(1)\frac{0.05}{1}\left[\frac{2+(0.4)0.05^2}{2.4}\right]^{1/2} = 0.516$$

Application of the temperature ratios yields answer (c),

$$T_2 = T_1\frac{T_1^*}{T_1}\frac{T_2^*}{T_1^*}\frac{T_2}{T_2^*} = 400\frac{2+(k-1)Ma_1^2}{2+(k-1)Ma_2^2} = 400\frac{2+(0.4)0.05^2}{2+(0.4)0.145^2} = 399$$

This example demonstrates what happens when the flow at the inlet to the duct is subsonic, the Mach number increases as the duct gets longer. When the inlet flow is supersonic, the Mach number decreases as the duct gets longer. A plot of the specific entropy of the fluid as a function of the duct Mach number (length) is presented in Figure 4 for both subsonic and supersonic flow.

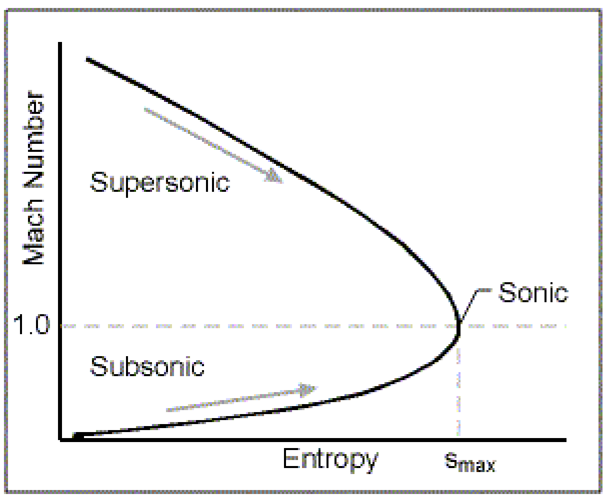


Figure 4

These results clearly illustrate that the Mach number in the duct approaches unity as the length of the duct is increased. Once the sonic condition exists at the duct exit, the flow becomes choked. This figure also demonstrates that the flow can never proceed from subsonic to supersonic (or supersonic to subsonic) flow, as this would result in a violation of the second law of thermodynamics.

Other compressible flows in constant area ducts such as isothermal flow with friction and frictionless flow with heat addition may be analyzed in a similar manner using the equations appropriate to each flow. Many of these flows also demonstrate choking behavior.

Oblique Shock Waves

Bodies moving through a compressible fluid at speeds exceeding the speed of sound create a shock system shaped like a cone. The half-angle of this *shock cone* is given by

$$\mu = \sin^{-1}\frac{1}{Ma}$$

This angle is known as the *Mach angle*. The interior of the shock cone is called the *zone of action*. Inside the zone of action, it is possible to hear any sounds produced by the moving body. Outside the Mach cone, in what is known as the *zone of silence*, sounds produced by the moving body cannot be heard.

An oblique shock wave at angle β with respect to the approaching compressible fluid whose Mach number is supersonic is shown in Figure 5. Observe that the streamlines (parallel to the velocity vector) have been turned by the deflection angle θ by passing through the oblique shock wave.

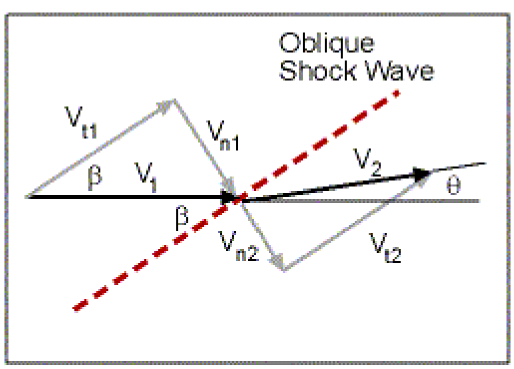


Figure 5

This flow is readily analyzed by considering the normal velocity components $V_{n1} = V_1 \sin\beta$ *and* $V_{n2} = V_2 \sin(\beta - \theta)$ and the tangential components V_{t1} *and* V_{t2}. Application of the momentum principle in the tangential direction (along which there are no pressure changes) verifies that

$$V_{t1} = V_{t2}$$

By defining the normal Mach numbers as

$$Ma_{n1} = \frac{V_{n1}}{a_1} = Ma_1 \sin\beta \quad and \quad Ma_{n2} = \frac{V_{n2}}{a_2} = Ma_2 \sin(\beta - \theta)$$

The simultaneous solution of the conservation of mass, momentum, and energy equations in the normal direction along with the ideal gas equation of state are the same as those of the normal shock wave with Ma_1 replaced with Ma_{n1} and Ma_2 replaced with Ma_{n2}. In this way, all the results developed in the normal shock wave section can be applied to two-dimensional oblique shock waves.

Oblique Shock Example:

A two-dimensional shock wave is created at the leading edge of an aircraft flying at Ma = 1.6 through air at 70 kPa, 300 K. If this oblique shock forms a 55^o angle with respect to the approaching air, what is (a) the Mach number of the flow after the oblique shock (this is not the normal Mach number) and (b) the streamline deflection angle θ?

The velocity of the fluid upstream of the oblique shock wave is

$$V_1 = Ma_1\, a_1 = Ma_1 \sqrt{k\,R\,T} = 1.6\sqrt{(1.4)(287)(300)} = 556\, m/s$$

whose components are

$$V_{n1} = V_1 \sin\beta = 556 \sin 55 = 455\, m/s$$

$$V_{t1} = V_{t2} = V_1 \cos\beta = 556 \cos 55 = 319\, m/s$$

The upstream normal Mach number is then

$$Ma_{n1} = Ma_1 \sin\beta = 1.6 \sin 55^\circ = 1.311$$

and the downstream normal Mach number is

$$Ma_{n2} = \left[\frac{(k-1)Ma_{n1}^2 + 2}{2\,k\,Ma_{n1}^2 - (k-1)} \right]^{1/2} = \left[\frac{(0.4)(1.311)^2 + 2}{2\,(1.4)\,(1.311)^2 - 0.4} \right]^{1/2} = 0.780$$

and the downstream temperature is

$$T_2 = T_1 \left\{ \left[(k-1)Ma_{n1}^2 + 2 \right] \frac{2\,k\,Ma_{n1}^2 - (k-1)}{(k+1)^2\,Ma_{n1}^2} \right\}$$

$$T_2 = 300 \left\{ \left[(0.4)1.311^2 + 2 \right] \frac{2(1.4)1.311^2 - 0.4}{(2.4)^2 1.311^2} \right\} = 359\,K$$

Now, the downstream normal velocity is

$$V_{n2} = Ma_{n2}\,a_2 = Ma_{n2}\sqrt{k\,R\,T_2} = 0.780\sqrt{(1.4)(287)(359)} = 296\,m/s$$

and the downstream fluid velocity is

$$V_2 = \sqrt{V_{n2}^2 + V_{t2}^2} = \sqrt{296^2 + 319^2} = 435\,m/s$$

and the downstream Mach number is

$$Ma_2 = \frac{V_2}{a_2} = \frac{435}{\sqrt{(1.4)(287)(359)}} = 1.15$$

According to the geometry of Figure 5,

$$\theta = \beta - \tan^{-1}\frac{V_{n2}}{V_{t2}} = 55 - \tan^{-1}\frac{296}{319} = 12.1$$

Other downstream properties can be calculated in the same way as the downstream temperature by using the normal Mach numbers in the normal shock relations.

Prandl-Meyer Expansion Waves

The preceding section demonstrated that when the streamlines of a supersonic flow are turned into the direction of the flow an oblique compression shock wave is formed. Similarly, when the streamlines of a supersonic flow are turned away from the direction of flow as illustrated in Figure 6, an expansion wave system is established. Unlike shock waves (either normal or oblique) which form a strong discontinuity to change the flow conditions, expansion waves are a system of infinitesimally weak waves distributed in such a manner as required to make the required changes in the flow conditions.

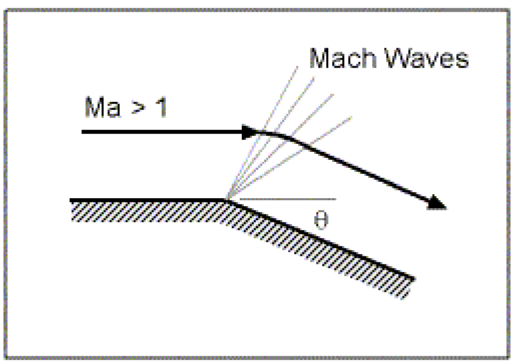


Figure 6

The Mach waves that accomplish the turning of supersonic flows form an angle with respect to the local flow velocity equal to the Mach angle $\mu = \sin^{-1}(1/Ma)$ and are isentropic. Application of the governing conservation equations and equation of state to an infinitesimal turning of the supersonic flow yields

$$-\theta(Ma) = \omega(Ma) = \left(\frac{k+1}{k-1}\right)^{1/2} \tan^{-1}\left[\frac{(k-1)(Ma^2-1)}{k+1}\right]^{1/2} - \tan^{-1}(Ma^2-1)^{1/2}$$

where $\omega(Ma)$ is the *Prandl-Meyer expansion function*. The overall change in the flow angle as a supersonic flow undergoes a Prandl-Meyer expansion is then

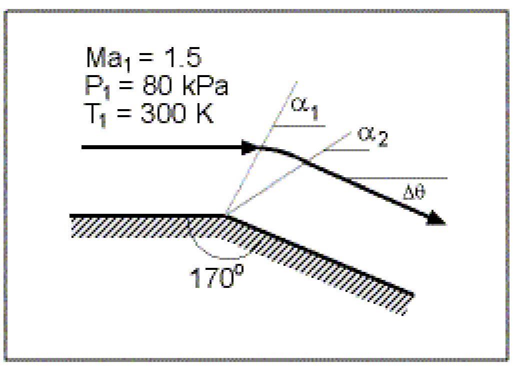
$$\Delta\theta = \omega(Ma_1) - \omega(Ma_2)$$

where 1 refers to the upstream condition and 2 refers to the downstream condition.

The flow through a Prandl-Meyer expansion fan is isentropic flow. The isentropic flow equations can then be used to relate the fluid properties upstream and downstream of the expansion fan.

Example:

Air at 80 kPa, 300 K with a Mach number of 1.5 turns the sharp corner of an airfoil as shown here. Determine the angles of the initial and final Mach waves, and the downstream pressure and temperature of this flow.



The initial angle between the flow velocity vector and the Prandtl-Meyer fan is the Mach angle.

$$\alpha_1 = \sin^{-1}\frac{1}{Ma_1} = \sin^{-1}\frac{1}{1.5} = 41.8^0$$

The upstream Prandtl-Meyer function is

$$\omega(Ma_1) = \left(\frac{k+1}{k-1}\right)^{1/2} \tan^{-1}\left[\frac{(k-1)(Ma_1^2 - 1)}{k+1}\right]^{1/2} - \tan^{-1}(Ma_1^2 - 1)^{1/2}$$

$$\omega(Ma_1) = \left(\frac{2.4}{0.4}\right)^{1/2} \tan^{-1}\left[\frac{(0.4)(1.5^2 - 1)}{2.4}\right]^{1/2} - \tan^{-1}(1.5^2 - 1)^{1/2}$$

$$\omega(Ma_1) = 11.90^0$$

The downstream Prandtl-Meyer function is then

$$\omega(Ma_2) = \omega(Ma_1) - \Delta\theta = 11.9^0 - 10^0 = 1.90^0$$

Solving the Prandtl-Meyer function gives the downstream Mach number $Ma_2 = 1.13$. The downstream Mach angle is then $\mu_2 = 62.2^0$. According to the geometry of the above figure,

$$\alpha_2 = \mu_2 - \Delta\theta = 62.2^0 - 10^0 = 52.2^0$$

Since T_0 and P_0 remain constant, the isentropic flow relations yield

$$T_2 = T_1 \frac{T_{01}}{T_1}\frac{T_2}{T_{02}} = T_1 \frac{1 + \frac{k-1}{2}Ma_1^2}{1 + \frac{k-1}{2}Ma_2^2} = 300\frac{1 + 0.2(1.5)^2}{1 + 0.2(1.13)} = 346\,K$$

$$P_2 = P_1 \frac{P_{01}}{P_1}\frac{P_2}{P_{02}} = P_1 \left[\frac{1 + \frac{k-1}{2}Ma_1^2}{1 + \frac{k-1}{2}Ma_2^2}\right]^{k/(k-1)} = 80\left[\frac{1 + 0.2(1.5)^2}{1 + 0.2(1.13)}\right]^{3.5} = 132\,MPa$$

Students are encouraged to examine the flow visualization photographs in Ch 9.