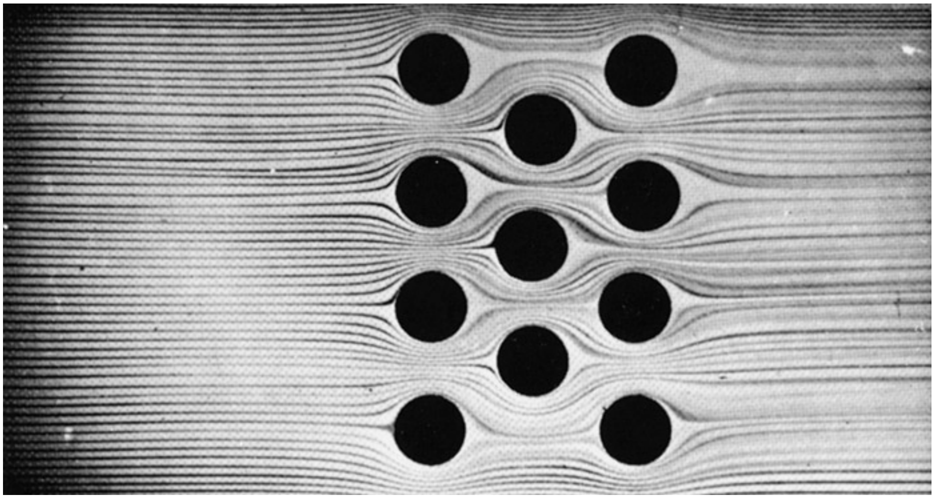


Inviscid potential flow past an array of cylinders. The mathematics of potential theory, presented in this chapter, is both beautiful and manageable, but results may be unrealistic when there are solid boundaries. See Figure 8.13b for the real (viscous) flow pattern. *(Courtesy of Tecquipment Ltd., Nottingham, England)*

# Chapter 4
# Differential Relations for a Fluid Particle

**Motivation.** In analyzing fluid motion, we might take one of two paths: (1) seeking an estimate of gross effects (mass flow, induced force, energy change) over a *finite* region or control volume or (2) seeking the point-by-point details of a flow pattern by analyzing an *infinitesimal* region of the flow. The former or gross-average viewpoint was the subject of Chap. 3.

This chapter treats the second in our trio of techniques for analyzing fluid motion, small-scale, or *differential*, analysis. That is, we apply our four basic conservation laws to an infinitesimally small control volume or, alternately, to an infinitesimal fluid system. In either case the results yield the basic *differential equations* of fluid motion. Appropriate *boundary conditions* are also developed.

In their most basic form, these differential equations of motion are quite difficult to solve, and very little is known about their general mathematical properties. However, certain things can be done which have great educational value. First, e.g., as shown in Chap. 5, the equations (even if unsolved) reveal the basic dimensionless parameters which govern fluid motion. Second, as shown in Chap. 6, a great number of useful solutions can be found if one makes two simplifying assumptions: (1) steady flow and (2) incompressible flow. A third and rather drastic simplification, frictionless flow, makes our old friend the Bernoulli equation valid and yields a wide variety of idealized, or *perfect-fluid*, possible solutions. These idealized flows are treated in Chap. 8, and we must be careful to ascertain whether such solutions are in fact realistic when compared with actual fluid motion. Finally, even the difficult general differential equations now yield to the approximating technique known as *numerical analysis*, whereby the derivatives are simulated by algebraic relations between a finite number of grid points in the flow field which are then solved on a digital computer. Reference 1 is an example of a textbook devoted entirely to numerical analysis of fluid motion.

## 4.1 The Acceleration Field of a Fluid

In Sec. 1.5 we established the cartesian vector form of a velocity field which varies in space and time:

$$\mathbf{V}(\mathbf{r}, t) = \mathbf{i}u(x, y, z, t) + \mathbf{j}\upsilon(x, y, z, t) + \mathbf{k}w(x, y, z, t) \tag{1.4}$$

This is the most important variable in fluid mechanics: Knowledge of the velocity vector field is nearly equivalent to *solving* a fluid-flow problem. Our coordinates are fixed in space, and we observe the fluid as it passes by—as if we had scribed a set of coordinate lines on a glass window in a wind tunnel. This is the *eulerian* frame of reference, as opposed to the lagrangian frame, which follows the moving position of individual particles.

To write Newton's second law for an infinitesimal fluid system, we need to calculate the acceleration vector field **a** of the flow. Thus we compute the total time derivative of the velocity vector:

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \mathbf{i}\frac{du}{dt} + \mathbf{j}\frac{dv}{dt} + \mathbf{k}\frac{dw}{dt}$$

Since each scalar component $(u, v, w)$ is a function of the four variables $(x, y, z, t)$, we use the chain rule to obtain each scalar time derivative. For example,

$$\frac{du(x, y, z, t)}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

But, by definition, $dx/dt$ is the local velocity component $u$, and $dy/dt = v$, and $dz/dt = w$. The total derivative of $u$ may thus be written in the compact form

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} + (\mathbf{V}\cdot\boldsymbol{\nabla})u \tag{4.1}$$

Exactly similar expressions, with $u$ replaced by $v$ or $w$, hold for $dv/dt$ or $dw/dt$. Summing these into a vector, we obtain the total acceleration:

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \underbrace{\frac{\partial \mathbf{V}}{\partial t}}_{\text{Local}} + \underbrace{\left(u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}\right)}_{\text{Convective}} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{V} \tag{4.2}$$

The term $\partial\mathbf{V}/\partial t$ is called the *local acceleration*, which vanishes if the flow is steady, i.e., independent of time. The three terms in parentheses are called the *convective acceleration*, which arises when the particle moves through regions of spatially varying velocity, as in a nozzle or diffuser. Flows which are nominally "steady" may have large accelerations due to the convective terms.

Note our use of the compact dot product involving **V** and the gradient operator $\boldsymbol{\nabla}$:

$$u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \mathbf{V}\cdot\boldsymbol{\nabla} \qquad \text{where} \qquad \nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

The total time derivative—sometimes called the *substantial* or *material* derivative—concept may be applied to any variable, e.g., the pressure:

$$\frac{dp}{dt} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} = \frac{\partial p}{\partial t} + (\mathbf{V}\cdot\boldsymbol{\nabla})p \tag{4.3}$$

Wherever convective effects occur in the basic laws involving mass, momentum, or energy, the basic differential equations become nonlinear and are usually more complicated than flows which do not involve convective changes.

We emphasize that this total time derivative follows a particle of fixed identity, making it convenient for expressing laws of particle mechanics in the eulerian fluid-field description. The operator $d/dt$ is sometimes assigned a special symbol such as $D/Dt$ as a further reminder that it contains four terms and follows a fixed particle.

## EXAMPLE 4.1

Given the eulerian velocity-vector field

$$\mathbf{V} = 3t\mathbf{i} + xz\mathbf{j} + ty^2\mathbf{k}$$

find the acceleration of a particle.

### Solution

First note the specific given components

$$u = 3t \qquad v = xz \qquad w = ty^2$$

Then evaluate the vector derivatives required for Eq. (4.2)

$$\frac{\partial \mathbf{V}}{\partial t} = \mathbf{i}\,\frac{\partial u}{\partial t} + \mathbf{j}\,\frac{\partial v}{\partial t} + \mathbf{k}\,\frac{\partial w}{\partial t} = 3\mathbf{i} + y^2\mathbf{k}$$

$$\frac{\partial \mathbf{V}}{\partial x} = z\mathbf{j} \qquad \frac{\partial \mathbf{V}}{\partial y} = 2ty\mathbf{k} \qquad \frac{\partial \mathbf{V}}{\partial z} = x\mathbf{j}$$

This could have been worse: There are only five terms in all, whereas there could have been as many as twelve. Substitute directly into Eq. (4.2):

$$\frac{d\mathbf{V}}{dt} = (3\mathbf{i} + y^2\mathbf{k}) + (3t)(z\mathbf{j}) + (xz)(2ty\mathbf{k}) + (ty^2)(x\mathbf{j})$$

Collect terms for the final result

$$\frac{d\mathbf{V}}{dt} = 3\mathbf{i} + (3tz + txy^2)\mathbf{j} + (2xyzt + y^2)\mathbf{k} \qquad \textit{Ans.}$$

Assuming that **V** is valid everywhere as given, this acceleration applies to all positions and times within the flow field.

## 4.2 The Differential Equation of Mass Conservation

All the basic differential equations can be derived by considering either an elemental control volume or an elemental system. Here we choose an infinitesimal fixed control volume ($dx$, $dy$, $dz$), as in Fig. 4.1, and use our basic control-volume relations from Chap. 3. The flow through each side of the element is approximately one-dimensional, and so the appropriate mass-conservation relation to use here is

$$\int_{\text{CV}} \frac{\partial \rho}{\partial t}\, d\mathcal{V} + \sum_i (\rho_i A_i V_i)_{\text{out}} - \sum_i (\rho_i A_i V_i)_{\text{in}} = 0 \qquad (3.22)$$

The element is so small that the volume integral simply reduces to a differential term

$$\int_{\text{CV}} \frac{\partial \rho}{\partial t}\, d\mathcal{V} \approx \frac{\partial \rho}{\partial t}\, dx\, dy\, dz$$

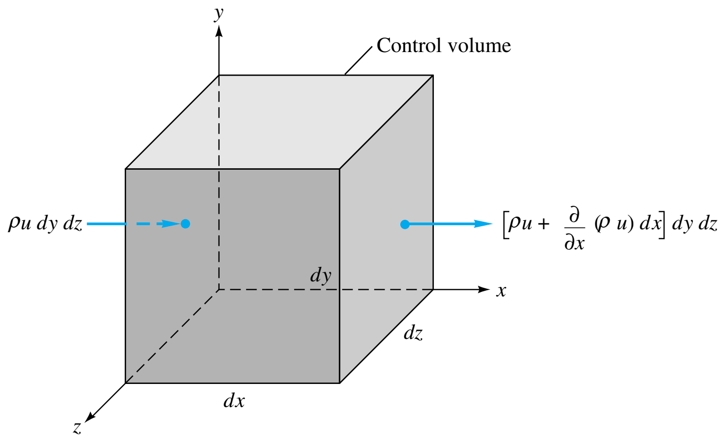



**Fig. 4.1** Elemental cartesian fixed control volume showing the inlet and outlet mass flows on the $x$ faces.

The mass-flow terms occur on all six faces, three inlets and three outlets. We make use of the field or continuum concept from Chap. 1, where all fluid properties are considered to be uniformly varying functions of time and position, such as $\rho = \rho(x, y, z, t)$. Thus, if $T$ is the temperature on the left face of the element in Fig. 4.1, the right face will have a slightly different temperature $T + (\partial T/\partial x)\ dx$. For mass conservation, if $\rho u$ is known on the left face, the value of this product on the right face is $\rho u + (\partial \rho u/\partial x)\ dx$.

Figure 4.1 shows only the mass flows on the $x$ or left and right faces. The flows on the $y$ (bottom and top) and the $z$ (back and front) faces have been omitted to avoid cluttering up the drawing. We can list all these six flows as follows:

| Face | Inlet mass flow | Outlet mass flow |
|---|---|---|
| $x$ | $\rho u\ dy\ dz$ | $\left[\rho u + \frac{\partial}{\partial x}(\rho u)\ dx\right] dy\ dz$ |
| $y$ | $\rho v\ dx\ dz$ | $\left[\rho v + \frac{\partial}{\partial y}(\rho v)\ dy\right] dx\ dz$ |
| $z$ | $\rho w\ dx\ dy$ | $\left[\rho w + \frac{\partial}{\partial z}(\rho w)\ dz\right] dx\ dy$ |

Introduce these terms into Eq. (3.22) above and we have

$$\frac{\partial \rho}{\partial t}\ dx\ dy\ dz + \frac{\partial}{\partial x}(\rho u)\ dx\ dy\ dz + \frac{\partial}{\partial y}(\rho v)\ dx\ dy\ dz + \frac{\partial}{\partial z}(\rho w)\ dx\ dy\ dz = 0$$

The element volume cancels out of all terms, leaving a partial differential equation involving the derivatives of density and velocity

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \tag{4.4}$$

This is the desired result: conservation of mass for an infinitesimal control volume. It is often called the *equation of continuity* because it requires no assumptions except that the density and velocity are continuum functions. That is, the flow may be either steady or unsteady, viscous or frictionless, compressible or incompressible.[1] However, the equation does not allow for any source or sink singularities within the element.

The vector-gradient operator

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

enables us to rewrite the equation of continuity in a compact form, not that it helps much in finding a solution. The last three terms of Eq. (4.4) are equivalent to the divergence of the vector $\rho\mathbf{V}$

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) \equiv \nabla \cdot (\rho\mathbf{V}) \tag{4.5}$$

so that the compact form of the continuity relation is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{V}) = 0 \tag{4.6}$$

In this vector form the equation is still quite general and can readily be converted to other than cartesian coordinate systems.

## Cylindrical Polar Coordinates

The most common alternative to the cartesian system is the *cylindrical polar* coordinate system, sketched in Fig. 4.2. An arbitrary point $P$ is defined by a distance $z$ along the axis, a radial distance $r$ from the axis, and a rotation angle $\theta$ about the axis. The three independent velocity components are an axial velocity $v_z$, a radial velocity $v_r$, and a circumferential velocity $v_\theta$, which is positive counterclockwise, i.e., in the direction

[1] One case where Eq. (4.4) might need special care is *two-phase flow*, where the density is discontinuous between the phases. For further details on this case, see, e.g., Ref. 2.

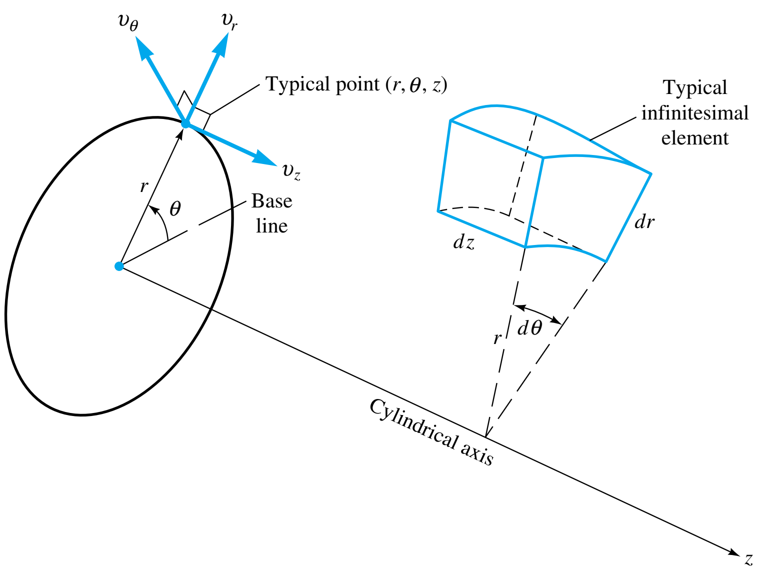



**Fig. 4.2** Definition sketch for the cylindrical coordinate system.

of increasing $\theta$. In general, all components, as well as pressure and density and other fluid properties, are continuous functions of $r$, $\theta$, $z$, and $t$.

The divergence of any vector function $\mathbf{A}(r, \theta, z, t)$ is found by making the transformation of coordinates

$$r = (x^2 + y^2)^{1/2} \qquad \theta = \tan^{-1}\frac{y}{x} \qquad z = z \tag{4.7}$$

and the result is given here without proof[2]

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(A_\theta) + \frac{\partial}{\partial z}(A_z) \tag{4.8}$$

The general continuity equation (4.6) in cylindrical polar coordinates is thus

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho \upsilon_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho \upsilon_\theta) + \frac{\partial}{\partial z}(\rho \upsilon_z) = 0 \tag{4.9}$$

There are other orthogonal curvilinear coordinate systems, notably *spherical polar* coordinates, which occasionally merit use in a fluid-mechanics problem. We shall not treat these systems here except in Prob. 4.12.

There are also other ways to derive the basic continuity equation (4.6) which are interesting and instructive. Ask your instructor about these alternate approaches.

## Steady Compressible Flow

If the flow is steady, $\partial/\partial t \equiv 0$ and all properties are functions of position only. Equation (4.6) reduces to

Cartesian:
$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho \upsilon) + \frac{\partial}{\partial z}(\rho w) = 0$$

Cylindrical:
$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho \upsilon_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho \upsilon_\theta) + \frac{\partial}{\partial z}(\rho \upsilon_z) = 0 \tag{4.10}$$

Since density and velocity are both variables, these are still nonlinear and rather formidable, but a number of special-case solutions have been found.

## Incompressible Flow

A special case which affords great simplification is incompressible flow, where the density changes are negligible. Then $\partial \rho/\partial t \approx 0$ regardless of whether the flow is steady or unsteady, and the density can be slipped out of the divergence in Eq. (4.6) and divided out. The result is

$$\nabla \cdot \mathbf{V} = 0 \tag{4.11}$$

valid for steady or unsteady incompressible flow. The two coordinate forms are

Cartesian:
$$\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{4.12$a$}$$

Cylindrical:
$$\frac{1}{r}\frac{\partial}{\partial r}(r\upsilon_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\upsilon_\theta) + \frac{\partial}{\partial z}(\upsilon_z) = 0 \tag{4.12$b$}$$

[2] See, e.g., Ref. 3, p. 783.

These are *linear* differential equations, and a wide variety of solutions are known, as discussed in Chaps. 6 to 8. Since no author or instructor can resist a wide variety of solutions, it follows that a great deal of time is spent studying incompressible flows. Fortunately, this is exactly what should be done, because most practical engineering flows are approximately incompressible, the chief exception being the high-speed gas flows treated in Chap. 9.

When is a given flow approximately incompressible? We can derive a nice criterion by playing a little fast and loose with density approximations. In essence, we wish to slip the density out of the divergence in Eq. (4.6) and approximate a typical term as, e.g.,

$$\frac{\partial}{\partial x}(\rho u) \approx \rho \frac{\partial u}{\partial x} \tag{4.13}$$

This is equivalent to the strong inequality

$$\left| u \frac{\partial \rho}{\partial x} \right| \ll \left| \rho \frac{\partial u}{\partial x} \right|$$

or

$$\left| \frac{\delta \rho}{\rho} \right| \ll \left| \frac{\delta V}{V} \right| \tag{4.14}$$

As we shall see in Chap. 9, the pressure change is approximately proportional to the density change and the square of the speed of sound $a$ of the fluid

$$\delta p \approx a^2 \, \delta \rho \tag{4.15}$$

Meanwhile, if elevation changes are negligible, the pressure is related to the velocity change by Bernoulli's equation (3.75)

$$\delta p \approx -\rho V \, \delta V \tag{4.16}$$

Combining Eqs. (4.14) to (4.16), we obtain an explicit criterion for incompressible flow:

$$\frac{V^2}{a^2} = \mathrm{Ma}^2 \ll 1 \tag{4.17}$$

where $\mathrm{Ma} = V/a$ is the dimensionless *Mach number* of the flow. How small is small? The commonly accepted limit is

$$\mathrm{Ma} \le 0.3 \tag{4.18}$$

For air at standard conditions, a flow can thus be considered incompressible if the velocity is less than about 100 m/s (330 ft/s). This encompasses a wide variety of airflows: automobile and train motions, light aircraft, landing and takeoff of high-speed aircraft, most pipe flows, and turbomachinery at moderate rotational speeds. Further, it is clear that almost all liquid flows are incompressible, since flow velocities are small and the speed of sound is very large.[3]

[3]An exception occurs in geophysical flows, where a density change is imposed thermally or mechanically rather than by the flow conditions themselves. An example is fresh water layered upon saltwater or warm air layered upon cold air in the atmosphere. We say that the fluid is *stratified*, and we must account for vertical density changes in Eq. (4.6) even if the velocities are small.

Before attempting to analyze the continuity equation, we shall proceed with the derivation of the momentum and energy equations, so that we can analyze them as a group. A very clever device called the *stream function* can often make short work of the continuity equation, but we shall save it until Sec. 4.7.

One further remark is appropriate: The continuity equation is always important and must always be satisfied for a rational analysis of a flow pattern. Any newly discovered momentum or energy "solution" will ultimately crash in flames when subjected to critical analysis if it does not also satisfy the continuity equation.

## EXAMPLE 4.2

Under what conditions does the velocity field

$$\mathbf{V} = (a_1x + b_1y + c_1z)\mathbf{i} + (a_2x + b_2y + c_2z)\mathbf{j} + (a_3x + b_3y + c_3z)\mathbf{k}$$

where $a_1$, $b_1$, etc. = const, represent an incompressible flow which conserves mass?

### Solution

Recalling that $\mathbf{V} = u\mathbf{i} + \upsilon\mathbf{j} + w\mathbf{k}$, we see that $u = (a_1x + b_1y + c_1z)$, etc. Substituting into Eq. (4.12*a*) for incompressible continuity, we obtain

$$\frac{\partial}{\partial x}(a_1x + b_1y + c_1z) + \frac{\partial}{\partial y}(a_2x + b_2y + c_2z) + \frac{\partial}{\partial z}(a_3x + b_3y + c_3z) = 0$$

or $$a_1 + b_2 + c_3 = 0 \qquad \textit{Ans.}$$

At least two of constants $a_1$, $b_2$, and $c_3$ must have opposite signs. Continuity imposes no restrictions whatever on constants $b_1$, $c_1$, $a_2$, $c_2$, $a_3$, and $b_3$, which do not contribute to a mass increase or decrease of a differential element.

## EXAMPLE 4.3

An incompressible velocity field is given by

$$u = a(x^2 - y^2) \qquad \upsilon \text{ unknown} \qquad w = b$$

where $a$ and $b$ are constants. What must the form of the velocity component $\upsilon$ be?

### Solution

Again Eq. (4.12*a*) applies

$$\frac{\partial}{\partial x}(ax^2 - ay^2) + \frac{\partial \upsilon}{\partial y} + \frac{\partial b}{\partial z} = 0$$

or $$\frac{\partial \upsilon}{\partial y} = -2ax \qquad (1)$$

This is easily integrated partially with respect to $y$

$$\upsilon(x, y, z, t) = -2axy + f(x, z, t) \qquad \textit{Ans.}$$

This is the only possible form for $v$ which satisfies the incompressible continuity equation. The function of integration $f$ is entirely arbitrary since it vanishes when $v$ is differentiated with respect to $y$.[†]

## EXAMPLE 4.4

A centrifugal impeller of 40-cm diameter is used to pump hydrogen at 15°C and 1-atm pressure. What is the maximum allowable impeller rotational speed to avoid compressibility effects at the blade tips?

### Solution

The speed of sound of hydrogen for these conditions is $a = 1300$ m/s. Assume that the gas velocity leaving the impeller is approximately equal to the impeller-tip speed

$$V = \Omega r = \tfrac{1}{2}\Omega D$$

Our rule of thumb, Eq. (4.18), neglects compressibility if

$$V = \tfrac{1}{2}\Omega D \le 0.3a = 390 \text{ m/s}$$

or
$$\tfrac{1}{2}\Omega(0.4 \text{ m}) \le 390 \text{ m/s} \qquad \Omega \le 1950 \text{ rad/s}$$

Thus we estimate the allowable speed to be quite large

$$\Omega \le 310 \text{ r/s } (18{,}600 \text{ r/min}) \qquad \textit{Ans.}$$

An impeller moving at this speed in air would create shock waves at the tips but not in a light gas like hydrogen.

## 4.3 The Differential Equation of Linear Momentum

Having done it once in Sec. 4.2 for mass conservation, we can move along a little faster this time. We use the same elemental control volume as in Fig. 4.1, for which the appropriate form of the linear-momentum relation is

$$\sum \mathbf{F} = \frac{\partial}{\partial t}\left(\int_{CV} \mathbf{V}\rho \, d\mathcal{V}\right) + \sum (\dot{m}_i \mathbf{V}_i)_{\text{out}} - \sum (\dot{m}_i \mathbf{V}_i)_{\text{in}} \tag{3.40}$$

Again the element is so small that the volume integral simply reduces to a derivative term

$$\frac{\partial}{\partial t}\left(\int_{CV} \mathbf{V}\rho \, d\mathcal{V}\right) \approx \frac{\partial}{\partial t}(\rho\mathbf{V}) \, dx \, dy \, dz \tag{4.19}$$

The momentum fluxes occur on all six faces, three inlets and three outlets. Referring again to Fig. 4.1, we can form a table of momentum fluxes by exact analogy with the discussion which led up to the equation for net mass flux:

[†]This is a very realistic flow which simulates the turning of an inviscid fluid through a 60° angle; see Examples 4.7 and 4.9.

| Faces | Inlet momentum flux | Outlet momentum flux |
|---|---|---|
| $x$ | $\rho u\mathbf{V}\, dy\, dz$ | $\left[\rho u\mathbf{V} + \frac{\partial}{\partial x}(\rho u\mathbf{V})\, dx\right] dy\, dz$ |
| $y$ | $\rho v\mathbf{V}\, dx\, dz$ | $\left[\rho v\mathbf{V} + \frac{\partial}{\partial y}(\rho v\mathbf{V})\, dy\right] dx\, dz$ |
| $z$ | $\rho w\mathbf{V}\, dx\, dy$ | $\left[\rho w\mathbf{V} + \frac{\partial}{\partial z}(\rho w\mathbf{V})\, dz\right] dx\, dy$ |

Introduce these terms and Eq. (4.19) into Eq. (3.40), and get the intermediate result

$$\sum \mathbf{F} = dx\, dy\, dz\left[\frac{\partial}{\partial t}(\rho\mathbf{V}) + \frac{\partial}{\partial x}(\rho u\mathbf{V}) + \frac{\partial}{\partial y}(\rho v\mathbf{V}) + \frac{\partial}{\partial z}(\rho w\mathbf{V})\right] \tag{4.20}$$

Note that this is a vector relation. A simplification occurs if we split up the term in brackets as follows:

$$\frac{\partial}{\partial t}(\rho\mathbf{V}) + \frac{\partial}{\partial x}(\rho u\mathbf{V}) + \frac{\partial}{\partial y}(\rho v\mathbf{V}) + \frac{\partial}{\partial z}(\rho w\mathbf{V})$$
$$= \mathbf{V}\left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{V})\right] + \rho\left(\frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}\right) \tag{4.21}$$

The term in brackets on the right-hand side is seen to be the equation of continuity, Eq. (4.6), which vanishes identically. The long term in parentheses on the right-hand side is seen from Eq. (4.2) to be the total acceleration of a particle which instantaneously occupies the control volume

$$\frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} = \frac{d\mathbf{V}}{dt} \tag{4.2}$$

Thus we have now reduced Eq. (4.20) to

$$\sum \mathbf{F} = \rho\frac{d\mathbf{V}}{dt}\, dx\, dy\, dz \tag{4.22}$$

It might be good for you to stop and rest now and think about what we have just done. What is the relation between Eqs. (4.22) and (3.40) for an infinitesimal control volume? Could we have *begun* the analysis at Eq. (4.22)?

Equation (4.22) points out that the net force on the control volume must be of differential size and proportional to the element volume. These forces are of two types, *body* forces and *surface* forces. Body forces are due to external fields (gravity, magnetism, electric potential) which act upon the entire mass within the element. The only body force we shall consider in this book is gravity. The gravity force on the differential mass $\rho\, dx\, dy\, dz$ within the control volume is

$$d\mathbf{F}_{\text{grav}} = \rho\mathbf{g}\, dx\, dy\, dz \tag{4.23}$$

where $\mathbf{g}$ may in general have an arbitrary orientation with respect to the coordinate system. In many applications, such as Bernoulli's equation, we take $z$ "up," and $\mathbf{g} = -g\mathbf{k}$.

The surface forces are due to the stresses on the sides of the control surface. These stresses, as discussed in Chap. 2, are the sum of hydrostatic pressure plus viscous stresses $\tau_{ij}$ which arise from motion with velocity gradients

$$\sigma_{ij} = \begin{vmatrix} -p + \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & -p + \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & -p + \tau_{zz} \end{vmatrix} \tag{4.24}$$

The subscript notation for stresses is given in Fig. 4.3.

It is not these stresses but their *gradients*, or differences, which cause a net force on the differential control surface. This is seen by referring to Fig. 4.4, which shows

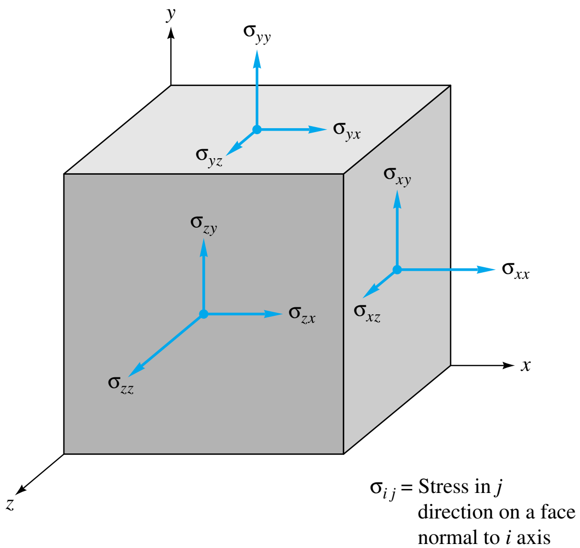



**Fig. 4.3** Notation for stresses.

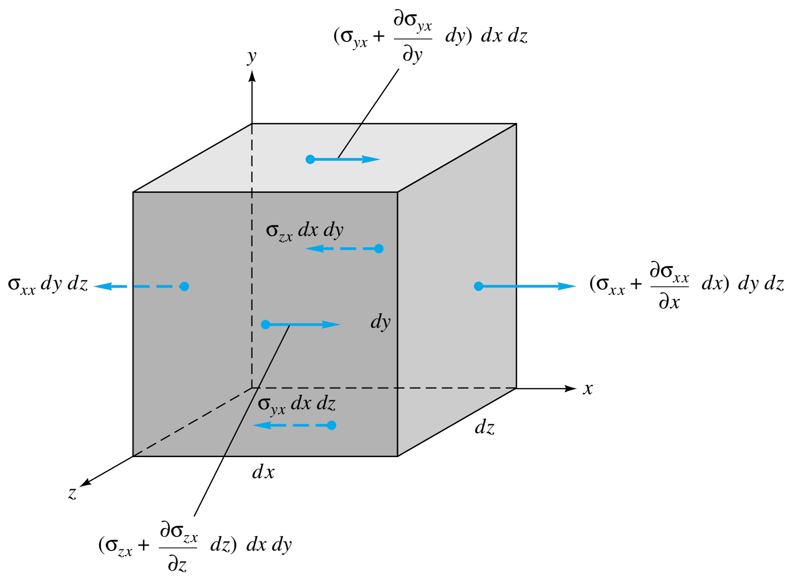



**Fig. 4.4** Elemental cartesian fixed control volume showing the surface forces in the $x$ direction only.

only the $x$-directed stresses to avoid cluttering up the drawing. For example, the leftward force $\sigma_{xx}\, dy\, dz$ on the left face is balanced by the rightward force $\sigma_{xx}\, dy\, dz$ on the right face, leaving only the net rightward force $(\partial \sigma_{xx}/\partial x)\, dx\, dy\, dz$ on the right face. The same thing happens on the other four faces, so that the net surface force in the $x$ direction is given by

$$dF_{x,\text{surf}} = \left[\frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\sigma_{yx}) + \frac{\partial}{\partial z}(\sigma_{zx})\right] dx\, dy\, dz \tag{4.25}$$

We see that this force is proportional to the element volume. Notice that the stress terms are taken from the *top row* of the array in Eq. (4.24). Splitting this row into pressure plus viscous stresses, we can rewrite Eq. (4.25) as

$$\frac{dF_x}{d\mathcal{V}} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}(\tau_{xx}) + \frac{\partial}{\partial y}(\tau_{yx}) + \frac{\partial}{\partial z}(\tau_{zx}) \tag{4.26}$$

In exactly similar manner, we can derive the $y$ and $z$ forces per unit volume on the control surface

$$\frac{dF_y}{d\mathcal{V}} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}(\tau_{xy}) + \frac{\partial}{\partial y}(\tau_{yy}) + \frac{\partial}{\partial z}(\tau_{zy})$$

$$\frac{dF_z}{d\mathcal{V}} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}(\tau_{xz}) + \frac{\partial}{\partial y}(\tau_{yz}) + \frac{\partial}{\partial z}(\tau_{zz}) \tag{4.27}$$

Now we multiply Eqs. (4.26) and (4.27) by **i**, **j**, and **k**, respectively, and add to obtain an expression for the net vector surface force

$$\left(\frac{d\mathbf{F}}{d\mathcal{V}}\right)_{\text{surf}} = -\boldsymbol{\nabla} p + \left(\frac{d\mathbf{F}}{d\mathcal{V}}\right)_{\text{viscous}} \tag{4.28}$$

where the viscous force has a total of nine terms:

$$\begin{aligned}\left(\frac{d\mathbf{F}}{d\mathcal{V}}\right)_{\text{viscous}} &= \mathbf{i}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\\ &\quad + \mathbf{j}\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right)\\ &\quad + \mathbf{k}\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right)\end{aligned} \tag{4.29}$$

Since each term in parentheses in (4.29) represents the divergence of a stress-component vector acting on the $x$, $y$, and $z$ faces, respectively, Eq. (4.29) is sometimes expressed in divergence form

$$\left(\frac{d\mathbf{F}}{d\mathcal{V}}\right)_{\text{viscous}} = \boldsymbol{\nabla} \cdot \boldsymbol{\tau}_{ij} \tag{4.30}$$

where

$$\tau_{ij} = \begin{bmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix} \tag{4.31}$$

is the viscous-stress tensor acting on the element. The surface force is thus the sum of the *pressure-gradient* vector and the divergence of the viscous-stress tensor. Substituting into Eq. (4.22) and utilizing Eq. (4.23), we have the basic differential momentum equation for an infinitesimal element

$$\rho \mathbf{g} - \nabla p + \nabla \cdot \boldsymbol{\tau}_{ij} = \rho \frac{d\mathbf{V}}{dt} \tag{4.32}$$

where

$$\frac{d\mathbf{V}}{dt} = \frac{\partial \mathbf{V}}{\partial t} + u \frac{\partial \mathbf{V}}{\partial x} + \upsilon \frac{\partial \mathbf{V}}{\partial y} + w \frac{\partial \mathbf{V}}{\partial z} \tag{4.33}$$

We can also express Eq. (4.32) in words:

Gravity force per unit volume + pressure force per unit volume

+ viscous force per unit volume = density × acceleration (4.34)

Equation (4.32) is so brief and compact that its inherent complexity is almost invisible. It is a *vector* equation, each of whose component equations contains nine terms. Let us therefore write out the component equations in full to illustrate the mathematical difficulties inherent in the momentum equation:

$$\begin{aligned}
\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} &= \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) \\
\rho g_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} &= \rho\left(\frac{\partial \upsilon}{\partial t} + u\frac{\partial \upsilon}{\partial x} + \upsilon\frac{\partial \upsilon}{\partial y} + w\frac{\partial \upsilon}{\partial z}\right) \qquad (4.35) \\
\rho g_z - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} &= \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + \upsilon\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)
\end{aligned}$$

This is the differential momentum equation in its full glory, and it is valid for any fluid in any general motion, particular fluids being characterized by particular viscous-stress terms. Note that the last three "convective" terms on the right-hand side of each component equation in (4.35) are nonlinear, which complicates the general mathematical analysis.

## Inviscid Flow: Euler's Equation

Equation (4.35) is not ready to use until we write the viscous stresses in terms of velocity components. The simplest assumption is frictionless flow $\tau_{ij} = 0$, for which Eq. (4.35) reduces to

$$\rho \mathbf{g} - \nabla p = \rho \frac{d\mathbf{V}}{dt} \tag{4.36}$$

This is Euler's equation for inviscid flow. We show in Sec. 4.9 that Euler's equation can be integrated along a streamline to yield the frictionless Bernoulli equation, (3.75) or (3.77). The complete analysis of inviscid flow fields, using continuity and the Bernoulli relation, is given in Chap. 8.

## Newtonian Fluid: Navier-Stokes Equations

For a newtonian fluid, as discussed in Sec. 1.7, the viscous stresses are proportional to the element strain rates and the coefficient of viscosity. For incompressible flow, the generalization of Eq. (1.23) to three-dimensional viscous flow is[4]

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} \qquad \tau_{yy} = 2\mu \frac{\partial \upsilon}{\partial y} \qquad \tau_{zz} = 2\mu \frac{\partial w}{\partial z}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial \upsilon}{\partial x}\right) \quad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \tag{4.37}$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial \upsilon}{\partial z} + \frac{\partial w}{\partial y}\right)$$

where $\mu$ is the viscosity coefficient. Substitution into Eq. (4.35) gives the differential momentum equation for a newtonian fluid with constant density and viscosity

$$\begin{aligned}
\rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) &= \rho \frac{du}{dt} \\
\rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 \upsilon}{\partial x^2} + \frac{\partial^2 \upsilon}{\partial y^2} + \frac{\partial^2 \upsilon}{\partial z^2}\right) &= \rho \frac{d\upsilon}{dt} \\
\rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) &= \rho \frac{dw}{dt}
\end{aligned} \tag{4.38}$$

These are the *Navier-Stokes equations*, named after C. L. M. H. Navier (1785–1836) and Sir George G. Stokes (1819–1903), who are credited with their derivation. They are second-order nonlinear partial differential equations and are quite formidable, but surprisingly many solutions have been found to a variety of interesting viscous-flow problems, some of which are discussed in Sec. 4.11 and in Chap. 6 (see also Refs. 4 and 5). For compressible flow, see eq. (2.29) of Ref. 5.

Equation (4.38) has four unknowns: $p$, $u$, $\upsilon$, and $w$. It should be combined with the incompressible continuity relation (4.12) to form four equations in these four unknowns. We shall discuss this again in Sec. 4.6, which presents the appropriate boundary conditions for these equations.

## EXAMPLE 4.5

Take the velocity field of Example 4.3, with $b = 0$ for algebraic convenience

$$u = a(x^2 - y^2) \qquad \upsilon = -2axy \qquad w = 0$$

and determine under what conditions it is a solution to the Navier-Stokes momentum equation (4.38). Assuming that these conditions are met, determine the resulting pressure distribution when $z$ is "up" ($g_x = 0$, $g_y = 0$, $g_z = -g$).

### Solution

Make a direct substitution of $u$, $\upsilon$, $w$ into Eq. (4.38):

$$\rho(0) - \frac{\partial p}{\partial x} + \mu(2a - 2a) = 2a^2\rho(x^3 + xy^2) \tag{1}$$

[4]When compressibility is significant, additional small terms arise containing the element volume expansion rate and a *second* coefficient of viscosity; see Refs. 4 and 5 for details.

$$\rho(0) - \frac{\partial p}{\partial y} + \mu(0) = 2a^2\rho(x^2y + y^3) \tag{2}$$

$$\rho(-g) - \frac{\partial p}{\partial z} + \mu(0) = 0 \tag{3}$$

The viscous terms vanish identically (although $\mu$ is *not* zero). Equation (3) can be integrated partially to obtain

$$p = -\rho g z + f_1(x, y) \tag{4}$$

i.e., the pressure is hydrostatic in the $z$ direction, which follows anyway from the fact that the flow is two-dimensional ($w = 0$). Now the question is: Do Eqs. (1) and (2) show that the given velocity field *is* a solution? One way to find out is to form the mixed derivative $\partial^2 p/(\partial x\, \partial y)$ from (1) and (2) separately and then compare them.

Differentiate Eq. (1) with respect to $y$

$$\frac{\partial^2 p}{\partial x\, \partial y} = -4a^2\rho xy \tag{5}$$

Now differentiate Eq. (2) with respect to $x$

$$\frac{\partial^2 p}{\partial x\, \partial y} = -\frac{\partial}{\partial x}[2a^2\rho(x^2y + y^3)] = -4a^2\rho xy \tag{6}$$

Since these are identical, the given velocity field is an *exact* solution to the Navier-Stokes equation. *Ans.*

To find the pressure distribution, substitute Eq. (4) into Eqs. (1) and (2), which will enable us to find $f_1(x, y)$

$$\frac{\partial f_1}{\partial x} = -2a^2\rho(x^3 + xy^2) \tag{7}$$

$$\frac{\partial f_1}{\partial y} = -2a^2\rho(x^2y + y^3) \tag{8}$$

Integrate Eq. (7) partially with respect to $x$

$$f_1 = -\tfrac{1}{2}a^2\rho(x^4 + 2x^2y^2) + f_2(y) \tag{9}$$

Differentiate this with respect to $y$ and compare with Eq. (8)

$$\frac{\partial f_1}{\partial y} = -2a^2\rho x^2 y + f_2'(y) \tag{10}$$

Comparing (8) and (10), we see they are equivalent if

$$f_2'(y) = -2a^2\rho y^3$$

or

$$f_2(y) = -\tfrac{1}{2}a^2\rho y^4 + C \tag{11}$$

where $C$ is a constant. Combine Eqs. (4), (9), and (11) to give the complete expression for pressure distribution

$$p(x, y, z) = -\rho g z - \tfrac{1}{2}a^2\rho(x^4 + y^4 + 2x^2y^2) + C \qquad \textit{Ans.} \tag{12}$$

This is the desired solution. Do you recognize it? Not unless you go back to the beginning and square the velocity components:

$$u^2 + v^2 + w^2 = V^2 = a^2(x^4 + y^4 + 2x^2y^2) \tag{13}$$

Comparing with Eq. (12), we can rewrite the pressure distribution as

$$p + \tfrac{1}{2}\rho V^2 + \rho g z = C \tag{14}$$

This is Bernoulli's equation (3.77). That is no accident, because the velocity distribution given in this problem is one of a family of flows which are solutions to the Navier-Stokes equation and which satisfy Bernoulli's incompressible equation everywhere in the flow field. They are called *irrotational flows*, for which curl $\mathbf{V} = \nabla \times \mathbf{V} \equiv 0$. This subject is discussed again in Sec. 4.9.

## 4.4 The Differential Equation of Angular Momentum

Having now been through the same approach for both mass and linear momentum, we can go rapidly through a derivation of the differential angular-momentum relation. The appropriate form of the integral angular-momentum equation for a fixed control volume is

$$\sum \mathbf{M}_O = \frac{\partial}{\partial t}\left[\int_{\mathrm{CV}} (\mathbf{r} \times \mathbf{V})\rho \, d\mathcal{V}\right] + \int_{\mathrm{CS}} (\mathbf{r} \times \mathbf{V})\rho(\mathbf{V} \cdot \mathbf{n}) \, dA \tag{3.55}$$

We shall confine ourselves to an axis $O$ which is parallel to the $z$ axis and passes through the centroid of the elemental control volume. This is shown in Fig. 4.5. Let $\theta$ be the angle of rotation about $O$ of the fluid within the control volume. The only stresses which have moments about $O$ are the shear stresses $\tau_{xy}$ and $\tau_{yx}$. We can evaluate the moments about $O$ and the angular-momentum terms about $O$. A lot of algebra is involved, and we give here only the result

$$\left[\tau_{xy} - \tau_{yx} + \frac{1}{2}\frac{\partial}{\partial x}(\tau_{xy})\, dx - \frac{1}{2}\frac{\partial}{\partial y}(\tau_{yx})\, dy\right] dx\, dy\, dz = \frac{1}{12}\rho(dx\, dy\, dz)(dx^2 + dy^2)\frac{d^2\theta}{dt^2} \tag{4.39}$$

Assuming that the angular acceleration $d^2\theta/dt^2$ is not infinite, we can neglect all higher-

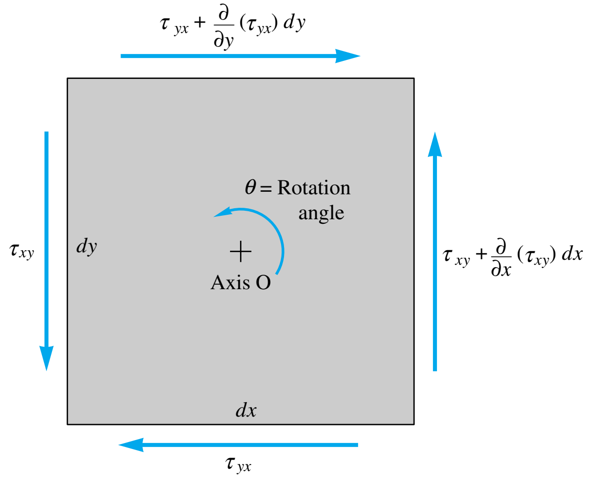



**Fig. 4.5** Elemental cartesian fixed control volume showing shear stresses which may cause a net angular acceleration about axis $O$.

order differential terms, which leaves a finite and interesting result

$$\tau_{xy} \approx \tau_{yx} \tag{4.40}$$

Had we summed moments about axes parallel to $y$ or $x$, we would have obtained exactly analogous results

$$\tau_{xz} \approx \tau_{zx} \qquad \tau_{yz} \approx \tau_{zy} \tag{4.41}$$

There is *no* differential angular-momentum equation. Application of the integral theorem to a differential element gives the result, well known to students of stress analysis, that the shear stresses are symmetric: $\tau_{ij} = \tau_{ji}$. This is the only result of this section.[5] There is no differential equation to remember, which leaves room in your brain for the next topic, the differential energy equation.

## 4.5 The Differential Equation of Energy[6]

We are now so used to this type of derivation that we can race through the energy equation at a bewildering pace. The appropriate integral relation for the fixed control volume of Fig. 4.1 is

$$\dot{Q} - \dot{W}_s - \dot{W}_v = \frac{\partial}{\partial t}\left(\int_{CV} e\rho \, d\mathscr{V}\right) + \int_{CS}\left(e + \frac{p}{\rho}\right)\rho(\mathbf{V}\cdot\mathbf{n})\, dA \tag{3.63}$$

where $\dot{W}_s = 0$ because there can be no infinitesimal shaft protruding into the control volume. By analogy with Eq. (4.20), the right-hand side becomes, for this tiny element,

$$\dot{Q} - \dot{W}_v = \left[\frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x}(\rho u \zeta) + \frac{\partial}{\partial y}(\rho v \zeta) + \frac{\partial}{\partial z}(\rho w \zeta)\right] dx\, dy\, dz \tag{4.42}$$

where $\zeta = e + p/\rho$. When we use the continuity equation by analogy with Eq. (4.21), this becomes

$$\dot{Q} - \dot{W}_v = \left(\rho \frac{de}{dt} + \mathbf{V}\cdot\nabla p\right) dx\, dy\, dz \tag{4.43}$$

To evaluate $\dot{Q}$, we neglect radiation and consider only heat conduction through the sides of the element. The heat flow by conduction follows Fourier's law from Chap. 1

$$\mathbf{q} = -k\,\nabla T \tag{1.29a}$$

where $k$ is the coefficient of thermal conductivity of the fluid. Figure 4.6 shows the heat flow passing through the $x$ faces, the $y$ and $z$ heat flows being omitted for clarity. We can list these six heat-flux terms:

| Faces | Inlet heat flux | Outlet heat flux |
|---|---|---|
| $x$ | $q_x\, dy\, dz$ | $\left[q_x + \frac{\partial}{\partial x}(q_x)\, dx\right] dy\, dz$ |
| $y$ | $q_y\, dx\, dz$ | $\left[q_y + \frac{\partial}{\partial y}(q_y)\, dy\right] dx\, dz$ |
| $z$ | $q_z\, dx\, dy$ | $\left[q_z + \frac{\partial}{\partial z}(q_z)\, dz\right] dx\, dy$ |

[5] We are neglecting the possibility of a finite *couple* being applied to the element by some powerful external force field. See, e.g., Ref. 6, p. 217.

[6] This section may be omitted without loss of continuity.

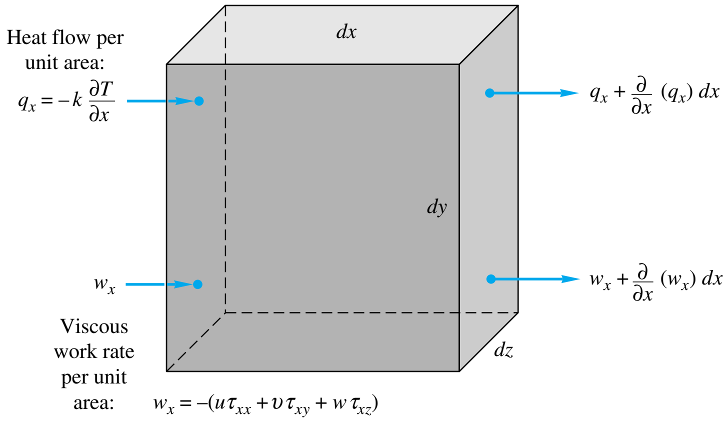



**Fig. 4.6** Elemental cartesian control volume showing heat-flow and viscous-work-rate terms in the $x$ direction.

By adding the inlet terms and subtracting the outlet terms, we obtain the net heat added to the element

$$\dot{Q} = -\left[\frac{\partial}{\partial x}(q_x) + \frac{\partial}{\partial y}(q_y) + \frac{\partial}{\partial z}(q_z)\right] dx\,dy\,dz = -\nabla \cdot \mathbf{q}\,dx\,dy\,dz \tag{4.44}$$

As expected, the heat flux is proportional to the element volume. Introducing Fourier's law from Eq. (1.29), we have

$$\dot{Q} = \nabla \cdot (k\,\nabla T)\,dx\,dy\,dz \tag{4.45}$$

The rate of work done by viscous stresses equals the product of the stress component, its corresponding velocity component, and the area of the element face. Figure 4.6 shows the work rate on the left $x$ face is

$$\dot{W}_{\upsilon,\mathrm{LF}} = w_x\,dy\,dz \qquad \text{where } w_x = -(u\tau_{xx} + \upsilon\tau_{xy} + w\tau_{xz}) \tag{4.46}$$

(where the subscript LF stands for left face) and a slightly different work on the right face due to the gradient in $w_x$. These work fluxes could be tabulated in exactly the same manner as the heat fluxes in the previous table, with $w_x$ replacing $q_x$, etc. After outlet terms are subtracted from inlet terms, the net viscous-work rate becomes

$$\begin{aligned}\dot{W}_\upsilon = -\bigg[&\frac{\partial}{\partial x}(u\tau_{xx} + \upsilon\tau_{xy} + w\tau_{xz}) + \frac{\partial}{\partial y}(u\tau_{yx} + \upsilon\tau_{yy} + w\tau_{yz})\\ &+ \frac{\partial}{\partial z}(u\tau_{zx} + \upsilon\tau_{zy} + w\tau_{zz})\bigg]\,dx\,dy\,dz\\ =\; &-\nabla \cdot (\mathbf{V} \cdot \boldsymbol{\tau}_{ij})\,dx\,dy\,dz\end{aligned} \tag{4.47}$$

We now substitute Eqs. (4.45) and (4.47) into Eq. (4.43) to obtain one form of the differential energy equation

$$\rho\,\frac{de}{dt} + \mathbf{V} \cdot \nabla p = \nabla \cdot (k\,\nabla T) + \nabla \cdot (\mathbf{V} \cdot \boldsymbol{\tau}_{ij}) \qquad \text{where } e = \hat{u} + \tfrac{1}{2}V^2 + gz \tag{4.48}$$

A more useful form is obtained if we split up the viscous-work term

$$\nabla \cdot (\mathbf{V} \cdot \boldsymbol{\tau}_{ij}) \equiv \mathbf{V} \cdot (\nabla \cdot \boldsymbol{\tau}_{ij}) + \Phi \tag{4.49}$$

where $\Phi$ is short for the *viscous-dissipation function*.[7] For a newtonian incompressible viscous fluid, this function has the form

$$\Phi = \mu\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial \upsilon}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial \upsilon}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial \upsilon}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2\right] \tag{4.50}$$

Since all terms are quadratic, viscous dissipation is always positive, so that a viscous flow always tends to lose its available energy due to dissipation, in accordance with the second law of thermodynamics.

Now substitute Eq. (4.49) into Eq. (4.48), using the linear-momentum equation (4.32) to eliminate $\nabla \cdot \boldsymbol{\tau}_{ij}$. This will cause the kinetic and potential energies to cancel, leaving a more customary form of the general differential energy equation

$$\rho \frac{d\hat{u}}{dt} + p(\nabla \cdot \mathbf{V}) = \nabla \cdot (k\, \nabla T) + \Phi \tag{4.51}$$

This equation is valid for a newtonian fluid under very general conditions of unsteady, compressible, viscous, heat-conducting flow, except that it neglects radiation heat transfer and internal *sources* of heat that might occur during a chemical or nuclear reaction.

Equation (4.51) is too difficult to analyze except on a digital computer [1]. It is customary to make the following approximations:

$$d\hat{u} \approx c_\upsilon\, dT \qquad c_\upsilon,\ \mu,\ k,\ \rho \approx \text{const} \tag{4.52}$$

Equation (4.51) then takes the simpler form

$$\rho c_\upsilon \frac{dT}{dt} = k\nabla^2 T + \Phi \tag{4.53}$$

which involves temperature $T$ as the sole primary variable plus velocity as a secondary variable through the total time-derivative operator

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + \upsilon\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} \tag{4.54}$$

A great many interesting solutions to Eq. (4.53) are known for various flow conditions, and extended treatments are given in advanced books on viscous flow [4, 5] and books on heat transfer [7, 8].

One well-known special case of Eq. (4.53) occurs when the fluid is at rest or has negligible velocity, where the dissipation $\Phi$ and convective terms become negligible

$$\rho c_\upsilon \frac{\partial T}{\partial t} = k\, \nabla^2 T \tag{4.55}$$

This is called the *heat-conduction equation* in applied mathematics and is valid for solids and fluids at rest. The solution to Eq. (4.55) for various conditions is a large part of courses and books on heat transfer.

This completes the derivation of the basic differential equations of fluid motion.

[7]For further details, see, e.g., Ref. 5, p. 72.

## 4.6 Boundary Conditions for the Basic Equations

There are three basic differential equations of fluid motion, just derived. Let us summarize them here:

Continuity:
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \tag{4.56}$$

Momentum:
$$\rho \frac{d\mathbf{V}}{dt} = \rho \mathbf{g} - \nabla p + \nabla \cdot \boldsymbol{\tau}_{ij} \tag{4.57}$$

Energy:
$$\rho \frac{d\hat{u}}{dt} + p(\nabla \cdot \mathbf{V}) = \nabla \cdot (k\, \nabla T) + \Phi \tag{4.58}$$

where $\Phi$ is given by Eq. (4.50). In general, the density is variable, so that these three equations contain five unknowns, $\rho$, $V$, $p$, $\hat{u}$, and $T$. Therefore we need two additional relations to complete the system of equations. These are provided by data or algebraic expressions for the state relations of the thermodynamic properties

$$\rho = \rho(p, T) \qquad \hat{u} = \hat{u}(p, T) \tag{4.59}$$

For example, for a perfect gas with constant specific heats, we complete the system with

$$\rho = \frac{p}{RT} \qquad \hat{u} = \int c_v\, dT \approx c_v T + \text{const} \tag{4.60}$$

It is shown in advanced books [4, 5] that this system of equations (4.56) to (4.59) is well posed and can be solved analytically or numerically, subject to the proper boundary conditions.

What are the proper boundary conditions? First, if the flow is unsteady, there must be an *initial condition* or initial spatial distribution known for each variable:

At $t = 0$:
$$\rho, V, p, \hat{u}, T = \text{known } f(x, y, z) \tag{4.61}$$

Thereafter, for all times $t$ to be analyzed, we must know something about the variables at each *boundary* enclosing the flow.

Figure 4.7 illustrates the three most common types of boundaries encountered in fluid-flow analysis: a solid wall, an inlet or outlet, a liquid-gas interface.

First, for a solid, impermeable wall, there is no slip and no temperature jump in a viscous heat-conducting fluid

$$\mathbf{V}_{\text{fluid}} = \mathbf{V}_{\text{wall}} \qquad T_{\text{fluid}} = T_{\text{wall}} \qquad \text{solid wall} \tag{4.62}$$

The only exception to Eq. (4.62) occurs in an extremely rarefied gas flow, where slippage can be present [5].

Second, at any inlet or outlet section of the flow, the complete distribution of velocity, pressure, and temperature must be known for all times:

Inlet or outlet:
$$\text{Known } \mathbf{V}, p, T \tag{4.63}$$

These inlet and outlet sections can be and often are at $\pm\infty$, simulating a body immersed in an infinite expanse of fluid.

Finally, the most complex conditions occur at a liquid-gas interface, or free surface, as sketched in Fig. 4.7. Let us denote the interface by

Interface:
$$z = \eta(x, y, t) \tag{4.64}$$

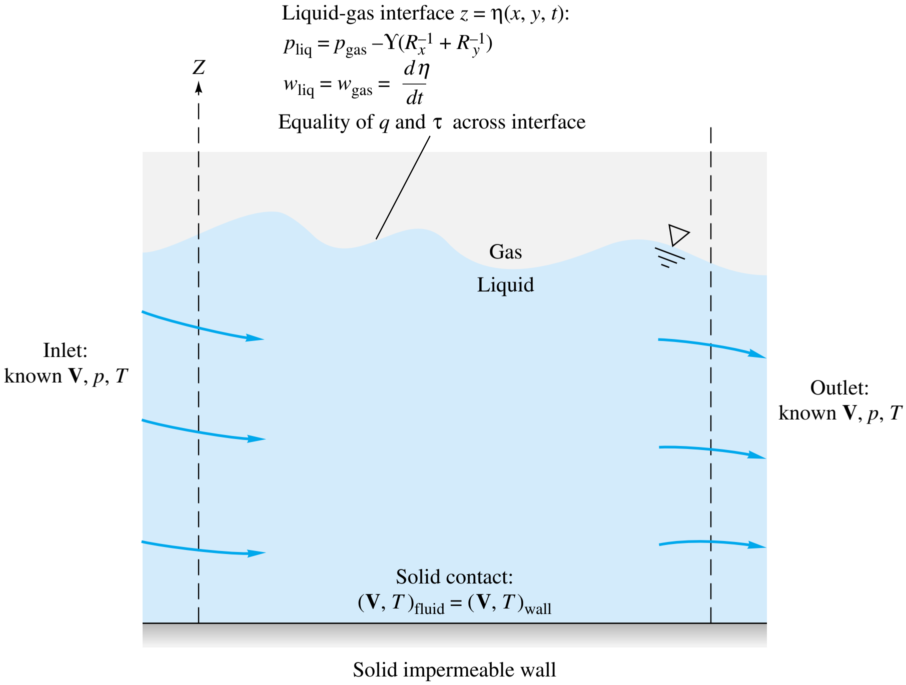



**Fig. 4.7** Typical boundary conditions in a viscous heat-conducting fluid-flow analysis.

Then there must be equality of vertical velocity across the interface, so that no holes appear between liquid and gas:

$$w_{liq} = w_{gas} = \frac{d\eta}{dt} = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + \upsilon\frac{\partial \eta}{\partial y} \tag{4.65}$$

This is called the *kinematic boundary condition*.

There must be mechanical equilibrium across the interface. The viscous-shear stresses must balance

$$(\tau_{zy})_{liq} = (\tau_{zy})_{gas} \qquad (\tau_{zx})_{liq} = (\tau_{zx})_{gas} \tag{4.66}$$

Neglecting the viscous normal stresses, the pressures must balance at the interface except for surface-tension effects

$$p_{liq} = p_{gas} - \Upsilon(R_x^{-1} + R_y^{-1}) \tag{4.67}$$

which is equivalent to Eq. (1.34). The radii of curvature can be written in terms of the free-surface position $\eta$

$$\begin{aligned} R_x^{-1} + R_y^{-1} = {} & \frac{\partial}{\partial x}\left[\frac{\partial\eta/\partial x}{\sqrt{1 + (\partial\eta/\partial x)^2 + (\partial\eta/\partial y)^2}}\right] \\ & + \frac{\partial}{\partial y}\left[\frac{\partial\eta/\partial y}{\sqrt{1 + (\partial\eta/\partial x)^2 + (\partial\eta/\partial y)^2}}\right] \end{aligned} \tag{4.68}$$

Finally, the heat transfer must be the same on both sides of the interface, since no heat can be stored in the infinitesimally thin interface

$$(q_z)_{\text{liq}} = (q_z)_{\text{gas}} \tag{4.69}$$

Neglecting radiation, this is equivalent to

$$\left(k \frac{\partial T}{\partial z}\right)_{\text{liq}} = \left(k \frac{\partial T}{\partial z}\right)_{\text{gas}} \tag{4.70}$$

This is as much detail as we wish to give at this level of exposition. Further and even more complicated details on fluid-flow boundary conditions are given in Refs. 5 and 9.

## Simplified Free-Surface Conditions

In the introductory analyses given in this book, such as open-channel flows in Chap. 10, we shall back away from the exact conditions (4.65) to (4.69) and assume that the upper fluid is an "atmosphere" which merely exerts pressure upon the lower fluid, with shear and heat conduction negligible. We also neglect nonlinear terms involving the slopes of the free surface. We then have a much simpler and linear set of conditions at the surface

$$p_{\text{liq}} \approx p_{\text{gas}} - \Upsilon\left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}\right) \qquad w_{\text{liq}} \approx \frac{\partial \eta}{\partial t}$$

$$\left(\frac{\partial V}{\partial z}\right)_{\text{liq}} \approx 0 \qquad \left(\frac{\partial T}{\partial z}\right)_{\text{liq}} \approx 0 \tag{4.71}$$

In many cases, such as open-channel flow, we can also neglect surface tension, so that

$$p_{\text{liq}} \approx p_{\text{atm}} \tag{4.72}$$

These are the types of approximations which will be used in Chap. 10. The nondimensional forms of these conditions will also be useful in Chap. 5.

## Incompressible Flow with Constant Properties

Flow with constant $\rho$, $\mu$, and $k$ is a basic simplification which will be used, e.g., throughout Chap. 6. The basic equations of motion (4.56) to (4.58) reduce to:

Continuity: $$\nabla \cdot \mathbf{V} = 0 \tag{4.73}$$

Momentum: $$\rho \frac{d\mathbf{V}}{dt} = \rho \mathbf{g} - \nabla p + \mu \nabla^2 \mathbf{V} \tag{4.74}$$

Energy: $$\rho c_v \frac{dT}{dt} = k \nabla^2 T + \Phi \tag{4.75}$$

Since $\rho$ is constant, there are only three unknowns: $p$, $\mathbf{V}$, and $T$. The system is closed.[8] Not only that, the system splits apart: Continuity and momentum are independent of $T$. Thus we can solve Eqs. (4.73) and (4.74) entirely separately for the pressure and velocity, using such boundary conditions as

Solid surface: $$\mathbf{V} = \mathbf{V}_{\text{wall}} \tag{4.76}$$

[8]For this system, what are the thermodynamic equivalents to Eq. (4.59)?

Inlet or outlet: Known $\mathbf{V}$, $p$ (4.77)

Free surface: $$p \approx p_a \qquad w \approx \frac{\partial \eta}{\partial t} \tag{4.78}$$

Later, entirely at our leisure,[9] we can solve for the temperature distribution from Eq. (4.75), which depends upon velocity $\mathbf{V}$ through the dissipation $\Phi$ and the total time-derivative operator $d/dt$.

## Inviscid-Flow Approximations

Chapter 8 assumes inviscid flow throughout, for which the viscosity $\mu = 0$. The momentum equation (4.74) reduces to

$$\rho \frac{d\mathbf{V}}{dt} = \rho \mathbf{g} - \nabla p \tag{4.79}$$

This is *Euler's equation*; it can be integrated along a streamline to obtain Bernoulli's equation (see Sec. 4.9). By neglecting viscosity we have lost the second-order derivative of $\mathbf{V}$ in Eq. (4.74); therefore we must relax one boundary condition on velocity. The only mathematically sensible condition to drop is the no-slip condition at the wall. We let the flow slip parallel to the wall but do not allow it to flow into the wall. The proper inviscid condition is that the normal velocities must match at any solid surface:

Inviscid flow: $$(V_n)_{\text{fluid}} = (V_n)_{\text{wall}} \tag{4.80}$$

In most cases the wall is fixed; therefore the proper inviscid-flow condition is

$$V_n = 0 \tag{4.81}$$

There is *no* condition whatever on the tangential velocity component at the wall in inviscid flow. The tangential velocity will be part of the solution, and the correct value will appear after the analysis is completed (see Chap. 8).

### EXAMPLE 4.6

For steady incompressible laminar flow through a long tube, the velocity distribution is given by

$$v_z = U\left(1 - \frac{r^2}{R^2}\right) \qquad v_r = v_\theta = 0$$

where $U$ is the maximum, or centerline, velocity and $R$ is the tube radius. If the wall temperature is constant at $T_w$ and the temperature $T = T(r)$ only, find $T(r)$ for this flow.

### Solution

With $T = T(r)$, Eq. (4.75) reduces for steady flow to

$$\rho c_v v_r \frac{dT}{dr} = \frac{k}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \mu\left(\frac{dv_z}{dr}\right)^2 \tag{1}$$

[9] Since temperature is entirely *uncoupled* by this assumption, we may never get around to solving for it here and may ask you to wait until a course on heat transfer.

But since $\upsilon_r = 0$ for this flow, the convective term on the left vanishes. Introduce $\upsilon_z$ into Eq. (1) to obtain

$$\frac{k}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)^2 = -\mu\left(\frac{d\upsilon_z}{dr}\right)^2 = -\frac{4U^2\mu r^2}{R^4} \tag{2}$$

Multiply through by $r/k$ and integrate once:

$$r\frac{dT}{dr} = -\frac{\mu U^2 r^4}{kR^4} + C_1 \tag{3}$$

Divide through by $r$ and integrate once again:

$$T = -\frac{\mu U^2 r^4}{4kR^4} + C_1 \ln r + C_2 \tag{4}$$

Now we are in position to apply our boundary conditions to evaluate $C_1$ and $C_2$.

First, since the logarithm of zero is $-\infty$, the temperature at $r = 0$ will be infinite unless

$$C_1 = 0 \tag{5}$$

Thus we eliminate the possibility of a logarithmic singularity. The same thing will happen if we apply the *symmetry* condition $dT/dr = 0$ at $r = 0$ to Eq. (3). The constant $C_2$ is then found by the wall-temperature condition at $r = R$

$$T = T_w = -\frac{\mu U^2}{4k} + C_2$$

or

$$C_2 = T_w + \frac{\mu U^2}{4k} \tag{6}$$

The correct solution is thus

$$T(r) = T_w + \frac{\mu U^2}{4k}\left(1 - \frac{r^4}{R^4}\right) \qquad \textit{Ans.} \tag{7}$$

which is a fourth-order parabolic distribution with a maximum value $T_0 = T_w + \mu U^2/(4k)$ at the centerline.

## 4.7 The Stream Function

We have seen in Sec. 4.6 that even if the temperature is uncoupled from our system of equations of motion, we must solve the continuity and momentum equations simultaneously for pressure and velocity. The *stream function* $\psi$ is a clever device which allows us to wipe out the continuity equation and solve the momentum equation directly for the single variable $\psi$.

The stream-function idea works only if the continuity equation (4.56) can be reduced to *two* terms. In general, we have *four* terms:

Cartesian:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho \upsilon) + \frac{\partial}{\partial z}(\rho w) = 0 \tag{4.82a}$$

Cylindrical:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho \upsilon_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho \upsilon_\theta) + \frac{\partial}{\partial z}(\rho \upsilon_z) = 0 \tag{4.82b}$$

First, let us eliminate unsteady flow, which is a peculiar and unrealistic application of the stream-function idea. Reduce either of Eqs. (4.82) to any *two* terms. The most common application is incompressible flow in the *xy* plane

$$\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} = 0 \tag{4.83}$$

This equation is satisfied *identically* if a function $\psi(x, y)$ is defined such that Eq. (4.83) becomes

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) \equiv 0 \tag{4.84}$$

Comparison of (4.83) and (4.84) shows that this new function $\psi$ must be defined such that

$$u = \frac{\partial \psi}{\partial y} \qquad \upsilon = -\frac{\partial \psi}{\partial x} \tag{4.85}$$

or

$$\mathbf{V} = \mathbf{i}\,\frac{\partial \psi}{\partial y} - \mathbf{j}\,\frac{\partial \psi}{\partial x}$$

Is this legitimate? Yes, it is just a mathematical trick of replacing two variables ($u$ and $\upsilon$) by a single higher-order function $\psi$. The vorticity, or curl **V**, is an interesting function

$$\text{curl } \mathbf{V} = 2\mathbf{k}\omega_z = -\mathbf{k}\nabla^2\psi \qquad \text{where} \qquad \nabla^2\psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \tag{4.86}$$

Thus, if we take the curl of the momentum equation (4.74) and utilize Eq. (4.86), we obtain a single equation for $\psi$

$$\frac{\partial \psi}{\partial y}\,\frac{\partial}{\partial x}(\nabla^2\psi) - \frac{\partial \psi}{\partial x}\,\frac{\partial}{\partial y}(\nabla^2\psi) = \nu\,\nabla^2(\nabla^2\psi) \tag{4.87}$$

where $\nu = \mu/\rho$ is the kinematic viscosity. This is partly a victory and partly a defeat: Eq. (4.87) is scalar and has only one variable, $\psi$, but it now contains *fourth*-order derivatives and probably will require computer analysis. There will be four boundary conditions required on $\psi$. For example, for the flow of a uniform stream in the $x$ direction past a solid body, the four conditions would be

$$\begin{aligned} &\text{At infinity:} && \frac{\partial \psi}{\partial y} = U_\infty \qquad \frac{\partial \psi}{\partial x} = 0 \\ &\text{At the body:} && \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial x} = 0 \end{aligned} \tag{4.88}$$

Many examples of numerical solution of Eqs. (4.87) and (4.88) are given in Ref. 1.

One important application is inviscid *irrotational* flow in the *xy* plane, where $\omega_z \equiv 0$. Equations (4.86) and (4.87) reduce to

$$\nabla^2\psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{4.89}$$

This is the second-order *Laplace equation* (Chap. 8), for which many solutions and analytical techniques are known. Also, boundary conditions like Eq. (4.88) reduce to

At infinity: $$\psi = U_\infty y + \text{const} \tag{4.90}$$

At the body: $$\psi = \text{const}$$

It is well within our capability to find some useful solutions to Eqs. (4.89) and (4.90), which we shall do in Chap. 8.

## Geometric Interpretation of $\psi$

The fancy mathematics above would serve by itself to make the stream function immortal and always useful to engineers. Even better, though, $\psi$ has a beautiful geometric interpretation: Lines of constant $\psi$ are *streamlines* of the flow. This can be shown as follows. From Eq. (1.41) the definition of a streamline in two-dimensional flow is

$$\frac{dx}{u} = \frac{dy}{v}$$

or $$u\,dy - v\,dx = 0 \qquad \text{streamline} \tag{4.91}$$

Introducing the stream function from Eq. (4.85), we have

$$\frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy = 0 = d\psi \tag{4.92}$$

Thus the change in $\psi$ is zero along a streamline, or

$$\psi = \text{const along a streamline} \tag{4.93}$$

Having found a given solution $\psi(x, y)$, we can plot lines of constant $\psi$ to give the streamlines of the flow.

There is also a physical interpretation which relates $\psi$ to volume flow. From Fig. 4.8, we can compute the volume flow $dQ$ through an element $ds$ of control surface of unit depth

$$\begin{aligned} dQ &= (\mathbf{V} \cdot \mathbf{n})\,dA = \left(\mathbf{i}\,\frac{\partial \psi}{\partial y} - \mathbf{j}\,\frac{\partial \psi}{\partial x}\right) \cdot \left(\mathbf{i}\,\frac{dy}{ds} - \mathbf{j}\,\frac{dx}{ds}\right) ds(1) \\ &= \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy = d\psi \end{aligned} \tag{4.94}$$

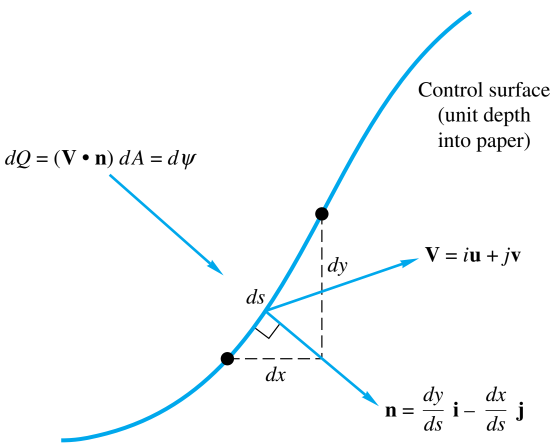



**Fig. 4.8** Geometric interpretation of stream function: volume flow through a differential portion of a control surface.

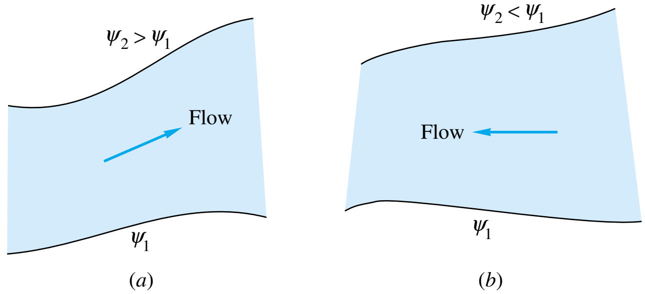



**Fig. 4.9** Sign convention for flow in terms of change in stream function: (*a*) flow to the right if $\psi_U$ is greater; (*b*) flow to the left if $\psi_L$ is greater.

Thus the change in $\psi$ across the element is numerically equal to the volume flow through the element. The volume flow between any two points in the flow field is equal to the change in stream function between those points:

$$Q_{1\rightarrow 2} = \int_1^2 (\mathbf{V} \cdot \mathbf{n})\, dA = \int_1^2 d\psi = \psi_2 - \psi_1 \tag{4.95}$$

Further, the direction of the flow can be ascertained by noting whether $\psi$ increases or decreases. As sketched in Fig. 4.9, the flow is to the right if $\psi_U$ is greater than $\psi_L$, where the subscripts stand for upper and lower, as before; otherwise the flow is to the left.

Both the stream function and the velocity potential were invented by the French mathematician Joseph Louis Lagrange and published in his treatise on fluid mechanics in 1781.

## EXAMPLE 4.7

If a stream function exists for the velocity field of Example 4.5

$$u = a(x^2 - y^2) \quad \upsilon = -2axy \quad w = 0$$

find it, plot it, and interpret it.

### Solution

Since this flow field was shown expressly in Example 4.3 to satisfy the equation of continuity, we are pretty sure that a stream function does exist. We can check again to see if

$$\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} = 0$$

Substitute: $$2ax + (-2ax) = 0 \qquad \text{checks}$$

Therefore we are certain that a stream function exists. To find $\psi$, we simply set

$$u = \frac{\partial \psi}{\partial y} = ax^2 - ay^2 \tag{1}$$

$$\upsilon = -\frac{\partial \psi}{\partial x} = -2axy \tag{2}$$

and work from either one toward the other. Integrate (1) partially

$$\psi = ax^2y - \frac{ay^3}{3} + f(x) \tag{3}$$

Differentiate (3) with respect to $x$ and compare with (2)

$$\frac{\partial \psi}{\partial x} = 2axy + f'(x) = 2axy \tag{4}$$

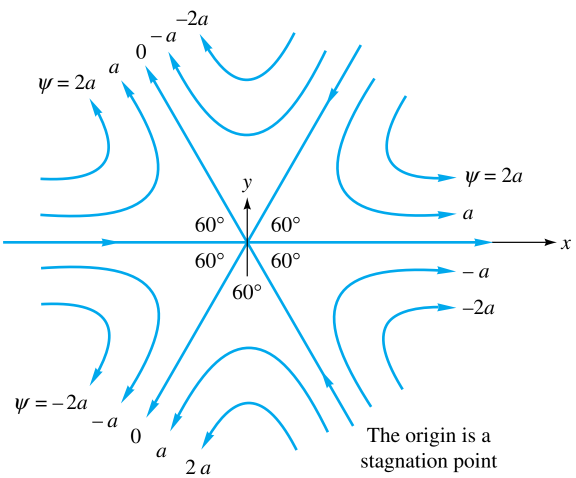
Therefore $f'(x) = 0$, or $f =$ constant. The complete stream function is thus found

$$\psi = a\left(x^2y - \frac{y^3}{3}\right) + C \qquad \textit{Ans.} \tag{5}$$

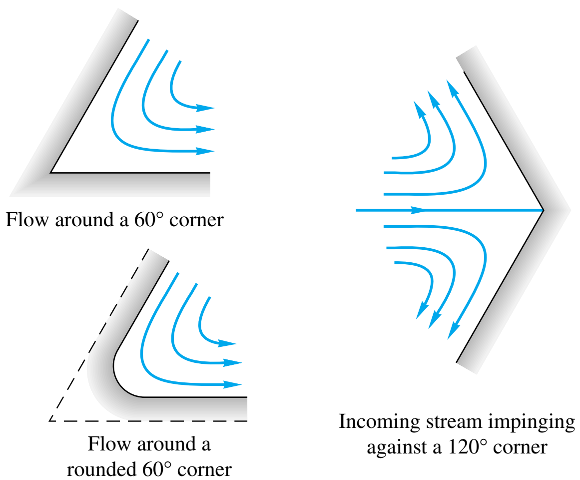
To plot this, set $C = 0$ for convenience and plot the function

$$3x^2y - y^3 = \frac{3\psi}{a} \tag{6}$$

for constant values of $\psi$. The result is shown in Fig. E4.7*a* to be six 60° wedges of circulating motion, each with identical flow patterns except for the arrows. Once the streamlines are labeled, the flow directions follow from the sign convention of Fig. 4.9. How can the flow be interpreted? Since there is slip along all streamlines, no streamline can truly represent a solid surface in a viscous flow. However, the flow could represent the impingement of three incoming streams at 60, 180, and 300°. This would be a rather unrealistic yet exact solution to the Navier-Stokes equation, as we showed in Example 4.5.




E4.7a




E4.7b

By allowing the flow to slip as a frictionless approximation, we could let any given streamline be a body shape. Some examples are shown in Fig. E4.7*b*.

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A stream function also exists in a variety of other physical situations where only two coordinates are needed to define the flow. Three examples are illustrated here.

**Steady Plane Compressible Flow**

Suppose now that the density is variable but that $w = 0$, so that the flow is in the $xy$ plane. Then the equation of continuity becomes

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{4.96}$$

We see that this is in exactly the same form as Eq. (4.84). Therefore a compressible-flow stream function can be defined such that

$$\rho u = \frac{\partial \psi}{\partial y} \qquad \rho v = -\frac{\partial \psi}{\partial x} \tag{4.97}$$

Again lines of constant $\psi$ are streamlines of the flow, but the change in $\psi$ is now equal to the *mass* flow, not the volume flow

$$d\dot{m} = \rho(\mathbf{V} \cdot \mathbf{n})\, dA = d\psi$$

or

$$\dot{m}_{1\to 2} = \int_1^2 \rho(\mathbf{V} \cdot \mathbf{n})\, dA = \psi_2 - \psi_1 \tag{4.98}$$

The sign convention on flow direction is the same as in Fig. 4.9. This particular stream function combines density with velocity and must be substituted into not only momentum but also the energy and state relations (4.58) and (4.59) with pressure and temperature as companion variables. Thus the compressible stream function is not a great victory, and further assumptions must be made to effect an analytical solution to a typical problem (see, e.g., Ref. 5, chap. 7).

**Incompressible Plane Flow in Polar Coordinates**

Suppose that the important coordinates are $r$ and $\theta$, with $v_z = 0$, and that the density is constant. Then Eq. (4.82*b*) reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) = 0 \tag{4.99}$$

After multiplying through by $r$, we see that this is the same as the analogous form of Eq. (4.84)

$$\frac{\partial}{\partial r}\left(\frac{\partial \psi}{\partial \theta}\right) + \frac{\partial}{\partial \theta}\left(-\frac{\partial \psi}{\partial r}\right) = 0 \tag{4.100}$$

By comparison of (4.99) and (4.100) we deduce the form of the incompressible polar-coordinate stream function

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = -\frac{\partial \psi}{\partial r} \tag{4.101}$$

Once again lines of constant $\psi$ are streamlines, and the change in $\psi$ is the *volume flow* $Q_{1\to 2} = \psi_2 - \psi_1$. The sign convention is the same as in Fig. 4.9. This type of stream function is very useful in analyzing flows with cylinders, vortices, sources, and sinks (Chap. 8).

**Incompressible Axisymmetric Flow**

As a final example, suppose that the flow is three-dimensional ($v_r$, $v_z$) but with no circumferential variations, $v_\theta = \partial/\partial\theta = 0$ (see Fig. 4.2 for definition of coordinates). Such a flow is termed *axisymmetric*, and the flow pattern is the same when viewed on any meridional plane through the axis of revolution $z$. For incompressible flow, Eq. (4.82*b*) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}(r\upsilon_r) + \frac{\partial}{\partial z}(\upsilon_z) = 0 \tag{4.102}$$

This doesn't seem to work: Can't we get rid of the one $r$ outside? But when we realize that $r$ and $z$ are independent coordinates, Eq. (4.102) can be rewritten as

$$\frac{\partial}{\partial r}(r\upsilon_r) + \frac{\partial}{\partial z}(r\upsilon_z) = 0 \tag{4.103}$$

By analogy with Eq. (4.84), this has the form

$$\frac{\partial}{\partial r}\left(-\frac{\partial \psi}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \psi}{\partial r}\right) = 0 \tag{4.104}$$

By comparing (4.103) and (4.104), we deduce the form of an incompressible axisymmetric stream function $\psi(r, z)$

$$\upsilon_r = -\frac{1}{r}\frac{\partial \psi}{\partial z} \qquad \upsilon_z = \frac{1}{r}\frac{\partial \psi}{\partial r} \tag{4.105}$$

Here again lines of constant $\psi$ are streamlines, but there is a factor $(2\pi)$ in the volume flow: $Q_{1\rightarrow 2} = 2\pi(\psi_2 - \psi_1)$. The sign convention on flow is the same as in Fig. 4.9.

## EXAMPLE 4.8

Investigate the stream function in polar coordinates

$$\psi = U \sin\theta\left(r - \frac{R^2}{r}\right) \tag{1}$$

where $U$ and $R$ are constants, a velocity and a length, respectively. Plot the streamlines. What does the flow represent? Is it a realistic solution to the basic equations?

### Solution

The streamlines are lines of constant $\psi$, which has units of square meters per second. Note that $\psi/(UR)$ is dimensionless. Rewrite Eq. (1) in dimensionless form

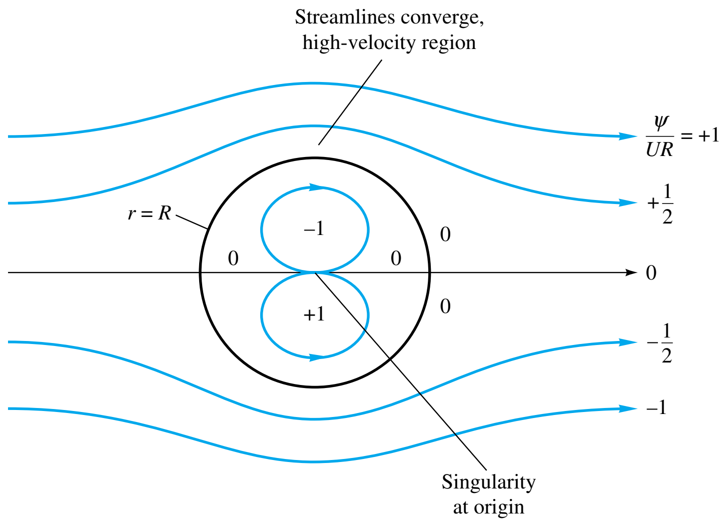
$$\frac{\psi}{UR} = \sin\theta\left(\eta - \frac{1}{\eta}\right) \qquad \eta = \frac{r}{R} \tag{2}$$

Of particular interest is the special line $\psi = 0$. From Eq. (1) or (2) this occurs when (*a*) $\theta = 0$ or 180° and (*b*) $r = R$. Case (*a*) is the $x$-axis, and case (*b*) is a circle of radius $R$, both of which are plotted in Fig. E4.8.

For any other nonzero value of $\psi$ it is easiest to pick a value of $r$ and solve for $\theta$:

$$\sin\theta = \frac{\psi/(UR)}{r/R - R/r} \tag{3}$$

In general, there will be two solutions for $\theta$ because of the symmetry about the $y$-axis. For example take $\psi/(UR) = +1.0$:




**E4.8**

| Guess $r/R$ | 3.0 | 2.5 | 2.0 | 1.8 | 1.7 | 1.618 |
|---|---|---|---|---|---|---|
| Compute $\theta$ | 22°<br>158° | 28°<br>152° | 42°<br>138° | 54°<br>156° | 64°<br>116° | 90° |

This line is plotted in Fig. E4.8 and passes over the circle $r = R$. You have to watch it, though, because there is a second curve for $\psi/(UR) = +1.0$ for small $r < R$ below the $x$-axis:

| Guess $r/R$ | 0.618 | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 | 0.1 |
|---|---|---|---|---|---|---|---|
| Compute $\theta$ | −90° | −70°<br>−110° | −42°<br>−138° | −28°<br>−152° | −19°<br>−161° | −12°<br>−168° | −6°<br>−174° |

This second curve plots as a closed curve inside the circle $r = R$. There is a singularity of infinite velocity and indeterminate flow direction at the origin. Figure E4.8 shows the full pattern.

The given stream function, Eq. (1), is an exact and classic solution to the momentum equation (4.38) for frictionless flow. Outside the circle $r = R$ it represents two-dimensional inviscid flow of a uniform stream past a circular cylinder (Sec. 8.3). Inside the circle it represents a rather unrealistic trapped circulating motion of what is called a *line doublet*.

## 4.8 Vorticity and Irrotationality

The assumption of zero fluid angular velocity, or irrotationality, is a very useful simplification. Here we show that angular velocity is associated with the curl of the local-velocity vector.

The differential relations for deformation of a fluid element can be derived by examining Fig. 4.10. Two fluid lines $AB$ and $BC$, initially perpendicular at time $t$, move and deform so that at $t + dt$ they have slightly different lengths $A'B'$ and $B'C'$ and are slightly off the perpendicular by angles $d\alpha$ and $d\beta$. Such deformation occurs kinematically because $A$, $B$, and $C$ have slightly different velocities when the velocity field $\mathbf{V}$

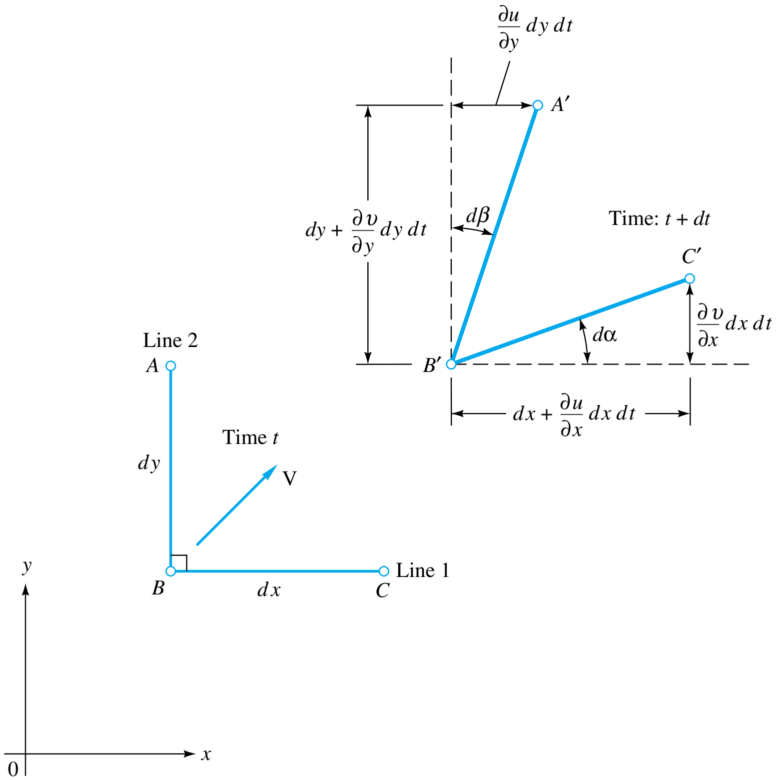



**Fig. 4.10** Angular velocity and strain rate of two fluid lines deforming in the $xy$ plane.

has spatial gradients. All these differential changes in the motion of $A$, $B$, and $C$ are noted in Fig. 4.10.

We define the angular velocity $\omega_z$ about the $z$ axis as the average rate of counterclockwise turning of the two lines

$$\omega_z = \frac{1}{2}\left(\frac{d\alpha}{dt} - \frac{d\beta}{dt}\right) \tag{4.106}$$

But from Fig. 4.10, $d\alpha$ and $d\beta$ are each directly related to velocity derivatives in the limit of small $dt$

$$\begin{aligned} d\alpha &= \lim_{dt\to 0}\left[\tan^{-1}\frac{(\partial v/\partial x)\,dx\,dt}{dx + (\partial u/\partial x)\,dx\,dt}\right] = \frac{\partial v}{\partial x}\,dt \\ d\beta &= \lim_{dt\to 0}\left[\tan^{-1}\frac{(\partial u/\partial y)\,dy\,dt}{dy + (\partial v/\partial y)\,dy\,dt}\right] = \frac{\partial u}{\partial y}\,dt \end{aligned} \tag{4.107}$$

Combining Eqs. (4.106) and (4.107) gives the desired result:

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{4.108}$$

In exactly similar manner we determine the other two rates:

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \tag{4.109}$$

The vector $\omega = \mathbf{i}\omega_x + \mathbf{j}\omega_y + \mathbf{k}\omega_z$ is thus one-half the curl of the velocity vector

$$\omega = \frac{1}{2}(\text{curl } \mathbf{V}) = \frac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & \upsilon & w \end{vmatrix} \tag{4.110}$$

Since the factor of $\frac{1}{2}$ is annoying, many workers prefer to use a vector twice as large, called the *vorticity*:

$$\boldsymbol{\zeta} = 2\boldsymbol{\omega} = \text{curl } \mathbf{V} \tag{4.111}$$

Many flows have negligible or zero vorticity and are called *irrotational*

$$\text{curl } \mathbf{V} \equiv 0 \tag{4.112}$$

The next section expands on this idea. Such flows can be incompressible or compressible, steady or unsteady.

We may also note that Fig. 4.10 demonstrates the *shear-strain rate* of the element, which is defined as the rate of closure of the initially perpendicular lines

$$\dot{\epsilon}_{xy} = \frac{d\alpha}{dt} + \frac{d\beta}{dt} = \frac{\partial \upsilon}{\partial x} + \frac{\partial u}{\partial y} \tag{4.113}$$

When multiplied by viscosity $\mu$, this equals the shear stress $\tau_{xy}$ in a newtonian fluid, as discussed earlier in Eqs. (4.37). Appendix E lists strain-rate and vorticity components in cylindrical coordinates.

## 4.9 Frictionless Irrotational Flows

When a flow is both frictionless and irrotational, pleasant things happen. First, the momentum equation (4.38) reduces to Euler's equation

$$\rho \frac{d\mathbf{V}}{dt} = \rho \mathbf{g} - \boldsymbol{\nabla} p \tag{4.114}$$

Second, there is a great simplification in the acceleration term. Recall from Sec. 4.1 that acceleration has two terms

$$\frac{d\mathbf{V}}{dt} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \boldsymbol{\nabla})\mathbf{V} \tag{4.2}$$

A beautiful vector identity exists for the second term [11]:

$$(\mathbf{V} \cdot \boldsymbol{\nabla})\mathbf{V} \equiv \boldsymbol{\nabla}(\tfrac{1}{2}V^2) + \boldsymbol{\zeta} \times \mathbf{V} \tag{4.115}$$

where $\zeta$ = curl $\mathbf{V}$ from Eq. (4.111) is the fluid vorticity.

Now combine (4.114) and (4.115), divide by $\rho$, and rearrange on the left-hand side. Dot the entire equation into an arbitrary vector displacement $d\mathbf{r}$:

$$\left[\frac{\partial \mathbf{V}}{\partial t} + \boldsymbol{\nabla}\left(\frac{1}{2}V^2\right) + \boldsymbol{\zeta} \times \mathbf{V} + \frac{1}{\rho}\boldsymbol{\nabla} p - \mathbf{g}\right] \cdot d\mathbf{r} = 0 \tag{4.116}$$

Nothing works right unless we can get rid of the third term. We want

$$(\boldsymbol{\zeta} \times \mathbf{V}) \cdot (d\mathbf{r}) \equiv 0 \tag{4.117}$$

This will be true under various conditions:

1. $\mathbf{V}$ is zero; trivial, no flow (hydrostatics).
2. $\boldsymbol{\zeta}$ is zero; irrotational flow.
3. $d\mathbf{r}$ is perpendicular to $\boldsymbol{\zeta} \times \mathbf{V}$; this is rather specialized and rare.
4. $d\mathbf{r}$ is parallel to $\mathbf{V}$; we integrate *along a streamline* (see Sec. 3.7).

Condition 4 is the common assumption. If we integrate along a streamline in frictionless compressible flow and take, for convenience, $\mathbf{g} = -g\mathbf{k}$, Eq. (4.116) reduces to

$$\frac{\partial \mathbf{V}}{\partial t} \cdot d\mathbf{r} + d\left(\frac{1}{2} V^2\right) + \frac{dp}{\rho} + g\, dz = 0 \tag{4.118}$$

Except for the first term, these are exact differentials. Integrate between any two points 1 and 2 along the streamline:

$$\int_1^2 \frac{\partial V}{\partial t}\, ds + \int_1^2 \frac{dp}{\rho} + \frac{1}{2}(V_2^2 - V_1^2) + g(z_2 - z_1) = 0 \tag{4.119}$$

where $ds$ is the arc length along the streamline. Equation (4.119) is Bernoulli's equation for frictionless unsteady flow along a streamline and is identical to Eq. (3.76). For incompressible steady flow, it reduces to

$$\frac{p}{\rho} + \frac{1}{2} V^2 + gz = \text{constant along streamline} \tag{4.120}$$

The constant may vary from streamline to streamline unless the flow is also irrotational (assumption 2). For irrotational flow $\zeta = 0$, the offending term Eq. (4.117) vanishes regardless of the direction of $d\mathbf{r}$, and Eq. (4.120) then holds all over the flow field with the same constant.

## Velocity Potential

Irrotationality gives rise to a scalar function $\phi$ similar and complementary to the stream function $\psi$. From a theorem in vector analysis [11], a vector with zero curl must be the gradient of a scalar function

$$\text{If} \qquad \nabla \times \mathbf{V} \equiv 0 \qquad \text{then} \qquad \mathbf{V} = \nabla\phi \tag{4.121}$$

where $\phi = \phi\,(x, y, z, t)$ is called the *velocity potential function*. Knowledge of $\phi$ thus immediately gives the velocity components

$$u = \frac{\partial \phi}{\partial x} \qquad \upsilon = \frac{\partial \phi}{\partial y} \qquad w = \frac{\partial \phi}{\partial z} \tag{4.122}$$

Lines of constant $\phi$ are called the *potential lines* of the flow.

Note that $\phi$, unlike the stream function, is fully three-dimensional and not limited to two coordinates. It reduces a velocity problem with three unknowns $u$, $\upsilon$, and $w$ to a single unknown potential $\phi$; many examples are given in Chap. 8 and Sec. 4.10. The velocity potential also simplifies the unsteady Bernoulli equation (4.118) because if $\phi$ exists, we obtain

$$\frac{\partial \mathbf{V}}{\partial t} \cdot d\mathbf{r} = \frac{\partial}{\partial t}(\nabla\phi) \cdot d\mathbf{r} = d\left(\frac{\partial \phi}{\partial t}\right) \tag{4.123}$$

Equation (4.118) then becomes a relation between $\phi$ and $p$

$$\frac{\partial \phi}{\partial t} + \int \frac{dp}{\rho} + \frac{1}{2} |\nabla \boldsymbol{\phi}|^2 + gz = \text{const} \tag{4.124}$$

This is the unsteady irrotational Bernoulli equation. It is very important in the analysis of accelerating flow fields (see, e.g., Refs. 10 and 15), but the only application in this text will be in Sec. 9.3 for steady flow.

### Orthogonality of Streamlines and Potential Lines

If a flow is both irrotational and described by only two coordinates, $\psi$ and $\phi$ both exist and the streamlines and potential lines are everywhere mutually perpendicular except at a stagnation point. For example, for incompressible flow in the $xy$ plane, we would have

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} \tag{4.125}$$

$$v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} \tag{4.126}$$

Can you tell by inspection not only that these relations imply orthogonality but also that $\phi$ and $\psi$ satisfy Laplace's equation?[10] A line of constant $\phi$ would be such that the change in $\phi$ is zero

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = 0 = u\, dx + v\, dy \tag{4.127}$$

Solving, we have

$$\left(\frac{dy}{dx}\right)_{\phi=\text{const}} = -\frac{u}{v} = -\frac{1}{(dy/dx)_{\psi=\text{const}}} \tag{4.128}$$

Equation (4.128) is the mathematical condition that lines of constant $\phi$ and $\psi$ be mutually orthogonal. It may not be true at a stagnation point, where both $u$ and $v$ are zero, so that their ratio in Eq. (4.128) is indeterminate.

### Generation of Rotationality

This is the second time we have discussed Bernoulli's equation under different circumstances (the first was in Sec. 3.7). Such reinforcement is useful, since this is probably the most widely used equation in fluid mechanics. It requires frictionless flow with no shaft work or heat transfer between sections 1 and 2. The flow may or may not be irrotational, the latter being an easier condition, allowing a universal Bernoulli constant.

The only remaining question is: *When* is a flow irrotational? In other words, when does a flow have negligible angular velocity? The exact analysis of fluid rotationality under arbitrary conditions is a topic for advanced study, e.g., Ref. 10, sec. 8.5; Ref. 9, sec. 5.2; and Ref. 5, sec. 2.10. We shall simply state those results here without proof.

A fluid flow which is initially irrotational may become rotational if

1. There are significant viscous forces induced by jets, wakes, or solid boundaries. In this case Bernoulli's equation will not be valid in such viscous regions.

[10] Equations (4.125) and (4.126) are called the *Cauchy-Riemann equations* and are studied in complex-variable theory.

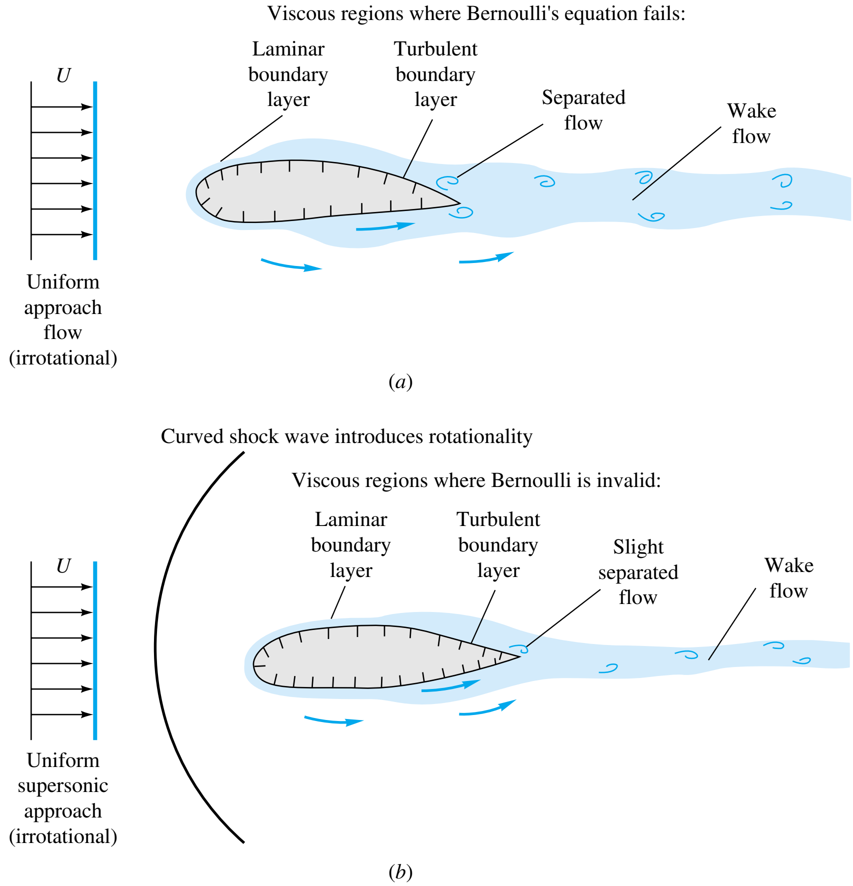



**Fig. 4.11** Typical flow patterns illustrating viscous regions patched onto nearly frictionless regions: (*a*) low subsonic flow past a body ($U \ll a$); frictionless, irrotational potential flow outside the boundary layer (Bernoulli and Laplace equations valid); (*b*) supersonic flow past a body ($U > a$); frictionless, rotational flow outside the boundary layer (Bernoulli equation valid, potential flow invalid).

2. There are entropy gradients caused by curved shock waves (see Fig. 4.11*b*).
3. There are density gradients caused by *stratification* (uneven heating) rather than by pressure gradients.
4. There are significant *noninertial* effects such as the earth's rotation (the Coriolis acceleration).

In cases 2 to 4, Bernoulli's equation still holds along a streamline if friction is negligible. We shall not study cases 3 and 4 in this book. Case 2 will be treated briefly in Chap. 9 on gas dynamics. Primarily we are concerned with case 1, where rotation is induced by viscous stresses. This occurs near solid surfaces, where the no-slip condition creates a boundary layer through which the stream velocity drops to zero, and in jets and wakes, where streams of different velocities meet in a region of high shear.

Internal flows, such as pipes and ducts, are mostly viscous, and the wall layers grow to meet in the core of the duct. Bernoulli's equation does not hold in such flows unless it is modified for viscous losses.

External flows, such as a body immersed in a stream, are partly viscous and partly inviscid, the two regions being patched together at the edge of the shear layer or boundary layer. Two examples are shown in Fig. 4.11. Figure 4.11*a* shows a low-speed subsonic flow past a body. The approach stream is irrotational; i.e., the curl of a constant is zero, but viscous stresses create a rotational shear layer beside and downstream of the body. Generally speaking (see Chap. 6), the shear layer is laminar, or smooth, near the front of the body and turbulent, or disorderly, toward the rear. A separated, or deadwater, region usually occurs near the trailing edge, followed by an unsteady turbulent wake extending far downstream. Some sort of laminar or turbulent viscous theory must be applied to these viscous regions; they are then patched onto the outer flow, which is frictionless and irrotational. If the stream Mach number is less than about 0.3, we can combine Eq. (4.122) with the incompressible continuity equation (4.73).

$$\nabla \cdot \mathbf{V} = \nabla \cdot (\nabla \boldsymbol{\phi}) = 0$$

or

$$\nabla^2 \phi = 0 = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \tag{4.129}$$

This is Laplace's equation in three dimensions, there being no restraint on the number of coordinates in potential flow. A great deal of Chap. 8 will be concerned with solving Eq. (4.129) for practical engineering problems; it holds in the entire region of Fig. 4.11*a* outside the shear layer.

Figure 4.11*b* shows a supersonic flow past a body. A curved shock wave generally forms in front, and the flow downstream is *rotational* due to entropy gradients (case 2). We can use Euler's equation (4.114) in this frictionless region but not potential theory. The shear layers have the same general character as in Fig. 4.11*a* except that the separation zone is slight or often absent and the wake is usually thinner. Theory of separated flow is presently qualitative, but we can make quantitative estimates of laminar and turbulent boundary layers and wakes.

## EXAMPLE 4.9

If a velocity potential exists for the velocity field of Example 4.5

$$u = a(x^2 - y^2) \qquad \upsilon = -2axy \qquad w = 0$$

find it, plot it, and compare with Example 4.7.

### Solution

Since $w = 0$, the curl of $\mathbf{V}$ has only one $z$ component, and we must show that it is zero:

$$(\nabla \times \mathbf{V})_z = 2\omega_z = \frac{\partial \upsilon}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}(-2axy) - \frac{\partial}{\partial y}(ax^2 - ay^2)$$

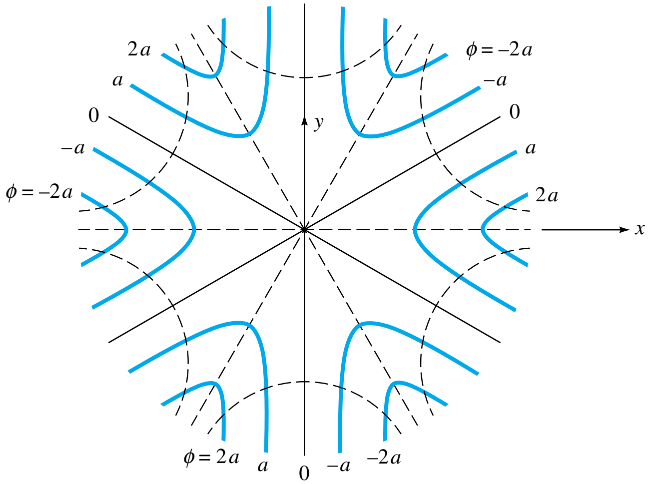
$$= -2ay + 2ay = 0 \qquad \text{checks} \qquad \textit{Ans.}$$

The flow is indeed irrotational. A potential exists.

To find $\phi(x, y)$, set

$$u = \frac{\partial \phi}{\partial x} = ax^2 - ay^2 \tag{1}$$

$$\upsilon = \frac{\partial \phi}{\partial y} = -2axy \tag{2}$$




**E4.9**

Integrate (1)

$$\phi = \frac{ax^3}{3} - axy^2 + f(y) \quad (3)$$

Differentiate (3) and compare with (2)

$$\frac{\partial \phi}{\partial y} = -2axy + f'(y) = -2axy \quad (4)$$

Therefore $f' = 0$, or $f =$ constant. The velocity potential is

$$\phi = \frac{ax^3}{3} - axy^2 + C \qquad \textit{Ans.}$$

Letting $C = 0$, we can plot the $\phi$ lines in the same fashion as in Example 4.7. The result is shown in Fig. E4.9 (no arrows on $\phi$). For this particular problem, the $\phi$ lines form the same pattern as the $\psi$ lines of Example 4.7 (which are shown here as dashed lines) but are displaced 30°. The $\phi$ and $\psi$ lines are everywhere perpendicular except at the origin, a stagnation point, where they are 30° apart. We expected trouble at the stagnation point, and there is no general rule for determining the behavior of the lines at that point.

## 4.10 Some Illustrative Plane Potential Flows

Chapter 8 is devoted entirely to a detailed study of inviscid incompressible flows, especially those which possess both a stream function and a velocity potential. As sketched in Fig. 4.11*a*, inviscid flow is valid away from solid surfaces, and this inviscid pattern is "patched" onto the near-wall viscous layers—an idea developed in Chap. 7. Various body shapes can be simulated by the inviscid-flow pattern. Here we discuss plane flows, three of which are illustrated in Fig. 4.12.

### Uniform Stream in the *x* Direction

A uniform stream $\mathbf{V} = \mathbf{i}U$, as in Fig. 4.12*a*, possesses both a stream function and a velocity potential, which may be found as follows:

$$u = U = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad v = 0 = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

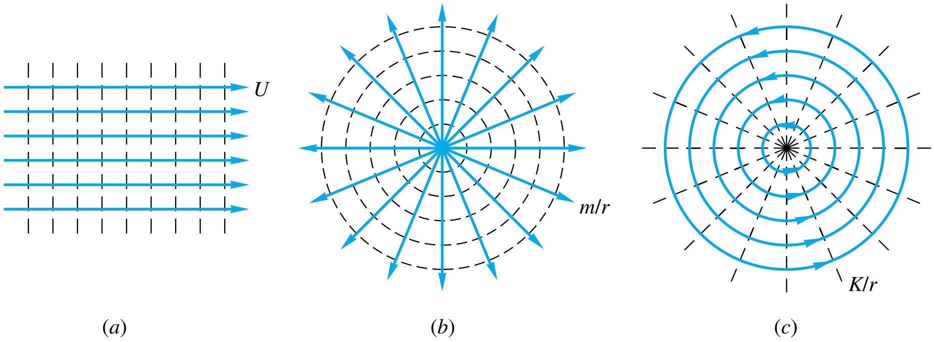



**Fig. 4.12** Three elementary plane potential flows. Solid lines are streamlines; dashed lines are potential lines.

We may integrate each expression and discard the constants of integration, which do not affect the velocities in the flow. The results are

Uniform stream $\mathbf{i}U$: $\quad \psi = Uy \qquad \phi = Ux \qquad$ (4.130)

The streamlines are horizontal straight lines ($y$ = const), and the potential lines are vertical ($x$ = const), i.e., orthogonal to the streamlines, as expected.

**Line Source or Sink at the Origin**

Suppose that the $z$-axis were a sort of thin-pipe manifold through which fluid issued at total rate $Q$ uniformly along its length $b$. Looking at the $xy$ plane, we would see a cylindrical radial outflow or *line source*, as sketched in Fig. 4.12$b$. Plane polar coordinates are appropriate (see Fig. 4.2), and there is no circumferential velocity. At any radius $r$, the velocity is

$$v_r = \frac{Q}{2\pi r b} = \frac{m}{r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r} \qquad v_\theta = 0 = -\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

where we have used the polar-coordinate forms of the stream function and the velocity potential. Integrating and again discarding the constants of integration, we obtain the proper functions for this simple radial flow:

Line source or sink: $\quad \psi = m\theta \qquad \phi = m \ln r \qquad$ (4.131)

where $m = Q/(2\pi b)$ is a constant, positive for a source, negative for a sink. As shown in Fig. 4.12$b$, the streamlines are radial spokes (constant $\theta$), and the potential lines are circles (constant $r$).

**Line Irrotational Vortex**

A (two-dimensional) line vortex is a purely circulating steady motion, $v_\theta = f(r)$ only, $v_r = 0$. This satisfies the continuity equation identically, as may be checked from Eq. (4.12$b$). We may also note that a variety of velocity distributions $v_\theta(r)$ satisfy the $\theta$-momentum equation of a viscous fluid, Eq. (E.6). We may show, as a problem exercise, that only one function $v_\theta(r)$ is *irrotational*, i.e., curl $\mathbf{V} = 0$, and that is $v_\theta = K/r$, where $K$ is a constant. This is sometimes called a *free vortex*, for which the stream function and velocity may be found:

$$v_r = 0 = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r} \qquad v_\theta = \frac{K}{r} = -\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

We may again integrate to determine the appropriate functions:

$$\psi = -K \ln r \qquad \phi = K\theta \tag{4.132}$$

where $K$ is a constant called the *strength* of the vortex. As shown in Fig. 4.12*c*, the streamlines are circles (constant $r$), and the potential lines are radial spokes (constant $\theta$). Note the similarity between Eqs. (4.131) and (4.132). A free vortex is a sort of reversed image of a source. The "bathtub vortex," formed when water drains through a bottom hole in a tank, is a good approximation to the free-vortex pattern.

## Superposition: Source Plus an Equal Sink

Each of the three elementary flow patterns in Fig. 4.12 is an incompressible irrotational flow and therefore satisfies both plane "potential flow" equations $\nabla^2\psi = 0$ and $\nabla^2\phi = 0$. Since these are linear partial differential equations, any *sum* of such basic solutions is also a solution. Some of these composite solutions are quite interesting and useful.

For example, consider a source $+m$ at $(x, y) = (-a, 0)$, combined with a sink of equal strength $-m$, placed at $(+a, 0)$, as in Fig. 4.13. The resulting stream function is simply the sum of the two. In cartesian coordinates,

$$\psi = \psi_{\text{source}} + \psi_{\text{sink}} = m \tan^{-1} \frac{y}{x + a} - m \tan^{-1} \frac{y}{x - a}$$

Similarly, the composite velocity potential is

$$\phi = \phi_{\text{source}} + \phi_{\text{sink}} = \frac{1}{2} m \ln [(x + a)^2 + y^2] - \frac{1}{2} m \ln [(x - a)^2 + y^2]$$

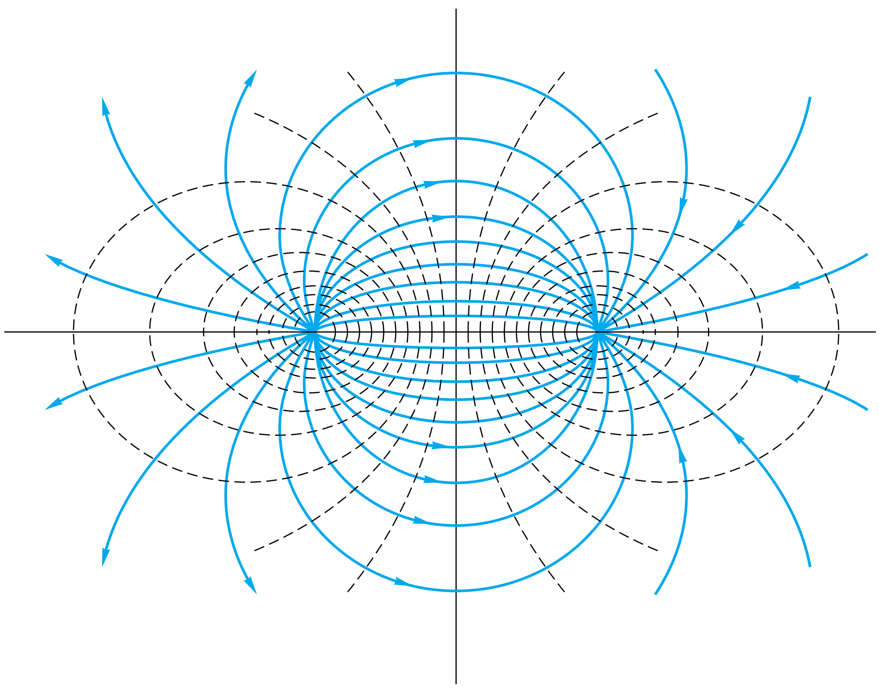


**Fig. 4.13** Potential flow due to a line source plus an equal line sink, from Eq. (4.133). Solid lines are streamlines; dashed lines are potential lines.

By using trigonometric and logarithmic identities, these may be simplified to

$$\text{Source plus sink:} \qquad \psi = -m \tan^{-1} \frac{2ay}{x^2 + y^2 - a^2}$$

$$\phi = \frac{1}{2} m \ln \frac{(x + a)^2 + y^2}{(x - a)^2 + y^2} \qquad (4.133)$$

These lines are plotted in Fig. 4.13 and are seen to be two families of orthogonal circles, with the streamlines passing through the source and sink and the potential lines encircling them. They are harmonic (laplacian) functions which are exactly analogous in electromagnetic theory to the electric-current and electric-potential patterns of a magnet with poles at ($\pm a$, 0).

**Sink Plus a Vortex at the Origin**

An interesting flow pattern, approximated in nature, occurs by superposition of a sink and a vortex, both centered at the origin. The composite stream function and velocity potential are

$$\text{Sink plus vortex:} \qquad \psi = m\theta - K \ln r \qquad \phi = m \ln r + K\theta \qquad (4.134)$$

When plotted, these form two orthogonal families of logarithmic spirals, as shown in Fig. 4.14. This is a fairly realistic simulation of a tornado (where the sink flow moves up the $z$-axis into the atmosphere) or a rapidly draining bathtub vortex. At the center of a real (viscous) vortex, where Eq. (4.134) predicts infinite velocity, the actual circulating flow is highly *rotational* and approximates solid-body rotation $\upsilon_\theta \approx Cr$.

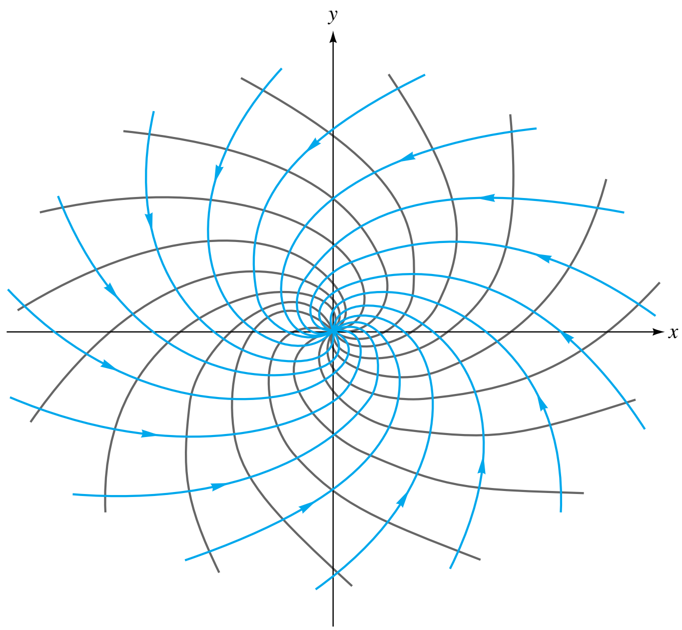



**Fig. 4.14** Superposition of a sink plus a vortex, Eq. (4.134), simulates a tornado.

### Uniform Stream Plus a Sink at the Origin: The Rankine Half-Body

If we superimpose a uniform $x$-directed stream against an isolated source, a half-body shape appears. If the source is at the origin, the combined stream function is, in polar coordinates,

$$\text{Uniform stream plus source:} \qquad \psi = Ur \sin \theta + m\theta \tag{4.135}$$

We can set this equal to various constants and plot the streamlines, as shown in Fig. 4.15. A curved, roughly elliptical, *half-body* shape appears, which separates the source flow from the stream flow. The body shape, which is named after the Scottish engineer W. J. M. Rankine (1820–1872), is formed by the particular streamlines $\psi = \pm\pi m$. The half-width of the body far downstream is $\pi m/U$. The upper surface may be plotted from the relation

$$r = \frac{m(\pi - \theta)}{U \sin \theta} \tag{4.136}$$

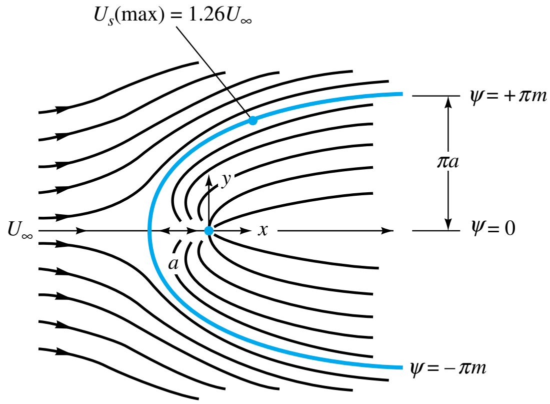



**Fig. 4.15** Superposition of a source plus a uniform stream forms a Rankine half-body.

It is not a true ellipse. The nose of the body, which is a "stagnation" point where $V = 0$, stands at $(x, y) = (-a, 0)$, where $a = m/U$. The streamline $\psi = 0$ also crosses this point—recall that streamlines can cross only at a stagnation point.

The cartesian velocity components are found by differentiation:

$$u = \frac{\partial \psi}{\partial y} = U + \frac{m}{r} \cos \theta \qquad \upsilon = -\frac{\partial \psi}{\partial x} = \frac{m}{r} \sin \theta \tag{4.137}$$

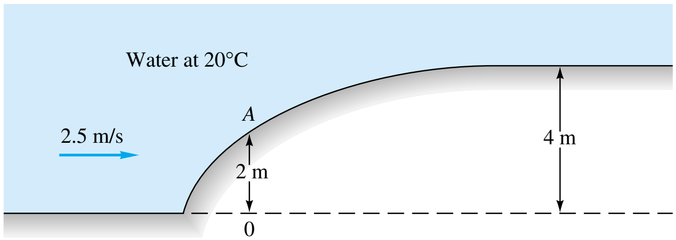
Setting $u = \upsilon = 0$, we find a single stagnation point at $\theta = 180°$ and $r = m/U$, or $(x, y) = (-m/U, 0)$, as stated. The resultant velocity at any point is

$$V^2 = u^2 + \upsilon^2 = U^2\left(1 + \frac{a^2}{r^2} + \frac{2a}{r} \cos \theta\right) \tag{4.138}$$

where we have substituted $m = Ua$. If we evaluate the velocities along the upper surface $\psi = \pi m$, we find a maximum value $U_{s,\max} \approx 1.26U$ at $\theta = 63°$. This point is labeled in Fig. 4.15 and, by Bernoulli's equation, is the point of minimum pressure on the body surface. After this point, the surface flow decelerates, the pressure rises, and the viscous layer grows thicker and more susceptible to "flow separation," as we shall see in Chap. 7.

## EXAMPLE 4.10

The bottom of a river has a 4-m-high bump which approximates a Rankine half-body, as in Fig. E4.10. The pressure at point $B$ on the bottom is 130 kPa, and the river velocity is 2.5 m/s. Use inviscid theory to estimate the water pressure at point $A$ on the bump, which is 2 m above point $B$.




E4.10

## Solution

As in all inviscid theories, we ignore the low-velocity boundary layers which form on solid surfaces due to the no-slip condition. From Eq. (4.136) and Fig. 4.15, the downstream bump half-height equals $\pi a$. Therefore, for our case, $a = (4 \text{ m})/\pi = 1.27$ m. We have to find the spot where the bump height is half that much, $h = 2 \text{ m} = \pi a/2$. From Eq. (4.136) we may compute

$$r = h_A = \frac{a(\pi - \theta)}{\sin \theta} = \frac{\pi}{2} a \qquad \text{or} \qquad \theta = \frac{\pi}{2} = 90°$$

Thus point $A$ in Fig. E4.10 is directly above the (initially unknown) origin of coordinates (labeled $O$ in Fig. E4.10) and is 1.27 m to the right of the nose of the bump. With $r = \pi a/2$ and $\theta = \pi/2$ known, we compute the velocity at point $A$ from Eq. (4.138):

$$V_A^2 = U^2\left[1 + \frac{a^2}{(\pi a/2)^2} + \frac{2a}{\pi a/2} \cos \frac{\pi}{2}\right] = 1.405U^2$$

or

$$V_A \approx 1.185U = 1.185(2.5 \text{ m/s}) = 2.96 \text{ m/s}$$

For water at 20°C, take $\rho = 998$ kg/m$^2$ and $\gamma = 9790$ N/m$^3$. Now, since the velocity and elevation are known at point $A$, we are in a position to use Bernoulli's inviscid, incompressible-flow equation (4.120) to estimate $p_A$ from the known properties at point $B$ (on the same streamline):

$$\frac{p_A}{\gamma} + \frac{V_A^2}{2g} + z_A \approx \frac{p_B}{\gamma} + \frac{V_B^2}{2g} + z_B$$

or

$$\frac{p_A}{9790 \text{ N/m}^3} + \frac{(2.96 \text{ m/s})^2}{2(9.81 \text{ m/s}^2)} + 2 \text{ m} \approx \frac{130{,}000}{9790} + \frac{(2.5)^2}{2(9.81)} + 0$$

Solving, we find

$$p_A = (13.60 - 2.45)(9790) \approx 109{,}200 \text{ Pa} \qquad \textit{Ans.}$$

If the approach velocity is uniform, this should be a pretty good approximation, since water is relatively inviscid and its boundary layers are thin.

## 4.11 Some Illustrative Incompressible Viscous Flows

The inviscid flows of Sec. 4.10 do *not* satisfy the no-slip condition. They "slip" at the wall but do not flow through the wall. To look at fully viscous no-slip conditions, we must attack the complete Navier-Stokes equation (4.74), and the result is usually not at all irrotational, nor does a velocity potential exist. We look here at three cases: (1) flow between parallel plates due to a moving upper wall, (2) flow between parallel plates due to pressure gradient, and (3) flow between concentric cylinders when the inner one rotates. Other cases will be given as problem assignments or considered in Chap. 6. Extensive solutions for viscous flows are discussed in Refs. 4 and 5.

### Couette Flow between a Fixed and a Moving Plate

Consider two-dimensional incompressible plane ($\partial/\partial z = 0$) viscous flow between parallel plates a distance $2h$ apart, as shown in Fig. 4.16. We assume that the plates are very wide and very long, so that the flow is essentially axial, $u \neq 0$ but $v = w = 0$. The present case is Fig. 4.16*a*, where the upper plate moves at velocity $V$ but there is no pressure gradient. Neglect gravity effects. We learn from the continuity equation (4.73) that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 = \frac{\partial u}{\partial x} + 0 + 0 \qquad \text{or} \qquad u = u(y) \text{ only}$$

Thus there is a single nonzero axial-velocity component which varies only across the channel. The flow is said to be *fully developed* (far downstream of the entrance). Substitute $u = u(y)$ into the $x$-component of the Navier-Stokes momentum equation (4.74) for two-dimensional $(x, y)$ flow:

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

or

$$\rho(0 + 0) = 0 + 0 + \mu\left(0 + \frac{d^2 u}{dy^2}\right) \qquad (4.139)$$

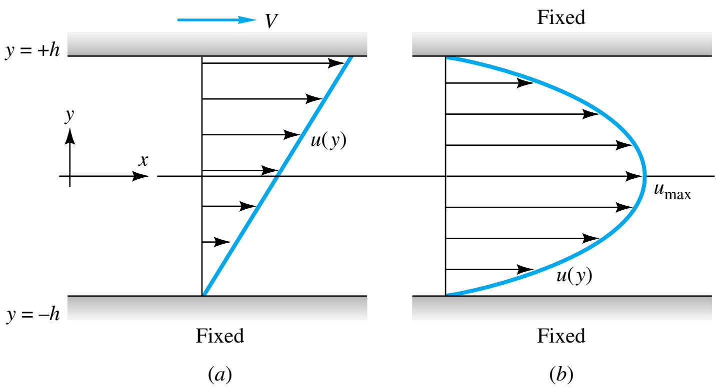



**Fig. 4.16** Incompressible viscous flow between parallel plates: (*a*) no pressure gradient, upper plate moving; (*b*) pressure gradient $\partial p/\partial x$ with both plates fixed.

Most of the terms drop out, and the momentum equation simply reduces to

$$\frac{d^2u}{dy^2} = 0 \qquad \text{or} \qquad u = C_1 y + C_2$$

The two constants are found by applying the no-slip condition at the upper and lower plates:

At $y = +h$: $\quad u = V = C_1 h + C_2$

At $y = -h$: $\quad u = 0 = C_1(-h) + C_2$

or $\quad C_1 = \dfrac{V}{2h} \quad$ and $\quad C_2 = \dfrac{V}{2}$

Therefore the solution for this case ($a$), flow between plates with a moving upper wall, is

$$u = \frac{V}{2h}\, y + \frac{V}{2} \qquad -h \le y \le +h \tag{4.140}$$

This is *Couette flow* due to a moving wall: a linear velocity profile with no-slip at each wall, as anticipated and sketched in Fig. 4.16$a$. Note that the origin has been placed in the center of the channel, for convenience in case ($b$) below.

What we have just presented is a rigorous derivation of the more informally discussed flow of Fig. 1.6 (where $y$ and $h$ were defined differently).

**Flow due to Pressure Gradient between Two Fixed Plates**

Case ($b$) is sketched in Fig. 4.16$b$. Both plates are fixed ($V = 0$), but the pressure varies in the $x$ direction. If $\upsilon = w = 0$, the continuity equation leads to the same conclusion as case ($a$), namely, that $u = u(y)$ only. The $x$-momentum equation (4.138) changes only because the pressure is variable:

$$\mu \frac{d^2u}{dy^2} = \frac{\partial p}{\partial x} \tag{4.141}$$

Also, since $\upsilon = w = 0$ and gravity is neglected, the $y$- and $z$-momentum equations lead to

$$\frac{\partial p}{\partial y} = 0 \qquad \text{and} \qquad \frac{\partial p}{\partial z} = 0 \qquad \text{or} \qquad p = p(x) \text{ only}$$

Thus the pressure gradient in Eq. (4.141) is the total and only gradient:

$$\mu \frac{d^2u}{dy^2} = \frac{dp}{dx} = \text{const} < 0 \tag{4.142}$$

Why did we add the fact that $dp/dx$ is *constant*? Recall a useful conclusion from the theory of separation of variables: If two quantities are equal and one varies only with $y$ and the other varies only with $x$, then they must both equal the same constant. Otherwise they would not be independent of each other.

Why did we state that the constant is *negative*? Physically, the pressure must decrease in the flow direction in order to drive the flow against resisting wall shear stress. Thus the velocity profile $u(y)$ must have negative curvature everywhere, as anticipated and sketched in Fig. 4.16$b$.

The solution to Eq. (4.142) is accomplished by double integration:

$$u = \frac{1}{\mu}\frac{dp}{dx}\frac{y^2}{2} + C_1 y + C_2$$

The constants are found from the no-slip condition at each wall:

At $y = \pm h$: $\quad u = 0 \quad$ or $\quad C_1 = 0 \quad$ and $\quad C_2 = -\dfrac{dp}{dx}\dfrac{h^2}{2\mu}$

Thus the solution to case (*b*), flow in a channel due to pressure gradient, is

$$u = -\frac{dp}{dx}\frac{h^2}{2\mu}\left(1 - \frac{y^2}{h^2}\right) \tag{4.143}$$

The flow forms a *Poiseuille* parabola of constant negative curvature. The maximum velocity occurs at the centerline $y = 0$:

$$u_{\max} = -\frac{dp}{dx}\frac{h^2}{2\mu} \tag{4.144}$$

Other (laminar) flow parameters are computed in the following example.

## EXAMPLE 4.11

For case (*b*) above, flow between parallel plates due to the pressure gradient, compute (*a*) the wall shear stress, (*b*) the stream function, (*c*) the vorticity, (*d*) the velocity potential, and (*e*) the average velocity.

### Solution

All parameters can be computed from the basic solution, Eq. (4.143), by mathematical manipulation.

(*a*) The wall shear follows from the definition of a newtonian fluid, Eq. (4.37):

$$\tau_w = \tau_{xy\ \text{wall}} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\bigg|_{y=\pm h} = \mu\frac{\partial}{\partial y}\left[\left(-\frac{dp}{dx}\right)\left(\frac{h^2}{2\mu}\right)\left(1 - \frac{y^2}{h^2}\right)\right]\bigg|_{y=\pm h}$$

$$= \pm\frac{dp}{dx}h = \mp\frac{2\mu u_{\max}}{h} \qquad \textit{Ans. (a)}$$

The wall shear has the same magnitude at each wall, but by our sign convention of Fig. 4.3, the upper wall has negative shear stress.

(*b*) Since the flow is plane, steady, and incompressible, a stream function exists:

$$u = \frac{\partial \psi}{\partial y} = u_{\max}\left(1 - \frac{y^2}{h^2}\right) \qquad v = -\frac{\partial \psi}{\partial x} = 0$$

Integrating and setting $\psi = 0$ at the centerline for convenience, we obtain

$$\psi = u_{\max}\left(y - \frac{y^3}{3h^2}\right) \qquad \textit{Ans. (b)}$$

At the walls, $y = \pm h$ and $\psi = \pm 2u_{\max}h/3$, respectively.

(*c*) In plane flow, there is only a single nonzero vorticity component:

$$\zeta_z = (\text{curl } \mathbf{V})_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{2u_{\max}}{h^2}\, y \qquad \textit{Ans. (c)}$$

The vorticity is highest at the wall and is positive (counterclockwise) in the upper half and negative (clockwise) in the lower half of the fluid. Viscous flows are typically full of vorticity and are not at all irrotational.

(*d*) From part (*c*), the vorticity is finite. Therefore the flow is not irrotational, and the velocity potential *does not exist*. *Ans. (d)*

(*e*) The average velocity is defined as $V_{av} = Q/A$, where $Q = \int u\, dA$ over the cross section. For our particular distribution $u(y)$ from Eq. (4.143), we obtain

$$V_{av} = \frac{1}{A}\int u\, dA = \frac{1}{b(2h)}\int_{-h}^{+h} u_{\max}\left(1 - \frac{y^2}{h^2}\right) b\, dy = \frac{2}{3}\, u_{\max} \qquad \textit{Ans. (e)}$$

In plane Poiseuille flow between parallel plates, the average velocity is two-thirds of the maximum (or centerline) value. This result could also have been obtained from the stream function derived in part (*b*). From Eq. (4.95),

$$Q_{\text{channel}} = \psi_{\text{upper}} - \psi_{\text{lower}} = \frac{2u_{\max}h}{3} - \left(-\frac{2u_{\max}h}{3}\right) = \frac{4}{3}\, u_{\max}h \text{ per unit width}$$

whence $V_{av} = Q/A_{b=1} = (4u_{\max}h/3)/(2h) = 2u_{\max}/3$, the same result.

This example illustrates a statement made earlier: Knowledge of the velocity vector **V** [as in Eq. (4.143)] is essentially the *solution* to a fluid-mechanics problem, since all other flow properties can then be calculated.

## Flow between Long Concentric Cylinders

Consider a fluid of constant ($\rho$, $\mu$) between two concentric cylinders, as in Fig. 4.17. There is no axial motion or end effect $v_z = \partial/\partial z = 0$. Let the inner cylinder rotate at angular velocity $\Omega_i$. Let the outer cylinder be fixed. There is circular symmetry, so the velocity does not vary with $\theta$ and varies only with $r$.

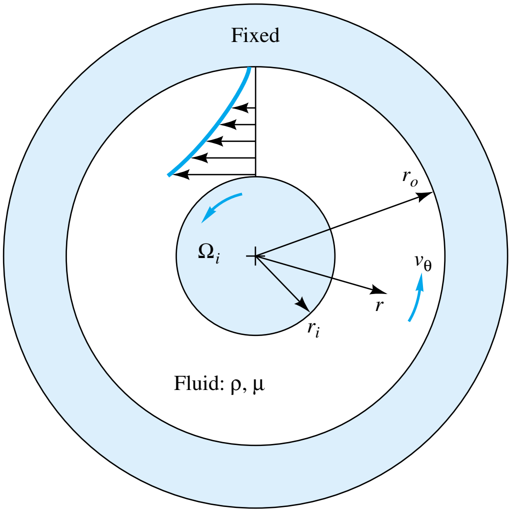



**Fig. 4.17** Coordinate system for incompressible viscous flow between a fixed outer cylinder and a steadily rotating inner cylinder.

The continuity equation for this problem is Eq. (D.2):

$$\frac{1}{r}\frac{\partial}{\partial r}(r\upsilon_r) + \frac{1}{r}\frac{\partial \upsilon_\theta}{\partial \theta} = 0 = \frac{1}{r}\frac{d}{dr}(r\upsilon_r) \qquad \text{or} \qquad r\upsilon_r = \text{const}$$

Note that $\upsilon_\theta$ does not vary with $\theta$. Since $\upsilon_r = 0$ at both the inner and outer cylinders, it follows that $\upsilon_r = 0$ everywhere and the motion can only be purely circumferential, $\upsilon_\theta = \upsilon_\theta(r)$. The $\theta$-momentum equation (D.6) becomes

$$\rho(\mathbf{V}\cdot\nabla)\upsilon_\theta + \frac{\rho\upsilon_r\upsilon_\theta}{r} = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \rho g_\theta + \mu\left(\nabla^2\upsilon_\theta - \frac{\upsilon_\theta}{r^2}\right)$$

For the conditions of the present problem, all terms are zero except the last. Therefore the basic differential equation for flow between rotating cylinders is

$$\nabla^2\upsilon_\theta = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\upsilon_\theta}{dr}\right) = \frac{\upsilon_\theta}{r^2} \qquad (4.145)$$

This is a linear second-order ordinary differential equation with the solution

$$\upsilon_\theta = C_1 r + \frac{C_2}{r}$$

The constants are found by the no-slip condition at the inner and outer cylinders:

Outer, at $r = r_o$:
$$\upsilon_\theta = 0 = C_1 r_o + \frac{C_2}{r_o}$$

Inner, at $r = r_i$:
$$\upsilon_\theta = \Omega_i r_i = C_1 r_i + \frac{C_2}{r_i}$$

The final solution for the velocity distribution is

Rotating inner cylinder:
$$\upsilon_\theta = \Omega_i r_i \frac{r_o/r - r/r_o}{r_o/r_i - r_i/r_o} \qquad (4.146)$$

The velocity profile closely resembles the sketch in Fig. 4.17. Variations of this case, such as a rotating outer cylinder, are given in the problem assignments.

**Instability of Rotating Inner Cylinder Flow**

The classic *Couette-flow* solution[11] of Eq. (4.146) describes a physically satisfying concave, two-dimensional, laminar-flow velocity profile as in Fig. 4.17. The solution is mathematically exact for an incompressible fluid. However, it becomes unstable at a relatively low rate of rotation of the inner cylinder, as shown in 1923 in a classic paper by G. I. Taylor [17]. At a critical value of what is now called the *Taylor number*, denoted Ta,

$$\text{Ta}_{\text{crit}} = \frac{r_i(r_o - r_i)^3\Omega_i^2}{\nu^2} \approx 1700 \qquad (4.147)$$

the plane flow of Fig. 4.17 vanishes and is replaced by a laminar *three-dimensional* flow pattern consisting of rows of nearly square alternating toroidal vortices. An ex-

[11]Named after M. Couette, whose pioneering paper in 1890 established rotating cylinders as a method, still used today, for measuring the viscosity of fluids.

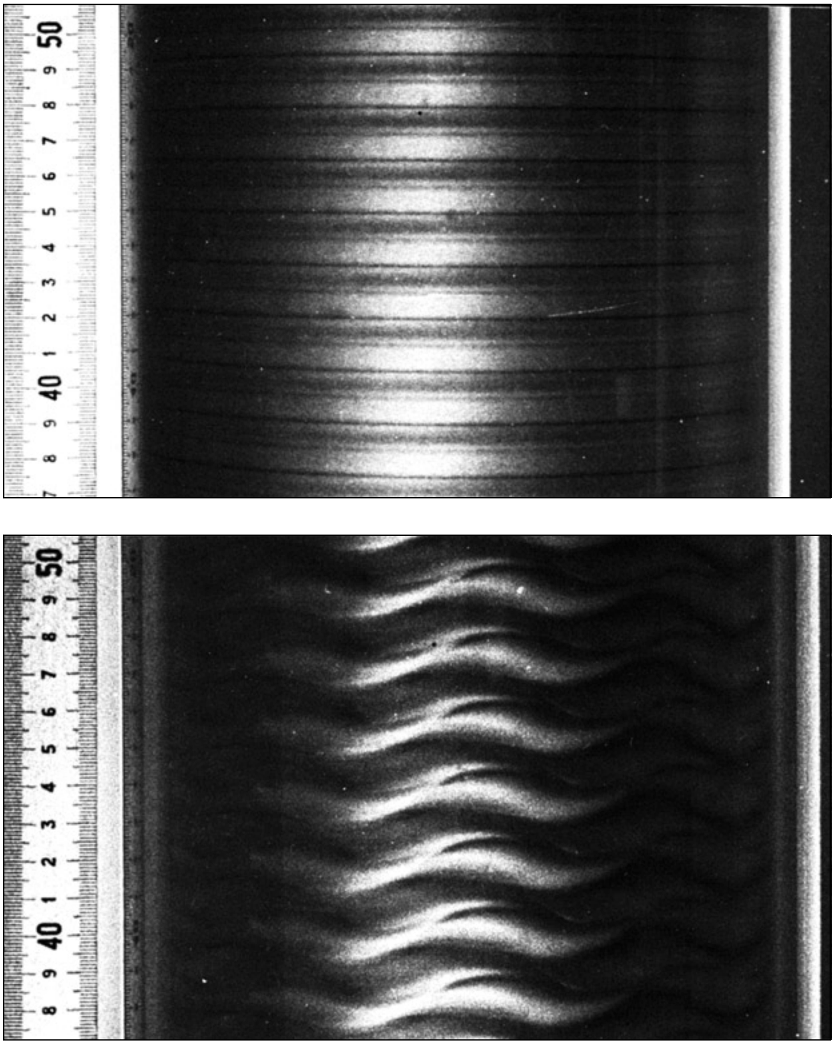



**Fig. 4.18** Experimental verification of the instability of flow between a fixed outer and a rotating inner cylinder. (*a*) Toroidal Taylor vortices exist at 1.16 times the critical speed; (*b*) at 8.5 times the critical speed, the vortices are doubly periodic. (*After Koschmieder, Ref. 18.*) This instability does not occur if only the outer cylinder rotates.

perimental demonstration of toroidal "Taylor vortices" is shown in Fig. 4.18*a*, measured at $\mathrm{Ta} \approx 1.16\ \mathrm{Ta}_{crit}$ by Koschmieder [18]. At higher Taylor numbers, the vortices also develop a circumferential periodicity but are still laminar, as illustrated in Fig. 4.18*b*. At still higher Ta, turbulence ensues. This interesting instability reminds us that the Navier-Stokes equations, being nonlinear, do admit to multiple (nonunique) laminar solutions in addition to the usual instabilities associated with turbulence and chaotic dynamic systems.

## Summary

This chapter complements Chap. 3 by using an infinitesimal control volume to derive the basic partial differential equations of mass, momentum, and energy for a fluid. These equations, together with thermodynamic state relations for the fluid and appropriate boundary conditions, in principle can be solved for the complete flow field in any given fluid-mechanics problem. Except for Chap. 9, in most of the problems to be studied here an incompressible fluid with constant viscosity is assumed.

In addition to deriving the basic equations of mass, momentum, and energy, this chapter introduced some supplementary ideas—the stream function, vorticity, irrotationality, and the velocity potential—which will be useful in coming chapters, especially Chap. 8. Temperature and density variations will be neglected except in Chap. 9, where compressibility is studied.

This chapter ended by discussing a few classical solutions for inviscid flows (uniform stream, source, sink, vortex, half-body) and for viscous flows (Couette flow due to moving walls and Poiseuille flow due to pressure gradient). Whole books [11, 13] are written on the basic equations of fluid mechanics. Whole books [4, 5, 15] are written on classical solutions to fluid-flow problems. Reference 12 contains 360 solved problems which relate fluid mechanics to the whole of continuum mechanics. This does not mean that all problems can be readily solved mathematically, even with the modern digital-computer codes now available. Often the geometry and boundary conditions are so complex that experimentation (Chap. 5) is a necessity.

## Problems

Most of the problems herein are fairly straightforward. More difficult or open-ended assignments are labeled with an asterisk. Problems labeled with an EES icon will benefit from the use of the Engineering Equation Solver (EES), while problems labeled with a computer disk may require the use of a computer. The standard end-of-chapter problems 4.1 to 4.91 (categorized in the problem list below) are followed by word problems W4.1 to W4.10, fundamentals of engineering exam problems FE4.1 to FE4.3, and comprehensive problem C4.1.

**Problem distribution**

| Section | Topic | Problems |
|---|---|---|
| 4.1 | The acceleration of a fluid | 4.1–4.8 |
| 4.2 | The continuity equation | 4.9–4.25 |
| 4.3 | Linear momentum: Navier-Stokes | 4.26–4.37 |
| 4.4 | Angular momentum: couple stresses | 4.38 |
| 4.5 | The differential energy equation | 4.39–4.42 |
| 4.6 | Boundary conditions | 4.43–4.46 |
| 4.7 | Stream function | 4.47–4.55 |
| 4.8 | Vorticity, irrotationality | 4.56–4.60 |
| 4.9 | Velocity potential | 4.61–4.67 |
| 4.10 | Plane potential flows | 4.68–4.78 |
| 4.11 | Incompressible viscous flows | 4.79–4.91 |

**P4.1** An idealized velocity field is given by the formula

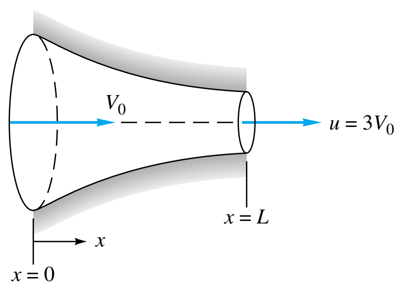
$$\mathbf{V} = 4tx\mathbf{i} - 2t^2y\mathbf{j} + 4xz\mathbf{k}$$

Is this flow field steady or unsteady? Is it two- or three-dimensional? At the point $(x, y, z) = (-1, 1, 0)$, compute (*a*) the acceleration vector and (*b*) any unit vector normal to the acceleration.

**P4.2** Flow through the converging nozzle in Fig. P4.2 can be approximated by the one-dimensional velocity distribution

$$u \approx V_0\left(1 + \frac{2x}{L}\right) \qquad v \approx 0 \qquad w \approx 0$$

(*a*) Find a general expression for the fluid acceleration in the nozzle. (*b*) For the specific case $V_0 = 10$ ft/s and $L = 6$ in, compute the acceleration, in $g$'s, at the entrance and at the exit.




**P4.2**

**P4.3** A two-dimensional velocity field is given by

$$\mathbf{V} = (x^2 - y^2 + x)\mathbf{i} - (2xy + y)\mathbf{j}$$

in arbitrary units. At $(x, y) = (1, 2)$, compute (*a*) the accelerations $a_x$ and $a_y$, (*b*) the velocity component in the direction $\theta = 40°$, (*c*) the direction of maximum velocity, and (*d*) the direction of maximum acceleration.

**P4.4** Suppose that the temperature field $T = 4x^2 - 3y^3$, in arbitrary units, is associated with the velocity field of Prob. 4.3. Compute the rate of change $dT/dt$ at $(x, y) = (2, 1)$.

**P4.5** The velocity field near a stagnation point (see Example 1.10) may be written in the form

$$u = \frac{U_0 x}{L} \qquad v = -\frac{U_0 y}{L} \qquad U_0 \text{ and } L \text{ are constants}$$

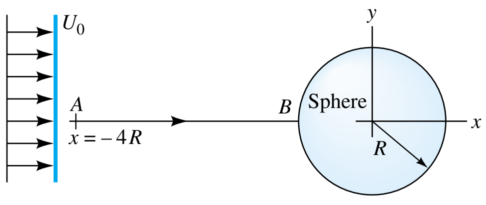
(*a*) Show that the acceleration vector is purely radial. (*b*) For the particular case $L = 1.5$ m, if the acceleration at $(x, y) = (1$ m, $1$ m$)$ is 25 m/s$^2$, what is the value of $U_0$?

**P4.6** Assume that flow in the converging nozzle of Fig. P4.2 has the form $\mathbf{V} = V_0[1 + (2x)/L]\mathbf{i}$. Compute (*a*) the fluid acceleration at $x = L$ and (*b*) the time required for a fluid particle to travel from $x = 0$ to $x = L$.

**P4.7** Consider a sphere of radius $R$ immersed in a uniform stream $U_0$, as shown in Fig. P4.7. According to the theory of Chap. 8, the fluid velocity along streamline $AB$ is given by

$$\mathbf{V} = u\mathbf{i} = U_0\left(1 + \frac{R^3}{x^3}\right)\mathbf{i}$$

Find (*a*) the position of maximum fluid acceleration along $AB$ and (*b*) the time required for a fluid particle to travel from $A$ to $B$.

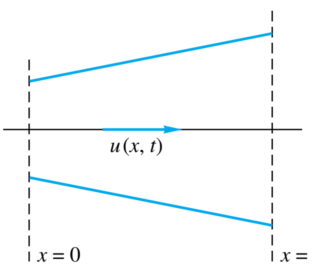



**P4.7**

**P4.8** When a valve is opened, fluid flows in the expansion duct of Fig. 4.8 according to the approximation

$$\mathbf{V} = \mathbf{i}U\left(1 - \frac{x}{2L}\right)\tanh\frac{Ut}{L}$$

Find (*a*) the fluid acceleration at $(x, t) = (L, L/U)$ and (*b*) the time for which the fluid acceleration at $x = L$ is zero. Why does the fluid acceleration become negative after condition (*b*)?




**P4.8**

**P4.9** A velocity field is given by $\mathbf{V} = (3y^2 - 3x^2)\mathbf{i} + Cxy\mathbf{j} + 0\mathbf{k}$. Determine the value of the constant $C$ if the flow is to be (*a*) incompressible and (*b*) irrotational.

**P4.10** Write the special cases of the equation of continuity for (*a*) steady compressible flow in the $yz$ plane, (*b*) unsteady incompressible flow in the $xz$ plane, (*c*) unsteady compressible flow in the $y$ direction only, (*d*) steady compressible flow in plane polar coordinates.

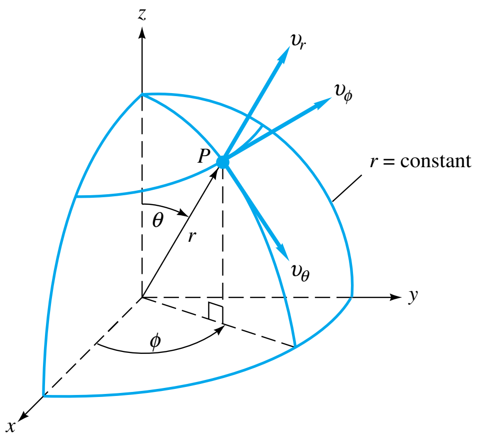
**P4.11** Derive Eq. (4.12*b*) for cylindrical coordinates by considering the flux of an incompressible fluid in and out of the elemental control volume in Fig. 4.2.

**P4.12** Spherical polar coordinates $(r, \theta, \phi)$ are defined in Fig. P4.12. The cartesian transformations are

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$




**P4.12**

The cartesian incompressible continuity relation (4.12*a*) can be transformed to the spherical polar form

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(v_\phi) = 0$$

What is the most general form of $v_r$ when the flow is purely radial, that is, $v_\theta$ and $v_\phi$ are zero?

**P4.13** A two-dimensional velocity field is given by

$$u = -\frac{Ky}{x^2 + y^2} \qquad v = \frac{Kx}{x^2 + y^2}$$

where $K$ is constant. Does this field satisfy incompressible continuity? Transform these velocities to polar components $\upsilon_r$ and $\upsilon_\theta$. What might the flow represent?

**P4.14** For incompressible polar-coordinate flow, what is the most general form of a purely circulatory motion, $\upsilon_\theta = \upsilon_\theta(r, \theta, t)$ and $\upsilon_r = 0$, which satisfies continuity?

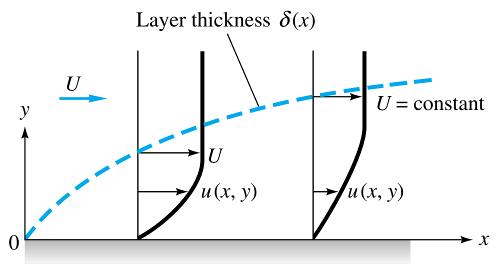
**P4.15** What is the most general form of a purely radial polar-coordinate incompressible-flow pattern, $\upsilon_r = \upsilon_r(r, \theta, t)$ and $\upsilon_\theta = 0$, which satisfies continuity?

**P4.16** An incompressible steady-flow pattern is given by $u = x^3 + 2z^2$ and $w = y^3 - 2yz$. What is the most general form of the third component, $\upsilon(x, y, z)$, which satisfies continuity?

**P4.17** A reasonable approximation for the two-dimensional incompressible laminar boundary layer on the flat surface in Fig. P4.17 is

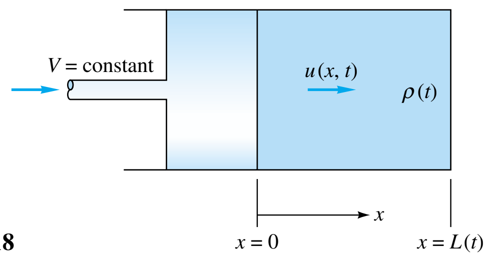
$$u = U\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right) \qquad \text{for } y \le \delta \qquad \text{where } \delta = Cx^{1/2},\ C = \text{const}$$

(*a*) Assuming a no-slip condition at the wall, find an expression for the velocity component $\upsilon(x, y)$ for $y \le \delta$. (*b*) Then find the maximum value of $\upsilon$ at the station $x = 1$ m, for the particular case of airflow, when $U = 3$ m/s and $\delta = 1.1$ cm.




**P4.17**

**P4.18** A piston compresses gas in a cylinder by moving at constant speed $V$, as in Fig. P4.18. Let the gas density and length at $t = 0$ be $\rho_0$ and $L_0$, respectively. Let the gas velocity vary linearly from $u = V$ at the piston face to $u = 0$ at $x = L$. If the gas density varies only with time, find an expression for $\rho(t)$.

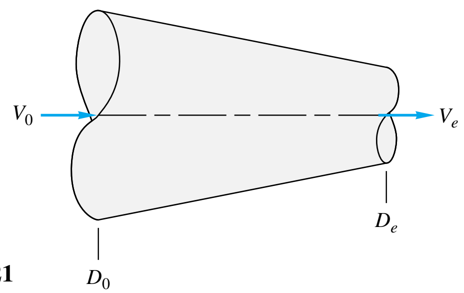



**P4.18**

**P4.19** An incompressible flow field has the cylindrical components $\upsilon_\theta = Cr$, $\upsilon_z = K(R^2 - r^2)$, $\upsilon_r = 0$, where $C$ and $K$ are constants and $r \le R$, $z \le L$. Does this flow satisfy continuity? What might it represent physically?

**P4.20** A two-dimensional incompressible velocity field has $u = K(1 - e^{-ay})$, for $x \le L$ and $0 \le y \le \infty$. What is the most general form of $\upsilon(x, y)$ for which continuity is satisfied and $\upsilon = \upsilon_0$ at $y = 0$? What are the proper dimensions for constants $K$ and $a$?

**P4.21** Air flows under steady, approximately one-dimensional conditions through the conical nozzle in Fig. P4.21. If the speed of sound is approximately 340 m/s, what is the minimum nozzle-diameter ratio $D_e/D_0$ for which we can safely neglect compressibility effects if $V_0 =$ (*a*) 10 m/s and (*b*) 30 m/s?




**P4.21**

**P4.22** Air at a certain temperature and pressure flows through a contracting nozzle of length $L$ whose area decreases linearly, $A \approx A_0[1 - x/(2L)]$. The air average velocity increases nearly linearly from 76 m/s at $x = 0$ to 167 m/s at $x = L$. If the density at $x = 0$ is 2.0 kg/m$^3$, estimate the density at $x = L$.

**P4.23** A tank volume $\mathcal{V}$ contains gas at conditions ($\rho_0$, $p_0$, $T_0$). At time $t = 0$ it is punctured by a small hole of area $A$. According to the theory of Chap. 9, the mass flow out of such a hole is approximately proportional to $A$ and to the tank pressure. If the tank temperature is assumed constant and the gas is ideal, find an expression for the variation of density within the tank.

***P4.24** Reconsider Fig. P4.17 in the following general way. It is known that the boundary layer thickness $\delta(x)$ increases monotonically and that there is no slip at the wall ($y = 0$). Further, $u(x, y)$ merges smoothly with the outer stream flow, where $u \approx U =$ constant outside the layer. Use these facts to prove that (*a*) the component $\upsilon(x, y)$ is positive everywhere within the layer, (*b*) $\upsilon$ increases parabolically with $y$ very near the wall, and (*c*) $\upsilon$ is a maximum at $y = \delta$.

**P4.25** An incompressible flow in polar coordinates is given by

$$\upsilon_r = K\cos\theta\left(1 - \frac{b}{r^2}\right)$$

$$\upsilon_\theta = -K\sin\theta\left(1 + \frac{b}{r^2}\right)$$

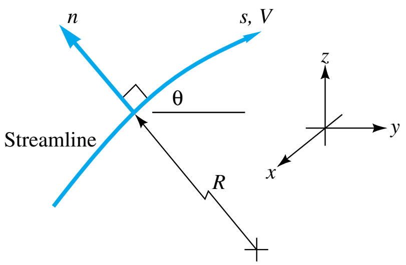
Does this field satisfy continuity? For consistency, what should the dimensions of constants $K$ and $b$ be? Sketch the surface where $\upsilon_r = 0$ and interpret.

***P4.26** Curvilinear, or streamline, coordinates are defined in Fig. P4.26, where $n$ is normal to the streamline in the plane of the radius of curvature $R$. Show that Euler's frictionless momentum equation (4.36) in streamline coordinates becomes

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial s} = -\frac{1}{\rho}\frac{\partial p}{\partial s} + g_s \qquad (1)$$

$$-V\frac{\partial \theta}{\partial t} - \frac{V^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial n} + g_n \qquad (2)$$

Further show that the integral of Eq. (1) with respect to $s$ is none other than our old friend Bernoulli's equation (3.76).




**P4.26**

**P4.27** A frictionless, incompressible steady-flow field is given by

$$\mathbf{V} = 2xy\mathbf{i} - y^2\mathbf{j}$$

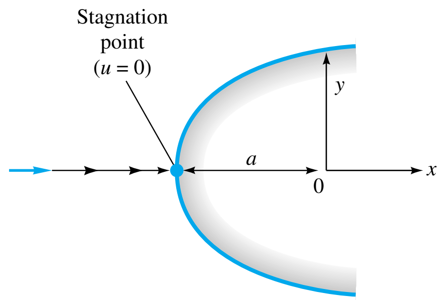
in arbitrary units. Let the density be $\rho_0$ = constant and neglect gravity. Find an expression for the pressure gradient in the $x$ direction.

**P4.28** If $z$ is "up," what are the conditions on constants $a$ and $b$ for which the velocity field $u = ay$, $\upsilon = bx$, $w = 0$ is an exact solution to the continuity and Navier-Stokes equations for incompressible flow?

**P4.29** Consider a steady, two-dimensional, incompressible flow of a newtonian fluid in which the velocity field is known, i.e., $u = -2xy$, $\upsilon = y^2 - x^2$, $w = 0$. (*a*) Does this flow satisfy conservation of mass? (*b*) Find the pressure field, $p(x, y)$ if the pressure at the point $(x = 0, y = 0)$ is equal to $p_a$.

**P4.30** Show that the two-dimensional flow field of Example 1.10 is an exact solution to the incompressible Navier-Stokes equations (4.38). Neglecting gravity, compute the pressure field $p(x, y)$ and relate it to the absolute velocity $V^2 = u^2 + \upsilon^2$. Interpret the result.

**P4.31** EES According to potential theory (Chap. 8) for the flow approaching a rounded two-dimensional body, as in Fig. P4.31, the velocity approaching the stagnation point is given by $u = U(1 - a^2/x^2)$, where $a$ is the nose radius and $U$ is the velocity far upstream. Compute the value and position of the maximum viscous normal stress along this streamline.




**P4.31**

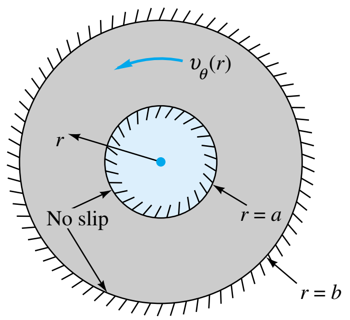
Is this also the position of maximum fluid deceleration? Evaluate the maximum viscous normal stress if the fluid is SAE 30 oil at 20°C, with $U$ = 2 m/s and $a$ = 6 cm.

**P4.32** The answer to Prob. 4.14 is $\upsilon_\theta = \mathrm{f}(r)$ only. Do not reveal this to your friends if they are still working on Prob. 4.14. Show that this flow field is an exact solution to the Navier-Stokes equations (4.38) for only two special cases of the function $\mathrm{f}(r)$. Neglect gravity. Interpret these two cases physically.

**P4.33** From Prob. 4.15 the purely radial polar-coordinate flow which satisfies continuity is $\upsilon_r = f(\theta)/r$, where $f$ is an arbitrary function. Determine what particular forms of $f(\theta)$ satisfy the full Navier-Stokes equations in polar-coordinate form from Eqs. (D.5) and (D.6).

**P4.34** The fully developed laminar-pipe-flow solution of Prob. 3.53, $\upsilon_z = u_{max}(1 - r^2/R^2)$, $\upsilon_\theta = 0$, $\upsilon_r = 0$, is an exact solution to the cylindrical Navier-Stokes equations (App. D). Neglecting gravity, compute the pressure distribution in the pipe $p(r, z)$ and the shear-stress distribution $\tau(r, z)$, using $R$, $u_{max}$, and $\mu$ as parameters. Why does the maximum shear occur at the wall? Why does the density not appear as a parameter?

**P4.35** From the Navier-Stokes equations for incompressible flow in polar coordinates (App. D for cylindrical coordinates), find the most general case of purely circulating motion $\upsilon_\theta(r)$, $\upsilon_r = \upsilon_z = 0$, for flow with no slip between two fixed concentric cylinders, as in Fig. P4.35.

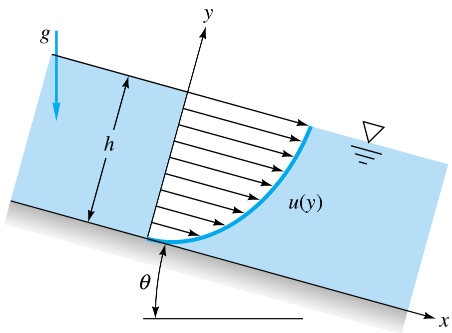



**P4.35**

**P4.36** A constant-thickness film of viscous liquid flows in laminar motion down a plate inclined at angle $\theta$, as in Fig. P4.36. The velocity profile is

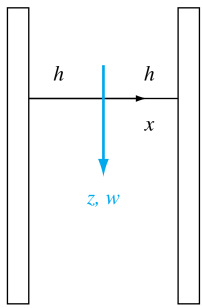
$$u = Cy(2h - y) \qquad v = w = 0$$

Find the constant $C$ in terms of the specific weight and viscosity and the angle $\theta$. Find the volume flux $Q$ per unit width in terms of these parameters.




**P4.36**

***P4.37** A viscous liquid of constant $\rho$ and $\mu$ falls due to gravity between two plates a distance $2h$ apart, as in Fig. P4.37. The flow is fully developed, with a single velocity component $w = w(x)$. There are no applied pressure gradients, only gravity. Solve the Navier-Stokes equation for the velocity profile between the plates.

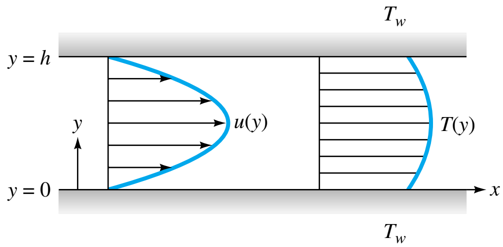



**P4.37**

**P4.38** Reconsider the angular-momentum balance of Fig. 4.5 by adding a concentrated *body couple* $C_z$ about the $z$ axis [6]. Determine a relation between the body couple and shear stress for equilibrium. What are the proper dimensions for $C_z$? (Body couples are important in continuous media with microstructure, such as granular materials.)

**P4.39** Problems involving viscous dissipation of energy are dependent on viscosity $\mu$, thermal conductivity $k$, stream velocity $U_0$, and stream temperature $T_0$. Group these parameters into the dimensionless *Brinkman number*, which is proportional to $\mu$.

**P4.40** As mentioned in Sec. 4.11, the velocity profile for laminar flow between two plates, as in Fig. P4.40, is




**P4.40**

$$u = \frac{4u_{\max}y(h - y)}{h^2} \qquad v = w = 0$$

If the wall temperature is $T_w$ at both walls, use the incompressible-flow energy equation (4.75) to solve for the temperature distribution $T(y)$ between the walls for steady flow.

***P4.41** The approximate velocity profile in Prob. 3.18 and Fig. P3.18 for steady laminar flow through a duct, was suggested as

$$u = u_{\max}\left(1 - \frac{y^2}{b^2}\right)\left(1 - \frac{z^2}{h^2}\right)$$

With $v = w = 0$, it satisfied the no-slip condition and gave a reasonable volume-flow estimate (which was the point of Prob. 3.18). Show, however, that it does *not* satsify the $x$-momentum Navier-Stokes equation for duct flow with constant pressure gradient $\partial p/\partial x < 0$. For extra credit, explain briefly how the actual exact solution to this problem is obtained (see, for example, Ref. 5, pp. 120–121).

**P4.42** In duct-flow problems with heat transfer, one often defines an average fluid temperature. Consider the duct flow of Fig. P4.40 of width $b$ into the paper. Using a control-volume integral analysis with constant density and specific heat, derive an expression for the temperature arising if the entire duct flow poured into a bucket and was stirred uniformly. Assume arbitrary $u(y)$ and $T(y)$. This average is called the *cup-mixing temperature* of the flow.

**P4.43** For the draining liquid film of Fig. P4.36, what are the appropriate boundary conditions ($a$) at the bottom $y = 0$ and ($b$) at the surface $y = h$?

**P4.44** Suppose that we wish to analyze the sudden pipe-expansion flow of Fig. P3.59, using the full continuity and Navier-Stokes equations. What are the proper boundary conditions to handle this problem?

**P4.45** Suppose that we wish to analyze the U-tube oscillation flow of Fig. P3.96, using the full continuity and Navier-Stokes equations. What are the proper boundary conditions to handle this problem?

**P4.46** Fluid from a large reservoir at temperature $T_0$ flows into a circular pipe of radius $R$. The pipe walls are wound with an electric-resistance coil which delivers heat to the fluid at a rate $q_w$ (energy per unit wall area). If we wish to analyze this problem by using the full continuity, Navier-Stokes, and energy equations, what are the proper boundary conditions for the analysis?

**P4.47** A two-dimensional incompressible flow is given by the velocity field $\mathbf{V} = 3y\mathbf{i} + 2x\mathbf{j}$, in arbitrary units. Does this flow satisfy continuity? If so, find the stream function $\psi(x, y)$ and plot a few streamlines, with arrows.

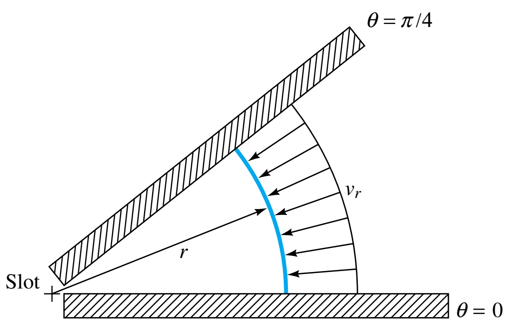
**P4.48** Determine the incompressible two-dimensional stream function $\psi(x, y)$ which represents the flow field given in Example 1.10.

**P4.49** Investigate the stream function $\psi = K(x^2 - y^2)$, $K$ = constant. Plot the streamlines in the full $xy$ plane, find any stagnation points, and interpret what the flow could represent.

**P4.50** Investigate the polar-coordinate stream function $\psi = Kr^{1/2} \sin \frac{1}{2}\theta$, $K$ = constant. Plot the streamlines in the full $xy$ plane, find any stagnation points, and interpret.

**P4.51** Investigate the polar-coordinate stream function $\psi = Kr^{2/3} \sin (2\theta/3)$, $K$ = constant. Plot the streamlines in all except the bottom right quadrant, and interpret.

**P4.52** A two-dimensional, incompressible, frictionless fluid is guided by wedge-shaped walls into a small slot at the origin, as in Fig. P4.52. The width into the paper is $b$, and the volume flow rate is $Q$. At any given distance $r$ from the slot, the flow is radial inward, with constant velocity. Find an expression for the polar-coordinate stream function of this flow.




**P4.52**

**P4.53** For the fully developed laminar-pipe-flow solution of Prob. 4.34, find the axisymmetric stream function $\psi(r, z)$. Use this result to determine the average velocity $V = Q/A$ in the pipe as a ratio of $u_{max}$.

**P4.54** An incompressible stream function is defined by

$$\psi(x, y) = \frac{U}{L^2}(3x^2y - y^3)$$

where $U$ and $L$ are (positive) constants. Where in this chapter are the streamlines of this flow plotted? Use this stream function to find the volume flow $Q$ passing through the rectangular surface whose corners are defined by $(x, y, z)$ = $(2L, 0, 0)$, $(2L, 0, b)$, $(0, L, b)$, and $(0, L, 0)$. Show the direction of $Q$.

***P4.55** In spherical polar coordinates, as in Fig. P4.12, the flow is called *axisymmetric* if $v_\phi \equiv 0$ and $\partial/\partial\phi \equiv 0$, so that $v_r = v_r(r, \theta)$ and $v_\theta = v_\theta(r, \theta)$. Show that a stream function $\psi(r, \theta)$ exists for this case and is given by

$$v_r = \frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta} \qquad v_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

This is called the *Stokes stream function* [5, p. 204].

**P4.56** Investigate the velocity potential $\phi = Kxy$, $K$ = constant. Sketch the potential lines in the full $xy$ plane, find any stagnation points, and sketch in by eye the orthogonal streamlines. What could the flow represent?

**P4.57** Determine the incompressible two-dimensional velocity potential $\phi(x, y)$ which represents the flow field given in Example 1.10. Sketch a few potential lines and streamlines.

**P4.58** Show that the incompressible velocity potential in plane polar coordinates $\phi(r, \theta)$ is such that

$$v_r = \frac{\partial\phi}{\partial r} \qquad v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

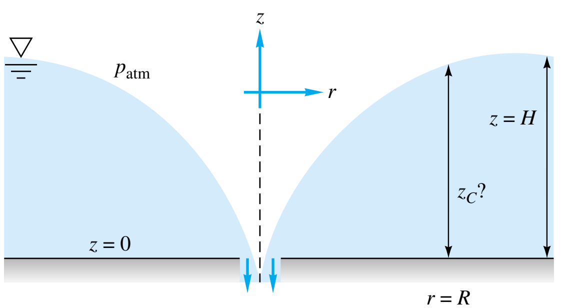
Further show that the angular velocity about the $z$-axis in such a flow would be given by

$$2\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) - \frac{1}{r}\frac{\partial}{\partial\theta}(v_r)$$

Finally show that $\phi$ as defined above satisfies Laplace's equation in polar coordinates for incompressible flow.

**P4.59** Consider the simple flow defined by $\mathbf{V} = x\mathbf{i} - y\mathbf{j}$, in arbitrary units. At $t = 0$, consider the rectangular fluid element defined by the lines $x = 2$, $x = 3$ and $y = 2$, $y = 3$. Determine, and draw to scale, the location of this fluid element at $t = 0.5$ unit. Relate this new element shape to whether the flow is irrotational or incompressible.

**P4.60** Liquid drains from a small hole in a tank, as shown in Fig. P4.60, such that the velocity field set up is given by $v_r \approx 0$, $v_z \approx 0$, $v_\theta = \omega R^2/r$, where $z = H$ is the depth of the water




**P4.60**

far from the hole. Is this flow pattern rotational or irrotational? Find the depth $z_C$ of the water at the radius $r = R$.

**P4.61** Investigate the polar-coordinate velocity potential $\phi = Kr^{1/2}\cos\frac{1}{2}\theta$, $K$ = constant. Plot the potential lines in the full $xy$ plane, sketch in by eye the orthogonal streamlines, and interpret.

**P4.62** Show that the linear Couette flow between plates in Fig. 1.6 has a stream function but no velocity potential. Why is this so?

**P4.63** Find the two-dimensional velocity potential $\phi(r, \theta)$ for the polar-coordinate flow pattern $\upsilon_r = Q/r$, $\upsilon_\theta = K/r$, where $Q$ and $K$ are constants.

**P4.64** Show that the velocity potential $\phi(r, z)$ in axisymmetric cylindrical coordinates (see Fig. 4.2) is defined such that

$$\upsilon_r = \frac{\partial \phi}{\partial r} \qquad \upsilon_z = \frac{\partial \phi}{\partial z}$$

Further show that for incompressible flow this potential satisfies Laplace's equation in $(r, z)$ coordinates.

**P4.65** A two-dimensional incompressible flow is defined by

$$u = -\frac{Ky}{x^2 + y^2} \qquad \upsilon = \frac{Kx}{x^2 + y^2}$$

where $K$ = constant. Is this flow irrotational? If so, find its velocity potential, sketch a few potential lines, and interpret the flow pattern.

**P4.66** A plane polar-coordinate velocity potential is defined by

$$\phi = \frac{K\cos\theta}{r} \qquad K = \text{const}$$

Find the stream function for this flow, sketch some streamlines and potential lines, and interpret the flow pattern.

**P4.67** A stream function for a plane, irrotational, polar-coordinate flow is

$$\psi = C\theta - K\ln r \qquad C \text{ and } K = \text{const}$$

Find the velocity potential for this flow. Sketch some streamlines and potential lines, and interpret the flow pattern.

**P4.68** Find the stream function and plot some streamlines for the combination of a line source $m$ at $(x, y) = (0, +a)$ and an equal line source placed at $(0, -a)$.

**P4.69** Find the stream function and plot some streamlines for the combination of a counterclockwise line vortex $K$ at $(x, y) = (+a, 0)$ and an equal line vortex placed at $(-a, 0)$.

***P4.70** Superposition of a source of strength $m$ at $(-a, 0)$ and a sink (source of strength $-m$) at $(a, 0)$ was discussed briefly in this chapter, where it was shown that the velocity potential function is

$$\phi = \frac{1}{2}m \cdot \ln\frac{(x + a)^2 + y^2}{(x - a)^2 + y^2}$$

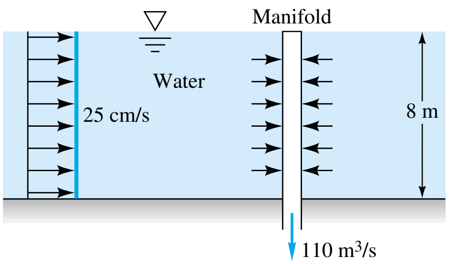
A *doublet* is formed in the limit as $a$ goes to zero (the source and sink come together) while at the same time their strengths $m$ and $-m$ go to infinity and minus infinity, respectively, with the product $a \cdot m$ remaining constant. ($a$) Find the limiting value of velocity potential for the doublet. *Hint*: Expand the natural logarithm as an infinite series of the form

$$\ln\frac{1 + \epsilon}{1 - \epsilon} = 2\left(\epsilon + \frac{\epsilon^3}{3} + \cdots\right)$$

as $\epsilon$ goes to zero. ($b$) Rewrite your result for $\phi_{\text{doublet}}$ in cylindrical coordinates.

**P4.71** Find the stream function and plot some streamlines for the combination of a counterclockwise line vortex $K$ at $(x, y) = (+a, 0)$ and an opposite (clockwise) line vortex placed at $(-a, 0)$.

**P4.72** A coastal power plant takes in cooling water through a vertical perforated manifold, as in Fig. P4.72. The total volume flow intake is 110 m$^3$/s. Currents of 25 cm/s flow past the manifold, as shown. Estimate ($a$) how far downstream and ($b$) how far normal to the paper the effects of the intake are felt in the ambient 8-m-deep waters.



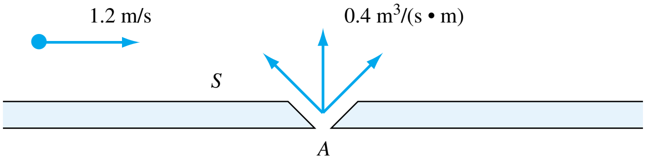

**P4.72**

**P4.73** A two-dimensional Rankine half-body, 8 cm thick, is placed in a water tunnel at 20°C. The water pressure far upstream along the body centerline is 120 kPa. What is the nose radius of the half-body? At what tunnel flow velocity will cavitation bubbles begin to form on the surface of the body?

**P4.74** Find the stream function and plot some streamlines for the combination of a uniform stream $\mathbf{i}U$ and a clockwise line vortex $-K$ at the origin. Are there any stagnation points in the flow field?

***P4.75** Find the stream function and plot some streamlines for the combination of a line source $2m$ at $(x, y) = (+a, 0)$ and a line source $m$ at $(-a, 0)$. Are there any stagnation points in the flow field?

**P4.76** Air flows at 1.2 m/s along a flat surface when it encounters a jet of air issuing from the horizontal wall at point $A$, as in Fig. 4.76. The jet volume flow is 0.4 m$^3$/s per unit depth into the paper. If the jet is approximated as an inviscid line source, ($a$) locate the stagnation point $S$ on the wall. ($b$) How far vertically will the jet flow extend into the stream?



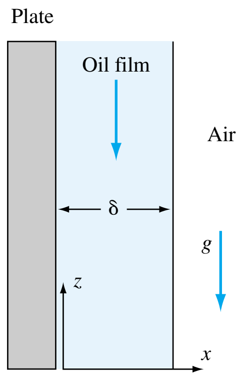

**P4.76**

**P4.77** A tornado is simulated by a line sink $m = -1000$ m$^2$/s plus a line vortex $K = +1600$ m$^2$/s. Find the angle between any streamline and a radial line, and show that it is independent of both $r$ and $\theta$. If this tornado forms in sea-level standard air, at what radius will the local pressure be equivalent to 29 inHg?

**P4.78** The solution to Prob. 4.68 (do not reveal!) can represent a line source $m$ at $(0, +a)$ near a horizontal wall ($y = 0$). [The other source at $(0, -a)$ represents an "image" to create the wall.] Find ($a$) the magnitude of the maxinun flow velocity along the wall and ($b$) the point of minimum pressure along the wall. *Hint*: Use Bernoulli's equation.

***P4.79** Study the combined effect of the two viscous flows in Fig. 4.16. That is, find $u(y)$ when the upper plate moves at speed $V$ and there is also a constant pressure gradient ($dp/dx$). Is superposition possible? If so, explain why. Plot representative velocity profiles for ($a$) zero, ($b$) positive, and ($c$) negative pressure gradients for the same upper-wall speed $V$.

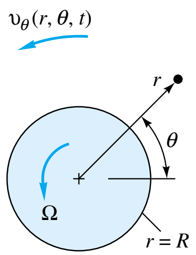
***P4.80** Oil, of density $\rho$ and viscosity $\mu$, drains steadily down the side of a vertical plate, as in Fig. P4.80. After a development region near the top of the plate, the oil film will become independent of $z$ and of constant thickness $\delta$. Assume that $w = w(x)$ only and that the atmosphere offers no shear resistance to the surface of the film. ($a$) Solve the Navier-Stokes equation for $w(x)$, and sketch its approximate shape. ($b$) Suppose that film thickness $\delta$ and the slope of the velocity profile at the wall $[\partial w/\partial x]_{\text{wall}}$ are measured with a laser-Doppler anemometer (Chap. 6). Find an expression for oil viscosity $\mu$ as a function of $(\rho, \delta, g, [\partial w/\partial x]_{\text{wall}})$.




**P4.80**

**P4.81** Modify the analysis of Fig. 4.17 to find the velocity $u_\theta$ when the inner cylinder is fixed and the outer cylinder rotates at angular velocity $\Omega_0$. May this solution be *added* to Eq. (4.146) to represent the flow caused when both inner and outer cylinders rotate? Explain your conclusion.

***P4.82** A solid circular cylinder of radius $R$ rotates at angular velocity $\Omega$ in a viscous incompressible fluid which is at rest far from the cylinder, as in Fig. P4.82. Make simplifying assumptions and derive the governing differential equation and boundary conditions for the velocity field $v_\theta$ in the fluid. Do not solve unless you are obsessed with this problem. What is the steady-state flow field for this problem?

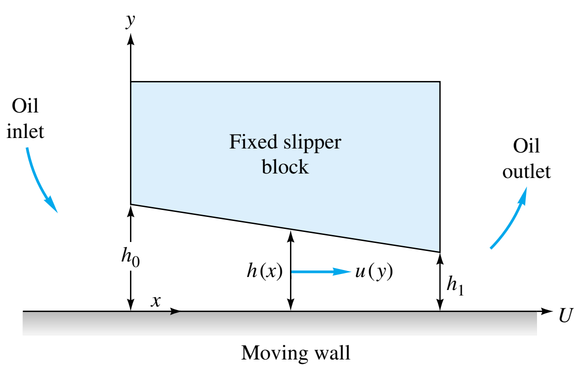



**P4.82**

**P4.83** The flow pattern in bearing lubrication can be illustrated by Fig. P4.83, where a viscous oil ($\rho$, $\mu$) is forced into the gap $h(x)$ between a fixed slipper block and a wall moving at velocity $U$. If the gap is thin, $h \ll L$, it can be shown that the pressure and velocity distributions are of the form $p = p(x)$, $u = u(y)$, $v = w = 0$. Neglecting gravity, reduce the Navier-Stokes equations (4.38) to a single differential equation for $u(y)$. What are the proper boundary conditions? Integrate and show that

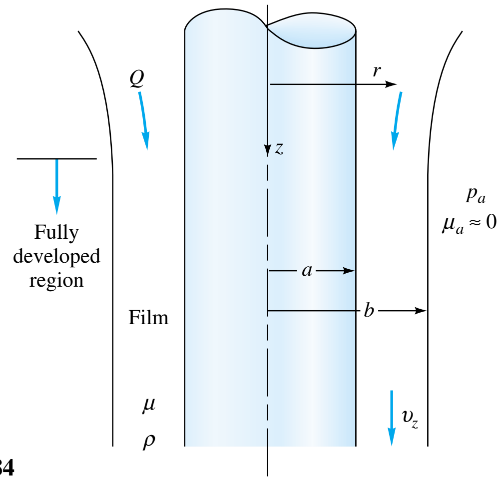
$$u = \frac{1}{2\mu}\frac{dp}{dx}(y^2 - yh) + U\left(1 - \frac{y}{h}\right)$$

where $h = h(x)$ may be an arbitrary slowly varying gap width. (For further information on lubrication theory, see Ref. 16.)



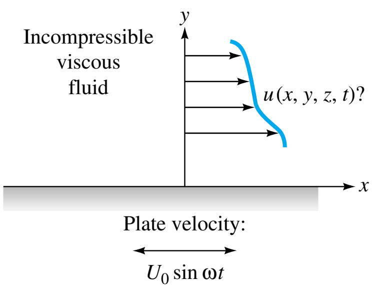

**P4.83**

***P4.84** Consider a viscous film of liquid draining uniformly down the side of a vertical rod of radius $a$, as in Fig. P4.84. At some distance down the rod the film will approach a terminal or *fully developed* draining flow of constant outer radius $b$, with $\upsilon_z = \upsilon_z(r)$, $\upsilon_\theta = \upsilon_r = 0$. Assume that the atmosphere offers no shear resistance to the film motion. Derive a differential equation for $\upsilon_z$, state the proper boundary conditions, and solve for the film velocity distribution. How does the film radius $b$ relate to the total film volume flow rate $Q$?




**P4.84**

**P4.85** A flat plate of essentially infinite width and breadth oscillates sinusoidally in its own plane beneath a viscous fluid, as in Fig. P4.85. The fluid is at rest far above the plate. Making as many simplifying assumptions as you can, set up the governing differential equation and boundary conditions for finding the velocity field $u$ in the fluid. Do not solve (if you *can* solve it immediately, you might be able to get exempted from the balance of this course with credit).




**P4.85**

**P4.86** SAE 10 oil at 20°C flows between parallel plates 8 cm apart, as in Fig. P4.86. A mercury manometer, with wall pressure taps 1 m apart, registers a 6-cm height, as shown. Estimate the flow rate of oil for this condition.

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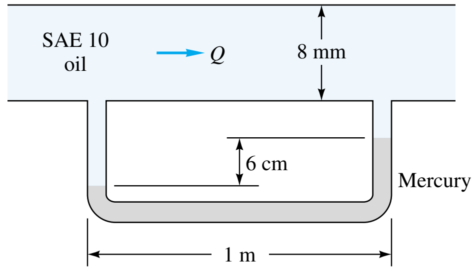
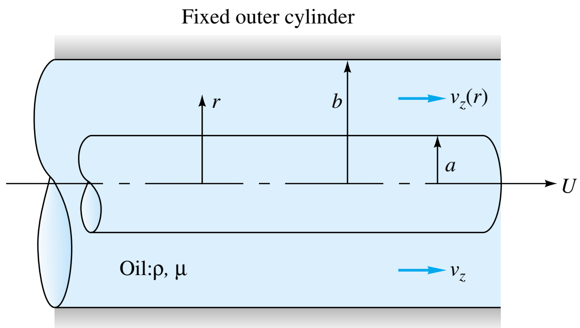



**P4.86**

**P4.87** Suppose in Fig. 4.17 that neither cylinder is rotating. The fluid has constant ($\rho$, $\mu$, $k$, $c_p$). What, then, is the steady-flow solution for $\upsilon_\theta(r)$? For this condition, suppose that the inner and outer cylinder surface temperatures are $T_i$ and $T_o$, respectively. Simplify the differential energy equation appropriately for this problem, state the boundary conditions, and find the temperature distribution in the fluid. Neglect gravity.

**P4.88** The viscous oil in Fig. P4.88 is set into steady motion by a concentric inner cylinder moving axially at velocity $U$ inside a fixed outer cylinder. Assuming constant pressure and density and a purely axial fluid motion, solve Eqs. (4.38) for the fluid velocity distribution $\upsilon_z(r)$. What are the proper boundary conditions?




**P4.88**

***P4.89** Modify Prob. 4.88 so that the outer cylinder also moves to the *left* at constant speed $V$. Find the velocity distribution $\upsilon_z(r)$. For what ratio $V/U$ will the wall shear stress be the same at both cylinder surfaces?

**P4.90** A 5-cm-diameter rod is pulled steadily at 2 m/s through a fixed cylinder whose clearance is filled with SAE 10 oil at 20°C, as in Fig. P4.90. Estimate the (steady) force required to pull the inner rod.

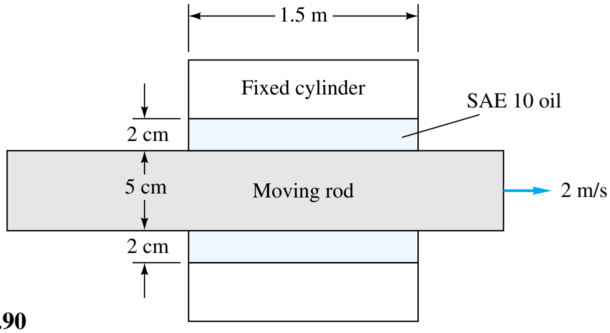
***P4.91** Consider two-dimensional, incompressible, steady Couette flow (flow between two infinite parallel plates with the upper plate moving at constant speed and the lower plate stationary, as in Fig. 4.16*a*). Let the fluid be *nonnewtonian*, with its viscous stresses given by

$$\tau_{xx} = a\left(\frac{\partial u}{\partial x}\right)^c \qquad \tau_{yy} = a\left(\frac{\partial v}{\partial y}\right)^c \qquad \tau_{zz} = a\left(\frac{\partial w}{\partial z}\right)^c$$

$$\tau_{xy} = \tau_{yx} = \tfrac{1}{2}a\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^c \qquad \tau_{xz} = \tau_{zx} = \tfrac{1}{2}a\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^c$$

$$\tau_{yz} = \tau_{zy} = \tfrac{1}{2}a\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^c$$

where $a$ and $c$ are constants of the fluid. Make all the same assumptions as in the derivation of Eq. (4.140). (*a*) Find the velocity profile $u(y)$. (*b*) How does the velocity profile for this case compare to that of a newtonian fluid?




**P4.90**

## Word Problems

**W4.1** The total acceleration of a fluid particle is given by Eq. (4.2) in the eulerian system, where **V** is a known function of space and time. Explain how we might evaluate particle acceleration in the lagrangian frame, where particle position **r** is a known function of time and initial position, **r** = fcn($\mathbf{r}_0$, $t$). Can you give an illustrative example?

**W4.2** Is it true that the continuity relation, Eq. (4.6), is valid for both viscous and inviscid, newtonian and nonnewtonian, compressible and incompressible flow? If so, are there *any* limitations on this equation?

**W4.3** Consider a CD compact disk rotating at angular velocity $\Omega$. Does it have *vorticity* in the sense of this chapter? If so, how much vorticity?

**W4.4** How much acceleration can fluids endure? Are fluids like astronauts, who feel that $5g$ is severe? Perhaps use the flow pattern of Example 4.8, at $r = R$, to make some estimates of fluid-acceleration magnitudes.

**W4.5** State the conditions (there are more than one) under which the analysis of temperature distribution in a flow field can be completely uncoupled, so that a separate analysis for velocity and pressure is possible. Can we do this for both laminar and turbulent flow?

**W4.6** Consider liquid flow over a dam or weir. How might the boundary conditions and the flow pattern change when we compare water flow over a large prototype to SAE 30 oil flow over a tiny scale model?

**W4.7** What is the difference between the stream function $\psi$ and our method of finding the streamlines from Sec. 1.9? Or are they essentially the same?

**W4.8** Under what conditions do both the stream function $\psi$ and the velocity potential $\phi$ exist for a flow field? When does one exist but not the other?

**W4.9** How might the remarkable three-dimensional Taylor instability of Fig. 4.18 be predicted? Discuss a general procedure for examining the stability of a given flow pattern.

**W4.10** Consider an irrotational, incompressible, axisymmetric ($\partial/\partial\theta = 0$) flow in ($r$, $z$) coordinates. Does a stream function exist? If so, does it satisfy Laplace's equation? Are lines of constant $\psi$ equal to the flow streamlines? Does a velocity potential exist? If so, does it satisfy Laplace's equation? Are lines of constant $\phi$ everywhere perpendicular to the $\psi$ lines?

## Fundamentals of Engineering Exam Problems

This chapter is not a favorite of the people who prepare the FE Exam. Probably not a single problem from this chapter will appear on the exam, but if some did, they might be like these.

**FE4.1** Given the steady, incompressible velocity distribution $\mathbf{V} = 3x\mathbf{i} + Cy\mathbf{j} + 0\mathbf{k}$, where $C$ is a constant, if conservation of mass is satisfied, the value of $C$ should be
(*a*) 3, (*b*) 3/2, (*c*) 0, (*d*) −3/2, (*e*) −3

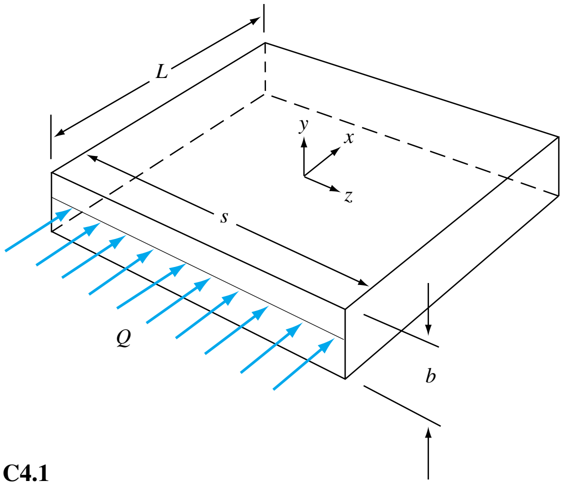
**FE4.2** Given the steady velocity distribution $\mathbf{V} = 3x\mathbf{i} + 0\mathbf{j} + Cy\mathbf{k}$, where $C$ is a constant, if the flow is irrotational, the value of $C$ should be
(*a*) 3, (*b*) 3/2, (*c*) 0, (*d*) −3/2, (*e*) −3

**FE4.3** Given the steady, incompressible velocity distribution $\mathbf{V} = 3x\mathbf{i} + Cy\mathbf{j} + 0\mathbf{k}$, where $C$ is a constant, the shear stress $\tau_{xy}$ at the point ($x$, $y$, $z$) is given by
(*a*) $3\mu$, (*b*) $(3x + Cy)\mu$, (*c*) 0, (*d*) $C\mu$, (*e*) $(3 + C)\mu$

## Comprehensive Problem

**C4.1** In a certain medical application, water at room temperature and pressure flows through a rectangular channel of length $L = 10$ cm, width $s = 1.0$ cm, and gap thickness $b = 0.30$ mm as in Fig. C4.1. The volume flow rate is sinusoidal with amplitude $\hat{Q} = 0.50$ mL/s and frequency $f = 20$ Hz, i.e., $Q = \hat{Q}\sin(2\pi f t)$.

(*a*) Calculate the maximum Reynolds number ($Re = Vb/\nu$) based on maximum average velocity and gap thickness. Channel flow like this remains laminar for Re less than about 2000. If Re is greater than about 2000, the flow will be turbulent. Is this flow laminar or turbulent? (*b*) In this problem, the frequency is low enough that at any given time, the flow can be solved as if it were steady at the given flow rate. (This is called a quasi-steady assumption.) At any arbitrary instant of time, find an expression for streamwise velocity $u$ as a function of $y$, $\mu$, $dp/dx$, and $b$, where $dp/dx$ is the pressure gradient required to push the flow through the channel at volume flow rate $Q$. In addition, estimate the maximum magnitude of velocity component $u$. (*c*) At any instant of time, find a relationship between volume flow rate $Q$ and pressure gradient $dp/dx$. Your answer should be given as an expression for $Q$ as a function of $dp/dx$, $s$, $b$, and viscosity $\mu$. (*d*) Estimate the wall shear stress, $\tau_w$ as a function of $\hat{Q}$, $f$, $\mu$, $b$, $s$, and time ($t$). (*e*) Finally, for the numbers given in the problem statement, estimate the amplitude of the wall shear stress, $\hat{\tau}_w$, in N/m$^2$.




**C4.1**

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