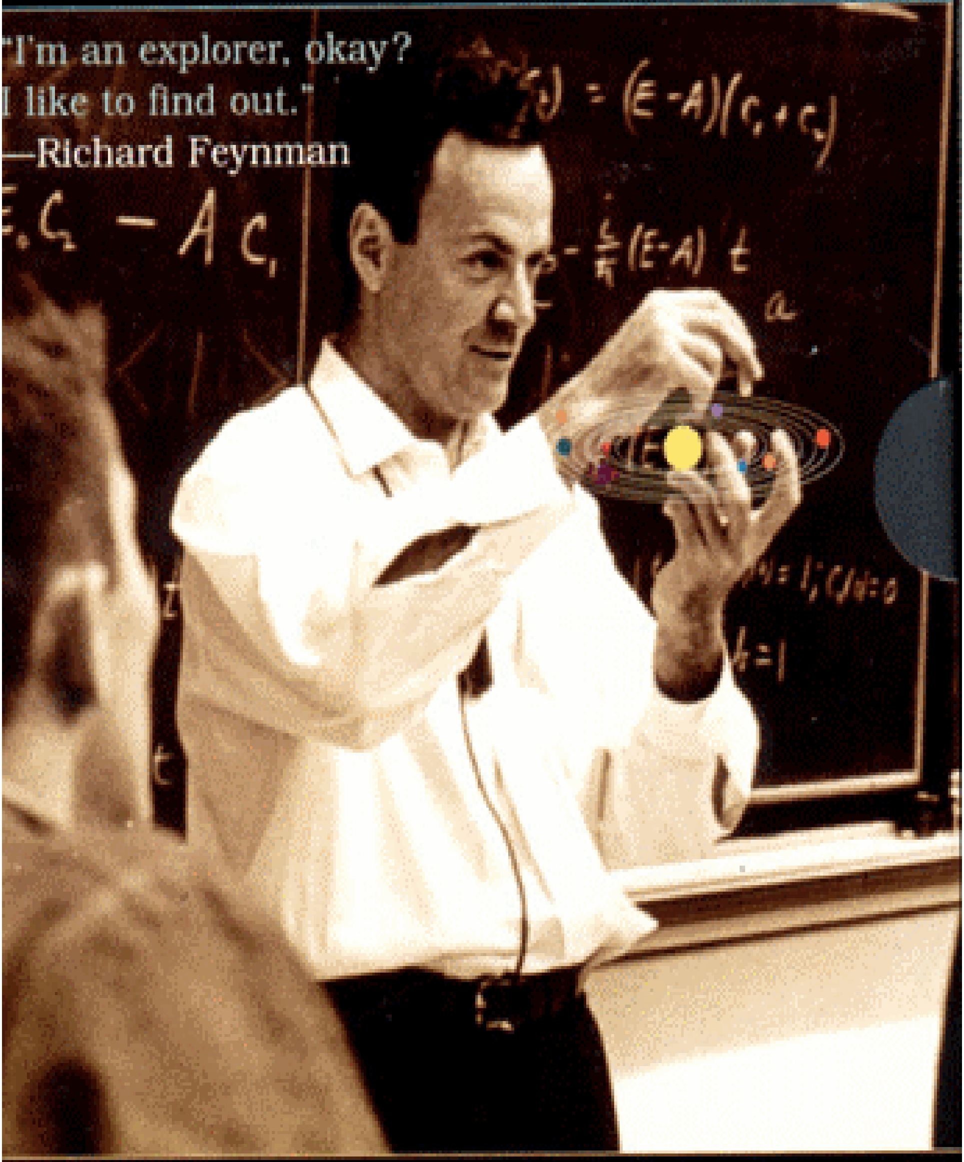
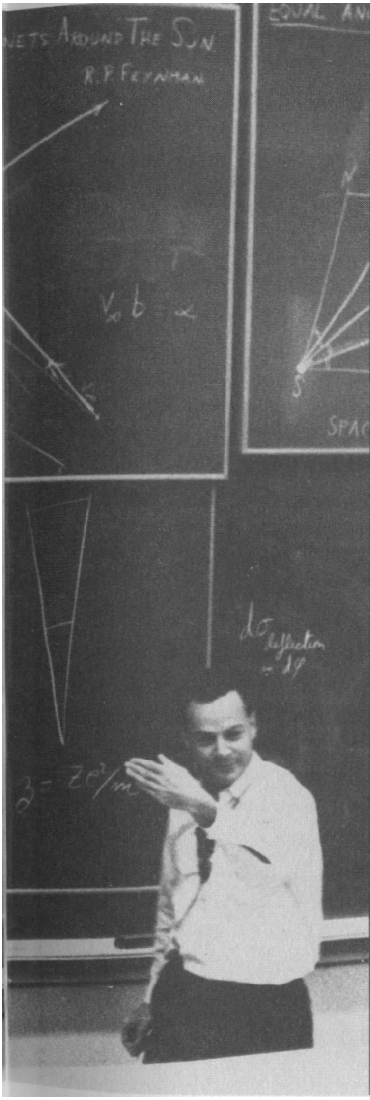
Feynman's Lost Lecture

The Motion of Planets Around the Sun



David L. Goodstein & Judith R. Goodstein



FEYNMAN'S LOST LECTURE

The Motion of Planets Around the Sun

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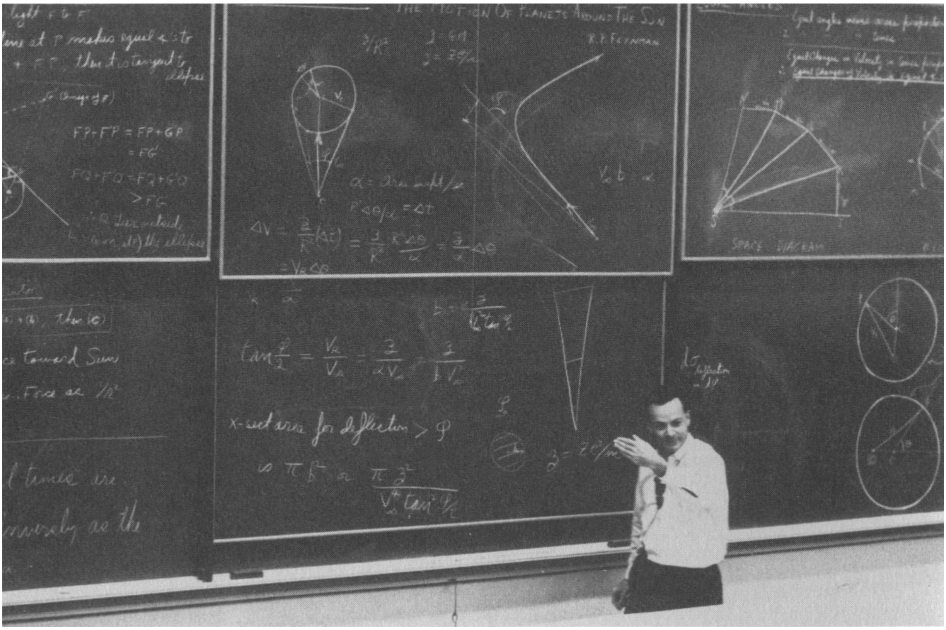

THE MOTION OF PLANETS AROUND THE SUN
R. P. FEYNMAN
SPACE DIAGRAM
x-sect area for deflection > φ

4

"The Motion of Planets Around the Sun"

(MARCH 13, 1964)

[Note: We advise that this chapter be read while listening to the recording of Professor Feynman's lecture.]

The title of this lecture is 'The Motion of Planets Around the Sun."

… After the bad news you just heard announced, I have some good news for the same reason, that since the exam is coming up Tuesday, nobody wants to give a lecture that you have to study, so I'm giving a lecture that's just for the fun of it, for your entertainment [applause]. All right, all right, I won't be able to give it. Save all that for the end and then make up your mind.

The history of our subject of physics [arrived] at one of the most dramatic moments when Newton suddenly understood so much from so little. And the history of this discovery is of course the long story about Copernicus, Tycho [Brahe] making his measurements of the positions of the planets, and Kepler finding the laws which empirically describe the motion of these planets. It was then that Newton discovered that he could understand the motion of the planets by stating another law. And you know all this from the lecture on gravitation, so I continue directly from there with a quick summary of that material.

In the first place, Kepler observed that the planets went in ellipses around the Sun, with the Sun as the focus of the ellipse. He also observed—he had three observations to describe the [orbits]—that *the* area that's swept out by a line drawn from the Sun to the orbit is proportional, this area here, is proportional to the time. Finally, to connect planets in different orbits, he discovered that the planets with different orbits have periods, or times of rotation around the complete orbit, which bear a 3/2 power ratio to the major axis of the ellipse. If there were circles (to make it easy), it would mean that the square of the time to go around the circle is proportional to the cube of the radius of the circle.

Now, Newton was able to discover two things from this. First he noticed that equal areas and equal times meant, from his point of view about inertia, that the material would continue in a straight line at a uniform velocity if it were not disturbed, that the deviations from the uniform velocity are always directed toward the Sun, and that equal areas and equal times is equivalent to the statement that the forces are toward the Sun. So he used one of Kepler's laws already to deduce that the forces were toward the Sun. And then it is easy to argue—especially for the special case of circles from the third law—that for such circles the force which would be directed toward the Sun would have to go inversely as the square of the distance.

The reason for that is something like this. Suppose that we take a certain fractional part of an orbit, some fixed angle, a small angle, and a particle has a certain velocity in one part of the orbit and another velocity later on. Then the changes in velocity for a fixed angle are evidently proportional to the velocity. And the change in velocity during an interval of time—during a fixed time—which is the force, is evidently proportional to the velocity in the orbit times the time that it takes to go across this fraction of the orbit. I mean, divided by the time. So the velocity changes proportional to the velocity. And the time over which that change has taken place is proportional to the time that it takes to go around the whole orbit—because it is a fixed angle, like one-hundredth of the orbit. Therefore the centripetal acceleration, or change per second of the velocity in the direction of the center, is proportional to the velocity on the orbit divided by the time that it takes to go around.[1]

You can put that in many different ways, because of course the time it takes to go around is related to the velocity by this relation. That the speed times the time is the distance around—or, rather, that the speed times the time is proportional to the radius. And so you can either substitute for the time, obtaining your famous v^2/R. Or better, I'll substitute for the velocity *R/T.* The velocity is evidently proportional to the radius divided by the time that it takes to go around, so that the centrifugal acceleration goes as the radius and inversely as the square of the time to go around. But Kepler tells us that the time to go around squared is proportional to the cube of the radius. That is, the denominator is proportional to the cube of the radius, and therefore the acceleration toward the center is inversely as the square of the distance. So Newton was able to deduce—in fact, [Robert] Hooke deduced earlier than Newton in the same way—that this force would be inversely as the square of the distance. So from two of Kepler's laws, we come [away] with only two conclusions. No one can verify anything that way. This may be of no particular interest, because the number of hypotheses entered is equal to the number of facts checked as the number of guesses used.

On the other hand, what Newton discovered—and which was the most dramatic of his discoveries—was that the third law (Feynman means the First Law] of Kepler was now a consequence of the other two. Given that the force is toward the Sun, and given that the force varies inversely as the square of the distance, to calculate that subtle combination of variations and velocity to determine the shape of the orbit and to discover that it is an ellipse is Newton's contribution, and therefore he felt that the science was moving forward, because he could understand three things in terms of two.

As you well know, he understood ultimately many more than three things—that the orbits in fact are not ellipses, that they perturb each other, that the motion of the Jupiter satellites is also understood, the motion of the Moon around the Earth nd so on, but let us just concentrate on this one

[1] Feynman is saying $\Delta v/\Delta t$ is proportional to *v/T.* See Chapter 3, page 108. He refers to $\Delta v/\Delta t$ as "the centripetal acceleration" above, and below he calls it "the centrifugal acceleration"

item, in which we disregard the interactions of one planet with another.

I can summarize what Newton said and in this way about a planet: that the changes in the velocity in equal times are directed toward the Sun, and in size they are inversely as the square of the distance. It is now our problem to demonstrate—and it is the purpose of this lecture mainly to demonstrate—that therefore the orbit is an ellipse.

It is not difficult, when one knows the calculus, and to write the differential equations and to solve them, to show that it's an ellipse. I believe in the lectures here—or at least in the book— [you] calculated the orbit by numerical methods and saw that it looked like an ellipse. That's not exactly the same thing as *proving* that it is exactly an ellipse. The Mathematics Department ordinarily is left the job of proving that it's an ellipse, so that they have something to do over there with their differential equations. [Laughter]

I prefer to give you a demonstration that it's an ellipse in a completely strange, unique, [and] different way than you are used to. I am going to give what I will call an elementary demonstration. [But] "elementary" does not mean easy to understand. "Elementary" means that very little is required to know ahead of time in order to understand it, except to have an infinite amount of intelligence. It is not necessary to have knowledge but to have intelligence, in order to understand an elementary demonstration. There may be a large number of steps that are very hard to follow, but each step does not require already knowing calculus, already knowing Fourier transforms, and so on. So by an elementary demonstration I mean one that goes back as far as one can with regard to how much has to be learned.

Of course, an elementary demonstration in this sense could be first to teach [you] calculus and then to make the demonstration. This, however, is longer than a demonstration which I wish to present. Secondly, this demonstration is interesting for another reason—it uses completely geometrical methods. Perhaps some of you were delighted in geometry in school with the fun of trying or having the ingenuity to discover the right construction lines. The elegance and beauty of geometrical demonstration is often appreciated by lots of people. On the other hand, after Descartes, all geometry can be reduced to algebra, and today all mechanics and all these things are reduced to analysis with symbols on pieces of paper and not by geometrical methods.

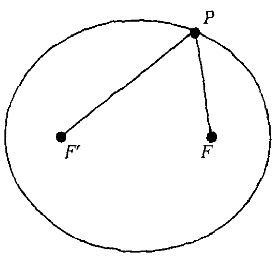
On the other hand, in the beginning of our science—that is, in the time of Newton—the geometrical method of analysis in the historical tradition of Euclid was very much the way to do things. And as a matter of fact, Newton's *Principia* is written in a practically completely geometrical way—all the calculus things being done by making geometric diagrams. We do it now by writing analytic symbols on the blackboard, but for your entertainment and interest I want you to ride in a buggy for its elegance, instead of in a fancy automobile. So we are going to derive this fact by purely geometrical arguments—well, by essentially geometrical arguments, because I don't know what that means, anything precise I don't know what it means, like purely geometrical arguments—but essentially geometrical arguments, and see how well we get on.

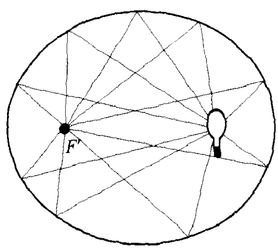
So our problem is to demonstrate that if this is true—that the changes in velocities are directed toward the Sun, and they are inversely as the square of the distance in equal times—that the orbit is an ellipse. We then have first to understand—we must start with something—we first must know what an ellipse is. If there is no available definition of an ellipse, it is going to be impossible to demonstrate the theory. And furthermore, if you cannot understand the meaning of this proposition, of course you also cannot demonstrate the theorem. So, many people have said, "Oh yeah, but you've got to know something about an ellipse." I know—you can't state the statement otherwise. And also you have to have some understanding of this idea. That's also true. But beyond that, I don't think we need much extra knowledge, but a large amount of attention, please, and careful thinking. That's not easy, and it's quite a job, and it's not worthwhile. It is much easier to do it by the calculus, but you're going to do it that way anyway, and you must remember that this is just to see how it would look.

There are several ways of defining an ellipse, and I have to choose one, and I will suppose that the one with which everyone is familiar is the fact that an ellipse can be made, or the ellipse is the curve that can be made, by taking one string and two tacks and putting a pencil here and going around. Or mathematically, it is the locus (nowadays they say the set of all points)

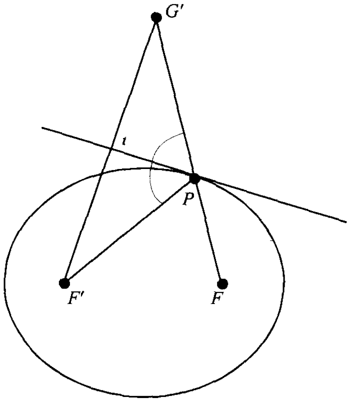
—all right, the set of all points—such that the sum of the distance *FP* and the distance *F'P* [*F* and *F'* being] the two fixed points, remains constant. I suppose you know that's the definition of an ellipse. You may have heard another definition of an ellipse: if you wish, these two points are called the foci, and this focus means that light emitted from *F* will bounce to *F'* from any point on the ellipse.

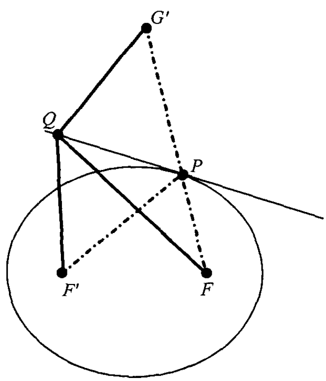
Let me just demonstrate the equivalence of those two propositions, at least. So the next step is to demonstrate that light will be reflected from *F* to *F'*. The light is reflected as though the surface here were a plane tangent to the actual curve. What I therefore have to demonstrate is this—and you know, of course, that the law of reflection for light from a plane is that the angle[s] of incidence and reflection are the same. Therefore, what I have to prove is this: that if I were to draw a line here, such that its angles made with the two lines *FP* and *F'P* are equal, that that line is then tangent to the ellipse.





Proof: Here's the line drawn as described. Make the image point of *F'* in this line. That is to say, extend the perpendicular from *F'* to the line the same distance on the other side, to obtain *G'*, the image of *F'*. Now connect the point *P* to *G'*. Notice [that] because of the equal angles, that this angle here is the vertical angle. Well, this angle is equal to this angle, because these two right triangles are exactly the same. It's an image, so this side is the same as that side, and these two angles are equal; this is a straight line. So that *PG'* here is exactly equal to the *F'P* part, and incidentally, *FG'* is a straight line, so that the *FP* + *F'P,* which is the sum of these two distances, is in fact *FP* + *G'P***,** because *F'P* = *G'P.* Now, the point is that if you take any other point on the tangent—say, Q—and you took the sum of these two distances to *Q,* it is easy to see that the distance *F'Q* is, again, the same as G'Q. So that the sum of these two distances, *F'Q* to *F,* is the same as the distance from *F* to *Q* and *Q* to G'. In other words, the sum of the distances from the two foci on any point on the line is equal to the distance from *F* to G', by going up to that point and across. Evidently larger, evidently always larger than going on the straight line across. In other words, the sum of the two distances to a point *Q* is greater than it is for the ellipse—for any point *Q* except for point *P.* For any point on this line, then, the sum of the distances to these two points is greater than it is for a point on the ellipse.

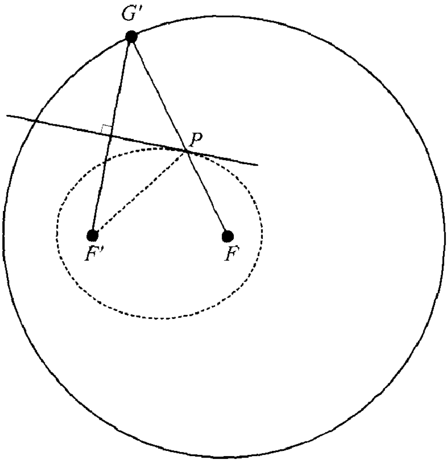




Now I take the following to be evident and perhaps you can devise a proof to satisfy you—that if the ellipse is the curve in which the sum of the two points is a constant, that the points outside the ellipse have the sum to the two points greater and the points inside the ellipse have the sum to the two points less; so that since these points on the line have a sum greater than a point on the ellipse, all this line lies outside the ellipse with the sole exception of the point *P,* whence it must be tangent and does not intersect at two points nor ever come inside. All right, so the thing is therefore tangent, and we know that the reflection law is right.

I have another property to describe about an ellipse, the reason for which will be completely obscure to you, but it's something which I will need later in this demonstration.

May I say that although the methods of Newton were geometrical, he was writing in a time in which the knowledge of the conic sections was the thing that everybody knew very well, and so he perpetually uses (for me) completely obscure properties of the conic sections, and I have, of course, to demonstrate my properties as I go along. I would like, however, for you to take the same diagram again, which I made here, and draw it over again. It's drawn exactly the same here: *F'* and *F,* there's that tangent line, here's the image point G' of *F'.* However, I would like for you to imagine what happens to the image point G' as the point *P* goes around the ellipse. It is evident, as I already indicated, that *PG'* is the same as *F'P,* so that *FP* + *F'P* is a constant, [and that] means that *FP* + *PG'* is a constant. In other words, that *FG'* is a constant. In short, the image point G' runs around the point *F* in a circle of constant radius. All right. At the same time. I draw a line from *F'* to G' and I find [that] my tangent is perpendicular to it. That's the same statement as all that was before. I just want to summarize that, to remind you of a property of an ellipse, which is this: that as a point G' goes around a circle, a line drawn from an eccentric point to this point G'— this is an off-center point to the point G'—will always be perpendicular to the tangent of the ellipse. Or the other way around: the tangent is always perpendicular to the line—or a line—drawn from an eccentric point. All right, that's all, [and] we'll come back to it and we'll remember, and we will review it again, so don't worry. That's just a summary of some of the



properties of an ellipse, starting from the facts. That's the ellipse.

On the other hand, we have to learn dynamics, we have to put them together. So now we have to explain what dynamics is all about. I want this proposition, that's the geometry; now the mechanics, what this proposition means. What Newton means by this is this: that if this is the Sun, for instance, the center of the attraction, and at a given instant a particle were to, say, be here, and let me suppose that it moves to another point, from *A* to *B,* in a certain interval of time. Then, [if] there were no forces acting toward the Sun, this particle would continue in the same direction and go exactly the same distance to a point *c.* But during this motion there's an impulse toward the Sun, which, for the purposes of analysis, we will imagine all the curves at the middle instant—in other words, at this instant. In other words, we concentrate all our impulses in an approximate way of thinking to this middle moment. And, therefore, the impulse is in the direction of the Sun, and this might represent the change in motion. That means that instead of this moving to here, it moves to a new point, which is *C,* which is different than *c,* because the ultimate motion is this motion compounded from the original plus the additional impulse given toward the

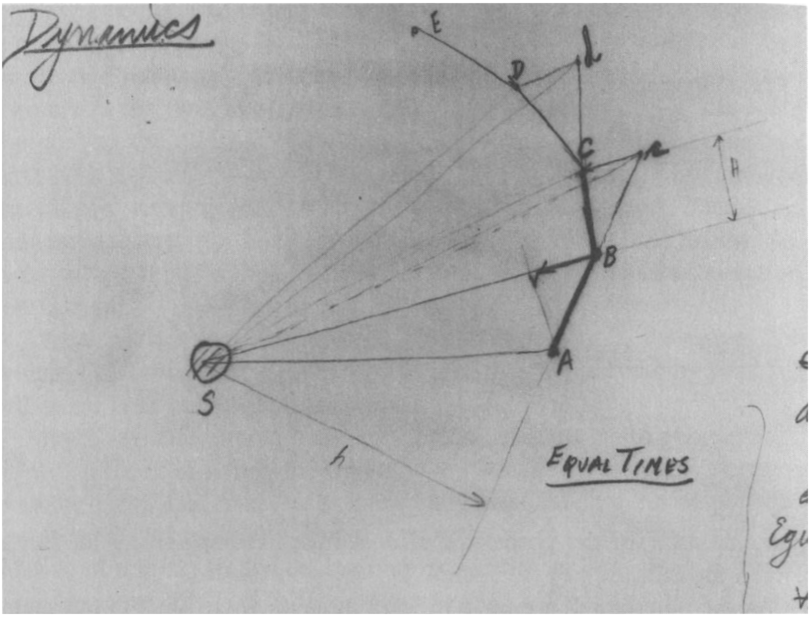


Diagram from Feynman's lecture notes.

center of the Sun. So that the ultimate motion is along the line *BC,* and at the end of the second interval of moment of time the particle will be at *C.* I emphasize that *Cc* is parallel to and equal to *BV,* let us say, the impulse given from the Sun. It is therefore parallel to a line from *B* to the center of the Sun. Finally, the rest of the statement is that the size of *BV* will vary inversely as the square of the distance as we go around the orbit.

I have drawn this same thing over again here—exactly the same way, no change at all, excepting color makes it more interesting. Here's the motion that the particle would have—has in the first instant of time—and the motion which it would continue to have if it were to continue for the second interval of time with no force. May I point out to you that the areas that would be swept through in that case would be equal during those two intervals of time. For these two distances, *AB* and *Bc,* are evidently equal, and therefore the two triangles *SAB* and *SBc,* which are the two areas, will be equal: for they have equal bases and a common altitude. If you extend the base and draw the altitude, it's the same altitude for both triangles; and since the bases are equal, the areas then swept through are equal.

On the other hand, the *actual* motion is not to the point *c* but to the point *C,* which differs from the position *c* by a displacement in the direction of the Sun at the moment *B,* that is, in the blue line parallel to the original blue line. Now I would like to point out to you that the area that would be most occupied—I mean, which would be swept out in that second interval of time even if there were a force: namely, the area *SBC*—*is* the same as the area that there would be if there were no force—namely, *SBc.* The reason is that we have two triangles which have a common base

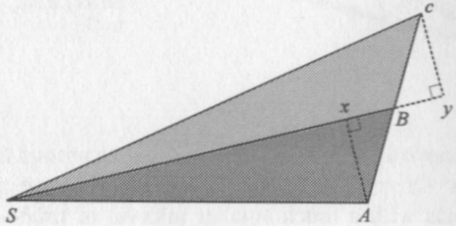
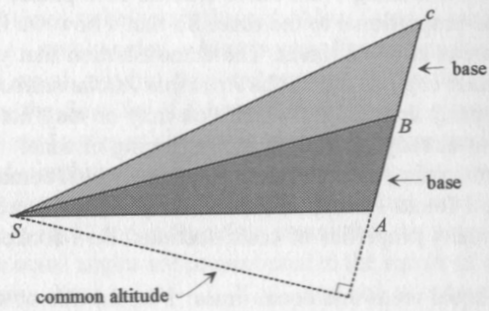
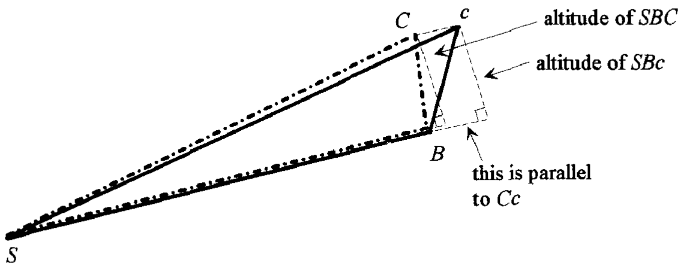


Figure in Chapter 3



Feynman does it this way instead

and who have an equal altitude, for they lie between parallel lines. Since the area[s] of the triangle *SBC* and the triangle *SAB* are equal—but since those points *A, B,* and *C* represented positions in succession at equal times in the orbit—we see that the area[s] moved through in equal times are equal. We can also see that the orbit remains a plane, that the point *c* being in the plane and the line *Cc* being in the plane of *ABS,* the remaining motion is in the plane ABS.



And I have drawn a succession of such impulses around this imaginary polygonal orbit. Of course, to find the actual orbit, we need to make the same analysis with a much smaller interval of time—and a much finer rate of impulsing—until we get the limiting case, in which we have a curve. And in the limiting case in which we have a curve—the area swept by this thing—the curve will lie in a plane, and the area swept will be proportional to the time. So that's how we know that we have equal areas in equal times. The demonstration that you have just seen is an exact copy of one in the *Principia Mathematica* by Newton, and the ingenuity and delight which you may or may not have gotten from it is that already existing in the beginning of time.

Now the remaining demonstration is not one which comes from Newton, because I found I couldn't follow it myself very well, because it involves so many properties of conic sections. So I cooked up another one.

We have equal areas and equal times. I would like now to consider what the orbit would look like if instead of using equal time, one were to think of the succession of positions which correspond to *equal angles* from the center of the Sun. In other words, I repicture the orbit with the succession of points, *J, K, L, M, N,* which correspond not to equal instants, like they did in the diagram before, but rather [to] equal angles of inclination from the original position. To make this a little bit simpler, although it is not at all essential, I have supposed that the original motion was perpendicular to the Sun at the first point—but that's not essential, it just makes the diagrams cleaner.

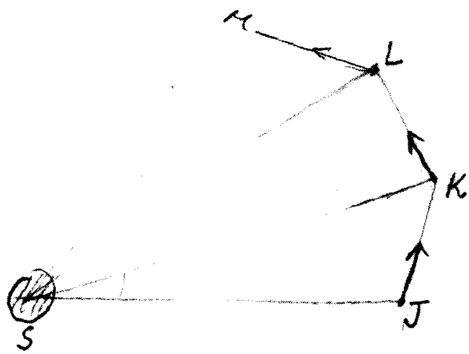


Diagram from Feynman's lecture notes.

Now we know from the proposition previously that equal [areas] occupy equal times to be swept through. Now listen: I would point out to you that . . . equal angles, which is what I'm aiming for, means that areas are not equal, no, but they are proportional to the square of the distance from the Sun; for if I have a triangle of a given angle, it is clear that if I make two of them that they are similar; and the proportional area of similar triangles is proportional to the square of their dimensions.[2] Equal angles therefore means—since areas are proportional to time—equal angles therefore means that the times to be swept through these equal angles are proportional to the square of the distance. In other words, these points—*J, K, L,* and so on—do not represent pictures of the orbit at equal times, no,

[2]This is the point explained in the footnote to Chapter 3, page 115.

but they represent pictures of the orbit with successions of times which are proportional to the square of the distance.

Now, the dynamical law is that there are equal changes in velocity, no—that the changes in velocity vary inversely as the square of the distance from the Sun—that is, the changes of velocity in equal times. Another way of saying the same thing is that equal changes of velocity will occupy times proportional to the square of the distance. It's the same thing. If I take more time, I get more change in the velocity, and, although they are falling off for equal times inversely as the square, if I make my times proportional to the square of the distance, then the changes in velocity will be equal. Or, the dynamical law is: equal changes in velocity occur in times proportional to the square of the distance. But look, equal angles were times proportional to the square of the distance. And so we have the conclusion, from the law of gravitation, that equal changes of velocity will occur in equal angles in the orbit. That's the central core from which all will be deduced—that equal changes in velocity occur when the orbit is moving through equal angles. So I now draw on this diagram a little line to represent the velocities. Unlike the other diagram, those lines are not the complete line from *J* to *K,* for in that diagram those were proportional to the velocities, for the times were equal, and the length divided by equal times represented the velocities. But here I must use some other scale to represent how far the particle would have gone in a given unit of time, rather than in the times which are, in fact, proportional to the square of the distance. So these represent the velocities in succession. It is quite difficult in that diagram to find out what the changes are.

I therefore make another diagram over here, which I'll call the diagram of the velocities, in which I draw a picture on a magnified scale only for convenience. These are supposed to represent exactly these same lines. This would represent the motion per second of a particle at *J* or in a given interval of time, at *J.* This would represent the motion that a particle would've made from the beginning in a given interval of time. And, I put them all at a common origin, so that I can compare the velocities. So I have then a series of the velocities for the succession of these points.

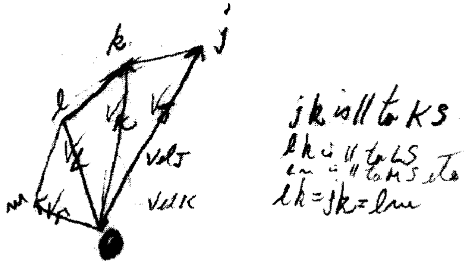
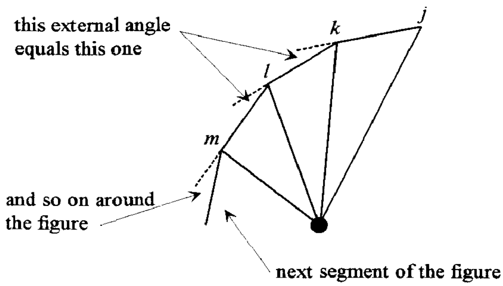


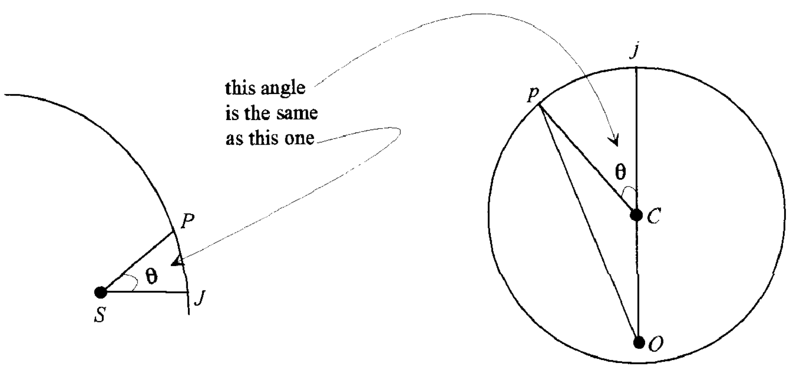
Diagram from
Feynman’s lecture notes.

Now, what are the changes in the velocity? The point is that in the first motion, this is the velocity. However, there is an impulse toward the Sun, and so there is a change in velocity, indicated by the green line that produces the second velocity, v_K. Likewise, there’s another impulse toward the Sun again, but this time the Sun is at a different angle, which produces the next change in the velocity, v_L, and so on. Now, the proposition that the changes in the velocities were equal— for equal angles, which is the one that we deduced—means that the lengths of these succession of segments are all the same. That’s what it means.

And what about their mutual angles? Since this is in the direction of the Sun at this radius, since this is at the direction of the Sun at that radius, and since this is the direction of the Sun at that radius, and so on, and since these radii each successively have a common angle to one another—so it is likewise true that these little changes in the velocity have, mutually to one another, equal angles. In short, we are constructing a regular polygon. A succession of equal steps, each turn through an equal angle, will produce a series of points on the surface underlying a circle. It will produce a circle. Therefore, the end of the velocity vector—if they call it that, the ends of these velocity points; you’re not supposed to know what a vector is in this elementary description—will lie on a circle. I draw the circle again.



I review what we found out. I take the continuous limit, where the intervals of angle are very tiny indeed, to obtain a continuous curve. Let 0 be the angle, total angle, to some point *P,* and let v_P represent the velocity of that point in the same way as before. Then the diagram of velocities will look like this. This is the origin of the velocity diagram, the same as over there, and this is the velocity vector corresponding to this point *P.* Then this lies on a circle, but always not necessarily the center of that circle. However, the angle that you've turned through in the circle is the same θ as here. The reason for that is that the angle turned through from the beginning by this thing is proportional to the angle turned through by the orbit, because it's the succession of the same number of small angles. And therefore, this angle in, here, is the same angle as in, here.

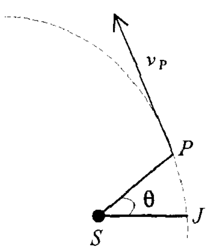


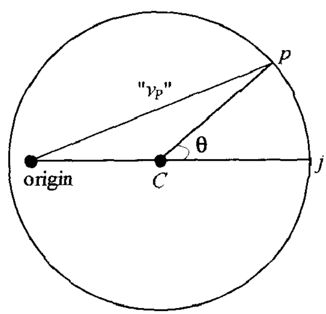
(orbit diagram)

(velocity diagram)

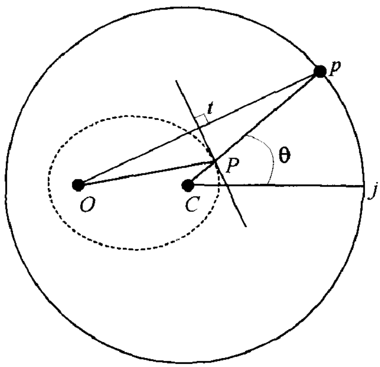
So here is the problem, here's what we have discovered: that if we draw a circle and take an off-center point, then take an angle in the orbit—any angle you want in the orbit—and draw the corresponding angle inside this constructed circle and draw a line from the eccentric point, then this line will be the direction of the tangent. Because the velocity is evidently the direction of motion at the moment and is in the direction of the tangent to the curve. So our problem is to find the curve such that if we draw a point from an eccentric center, the direction of the tangent of that curve will always be parallel to that when the angle of the curve is given by the angle in the center of that circle.

In order to make still clearer why it is going to come out in this thing, I'll turn the velocity diagram 900, so that the angles correspond exactly and are parallel to each other. This diagram under here, then, is precisely the same diagram as the one you see above, but turned 90^0—only to make it easier to think. This, then, is the velocity vector, except that it's turned 900 because the whole diagram is turned 90~. That is, this is perpendicular evidently to that, and therefore this is evidently perpendicular to that. In short, we must find *the curve such* that if we put the orbit in it, I think I've started—yes, so I'll just say it and then I'll draw it again—if we put the orbit in it at a given point, here, where this line intersects the orbit (never mind the scales, they're all imaginary, I mean, it's all in proportion), where this line intersects the orbit, the tangent should be perpendicular to that line *from an* eccentric point.





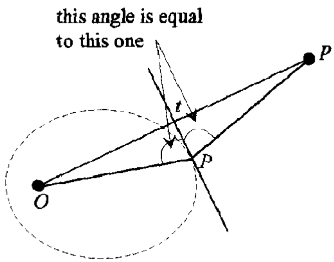
I draw it again, to show you how it is. You know now what the answer is. But here's a picture again of the same velocity circle, but this time the orbit is drawn inside at a different scale, so that we can see this picture laid right over this picture, so the angles correspond. So since the angles correspond, I can draw the single line to represent both the point *P* on the orbit and the point p on the velocity circle. Now what we have discovered is that the orbit is of such a character that a line drawn from the eccentric point—here, from an extension of this point onto a circle outside—will always be perpendicular to the tangent to the curve. Now that curve is an ellipse, and you *can* find that *out* by the following construction.

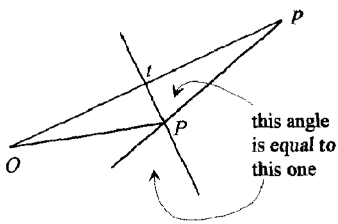


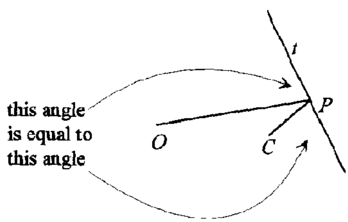
Construct the following curve. The curve I'm going to construct will satisfy all the conditions. Construct the following curve. Always take the perpendicular bisector of this line and ask for its intersection with the other line, *Cp,* and call that intersection point *P.* This is the perpendicular bisector. Now I'll prove two things. First, that the locus of this point that's been generated there is an ellipse, and, second, that this line is a tangent there, too—that is, to the ellipse—and therefore satisfies the conditions, and all is well.

First, that it's an ellipse: Since this was the perpendicular bisector, it is at equal distances from *0* and p. It is therefore clear that *Pp* is equal to *P0.* That means that $CP \pm P0$, which is therefore equal to $CP + Pp$, is the radius of the circle, which is evidently constant. So the curve is an ellipse, or the sum of these two distances is a constant.

And next, this line is tangent to the ellipse because, since . . . the two triangles are congruent, this angle here is equal to this angle here. But if I







extend this line on the other side, [then] also is that angle equal. So therefore the line in question makes an equal angle with the two lines to the foci. But we proved that that was one of the properties of an ellipse—the reflection property. Therefore, the solution to the problem is an ellipse—or the other way around, really, is what I proved: that the ellipse is a possible solution to the problem. And it is this solution. So the orbits are ellipses. Elementary, but difficult.

I have considerable more time, and so I will say a few things about this. In the first place, I would like to say how I got this demonstration— the fact that the velocities went in a circle. The demonstration Lof3 this point was due to Mr. Fano and I read it. And after that, to prove that it was an ellipse took me an awful long time: that is, the obvious, simple step—you turn it this way, and you draw that and all that. Very hard, and like all these elementary demonstrations they require a large amount—like any geometrical demonstration—of ingenuity. But once presented, it's elegantly simple. I mean, it's just finished. But the fun of it is that you've made a kind of a carefully put-together piece of pieces.

It is not easy to use the geometrical method to discover things. It is very difficult, but the elegance of the demonstrations after the discoveries are made is really very great. The power of the analytic method is that it is much easier to discover things than to prove things. But not in any degree of elegance. it's a lot of dirty paper, with x's and y's and crossed out, cancellations and so on.

I would like to point out a number of interesting cases. It of course can happen that the point *0* lies on the circle, or even that the point *0* lies outside the circle. It turns out that the point *0* lying on the circle does not produce, of course, an ellipse; it produces a parabola. And the point *0* lying outside the circle, which is another possibility, produces a different curve, a hyperbola. I leave some of those things for you to play with. On the other hand, I would like now to make some application of this and to continue the argument that Mr. Fano originally made, for another purpose. He was going in a different direction, and I'd like to show you that.

What he [Fano] was trying to do was to make an elementary demonstration of a law which was very important in the history of physics in

1914. And that had to do with the so-called Rutherford's law of scattering. If we have an infinitely heavy nucleus—which we don't have, but suppose—and if we shoot a particle by that nucleus, then it will be repelled by an inverse-square law, because of the electrical force. If q~ is the charge on an electron, then the charge on the nucleus is *Z* times
q~ when Z is the atomic number. Then the force between the two things is given by 4-rr~ times the square of the distance, which for simplicity I will write temporarily as z/R^2—*the* constant over R^2. I don't know whether you've done this in the class or not; but I'll suppose, I'll define another thing because, q~^{2} *1* ~ will *be* written e^2 for short. *Then* this thing is just Ze^2 / R^2. Anyway, that's the force inversely as the square of the distance, but it's a repulsion. And now the problem is the following: If I shoot a lot of particles at these nuclei, where I can't see the nuclei, how many of them will be deflected through various angles? What percentage will be deflected more than 300? What percentage will be deflected more than **450?** And how are they distributed in angles? And that was the problem *that* Rutherford wanted solved, and when he had the correct solution, he then checked it against experiment.

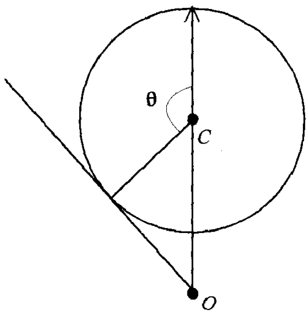
[At this point. Feynman goes off in the wrong direction. He'll correct himself in a moment.]

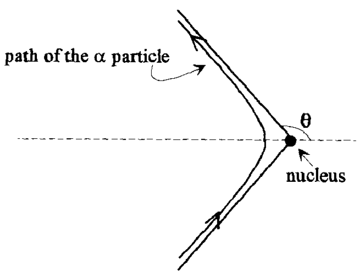
And he found that the ones that were supposed to be deflected through large angles were not there. In other words, the number of particles deflected through large angles was much less than you would think, and he therefore deduced that the force was not as strong as hR^2 for small distances. Because it is obvious that to get the large angle, you need a lot of force, and it corresponds to the [particles] that hit [the nucleus] almost head-on. So those which come very close to the nucleus do not seem to come out the way they ought to, and the reason is that the nucleus has a size . . . I've got the story backwards. If the nucleus had a big size, then those which were supposed to come out at large angles wouldn't get their full force, because they would get inside the charge distribution and would be deflected less. I got mixed up. Excuse me. I start again.

Rutherford deduced how it should go if all the forces were concentrated at the center. In his day, it was supposed that the charge in an atom was distributed uniformly over the atom, and in order to discover this distribution, he thought that if he scattered these particles, they would show a weaker deflection—they would never show a very large deflection corresponding to a very close approach to the repulsion center because (in] the close approach there's no center. He, however, did find the large-angle deflections, and deduced that the nucleus was small and that the atom had all its mass at a very small central point. I got it backwards. It was later that it was demonstrated, by the same thing again, that the nucleus has a size. But the first demonstration was that the atom is not as big, for this kind of electrical purposes, as the whole atom is known to be: that is, all the charge was concentrated at the center, and thus the nucleus was discovered. However, we need now to understand this: we need to know what the law is for the angle of deflection here, and that we can obtain in this way.

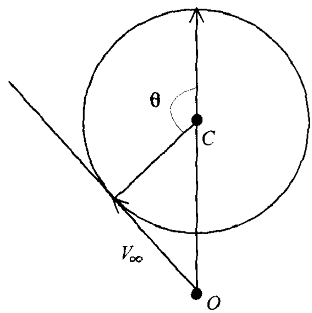
Suppose that we do the same thing as we did before, and we draw the orbit. Here is the charge, and here is the motion of a particle going around, only this time it's repulsion. I start the picture at this point, for the fun of it, and I draw my velocity circles as before. This is the velocity. We know that the velocity, the initial velocity at this point— I should use the same colors so you know what I'm doing, this should be blue, this orbit is red— now the velocity changes lie on a circle. But the changes in the velocity this time are repulsions, and the sign is reversed. And after some minor thought, you can see that the deflections go like that, and that the center of the calculation [which] used to be called the origin of the velocity space *0,* lies on the outside of the circle. And the succession of small velocity changes lie on the circle, and the succession of velocities then in the orbit are these lines, until a very interesting point comes: until we get to this tangent.

At this tangent point to the curve—what does it mean? It means that all the changes in velocity are in the direction of the velocity. But the changes in the velocity are in the direction of the Sun, and that means that this velocity, in this part of the diagram, is in the direction of the Sun, because it is in the direction of the changes. That is to say, this point here, as we approach this point here—which I could call x, say— corresponds to coming from infinity toward the Sun along a line here. That is, very far out we are directed toward the Sun very closely (not the Sun, but the nucleus) and then as it comes around here—this diagram should be the other way,





("orbit" diagram)



(velocity diagram)

the other way, the arrows should be here, I got the changes the wrong way in time—comes around here and goes out this way and, going out that way, corresponds to going with the velocity off in this direction.

Now, if we draw then the orbit more carefully, it will look very much like this. It goes around like this. If I call this point, here, V_∞, then the velocity that the particle has at the beginning is V_∞. If, on the same scale, I call the radius of this circle V—the velocity corresponding to the radius of the circle—I'm going to make up some equation, I'm not going to do it completely geometrically, but to save time and so on, I've done all the work. One should not ride in the buggy all the time. One has the fun of it and then gets out. Now first I want to find the velocity of the center, the radius of the velocity circle. In other words, I'm now going to come down and make some of these geometric things more analytic.

I will suppose that the force is some constant: the force—the acceleration, rather—is some constant over R^2. For gravity, this constant is *GM* and, for electricity, it is Ze^2Im, over *m* because of the acceleration. That is to say, the changes in velocity are always equal to zIR^2 times the time. Now let us suppose that we call **~**, which is a constant for the motion, the area swept by the orbit per second. That is then this way:
that the time—if I wanted to change this to angle, I have the following—$R^{2\sim}O$ would be the area. ff1 divide that by the rate that area is swept through—this tells me how much time it takes to sweep an angle. The time is, then, for given angles, proportional to the square of the distance. All this I'm saying now analytically, where I said in words before. Substitute this ~t, in here, to find out how the changes in the velocity are with respect to angle, and one obtains $R^2L19Io\sim$, or the R^2*'s* cancel, and it means that the changes in velocity are as advertised: for equal angles, equal.

Now then, the velocity diagram—although this isn't the piece of the orbit that you can get to, never mind—these are changes in the velocity and these are changes in the angle in the orbit. So *~V* is also equal by the geometry of that circle to the radius of the circle, which I call *VR* **X** ~O. In other words, we have that the radius of the velocity circle is equal to *zia,* where ct is the rate of area swept per second and *z* is a constant having to do with the law of force. Now, the angle through which this planet has deflected is this one, here, and I call it, the angle of deflection from the planet—I mean the charged particle from the nucleus. It is evident from my discussion that it's the same as this angle in here, 4, because these velocities are parallel to the two original directions. It is clear, therefore, that we can find ~ if we can get the relation with ½~ and VR. You see, look, tangent of ~/2 = *VR/V.X,* and that gives us the angle. The only thing is that we need—we have to substitute for *VR, zIaR,* and we have that much.

Now, it doesn't do us much good until we know a for this orbit. An interesting idea is this: think of this thing as approaching this, so that if there were no force it would miss by a certain distance, *b*. This is called the impact parameter. We imagine that the thing comes from infinity aimed for the force center, but is missing—because it misses, it is deflected. By how much is it deflected, if it was aimed to miss by *b?* That's the question. If it's aimed to miss by a distance *b,* how much will it get deflected?

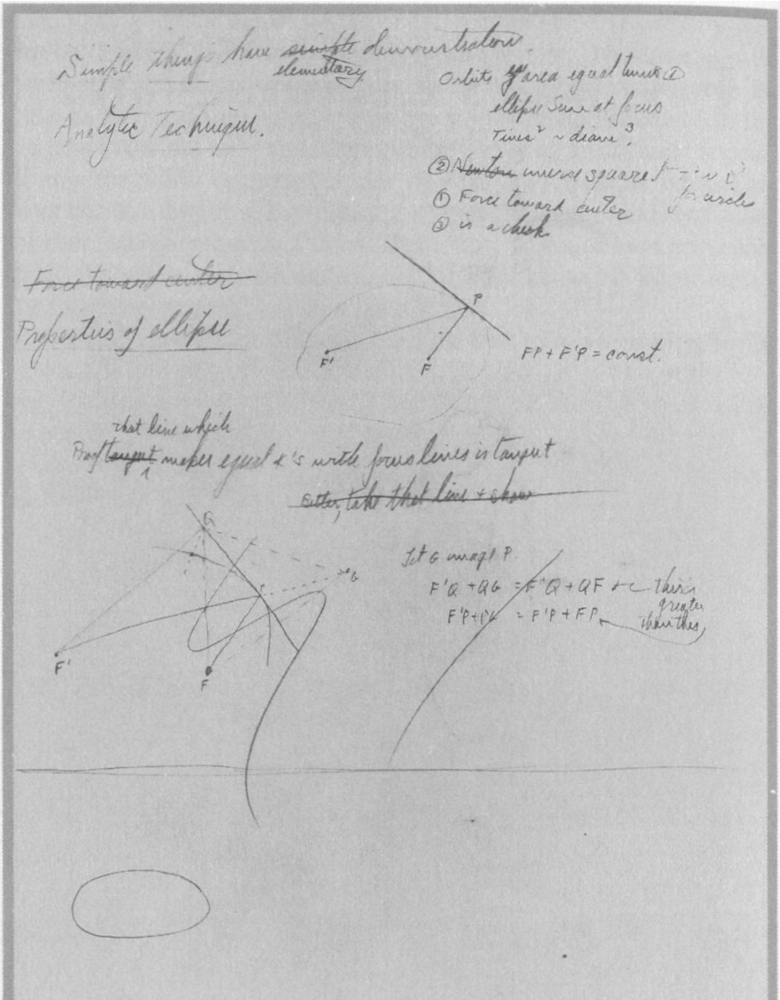
So I need now only determine how a is related to *b*. *V,.* is the distance gone in 1 second, so if I were to draw way out here a horrible-looking area, a triangle—a terrible-looking triangle, then the—I got a factor of 2 somewhere, yeah, the area of a triangle is 1/2 R^2. There are two factors, two, which you will straighten out please when the time comes. There is 1/2 in here and, there is 1/2 somewhere else, which I'm now going to make. The area of this triangle is the base *V.,.* times the height *b* times 1/2. Now that triangle is a triangle through which a particle would sweep—the radius would sweep in I second. And this is, therefore, a. So, therefore, we have that this goes as *zIbV,.*^{2}. That tells us that given the impact distance, the aiming accuracy, what angle we would find in the deflection in terms of the speed at which the particle approaches and the known law of force. So it's completely finished.

One more thing that is rather interesting. Suppose that you would like to know with what probability, what chance is there of getting a deflection more than a certain amount. Let's say you pick a certain 4— **~**, say—and you want to make sure that you get greater than fr~. That only means that you have to hit inside an area closer than the *b* which belongs to that ~. Any collision closer than *b* will produce a deflection bigger than **~**, where *b* is b_0, belonging to ~ through this equation. If you come further away, I have less deflection, less force. So, therefore, the so-called cross section of area that you have to hit for deflection, to be greater than 4 (I'll leave off the naught), is *-rrb*^{2}*,* where *b* is *Z/Va$* tan^2 *412*. In other words, it is *irz*^{2}*IV.,*^{4} tan^2 *~I2*. And that's the law of Rutherford's scattering. That tells you the probability of the area you have to hit—the effective area that you have to hit—in order to get a deflection more than a certain amount. This *z* is equal to *Ze*^{2}*Im;* this is a fourth power, and it is a very famous formula.

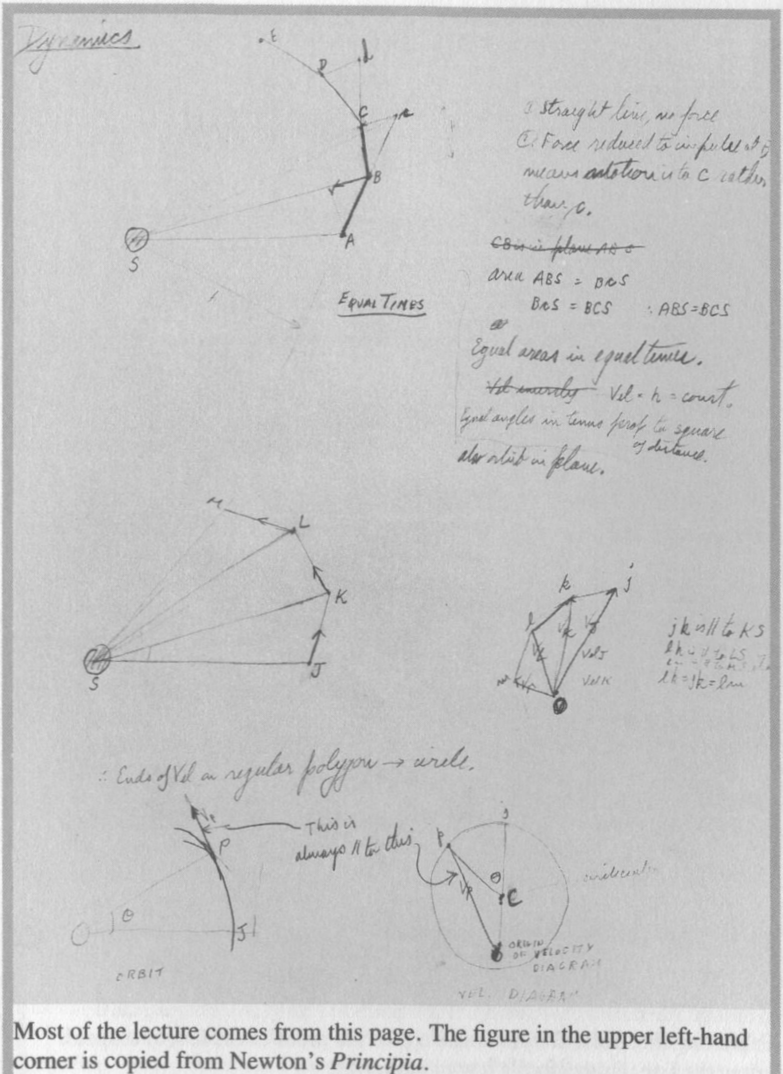
It is so famous that, as usual, it was not written in this form when it was first deduced, and so I, just for the famousness of it, will write it in a form—well, I'll leave you to write it in a form. I'll write just the answer, and I'll let you see if you can show it. Instead of asking for the cross section for a deflection greater than a certain angle, we can ask for the piece of cross section, *du,* that corresponds to the deflection in the range *d~* that the angle should be between, here, and there. You just have to differentiate this thing, and the final result for that thing is given as the famous formula of Rutherford, which is 4 Z^2e^4 times 2rr sin~ *d~* divided by $4m^2\ V,.^4$ times the sine of the fourth power of ~!2. This I write only because it's a famous one that comes up very much in physics. The combination 2ir sin4 *d~* is really the solid angle that you have in range *d4~*. So in a unit of solid angle, the cross section goes inversely as the fourth power of the sine of 4/2. And it was this law which was discovered to be true for scattering of a particles from atoms, which showed that the atoms had a hard center in the middle…a nucleus. And it was by this formula that the nucleus was discovered.

Thank you very much.

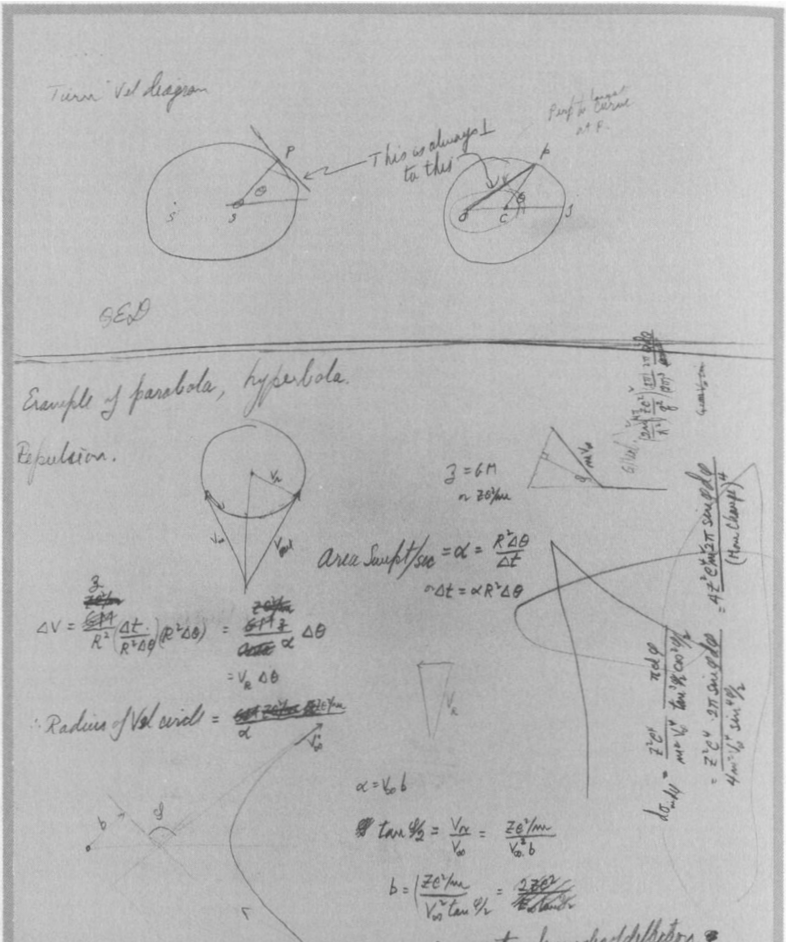
Feynman's Lecture Notes



Feynman's notes for his introductory remarks, 1964.



Most of the lecture comes from this page. The figure in the upper left-hand corner is copied from Newton's *Principia*.



Above the line, notes for the final steps of the proof of the law of ellipses.
Below the line, Rutherford's law of scattering.