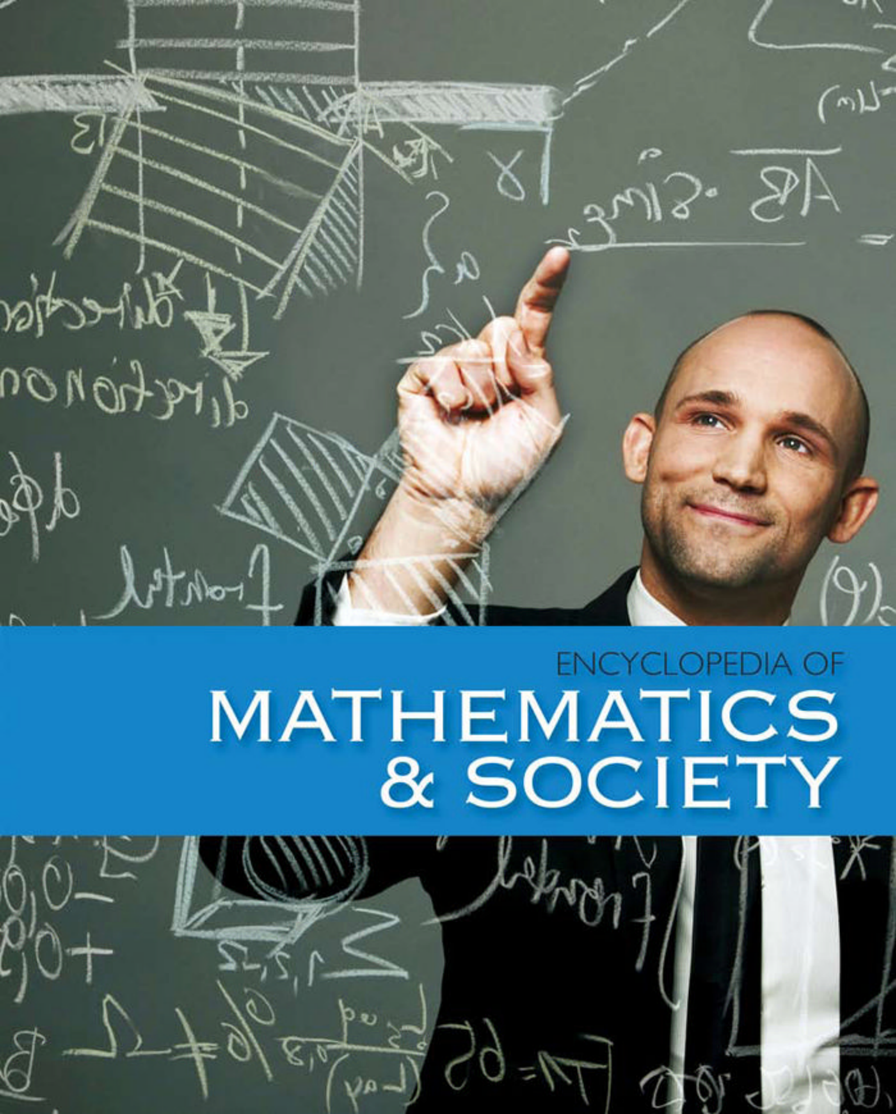

ENCYCLOPEDIA OF

MATHEMATICS & SOCIETY

ENCYCLOPEDIA OF

Mathematics and Society

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Mathematics and Society

Sarah J. Greenwald
Jill E. Thomley
Appalachian State University

VOLUME 1

Salem Press

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Contents

Publisher's Note

The *Encyclopedia of Mathematics and Society* (three volumes) explains how mathematics is at the root of modern civilization, from measuring temperature on a frigid day to driving a car to using a digital camera; enthusiasts might say applied mathematics rules the world. The set includes 478 articles, all of which were written specifically for the work.

Scope of Coverage

The *Encyclopedia of Mathematics and Society* is designed to provide students at the high school and undergraduate levels with a convenient source of information on the fundamental science and the mathematics behind our daily lives, explaining to students how and why mathematics works, and allowing readers to better understand how disciplines such as algebra, geometry, calculus, and others affect what we do every day. This academic, multiauthor reference work serves as a general and nontechnical resource for students and teachers to understand the importance of mathematics; to appreciate the influence of mathematics on societies around the world; to learn the history of applied mathematics; and to initiate educational discussion brought forth by the specific social and topical articles presented in the work.

The articles in the set fall into one or more of the following broad categories: architecture and engineering (35 articles); arts, music, and entertainment (41): business, economics, and marketing (32); communication and computers (22); friendship, romance, and religion (18); games, sport, and recreation (42); government, politics, and history (43); history and development of curricular concepts (63); mathematics around the world (21); mathematics culture and identity (27); medicine and health (34); school and society (19); space, time, and distance (25); travel and transportation (18); and weather, nature, and environment (35).

Rationale for Choice of Topics

Mathematics is a fundamental part of society, yet many people may not be aware of the interconnections between what they have learned in school and their everyday lives. In its Curriculum Guide (MAA, 2004), the Mathematical Association of America's Committee on the Undergraduate Program in Mathematics (CUPM) recommends that mathematics programs lead people "to learn mathematics in a way that helps them to better understand its place in society: its meaning, its history, and its uses." In keeping with this philosophy, the editors chose topics for inclusion based on one or more of the following criteria:

- The topic is timely and likely to remain so.
- The topic can be tied to mathematical concepts that people likely have been exposed to.
- The topic is related to concepts and connections that professional mathematical organizations have suggested are important.
- The topic is one that the general public has expressed interest in.
- The topic is one we have successfully used or that we know has been successfully used in other contexts.

Article Length and Format

Articles in the encyclopedia range in length from 500 to 3500 words. Each is first presented with the category to which it belongs (for example, architecture and engineering), an article summary, and fields of study for the article. The fields of study include the following:

- Algebra
- Calculus
- Communication
- Connections
- Data Analysis and Probability
- Geometry
- Measurement
- Number and Operations
- Problem Solving
- Reasoning and Proof
- Representations

Each article is then followed by "See Also" cross-references to other relevant articles and "Further Reading" sources that include bibliographic citations. Many articles are richly illustrated with photos and captions, and charts, graphs, and tables. Finally, each article is signed by the contributor to the encyclopedia.

Frontmatter and Backmatter

Volume 1 of the *Encyclopedia of Mathematics and Society* begins with "About the Editors" and then presents their introduction to the encyclopedia. The "List of Articles," repeated in all three volumes, features all the articles in alphabetical order with page numbers as they are listed in the encyclopedia. A "Topic Finder" shows all the articles organized by category to enable readers to find related article by topic. The "List of Contributors" presents all the writers for the encyclopedia along with their academic or institutional affiliations.

The backmatter of the encyclopedia at the end of Volume 3 has the "Chronology of Mathematics," a timeline of major milestones in the discoveries and development of mathematics. Next is the "Resource Guide" for further research that includes books that are major works in the history of mathematics as well as current editions of new works, journals in the mathematics field, and Internet sites that pertain to mathematics. A "Glossary" provides mathematical definitions for terms encountered in the articles. Lastly, a comprehensive subject index references all concepts, terms, events, persons, places, and other topics of discussion.

Online Access

Salem provides access to its award-winning content both in traditional printed form and online. Any school or library that purchases this three-volume set is entitled to complimentary access to Salem's online version of the content through our Salem Science Database. For more information about our online database, please contact our online customer service representatives at (800) 221-1592.

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- E-books are also available

About the Editors

Sarah J. Greenwald is a professor of mathematics and a women's studies core faculty member at Appalachian State University in Boone, North Carolina. She obtained her Ph.D. in mathematics from the University of Pennsylvania in 1998 and since then has published more than 35 articles. Her areas of expertise include Riemannian geometry, popular culture as it pertains to mathematics, and women and minorities in mathematics. Dr. Greenwald has discussed the impacts of scientific popular culture representations on NPR's *Science Friday.* She has spoken all over the country, and her interactive mathematics lecture appears on 20th Century Fox's *Futurama* movie *Bender's Big Score.*

Dr. Greenwald has won numerous teaching awards including a Mathematical Association of America Alder Award for Distinguished Teaching and an Appalachian State University Wayne D. Duncan Award for Excellence in Teaching in General Education. Dr. Greenwald has also been active in professional service as a member of the Joint Policy Board for Mathematics Advisory Panel for Mathematics Awareness Month and as the associate editor for the Association for Women in Mathematics, just to name a few. Her husband, Joel Landsberg, is the bassist for the Kruger Brothers.

Jill E. Thomley is an associate professor of statistics in the Department of Mathematical Sciences at Appalachian State University. Her education and scholarly interests are diverse, generally focusing on mathematics and science applications. She earned a Ph.D. in Decision Sciences from Rensselaer Polytechnic Institute and was awarded the Del and Ruth Karger Dissertation Prize and a Rensselaer Founders Award of Excellence. Additional degrees include an M.S. in industrial/organizational psychology from Rensselaer and an A.B. in psychology from Harvard University.

Along with teaching, Dr. Thomley consults on statistical design and analysis of scientific research and evaluates the results of federal education grants. Areas of focus include computational science, a discipline arising from the intersection of science, mathematics, and computer science, and the adoption and diffusion of educational innovations in mathematics and science. She presented at the first Science in Society Conference in 2009 and was published in *The International Journal of Science in Society*. Additional interests include history of statistics and statistics in popular culture.

Introduction

Mathematics is pervasive in modern society, and on some level we all use mathematics in our daily lives. At the same time, many people are not fully aware of the diverse interactions and connections between mathematics and society. Mathematics takes a readily apparent starring role in highly technological fields like engineering, computer science, and the natural sciences. Outside these fields, however, there are countless ideas, inventions, and advances that cannot be fully realized without the involvement of mathematics.

Organizations like the National Council of Teachers of Mathematics and the Mathematical Association of America recommend that mathematics be explored in the context of contemporary society. To examine these connections, we approach them from different angles. We can look at mathematics through the lens of larger societal structures like nations, cultures, and educational systems, or we can turn this method around to explore the societal structures within mathematics, such as the culture of mathematicians and notions of proof, certainty, and success.

Connections are also found in the countless applications of mathematics to society. Overall, definitions and applications of mathematics are inherently dependent on context: the socio-historical events during which they developed; the people who created or discovered concepts, who built upon the work of others, or who passed their knowledge on to the next generation; the fundamental connections to daily tasks of living; the ethics, controversies, and philosophies surrounding mathematics; the public's perceptions of mathematics and mathematicians; the way current society uses mathematics to solve problems and educate its citizens; and the way mathematics draws from society in order to grow and evolve.

Mathematics shapes the world in which we live. In the twenty-first century it is almost impossible to find an academic field of study that does not use mathematics, either directly or via tools and technology in which mathematics plays a vital role. The world in turn shapes the discipline of mathematics by inspiring mathematicians to formulate new questions, solve new problems, develop new theories, and use new technologies. Each successive generation of mathematicians brings fresh perspectives, expectations, and ways of thinking and working into the culture of mathematics. These mathematicians are influenced by the home, school, and play environments in which they were raised.

However, despite the mathematics all around us, people's exposure may be limited. Representations in the media or in popular culture may portray mathematics and mathematicians in highly stereotypical ways that do not reflect the true depth, breadth, diversity, and culture of the mathematics community.

The goal of *The Encyclopedia of Mathematics and Society* is to weave multilayered connections between society, history, people, applications, and mathematics. These connections address both mathematical concepts that our readers likely have been exposed to at school, work, or through other sources, as well as advanced topics that are built upon these fundamental ideas. The articles in the *Encyclopedia*, which were contributed by a broad spectrum of authors in many fields, also include connections to multiple disciplines within and outside of mathematics.

In general, the articles do not teach or present detailed mathematical theory, derivations, and equations. There is already a vast array of textbooks and other works better able to accomplish that important task. Instead, we intended them to serve as a foundation and jumping-off point for additional explorations. As mathematics professor and educator Art Johnson has noted in other settings, we hope that this type of contextualization helps people to "see mathematics as a discipline that transcends culture, time, and gender, and as a discipline for everyone, everywhere."

In keeping with this focus on linkages and interdisciplinarity, we have organized the articles not by mathematics topic but according to various connecting themes. For example, there are few stand-alone articles about individual people within the encyclopedia. Instead, we encouraged our authors to include significant mathematical contributors within the associated context of one or more topics or applications. The people we did choose to include as stand-alone articles serve to highlight the diversity of individuals who have produced great achievements with mathematics.

Further, our intent was to discuss, via these individuals and other articles in the Mathematics Culture and Identity theme, the community of mathematicians today: who mathematicians are, as professionals and people; the type of work mathematicians do; the different ways in which mathematicians describe mathematics and where their ideas come from; and mathematicians' personal processes when working with mathematics. We also wanted to address in these articles how the mathematics community perceives itself and how it is in turn perceived by society.

Articles within the History and Development of Curricular Topics theme highlight many of the earliest known uses, both ancient and modern advances, and people who have contributed to the development and spread of the concept or field. In contrast, the articles within the School and Society theme examine the importance of broad fields inside and outside of school, primarily in the United States. These articles showcase, for example, what jobs use particular skills and why the field is a fundamental part of current school curricula and society. The Mathematics Around the World theme extends the discussion of cross-cultural attitudes and perspectives on mathematics, with geographic regions grouped according to current United Nations standards. Other themes that center on mathematics application are Games, Sport, and Recreation; Government, Politics, and History; and Space, Time, and Distance.

Why did we choose to focus on connections? In modern society, widespread Internet access has placed data about a broad spectrum of people, objects, and events essentially at our fingertips, yet mathematics content may be buried among other discussions rather than brought to the forefront.

Both Internet and other types of library searches can result in a potentially overwhelming number of results, many of which contain almost nothing of mathematical relevance, though important connections exist. Too often, regardless of the amount of data or sources returned, connections between mathematics, people, objects, and events are missing, or they are presented in isolation from their broader historical context. Such connections are critical components of knowledge acquisition, creation, and dissemination. They are what allow people to extrapolate from what they already know to new situations, to create new knowledge or new applications, to overcome existing negative stereotypes about mathematics, and to fully understand the timeline of human events from multiple perspectives.

Even several hundred articles cannot provide an exhaustive examination of mathematics and society. At best, we can perhaps provide a snapshot of the history, people, applications, and mathematical connections as they exist at the time of publication, with some discussion of the rich history and speculations about future directions. Hopefully, this encyclopedia is a representative sampling of articles that, with the accompanying further readings, will allow a reader to follow the path to related topics of interest.

In making the very difficult decision regarding what topics to include, given that time and space were not unlimited, we used an array of selection criteria, such as: the topic was timely and likely to remain so for a reasonable period of time; the topic was tied to mathematical concepts to which people likely have been exposed; the topic was related to concepts and connections that professional mathematical organizations suggested are important; the topic is one that the general public has expressed interest in; or the topic was one that we ourselves have successfully used or that we know has been successfully used in other educational or professional contexts.

When embarking on this work, we already knew in a general sense how pervasive mathematics is in society, and we were eager to share these ideas with others. However, even though we are mathematicians with diverse interdisciplinary backgrounds, research and teaching interests, we were surprised to discover so many interesting and amazing connections. We learned more than we ever imagined we would. It was regrettably impossible to include everything we thought was interesting or important, and we have accumulated a long list of items that we want to explore in the future, on our own or with our students and colleagues.

The creation of this encyclopedia has been an intellectual pleasure and a profound learning experience, and we hope that our readers find the same kind of enjoyment and wonder that we experienced.

Sarah J. Greenwald
Jill E. Thomley
General Editors

List of Articles

Topic Finder

The following list is provided for readers to find articles related by topic.

Architecture and Engineering

Aircraft Design
Bridges
Canals
Carpentry
Castles
City Planning
Dams
Domes
Electricity
Elevators
Engineering Design
Fireworks
Green Design
Highways
Jackson, Shirley Ann
Landscape Design
Levers
Light Bulbs
Mattresses
Microwave Ovens
Nanotechnology
Packing Problems
Pulleys
Robots
Schools
Skyscrapers
Solar Panels
Symmetry
Televisions
Thermostat
Toilets
Tunnels
Vending Machines
Water Distribution
Windmills
Wright, Frank Lloyd

Arts, Music, and Entertainment

Animation and CGI
Ballet
Ballroom Dancing
Closed-Box Collecting
Comic Strips
Composing
Contra and Square Dancing
Cooking
Crochet and Knitting
Digital Cameras
Digital Images
Escher, M.C.
Geometry of Music
Golden Ratio

Business, Economics, and Marketing

Communication and Computers

Friendship, Romance, and Religion

Games, Sport, and Recreation

Government, Politics, and History

Medicine and Health

School and Society

Space, Time, and Distance

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A

Accident Reconstruction

Category: Travel and Transportation.
Fields of Study: Algebra; Data Analysis and Probability; Measurement.
Summary: Accidents can be mathematically reconstructed to model accident risk and to improve safety equipment designs.

Accident reconstruction is important for understanding how accidents happen and for preventing accidents in the future. Principles and techniques from physics, mathematics, engineering, and other sciences are used to quantify critical variables and calculate others. For example, the initial speed of a suddenly braking vehicle can be determined by mathematically analyzing tire skid and yaw marks. The length of skid marks is a function of vehicle velocity and the amount of friction between the wheels and the road surface. In the case of yaw or circular motion, the radius of the yaw mark is also a factor in the calculation, as well as the elevation of the road. Speed can also be calculated from the trajectories, angles, and other characteristics of objects struck by a speeding vehicle, or between two or more colliding vehicles. Investigators may use distances and angles to determine the original positions of passengers who have been ejected from a vehicle. For more complex modeling, mathematicians, engineers, and other accident reconstructors rely on principles and equations from physics, such as those governing energy and momentum, as well as vehicle specifications, mechanical failure analyses, geometric characteristics of highways, and quantification of visibility, perception, and reaction. Data from both real accidents and staged collisions, along with statistically designed safety analyses and other methods such as stochastic modeling, are often used to construct accident simulations and visualizations for use in a wide variety of contexts, including legal proceedings. Actuaries use accident data to model accident risk, which in turn influences insurance rates and public policy, such as seat belt and helmet laws.

Modeling Accident Reconstructions

Accidents related to travel and transportation can have a variety of negative consequences including personal injury and death. The analysis of accidents can lead to improved designs of vehicles and reduced fatalities as well as warning travelers about potential risks of travel. In reconstructing accidents, evidence from photographs, videos, eyewitnesses, or police reports is collected. Decision trees are used to ask questions at each stage of reconstruction and help decide the closest accident scenario dictated by the available evidence. In such reconstructions, probability must be assigned for the likely cause of the accident and for the particular accident type among the possible accident scenarios based on the available evidence. Stochastic modeling is used to help solve such problems in accident reconstructions.

Uses of Accident Reconstructions

Another important aspect of accident reconstructions is to estimate the probability of occurrence of various types of injuries one may suffer in accidents. Such probability estimates are used to help calculate travel insurance. By nature, accidents happen randomly and—since the types of injuries suffered in accidents also vary randomly—it is important to model accident types and predict the kinds of injuries one may suffer in different accident types. Such models can help prepare communities with the optimal number of emergency services and also help doctors prepare for any unique types of injuries they are likely to deal with.

A typical problem is determining the types of special medical facilities that should be established to deal with travel-related accidents in a city. Such problems require stochastic modeling based on past data, which will help in simulating different types of accidents. Simulations help in planning emergency services to deal with accidents. Accident reconstructions may also help in forecasting the number of accidents of different types likely to happen in the near future, which may lead to better planning of the health, emergency, and disaster management facilities in the city.

Safety and Design Using Accident Reconstructions

Accident reconstructions also may help in improving vehicle design. Incorporating safety devices in vehicles is also a very important aspect of design. Safety devices, which help in avoiding severe injuries to passengers because of accidents, are designed with the help of accident reconstruction and are always a matter of high priority. Simulations can be used to develop sensors that can give an early warning about impending accidents or reduce the speeds of vehicles—thereby reducing the severity of an accident. In creating such designs, mathematical optimization methods are used to determine the optimal cost and space to be allotted. Another crucial application of accident reconstruction and accident modeling is driver training. Sophisticated simulators can be used to simulate different accident scenarios and train drivers to react appropriately to each situation in real time. These simulators are based on algorithms and use random number generators to simulate accident situations. Well-developed algorithms that closely simulate real accidents are needed to reduce—or even eliminate—major accidents.

Further Reading

Brach, Raymond, and R. Matthew Branch. *Vehicle Accident Analysis and Reconstruction Methods*. Warrendale, PA: SAE International, 2005.

Franck, Harold, and Darren Franck. *Mathematical Methods for Accident Reconstruction: A Forensic Engineering Perspective*. Boca Raton, FL: CRC Press, 2009.

Ravi Sreenivasan

See Also: Animation and CGI; Crime Scene Investigation; Data Mining; Insurance; Probability.

Accounting

Category: Business, Economics, and Marketing.
Fields of Study: Algebra; Number and Operations.
Summary: Accounting applies mathematics to the recording and analysis of a business's financial status.

Accounting is the recording, interpretation, and presentation of financial information about a business entity, typically with the goal of producing financial statements that describe the business's economic resources in standardized terms. Formal accounting began with the work of Franciscan friar Luca Pacioli, who introduced accounting techniques in his 1494 mathematical work *Summa de Arithmetica*, *Geometria*, *Proportioni et Proportionalita*. During the Industrial Revolution, Josiah Wedgwood introduced cost accounting, a technique to ensure a profit margin by calculating the costs of materials and labor at every stage of production and setting the price accordingly. The needs of stockholders and other interested parties within the business, and an increasingly complex business environment, have increased the need for financial record-keeping techniques that are thorough and produce useful financial statements. Modern accounting is assisted by a variety of software packages, but the accountant must still be well-versed in mathematics in order to interpret the information. The fundamental accounting equation can be stated as the following:

Assets = Liabilities + Owners' Equity.

For any given company, assets can be thought of as what the company owns. This includes cash (actual cash and bank accounts), money that is owed to the business (called accounts receivables), inventory, buildings, land, equipment, and intangibles like patents and goodwill. Liabilities are what the company owes. This includes money owed to a bank (notes payable), suppliers (accounts payable), or the government (taxes payable). Owners' equity can take several forms depending on who the owners are: a single person (sole proprietor), a few people (partnership), or shareholders (corporation). Each method of ownership has advantages and disadvantages, but regardless of the method, the owners' equity can be thought of as a net asset since it can be found by subtracting liabilities from assets.


Accounting is the process of keeping track of the operations and financial status of a business.

Accounting as Record Keeping

Whenever a financial transaction takes place, it must be recorded in at least three locations. First, it will be recorded in the general ledger (a book of entry summarizing a company's financial transactions). When recorded, the entry should contain the date of the transaction, a brief description of the transaction, and the monetary changes to all accounts affected (which will be at least two).

From there, the transaction gets recorded a second time in a secondary (or subsidiary) ledger for each of the accounts affected. When the amounts are recorded, they are put into the left (debit) column or the right (credit) column of the ledger. (In bookkeeping, "debit" and "credit" mean left and right, respectively; they are not related to debit or credit cards in this situation.) The total of each column of the general ledger record must add to the same sum. In that manner, all money can be accounted for as going into or out of an account.

In order to determine whether to credit or debit an account, a general rule that works for most accounts is to first look at the fundamental accounting equation. Since assets are listed on the left, to increase assets, the transaction is recorded in the left column (debiting the account) and to decrease assets, the transaction is recorded in the right column (crediting the account). Similarly, since liabilities and owners' equity are listed on the right-hand side of the equation, to increase liabilities and owners' equity, the transaction is recorded in the right column (crediting the account) and to decrease liabilities and owners' equity, the transaction is recorded in the left column (debiting the account). For example, suppose a company needed to purchase $100 worth of office supplies. Furthermore, suppose the company pays $40 with cash and puts the remaining $60 on account (store credit). The general ledger may look like the following:

Figure 1. Purchased office supplies.

04-31-2017	Office supplies	$100	
	Cash		$40
	Accounts payable		$60

In Figure 1, notice that both the right and left columns add up to $100; this shows that no money was lost in the process. Office supplies are considered an asset, so since the company increased the amount of office supplies, that account was recorded on the left—in other words, it debited office supplies for $100. Cash is also an asset, but the company decreased the amount of cash it had. As a result, cash was credited (the transaction was recorded on the right for that account). Accounts payable is a liability the company owes to the retailer it purchased the products from. Since the company increased the amount it owed the retailer, that account was recorded on the right as an increase to the company's liabilities—accounts payable was credited.

Once this transaction was recorded in the general ledger, the company would also need to record this transaction in the Office Supplies ledger, the Cash ledger, and the Accounts Payable ledger. Accounts are debited or credited in their specific ledgers in the exact same manner that they are debited or credited in the general ledger. In a similar manner, the retailer who sold the office supplies would need to record this same transaction into his or her general and secondary ledgers. However, the retailer's transaction would use the opposite side to denote the sale as follows:

Figure 2. Sold office supplies.

04-31-2017	Cash	$40	
	Account Receivable	$60	
	Inventory		$100

Again, the right and left columns add up to the same amount. Contrary to the purchasing company, the receiving company lists three assets to record the transaction. Cash and accounts receivable are both being increased, so debited. The asset "inventory" is being decreased and results in a credit to inventory. If this were a large company, rather than record each individual transaction, the retailer would most likely record an entire day's transactions as a single entry at the end of each business day. Once the general ledger has been recorded, the secondary ledgers need adjusting entries as well to denote the transaction(s).

Benford's Law

Benford's law, named after physicist Frank Benford, gives the probability with which the numbers 1 through 9 will occur as the first digit in many types of real-life data. For example, in a list of actual bank account deposits in a given day, about 30% of the time the first digit of the deposit amount will be a 1. Fraudulent data that has been created by people often does not match the expected probabilities.

In very large modern data sets, highly focused tests use this principle to find deviations in selected subsets; for example, the occurrence of a suspiciously large frequency of $24 receipts submitted in a company that has a $25 maximum meal allowance.

Accounting as Record Sharing

In addition to keeping records of transactions for a business, accounting is responsible for creating reports that summarize the journals to share with others. To learn about the reports and how to create reports intended for people outside the business (such as shareholders, creditors, or government agencies), a person can take a class in financial accounting. To learn about the reports and how to create reports intended for people inside the business (such as managers), a person can take a class in managerial accounting.

The most common reports created for people outside the business are balance sheets, income statements, cash flow statements, and retained earnings statements. Of the four statement types, the balance sheet is written as a snapshot of the company at a point in time. In contrast, the other three statements are created to show what happened over a period of time such as a month, quarter, or year. When creating these reports, the income statement is usually completed first. As its name implies, the income statement is created to determine the company's income during a specific time period. The income statement is also known as a profit and loss statement (P&L) or earnings statement. Information from the income statement is then used to create the retained earnings statement. Finally, the information from the retained earnings statement is used on the balance sheet.

The balance sheet first lists all of the company's assets in order of liquidity (the ability to turn the asset into cash easily) from the most liquid to the least liquid. The assets are then added together to find the total assets of the company. The balance sheet next lists all of the company's liabilities in order of due date from the soonest due to the latest due. Below the liabilities is listed the owners' equity (which includes retained earnings from the retained earnings statement). The liabilities and owners' equity are added together. Referring back to the fundamental accounting equation, both of these amounts (the total assets and the sum of the liabilities and owners' equity) should equal one another.

Reports created for internal users vary widely depending on the reasoning and the need for the report. Internal reports are usually created and specifically designed for making decisions within the company. For example, manufacturers could use internal reports to determine the optimal price of their product.

Manufacturers may also use internal reports to determine if it is more cost effective to create a needed part or to purchase the part from another company. They may need to consider continuing or eliminating a division of their company. Managerial accounting is also responsible for budgeting and forecasting.

Mathematical Models

Many areas in financial accounting rely on mathematical models for explanation and prediction. For example, models have played important roles in applications such as understanding the consequences of public disclosure, formalizing market efficiency or competition, measuring income, and evaluating equilibrium pricing for goods and services. Some important mathematical techniques used in accounting models include linear regression, systems of simultaneous equations, equilibrium notions, and stochastic analysis. In the latter, random rather than constant inputs are used to model scenarios where decisions must be made under realistic conditions of uncertainty. The data used in these models may be cross-sectional (representing a single snapshot in time) or longitudinal (one or more variables are measured repeatedly to detect trends and patterns). Probability theory is also used to detect instances of accounting fraud.

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See Also: Budgeting; Payroll; Shipping.

Acrostics, Word Squares, and Crosswords

Category: Games, Sport, and Recreation.
Fields of Study: Geometry; Number and Operations; Problem Solving.
Summary: Mathematics and symmetry come into play in creating and solving word puzzles.

Acrostics, word squares, and crossword puzzles are the most common forms of word puzzles in English. Acrostics and word squares are over 2000 years old and call for the solver to discover words hidden either covertly (acrostics) or overtly (word squares). The crossword puzzle premiered in 1913 and is similar to a word square expanded onto a larger grid, with gaps. Word puzzles have been used as mnemonics, ciphers, literary devices, educational exercises, and as simple games. Their construction, especially in the case of crossword puzzles, is informed by geometry; their solution can be pursued through probability theory. In a sense, the construction and solving of word puzzles provide pleasures very similar to those of doing mathematics.

Historic Examples

The earliest examples of acrostics are in the Old Testament of the Bible. The Lamentations of Jeremiah and 12 Psalms are arranged so that the first letters of each verse spell out the Hebrew alphabet.

In Greece in 400 B.C.E., Dionysius forged a Sophoclean text titled *Parthenopaeus* with the intention of mocking his rival, Heraclides. Having declared the author to be Sophocles, Heraclides was referred to in one of the several acrostics that Dionysius had included, which read, "Heraclides is ignorant of letters."

In more contemporary times, novelist Vladimir Nabokov enjoyed chess problems, and one can find acrostics, number puzzles, cryptic references, and puns in several of his novels and stories. The last paragraph of his 1951 short story "The Vane Sisters," for example, can be read both as the narrator's confusion and acrostically (taking the first letter of each word) as a message from the dead sisters.

Acrostics are often found in poetry because of its greater flexibility in syntax and phrasing. Former U.S. President George Washington is known to have constructed at least one acrostic when he was 15—a love poem for a girl about whom nothing is known other than her name, Frances.

Another good example of an acrostic poem is to be found at the end of Lewis Carroll's 1871 book *Alice Through the Looking Glass*; each letter of the name Alice Pleasance Liddell begins a new line in the poem about childhood innocence.

Word Squares

If the first acrostics appeared in the Old Testament, word squares were not far behind. One of the most well known is a Latin word square from about 2000 years ago:

S A T O R
A R E P O
T E N E T
O P E R A
R O T A S

This word square is called a 5-by-5 symmetric word square because there are five words that can be read either down or across. The words "TENET," "OPERA," and "ROTAS" will be familiar to speakers of languages descended from Latin. SATOR is a Latin word for planter or creator. AREPO is a contentious word; it can be assumed that it was at some time used in Latin. This particular word square is unique in another way—SATOR reversed is ROTAS, AREPO is OPERA reversed, and TENET is palindromic (reads the same forward and backward).

Below is an example of an ordinary symmetrical 4-by-4 word square using English words

B A S E
A W A Y
S A L E
E Y E S

Many 5-by-5 and 6-by-6 squares exist in English. There are even a few 9-by-9 word squares, though many of the constituent words are extremely unfamiliar.

Those with an interest in algebra will notice that symmetry in word squares is equivalent to symmetry in matrices. If one transposes—swaps the rows and columns—a symmetrical word square, the resulting word square is the same as the original. A non-symmetrical word square does not have this property. A 4-by-4 double word square, like the one below, is not symmetrical. It is a double word square because it contains twice the number of words of a 4-by-4 symmetrical square, that is, eight:

D A R T
O B O E
C L A M
K E M P

Crosswords

Word squares can be entertaining in themselves. However, simply by expanding a word square onto a larger grid and using gaps to section long words into shorter ones, one can create a puzzle of an altogether different kind. By doing so, puzzle creator Arthur Wynne turned the largely esoteric practice of crafting word squares into a puzzle for the masses—the crossword.

The first published crossword appeared in December 1913 in the newspaper *New York World*. Wynne wrote definitions for each of the words he had used to complete a diamond-shaped grid, and it was up to the solvers of the newspaper's puzzle page to fill in the blanks.

Wynne's grid was almost fully "checked," which means that most letters were part of two words—a white square is "unchecked" when it is part of only one word. In U.S. crosswords, it remains the norm to have very heavily—if not fully—checked grids. For other crossword types, particularly cryptic crosswords, grids may be only 50% to 60% checked. Having a fully checked grid means that it is possible to complete the crossword by entering only the across (or down) words. As the number of unchecked squares increases, however, the ability to build on one's correct answers decreases. Most crosswords have a 15-by-15 grid and twofold rotational symmetry (they look the same after 180 degrees of rotation), but differences in the number of checked squares can produce as many as 80 words or as few as 30.

PROVERB, a computer program designed to solve crosswords, relies on the heavily checked nature of American-style grids. Computer scientist Michael Littman and others report that PROVERB averaged more than 95% correct answers in less than 15 minutes per puzzle on a sample of 370 puzzles. This result is better than average human solvers but not better than the best. If nothing else, the complexity of the PROVERB program serves to highlight the vast computing power humans naturally possess.

Instinctively, many people may not be aware that the five most frequently used letters in the English language are E, T, A, O, and I. Crosswords setters (and PROVERB), on the other hand, are acutely aware of this and aim to use letters in their longer words that will be easy to intersect with the shorter ones. It is therefore worth bearing in mind that, for example, "Erie" and "Taoist" will appear in crosswords much more often than "jazz" and "Quixote." Incidentally, the five least frequently used letters are K, J, X, Z, and Q.

Estimates suggest that fewer than 100 people construct crossword puzzles for a living in the United States. Mathematician Byron Walden has been called "one of the best" by a *New York Times* crossword editor. For some, he may be most well known for writing the puzzle that was used in the championship round of the American Crossword Puzzle Tournament, later featured in the film *Wordplay*. He has also analyzed and given talks on symmetry and patterns associated with conventional crossword construction, with the aim of helping people become more skilled puzzle solvers.

Mathematician Kiran Kedlaya is also a well-known puzzle solver and creator. He believes that the brain processes required for computer science, mathematics, music, and crossword puzzles are similar, and he pursues all of these activities professionally and recreationally. One puzzle he created was published on the well-known *New York Times* crossword page, and he regularly contributes mathematics puzzles to competitions like the USA Mathematical Olympiad. He has been quoted as saying, "It's important to tell kids who are interested in math as a career that there are many venues to do it, not just in the academic area within math departments."

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Eoin O'Connell

See Also: Literature; Poetry; Puzzles; Religious Writings; Sudoku.

Actors

See *Writers, Producers, and Actors*

Addition and Subtraction

Category: History and Development of Curricular Concepts.
Fields of Study: Communication; Connections; Number and Operations; Representations.
Summary: Addition and subtraction are binary mathematical operations, each the inverse of the other, and are among the oldest mathematical concepts.

Addition can be thought of as a process of accumulation. For example, if a flock of 3 sheep is joined with a flock of 4 sheep, the combined flock will have 7 sheep. Thus, 7 is the sum that results from the addition of the numbers 3 and 4. This can be written as $3 + 4 = 7$ where the sign "+" is read "plus" and the sign "=" is read "equals." Both 3 and 4 are called addends. Addition is commutative; that is, the order of the addends is irrelevant to how the sum is formed. Subtraction finds the remainder after a quantity is diminished by a certain amount. If from a flock containing 5 sheep, 3 sheep are removed, then 2 sheep remain. In this example, 5 is the minuend, 3 is the subtrahend, and 2 is the remainder or difference. This can be written as $5 - 3 = 2$ where "−" is read "minus." Subtraction is not commutative and therefore the ordering of the minuend and subtrahend affects the result: $5 - 3 = 2$, but $3 - 5 = -2$.

The concept of addition can be extended to have meaning for fractions, negative numbers, real numbers, measurements, and other mathematical entities. The algorithms used for computing the sum or difference, some of which have been taught for millennia, ultimately depend on the representation used for the numbers. For example, the approach used for adding Roman numerals is different from that used to add Hindu-Arabic numbers. Computers perform subtraction using the same circuits they use for addition.

History and Development of Addition and Subtraction

Human beings' ability to add and subtract small whole numbers is probably innate. Some of the earliest descriptions of techniques for handling large numbers come from ancient China during the Warring States period (475–221 B.C.E.), when arithmetic operations were performed by manipulating rods on a flat surface that was partitioned by vertical and horizontal lines. The numbers were represented by a positional base-10 system. Some scholars believe that this system—after moving westward through India and the Islamic Empire—became the modern system of representing numbers.

The Greeks in the fifth century B.C.E., in addition to using a complex ciphered system for representing numbers, used a system that is very similar to Roman numerals. It is possible that the Greeks performed arithmetic operations by manipulating small stones on a large, flat surface partitioned by lines. A similar stone tablet was found on the island of Salamis in the 1800s and is believed to date from the fourth century B.C.E. The word "calculate" was derived from the Latin word for "little stone."

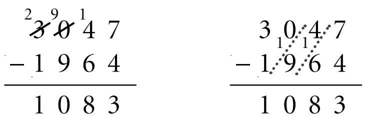
The Romans had arithmetic devices similar in appearance to the typical Chinese abacus. It is difficult to use modern paper-and-pencil techniques for adding and subtracting Roman numerals (with I as one, II as two, V as five, X as ten, L as fifty, C as one hundred, D as five hundred, M as one thousand)—but it worked well in its time, since it was devised for use with an abacus.

During the Middle Ages, counting boards were used to perform arithmetic. A counting board consisted of a series of actual or virtual horizontal lines that were labeled from the bottom by I, X, C, M, and so on. The system borrowed the symbols used for core numbers from the Roman system. The spaces between the lines were labeled starting from the bottom by V, L, and D. A number like MMDCCXXXVIIII (2739) would be represented by placing the appropriate number of counters on each line. The line labeled M would have 2 counters (for 2000, or two thousands). The space just below, labeled D, would have 1 counter (500, or one five-hundreds); the line labeled C, 2 counters (200, or two hundreds); the space labeled L, 0 counters; the line labeled X, 3 counters (for 30, or three tens); the line labeled V, 1 counter (5); and the line labeled I, 4 counters (4, or four ones). The total of all these numbers is 2739. Note that accountants used VIIII (denoting five plus four) to represent 9, whereas stonemasons used "IX"(denoting 10 less 1). To compute the sum MMDCCXXXVIIII + MCLXI, a person would simply transcribe the numbers to the counting board and then combine the counters following rules of carrying to ensure that no more than 4 counters were on any line and 1 counter on any space. This representation was then easily transcribed back into Roman numerals.

Many early books on arithmetic claim that this method of performing arithmetic was especially preferred by women, who at times had the responsibility for keeping the books for small family businesses. Hindu-Arabic numerals and paper-and-pencil methods for performing arithmetic began to appear in Europe in the twelfth century and replaced Roman numerals and the counting board by the nineteenth century.

Two Methods for Subtracting by Hand

Two popular methods for handling "borrowing" that are taught today are shown below. The method shown in the figure below on the left is popular in Italy, England, and the United States, while the one on the right is popular in Spain, France, and parts of Latin America. The example is to compute 3047 − 1964. Starting with the method on the left, first begin with the rightmost column and subtract 4 from 7. Write the result, 3, below the 4. Moving one column to the left, try to subtract 6 from 4, which cannot be done without using negative numbers. The method is thus to attempt to "borrow" 1 from 0, which is the digit to the left of the 4. Again, this cannot be done without using negative numbers. Therefore, the method is to borrow 1 from 3, which is the digit to the left of the 0 resulting in crossing out the 3 and replacing it with a 2. Then the zero becomes a 10, and it in turn can be replaced by a 9 so the borrowed 1 can be placed in front of the 4 to make it 14. Now, one can subtract 6 from 14 to get 8, which is written below the 6. Moving left to the next column, one can subtract 9 from 9 to get a 0, which is written below the 9. Finally, 1 is subtracted from 2 to get a 1, which is written below.



To solve the problem using subtraction with carry, use the example on the right. The carrying numbers (the small 1s) affect the numbers on a diagonal, as shown in the example. The number 1 adds 10 to the integer in the top row and adds 1 to the integer in the bottom row. Starting from the rightmost column, 4 is subtracted from 7, resulting in 3, which is written below. Then, try to subtract 6 from 4, which cannot be done, so insert a small 1 to the left of the space between the 4 and the 6. This is interpreted to mean that the 4 has become 14. Subtract 6 from 14 and record the answer, 8, below. Move left to the next column containing 0 and 9. The small 1, written above and to the right of the 9, is added to the 9 to get 10. Attempt to subtract the 10 from the 0 above, which cannot be done. Instead, write a small 1 just to the left of the space between the 0 and 9, and interpret this to mean that the 0 has become a 10. Now, 10 minus 10 is 0, which is written below. Move left to the next column. The small 1, written above and to the right of the 1, is added to the 1 giving 2, which is subtracted from 3 resulting in 1, which is written below.

Adding and Subtracting on a Computer

At the most basic level, whole numbers are represented in a computer in base-two by a sequence of the binary states "Hi" and "Lo" interpreted as "1" and "0." The circuits that perform addition are implemented by sequences of logical gates. Typically a "1" in the leftmost bit indicates that the number is negative, with the remaining bits indicating the magnitude of the number. Subtraction can be performed by the same circuits that perform addition. Two popular approaches are designated as "one's complement" and "two's complement." "One's complement" can best be explained by performing subtraction in base-10 using "nine's complement." Assume a computation of 3047 − 1964. To find the "nine's complement" of 1964, subtract each digit from 9 to obtain 8035. This is added to 3047 resulting in 11,082. The leftmost 1 is viewed as a "carry" and brought around and added to the rightmost digit in an operation called "end-around carry" to obtain the final result: 1083.

Generalizing Addition and Subtraction

The sum of two fractions a/b and c/d is defined to be

$$\frac{ad+bc}{bd}.$$

The sum of irrational numbers (numbers that cannot be represented as fractions of whole numbers) can be approximated only by adding their approximating rationals. The exact sum of two irrational lengths, a and b, can be found exactly using geometry by first extending the segment representing a sufficiently on one end so that the length b can be marked off from that end with a compass.

Addition can be generalized to other mathematical objects, such as complex numbers and matrices. One of these objects, typically called the additive identity and denoted by "0," has the property such that if "a" is any object then the sum of 0 and a is a. The additive reciprocal of an object a is denoted by $-a$ and is defined to the object so that the sum $a+(-a)$ is 0. The difference $a-b$ is defined to be $a+(-b)$.

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See Also: Multiplication and Division; Number and Operations; Number and Operations in Society.

Advertising

Category: Business, Economics, and Marketing.
Fields of Study: Algebra; Data Analysis and Probability; Number and Operations.
Summary: Mathematics is used to weigh the costs and gains of advertising and to profile and target consumers.

Advertising delivers product information from suppliers to consumers—suppliers may be manufacturers, hospitals, software developers, educators—and is critical to the success of a business in marketing development. Advertising media may be traditional (such as television, newspapers, and posters) or technological (via Internet and e-mail), as well as commercial (to sell products for profit) or noncommercial (in political campaigns or for religious purposes). The annual advertising cost in the United States amounts to more than $100 billion.

Advertising includes two stages: the planning stage for marketing strategies, whose goal is business development, and the analysis stage of cost analysis involved with the forms and the contents of communication between suppliers and potential customers. Mathematics and statistics play critical roles in both stages of advertising.

Market Shares

In the planning stage, the analysis of market shares for advertising necessitates matrix operations and multivariate probability inequalities to portray the dynamics of market shares over time. The following is an example of matrix operations, which bridge advertising with market shares. Consider the market shares of General Motors (GM) and Ford in the U.S. automobile industry. Assume that the current market shares distribute as follows:

General Motors:	21%
Ford:	17%
Other Manufacturers:	62%

If GM starts an advertising campaign with the goal of increasing the market share to 29% in three years, GM may count on customers to switch from Ford or other manufacturers to GM. However, in reality, some of the GM customers may switch to Ford or to other manufacturers.

Let a_1, a_2, a_3 be the percentages of original GM users who, at the end of the advertising campaign, remain with GM, who switch to Ford, and who switch to other manufacturers, respectively. Let b_1, b_2, b_3 be the percentages of original Ford users who switch to GM, who remain with Ford, and who switch to other manufacturers, respectively. Let c_1, c_2, c_3 be the percentages of the other customers who switch to GM, who switch to Ford, and who remain with their manufacturers, respectively. Then, the market shares x_{GM}, x_{Ford}, and x_{Others} at the end of the three years are determined by the following simple matrix equation:

$$\begin{bmatrix} x_{\text{GM}} \\ x_{\text{Ford}} \\ x_{\text{Others}} \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} 21\% \\ 17\% \\ 62\% \end{bmatrix}.$$

If GM intends to increase x_{GM} to 29%, GM should advertise specifically to different groups of customers. This is mathematically equivalent to manipulating the elements in the 3×3 matrix above within plausible ranges of the elements.

The foregoing scenario is a simplified example to illustrate the role of matrix operations in advertising. In reality, the story is more complex. For example, the 3×3 matrix above will become an $n \times n$ matrix, where n is the number of competing suppliers in the market. Also, the stochastic feature of the supply-demand market, the market shares, and the corresponding elements for the $n \times n$ matrix change constantly under the influence of the advertising campaign.

Thus, it is more appropriate to treat the market shares as a vector consisting of random variables. In this case, one of the convenient approaches to evaluating the market shares is the method of multivariate probability inequalities in conjunction with the construction of Hamilton-type circuits.

Advertising Costs and Effects

The analysis stage examines costs and effects associated with various communication channels and advertising media. For instance, in Internet advertising, typical cost considerations are cost per mile (CPM), cost per click (CPC), and conversion rate. These terms have strong connections with mathematics and statistics.

For Web advertising, CPM usually refers to the cost for every thousand visits to the publisher's Web site. For example, assume that an ad network offers a \$5 CPM for a banner, which was put on three Web sites for three months. If the total page views for the three Web sites are 80,000, 110,000, and 140,000 during the three-month period, the total cost of Web advertising for the ad network is

$$\$5\,\frac{80{,}000}{1000} + \$5\,\frac{110{,}000}{1000} + \$5\,\frac{140{,}000}{1000} = \$1{,}650$$

In general, if an ad is posted in n Web sites, the total cost is

$$\sum_{i=1}^{n} \text{CPM} \times (W_i / 1000)$$

where W_i is the number of Web impressions (visits) to the i^{th} publisher's Web site for the same period of time.

Consider that the number of Web impressions on each publisher's Web site depends on many continuously changing factors; then W_i is a random number. Let $E(W_i)$ be the expected value of W_i, which measures the long-term average of the number of Web impressions of the banner on the i^{th} publisher's Web site. The long-term average cost is

$$\sum_{i=1}^{n} \text{CPM} \times (E(W_i)/1000).$$

CPC refers to the amount that the advertiser pays for each click generated from the Web publisher. For example, if the cost per click is \$0.04, and three Web publishers generate 1700, 1600, and 900 clicks in three months, the cost of Web advertising is

$$\$0.04(1700) + \$0.04(1600) + \$0.04(900) = \$168$$

In general, if a Web ad is posted in m Web sites, the total cost is

$$\sum_{i=1}^{m} \text{CPC} \times C_i$$

where C_i is the number of clicks generated on the i^{th} publisher's Web site for a given period of time.

Consider the fact that the number of clicks on each publisher's Web site depends on various unexpected factors: C_i is actually a random variable. Let $E(C_i)$ be the expected value of C_i, which measures the long-term average of the number of clicks generated from the i^{th} publisher's Web site over a given period of time. The long-term average ad cost is then

$$\sum_{i=1}^{m} \text{CPC} \times E(C_i).$$

The foregoing two concepts, CPM and CPC, measure the potential impact of the internet ad only in terms of clicks or Web visits. However, these two concepts are unable to provide the advertiser with information regarding whether the Web impression has been transferred into the desired action (such as buying the advertised product). A useful measurement in Web advertising to help account for the advertising effect is the "conversion rate" (or CR, the average number of people taking the action encouraged by the ad per 100 visits to the publisher's Web site). For example, if out of 2000 clicks on an ad posted on a publisher's Web site, 12 people end up buying the product, the conversion rate of the ad for this Web site is then

$$\left(\frac{12}{2000}\right) \times 100 = 0.6\%.$$

Being highly associated with key factors such as the design of the publisher's Web site, the conversion rate is an index that directly measures the final impact of the ad for the Web site.

Since the conversion rate directly reflects the performance of the Web site, it can be used to compare

Figure 1.

	May	June	July	August
Google AdSense	5%	6.1%	4.3%	7.5%
Chitika	7.3%	5.2%	5.7%	6.4%

advertising effects of two or more Web sites. However, it is risky to compare conversion rates directly. The example in Figure 1 helps illustrate this point. Consider two Web sites: Google AdSense and Chitika. If the conversion rates of the two Web sites are as follows in the past four months, it is impossible to claim which site has better performance on Web advertising.

In fact, the raw values shown in Figure 1 include the stochastic influence of many online factors. In this case, to evaluate the monthly advertising effect of different Web sites accurately, statistical data analysis is needed.

Because of random effects, the expected value of the conversion rate of each Web site should be considered when comparing two or more publishers' Web sites in terms of the conversion rates. Given a set of historical data involving all the Web sites of interest, one of the statistical estimation approaches is the method of "simultaneous confidence intervals," which compares the ranges of expected conversion rates with a prespecified confidence level. For example, with a set of data for the conversion rates of three Web sites over a period of time, if a 95% simultaneous confidence interval reads

$$0.5\% < CR_{\text{Google}} - CR_{\text{Chitika}} < 2\%$$

and

$$1.3\% < CR_{\text{Google}} - CR_{\text{Yahoo}} < 3.4\%$$

it means that at 95% confidence level, the advertising performance (in terms of conversion rate) of Google is better than that of Chitika and Yahoo.

To enhance the accuracy of the simultaneous confidence ranges, or to improve the power of testing multiple advertising effects, the two-stage estimation procedure can be considered. When the underlying distribution of the monthly conversion rates is skewed, the two-stage estimation procedure can be used with nonparametric tests to make inferences on the performance of multiple Web sites.

Data Mining and Advertisements

Masses of personal data being collected every day about consumers, via mechanisms like credit card applications, consumer discount cards, and product views and ratings on shopping Web sites are poised to revolutionize the field of advertising. Data mining is the mathematical and statistical method for sifting through large volumes of data to find patterns and create prediction models, in this case of consumer behavior. In 2009, the online video rental company Netflix awarded a $1 million prize to the winners of its three-year contest to develop a better algorithm to predict what movies its users would prefer, based on ratings data provided by the company.

Finally, mathematics is used not only to decide when, where, and how to advertise products and services but also to determine what to emphasize within the advertisements themselves: discounts on pricing or the number of calories per serving, just to name two. However, it is often difficult to verify those numbers. Many will remember Trident Gum's 1960s slogan, "Four out of five dentists surveyed would recommend sugarless gum to their patients who chew gum." Although the statement was popular at the time, its legitimacy was later questioned, since it came from a survey whose details have never been released.

IBM has initiated a Smarter Planet campaign focused on dispersed or cloud computing (Internet-based computing). Its "Smarter Math Builds Equations for a Smarter Planet" commercial cites mathematics as the universal language and gives a number of ways in which mathematics will be used to create a "smarter planet."

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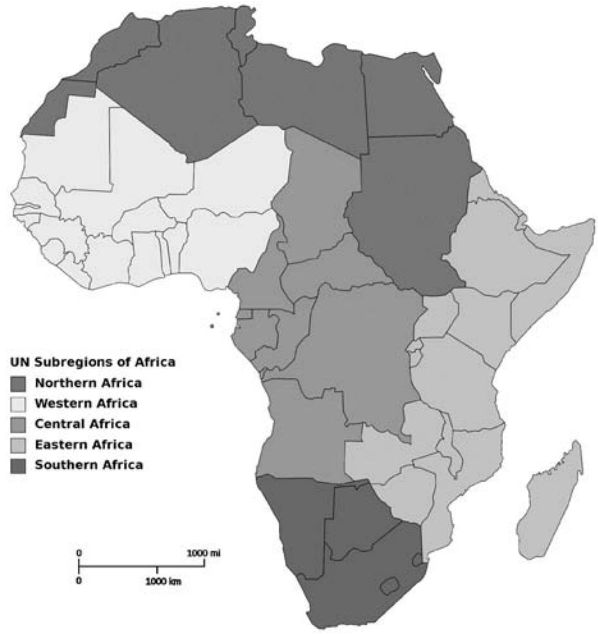
John T. Chen

See Also: Expected Values; Market Research; Matrices.

Africa, Central

Category: Mathematics Around the World.
Fields of Study: All.
Summary: Central African contributions include counting games and decorative geometric patterns.



Central Africa is composed of eight countries and is shown in the medium gray shaded area.

Central Africa comprises Angola, the Central African Republic, Chad, Congo, the Democratic Republic of the Congo, Equatorial Guinea, Gabon, and Sao Tome and Principe. Mathematical concepts developed in central Africa include variations of the counting game Mancala and the sophisticated geometric patterns used in traditional art. These patterns, in sand art and pottery, woven into mats and baskets, and displayed in tattoos, include complex symmetries and fractals. Some educators have advocated incorporating these indigenous African manifestations of mathematics into school curriculums.

Mancala

As with much of Africa, variations of the mathematical counting game Mancala were played throughout the region. The mathematics of Mancala games are discussed in more detail in the entry "Africa, East," but some description here is warranted. *The Complete Mancala Games Book* gives rules for 28 different versions of this game played in central Africa. These variations arise throughout much of central Africa but especially in Cameroon and the Congo. While the version of Mancala best known in the United States is a two-row version (also called Wari or Oware), many of the variations played in the Congo have four rows, which adds substantially to the complexity of the game, as well as the complexity of the arithmetic calculations and logical thinking required to play them well. Even with the two-row version, the Congolese variation Mbele uses a complicated game board (a two-row version with many holes in each row, with the rows pinched together near the ends). Again, this adds mathematical complexity to the game.

Geometric Patterns

Many of the most interesting mathematics developed by the peoples of central Africa have been geometric in nature. A significant part of African art traditions include quite complex—and mathematically sophisticated—geometric patterns. These patterns include symmetries in various combinations, between different elements, and between various colors. Claudia Zaslavsky writes: "If one wanted to survey the whole field of geometric design in Africa, one would have to catalogue almost every aspect of life." In central Africa, such geometric patterns are found on pottery, cloths, mats, carvings, baskets, bowls, tattoos, and other objects of daily use.

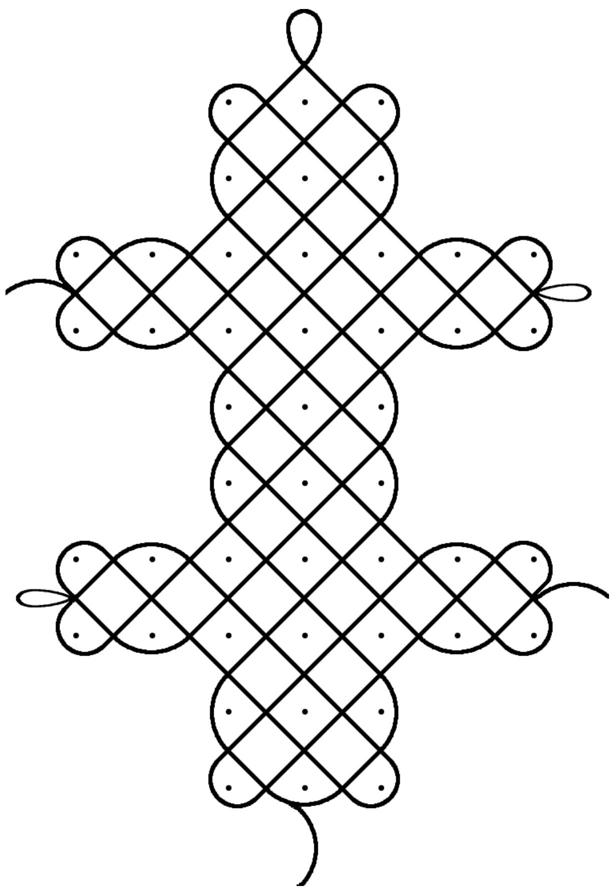
The Kuba people of the Congo are particularly famous for such art, especially their raffia embroidered cloth. Both *Africa Counts* and *Geometry From Africa* show many examples of Kuba artwork, along with artwork of other African peoples. The woven mats of the Yombe women of the Congo are another example of complex geometric design. Paulus Gerdes has studied these mat designs as an interplay between cultural values and mathematics.

The art of the Chokwe people of the Congo and Angola includes a mathematically challenging art form called "sona," usually drawn in the sand. These drawings are made with a single line continuously weaving through an arrangement of dots, such as the "Lion With Cubs" drawing of the accompanying figure. The heads and tails of the animals are added after the principal line is drawn. These drawings represent stories, morals, or values of the Chokwe, or just an animal or object from their environment. The techniques for determining which dot arrangements will generate such one-line drawings are fundamentally mathematical in nature. Drawings that can be done in a single line, without retracing, are a mathematics topic known as Eulerian Graphs. This artwork of the Chokwe is strongly connected to this mathematical idea, and was being investigated by the Chokwe artists about the same time that the idea was first studied by European mathematicians in the mid-eighteenth century.

The geometric patterns of central Africa extend to include fractal designs. Fractals are a mathematical structure that can be viewed as a repetition of the same shapes at many different sizes or scales. For example, trees have branches, each with smaller branches, and then even smaller branches. Western architecture often has rectangular blocks with rectangular houses, but rarely are such shapes repeated at more than two scales, and rarely is this a conscious shape imitation. African fractals often use circular, oval, or diamond shapes at several scales, with smaller shapes inside or around the larger shapes. There is substantial evidence that at least some of these fractal designs are a conscious choice of the artists and builders, and not accidental. *African Fractals* shows several Cameroonian examples of fractal designs in cities and villages, and even in hair braiding. This book also shows a similar style of pattern, using increasingly smaller but otherwise identical shapes in the art of the Mangbetu people of the Congo.

Education

Several African educators have suggested incorporating these traditional mathematical elements into their schools. The Cameroonian educator A. N. Boma writes: "In African traditional education, the curriculum was organized holistically rather than in discipline areas such as mathematics, history. . . .Education for all cannot afford the luxury of isolating education in terms of disciplines, rather it should take the holistic approach in developing a total person. . . ." The ideas described here integrate mathematics with cultural,



"Lion With Cubs" drawing made with a single line weaving through an arrangement of dots.

artistic, and other elements to achieve this holistic approach. Unfortunately, the schools in Central Africa cannot easily incorporate such ideas. The 2009 *Mathematics in Africa* report describes low percentages of the population attending schools, high student-to-teacher ratios, heavy use of recycled European mathematics textbooks, and few prepared teachers in most of central Africa outside of Cameroon. All of these facts make it difficult to customize mathematics education for African students. Cameroon has a more developed education system, but at the college level it is struggling with filling the mathematics faculty positions that have been approved, and most mathematics teaching there is done in large classes by low-level staff. Nevertheless, with more than half of the central African Ph.D.s in mathematics, Cameroon may become a leader in mathematics education for the region.

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Darrah Chavey

See Also: Africa, Eastern; Africa, Southern; Africa, West; African Mathematics; Board Games; Graphs.

Africa, Eastern

Category: Mathematics Around the World.
Fields of Study: All.
Summary: East African contributions include Mancala, logic games, and games similar to Tic-Tac-Toe.

Eastern Africa is the birthplace of the human species, and includes Burundi, Comoros, Djibouti, Eritrea, Ethiopia, Kenya, Madagascar, Malawi, Mauritius, Mayotte, Mozambique, Reunion, Rwanda, Seychelles, Somalia, Tanzania, Uganda, Zambia, and Zimbabwe. Mancala, an ancient counting game with many variations throughout the continent, originates in East Africa, which is also home to complicated geometric patterns in woven art and a number of logic puzzles and other mathematical games. The quality of mathematics education continues to be a serious concern.

Mancala

Eastern Africa is home to an impressive variety of mathematically based games. The most well known are the many variations of Mancala, often called the "African national game." Although there are hundreds of variations, the general idea is: (1) stones or seeds are placed in pits laid out with two to four rows and several pits per row; (2) players collect the seeds from one pit and "sow" them one at a time into other pits around the board; (3) under some circumstances, the player picks up the seeds from the final pit and continues sowing those seeds; (4) when the move ends, the player will, in some cases, capture seeds from his or her opponent. These games generally involve a substantial amount of counting, adding, and subtracting (for example, to determine where the final seed will land), as well as consideration of multiple possibilities, analysis to calculate where an opponent can move afterward, strategy, and logic. It is no wonder that some leaders (including Tanzanian president Julius Nyerere) were first noticed as good Mancala players. It is uncertain where the game originated, but the oldest dated game boards come

The game of Mancala has many variations, one of which is played on this board in Zanzibar.

from Ethiopia and Eritrea about 1300 years ago. The game is surely older than that, possibly as much as 3300 years old. *The Complete Mancala Games Book* includes 61 different variations of this game played in eastern Africa, including variations specific to every country except Burundi.

Other Puzzles and Games

Logic puzzles come in many forms. One puzzle type common to eastern Africa is the river-crossing puzzle. For example, a man with a wolf, a goat, and a cabbage must use a boat to cross a river except that (1) he can take only one item across at a time; and (2) the goat cannot be left alone with the wolf (who would eat it) or the cabbage (which it would eat). These kinds of puzzles are mathematical because, as Marcia Ascher writes, "A stated goal must be achieved under a given set of logical constraints." Variations of this puzzle, with different logical constraints, appear in Ethiopia, Zambia, and Mozambique.

Several "three-in-a-row" games, related to Tic-Tac-Toe, are played in eastern Africa. In Shisima, from Kenya, players start with an octagonal board, the eight corners, a center point, and lines connecting opposite corners through that center. Players start with three stones each, on the corners closest to them. During a turn, players move one of the stones to one of the nine points (eight corners and the center) connected to it, if it is empty. The goal is to get three stones in a row (a straight line), which must include the center and two corners opposite each other. *Africa Counts* describes two other three-in-a-row games from Zimbabwe, each of which begins like Tic-Tac-Toe where players place stones on points on the board, then continues like Shisima with players moving their stones to get a triple. In Tsoro Yematatu, the board has seven spots, each player has three stones, and one spot is always empty. In African Morris, there are 24 spots, and each player has 12 stones. Here, it could happen that the board becomes filled, but if there is a three-in-a-row during that stage, the player does not win; instead, the player captures an opponent's stone. Hence, the game usually continues into the second phase. These three-in-a-row games are logic puzzles and are examples of games of position, which have been widely studied in mathematics.

Geometric Patterns

The geometric patterns of art from eastern Africa contain a great deal of mathematical and geometric structure and symmetry. Some of the most well known of such crafts are the woven *sipatsi* baskets of Mozambique, and other types of woven baskets and mats from Mozambique, Kenya, Tanzania, Uganda, and Madagascar. This artwork contains varied types of symmetries and dramatic patterns. Paulus Gerdes writes that this art "reveals the force of the imagination and the artistic and geometric creativity of the women and men who weave [these baskets]." Examples exist in the Ba-ila settlement in Zambia and in Ethiopian processional crosses.

Mathematical Education

Mathematical education in eastern Africa shares many of the challenges that exist throughout the continent, especially the lack of prepared teachers at the secondary level. As the South African mathematics educator Jan Persons writes, "At the departure of the Portuguese from Mozambique in the early 1970s, there were only a handful of qualified secondary mathematics teachers. In general, starving the local population of decent and effective education was used as a weapon to halt or, at least, retard development."

This issue has been a major problem in eastern and central Africa, which combined have 48% of Africa's population but have produced less than 8% of Africa's mathematics Ph.D.s. Kenya has a strong college-level mathematics program, having produced nearly half of all Ph.D.s in eastern Africa. Unfortunately, as also happens in central Africa, most of the mathematics students are attracted into professions other than teaching because of the low salaries for teachers. There are several efforts in place to improve mathematics education in these countries, but much work on the educational structures remains to be done throughout this region.

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Darrah Chavey

See Also: Africa, Central; African Mathematics; Basketry; Board Games; Mathematical Puzzles; Tic-Tac-Toe.

Africa, North

Category: Mathematics Around the World.
Fields of Study: All.
Summary: North Africa has been a major contributor to mathematics, particularly in ancient Egypt and the Islamic Golden Age.

North Africa, comprised of Algeria, Egypt, Libya, Morocco, Sudan, Tunisia, and Western Sahara, has long been geographically and culturally distinct from the rest of the continent because of the Sahara desert (which includes most of the region) and the proximity to southern Europe and the Middle East. The mathematics of ancient Egypt is among the oldest known mathematics traditions, and the Egyptian city of Alexandria was an important center of learning in the ancient world. Centuries later, Egyptian mathematicians were among the contributors to the Islamic Golden Age, translating classical works, which also helped bring about the Renaissance and Age of Enlightenment.

Mathematics historians and teachers have explored a variety of historical mathematics in the area, such as string figures and precolonial mathematics in Sudan, or the work of Gaston Julia, who was born in Algeria at the end of the nineteenth century and is known for his investigations on dynamical systems. The Julia set is named for him. Modern mathematicians and scholars in North Africa continue to take part in mathematics research and teaching.

Ancient Developments

Papyrus scrolls predating 1500 b.c.e. have been found in Egypt that discuss mathematical topics. One of the more famous is the Ahmes scroll (after the name of the scribe to whom it is attributed), currently held in the British Museum, which describes many problems in algebra and geometry and demonstrates their solutions. It is of particular interest for its use of unit fractions (fractions with a numerator of 1, such as 1/8) and for demonstrating a method of calculating circular areas.

In the Hellenistic period (c. 323–146 B.C.E.), and in the Roman period that followed, the city of Alexandria in Egypt was a center of learning, and the Great Library of Alexandria was the most important library in the ancient world. Euclid (c. 300 B.C.E.), a Greek mathematician who worked in Alexandria, is best known for his treatise *Elements*, which formed the basis for how geometry has been understood and taught for more than 2000 years. Eratosthenes of Cyrene (276–194 B.C.E.) was born in what is now Libya. He estimated the circumference of Earth and is known for the Sieve of Eratosthenes, which is useful in number theory.

One of the best-known Egyptian mathematicians from the Roman period was Ptolemy (c. 90–168 C.E.), a Roman citizen who lived in Egypt. One of his well-known works is the *Almagest*, the most comprehensive surviving ancient treatise on astronomy. Hypatia (c. 350–415), a Greek who lived in Alexandria, was a female mathematician who wrote commentaries and was also known as a teacher of astronomy and philosophy.

Islamic Period

Mathematics flourished during the Islamic Golden Age (c. mid-eighth to mid-thirteenth century). One impetus to this development was the translation of classical Greek works, such as Ptolemy's *Almagest* and Euclid's *Elements*. These translations were often the only surviving copies and their preservation by Islamic scholars allowed them to be reintroduced into Western thought. Besides the appreciation of knowledge for its own sake, the development of mathematical sciences had practical uses in the Islamic world; for instance, knowledge of astronomy was required to understand the phases of the moon and thus correctly observe Islamic holy days, while algebraic notation was developed in part to solve problems relating to the laws of inheritance. Geometric motifs are very common in Islamic art and design, in part because, for religious reasons, Islamic artists did not create representational art, such as portraits. Instead, complex patterns such as tessellation figures (tilings) were developed for artistic use.

Many mathematicians worked in Egypt during the Islamic Golden Age. Ahmed ibn Yusuf (c. 835–912) was born in what is now Iraq but moved to Egypt and died in Cairo. He worked with his father, Yusuf ibn Ibrahim, on mathematics and wrote a book on ratio and proportion, which commented on Euclid's *Elements* and was translated into Latin in the twelfth century. Abu Kamil Shuja ibn Aslam (c. 850–930) was a mathematician who made important contributions to the study of real numbers, irrational numbers, and combinatorics, and some of whose techniques were adopted by the thirteenth-century Italian mathematician Fibonacci. Ibn Yunus (c. 950–1009) was an Egyptian astronomer and mathematician whose most famous work is a handbook of astronomical tables, which is notable for the accuracy of his observations and for his meticulous description of numerous planetary conjunctions and lunar eclipses. Abu Ali al-Hasan ibn al-Hasan ibn al-Haytham (c. 965–1039) was born in Persia but lived primarily in Egypt and died in Cairo. He worked as an engineer, reportedly attempting to develop a method to dam the Nile River, and made important contributions to optics and to the development of the scientific method. Al-Marrakushi ibn Al-Banna (c. 1256–1321) lived in Morocco and may have been born there. He worked on Euclid's *Elements* and texts on algebra and arithmetic operations.

Besides being an Egyptian mathematician, Ptolemy was also an astronomer, a geographer, and an astrologer.

Modern Developments

In the early twenty-first century, mathematical study and research continues in North Africa. Mathematicians belong to professional organizations like the Association Mathématique Algérienne, the Egyptian Mathematical Society, the Tunisian Mathematical Society, and the Société des Sciences Naturelles et Physiques du Maroc. Egypt and Tunisia are members of the International Mathematical Union, which is a worldwide organization designed to promote mathematics. North African countries have participated in the International Mathematical Olympiad, an annual competition held since 1959 for high school students. Algeria first participated in 1977, Morocco in 1983, and Tunisia in 1981.

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Sarah Boslaugh

See Also: Africa, West; African Mathematics; Arabic/Islamic Mathematics; Egyptian Mathematics.

Africa, Southern

Category: Mathematics Around the World.
Fields of Study: All.
Summary: Southern Africa is the home of ancient mathematical artifacts and modern mathematical innovations.

Southern Africa comprises the five nations of the Southern African Customs Union: Botswana, Lesotho, Namibia, South Africa, and Swaziland. Colonization led to significant European populations, especially in South Africa and Namibia.

The oldest known mathematical artifact is the Lebombo bone, discovered in a rock shelter in the Lebombo Mountains near the South Africa/Swaziland border. There is evidence of the cave having been inhabited continuously beginning some 200,000 years ago, and the bone itself is estimated to be 35,000 years old. The Lebombo bone is a fragment of baboon fibula with 29 notches, most likely used as a tally stick—a notched object used to keep track of quantities. In this case it may have been a menstrual calendar.

Historically, the Dutch and British were particularly influential in this region. For example, the nineteenth-century Boer (also known as Afrikaner) community established the Boer States, including Transvaal and the Orange Free State. It has been documented that the Boer farmers, who were largely descendants of Dutch and some other European settlers, relied heavily on education at home. The migration of large numbers of predominantly British settlers into South Africa in the nineteenth century saw the establishment of more schools and later, universities in the European style. The mathematics heritage of southern Africa reflects both the diversity of the native cultures and the effects of this European colonialism.

South African Mathematicians

One early South African mathematician was Francis Guthrie (1831–1899), who proposed the Four Color Problem. It stemmed from a problem he first explored as a student in which only four colors could be used to denote the counties of England, and no two counties sharing a border could have the same color. Guthrie was born in London but immigrated to South Africa, where he worked as both a mathematician and a botanist. Mathematician Stanley Skewes (1899–1988), who was a faculty member at the University of South Africa and grew up near Johannesburg, postulated his Skewes number, which is an important concept in number theory.

Within South Africa, one well-known mathematician is Chris Brink, who grew up in a town on the edge of the Kalahari Desert and studied at Johannesburg. He earned a degree in mathematics before earning a scholarship to Cambridge University in England, where he completed his doctoral thesis on algebraic logic.

Returning to South Africa, he worked on Boolean modules and was vice-chancellor of the University of Stellenbosch from 2002 until 2007. Outside of the country of South Africa, another early mathematics Ph.D. from the southern Africa region is Abraham Busa Xaba. He was born in Swaziland in 1938 and earned his Ph.D. in 1984. His doctoral dissertation was titled "Maintaining an optimal steady state in the disturbances."

During the latter years of the twentieth century, some South African mathematicians also became known for their work overseas. For example, Lionel Cooper (1915–1979) left the country for political reasons. He grew up in Cape Town and won a Rhodes scholarship to study mathematics at Oxford University. Afterward, he served as a lecturer at Birkbeck College, London, and at Cardiff University, then became head of the Mathematics Department at Chelsea College, London. Abraham Manie Adelstein (1916–1992) was born in South Africa but left to live in England in 1961, where he became a leading medical statistician.

Organizations

As well as these important role models, there have been many attempts to encourage collaboration and development of mathematics in the southern African region. The Southern Africa Mathematical Sciences Association was founded in 1981 and is headquartered in Botswana. Its serves as a forum for the sharing of mathematical ideas for the countries in southern Africa as well as some neighboring countries that may be more broadly defined as being in the southern portion of the African continent.

The African Institute for Mathematical Sciences was founded in 2003 as a partnership of six universities: Cambridge University (England), University of Cape Town (South Africa), Oxford University (England), Université Paris-Sud XI (France), Stellenbosch University (South Africa), and University of the Western Cape (South Africa). Its three primary goals are: promoting mathematics and science in Africa; recruiting and training talented students and teachers of science and mathematics; and building capacity for educational, research, and technological initiatives in Africa. The South African Mathematics Olympiad is held each year for high school students, and teams from southern Africa have participated in the International Mathematical Olympiad since 1992.

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Justin Corfield

See Also: Africa, Central; Africa, Eastern; Africa, West; African Mathematics.

Africa, West

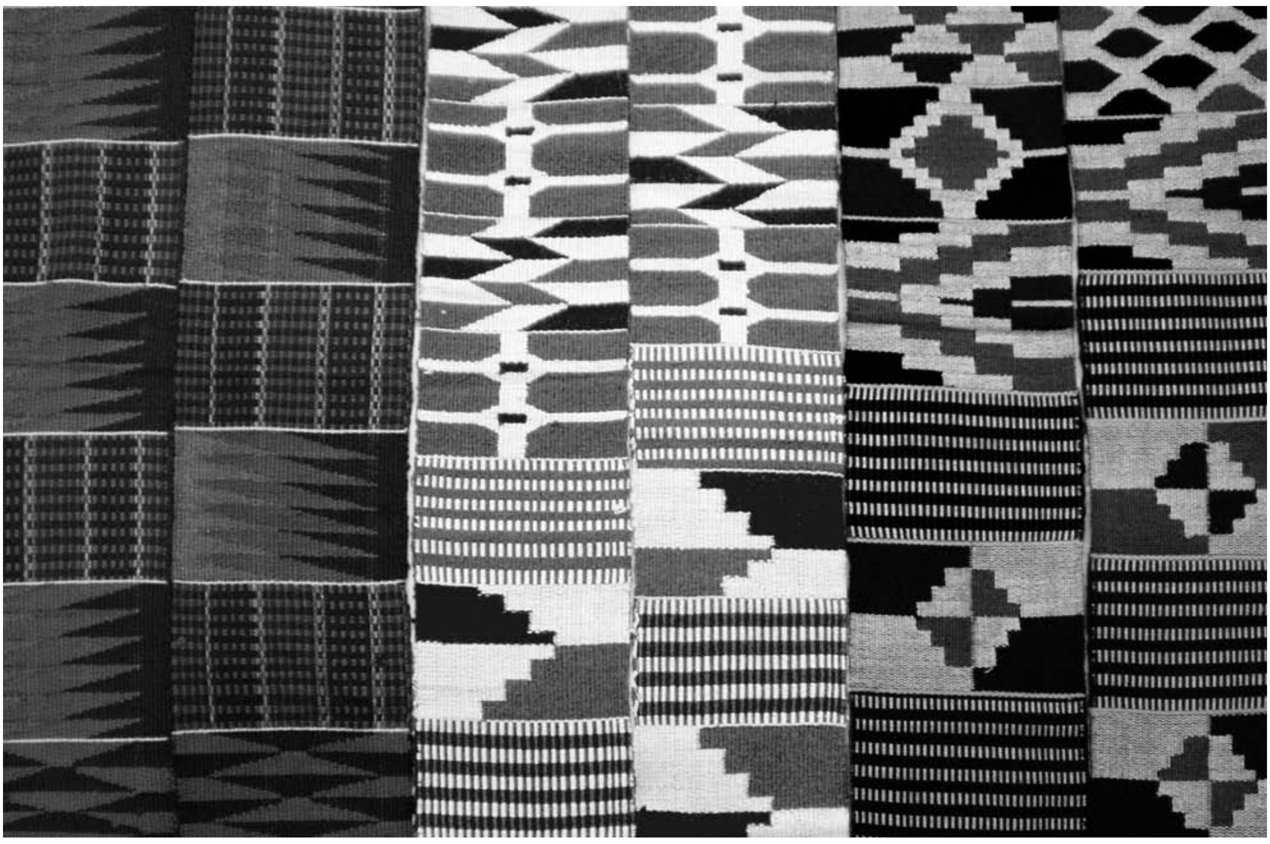
Category: Mathematics Around the World.
Fields of Study: All.
Summary: Mathematics has long been used in west African art, architecture, industry, and music.

The peoples of west Africa have a long history of using mathematics. Everyday uses were similar to mathematics in other traditional societies around the world. Farmers measured their fields and counted their crops, anticipating the production figures. Fishers designed boats to carry them off the coast and prepared nets for catching fish. For both, there were processes to handle their products, either for immediate consumption or—with additional mathematics—for sale in local or distant markets. Markets served as centers of trade and also as centers of mathematical calculations of quantities and sizes, profits and losses. Everyone designed and built houses, often round in shape, which calculus shows to provide the maximum area for a given perimeter. As larger societies and governing units grew beyond the villages, mathematics played a role in governments, from taxation and salaries to the design of palaces and warehouses.

Mathematics in West African Art

West Africa has long been known for its art, textiles, music, and dance. Mathematics is central to the creative and performing arts. Some particular west African examples include carved sculptures, wall paintings, tie-dyed textiles, and woven cloth. Sculptures often show symmetries, not only of human features but also of geometrical designs and proportions of animals, village scenes and daily life, and abstractions of circles, rhombi, stars, and repeating patterns. Often, the palaces of chiefs or emirs became sites of art, especially with designs on the walls or in the architecture of the structure—all incorporating geometrical designs.

Throughout west Africa, textiles have been a central part of culture. From the multicolored patterns in Sierra Leone to the deep blues and indigos of the Hausas, the techniques of dyeing cloth have been popular, especially with tied or sewn folds of the cloth to yield intricate patterns of dyed and nondyed areas of the material. Often, the use of symmetries and Euclidean geometric constructions is necessary to produce the desired circular, radial, rhombic, and zigzag patterns. Woven cloth includes the brightly colored *kente* of Ghana, the metallic shine of the Okenne cloth of western Nigeria, and others. Weaving requires engineering mathematics to design and build looms, and then careful planning so that the strips of material that come



Kente *weaving is traditional among the Ashanti and Ewe people of Ghana, Africa. A* kente *cloth is sewn together using narrow strips of brightly colored cloth with different geometric patterns.*

off the looms will fit together in two-dimensional symmetrical arrangements.

These traditional artistic products have been carried into the present day. Traditional designs are now seen in modern buildings throughout the region. Fashionable textiles sometimes use new materials or printed cloth but continue the geometric traditions. *Kente* has become a popular material not only in Ghana but also in the United States, especially the symmetrical strips used as wraps and ties. Recent studies by ethnomathematician Ron Eglash have demonstrated a variety of uses of fractal patterns in traditional west African arts, ranging from repeating smaller patterns in the geographical arrangement of savannah villages, to necklaces and bracelets, carvings from Mali of increasingly small antelopes, and even corn-row hair braids that repeat smaller shapes as the pattern goes from the forehead and temples to the rear of the head.

Music and dance from west Africa are famous to both ethnomusicologists and jazz aficionados—and to ethnomathematicians. The rhythm patterns, especially from complex drumming structures, often involve unusual time signatures and alternations of loud and soft sounds. The three-dimensional movements of dance, like the carvings and textiles, show complex symmetries and geometrical arrangements of the dancers.

Early in the second millennium, Islam was introduced in west Africa, along with Islamic mathematical studies. This introduction added to the original practical base of west African mathematics, as west Africans adapted Islamic counting methods, reflected not only in the languages of west Africa but also in theoretical mathematics studied at scholarly centers such as Timbuktu (in modern Mali) and Katsina (in modern Nigeria).

Mathematics and West African Development

Since gaining independence, mostly in the 1960s, west African countries have moved rapidly to modernize. In the process, they have shown a dynamic use of mathematics—on a smaller scale than but similar to the technical mathematics of the developed world. Oil production in Nigeria, gold mining in Ghana, and diamond mining in Sierra Leone all use modern mathematical techniques, including those employed by geological surveys, sophisticated industrial equipment design, accounting, marketing, and business management. New businesses are being established to work with cell phones, the Internet, automatic teller machines, television and film production, and other industries that rely on technical mathematics and engineering. Modern freeways connect the larger cities and are designed by civil engineers and urban planners.

Education and West African Mathematics

Education throughout west Africa has grown dramatically since independence—universal primary education remains elusive, but the percentage of children attending school is approaching that goal in several countries. Political independence also brought educational independence, including national curricula offered by the Ministries of Education, the West African Examinations Council's system of standardized examinations, and locally produced textbooks and teaching materials, using familiar names, places, and situations in examples. Local researchers are studying their own cultures, seeking examples of ethnomathematics in traditional life, often with the goal of using these findings to strengthen the content of school mathematics curricula. With only a few universities in existence at the time of independence, west African countries now have numerous universities. These are often managed by the national governments—though some states of Nigeria operate their own universities and research centers, and the number of private universities is growing. These universities offer degree programs in mathematics, the sciences, engineering, and computer science, all with curricula based on the accepted world standards of these fields. Most countries have professional and scholarly organizations of mathematicians and mathematics educators, and periodically there are regional and continent-wide conferences, such as the meetings of the African Mathematical Union (AMU). The AMU's activities include the Commissions on Mathematics Education in Africa, Women in Mathematics in Africa, the African Mathematics Olympiads, and publishing the journal *Afrika Matematica*. Thus, even as west Africa maintains its traditional uses of mathematics in the arts and music, it has also become a part of the modern world mathematics community.

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Lawrence H. Shirley

See Also: African Mathematics; Arabic/Islamic Mathematics; Textiles.

African Mathematics

Category: Government, Politics, and History.
Fields of Study: Connections; Geometry; Measurement; Number and Operations; Representations.
Summary: Throughout African history, mathematics has been used in the arts, in engineering and business dealings, and in games.

As in all societies, mathematics has always been a part of the cultures and daily life of people in Africa. One difficulty of studying African mathematics is that for much of the history of Africa, the societies were non-literate, relying on oral traditions to pass their stories to the coming generations.

The wet tropical and subtropical climates of most African civilizations destroyed whatever records may have been kept—or at least hid them from the eyes of future probing historians. Hence, when assertions like the first African mathematical achievement are proclaimed, there should be a caveat that this is the first "that we know of," for similar earlier achievements may well have been lost to history. The African Mathematical Union hosts a Commission on History of Mathematics in Africa—and readily recognizes the difficulty of its charge when even details of the social, political, and military history of precolonial Africa remain difficult to find. Discovering the history of African mathematics is an even greater challenge. Hence, much mathematics history in Africa remains speculative, based on general understandings of how mathematics works in other societies past and present, and fitted into the growing framework of bits and pieces of the history of Africa and African society.

Modern Western mathematics (now used around the world) has indeed come from the developments in the European academy, but it is only the formalized structures of pure theoretical mathematics and their applications in science, industry, and technology that grew from this theoretical work. However, mathematical thinking is much broader than the tightly logical structures of academic mathematics. Everyone who thinks about counting, arranging, or designing—anyone who makes strategic plans for achieving a goal—is thinking in mathematical terms. These examples of mathematics have occurred in Africa as much as anywhere else in the world.

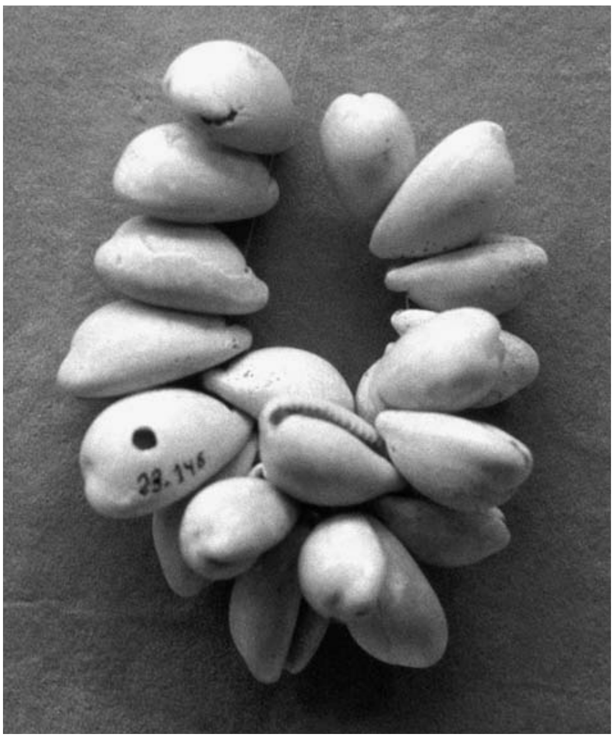
Development of African Mathematics

Before recorded history, Africans herded their animals, planted and harvested crops, and built homes and other structures. All these activities required mathematics. Farming required finding the best time to plant and the appropriate time for harvest. Over time, it is likely that this led to formal or informal calendars, so the farmers would be prepared to do their tasks at the right time. They applied measurements and design as they laid out their fields, including sorting out boundary disputes with neighbors. Anthropologists have even studied the variations in the arrangements of fields in farming communities. When the time for harvesting came, several other mathematical issues arose. Initially, there would be a need for containers and storage bins for the produce, requiring geometrical design.

Later, business mathematics would be used in the markets—even those using barter systems—to determine the comparative values of the products, the gains and losses, and the purchases of other products. Some societies developed currencies—a famous example is the use of strings and bundles of cowry shells by the Yorubas. This probably contributed to the complex numeration system of the Yoruba language, which can handle very large numbers. It has even been suggested that the use of higher numbers came as a result of inflation requiring higher prices. Also, the use of strings and bundles easily flows into the grouping used in place-value of counting systems.

Village life also measured the times of human life, from the diurnal movement of the sun and language of timekeeping, to the much longer periods of milestones of maturity—birth, initiation as an adult, old age, and death. These time markers sometimes went beyond the individual and family, such that entire age cohorts measured time and followed the appropriate customs of their ages together. Kinship relations sometimes were built into mathematical structures, attempting to avoid disputes and maintain a smoothly functioning society.

Over the past one to two millennia, villages grew and coalesced into larger units. As societies grew beyond the size of villages, the mathematics correspondingly grew. The savanna of west Africa saw Songhai, ancient Mali and Ghana, and the Hausa States. The Swahili civilization grew along the coast of the Indian Ocean in east Africa. Also, trading links reached to increasingly distant targets across the Sahara and along distant stretches of ocean coastline. Although few records have survived, it is acknowledged that large governmental and trading organizations required complex record keeping and accounting. A trader would certainly want to keep careful records of items being traded to avoid being cheated by faraway customers. Governments had to handle administrative and logistical details of the equivalent of civil servants and the king's retinue, and, especially, of armies. Longer trade routes required the design of stronger boats for coastal travel and navigational skills for caravan travel across empty desert landscapes. Also, the needs for currencies grew far beyond those of local markets, as traders had to convert the prices of the sellers to those of the buyers and still control costs and profits.



A string of cowry shells on display in a museum. The Yorubas used these shells as currency.

Reaching out from local roots also put Africans in contact with others—and often, the reverse happened as outside groups came into Africa. Either way, this led to a mixing of culture and a growth of experience. Mathematical ideas jumped from culture to culture, contributing a growth of power and sophistication of mathematics. It is reported that when king Mansa Musa of Mali accepted Islam and traveled across the Sahara to make the hajj pilgrimage to Mecca in 1324–1325 C.E., he brought so much of the golden riches of his empire that he upset the economy of Egypt as he passed through! The flow of the Arabs into both west Africa and east Africa brought the intellectual riches of Islamic mathematics. Even in the terminology of counting words, Arabic influence can be seen in the words for the decade numbers (20, 30, 40, and so on) in both the Hausa language of west Africa and Swahili of east Africa. Arab mathematics, which would later also make fundamental contributions to European mathematics, was taught in Qur'anic schools, and scholarly centers were established in various place including Timbuktu and Mombasa. One of the few documented examples of precolonial history of mathematics in west Africa was the work of Muhammad ibn Muhammad, who worked in Katsina—now in northern Nigeria—in the early 1700s. Interestingly, part of his work became controversial—his calculations of "magic squares," which some of the Islamic authorities considered as flirting with the occult. The astronomical calculations required to maintain the calendar of Islamic festivals led to a growth of formalized geometry and trigonometry.

Mathematics in Egypt

In addition to the mathematics of subsistence, daily life, government, and trade, there was also considerable mathematics used in the arts and recreation. Probably the most famous and spectacular mathematics of the arts and architecture on the African continent is the mathematics of early Egypt. Beyond the famous hieroglyphic mathematics of ancient Egyptian numerals and the arithmetic of the problems found in rolls of papyrus, the mathematics of Egyptian architecture reached the level of "wonders of the world." Notably, the famous pyramids are built with precise lengths, angles, and alignments. They fit into near-perfect geometrical shapes—all the more impressive given their massive size and the belief they were actually constructed by uneducated laborers working under the supervision of masters of labor. The mathematical history questions remain: Who did the design work? How were the designs communicated to the individual laborers?

Mathematics in Sub-Saharan Africa

In sub-Saharan Africa such spectacular wonders are not often seen, but the mathematics of the arts remains impressive. Other architectural examples include the massive structure of the Zimbabwe fortress as well as decorative design in chiefs' palaces and public structures throughout the continent. Walls are often decorated with geometrical patterns—some to be washed off for new work when a new king would arrive.

On a smaller scale, many parts of Africa are known for their textile designs. Sierra Leone has intricate tie-and-dye patterns in cloth. Akan weavers in Ghana produce long strips of woven *kente* cloth in bright colors of red, blue, green, and gold, and then align them side by side to create broad sheets used as toga-like robes in traditional dress. Okenne weavers also make cloth, often with metallic threads giving a shiny appearance to the design. All of these patterns require mathematics in their design—especially considerations of symmetry. Tie-and-dye requires careful planning of the ties so that the resulting dye pattern reflects the design pattern. *Kente* and *Okenne* cloth show symmetry both along the initial woven strips and also across the strips in the full cloth of the robe.

The sculptures from many parts of Africa contributed to some of the designs of modern Western art. They show much use of symmetry, scale distortion, and even repetitive fractal-like patterns. Similarly, African music and dance, especially from west Africa, show mathematically complex rhythm structures in drumming and in the use of a variety of plucked and strummed musical instruments. Like African art, the music of Africa has contributed much to Western music, especially via the music the African slaves brought to the Americas, which formed the roots of jazz.

Beyond the arts, recreational mathematics is seen in numerous African games and pastimes. The best example is the many varieties of the mancala games (known under various names in different countries), which involve sharing seeds into pits in a game board, trying to capture the seeds of the opponent. There are many variations of the rules but all require a careful strategy of play and mathematical problem solving. Some game experts have listed mancala among the great games of the world.

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LAWRENCE H. SHIRLEY

See Also: Africa, Central; Africa, Eastern; Africa, Southern; Africa, West; Arabic/Islamic Mathematics; Egyptian Mathematics.

AIDS

See *HIV/AIDS*

Aircraft Design

Category: Architecture and Engineering.
Fields of Study: Geometry; Number and Operations.
Summary: Mathematics plays a pivotal role in designing, manufacturing, and enhancing aircraft components and launch platforms.

Achieving flight has been a dream of mankind since prehistory, one never abandoned. As early as Leonardo da Vinci, mathematics—the cornerstone of engineering and physics—was recognized as the key to realizing the dream. Da Vinci's 1505 "Codex on the Flight of Birds," for instance, is a brief illustration-heavy discussion attempting to discover the mechanics of birdflight in order to replicate those mechanics in manmade flying machines. Da Vinci considered not simply the wingspan and weight of birds but a fledgling notion of aerodynamics. He was the first to note that in a bird in flight, the center of gravity—the mean location of the gravitational forces acting on the bird—was located separately from its center of pressure where the total sum of the pressure field acts on the bird. This fact would be important in later centuries when aircraft were designed that are longitudinally stable. Today, mathematics is used in the study of all aspects of flight, from launch platform design to the physics of sonic booms.

Complex Analysis and the Joukowski Airfoil

Abstract mathematics can find its place in physical applications people experience quite often. For example, complex analysis and mappings play a vital role in aircraft. In layman's terms, complex analysis essentially amounts to reformulating all the concepts of calculus using complex numbers as opposed to real numbers. This formulation leads to new concepts that cannot be achieved with only real numbers. In fact, the very notion of graphing complex functions, rather than real functions, is quite different—mathematicians often call the graphing of complex functions a "mapping." Taking a simplistic geometric figure, like a circle, and then applying a complex function transforms the figure into a more complicated geometric structure. One figure that results from such a transformation looks like an airplane wing. Furthermore, one can consider the curves surrounding the circle as fluid flow, that is, air currents, and we obtain a rudimentary model of airflow around an airplane wing. This transformation is entitled the Joukowski Airfoil, which is named after the Russian mathematician and scientist Nikolai Joukowski (1847–1921), who is considered a pioneer in the field of aerodynamics. Variations of this transformation have been utilized in applications for the construction of airplane wings.

Nature-Inspired Algorithms

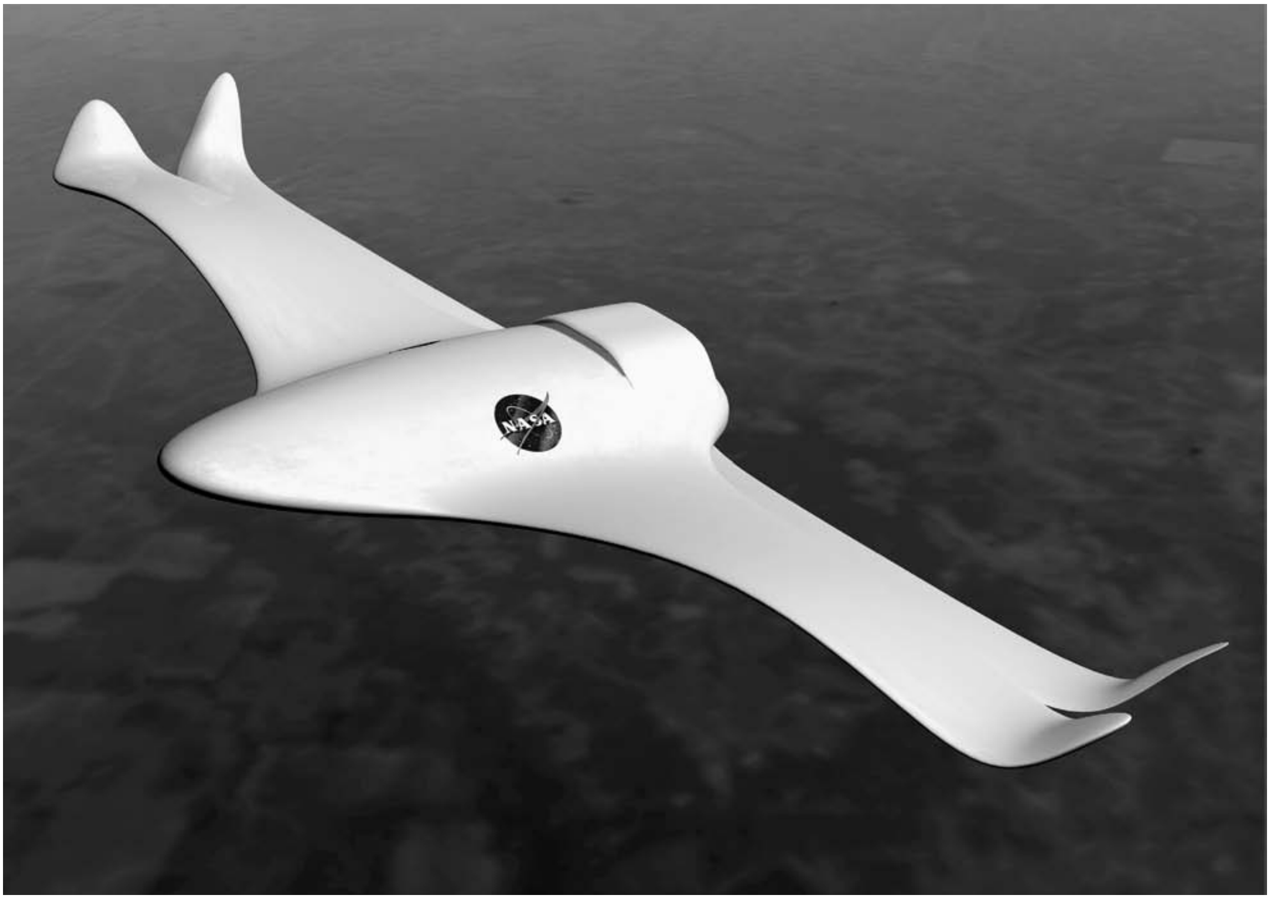
An example of how various fields of mathematics, science, and engineering coalesce is epitomized at the Morpheus Laboratory, where applications of methods and systems found in nature are applied to the study and design of various types of aircraft. For example, biologically inspired research is conducted by studying an assortment of details related to the mechanics of birds in flight.

Birds are an example of near perfection in flight, a fact that humans have long observed. Birds have been evolving for millions for years and have adapted to various environmental changes, thus altering their flight mechanics accordingly. By studying the mathematical properties related to their wing morphing, surface pressure sensing, lift, drag, and acceleration, among other aspects, the researchers at Morpheus Laboratory can use the knowledge they have gleaned and apply it to several different types of aircraft. In order to accomplish this feat, mechanical models of actual birds are constructed and analyzed. Morpheus researchers utilize an assortment of mathematics and physics, including fluid mechanics (the study of air flow in this case) and computer simulations, to analyze the data that result from studying the mechanical birds in flight. The analysis, in turn, results in novel perspectives in flight as well as the design of innovative types of planes.

In addition, many of the problems that arise regarding the machinery and components that comprise an aircraft carrier can also be potentially solved via Darwinian-inspired mathematical models. For example, the structural components of aircraft are constantly being optimized, as numerical performance is attempted to be maximized while cost is minimized.

The managing of cabin pressurization has made it possible for aircraft to fly safely under various weather conditions and landscape formations. This ability is due in large part to devices known as "pressure bulkheads," which close the extremities of the pressurized cabins. Because of the wealth of physical phenomena that influence the stability of these bulkheads, such as varying pressures, it has been a challenge to optimize their design. In the early twenty-first century, it was proposed that the bulkheads should have a dome-like shape, as apposed to a flat one, which was suggested by both mathematical and biological evidence. Interestingly, these two structures demonstrate completely dissimilar mechanical behaviors, which lead researchers to consider different approaches to modeling the dome-like bulkheads.

The dome-like structured bulkheads are analogous to biological membranes and can be mathematically modeled in a similar fashion. In addition to the imple-



Scientists at the U.S. National Aeronautics and Space Administration (NASA) envisioned this design as a twenty-first-century aerospace vehicle. The "Morphing Airplane" is part of NASA's vision for aircraft of the future.

mentation of these membrane-like designs, the minimization of the cost of their construction and the assurance of their durability is mathematically modeled.

Simulating Sonic Booms

Every time an aircraft travels faster than the speed of sound, a very loud noise is produced called a "sonic boom." The boom itself results when an aircraft travels faster than the speed of the corresponding sound waves. The boom is a continuous event, as opposed to an instantaneous sound, which is a result of the compression of the sound waves. Other fast-moving projectiles like bullets and missiles also produce sonic booms.

Mathematically, this concept means that the velocity of an aircraft (v_a) exceeds the wave velocity of sound (v_s). The Mach Number (M), named after the Austrian physicist and philosopher Ernst Mach (1838–1916), is defined as the ratio of the velocity of an aircraft to the velocity of sound. This ratio is expressed mathematically as

$$M = \frac{v_a}{v_s}.$$

When $v_a < v_s$, $M < 1$, the object is moving at what is often referred to as "subsonic speed." If $v_a = v_s$, $M = 1$, and the object is moving at what is frequently called "sonic speed." Whenever $v_a > v_s$, $M > 1$, and the object is moving at what is titled "supersonic speed." Furthermore, whenever $v_a > v_s$, a shock wave is produced.

The shock waves from jet airplanes that travel at supersonic speeds carry a great amount of concentrated energy resulting in great pressure variations. In fact, two booms are often produced when jets fly at supersonic speeds. Usually, these two booms coalesce into an *N*-shaped sound wave that propagates in the atmosphere toward the ground. Although shock waves are exceedingly interesting, they can be unpleasant to the human ear and can also cause damage to buildings including the shattering of windows.

However, there is increasing economical interest in designing aircraft carriers that can travel at supersonic speeds with a low sonic boom. To demonstrate, the flight time for a trip from New York to Los Angeles can essentially be cut from 10% to 50% if the plane flies at a supersonic cruise speed instead of subsonic speed. Therefore, physicists are currently developing adaptive methods that model sonic booms in order to ultimately develop aircraft that can travel at supersonic speeds without causing structural damage—aircraft that create a low sonic boom. Aspects such as near-field airflow as well as pressure distribution have been analyzed in these models by utilizing techniques of mathematical analysis.

Aircraft Carriers

Airplanes were a major evolution in modern warfare. World War II aircraft carriers that moved airplanes closer to targets that would otherwise be well beyond their fuel ranges proved to be pivotal to many battles, especially in the Pacific. They continue to be a key component of many countries' navies for rapid deployment of aircraft for surveillance, rescue, and other military uses. Launching from and landing airplanes on aircraft carriers is considered one of the most challenging pilot tasks because of the restricted length of the deck and the constant motion of the deck in three dimensions. A catapult launch system gives planes the added thrust they need to achieve liftoff and requires calculations that take into account mass, angles, force, and speed. Similar issues apply to the tailhook capture system that stops planes when they land.

There are also significant scheduling issues for multiple aircraft on a carrier, fuel use, weapons logistics, and radar systems used to monitor both friendly and enemy planes. Aircraft carriers are like large, self-contained floating cities. Mathematicians work in the nuclear or other power plants that provide electricity for the massive aircraft carriers of the twenty-first century and in many other logistics areas beyond direct flight launch and control. They also help design and improve aircraft carriers. For example, mathematician Nira Chamberlain modeled the lifetime running costs of aircraft carriers versus operating budgets to develop what are known as "cost capability trade-off models," which were used to help make decisions about operations. He also worked on plans for efficiently equipping ships to optimize speedy access to spare components. Some of the mathematical methods he used include network theory, Monte Carlo simulation, and various mathematical optimization techniques.

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Daniel J. Galiffa

See Also: Airplanes/Flight; Mathematics, Applied; Numbers, Complex; Weightless Flight.

Airplanes/Flight

Category: Travel and Transportation.
Fields of Study: Algebra in Society; Geometry in Society; Number and Operations.

Summary: Aerodynamics is necessary to understanding the flight of objects through three-dimensional space and the forces acting upon them.

Human flight involves moving in a three-dimensional environment within the atmosphere in a stable, controlled way. Aerodynamics is the study of forces and the resulting motion of objects through air. It comes from Greek *aerios*, meaning "air," and *dynamis*, meaning "force." Mathematics is fundamental to understanding flight and in the design of different flying devices and machines, including kites, balloons, helicopters, and airplanes. From Orville and Wilbur Wright's initial experiments with gliders at the beginning of the twentieth century, to the breaking of the sound barrier in the middle of the century, to the development of suborbital craft at the start of the twenty-first century, airplanes have been constructed in many different forms.

However, the ability to fly for all fixed-wing aircraft ultimately depends on a differential movement of air above and below the wings to generate positive lift. Control depends on three parameters, known as "pitch," "yaw," and "roll," that are angles of rotation in three dimensions or axes about the plane's center of mass. Mathematicians and others continue to study flight in order to more fully understand the mathematical and scientific principles that keep heavier-than-air craft in the air and to produce designs that are faster, safer, and more efficient. They also explore related issues in air travel, such as optimal strategies for loading passengers onto planes and the scheduling of aircraft flight crews.

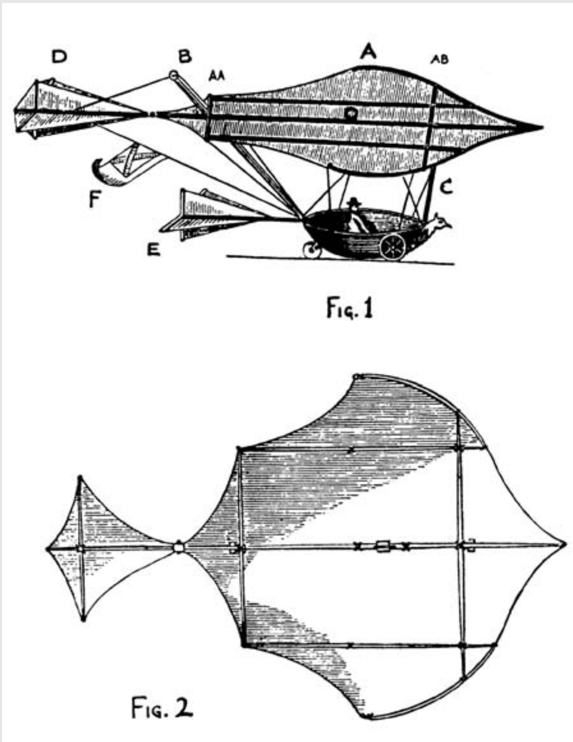
Mathematical History

Stories from many cultures around the world suggest that humans have been interested in flight for thousands of years. There is evidence that the Chinese used kites well before the first century C.E. Leonardo da Vinci

The Father of Aviation

Engineer George Cayley (1773–1857), working in the eighteenth and nineteenth century, is often called the "father of modern aviation" for his research, which helped identify the aerodynamic forces of flight: weight, lift, drag, and thrust. Though Cayley experimented with manned gliders, modern heavier-than-air flight is generally traced to the 1903 launch of the Wright Flyer, a twin propeller biplane with a single motor to provide thrust and mechanisms so that the pilot could control for pitch, roll, and yaw.

Their design helped overcome previous obstacles to sustained stable and controlled flight by adding ailerons to the wings, elevators to the tail surfaces, and rudders to the fuselage to manage airflow. By common convention, roll is motion about the longitudinal axis of the plane. Yaw is movement about the vertical body axis. Pitch is movement about an axis that is perpendicular to the longitudinal plane of symmetry. Pilots require a firm grasp of this three-dimensional geometry to navigate aircraft and follow directional headings.



George Cayley's "Governable Parachute" design was printed in Mechanics' Magazine *in 1857.*

recorded his studies of flight in the fifteenth century with more than 100 drawings, including his theoretical ornithopter. Air is a fluid, and so much of the mathematics of flight science derives from fluid force studies, such as those performed by mathematician Daniel Bernoulli in the seventeenth century. Bernoulli's principle is one foundation of flight mechanics.

Mathematical models for flight rely on the Navier-Stokes equations, named for mathematicians Claude-Louis Navier and George Stokes, which are fundamental partial differential equations describing fluid flow. They have many extensions. The Darcy–Weisbach equation, derived by dimensional analysis and named for engineer Henry Darcy and mathematician Julius Weisbach, is important to understanding the dissipation of energy because of friction, such as drag. Working in the early twentieth century, mathematician Otto Blumenthal studied the theory of complex functions, which he also applied to problems such as stress in airplane wings. Mathematician Selig Brodetsky studied equations of airplane motion, including three-dimensional phugoids, which are extensions of common, undesirable oscillatory motions where a plane pitches up and climbs, then pitches down and descends, with changes in airspeed. Peter Lax studied a class of nonlinear equations that can develop singularities, which have applications in aerodynamics that are related to phenomena like the shock waves that result from breaking the sound barrier.

Principles of Flight

Balloons are an example of lighter-than-air craft that use buoyancy to ascend and descend within the atmosphere, and hot air balloons are known to have been explored and used in the eighteenth century. There is also evidence that miniature hot air balloons were used in China for several centuries.

Heavier-than-air craft use the principle of lift to overcome gravity. There have been various mathematical and physical theories posed regarding how lift in airplane wings is accomplished. Aerodynamicists have analyzed how the motion of the air over an airplane wing creates circulation and differential pressure above and below the wing, which creates lift. Lifting forces on the airfoil are perpendicular to the motion of the lifting surface through the air and, in level flight, they counteract gravity. An observable example is the "sing" or hum that occurs in telephone wires in a steady wind, which is a repeating pattern of swirling vortices. This effect is because of the oscillations induced by a phenomenon called "vortex shedding," which causes the wires to oscillate perpendicular to the wind flow.

Studies and models suggest that an airfoil produces circulation in a similar manner. Airfoils can be optimally designed to take advantage of this effect by allowing a smooth flow to develop over the surface of the airfoil, called "laminar flow." The Reynolds number, named for mathematician Osborne Reynolds, quantifies laminar flow. Without laminar flow over an airfoil, turbulence is produced and vortex shedding occurs. Others suggest that aircraft lift is a Newtonian reaction force, named for Isaac Newton, coupled with the Coandă effect, named for engineer Henri Coandă, which is the tendency of a fluid to be attracted to a surface, like an airplane wing. The wing pushes the air down, so the air pushes the wing up.

Lift and Thrust

In general, a pilot taking off from the ground initially accelerates directly into oncoming wind whenever possible, since there is agreement based on observation and mathematics that relative forward motion of the plane's wings with respect to the air is required for flight. Usually, the plane itself is in motion, though a strong wind over a stationary wing can also generate some lift. To maintain a steady, level flight path after takeoff, without any added acceleration, two mathematical relations must be maintained: thrust = drag and lift = weight. Early aircraft engines were powered by gasoline, similar to automobile combustion engines. A fundamental problem of weight, which inhibited lift, was solved by using aluminum as a construction material. Although oxygen is needed to burn gasoline, it is not carried by the aircraft but extracted from the atmosphere so that it does not add to the mass of the aircraft. Jet engines compress and discharge a fast-moving jet of air to generate thrust, using the same principles of fluid dynamics that govern other aspects of aircraft flight, according to Newton's third law of motion. In contrast, a rocket must carry propellants, both fuel and oxidizer, and can thus fly outside of the atmosphere. The added force helps compensate for the extra weight.

Flight Speed

The types of speeds of flight are typically classified as slow subsonic flight, fast subsonic flight, trans-sonic flight, and supersonic flight. The Bell X-1 rocket-propelled airplane is credited as the first piloted aircraft in the world to break the sound barrier, under control of test pilot Charles Yeager. Other planes have been thought to have broken the sound barrier during steep dives, which many do not consider flight. The joint United Kingdom and France plane known as the *Concorde*, which flew from the 1970s until its retirement in 2003, was the only commercial supersonic aircraft. Commercial jets of the early twenty-first century typically achieve speeds in the range of 80% to 85% of the speed of sound, the slower end of trans-sonic flight.

The design speeds tend to avoid compressibility effects in air, which occur above roughly 80% of Mach 1. The Mach number is a ratio of the speed of the aircraft to the speed of sound at the aircraft's altitude. Supersonic flight requires much more energy to sustain, and generally only military aircraft conduct sustained supersonic flight within the atmosphere. The Prandtl-Glauert equation, named for scientists Ludwig Prandtl and Hermann Glauert, is used to help correct computations of fluid flow at high speeds a function of compressibility, while the Prandtl-Glauert singularity is observed as a visible cloud of vapor that results from air pressure changes around a trans-sonic airplane. The pressures can be modeled as an N-wave, named because a plot of pressure versus time resembles the letter N.

A mode of atmospheric flight explored with experimental aircraft at the beginning of the twenty-first century is hypersonic flight, which starts at speeds approximately 5–10 times the speed of sound. Special engines must be developed to make this speed possible. Previously, the Lockheed Aircraft SR-71 held the speed record at greater than Mach 3. It was powered by a special fuel and was air breathing. In 1974, the SR-71 set a speed record flying across the Atlantic from Beale Air Force Base in Louisiana to London in less than two hours. This flight occurred many decades after aviator Beryl Markham's speculations about flying the Atlantic in an hour. Hypersonic aircraft flying at speeds greater than Mach 5 likely will be powered by different forms of air breathing propulsion systems, such as turbine-free engines known as "scramjets," which at very high speeds use ram air compression to ignite a fuel in the engine. In principle, such designs have the capability of going at very high speeds at high altitude and form a transition to spaceflight.

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See Also: Aircraft Design; Skydiving; Weightless Flight; Wheel; Wind and Wind Power.

Algebra and Algebra Education

Category: History and Development of Curricular Concepts.

Fields of Study: Algebra; Communication; Connections; Geometry.

Summary: Algebra and algebra education have undergone many radical changes and remain highly adaptable mathematical disciplines with many real-world applications.

When they hear the term "algebra," many people may think only of solving an equation for an unknown variable *x*. In reality, algebra is a broad mathematical discipline that includes a range of theories and methods and which has no single agreed-upon definition. Even young children may engage in algebraic reasoning, such as understanding the relationships between quantities or manipulating symbols, without referring to it by name. For much of human history, computations were likely performed using a variety of words and symbols to meet needs such as accounting, taxation, and planting. There is evidence of algebraic problem solving in Egypt and Babylonia. Their techniques appear to have relied a great deal on spoken rhetoric rather than symbol manipulation, though the Babylonians solved quadratic equations using methods similar to those taught in the twenty-first century.

Algebraic thinking is also found in works from ancient China. Greeks, Hindus, Arabs, Persians, and Europeans all made advances and contributions to algebra, and the term is derived from an Arabic word. In the nineteenth century, mathematicians began to expand the notions of algebraic form and structure to encompass more types of mathematical objects such as vectors and matrices as well as operations that could be carried out upon these objects. Also, algebra was not constrained to the ordinary systems of numbers, and noncommutative algebras emerged. The discipline of abstract or modern algebra has grown even further to encompass concepts like groups, rings, and fields.

Concurrently, algebra has become increasingly more important in education at all levels. One of the perceived advantages of algebra and algebraic thinking is that problem solving can be accomplished by symbolic manipulation rules without constant reference to meaning, and these generalized problem-solving skills are viewed as advantageous for students in a wide range of life and occupations skills. This notion has led to a somewhat controversial "algebra for all" approach in many K–12 educational systems in which all students must take an algebra course before graduating, and basic algebraic concepts are introduced as early as the primary grades.

Early History

Even in the classical and ancient period, people had started to use numerals such as 1, 2, and 3 (or I, II, and III, for example) to represent quantities. Numerals, however, bore a direct relation to the quantity being counted. The numeral 1, for instance, only ever referred to a quantity of one. In ancient Egypt, some mathematicians had started to use other symbols, called *ahau*, to represent unknown quantities. These symbols are called "variables" in the twenty-first century because the quantity or number they represented could vary. But the variation in quantity that the Egyptians allowed for was much more restricted than what is allowed for in modern algebra. For example, the symbol *x* can refer to any number (whole, integer, or other) depending on the mathematical context in which it is used.

Thus, while ancient Egyptians and mathematicians in other ancient civilizations may have used symbols to represent quantities, they did not use symbols in the generalized way in which they are used today. In fact, it was only in the third century C.E. that a Greek mathematician, Diophantus of Alexandria, first used letters of the alphabet to stand in for numbers. It is because of Diophantus's works that mathematicians started to express "an unknown quantity" using symbols such as *x* and *y* rather than written words.

Al-Khowarizmi

Diophantus's symbolical technique was not widespread, however. In fact, the term "algebra" actually stems from a period much later than that of Diophantus. It comes from the work of the eighth-century Muslim scholar Muhammad Ibn Musa Al-Khowarizmi (there are various spellings of his name). Al-Khowarizmi worked as a scholar and intellectual during the reign of the Caliph al-Ma'mun (r. 813–833 C.E.).

Al-Khowarizmi was a prominent member of the Bayt al-Hikma, the "House of Wisdom," which the

Pioneers in algebra include Muslim scholar Muhammad Ibn Musa Al-Khowarizmi, Italian mathematician and friar Luca Pacioli, and French mathematicians René Descartes and Évariste Galois.

Caliph had created as an academy and library to promote science. Al-Khowarizmi's book, *Al-Kitab al-mukhtasar fi hisab al-jabr w'al-muqabalah* (an abridged book on the operations of al-jabr and al-muqabalah) is the oldest surviving Arabic book on mathematics. Al-Khowarizmi was also one of the first algebra teachers, as he taught algebra within the Bayt al-Hikma as a subject on its own. Although the ancient Egyptians and Babylonians did produce texts on arithmetic, algebraic, and geometric problems as early as 2000 B.C.E., Al-Khowarizmi was among the first to teach algebra as a science on its own rather than as a subbranch of other branches of mathematics.

The word *al-jabr*, from which the modern-day term "algebra" is derived, first appeared in the title of Al-Khowarizmi's book. Some historians have interpreted it to mean "the restoration of a broken bone" or, in mathematical terms, "the removal of the negative quantity from the equation," while the word *al-muqabalah* has often been interpreted to mean the removal of positive quantities. Gandz has contested these interpretations, however, to argue that a better translation of *al-jabr* is simply "the science of equations."

Europeans

Europeans first became acquainted with Al-Khowarizmi's works through Latin translations by Gerhard of Cremona (1114–1187) and Robert of Chester (c. 1150), both of which first appeared in the twelfth century. Historians have often accredited these Latin translations of Arabic mathematics with the origins of European algebra. One of the first European treatises on algebra to emerge in the Renaissance period was written by the Italian mathematician and friar Luca Pacioli in 1494. Other Italians worked on varied algebraic problems in subsequent years, including Scipione del Ferro (1465–1526), who was able to derive the solution to a cubic equation in the early sixteenth century. The Italian mathematician, Niccolò Tartaglia (1499–1557), derived a general solution to cubic equations a few years later.

In the same century, the French mathematician René Descartes (1596–1650) began to combine algebra (and algebraic rules) with geometry. Descartes was the first to apply algebra to the study of geometric curves. In 1637, he published a work in which he represented curves by means of algebraic equations. Descartes' innovation was to study curves in their algebraic form rather than in their geometric form. The result was a field of mathematics known as "analytic geometry" (also called "geometric analysis") according to some eighteenth and nineteenth practitioners. Analytic geometry allowed mathematicians to use symbols, along with the rules that govern the combination and interaction of symbols, to solve problems related to the motion of bodies in space and the behavior of geometrical objects, such as circles, parabolas, and hyperbolas.

Solving Equations

Algebra could therefore be used to find solutions to linear equations such as $ax + by = 0$, which describe lines in space; quadratic equations, such as $ax^2 + bx + c = y$, which describe parabolas in space; cubic equations, such as $ax^3 + bx^2 + cx + d = 0$, which describe cubic relations in space; and other higher-order equations, such as $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, which describe various curves. The upshot of the Cartesian use of algebra in geometry was that algebraic manipulations could be used to also solve "systems of equations," such as

$$\begin{aligned} ax + by &= c \\ dx + by &= f. \end{aligned}$$

Another outcome of the rise of analytic geometry was the development of the calculus in the seventeenth century. However, although calculus uses the tools of algebra—including symbolic representation and algebraic manipulation—to compute its solutions, it is not the same as algebra. Algebra is generally understood to include only those expressions that possess a finite number of terms and factors. This means that the computation of solutions to algebraic equations terminates after a certain number of steps. In calculus, on the other hand, the concept of a "limit" means that the process of differentiation can be repeated *ad nauseam* and therefore never terminate.

Modern Period

Over the course of the past 1000 years, algebra has thus expanded from a basic use of symbols in simple numerical reasoning to the analysis of structures called algebraic "fields" and "groups" in the nineteenth and twentieth centuries. In fact, the "modern" period in algebra is typically understood as having begun in the early nineteenth century with the work of mathematicians such as the French mathematicians Joseph

Louis Lagrange (1736–1813) and Évariste Galois (1811–1832), as well as the Norwegian mathematician Niels Henrik Abel (1802–1829). Galois, for instance, worked on the concept of an algebraic field. Though Galois died prematurely young at the age of 20 (as the result of a duel in which he was shot), his work later culminated in what is called Galois Theory.

Another important change in the field of algebra occurred in the mid-nineteenth century with the algebraic-geometric work of the Irish natural philosopher Sir William Rowan Hamilton (1805–1865). Hamilton started to work on couples (number-pairs that can be represented as (x, y) on a Cartesian graph) to understand the algebra that could be used to describe their behavior.

In trying to extend the algebra of couples to the algebra of triplets (numbers that could be represented by the point (x, y, z) on a three-dimensional axis system), Hamilton generated an interesting mathematical operator known as the "quaternion." The quaternion can be represented as $w + xi + yj + zk$, where w, x, y, and z represent real numbers, and i, j, and k represent imaginary numbers. To get his quaternion algebra to work, however, Hamilton had to manipulate the standard rules of algebra as they were conceived of at the time. While it is the case in normal arithmetic that $1 \times 2 = 2 \times 1$, such that the order of the numbers does not affect the outcome of the operation of multiplication, Hamilton's quaternions did not follow this rule. Hamilton found that when numbers are represented as directed lines in space (called "vectors"), the order in which the numbers are multiplied with one another does matter. In Hamilton's algebra, therefore, $1 \times 2 \neq 2 \times 1$. Rather, $1 \times 2 = -(2 \times 1)$.

Hamilton is often seen as a pioneer in the study of algebras. Based on his work in quaternion algebra, other mathematicians developed the idea that by changing the rules of the game—by playing around with the standard rules of algebra and arithmetic, such as the commutative principle in multiplication—one could generate new algebraic systems in which the component parts—the variables being manipulated and the objects they represent—do not necessarily follow the same rules as normal algebra.

Another mathematician who developed a similarly new algebraic system was Hermann Grassmann (1809–1877). However, Grassmann's works were largely unknown across Europe until the mid-nineteenth century, by which point Hamilton had already published his major works on quaternions. A British mathematician who attempted to extend Hamilton and Grassmann's new algebraic systems to n-dimensional space was William Kingdon Clifford (1845–1879). Clifford died young, and, as a result, it took many years for his bi-quaternion algebraic operator to become widely known, understood, or used.

Fermat's Theorem

The history of algebra is therefore replete with breakthroughs. In the seventeenth century, a French mathematician, Pierre de Fermat, worked on a problem in number theory that he had picked up while studying the works of Diophantus. Fermat was interested in studying the Pythagorean numbers. Pythagorean numbers are sets of three numbers, such as a, b, and c, which satisfy the equation $a^2 + b^2 = c^2$. Students often learn about Pythagorean numbers through the Pythagorean theorem, which describes the length of sides in right-angle triangles in geometry. Fermat, however, was not interested in triangles so much as he was interested in the consequences of slight manipulations to the Pythagorean theorem.

He attempted to determine the consequence of manipulating the exponents in the Pythagorean numbers from 2 to n. In so doing he wrote, "I have discovered a truly remarkable proof" Fermat explained that when the Pythagorean theorem is made to read $a^n + b^n = c^n$, the new equation has no integer solutions for any value of n greater than 2. In other words, it is impossible to find numbers a, b, and c that satisfy the equation $a^5 + b^5 = c^5$. Fermat never offered a full proof of this claim and mathematicians ever since have struggled to generate it. This bit of algebra is still called a "theorem" to indicate that, although it is believed to be true, one cannot be sure that it actually holds true for all integer values of n.

Mathematicians who have tried to prove Fermat's Theorem over the years have been led to develop other branches of algebra along the way. One example is Eduard Kummer (1810–1893), who created the concept of "ideals" in algebra. The theory of ideals remains an important tool in algebraic systems. An "ideal" A is a (nonempty) subset of a ring R whenever the sum of two elements of A is an element of A as well. In addition, if a is any element of the subset A, and r is any element of the ring R, the products ar and ra are both in the subset A.

An example of this is the integer 3. All of the multiples of 3 form an "ideal" in the ring of the set of integers.

Later Developments

By the late nineteenth century, mathematicians in Europe, Great Britain, and the United States also became interested in studying the structure of certain algebraic equations. Rather than concerning themselves with particular solutions to individual equations, these mathematicians wanted to identify the axioms (or laws) that governed the behavior of differing algebraic equations. These mathematicians focused on the structure of algebraic systems, where a system consists of a set of elements and a set of operations that abide by certain axioms (or rules). The simplest example of an algebraic structure is called a "group." The French mathematician Camille Jordan (1833–1922), the German mathematician Felix Klein (1849–1925), and the Norwegian Sophus Lie (1842–1899) studied groups and did much to establish this area of algebraic research, although older mathematicians such as the eighteenth-century German mathematician Leonhard Euler and the nineteenth-century German mathematician Carl F. Gauss (1777–1855) had already developed some foundational notions that related to abstract groups. Groups are a fundamentally nineteenth-century idea. By the mid-twentieth century, the notion of a group had become widely accepted and had even come to form the core of abstract algebra.

Throughout the nineteenth and twentieth centuries, mathematicians who worked on various aspects of algebraic structure included people such as Benjamin Pierce, Eduard Study, Karl Weierstrass, Richard Dedekind, Theodor Molien, Élie Cartan, Emil Artin, and the twentieth-century female mathematician Emmy Noether (and the entire "school" of mathematicians that she fostered). Some of the "groups" that they helped to define, use, and develop include semigroups, loops, rings, integral domains, fields, lattices, modules, Boolean algebras, and linear algebras, among others.

In the twentieth and twenty-first centuries, abstract algebra has come to include a wide variety of subject topics, including negative and complex numbers, proportions, theory of exponents, finite arithmetic progression, geometric progression, mathematical induction, the binomial theorem, permutations and combinations, the theory of equations, partial fractions, inequalities, and determinants.

Algebra Instruction

Algebra developed because of the need to solve real-life questions and as an extension of mathematical investigations, but in the eighteenth century, mathematicians such as Colin Maclaurin and Euler thought of algebra as a universal arithmetic, and education focused on solving equations for unknown quantities by symbol assignment and manipulation. The focus on symbol manipulation and transformational activities such as collecting like terms, factoring, and simplifying equations continued in school algebra until the mid-1960s, when educators experimented with ways to make algebra more meaningful to students. By the early 1990s, generational activities that included algebra as a way to describe numerical or geometric patterns replaced transformational activities in some countries. Teachers also investigated the effectiveness of a wide variety of teaching strategies such as computer algebra software, historical perspectives, or active learning methodologies, and there were also many algebra survival books marketed such as *Hot X: Algebra Exposed* by actress Danica McKellar, who majored in mathematics.

Teachers continue to experiment with ways to help students understand algebraic equations and models as well as the process of manipulating them. There is also a long history of debate about when to begin teaching algebra. Before 1700, algebra was not routinely part of the U.S. curriculum at any level of schooling, though evidence suggests it was taught in some places, such as Harvard University, in the early part of the 1700s. By 1820, Harvard required algebra for admission, and several other Ivy League schools adopted this standard over the next three decades. Massachusetts also passed a law in 1827 requiring algebra to be taught in many high schools. As early as the first part of the twentieth century, some educators such as Claude Turner suggested that algebra should be taught in eighth grade to help students understand concepts like cube roots. Some educators pointed to developmental theories such as Jean Piaget's theory of cognitive development in order to resist teaching algebra any earlier than eighth grade. In the twenty-first century, many states in the United States have adopted an "algebra for everyone" approach to teaching, and several states require students to pass an algebra test to graduate from high school. This emphasis is due in part to the increased focus on problem-solving skills believed to develop a wide range of life and occupational skills.

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Josipa G. Petrunic

See Also: Algebra in Society; Cubes and Cube Roots; Polynomials; Squares and Square Roots; Wiles, Andrew.

Algebra in Society

Category: School and Society.
Fields of Study: Algebra; Connections.
Summary: Algebra provides tools for orderly thinking and problem solving, applicable across a spectrum of pursuits.

Among the many discussions in his 1961 book *The Realm of Algebra*, science fiction author and biochemist Isaac Asimov described the real-life uses of algebra; explored the role it played in the discoveries of scientists and mathematicians such as Galileo Galilei and Sir Isaac Newton; and suggested the idea that "the real importance of algebra, and of mathematics in general, is not that it has enabled man to solve this problem or that, but that it has given man a new outlook on the universe." This notion underlies many of the perspectives on algebra in the twenty-first century.

Knowledge of algebra is seen as important not only for scientific research and the workplace but also for teaching general logical thinking and for making decisions that are important to personal well-being and society as a whole. For example, some functional relationships among people's day-to-day activities that may affect personal decisions include the relationship between how much food a person eats and weight; the amount of exercise and weight loss; and calculations for loans, interest, and other financial matters.

Some would say that the ramifications of these relationships and a lack of understanding of them mathematically are found in the housing crisis of the early twenty-first century and the increase in obesity. Algebra is reported as being a challenging subject for some people.

Many consider algebra to be a major gateway into higher mathematics in both high school and college, and it is thus critical to careers in engineering, science, mathematics, and other disciplines that require advanced mathematics training. Performance of primary and secondary students on algebra tests is one common comparison measure used to evaluate the relative standing of countries with regard to education. Professional organizations like the National Council of Teachers of Mathematics (NCTM) continue to examine the role of algebra in society and make recommendations. Some of the numerous careers that have been cited as requiring algebra include architecture, banking, carpentry, dentistry, civil engineering, nursing, pharmacy, and plumbing.

How Is Algebra Useful?

In 2003, the RAND Corporation's Mathematics Study Panel underscored the key role of algebra in education by choosing it as one of the panel's main areas of focus, explaining their decision in part by saying, "Algebra is foundational in all areas of mathematics because it provides the tools (i.e., the language and structure) for representing and analyzing quantitative relationships, for modeling situations, for solving problems, and for stating and proving generalizations." In algebra, there are general laws or algebraic models that can be used to represent a given scenario.

Algebra is sometimes noted as a type of language that provides answers to all cases at all times and models the relationships between quantities, reducing the need for repeated or inefficient computation. For example, in order to determine the savings in an interest-bearing account after a given period of time, one could compute the savings each month or year by

Clifford Ho, 2010 Asian American Engineer of the Year in Sandia National Laboratory's solar power heliostat field. Studying algebra is critical to pursuing careers in engineering, science, and mathematics.

multiplying by the interest rate. However, this computation is cumbersome after many compounding cycles. Instead, the algebraic formula

$$A = P\left(1+r\right)^t$$

can be applied directly, where P is the initial investment, r is the interest rate per period, t is the number of periods, and A is the amount of money in the bank after t periods. People may want to know if it is profitable to leave money in a bank subjected to the stated formula. On the other hand, people may want to determine the present and future value of the money they have invested because of the effect of inflation. In other instances, such as taking a car or home loan, similar algebraic laws exist. These laws help people know how much money, for instance, they may save if they pay off their loan earlier than the due date. In the eleventh century, scholar, poet, and mathematician Omar Khayyam explained the following:

> . . . Algebra is a scientific art. The objects with which it deals are absolute numbers and measurable quantities which, though themselves unknown, are related to "things" which are known, whereby the determination of the unknown quantities is possible. . . . What one searches for in the algebraic art are the relations which lead from the known to the unknown. . . . The perfection of this art consists in knowledge of the scientific method by which one determines numerical and geometric unknowns.

Early History

Algebra definitions and applications have evolved over time, though many aspects of algebraic thinking and methods that are taught in twenty-first-century schools can be traced back to antiquity. The Babylonians and Egyptians used algebraic techniques to solve problems directly related to the everyday needs of society, such as dividing land and keeping financial records. One such example from Babylonian mathematics is an alternative method for solving cubic equations of the form $x^3 + x^2 = b$, via tabulated numerical values of squares and cubes. The Babylonians were able to solve this polynomial by using the table that gave the values of $x^3 + x^2$ or $x^2(x+1)$. They constructed the table to solve: $x^2(x+1) = 1;30$ in sexigesimal notation. The "periods" below are used to represent multiplication.

x	$x^3 + x^2$
1	1.2 = 2
2	4.3 = 12
3	9.4 = 36
4	16.5 = 80
5	25.6 = 150
6	36.7 = 252
7	49.8 = 392
8	64.9 = 576
9	81.10 = 810
10	100.11 = 1100
.	
30	900.31 = 27900

The algorithm used by the Babylonians to find the roots of cubic equations is different from the modern approach, although it can be explained using modern language.

For example, in modern notation, in solving the equation $x^3 + 2x^2 - 3136 = 0$ set $x = 2y$. Then the equation can be rewritten as the following:

$$(2y)^3 + 2(2y)^2 - 3136 = 0$$
$$8y^3 + 8y^2 - 3136 = 0$$
$$y^3 + y^2 = 392.$$

From the table, $y = 7$. Since $x = 2y$, then $x = 14$.

Topics that are viewed as algebra in contemporary mathematics were often numerical or geometric in nature. The Pythagorean theorem, named for Pythagoras of Samos, can be expressed in terms of the algebraic equation that relates the sum of the sides surrounding a right angle in a triangle squared to the square of the hypotenuse. However, historically, there is evidence that the Babylonians explored numerical versions of the theorem, while the Greeks examined the areas of the geometric squares that sat on the edges of the triangle.

The Pythagorean theorem can be found in twenty-first-century algebra classrooms, and it is useful in setting right angles in constructions and in measuring distance in flat objects. Symbolic notation for algebra was developed in India and became popular in Europe in the seventeenth and eighteenth centuries. Historical methods reflect the unique construction of understanding, indicative of the localized culture at that time. Algebraic methods have also been found in some ancient Chinese works.

Greeks, Hindus, Arabs, Persians, and Europeans all contributed to the development of algebra. The term itself comes from the Arabic word *al-jabr*, which has several translations including "the science of equations." The word appears in the title of the early algebra text written by Muhammad Ibn Musa Al-Khowarizmi in the ninth century.

Applied Algebra

For a long time, one major emphasis in algebra was solving polynomial equations, but in the eighteenth century, algebra went through a transformation that broadened the field to include study of other mathematical structures. Around that time, textbooks defined algebra in many different ways. According to mathematician Colin Maclaurin, "Algebra is a general method of computation by certain signs and symbols which have been contrived for this purpose, and found convenient.

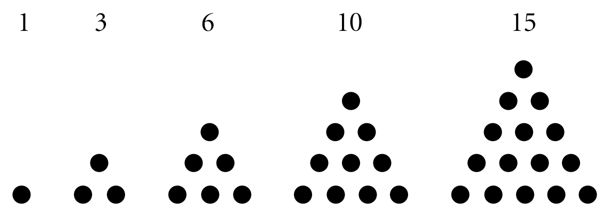
It is called an universal arithmetic, and proceeds by operations and rules similar to those in common arithmetic, founded upon the same principles." Leonhard Euler defined algebra as: "The science which teaches how to determine unknown quantities by means of those that are known." As the concept of variables was further developed, many physical properties, including time, mass, density, pressure, temperature, charge, and energy, were expressed algebraically.

For instance, Albert Einstein's equation relates energy to mass times the speed of light squared. In the twenty-first century, defining algebra commonly requires a broader approach. First, one could say that early or elementary algebra is essentially the study of equations and methods for solving them; and second, that modern or abstract algebra is the study of various mathematical structures. High school algebra textbooks typically contain a breadth of topics, such as polynomials and systems of linear equations. These are important in modeling many relationships in society. For example, parabolas represent the paths of ball or bullet trajectories, and systems of linear equations and matrices give rise to digital images. At the college level, students continue their study of algebraic equations in virtually every mathematics and statistics class. Students in a broad range of majors, including the sciences and mathematics, may further their understanding of systems of linear equations and their applications in a linear algebra class.

Mathematics majors in modern or abstract algebra study topics like groups, rings, and fields, and graduate students further explore these and other algebraic structures. These concepts have been useful in chemistry, computer science, cryptography, crystallography, electric circuits, genetics, and physics. Algebra is a core area from the middle grades and high school to undergraduate and graduate mathematics. Research fields include the connections of algebra with other subdisciplines, like algebraic geometry, algebraic topology, or algebraic number theory, and the abstract structures and notions in pure algebra have been applied in many contexts. Some algebraists work for the National Security Agency and others work as professors.

In general, mathematicians and scientists often algebraically derive laws for a given scenario or relationship from patterns. For example, consider a triangle number pattern. It is fairly simple to find the next number recursively but finding larger values such as the 1000th triangular number without a general rule can be more challenging. (See Figure 1 and Table 1.)

Figure 1.



Algebra can be used to generalize the preceding case and derive that

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

so the general law will be

$$a_n=\frac{n(n+1)}{2}.$$

Hermann Weyl noted "The constructs of the mathematical mind are at the same time free and necessary. The individual mathematician feels free to define his notions and set up his axioms as he pleases. But the question is will he get his fellow mathematician interested in the constructs of his imagination. We cannot help the feeling that certain mathematical structures which have evolved through the combined efforts of the mathematical community bear the stamp of a necessity not affected by the accidents of their historical birth. Everybody who looks at the spectacle of modern algebra will be struck by this complementarity of freedom and necessity."

Many algebraic equations are used in everyday life to meet societal needs. For example, the area of a rectangle is given by the length times the width. There are algebraic equations like finding the area of a square or circle, and also finding volume, which are used in applications like home decorating, cooking, landscaping, and construction. Building houses and fences, determining amounts of material needed for a project, and completing everyday chores use algebra to make work accurate and efficient. Economists use algebraic laws to project business profits or losses and to advise investors and other decision makers. In other instances such as taking a car or home loan similar algebraic laws help people know how much money, for instance, they may save if they pay off their loan earlier than the due date.

Many formulas are easy to use and can easily be entered in a hand calculator or computer to generate the required result. Such formulas have been adapted

Table 1.

1st term	2nd term	3rd term	4th term	5th term	*n*th term
1 = 1	3 = 1 + 2	6 = 1 + 2 + 3	10 = 1 + 2 + 3 + 4	15 = 1 + 2 + 3 + 4 + 5	$= 1 + 2 + 3 + \cdots + n$

to Web-based applets and software like spreadsheets to track financial records, making them widely accessible and often easy to use.

Mathematician Roger Cooke explained, "Algebra provided more than just a compact notation for writing down relations among variables. Its rules made it possible to manipulate those laws on paper and derive some of them from others. For example, a consequence of Kepler's third law is that the ratio . . . of the square of a planet's period to the cube of its distance from the sun is the same for all planets . . . Kepler's third law and Newton's law of gravitation are equivalent statements, given certain basic facts of mechanics." Kepler's laws were named for Johannes Kepler and Newton's for Sir Isaac Newton. The ability to express algebraic relationships using variables and rates of change in calculus increased the applicability of equations in a wide variety of contexts.

The U.S. Bureau of Labor Statistics highlights the importance of coursework in algebra for numerous careers, including brickmasons, blockmasons, and stonemasons; carpenters; computer control programmers and operators; construction and building inspectors; engineers and engineering technicians; line installers and repairers; machine setters, operators and tenders in metal and plastic; machinists; opticians; physical therapist assistants; power plant operators, distributors and dispatchers; sheet metal workers; surveyors, cartographers, photogrammetrists, and surveying and mapping technicians; radiation therapists; tool and die makers; and veterinarians.

Algebra's Role as a Gateway

Some would argue that in the United States, mathematics achievement has not met the same standards of excellence as in other developed countries, and that, as a result, students may not be prepared to enter college. Some historians trace the growing need for mathematics education to the turn of the twentieth century or the Industrial Revolution, when there were debates about the appropriate level of mathematics for high school education. Historically, popular opinion was often against algebra as a subject of widespread study in secondary schools, since many did not see clear connections between algebra and real-world needs. Mathematics educator W. D. Reeve cited one newspaper editorial as an example of such an attitude in a 1936 National Council of Teachers of Mathematics report, *The Place of Mathematics in Modern Education*, saying the following:

> Quite frankly, I see no use for algebra except for the few who will follow engineering and technical lines. . . . I cannot see that algebra contributes one iota to a young person's health or one grain of inspiration to his spirit. . . . I can see no use for it in the home as an aid to a parent, a citizen, a producer, or a consumer."

The same report noted deficits in algebra skills even among graduate students and relatively high failure rates for algebra students in some high schools, such as in New York City, which were used by some as additional arguments against algebra's broad inclusion in the high school curriculum. With regard to who should and should not study algebra, Reeve countered: ". . . no one, I think, has the wisdom to decide who will profit most by its study or predict who the future Newtons and Einsteins are to be."

Mathematician and philosopher Alfred Whitehead stated the following:

> Quadratic equations are part of algebra, and algebra is the intellectual instrument which has been created for rendering clear the quantitative aspects of the world. There is no getting out of it. Through and through the world is infected with quantity. To talk sense, is to talk in quantities. It is no use saying that the nation is large. . . . How large? It is no use saying that radium is scarce. . . . How scarce? You cannot evade quantity. You may fly to poetry and to music, and quantity and number will face you in your rhythms and your octaves. . . . This question of the degeneration of algebra into gibberish, both in word and in fact, affords a pathetic instance of the uselessness of reforming educational schedules without a clear conception of the attributes which you wish to evoke in the living minds of the children. . . . First, you must make up your mind as to those quantitative aspects of the world which are simple enough to be introduced into general education; then a schedule of algebra should be framed which will about find its exemplification in these applications.

Other newspapers like the *Columbus Dispatch* supported broad high school mathematics education during that time period, asserting that schools should provide the "mathematical key" to the "gateways of a larger life."

Algebra eventually became commonplace in high schools and some middle schools, with basic algebraic concepts often introduced even in the primary grades, and yet questions about how to teach algebra continued. Algebra is usually a prerequisite for all higher mathematics courses in both high school and college, and, in some cases, it is required for high school graduation. Students will not advance in many majors or career paths unless they pass algebra, and the result is that some students change majors or abandon education altogether. Students requiring remediation courses at the college level are fairly common.

The result is that in the twenty-first century, algebra is still viewed by many as a major gatekeeper to educational and career advancement, and learning algebra has been promoted as a civil rights issue for every U.S. citizen, though many of the same arguments from past decades continue to be debated. In the latter twentieth century, algebra education became a renewed topic of discussion from local school districts all the way to the White House. The RAND Corporation's panel further explained its decision to focus on algebra by saying, "Without proficiency in algebra, students cannot access a full range of educational and career options, and this curtailment of opportunities often falls most directly on groups that are already disadvantaged."

At the same time, naysayers continue to publish counterpoints regarding algebra's lack of utility. One 2006 *Washington Post* article about a student named Gabriella, who purportedly dropped out of high school after failing her algebra course many times, asserted that writing teaches logical reasoning more effectively than algebra and stated that many students will "never need to know algebra" in the real world, since most mathematics can now be done by computer or calculator. It concluded that having an algebra requirement for high school graduation is potentially more detrimental than helpful because it may spur students to drop out who otherwise might have graduated.

This article spurred many further discussions, and it appeared to reflect the author's own difficult experiences with algebra, a phenomenon that has been reported by many educational researchers and that drives further curricular revisions. Authors of algebra textbooks and self-help books have explored different ways to help students connect to algebra. For instance, actress Danica McKellar has written algebra readiness and algebra books that include stories and characters in order to express equations and solutions in contextual situations. Some educators incorporate mnemonics, songs, or other memory techniques such as First, Outside, Inside, Last (FOIL) in order to teach the multiplication of two binomials. Other authors highlight real-life applications, historical connections, or solutions using technology.

Many national reports have indicated that education in the United States is in a critical period, and some would say particularly in mathematics and science. Educators and politicians have proposed changes to the mathematics education curriculum to prepare U.S. students. The number of students entering college and requiring courses that enable them to be effective in the workplace is rising. Further, engineering and other technical fields that were once seen as elite or remote are increasingly a part of daily life, including computing, electronics, business, and architecture. Technology is changing every day, which has changed society, including mathematics. As a result, there is an increased need for people who can adapt to the changes and continue being effective in society. In this context, there has been a movement to reform algebra education so that it can be more readily accessed by everyone. The "algebra for all" movement has been a central point within the reform initiatives. National standards such as those published by the NCTM have stressed the need to make algebra more accessible to students, and they often outline both the content to be covered and instruction expectations. Some research has shown that students who take algebra by eighth or ninth grade are more likely to pursue higher mathematics, though this cannot be interpreted as a cause-and-effect relationship.

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Samuel Obara

See Also: Algebra and Algebra Education; Equations, Polar; Exponentials and Logarithms; Function Rate of Change; Polynomials; Squares and Square Roots; Vectors; Wiles, Andrew.

Anesthesiologists uses mathematics for tasks from calculating drug dosages to monitoring patients.

Analytic Geometry

See *Coordinate Geometry*

Anesthesia

Category: Medicine and Health.
Fields of Study: Algebra; Measurement.
Summary: Anesthesia dosages must be precisely determined and the patient must be monitored for signs of too high or low a dosage.

The word "anesthesia" was coined from the Greek word *anaisthesis* meaning "insensibility." As early as 4200 b.c.e, opium poppies were used as an herbal remedy in Samaria and, later, in Cyprus, India, and China. The three main types of anesthesia are local (loss of sensation in a small area of the body by the blockage of nerve signals), regional (loss of sensation in a larger area of the body), and general (loss of consciousness), are used to relieve the feeling of pain during medical and dental procedures. Anesthesia uses mathematics in variety of ways including the calculation of appropriate drug dosages, the monitoring of patients under general anesthetics during surgery and recovery, and the design and use of anesthetic equipment including vaporizers, ventilators, and pressure gauges.

Non-pharmacological anesthetic techniques historically have included local anesthetics such as ice and rum. Nitrous oxide ("laughing gas"), ether, and chloroform were used as general anesthetics in the 1800s during childbirth and surgery.

Applications of Mathematics

Anesthetic drug dosages per minute are based on milligrams of drug per kilogram weight of the patient. The rate of elimination of drugs from the body per unit of time is proportional to the amount of drug in the body. The time taken for the drug concentration in the plasma to be reduced by 50% is called the elimination half life.

The measurements monitored while a patient is under general anesthetics can include data such as temperature; heart rate via ECG (electrocardiogram);

oxygen saturation via pulse oximetry; ratio of oxygen, carbon dioxide, and nitrous oxide from the patient's inspired and expired gases; urine output; arterial blood pressure; central venous pressure; pulmonary artery occlusion pressure; cerebral activity via EEG (electroencephalogram); and neuromuscular function.

A major concern is keeping the patient at the appropriate level of anesthesia. The cardiovascular system is threatened if the anesthetic is too deep, but if the anesthetic is too light, the patient may experience pain or regain consciousness.

Researchers have attempted to measure the depth of anesthesia by monitoring on a graph and on a time scale EEG signals generated by electrical discharges of neurons near the brain surface. One method is to administer a gaseous anesthetic drug and hypothesize that the concentration of the drug in the expired air is proportional to the blood-plasma concentration. An alternative research technique is to observe respiratory sinus arrhythmia (RSA), which is the variation in heart rate during a breathing cycle. The heart rate increases during inhalation and decreases during exhalation. On the graph of an ECG, each heartbeat is referred to as an R peak. The difference between two consecutive R peaks is an RR interval, which is shortened during inspiration and lengthened during expiration.

Anesthesia Providers' Educational Backgrounds

The academic and clinical preparation for an anesthesiologist in the United States consists of four years of college, four years of medical school, one year of internship, and three years of anesthesiology residency. A description of Steven Cruickshank's 1998 book *Mathematics and Statistics in Anaesthesia* states that anesthesia residents are required to study and understand pharmokinetics (the study of what the body does to a drug) and statistics as "a core part of their training." In addition to physician anesthesiologists, anesthesiologist assistants or Certified Registered Nurse Anesthetists (CRNAs) can apply anesthesia or sedation while working with healthcare professionals. CRNAs complete four years of college, at least one year of acute-care nursing, and a 24- to 36-month master's degree program before passing the required certification examination. Anesthesiologist assistants (AAs) with master's degrees may practice under the supervision of an anesthesiologist in several states.

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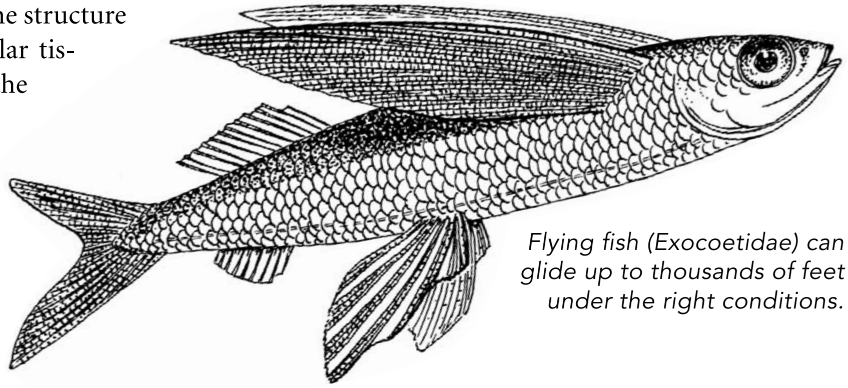
Karen Doyle Walton

See Also: Drug Dosing; LD50/Median Lethal Dose; Medical Imaging; Surgery.

Animals

Category: Weather, Nature, and Environment.
Fields of Study: Algebra; Communication; Geometry; Measurement.
Summary: Principles of engineering, physics, and mathematics are demonstrated by the physiology, movement, and behavior of animals.

Animals, including human beings, are living organisms that belong to the domain Eukaryota (having complex cellular structures enclosed with membranes) and the kingdom Animalia. Within this taxonomy, the kingdom is defined by several characteristics, including internal digestion of food (called "heterotrophism") and the ability to move using its own energy in at least some stages of life (called "motility"). Some say that what distinguishes humans from other animals is mathematical ability. However, researchers have studied a diverse range of mathematical concepts as they relate to animal behavior and have found evidence of abilities such as symbolic calculation, efficiency in locomotion, and synchrony. There are questions about whether these findings are biased perceptions of mathematical significance. Many mathematical patterns and symmetry can also be found in the structure of animals, ranging from their cellular tissue to their coat patterns. Some of the motivation behind the development of many statistical measures and methods, such as standard deviation and regression, was to characterize natural variability and associations in animal species.



Flying fish (Exocoetidae) can glide up to thousands of feet under the right conditions.

Biological Systematics, Set Theory, and Logic

Biological systematics is the field that describes and names living organisms, provides their classifications and keys for identification, and situates classes of organisms within evolutionary history and modern adaptations. In particular, classification of organisms (called "taxonomy") is an empirical science, where description of classes is the final step in the discovery and description of organisms. Existing biological classifications may include the ranks of domain, kingdom, phylum, class, order, family, genus, and species.

The definition of the kingdom Animalia is intensional definition—it specifies necessary and sufficient conditions for belonging to the set of animals. The particular subclass of definitions used in systematics to define animals is called definition by genus and differentia. Such definitions rely on a structure of sets, subsets, and supersets as well as their differentiating conditions. For example, defining negative numbers as the set of rational numbers that are less than zero, mathematicians use the superset of rational numbers (defined elsewhere) and the differentiating condition of being less than zero. Animalia is one of several kingdoms (subsets) of the domain Eukaryota, differentiated from other kingdoms by particular conditions.

Careful decisions are made in the organization of kingdoms and in defining differentiated conditions. For example, if only the conditions of internal digestion of food and motility were used, the Venus flytrap would be considered an animal rather than a plant. However, plants are also differentiated by the sufficient condition of having plastids, such as chloroplast, in their cells. Internal digestion of food and motility are necessary but not sufficient conditions for declaring an organism an animal. There are historical and modern systems defining anywhere from two to eight kingdoms of living organisms, depending on the necessary and sufficient conditions used for definitions.

Animal Tissue Structures

All animal cells have extracellular matrix, the boundary that can serve many functions, including exchanging substances between cells, segregating tissues, and anchoring cells. Animal cells typically form tissues; groups of cells carry out particular functions within animal bodies. There are four types of animal tissues, defined by their functions: muscle, nervous, epithelial, and connective. Cells within tissues and tissues within organs may be tessellated (filling space or surface infinitely, without gaps).

Tissue engineering is an interdisciplinary field combining biology, material science, chemistry, and engineering to re-create, change, or replace tissues. It pays special attention to the mechanical and structural properties of tissues, often modeled mathematically before being implemented in the lab.

Technological Metaphors and Models

Beginning in the Renaissance, it was common for people to conceptualize living organisms in terms of human-made machines. This phenomenon worked both ways, since human constructions were informed by new understandings and observations of nature. During the Renaissance, animal tissues and organs were seen as combinations of relatively simple mechanisms such as levers. Attempts were made to imitate some functions of animals in construction, such as making bird-like wings. This analytic approach informed the development of scientific methods in biology—in contrast with a holistic view of living things as having a completely different nature from human-made mechanisms. In the seventeenth century, this philosophical approach of modeling animals on machines was supported by such influential scientists as Galileo Galilei, René Descartes, and Isaac Newton.

Engineering and mathematics developed along with explanations in biology. Developments in steam technology introduced the ideas of energy and work, which, in turn, led to the analysis of gas and liquid pressures as explanations of the interaction of tissues and organs in animal bodies. The metaphors of heart or cellular structures as pumps—or kidneys and the liver as filters—persist to this day. When electricity and magnetism were first discovered, there were numerous attempts to apply them directly to explanations of animal bodies, but many of these early models were discarded later. In the twentieth century, animal processes are often conceptualized as computer entities, such as nervous system as a computational network. Likewise, animal brains are observed for the purpose of building the artificial intelligence. Mathematical models in biology developed from simple measurements of weight, length, and proportion to those incorporating calculus, differential equations, statistics, computational science, and other areas of modern mathematics.

Animal Motility, Field Perception, and Gradients

Animals can move under their own power. Animals movement in response to external stimuli or gradients of stimuli is called taxis. In calculus, the gradient is a vector field; its vectors point in the direction of the greatest rate of increase in a variable and have the magnitude equal to that rate. Depending on the nature of the variable in the gradient, animals or animal cells can exhibit different types of taxis, such as thermotaxis along temperature gradients or phototaxis along light gradients. Mathematical models of taxis are based on calculus, differential equations, and statistics.

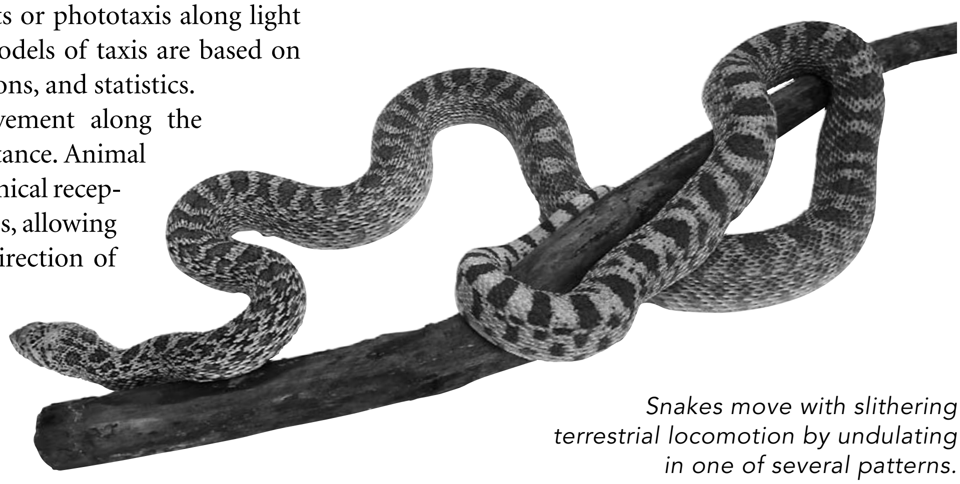
Chemotaxis is the movement along the gradient in a chemical substance. Animal cells may have multiple chemical receptors around their boundaries, allowing the cell to determine the direction of chemical gradient vectors. Animal cells can move toward chemoattractors, such as immune cells arriving where they need to be, or away from chemorepellents. The development of animal embryos involves the movement of cells and is regulated by gradients in signal chemicals. Sperm movement occurs because of chemotaxis and thermotaxis.

Magnetoperception (the ability to detect magnetic fields) is observed in migrating birds, sharks, rays, honeybees, and other animals. It is an important factor in regulating animal movement and navigation—for example, during bird migrations. Experiments and applications in magnetoperceptions usually involve attaching magnetic substances to animals and observing effects. For example, cows and deer grazing under power lines orient themselves differently. The mechanisms of magnetoperception continue to be actively investigated.

Animal Locomotion

The way animals move, in addition to being a matter of biological interest, is a source of engineering ideas. Until the twentieth century, the main source of data on animal movement was observation and, sometimes, experiments with animals or their body parts. Photography and videography added details to the observation. Animals may be equipped with miniature devices that track their positions in space, as well as the electric activity within muscles, the contraction of muscles, or the forces exerted by muscles. These devices allow the development of detailed models of animal bodies during movement.

Every type of locomotion has been modeled in physics, with a variety of relevant equations. There are three major types of terrestrial locomotion (movement on



Snakes move with slithering terrestrial locomotion by undulating in one of several patterns.

solid surfaces): legged movement, slithering, and rolling. Legged animals may have from two to 750 legs, with the geometry of leg and joint position defining posture, and the pattern and pace of leg use defining gaits. Snakes move by undulating in several patterns, such as sidewinding, or by lifting parts of their belly slightly off the ground, moving them forward relative to their ribs, and then pulling the body to them (rectilinear motion). These movements on land are described by kinematic equations, in water by hydrodynamic equations. Rolling animals, such as pangolins, can briefly achieve great speed, usually by forming a wheel or a ball out of their whole body and using gravity to escape predators.

Swimming is accomplished by body movement propulsion in fish, jet propulsion in mollusks, undulation in several types of animals, and limb movement in some birds and mammals. Jet propulsion requires relatively high energy but can provide animals with an occasional burst of speed. Models of swimming include such measures as buoyancy and are modeled with fluid dynamics and mechanics.

Gliding, soaring, and flying are energy-efficient ways of locomotion, and attract much interest in biomechanics and aerodynamics. Scientists study concepts like lift and drag as well as ratios of wing measurements such as loading (weight to area). Animals use different types of motion through the air, which are defined by a combination of timing and geometry. For example, falling with increased drag forces that prolongs the fall can be either parachuting (when the angle to earth is more than 45 degrees) or gliding (when the angle is less than 45 degrees). Gliding animals such as fish and squirrels have aerodynamic adaptations including streamlining. The variable glide ratio is the ratio between the horizontal and the vertical speed components (lift to drag). A flying squirrel has a glide ratio of about two, and a human in a glider windsuit modeled after gliding animals has a glide ratio of about two and a half. Soaring birds glide during parts of their flight.

The properties of winged flight in birds and bats depend on proportions of the animal's body. Wingspan is the distance between wingtips, and the mean wing chord is the average of the distances between the front and the back edge of the wing, found using calculus. Aspect ratio of a wing is the ratio of wingspan to mean chord. Fast birds such as falcons have pointy short wings with high aspect ratio (narrow wings). Long wings with high aspect ratios such as the wings of albatrosses, on the other hand, can produce slow soaring and gliding flight. Wide, rounded wings with medium aspect ratios can be used for a variety of flight types, for example, in storks or sparrows.

Biophysicists first attempted to explain insect flight using bird flight mechanics. They found that the resulting forces were several times less than what would be needed to lift and to propel an insect. Current theories of insect flight are still controversial. The theories use computational differential equations to model effects such as vortexes created in front of wings. When wings flap with high enough frequency, such a vortex can provide significant additional suction force.

Relatively rare types of animal locomotion depend on surface tension and capillary forces for walking on the water surface, or moving faster over released liquid (Marangoni effect). These forces are studied in fluid dynamics and thermodynamics.

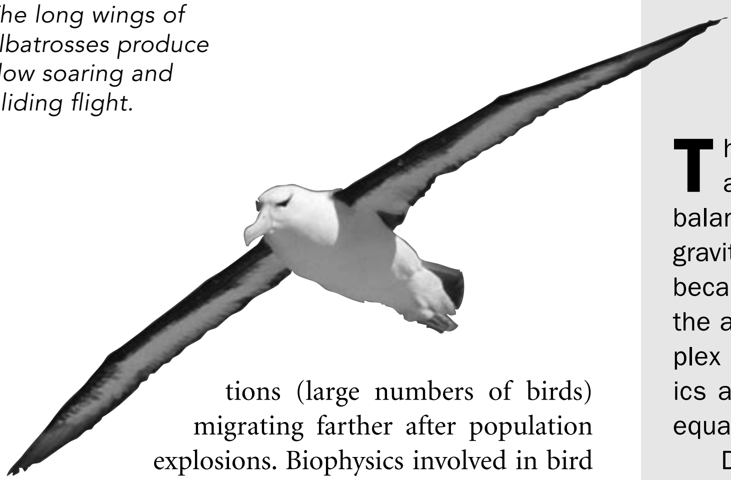
Researchers debate why the wheel, which provides several mechanical advantages in terrestrial locomotion, has never evolved in any animal. The relevant mathematical model is a graph measuring fitness of organisms to the environment, called fitness landscape. Fitness peaks are stable states, with genetic modifications meaning worse fitness. While wheel locomotion may be a fitness peak, it is surrounded by fitness valleys too deep to be crossed by evolutionary means.

Migrating Animals

Many animals migrate—periodically travelling among habitats—sometimes over long distances. Models of migration take into account the time of each leg of the journey as well as the full period of migration. These times can be synchronized with seasonal milestones, developmental stages in the life of each animal, and other natural events. Because migrations can take place across international boundaries, they can help promote international efforts in research and conservation. The Convention on the Conservation of Migratory Species of Wild Animals, for example, covers several endangered species of birds and fish, as well as migratory bats and turtles.

About a fifth of all bird species in the world migrate. Typically, birds migrate closer to the equator in winters, and farther from the equator in summers. Mathematical models of bird migration include the overall patterns for particular populations such as migration corridors as well as random events such as irrup-

The long wings of albatrosses produce slow soaring and gliding flight.



The Physics of Winged Flight

The physics of winged flight in birds, bats, and extinct dinosaurs focuses on the balance of four forces: lift, drag, thrust, and gravity. Air moving around wings produces lift because of speed and pressure differences in the airflow on either side of the wing—a complex process still being studied in aerodynamics and modeled with systems of differential equations.

Drag comes from air resistance to the flying body, and by air turbulence created by wing movements. Newton's second and third laws of physics explain thrust (the force created when a wing flaps). The vector of thrust points in the direction opposite to where the wing is moving. In other words, the flapping wing propels the animal forward by by pushing against the air. The force of gravity is proportional to the mass of the animal.

tions (large numbers of birds) migrating farther after population explosions. Biophysics involved in bird migration includes theories of energy efficiency, and various mechanical effects, such as wear on feathers that necessitates periodic molting synchronized with the migration period.

During migration, birds navigate by using the landscape clues they learn while young, orienting by the sun, or using magnetoperception. In some bird species, navigating is mostly a learned behavior; in others, it is mostly coded genetically. Sometimes the coding goes wrong, reversing the migration direction 180 degrees, thus causing birds to reverse-migrate in the opposite direction from the majority of their flock. Bird species that learn their migration routes from their elders, such as cranes, can be taught to use safer routes by following light aircraft of animal preservation specialists.

Shorter migration routes also exist. For example, many fish species rise to the water surface to feed at night—a type of diel vertical migration. Many fish species high in the food chain migrate to follow their prey, with varying times and lengths of migration journeys.

Because many insects are relatively short-lived, their migrations may involve multiple generations being born along the route. In these cases, none of the individual insects travels the full migration route. Some migrating insects, such as locusts, swarm for the purpose of migration. A swarm can be modeled using a system of differential equations where pairs of individuals move closer if they are too far, move away if they are too close, and orient themselves toward the same direction. However, studies of insects, including locusts, show complex mechanisms that include chemoregulation, physiological change in response to overcrowding (measured in contacts per unit of time), emission, and responsiveness to sounds and other variables involved in swarming.

Herds of animals, schools of fish, and flocks of birds can be modeled as groups of particles, with interactions among individuals determined by differential equations with some fixed and some random parameters to account for individual behavior variations. Such mathematical models (called "interacting particle models") can describe flock behavior or predict school migration routes. To observe animal migration, researchers use tracking devices, satellite observation, and echolocation for marine species.

Food Webs

Food webs and food chains map food relationships in ecosystems. The key measurement of the position within the food web is called "trophic level." Autotrophs (producers) are at trophism level one. Autotrophs are organisms that do not consume other organisms or carbon produced by them, and therefore are not animals. Two mechanisms of autotrophism are photosynthesis in plants, and chemosynthesis in archae and bacteria. The first organisms to evolve on Earth used chemosynthesis. A third mechanism, radiotrophy, is being researched in fungi in high-radiation areas. All food chains within all food webs on Earth start with level one autotrophs. Predator species that no other species predate upon are called apex predators.

More specifically, classes of organisms are named according to the flow chart with three branchings. The first branching determines the source of energy, either light (photo-) or chemical (chemo-). The second branching determines the source of extra electrons in reduction-oxidation reactions, either organic (-organo-) or inorganic (-litho-). The third branching defines the source of carbon, either organic (-heterotroph) or carbon dioxide (-autotrophs). For example, fungi are chemoorganotrophic. All eight combinations resulting from these three branchings exist in nature. Heterotrophic organisms that break down other dead organisms into simpler organic or inorganic compounds are called decomposers. Consumer organisms use other living organisms as their source of energy. Simplistically, the second trophic level comprises primary consumers that eat plants (herbivores) or chemosynthesizing creatures. The third trophic level, secondary consumers or predators, consists of animals that eat primary consumers. Animals that eat those at the third trophic level are said to have the fourth trophic level, and so on. However, most existing animal species obtain energy from several sources. For example, foxes eat rabbits and berries; chickens eat grains and insects.

To address the complexity of food chains, the trophic level of an animal is determined by the formula of adding all products of levels of its food by the fraction of that food in the animal's diet, and adding 1. For example, if a chicken's diet consists of 30% worms (level 2) and 70% grain (level 1), its trophic level is equal to

$$0.3\,(2) + 0.7\,(1) + 1 = 2.3.$$

Statistical analysis is used to determine the mean trophic level of a species in a particular ecosystem.

Changes in any part of the food web affect all other parts. For example, the effect of introducing predators that reduce the numbers of the prey and cause abundance in the next trophic level down is an example of a "trophic cascade event." The ability of an ecosystem to withstand disturbances is measured by an index called ascendency, and is derived by formulas from the information theory field of mathematics. Variables in ascendency formulas include both the amounts of energy and matter circulated within an ecosystem and the information shared among members of the system. Low ascendancy values make ecosystems internally unstable; high ascendancy values make ecosystems oversensitive to external disturbances. Ascendency values corresponding to stable systems are called "the window of vitality."

Ascendency is an example of using multiple indices and metrics to model, evaluate, and predict changes in food webs. For example, consider energy or biomass transfer from one feeding level to the next feeding level. The efficiency of this transfer is a measure of an ecosystem called ecological efficiency. For example, in a food chain that consists of four levels, with mean ecological efficiency of 1/10, the apex predator has the ecological efficiency of converting sunlight into its biomass of

$$\left(\frac{1}{10}\right)^4 = 0.0001.$$

Ecological efficiency restricts the number of possible trophic levels.

Fantastic Animals, Hybrids, and Genetic Chimeras

A variety of cultures describe fantastic animals or humanoids with animal traits. These animals—especially those invented before the nineteenth century—are used in mathematics education to help students understand concepts related to combinatorics because they are made by combining parts of existing animals. For example, ancient Greeks invented a chimera that had the body of a lion, the heads of a goat and a lion, and a snake for the tail. In genetics, chimeras are animals that have genetic material from more than one zygote—from four or more parents. Chimeras of different animals of the same species happen naturally when several eggs in one female are fertilized by sperm from different males and then fused. They may also happen artificially, in which case different animals species can be used. For example, a goat-sheep chimera called "geep" was first produced in the 1970s.

Hybrid animals are different from chimeras in that they have two parents, but the parents are of different species. Hybridization has been recognized and used for millennia. For example, humans have produced large populations of mules since ancient times. The mathematics of hybrids involves tracking the amount of genetic material from each species through generations, and calculating the probabilities of achieving particular traits in offspring. For example, a single-cross hybrid has 50% genetic material from either line of parents. Crossing such hybrids with the line of one of the parents (called backcrossing) produces hybrids with roughly 75% genetic material from that parent's species—averaged across a species, as individuals will have either pure or half-and-half genetic material.

Symmetry and Fractals

Most animal bodies exhibit either rotational (radial) or reflection symmetry. Animals with bilateral reflection symmetry (having a plane separating bodies into roughly reflected halves) form the taxon Bilateria. Observation of symmetry is a major tool of evolutionary theory. For example, it is hypothesized that all Bilateria animals evolved from a common ancestor species, Urbilaterian, that lived around six hundred million years ago. This makes Bilateria a clade (a group of animals that come from a common ancestor). Bilaterians have the front end with the mouth and the back end with the anus, defined by the plane of symmetry.

Rotationally symmetric animals such as sea anemones and sea stars usually have the mouth on the axis of the symmetry. When animals have a certain number of body regions positioned around the axis symmetrically, they are called by the number of regions. For example, five-armed stars exhibit pentamerism, and many coral polyps exhibit hexamerism, or six-part rotational symmetry.

Combinations of reflections, rotations, and translations can produce repeated geometric patterns called tessellations or wallpaper groups in plane and crystallographic groups in space. There are 17 types of wallpaper groups and 230 types of crystallographic groups described by the area of mathematics called group theory. Wallpaper and crystallographic groups can be found in colonies of animals such as corals or in arrangements of animal body parts such as fish scales.

Fractals are shapes that can be split into parts that are copies of the whole. Fractals frequently occur in the living nature. For example, feathers are fractal-like structures of the tree type, with three or four levels. Nervous systems and lungs of mammals are also tree-type fractals. Beyond the literal meaning as a geometric shape, the idea of a fractal as a self-repeating structure is applied to many areas related to animals to describe patterns within systems behavior, evolution, migration, and development.

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Maria Droujkova

See Also: Airplanes/Flight; Bees; Genetics; Joints; Nervous System; Symmetry; Synchrony and Spontaneous Order.

Animation and CGI

Category: Arts, Music, and Entertainment.
Fields of Study: Geometry; Measurement; Representations.
Summary: Animators have become adept at creating realistic products with the help of mathematics.

Animation is the process of creating the illusion of fluid movement from a series of static images. When these images are viewed in sufficiently fast succession, the human eye sees them as continuous motion rather than a sequence of discontinuous still images. From the earliest mechanical devices, through hand-drawn, stop-motion, and computer-assisted film techniques of the twentieth century, up to the latest computer-generated imagery (CGI), the quest has been to create interesting representations of movement and action.

Early Animation Devices

Historically, there have been several mechanical devices that were developed to simulate movement using still pictures. The Phenakistoscope, invented in 1832 by the Belgian physicist Joseph Plateau, consisted of a spindle with two mounted discs, one with slots around its edge and the other with pictures of successive action. With the discs spinning in unison and the picture side facing a mirror, the view through the slots appeared to show a moving drawing. In 1834, a British mathematician named William Horner produced the Zoetrope, a cylinder cut with vertical slits. Pictures of successive action were positioned on the inside opposite the slits. With the cylinder rotating, the image seen through the slits appeared to be in motion. As the Zoetrope used more pictures, and could be rotated more quickly, this gave a better illusion of movement. Even in the early twenty-first century, the Zoetrope is used to illustrate the basic idea of animation.

Animation Principles

By the start of the twentieth century, these mechanical devices were superseded by animated films. The principal technique was to hand-draw each frame. In the 1930s, animators at Walt Disney Studios developed what became known as the "12 principles of animation," many of which remain pertinent in an era of CGI. To illustrate, consider someone throwing a ball so that it bounces along the ground. A thrown ball is known to follow a parabolic path, a form of arc. The "arc" principle of animation is that almost all actions follow some form of arc. Arcs, as the Disney animators were well aware, give animation a more natural appearance. Another principle, "slow in and slow out," relates to the ball taking time to accelerate and decelerate. The animation looks most realistic if there are more frames near the beginning and end of a movement, and fewer in the middle. The flight of the ball and its bounce involves the principle of "squash and stretch." As the ball falls, animating a slight stretch gives the impression of the ball having speed. Dilation is the mathematical transformation for stretching and shrinking. Animating a squash to the ball as it bounces gives the impression of weight. For the ball to seem real, the animator uses the principle of "solid drawing" by taking into account the form of the ball in three-dimensional space as well as using the geometry of light and shadow.

CGI and Mathematics

CGI is even more mathematically based than hand animation because the images must be mathematically represented in order to be manipulated in the computer environment. Oscar-winning computer scientist Tony DeRose, who has worked for Pixar Animation Studios, said, ". . . different kinds of mathematics are used for different aspects of a film, from the simulation of how light bounces around in an environment (integral calculus) to obtaining smooth surfaces efficiently (subdivision surfaces) and making characters move in a realistic fashion (harmonic coordinates)." Trigonometry and vector algebra are widely used in CGI algorithms for creating and manipulating images. Matrices are a standard algebraic way of representing various transformations. Dilation makes objects larger or smaller in addition to stretching; translations move objects; and rotations turn objects.

One classic CGI method for creating three-dimensional animated objects involves using polygonal meshes, which are collections, or grids, of polygons. This method makes use of the geometry of smooth surfaces. Like animated motion, this method relies on the human eye's tendency to smooth discon-

When the cylinder of a Zoetrope rotates, the images on the inside appear to be moving when viewed through a succession of slits.

tinuous regions. Locally, smooth surfaces look flat, so they can be approximated with small, flat polygons such as triangles or quadrilaterals. Basic three-dimensional shapes such as cubes, cylinders, spheres, and cones may be joined to form composite three-dimensional objects. Interpolation is also used. More complex and smoother-looking three-dimensional objects can be modeled using sophisticated mathematics like spline patches and non-uniform rational basis splines, where a spline is a mathematical function defined piecewise by polynomials. Such techniques have become standard practice in CGI. The mathematical representation of three-dimensional shapes, including layout and materials, is used to compute a two-dimensional image from a given viewpoint, a process called "rendering." This process entails addressing issues such as visibility from selected viewer angles (including which parts of objects in the scene are hidden) and appearances, and how objects look different as the lighting varies. Finally, the motion of each object in the scene has to be specified.

Lucasfilm LTD and animator Kecskemeti B. Zoltan of Ste-One provided mathematician Timothy Chartier with digital models of Yoda from the *Star Wars* movies to explore in linear algebra classrooms. One of the models had 53,756 vertices, 4040 triangles, and 49,730 quadrilaterals, illustrating that realistic images and their transformations have many more data points and matrix multiplications than is typical as classroom examples. Chartier noted, "More recently, computer animation produced the character's movement, which required mathematical concepts from such areas as linear algebra, calculus, differential equations, and numerical analysis. Drawing on these popular culture ties in appropriate coursework can pique students' curiosity and compel further learning."

Despite the many available mathematical techniques and advances in the computational and visualization power of computer systems, convincing simulation of some physical features, like hair, continues to be challenging. Pixar noted that it took up to 12 hours to render a single frame of the character Sulley in the 2001 movie *Monsters, Inc.* because of his nearly 3 million individually animated hair strands. Each hair was mathematically modeled as a series of springs connected via hinges.

CGI has come a long way since the 1976 movie *Futureworld*, which many acknowledge as the first use of three-dimensional computer imagery. Even though the first CGI film to win an Oscar was Pixar's short movie *Tin Toy* in 1988, the 1995 movie *Toy Story* was the first full-length, fully CGI feature film. Many challenges and problems remain to be solved in the quest for photorealism in CGI. Examples include more accurate modeling and representation of physical actions, such as swallowing, as well as textural and other properties of materials like skin, including wrinkles. Animators also seek to better differentiate faces for people of varying ages, such as children or the elderly.

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See Also: Digital Images; Movies, Making of; Optical Illusions; Painting; Transformations.

Apgar Scores

Category: Medicine and Health.
Fields of Study: Algebra; Measurement; Representations.
Summary: The Apgar score is a simple prognostic device for neonatal care.

Throughout the developed world and in many other countries, every newborn baby is assessed according to various factors, each of which is assigned a score that is aggregated to quantify the baby's condition and prognosis. The system was introduced in the 1950s by Dr Virginia Apgar, whose last name has come to serve as a mnemonic for the assessed categories: activity, pulse, grimace (reflex), appearance (skin color), and respiration. The Apgar score indicates the health of the newborn and the likelihood that medical treatment or special intervention will be necessary much more quickly and more accurately than any system that had previously been in use.

In medicine, there are many scoring systems designed to predict and identify clinical situations in which the potential value of intensive care is low, while the burden of therapy is high, providing a numerical prediction of mortality. In gastroenterology, the Child-Pugh score is a scoring system to assess the prognosis of chronic liver disease; for vascular patients, the Eagle score allows estimation of a patient's risk of dying during heart surgery; the probability of pulmonary embolism is estimated by the Geneva score; the Gleason Grading system is used to help evaluate the prognosis of men with prostate cancer; and for pediatric end-stage liver disease, a scoring system exists for prioritizing allocation of liver transplants for children under 12 years of age.

Development and Effectiveness

Dr. Virginia Apgar was the first woman at Columbia University College of Physicians and Surgeons to be named a full professor. She developed a practical method for measuring the status of probable survival of newborn infants. Initially, she listed all objective signs that pertained in any way to the condition of the infant birth. Then she observed that five of these signs could be easily determined one minute after the complete birth of the baby. Depending on if the sign was absent, weak, or present, a rating of 0, 1, or 2 was given for each signal. The signs are heart rate (slow, normal, fast, or irregular), respiratory effort (from normal to distressed), reflex response (from over- to under-reactive), muscle tone, and color (pale, normal, or blue). In this system, infants in poor condition scored 0–2, infants in fair condition scored 3–7, and a score of 10 indicated a baby in the best possible condition.

In 1953, she observed the mortality rates of 2096 newborn infants with low, moderate, and high Apgar scores within 60 seconds after complete birth. This evaluation was rapidly adopted in delivery rooms throughout the United States and elsewhere. In 1959, a study with 15,348 infants established the predictive value of the Apgar score. The death rate among infants scoring 2, 1, or 0 was about 15%, while the rate for infants scoring 10 was about 0.13%. This prediction is especially useful in judging the urgency for resuscitative measures, such as respiratory assistance. It can be used to guide care, including intensive care. The score is generally determined by doctors and nurses at one minute and at five minutes after delivery. The five-minute score is generally accepted as the best predictor for newborn infant survival. A low score on the one-minute test may show that the neonate requires medical attention but is not necessarily an indication that long-term problems will occur, particularly if there is an improvement for the five-minute test.

Prediction

Probability is used to express knowledge or belief that an event will occur or it has occurred. A prediction is a statement that tells what might happen in the future based upon the given information. Prediction methods are important in various fields, including medicine, physics, and finance. Mathematics can be used to develop predictions, which are based on a careful analysis of patterns and collected data. Apgar recognized the patterns related to a baby's health signs and used them as a basis to make subsequent predictions. This example provides a clear idea of a credible prediction that was based on some form of empirical evidence. Thanks to this predictor approach, thousands of babies with special needs get the care they need immediately. Although it is not possible to make a 100% accurate prediction, predictions based on solid data and statistical analysis can increase the likelihood of accuracy.

Before 1952, the way to judge the condition of a newborn baby quickly and accurately shortly after birth was based on "breathing time" and "crying time." Apgar's accurate observations between 1949 and 1952 allowed the development of the automatic method of one-minute observation covering several signs easily. Thus, using some mathematical tools it is possible to transform qualitative values, such as physiological signs of babies, into quantity values—Apgar scores. By making predictions using Apgar scores it is also possible to perform the reverse: using the quantitative values (scores) to predict future qualitative values (health of babies).

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See Also: Data Analysis and Probability in Society; Diagnostic Testing; Medical Imaging.

Arabic/Islamic Mathematics

Category: Government, Politics, and History.
Fields of Study: Algebra; Connections; Geometry; Measurement; Number and Operations; Representations.
Summary: Arabic and Islamic mathematicians popularized the decimal system and Arabic numerals and also developed algebra.

Mathematicians living in Islamic lands and writing in Arabic have played a central role in the development of mathematics, particularly during the 700-year period from around the year 750 C.E to around 1450 C.E. These scholars lived in an area that not only includes the present-day Middle East but stretches into the western parts of India, the major cities of central Asia, all of northern Africa, and most of the Iberian Peninsula. Most of the influential mathematicians of this seventh-century era were Muslim, and most wrote in Arabic. However, the lands ruled by Muslim rulers included many ethnicities, cultures, languages, and religions. Muslims, Christians, Jews, Zoroastrians, Manichaeans, Sabians, Buddhists, Hindus, Persians, Turks, Sogdians, Mongols, Arabs, Berbers, Egyptians, and many others contributed to a remarkable multiethnic, multicultural civilization. Mathematics was not an exception. The full story of mathematics in this era has yet to be told. Hundreds of manuscripts await examination, translation, and a critical edition. Undoubtedly, in the years to come, our understanding of the extent, the import, and the influence of the mathematics of this period will change dramatically.

While their knowledge of what came before them was incomplete and uneven, the mathematicians of the Islamic era were aware of—and in some ways heirs to—ideas, methods, and points of view that originated in India, Persia, and—especially—Greek Alexandria. A remarkable translation movement coupled with a scholarly tradition of writing commentaries on previous works meant that mathematicians of this era were comfortable with the contents and the methodology of the works of, among others, Euclid, Archimedes, Apollonius, Ptolemy, and Diophantus as well as the basics of Indian decimal arithmetic and trigonometry. They also had access to Persian astronomical tables. They accomplished a great deal with this heritage. What the mathematicians of the Islamic era bequeathed to those who came later was very different in content, style, and approach than what had come before them. (A note on names: names of mathematicians and places can be transliterated to English based on their Arabic, Persian, or Turkish versions. For the most part, we have chosen what is currently most common in English. The one exception is that we have often omitted the Arabic definite article "al" that precedes titles and nicknames.)

The Decimal System and the Concept of Number

For Euclid—the preeminent mathematician of Greek Alexandria—"number" meant a rational number. In his work, irrational numbers were called magnitudes and were treated quite differently from numbers. In fact, Euclid's very influential book *Elements* contains few numbers and hardly any calculations. Starting with Khwarizmi of Khwarizm (c. 780–850 C.E.), the principles of the positional decimal system that had originally come from India were organized and widely disseminated. Hence, with the use of 10 symbols it was possible to carry out all arithmetic operations. Over the following centuries, the methods for these arithmetical operations were improved and included working with decimal fractions and with large numbers. In fact, in the process, the Euclidean concept of number was gradually enlarged to include irrational numbers and their representation as decimal fractions. The mathematician Kashani (c. 1380–1429), also known as al-Kashi, worked comfortably with irrational numbers and, for example, was able to produce an approximation that was correct to 16 decimal places. The Arabic texts on the decimal number system were translated to Latin and were the basis for what are now called the Hindu-Arabic numerals.

Algebra

While it is possible to recognize algebraic problems in ancient mathematics, algebra as a discipline distinct from geometry and concerned with solving of equations was developed during the Islamic period. The first book devoted to the subject was Khwarizmi's *Al-kitab al-muhtasar fi hisab al-jabr wa-l-muqabala* (*Compendium on Calculation by Completion and Reduction*).

In this title, "al-jabr"—the origin of the word "algebra"—means "restoration" or "completion" and refers to moving a negative quantity to the other side of an equation where it becomes positive. *Al-muqabala* means "comparison" or "reduction" and refers to the possibility of subtracting like terms from two sides of an equation. While all algebra problems were stated and solved using words and sentences—symbolic algebra did not arise until much later in the fifteenth century in Italy—an algebra of polynomials was developed by Abu Kamil (c. 850–930), Karaji (c. 953–1029), and Samu'il Maghribi (c. 1130–1180+, also known as al-Samaw'al). Powers, even negative powers, of unknowns were considered and many algebraic equations were classified and solved. Khwarizmi gave a full account of second-degree equations, and Khayyam (1048–1131) gave a geometric solution to equations of degree three using conic sections. Here, we give a problem—translated to modern notation—solved by Abu Kamil. Some 300 years later, this exact same problem appeared in Chapter 15 of the 1202 text *Liber Abaci* by Leonardo Fibonacci. Abu Kamil gave a solution to the following system of three equations and three unknowns:

$$x + y + z = 10$$
$$x^2 + y^2 = z^2$$
$$xz = y^2.$$

Abu Kamil first started with the choice of $x = 1$ and solved the latter two equations for y and z. Since, for the latter two equations, any scalar multiple of the solutions continues to be a solution, he then scaled the solutions so that the first equation was also satisfied. He simplified the answer to get:

$$x = 5 - \sqrt{\sqrt{3125} - 50}.$$

Geometry

Geometrical methods and problems were ubiquitous in the Islamic era. While algebraic problems were solved using the newly developed algebraic algorithms (the word "algorithm" itself is derived from *algorismi*, the Latin version of the name of the mathematician al-Khwarizmi), the justification for the algebraic methods was usually given using geometrical arguments and often relying on a distinctively Euclidean style. Guided by problems in astronomy and geography (for example, finding, from any place on Earth, the direction of Mecca for the purpose of the Islamic daily prayers), spherical geometry was developed.

But new work in plane geometry was also carried out. Khayyam and Nasir al-din Tusi (1201–1274), for example, studied the fifth postulate of Euclid and came close to ideas that much later on led to the development of non-Euclidean geometries in Europe. However, as is the case with much of the mathematics of this era, applications play an important role in the choice of questions and problems.

For example, Abu'l Wafa Buzjani (940–997) reports on meetings that included mathematicians and artisans. A problem of interest to tile makers is how to create a single square tile from three tiles. A traditional mathematician, Abu'l Wafa explains, translates this problem into a ruler and compass construction and gives a method for constructing a square of side $\sqrt{3}$.

While logically correct, this construction is of little use to the tile maker, who is confronted with three actual tiles and wants to cut and rearrange them to create a new tile. Abu'l Wafa also gives the customary practical method that is actually used by tile makers to solve this problem, and proves that their method, while practical, is not precise, and the final object is not exactly a square. While stressing the importance of being both practical and precise, and the virtues of Euclidean proofs, he presents his own practical and correct methods for solving this and related problems.

Trigonometry

The origins of trigonometry begin with the Greek study of chords as well as the Indian development of what is now called the "sine function." Claudius Ptolemy's table of chords and Indian tables of sine values were powerful tools in astronomy. However, a systematic study and use of all the trigonometric functions motivated by applications to astronomy, spherical geometry, and geography begins in the Islamic era. Abu'l Wafa had a proof of the addition theorem for sines and used all six trigonometric functions; Abu Rayhan Biruni (973–1048) used trigonometry to measure the circumference of Earth; and Nasir al-din Tusi gave a systematic treatment in his *Treatise on the Quadrilateral* that helped establish trigonometry as a distinct discipline.

Combinatorics

One of the earlier known descriptions and uses of the table of binomial coefficients (also known as the Pascal triangle) is that of Karaji. While his work on the subject is not extant, his clear description of the triangle survives in the writings of Samu'il Maghribi. Binomial coefficients were used extensively, among other applications, for extracting roots. Kashani, for example, used binomial coefficients to give an algorithm for extracting fifth roots. He demonstrated it by finding the fifth root of 44,240,899,506,197. Other combinatorial questions were treated as well. Ibn al-Haytham (c. 965–1039, also known as Alhazen) gave a construction of magic squares of odd order, and Ibn Mun'im (died c. 1228) devotes a whole chapter of his book *Fiqh al-Hisab* to combinatorial counting problems.

Numerical Mathematics

The prominence of applied problems, the development of Hindu-Arabic numerals and calculation schemes, and the development of algebra and trigonometry led to a blossoming of numerical mathematics. One prime example is Kashani's *Miftah al-Hisab* or *Calculators' Key*. In addition to his approximation of 2π and his extraction of fifth roots, he also gave an iterative method for finding the root of a third-degree polynomial in order to approximate the sine of one degree to as close as an approximation as one wishes.

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Shahriar Shahriari

See Also: Babylonian Mathematics; Greek Mathematics; Measurement, Systems of; Number and Operations; Numbers, Rational and Irrational; Numbers, Real; Ruler and Compass Constructions; Squares and Square Roots; Zero.

Archery

Category: Games, Sport, and Recreation.
Fields of Study: Algebra; Geometry; Measurement.
Summary: Mathematics is essential in modeling and predicting a bow's performance.

Archery is the practice of propelling an arrow with a bow for the purpose of hunting, warfare, or sport. A bow is a pair of elastic limbs connected at the tips by a string. A bow acts as a spring and stores in the limbs the energy applied by the archer. As the archer releases the string, the arrow is propelled with a force proportional to the tension on the string. The path of the arrow is a parabola whose shape is determined in part by the angle of release from the bow, measured with reference to the ground.

The origins of archery are lost in the beginning of civilization and probably will never be determined with precision. The earliest bows known today were found in the Holmegaard area of Denmark and were made of elm and yew. The Holmegaard bows date form the Mesolithic period (10,000–3000 B.C.E.); however, there is archaeological evidence of projectile wounds—possibly caused by bows—from the Upper Paleolithic (40,000–10,000 B.C.E.) in all continents. It is speculated that archery was first used for hunting and, later, for warfare as social structures became increasingly complex. By the twelfth century B.C.E., archery was a decisive branch of military power. For example, the wall of the Theban temple of Ramses III depicts the Aegean fugitive fleet—driven from Crete by the Greek immigration—engaging in and losing a battle against the Egyptian fleet, whose primary weapon is shown to be archery. Archery remained the weapon of choice in the West for distance combat until the introduction of gunpowder toward the fourteenth century C.E. Archers in the medieval era would fire in a high arc, achieving accuracy by volume rather than deliberate aim. Today, archery is practiced as a precision sport and for hunting. Men's archery was one of the events of the second modern Olympics in 1900. The first Olympic archery event for women was held in 1904.

Mathematical Modeling of Bows

Since the 1930s, engineers and scientists have studied the design of bows. In 1947, C. N. Hickman made the first accurate mathematical model for flat bows, consisting of an idealized representation of two linear elastic hinges and rigid limbs with point mass (an idealized representation of a body used to simplify calculations) at the tip. More recent modeling efforts by B. W. Kooi and C. A. Bergman consider the limbs as beams that store elastic energy by bending.

An archer's body should be perpendicular to the target and his or her feet should be shoulder-width apart.

The Bernoulli-Euler equation, named for Daniel Bernoulli and Leonhard Euler, describes the change in the curvature of a beam as a function of the "bending moment" (tendency to rotate about an axis) and is used to estimate the force in the string. When the archer draws the bow, the force exerted at the middle of the string causes an increase in the bending of the limbs, thus increasing the momentum and storing more energy for the shot. The elasticity modulus of the bow's material—the proportionality constant that relates limb deformation versus energy stored—determines the force with which the limbs recover their original shape after being deformed.

Mathematical modeling is a viable alternative for the evaluation of the performance of old bow models. As time passes, environmental conditions and natural processes cause considerable degradation within the cell structure of the wood used in ancient bows, which prevents a realistic assessment of the original density of the material and precludes direct testing.

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Juan B. Gutierrez

See Also: Artillery; Firearms; Learning Models and Trajectories; Middle Ages.

Archimedes

Category: Government, Politics, and History.
Fields of Study: Calculus; Connections; Geometry; Measurement; Representations.
Summary: Archimedes contributed to many areas of mathematics, demonstrated the law of buoyancy, and designed a number of marvelous devices.

Archimedes of Syracuse (287–212 B.C.E.) was born and lived most of his life in Syracuse on the Greek island of Sicily in the Mediterranean. Archimedes possessed an incredibly versatile intellect—today he is remembered as one of the most important mathematicians, astronomers, and engineers in history.

He is credited with numerous inventions, such as the mechanical pump known as the Archimedes screw, the compound pulley, and various engines of war (including advanced catapults, ship-destroying mechanical arms, and the famous Archimedes "death ray"). As important as Archimedes' legacy is to engineering and astronomy, perhaps his most important work was in mathematics. His contributions in geometry, conic sections, and number theory along with his work in computational mechanics, his discovery of the law of buoyancy, and his contributions to the field of mathematics that would become known as calculus almost two millennia later, secure Archimedes' place in mathematical history.

Early Life

Most of what is known about Archimedes was written long after his death by Roman historians such as Plutarch. This lack of contemporary sources—coupled with the fact that surviving works of Archimedes himself are copies made many centuries later—make some of the popular stories about the Greek mathematician questionable.

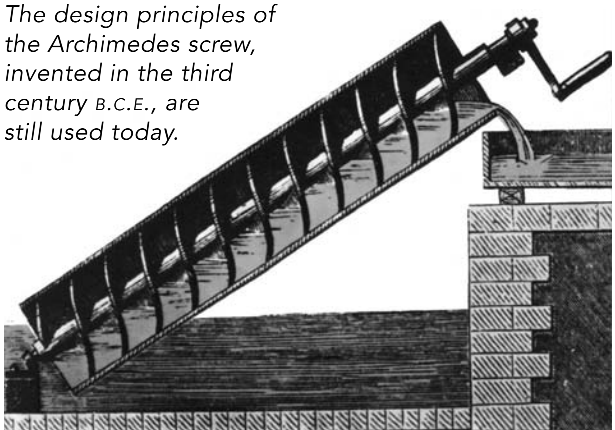
Archimedes' father was an astronomer named Phidias. According to some authors, young Archimedes was sent to Alexandria, Egypt, to study. The library at Alexandria was the center of learning for the Greek world, containing the mathematical and astronomical manuscripts produced by scholars such as Euclid of Alexandria, the "father of geometry." It is very likely that Archimedes studied mathematics with the students of Euclid. While in Alexandria, Archimedes may have produced his first important invention, a pump now known as the Archimedes screw (some historians claim the device was invented by Archimedes at a later date at the request of King Hieron II to be used as a bilge pump for removing water from ships). Whatever Archimedes' motivation for developing it, the Archimedes screw is a simple mechanical device that is used to move water. In fact, the device continues to be used today in various applications.

Although there is no indisputable evidence that Archimedes studied in Alexandria, there are several indicators that he was friendly with the mathematicians there. For instance, the famous Archimedes' Cattle Problem is found in a letter Archimedes sent to the Alexandrian astronomer and mathematician Eratosthenes. Archimedes challenged the mathematicians of Alexandria to solve a complicated mathematical riddle concerning the number of cattle in the herd of the sun god. Written in verse, the problem involves extremely large numbers and was not solved by the mathematicians of Alexandria, or by anyone else for that matter (including Archimedes), until the late nineteenth century.

Archimedes' Inventions

In addition to the Archimedes screw, Archimedes is credited with the invention of the compound pulley. A compound pulley is a system of movable pulleys that provides a substantial mechanical advantage for doing work. Evidently very confident in his knowledge and abilities, Archimedes once asserted, "Given a place to stand on, I can move the Earth." King Hiero II decided to test Archimedes' boast by assigning him the job of

The design principles of the Archimedes screw, invented in the third century B.C.E., are still used today.



The Archimedes Palimpsest

One of the primary duties of historians is to use old letters, manuscripts, and other documents to try to understand a people or culture. In most cases, the existence of these documents is already known to others, and the historian is simply trying to shed new light on the past through the interpretation of the existing manuscripts. Occasionally, a new document is discovered and serves to excite the historical community.

In the tenth century, a manuscript containing several of the works of Archimedes was produced in Greek, the language Archimedes had used to compose the original works thirteen centuries earlier. A few centuries later, this manuscript and several other unrelated manuscripts were reused to produce a Byzantine prayer book.

Since the manuscripts were written on parchments—or animal skins—the words were literally scraped off the page (the word "palimpsest" means "scraped again"), making the parchment ready for its new authors. In this way, the monks saved the expense of new parchment. This prayer book survived the ensuing centuries until the early twentieth century, when a leading Archimedes scholar, John Ludwig Heiberg, determined that underneath the prayers and barely visible in a few places were works of the Greek mathematician Archimedes.

The location of the Archimedes Palimpsest throughout the rest of the twentieth century is a bit of a mystery, but in 1999 a wealthy collector, wishing to remain anonymous, bought the manuscript at auction and entrusted it to the Walters Art Museum in Baltimore for conservation and study. At the museum, experts in various fields have worked with scholars to uncover the hidden text. What they uncovered proved to be one of the most exciting discoveries in the history of mathematics. One of the Archimedean treatises on the palimpsest represents the only known copy of that work in Greek. Two others are manuscripts previously thought lost by scholars.

Although Greek and Roman writers attributed both of these works to Archimedes, no one in the modern age had ever laid eyes on either one—until the secrets of the palimpsest were unlocked.

moving a heavily laden ship from its dock, a job that would have required a great effort from a substantial number of men. Archimedes constructed a system of pulleys by which, with very little effort on his part, he was able to move the heavy ship.

Perhaps the best-known inventions of Archimedes were the engines of war used by the defenders of Syracuse against the invading Roman army. Although the Syracusans were badly outnumbered, Archimedes' ingenious devices kept the attackers at bay, even striking fear into the hearts of the Roman soldiers. In addition to making improvements on existing weapons such as the catapult, Archimedes developed new and frightening methods for defending his home. One of these inventions reportedly involved the use of great mirrors to focus the sun's rays on the Roman ships blockading Syracuse, setting fire to the helpless ships and the soldiers contained in them. This particular story of Archimedes' exploits has been questioned for centuries. In fact, it was a topic of the Discovery Channel television show, *MythBusters*, which concluded that the likely success of such a device was very small.

Archimedes also built, according to Roman historians, gigantic mechanical arms that swung out over the enemy ships. Some of these arms dropped massive stones and other weapons, sinking the ships. Another mechanical device, today known as the Archimedes claw, was used to pluck ships out of the water and bash them against the rocks or simply shake them and drop them from great heights so that they would sink to the bottom of the sea.

Archimedes' inventions were so effective, and so terrifying, that reports claim that the Roman ships and their invading armies would flee in terror at the slightest sound or movement emanating from the Syracuse defenses that hinted at another attack.

Archimedes' Mathematics

Although renowned for his engineering achievements and machines of war, Archimedes was at heart a pure mathematician. His insights and discoveries in many fields of mathematics cause historians today to consider him one of the greatest and most original mathematicians who ever lived. Archimedes was reportedly obsessed with mathematics, and stories abound regarding this obsession. While lounging in the public baths—as was the custom at the time—Archimedes would often draw geometric figures in the chimney embers. This single-mindedness eventually led to his demise. Two stories emerged from biographies of Archimedes by the Roman historian Plutarch, both of which occurred when the Romans finally overran Syracuse. In the first and most popular story (probably because it illustrates the romantic idea of the dedicated but absent-minded scientist), Archimedes is contemplating some geometric figures when a Roman soldier comes upon him and orders Archimedes to come with him. Archimedes' response to the soldier is that the soldier should leave him alone until he has finished the proof to the problem he is contemplating—a response not appreciated by the soldier, as he slew Archimedes with a sword. The other story involves a soldier coming upon Archimedes as he carried various mathematical instruments to General Marcellus, the Roman general in charge of the invasion. The soldier, thinking the instruments were valuable—perhaps even vessels filled with gold—slew Archimedes for the treasure. In either account, General Marcellus is very unhappy, as he had ordered Archimedes brought to him alive.

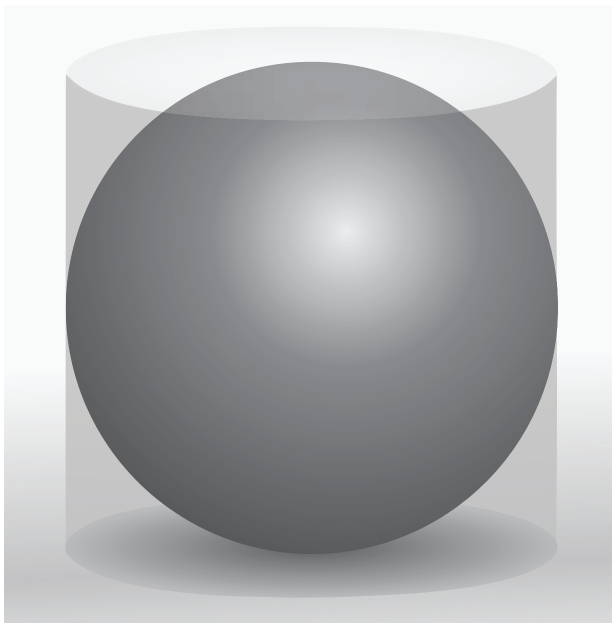
Today, Archimedes the mathematician is remembered for much, but his discovery of the methods that one day would be called integral calculus is at the forefront. Archimedes used a technique called the method of exhaustion to approximate the area of a circle and thus the value of pi. In this method, Archimedes inscribed a polygon in a circle and calculated the area of the polygon. Inscribing a polygon involves drawing a polygon—a multisided, closed figure such as a pentagon or an octagon—so that each vertex just touches the inside of a circle. He then circumscribed a polygon around the outside of the circle and calculated its area. He knew the actual area of the circle must be somewhere between the areas of the two polygons. By calculating the areas of polygons with more and more sides—eventually using a 96-sided figure—he was able to approximate the area of the circle closely and conclude that the value of π lay somewhere between the following two fractions:

$$3\frac{1}{7} \text{ and } 3\frac{10}{71}.$$

Archimedes used the method of exhaustion to find many other interesting mathematical theorems. Most of these theorems are geometric in nature and address methods for finding areas of plane figures and volumes of three-dimensional solids. For example, Archimedes proved that the surface area of a sphere is four times the area of a great circle.

He also compared a sphere and a cylinder circumscribed around the sphere and found that the sphere's volume and surface area were two-thirds those of the cylinder. Many scholars believe that Archimedes considered his most important work to be in this area. A little more than century after the death of Archimedes, the Roman senator and orator Cicero discovered a grave he believed to belong to Archimedes, marked with sphere inscribed in a cylinder along with the related theorem.

Interestingly, although Archimedes used geometric methods like the method of exhaustion to prove his theorems, he used other methods in the discovery



Archimedes found that a sphere's volume and surface area were two-thirds those of a circumscribed cylinder.

of some of those same theorems. One such method involved the use of infinitesimals. An infinitesimal is an indefinitely small number that proved to be critical to the development of calculus many centuries later. Ironically, Archimedes did not accept the use of infinitesimals—or other mechanical methods he used to uncover interesting mathematical truths—in a rigorous proof. Archimedes accepted the Greek tradition that only pure geometric demonstrations contained the rigor demanded in mathematical proof. According to Archimedes, as translated by British mathematician Thomas Heath, "Certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards.... But it is of course easier, when we have previously acquired by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge."

Archimedes made many other mathematical discoveries. He found methods for calculating the center of gravity of plane figures, methods for summing infinite series, techniques for finding tangents to curves (a forerunner of differential calculus), and a method for finding the weight of a solid body immersed in liquid. In addition to these geometric discoveries, Archimedes found many interesting results in arithmetic, or the theory of numbers. He developed methods for estimating the value of square roots, and in a work called *The Sand Reckoner*, Archimedes invented a new number system capable of representing impossibly large numbers—numbers large enough, according to Archimedes, to count the number of grains of sand in the universe.

Today, we know of Archimedes' great mind through copies of his own works made centuries after his death as well as books from other authors who attribute results to Archimedes. Several works of Archimedes no longer exist, and we know only of their one-time existence from references in other books. Until the twentieth century, an Archimedes work called *The Method* belonged in this category of lost works. Other authors related that *The Method* contained explanations of the techniques used by Archimedes to discover many of his mathematical theorems. Historians and mathematicians alike lamented the fact that this potential insight into the mind of the great man would never be known. A copy of this work has recently come to light and has opened a treasure trove of new information regarding the creative processes used by Archimedes.

Archimedes' Legacy

The esteem with which history holds a figure like Archimedes may be tied to the stories—substantiated or otherwise—that become a part of the folklore surrounding that figure. The most famous story concerning Archimedes is another example of this historical perception. It seems that King Hiero II was concerned that a greedy goldsmith had used a certain amount of silver in a crown that was intended to be pure gold.

The king asked Archimedes to determine the purity of his crown without destroying it. While bathing one day—as the story goes—Archimedes realized that the volume of water displaced by his (or any other) body could be used to calculate the density of the body—a method that could measure the density of the crown and thus its content. Archimedes immediately sprang from his bath and ran naked through the streets of Syracuse yelling, "Eureka!" ("I have found it!")

Although this Archimedean anecdote does not appear in print until several centuries after his death, "Eureka!" remains the first thing that comes to the minds of many modern readers when they encounter the name of Archimedes. If indeed Archimedes did actually solve the problem of the impure crown for his king, it seems much more likely that he used a method that is now called the Archimedes principle. This method, which actually appears in Archimedes' writings, involved weighing an object while it is submerged in water to determine its buoyant force.

In his own lifetime, Archimedes was renowned as an inventor and military engineer rather than as a mathematician. History, however, remembers Archimedes as one of the most brilliant and original mathematical thinkers who ever lived. In today's modern world, "pure" and "applied" mathematics are often scrupulously separated by their practitioners. In his time, however, Archimedes was both a brilliant pure mathematician—whose work involving integral calculus predated Isaac Newton and Gottfried Leibniz by almost 2000 years—as well as a gifted applied mathematician—who used geometric techniques to find, among other things, the center of gravity of solid objects. Certainly, Archimedes is a part of the small pantheon of scientific geniuses like Newton and Albert Einstein whose brilliance changed the way in which we see our world.

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Todd Timmons

See Also: Calculus and Calculus Education; Greek Mathematics; Infinity; Levers; Limits and Continuity; Mathematics, Applied; Pi; Polygons.

Arenas, Sports

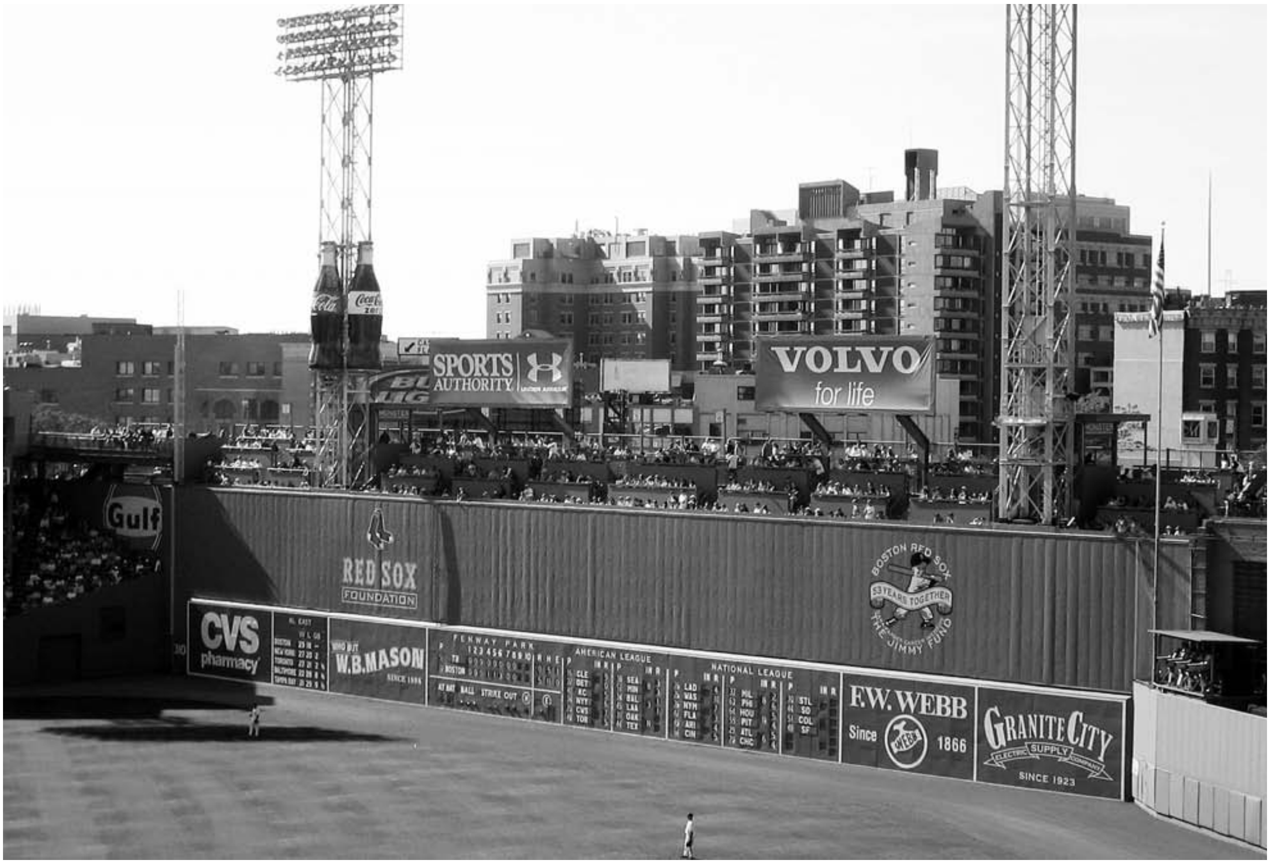
Category: Games, Sport, and Recreation.
Fields of Study: Algebra; Geometry.
Summary: Modern arena designers consult mathematicians to determine the effects of design on play and crowd behavior.

A sports arena is essentially an enclosed area consisting of a large open space where a sport is played, surrounded by seating for spectators. It may also include various facilities for athletes, spectators, and the press. Many sports use specific terms for arenas, like "park" for baseball and "stadium" for football. Some sports arenas are open-air while others are roofed. The word "stadium" comes from *stadion*, an ancient Greek unit of length. Mathematics plays a significant role in the design and maintenance of modern sports arenas, including not only the geometrically shaped playing surfaces but also the optimization of seating, sightlines, acoustics, lighting, spectator traffic flow, and placement of restrooms and concessions. Features such as retractable roofs and convertible forms to accommodate multiple sports require careful design as well. Mathematicians also analyze and model features of sports arenas to determine their potential effect on the game play.

The rules of each sport dictate dimensions for the field of play. Some such as hockey, football, basketball, and soccer specify exact dimensions for the playing surface and delineate areas for specific activities, like the rectangular key in basketball or the half-circle goal crease for amateur hockey. Baseball, on the other hand, standardizes the dimensions of some features such as the distance between bases and the distance between the pitcher and home plate, but the outfield varies depending on the positions of the outfield walls. Further, aspects of game play can be affected by design choices. Fenway Park's outfield wall known as the "Green Monster" is notorious for stopping home runs, yielding more doubles and triples. When the new Yankee Stadium produced a higher rate of home runs, there was speculation about a "wind tunnel" effect. Statistical analyses suggested that curvature and height of the right field wall were more important than wind speeds or patterns. Statistician George "Bill" James developed the concept of park factors, which attempts to measure how park characteristics influence game outcomes.

Robert F. Kennedy Memorial Stadium in Washington, D.C., which opened in 1961, was the first multiuse stadium. It was widely decried for being a "concrete donut." Some critics suggested its wavy shape and curvature optimized it for baseball seating, though the widely replicated design has deficiencies for baseball and football. Some critical seats were too low for football and too high for baseball, resulting in poor sightline angles.

The baseball configuration was also more symmetrical than most baseball-only fields. Modern designers use mathematical techniques and tools (such as Mathcad software), simulations, and three-dimensional modeling for their designs, resulting in unique facilities like The Float in Singapore, which is literally floating on Marina Bay. Similar methods are involved in the design of arena roofs or domes, some of which are retractable. Calculating the amount of material



The Green Monster is the nickname for the left field wall at Fenway Park, home to the Boston Red Sox. The 37-foot-high wall is famous for preventing home runs that would clear the walls of most other ballparks.

needed to construct a curved dome, as well as calculating the weights, forces, and stresses, typically involves the use of calculus. These calculations, in turn, partially determine the type of support required.

Geometry and graph theory also contribute to dome design. R. Buckminster Fuller suggested that domes are strongest when the edges lie along great circles. Triangles are often used to give great strength with minimal weight, while other support structures resemble the latitude and longitude configuration on a globe. Fibonacci sequences and plane tilings also are used in the design of some domes. Veltins Arena in Germany uses features like hinged columns with ball-bearing edges that move in three dimensions. Both Veltins Arena and University of Phoenix Stadium in the United States feature sliding roofs and retractable natural-grass playing surfaces weighing millions of pounds. These were mathematically modeled extensively before construction. Transformative structures of this type have become known as "kinetic architecture."

Mathematicians continue to investigate questions related to sports arenas, some of which have wider applications. Researchers have considered the impact of sports arenas on land values using hedonic regression models. Mathematical analyses of crowd sequence videos (frequently taken from sports venues) benefit research in areas including surveillance, designs of densely populated public spaces, and crowd safety. In some cases, people are conceptualized as a "thinking fluid" to which fluid dynamic and stochastic models may be applied. Unusual events like emergency evacuations are fairly rare, and there are legal barriers to obtaining extensive live footage. As such, computer scientists and mathematicians have developed detailed simulations for both "normal" behavior and unusual crowd events. Some have suggested that topology optimization would be beneficial for investigating arena evacuation plans.

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Bill Kte'pi

See Also: Baseball; City Planning; Domes.

Artillery

Category: Government, Politics, and History.
Fields of Study: Algebra; Calculus; Measurement; Number and Operations.
Summary Mathematics is essential to the design and firing of artillery pieces.

Mathematics has had numerous military applications, including the development of artillery pieces after the invention of gunpowder in China in the fourteenth century. Mathematical formulas and calculations are critical to the design and use of artillery. The science of ballistics, which relies on mathematical formulas to study the flight paths of projectiles, also plays a major role in artillery development.

Engineer Benjamin Robbins invented the ballistic pendulum and is referred to by some as the founder of modern ballistics. Technological and scientific developments resulted in the modern use of artillery firing tables and computer-based firing calculation programs. Mathematics also plays a significant role in the ability to centralize fire control command centers and the use of indirect fire in which targets are not visible through a weapon's sightlines.

Many mathematicians have worked in places such as the Ballistics Research Laboratory at Aberdeen Proving Ground, such as Gilbert Bliss, who worked on firing tables for artillery.

Early artillery weapons relied on mechanical energy to fire projectiles and were not of uniform design—making them large, cumbersome, and inaccurate. Technological innovations in the eighteenth and nineteenth centuries led to the development of standardized artillery with increased accuracy and mobility.

In the late eighteenth century, British Royal Artillery Lieutenant Henry Shrapnel created a shell (container) that held multiple musket balls and a time fuse that allowed the shell to travel longer distances before exploding, increasing the cannon's range. High-explosive fragmentation shells and improved conventional munitions replaced shrapnel shells beginning in the early twentieth century.

Military scientists, weapon and projectile engineers, and soldiers have utilized the science of ballistics (the study of the flight of projectiles as they exit the weapon, travel through the air, and penetrate the target) since its early development in the fourteenth century to improve the accuracy and range of artillery.

Cannons, which first appeared in the early fourteenth century, spurred the development of ballistics. Early artillery crews used mathematics to determine the optimal angles at which to elevate their weapons for improved accuracy and range. Engineers also used mathematics to determine the angles at which

Artillery Categories

The three main categories of modern artillery are guns (excluding personal weapons such as handguns and rifles), howitzers, and mortars. Guns rely on stored kinetic energy to force a solid projectile through both the weapon's barrel and the air to its intended target, while howitzers and mortars use stored chemical energy and explosive (non-solid) projectiles. Howitzers and mortars generally fire shorter ranges along parabolic arcs while guns fire longer ranges along sightlines. Tanks, anti-aircraft batteries, rockets, and missiles can also be considered part of modern artillery. Artillery can be either self-propelled or towed.

A U.S. M777 Light Towed Howitzer being fired in 2009 in Afghanistan. Howitzers have relatively short barrels and are used to shoot projectiles at high trajectories with a steep angle of descent.

to build fortifications to best defend against artillery bombardments.

Calculations of elevation, distance to target, weather conditions, projectile weight, and flight trajectory are necessary to achieve accuracy. Scientists and mathematicians, beginning with Italian mathematician Niccolo Franco Tartaglia, sought to improve the accuracy and reliability of early artillery pieces through ballistics. Tartaglia's studies on a variety of cannons led to his determination that a 45-degree angle was ideal for firing—with the caveat that external factors such as air drag would affect the results. Tartaglia is also credited with the development of the first ballistics firing tables based on standardized weapons and projectiles.

Other notable mathematical advances in early ballistics included the theories of Galileo Galilei on the effects of the forces of gravity and air drag on the projectile's velocity and flight path, as well as the parabolic nature of ballistic trajectories. In the early eighteenth century, English scientist Benjamin Robbins invented the Ballistic Pendulum, which allowed the measurement of a projectile's velocity and the effects of air drag on that velocity. He also determined that air drag plays a much greater role in affecting a projectile's velocity than gravity does. Sir Isaac Newton is credited with the development of formulas used to calculate aerodynamic drag, which he determined was proportional to air density, the projectile's cross-sectional area, and the approximate square of the projectile's velocity. However, Newton's solution was incomplete, and mathematician Johann Bernoulli produced a more general solution. Mathematician Leonhard Euler integrated the various stages of a projectile's flight to reduce the difficulty of the equations utilized in ballistics.

Artillery projectile designers use ballistics studies that calculate projectile properties, such as mass and diameter, based on the design specifications of the weapon in order to ensure the projectile will fit inside the barrel and generate enough energy to propel the projectile without damaging the weapon. Mathematical formulas are used to determine projectile design based on various input data including the force of aerodynamic drag, the ratio of the projectile's velocity to that of the sound in the medium it will traverse, the properties of the medium, the projectile's caliber (diameter), and the velocity at which it travels.

The mathematics of ballistics can be further broken down into internal, external, and terminal ballistics. Internal ballistics studies the flight properties of a projectile as it travels through the barrel of the weapon. A firing mechanism lights the gunpowder, which creates energy through the pressure generated by expanding gases. The energy is equal to the force times the barrel length. This energy forces the projectile through and out of the barrel. External ballistics studies the flight properties of a projectile as it travels through the air from the weapon to its intended target. Various formulas can be used to determine the kinetic energy of the projectile as it leaves the muzzle. Other calculations are then used to determine ballistic coefficient (a measure of a body's ability to overcome air resistance in flight.). The distance and direction of artillery projectiles is affected by aerodynamic drag caused by a combination of air pressure (the disturbance of air around a projectile creating an area of low pressure behind it) and skin friction (the contact between the air and the projectile's surface). Retardation is the measurement of the degree to which drag will slow a projectile's flight speed and can be calculated by the following formula:

$$R = \frac{D}{M}$$

where R is retardation, D is drag, and M is the projectile's mass. The ballistic coefficient is often used in place of drag because of the greater difficulty in calculating drag, which reduces along the flight path in relation to the decrease in velocity.

External ballistic formulas must also account for the fact that projectiles do not travel along straight flight paths. Physical and meteorological forces must be taken into account when determining or predicting a projectile's flight path. These forces include yaw (caused when the nose of the bullet rotates away from a straight trajectory) and precession (caused when the bullet rotates around the center of mass). Terminal ballistics studies the impact of the projectile as it hits the target. Mathematical calculations can be used to study how a projectile's design and flight features, such as velocity, shape, and mass, will affect its damage and wound capabilities.

Artillery firing requires the use of mathematical equations to determine range, elevation, and deflection, as well as the arc of fire and the probability of hitting the intended target. Artillery equation data also include the projectile's initial velocity, which is further divided into vertical and horizontal velocity components. Calculating the distance a projectile travels is performed by multiplying the time the projectile is in the air by the velocity's horizontal component. The needed angle to achieve a certain distance can then be determined by solving the equation for distance as a function of the angle.

Modern artillery crews rely on indirect fire, a technique developed in the early twentieth century in which a target is fired upon despite not being visible along sightlines. Indirect fire required more complex mathematical formulas and calculations, increasing the importance of specialized trained military personnel. These personnel calculated the range and bearing to the target. New techniques of determining the locations of enemy artillery batteries and subsequent firing data included flash spotting, sound ranging, air photography, and registration point. Indirect fire led to the development of graphical or tabular firing tables and the maintenance of a command center. Technological developments also allowed for greater adjustments to firing data based on such variable conditions as wind speed and weather. Initially, firing data derived from these tables was placed on the weapon's sights.

Use of Computers

Battlefield computers began to appear by the 1960s and were in use by the British and U.S. military by the following decade. Computerized firing tables utilize input data to determine the angle and position of artillery, which weapons will fire, and how many rounds will be fired (although some military forces still rely on older instruments and human calculations as backups).

Firing data such as quadrant elevation, azimuth (an angular measurement in a spherical coordinate system), fuse setting, and projectile properties are inputted into the software program spreadsheets based on established data and standard conditions, which determine ideal firing information. The firing information is then corrected for deviations from standard conditions, such as meteorological conditions. Further technological developments include computer-based surveillance and target acquisition systems, global positioning systems, and laser rangefinders.

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Marcella Bush Trevino

See Also: Archery; Infantry (Aerial and Ground Movements); Missiles.

Asia, Central and Northern

Category: Mathematics Around the World.
Fields of Study: All.
Summary: The contributions of central Asia have included algebra and its great houses of wisdom.

Throughout history, countries in Asia have had shifting sociopolitical boundaries. The names of some countries have changed, influenced by the Arab and Islamic empires as well as European colonialism in the eighteenth and nineteenth centuries. Though not widely used, Northern Asia sometimes refers to the part of the Asia occupied by the transcontinental country of Russia, which is commonly included in eastern Europe. Central Asia includes the former Soviet satellites of Kazakhstan, Kyrgyzstan, Tajikistan, Turkmenistan, and Uzbekistan. Mongolia, typically considered part of central Asia by historians, is in the modern world classified as part of Eastern Asia by the United Nations. "Northern Asia" is a term that is not commonly used, thus the transcontinental country of Russia is usually thought of as part of Eastern Europe. Knowledge of the contributions of mathematicians around the world is constantly changing as historians discover and translate written materials in many languages. Further, the breakup of the Soviet Union and shifting alliances have given researchers access to documents from decades in which many Eastern Bloc nations kept themselves in isolation, as well as even older works contained in the libraries and educational institutions of these nations. For example, medieval Islamic texts in Uzbekistan have helped shed light on the rich mathematics culture of central Asia. However, the mathematics contributions and achievements of some people from central Asia may be included in the histories of other areas, countries, or cultures.

In the seventh century, the great Library of Alexandria in Egypt was captured by a Muslim army, and there are some historians who believe some contents of the library were taken into Muslim lands. Many cities in central Asia became famous in the medieval period for their own libraries, which contained original works and translations of texts from Greek and Sanskrit, some of which became the only surviving copies of these earlier works. Houses of wisdom provided places for scholars to gather, as well as scientific centers such as the fifteenth-century Samarkand Observatory in what is now Uzbekistan, which was founded by astronomer Muhammad Taragay Ulughbek. This observatory reputedly served as a model for later observatories in India. Astronomer and mathematician Ala al-Din Ali ibn Muhammed, also known as Ali Kushji, later preserved and disseminated some of the knowledge gathered by the observatory when it was destroyed. This catalogue of stars, containing the most accurate mathematical measurements of location known prior to the invention of the telescope, is still studied.

Significant Central Asian Mathematicians

In the same way that mathematicians in central Asia studied and developed many concepts that were first introduced by other cultures, other concepts and techniques in twenty-first-century mathematics were first brought to Europe by mathematicians who worked in or came from central Asia. The word "algorithm" derives from a Latin transliteration of the name of eighth- and ninth-century mathematician Abu Abdallah Muhammad ibn Musa al-Khwarizmi (sometimes written as Al-Khoresmi). The Khwarizm (or Koresm) region included portions of what are now Turkmenistan and Uzbekistan. The word "algebra" comes from the term *al-jabr*, which was found in al-Khwarizmi's treatise on that subject. Another of his mathematical writings, the *Book of Addition and Subtraction by the Indian Method*, helped promote the Hindu base-10 decimal system within the Arabic world. This system spread to Europe and revolutionized mathematics around the world in subsequent centuries.

Historical evidence suggests that tenth-century astronomer and mathematician Abu Mahmud Hamid ibn al-Khidr Al-Khujandi was born in the city of Khudzhand, in what is now Tajikistan. His mural sextant produced some of the most accurate astronomical observations of the day, and he may have contributed to trigonometry. The tenth- and eleventh-century mathematicians Abu Rayhan al-Biruni and Abu Nasr Mansur are also cited as being natives of Khwarizm. Al-Biruni studied a diversity of topics in mathematics and science, including cartography and map projections, trigonometry, combinatorial analysis, ratio theory, algebraic problem solving, geometry, Archimedes of Syracuse's theorems, conic sections, and spherical triangles. Along with his own prolific body of writings, he was also a translator of Sanskrit texts. Abu Nasr Mansur taught and collaborated with al-Biruni—the two frequently cited one another's contributions to their own work.

Many consider Mansur's primary mathematical contributions to be his commentary on Menelaus of Alexandria's *Sphaerica*, his development of trigonometry, and his tables for numerical solutions to problems in spherical astronomy. In the same time period, Abu Ali al-Husain ibn Abdallah ibn Sina, also known by the Latin name Avicenna, wrote on many topics, including medicine and mathematics. Some of his investigations included ruler and compass constructions, areas of circles, and geometric algebra. He also considered music to be a subdiscipline of mathematics, and some believe that his studies led to musical tuning by the method of just intonation, where the note frequencies are related by ratios of small whole numbers, rather than Pythagorean tuning, named for Pythagoras of Samos.

Beginning in about the twelfth century, central Asia underwent a great deal of social and political disruption, and there is often little surviving evidence regarding mathematics and science during those eras. During the Soviet period, mathematicians from Kazakhstan, Kyrgyzstan, Tajikistan, Turkmenistan, and Uzbekistan may have been drawn to some of the central academic centers in Russia and other parts of the Soviet Union. Since the fall of the Soviet Union, these countries are reestablishing themselves as independent nations, and achievement in mathematics continues. For example, students from central Asia have participated in and won numerous medals in the International Mathematical Olympiad, an annual competition for high school students in which individual medals are awarded based on each student's success in solving a set of mathematics problems. Countries send six-member teams.

Kazakhstan, Kyrgyzstan, and Turkmenistan first participated in 1993, Uzbekistan in 1997, and Tajikistan in 2005. In 2010, Kazakhstan hosted the 51st Olympiad in its capital of Astana. Students from 98 countries around the world participated. Professor Askar Dzhumadildayev noted, "Mathematics is one of the most important indexes of the education level in the country. Gathering the best young mathematicians in Astana is a great honor for us." A news report regarding the Olympiad acknowledged the rich history of central Asia: ". . . we should not forget that our country is an heiress of the mathematical school founded by great scientists of the Middle Ages. . . . who greatly contributed to development of mathematics long before the modern countries of the West appeared."

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Sarah Boslaugh

See Also: Arabic/Islamic Mathematics; Asia, Western; Europe, Eastern.

Asia, Eastern

Category: Mathematics Around the World.
Fields of Study: All.
Summary: Across eastern Asia, mathematics education is given a high priority, with the goal of continuing the region's tradition of excellence.

Eastern Asia is one of the most populated regions of the world, lagging behind only southern Asia, and includes the Chinese cultural sphere once called the "Far eastern civilizations": China, Hong Kong, Macau, Taiwan, Japan, Mongolia, North Korea, and South Korea. The region is by no means homogeneous but has certainly been influenced to varying degrees by China in its writing systems, its cuisine, its architecture, and its religion. These influences are principally historical, cultural exchange being less centralized now, and influences like the Western world and the Soviet Union (in the case of Mongolia and North Korea) having been significant in the recent past. The technology sector is important in much of this region and mathematics education is a priority. Mathematics education in most of east Asia follows the Confucian model.

Number System

The number system in all Chinese-based east Asian languages centers on the same decimal system as the West but with stricter adherence to simple place-value patterns. For example, employing literal translations, the word for the number 12 is *ten-two*, 20 is *two-ten*, 37 is *three-ten, seven*, and 533 is *five-hundred, three-ten, three*. This system, along with the use of an abacus, facilitates the understanding of place value among east Asian elementary students. east Asian countries also follow the Chinese myriad-grouping system, which groups large numbers by ten thousands, rather than thousands. In other words, these languages have single words for the numbers "ten thousand" and "one-hundred million," but not for "million" or "billion."

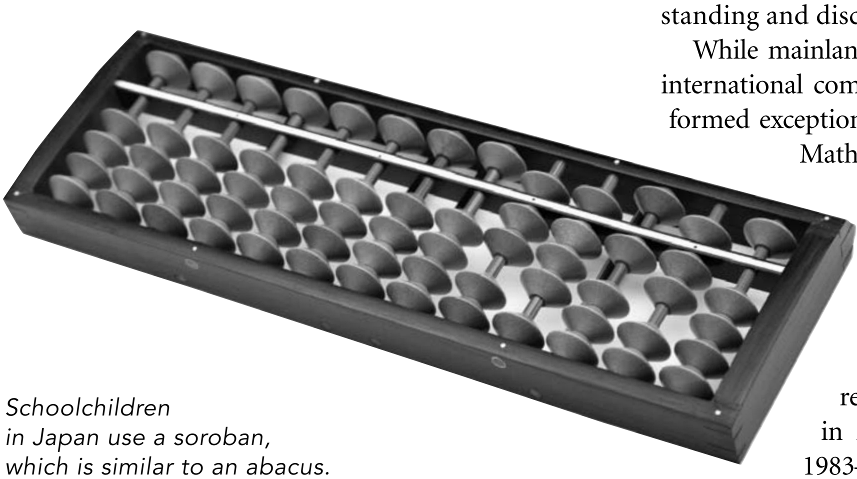
Educational Philosophy

Historically, public east Asian mathematics classrooms could be generalized as teachers delivering lectures to large classes of students who are expected to master calculations and grasp theory through repetition and memorization. Inherent in this Confucian approach is the assumption among students, parents, and teachers that mathematical success results more from diligent studying than natural talent. Student-centric and practical applications of mathematics are not a primary focus in east Asia, as they sometimes are in the West. This educational philosophy is true not only of the textbooks, which in east Asia are succinct and cover the minimal core set forth by each of the national governments but also of the classrooms, which must closely follow the textbooks. However, since the international test results illuminated relative weaknesses in problem solving, creativity, and practical applications, the east Asian governments have been working to adapt curricula in various ways.

China

Chinese children's task of memorizing thousands of Chinese characters naturally seems to transfer to the subject of mathematics where memorization of formulas and processes is assumed to lead to understanding and discovery.

While mainland China did not participate in some international comparisons, the Chinese team has performed exceptionally well in the annual International Mathematical Olympiad (IMO), a competition among high school students, where it placed first almost every year between 1990 and 2010. But these achievements in mathematics are not limited to Chinese students; two Chinese mathematicians have received the distinguished Wolf Prize in Mathematics: Shiing-Shen Chern in 1983–1984 and Shing-Tung Yau in 2010.



Schoolchildren in Japan use a soroban, which is similar to an abacus.

Hong Kong

The mathematics education system in Hong Kong employs elements both from mainland China and Great Britain. Despite the fact that international test scores ranked students from Hong Kong as years ahead of many Western countries, there is widespread concern about students viewing mathematics as irrelevant beyond testing. This concern has been leading to a curriculum that emulates the Western approach to teaching more mathematics related to problem solving and practical abilities.

Japan

While Japan distinguished itself in mathematics from the other east Asian countries during the Edo period (1603–1868), modern Japanese mathematics carries few remnants of this period. One such remnant is the *soroban*, a Japanese modification of an abacus. Japanese schoolchildren continue to use this beaded calculation device as a means of mastering the decimal system. Like in all east Asian countries, private schools (called *juku*) are attended widely by Japanese students. Japan has produced some of Asia's best mathematicians of the past century, including three winners of the Wolf Prize (Kunihiko Kodaira in 1984–1985, Kiyoshi Ito in 1987, and Mikio Sato in 2002–2003), and three winners of mathematics' most revered award, the Fields Medal (Kunihiko Kodaira in 1954, Heisuke Hironaka in 1970, and Shigefumi Mori in 1990).

Mongolia

Geographically, Mongolia lies between China and Russia. Until the early twentieth century, it was largely under the control of China and was later strongly influenced by Russia and the Soviet Union, adopting a Soviet-style government until 1990. Mongolian teams began participating in the International Mathematical Olympiad in 1964. Ming Antu was a Mongolian mathematician and astronomer, though he has been referred to as Chinese in the past. He worked on infinite series in the eighteenth century, among other accomplishments.

North Korea

While North Korea has the same Confucian background as the other east Asian countries, the former Soviet Union played a significant role in sculpting the modern approach to mathematics education. As do most countries around the world, the North Korean education system upholds mathematics as a central focus for both primary and secondary students, although North Korean story problems tend to be phrased in a nationalistic context. Students who excel in mathematics during their secondary school education may be admitted into the esteemed Kim Il-Sung University. In terms of global rankings, North Korea has sporadically entered a team into the International Mathematics Olympiad, some of which placed in the top 10.

South Korea

From childhood, South Koreans grow up using two separate number systems in their daily lives. The first one, a purely Korean system, is used mainly for counting objects, animals, and people and is no longer used for numbers larger than 99. It is worth noting that the numerals in this Korean system do not follow the same simple place-holding constructions as the number systems rooted in the Chinese language. The Sino-Korean number system, on the other hand, does follow these rules, and is most commonly used with money and large numbers. In school, many South Korean students receive just as much, if not more, of their mathematics instruction from private tutors or *hagwons* (academies) as from the public school environment. This system stems from the inextricable link between a student's mathematics performance on entrance exams and his or her eventual place in society. Some people cite this pressure as an explanation for why South Korean and Japanese students, despite performing exceptionally well on international tests, also rank the highest in their professed dislike for mathematics.

Taiwan

Private mathematics academies in Taiwan are referred to as *buxiban* (cram schools), suggesting their primary, but not exclusive, role of preparing Taiwanese students for entrance examinations. With electronics as a major industry, there has been a recent overhaul of the Taiwanese education system to focus on practical applications of mathematics instead of only theoretical mathematics.

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Daniel Showalter

See Also: Asia, Southeastern; Chinese Mathematics; Europe, Eastern.

Asia, Southeastern

Category: Mathematics Around the World.
Fields of Study: All.
Summary: Mathematics in the region has long been intertwined with religion and astrology and in recent generations has been impacted by colonialism.

The United Nations classification of southeastern Asia includes Brunei Darussalam, Cambodia, Indonesia, Lao People's Democratic Republic, Malaysia, Myanmar, Philippines, Singapore, Thailand, Timor-Leste, and Vietnam. Throughout history, the countries of Asia have had shifting political and social boundaries, and the names of many countries have changed over time, especially from the European colonial eras of the eighteenth and nineteenth centuries—when Western historians often began to study and document these countries—into the twenty-first century. For example, Burma became known as Myanmar; Siam became Thailand; Malay or Malaya became Malaysia; the Dutch East Indies or Netherlands East Indies and Java became Indonesia; and French Indochina included Laos, Cambodia, and Vietnam. Singapore was also part of Malaysia for a brief time in the 1960s, and the two regions share many historical developments. China and India, which have long histories of mathematics work and achievement, also had an influence in this region of the world. Therefore, mathematics contributions of some people from southeastern Asia, may be included within the histories of other regions or countries.

Early History

The great architectural feats found in places such as Borobudur, built in the ninth century on the island of Java, now part of Indonesia, and Angkor Wat, constructed three centuries later in Cambodia, suggest to scholars and historians that the architects and the builders must have had considerable mathematics knowledge. Some mathematics was probably brought to the region from India and China, as also happened in Europe and other areas, but there were almost certainly local mathematicians as well. The geometry involved in the design of both Borobudur and Angkor Wat has amazed generations of scholars who have discovered many complex ratios and formulas in the designs. Historians have also discussed the interconnection between religion, astronomy, mathematics, and astrology in southeastern Asia. Often there was little distinction made between mundane and divine matters, and some sequences of numbers (for example, 4, 8, 16, and 32) had religions connotations.

These numbers were used in both government and spiritual matters, such as the number of chiefs and territories in some Malay courts. Numerical systems emerged for the Burmese, Siamese, Cambodian, Laotian, Vietnamese, and Javanese languages. When Europeans began to explore and colonize southeastern Asia, they brought with them their own formal methods of school structure and mathematics teaching, which were documented by historians. Colonial influence saw the Vietnamese language develop a Romanized script, along with Western systems of counting, but the other scripts kept their systems of numerals. The introduction to southeastern Asia of a European-style school education, which replaced previous systems of instruction at pagodas or mosques, was a contributing factor in mathematics education. Much of this education came from the commercial needs of colonial powers to educate boys for work as bookkeepers and businessmen, so Western accounting systems were introduced to these populations—though many merchants continued to use Chinese systems, including the abacus, up through the twenty-first century.

Singapore and Malaysia

Singapore and Malaysia have active mathematics programs. The Raffles Institution in Singapore has a mathematics club whose members compete in events like the Singapore Mathematical Olympiad. The school was established in 1823 and named for (Thomas) Stamford Raffles, who is known as the founder of the British colony in Singapore. The Singapore Mathematical Society was founded in 1952. In the twenty-first century, it organizes participation in events like the national and international mathematics olympiads and the Singapore Mathematics Project Festival, among other educational and professional activities. Singapore first participated in the International Mathematical Olympiad (IMO) in 1988. While many twentieth-century textbooks on mathematics were imported into Singapore, the "Singapore Math Method," first developed in the 1980s and used in the national curriculum in Singapore, is now used in several places in the United States and elsewhere.

One of Singapore's well-known mathematicians is Tony Tan, who completed his doctorate, with a dissertation on "Mathematical models for commuter traffic in cities," at the University of Adelaide, South Australia. He taught mathematics before going into banking, and then into politics, ultimately becoming his country's deputy prime minister. Raffles College in Singapore taught mathematics from the time it started operations in 1928. Relations between Singapore and Malaysia in the twentieth century led to its transformation into the University of Malaya, then the University of Singapore, and the National University of Singapore. Sir Alexander Oppenheim, the vice-chancellor of the University of Malaya 1957–1965, was a prominent mathematician who had taught at Raffles College.

The Malaysian Mathematical Sciences Society, founded in 1970, was formerly known as the Malaysian Mathematical Society. It hosts events like the National Mathematical Olympiad in Malaysia; Malaysia first participated in the IMO in 1995. The Penang Free School, established in Malaysia in 1816, has taught mathematics from its inception. The Institute of Mathematical Sciences at the University of Malaya, founded in 1959 as the Department of Mathematics, continues to provide education for many Malaysian and overseas students and is an important mathematical institute in that country.

Thailand

Historically, Thailand was the only country in southeastern Asia never to be colonized by a foreign nation. Rulers such as the nineteenth-century King Mongkut, the inspiration for the 1946 movie *Anna and the King of Siam* and often called "the father of science and technology," embraced Western innovations. Assumption College, Bangkok, founded in 1885, had an extensive program of mathematics. The Mathematical Association of Thailand publishes the *Thai Journal of Mathematics* and hosts conferences and contests. Thailand has been participating in the IMO since 1989. The Center for Promotion of Mathematical Research of Thailand was established in 1978. Mathematician Yupaporn Kemprasit is an acknowledged world expert on algebraic semigroup theory, ring theory, and algebraic hyperstructure theory.

Cambodia, Laos, and Vietnam

In French Indochina, mathematics was encouraged for commerce. The Quoc Hoc or National Academy, established in 1896, included mathematics in its curriculum, with French as the language of instruction. Until the 1950s, most secondary schools in this region used French and French-language mathematics books—this was done in Cambodia until the early 1970s. Growth in the education system in the late twentieth century produced new native mathematics teachers, including Cambodian Communists Saloth Sar (Pol Pot), Khieu Samphan, and Gaing Kek Ieu (called "Comrade Deuch").

The Vietnamese Mathematical Society was founded in 1965, roughly the same time as one major buildup of American troops during the Vietnam conflict. Many educational institutions were closed for many years because of the war, but the society continued to support regional mathematical research. Vietnam first participated in the IMO in 1974 and hosted the competition in 2007. Mathematics researchers and students from Lao People's Democratic Republic (Laos) also participate in conferences and competitions. For example, in 2010, two high school students won a mathematics prize in a competition that included students from Brunei, Cambodia, Indonesia, Malaysia, the Philippines, Singapore, Vietnam, Thailand, and Laos.

Indonesia

The Dutch in the Netherlands East Indies operated a system of European schools, so-called "native schools," and vocational schools, teaching primarily in Dutch with Dutch-language textbooks. Many of the applied mathematics courses were directed toward engineering. After independence, with the expansion of the education system in Indonesia, there are mathematics departments in all schools and most universities in the country. Indonesia first participated in the IMO in 1988.

Brunei, Myanmar, and the Philippines

Elsewhere in the region there is also mathematical activity. The study of mathematics in the Philippines has been influenced by its close connections with the United States. The Mathematics Society of the Philippines was established in 1973, and the Philippines began participating in the IMO in 1988. Brunei participated in the IMO in 2000. The country of Myanmar has been isolated for much of the period since its independence in 1948. At the start of the twenty-first century, it initiated a 30-year plan for educational reform to address the challenges of the information age. Traditionally, state schools focused on writing, reading, and speaking in Myanmar and English, as well as mathematics, science, and Myanmar geography and history. Newer programs offer increased access to computer skills, as well as courses on information technology, medicine, and engineering, which require more advanced mathematics skills.

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Justin Corfield

See Also: Asia, Eastern; Asia, Southern; Chinese Mathematics; Europe, Northern; Europe, Western; Vedic Mathematics.

Asia, Southern

Category: Mathematics Around the World.
Fields of Study: All.
Summary: Southern Asia's history of mathematics reaches back thousands of years and mathematics continues to be a priority.

Southern Asia has a rich tradition in mathematics. Persian, Hindu, and Vedic scholars, among others in this area, contributed to the body of mathematics knowledge. Some of the achievements that have been historically credited to Arabic or Islamic mathematicians may have been influenced by pre-Islamic Persia. From ancient times, the rise and fall of various empires, wars, migration, and colonial influences have resulted in shifting cultural and geographical boundaries. As a result, many countries and regions in southern Asia have changed over time. The United Nations statistical classification for southern Asia contains Afghanistan, Bangladesh, Bhutan, India, Iran (Islamic Republic of), the Maldives, Nepal, Pakistan, and Sri Lanka. In the twenty-first century, these Asian nations continue to make advances in mathematics and mathematics education.

History

Construction of many ancient temples or monuments in southern Asia clearly involved mathematical knowledge, and mathematicians from this time period made various contributions to mathematics. One example is Indian scholar Baudhayana, who lived around 800 B.C.E. and is credited by some with developing the Pythagorean theorem, although others feel he was reflecting Babylonian work. The Vedic priest Katyayana, who lived approximately six centuries later, appears to have been interested in mathematics for religious purposes. Panini (520–460 B.C.E.), born in Shalatula, now part of Pakistan, wrote a scientific theory of Sanskrit. Some historians have theorized that development of algebraic structures and number systems in this region may be tied to the linguistic structure of

Sanskrit. Panini's work also influenced computer languages. Aryabhata (476–550) wrote a mathematical text known as the *Aryabhatiya*. It is composed of 123 metrical stanzas, whose organization has been studied by mathematicians because it differs from later mathematical works in many traditions. Some historians believe that it was influenced by Mesopotamia, while others suggest that it might be an anthology of works by earlier mathematicians. Another text, the *Bakhshali* manuscript, discovered in 1881 near Peshawar in present-day Pakistan, is believed to date from the seventh century, although some experts have dated it to up to eight centuries earlier or five centuries later.

By medieval times, Indian mathematicians had developed the notion of zero as a number, the use of negative numbers, and the definitions of sine and cosine. Some early Indian poetry also shows evidence of the binary number system and the use of decimal numbers. Mathematician Abd Al-Hamid ibn Wasi ibn Turk Al-Jili (c. ninth century) is believed to have been born in Iran, Afghanistan, or Syria. He wrote an algebra book. Persian mathematician, poet, and astronomer Omar Khayyam (1048–1141) wrote books on arithmetic and algebra by the age of 25 and contributed to many mathematical areas. Mathematician Nasir al-Din al-Tusi (1201–1274) was born in the city of Tus, now in Iran. He wrote Arabic translations of several Greek mathematical texts and is also credited with developing planar and spherical trigonometry from what many considered an astronomical tool into a separate mathematical discipline. Ghiyath al-Din Jamshid Mas'ud al-Kashi (1380–1429) was born and worked in Kashan, now in Iran. His *Treatise on the Circumference* included a calculation of π, which exceeded any known precision at the time. He also authored a teaching text called *The Key to Arithmetic*.

Education

Mathematics education has long been a focus in southern Asia. Mathematics was a part of *garakula* residential schools in ancient Nepal and India. From the fourteenth century, what became known as the "Kerala school of astronomy and mathematics" emerged in southern India. There was a flourishing of new discoveries, including the use of calculus long before it was developed by Isaac Newton and Gottfried Leibniz. These developments continued under mathematicians such as Citrabhanu (c. 1530) and Jyesthadeva (c. 1500–1575). English scholar Charles Whish (1794–1833) publicized many of the Kerala achievements to the rest of the world. Even then, the work of Whish—primarily a collector of Sanskrit manuscripts—was largely unknown beyond the scholarly community until the Indian mathematicians K. M. Marar and C. T. Rajagopal were able to demonstrate the advances made in Kerala just prior to the establishment of the European colonial empires in India.

British colonialism brought some European teaching styles into areas of southern Asia, and many universities were founded in the nineteenth century. Also in the nineteenth century, some Nepalese students traveled to India to study, where they were exposed to texts like Bhaskaracharya II's (1114–1185) *Siddhanta Siromani*. French mathematics traditions were introduced to southern Asia by Father Racine (1897–1976), a Jesuit missionary who had previously earned a doctorate in mathematics. With Indian colleagues such as Ramaswamy Vaidyanathaswamy (1894–1960), he promoted "modern" or contemporary mathematics teaching versus solely classical mathematics in the twentieth century. Indo-French collaborations continue to flourish into the twenty-first century and have been cited as contributing to development of areas like algebraic geometry and theoretical partial differential equations in southern Asia. There were other well-known collaborations, such as that between Indian mathematician Srinivasa Ramanujan and British mathematician Godfrey (G. H.) Hardy. In the 1980s, the Maldives introduced a new school curriculum that increasingly emphasized the importance of a variety of subjects, including mathematics.

Researchers in southern Asia have investigated a wide variety of different curricular issues such as gender differences in mathematics in Pakistan. King of Bhutan Jigme Khesar Namgyel Wangchuck noted in 2009:

> In all the countries where progress has been strong in the areas we strive to develop, the strength of the education system has been in Math and Science. In fact in India, the favourite subject for most students is Mathematics. In Bhutan, Mathematics is one of our main weaknesses.

Students from Bangladesh, India, Iran, Pakistan, and Sri Lanka have competed in the International Mathematics Olympiad: Iran since 1985, India since 1989, Sri Lanka since 1995, and Bangladesh and Pakistan since 2005. Mumbai, India, hosted the Olympiad in 1996.

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Justin Corfield

See Also: Arabic/Islamic Mathematics; Asia, Western; Babylonian Mathematics; Vedic Mathematics.

Asia, Western

Category: Mathematics Around the World.
Fields of Study: All.
Summary: The people of western Asia have long studied and influenced mathematics.

Ancient western Asia, including Anatolia, Syria, Mesopotamia, and the Iranian plateau, along with Egypt, is regarded by many as the cradle of civilization. Activities that shaped numerous civilizations are traced historically to this region, including the invention of the wheel, practice of agriculture, first writing systems, and first administrative structures. Many intellectual and scientific disciplines flourished. The development of mathematics followed and was affected by the rise and decline of the civilizations of western Asia. Throughout history, the territory has been settled or invaded by many ethnic groups, including the Babylonian, Persian, Hellenistic, Roman, and Islamic cultures. Some countries were also part of the Soviet Union. It is not always possible to determine the exact origin of historical figures, and, as such, people may be included in the histories of many regions or identified by cultural heritage and the location where they did their work. Further, many of their accomplishments are named for later mathematicians. The twenty-first-century United Nations grouping for western Asia is listed as Armenia, Azerbaijan, Bahrain, Cyprus, Georgia, Iraq, Israel, Jordan, Kuwait, Lebanon, Occupied Palestinian Territory, Oman, Qatar, Saudi Arabia, Syrian Arab Republic, Turkey, the United Arab Emirates, and Yemen.

Babylon

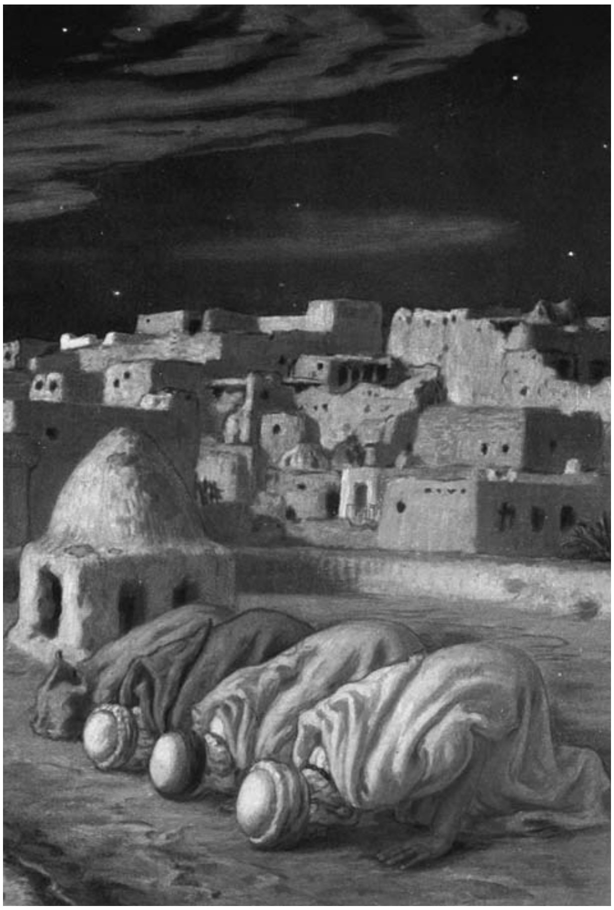
Historical knowledge of Babylonian mathematics is largely limited to translations of the surviving clay tablets that have been unearthed by archaeologists, but even this evidence suggests a rich depth and breadth of mathematics scholarship, largely focused on practical problems. Subsequent cultures that came to the region also left parts of their mathematical legacies. With the emergence of Islam at the end of the sixth century, many of the nomadic tribes living in the Arabian Peninsula joined together to form a significant power.

By the early eighth century, a sociopolitical entity often called the Islamic Empire, which was ruled mostly by a series of government entities known as *caliphates*, spanned from Spain and north Africa to southeastern Anatolia, Persia, and the western portion of central Asia. On the east, the region shared a long border with India, and hence many Muslim intellectuals were also cognizant of Indian culture and mathematical accomplishments. Many local rulers encouraged scholarship, building on the legacy left by the Hellenic and Roman periods.

The House of Wisdom in Baghdad, in what is now Iraq, became the main hub of research and intellectual activity, rivaling Alexandria at its zenith. Works of Hellenistic mathematicians were translated into Arabic—the only surviving copies of certain works. Mathematicians also extended and introduced new ideas and fields. Social factors were another motivating influence in mathematics scholarship in Muslim lands, such as the calculation of the local daily prayer times, the direction of the prayer (toward Mecca), and the determination of the local first day and the end of the holy month of Ramadan. Since the commonly used lunar calendar was 11 days shorter than the solar year, this problem added complexity for numerous peoples and religions in the area. Observing the heavens and predicting the astronomical events was a major field of research for mathematicians and astronomers.

Ottoman Empire and Turkey

Wars brought turmoil to the area, and scholarly activities suffered. Following the conquest of Istanbul in 1453, Ottoman Sultan Mehmed-II built madrasas (buildings used for teaching Islamic theology and religious law, often including a mosque) and encouraged scholars to congregate. However, later events negatively impacted mathematics in western Asia; for example, the destruction of centers of learning such as the Istanbul observatory and the spread of religious scholasticism (a philosophy of teaching that follows a relatively a narrow set of traditional methods heavily influenced by religious teachings), which also occurred in medieval Europe. Some scholars indicate the passage of mathematical leadership over to Europe after about the fifteenth century.



Mathematics was used in Muslim lands to calculate the direction of prayer toward Mecca.

Ottoman Empire efforts of the early nineteenth century reenergized mathematics efforts. Vidinli Hüseyin Tevfik Pasa (1832–1901) contributed to linear algebra and Mehmet Nadir (1856–1927) worked on the theory of Diophantine equations, named for Diophantus of Alexandria. The Ottoman Empire faded after World War I, but the Turkish Republic continued its efforts. A well-known Turkish mathematician is Cahit Arf (1910–1997), known for the Arf invariant in algebraic topology, Arf semigroups, and Arf rings, among others. The *Turkish Journal of Mathematics* is one of the many scientific journals published by the Scientific and Technological Research Council of Turkey. The Turkish Mathematical Society was founded in 1948, and the country is a member of the International Mathematical Union (IMU), a worldwide association that promotes mathematics research and activity. In 1978, Turkey began participating in the International Mathematical Olympiad (IMO), a competition for high school students. Turkey hosted the IMO in 1993.

Israel

Mathematical activity in Israel dates back to antiquity, and it is one of the countries in western Asia with a thriving mathematics community. This fact is due in part to researchers like algebraist Shimshon Avraham Amitsur, who was one of the 1963 founders of the *Israel Journal of Mathematics*. Some other notable Israeli-born mathematicians include Oded Schramm, Saharon Shelah, and 2010 Fields Medal winner Elon Lindenstrauss. The Einstein Institute of Mathematics, named for Albert Einstein, was founded in the 1920s. Israel is a member of the IMU, and Israeli high school students first participated in the IMO in 1979. The Israel Mathematical Union is an organization that offers opportunities for students, teachers, and researchers. In the twenty-first century, there were some calls to boycott Israeli scholars over disputed territories. In response, numerous mathematical organizations worldwide, including the IMU, passed resolutions that stressed the importance of open international scientific exchange.

Other Countries

A revitalization of mathematical activity took place in many other western Asian countries in the twentieth century often connected with professional organizations or national institutes of science. For example, the development of contemporary mathematics in Armenia is tied to the 1944 beginnings of the Institute of Mathematics of the National Academy of Sciences of the Republic of Armenia. The country began participating in the IMO in 1993, the same year as Azerbaijan.

The first issue of the *Azerbaijan Journal of Mathematics* was published in January 2011. The Kuwait Foundation for the Advancement of Sciences supports the Kuwait Mathematics Program at the University of Cambridge, which underscores the relationships between western Asia and universities in other areas of the world. Kuwait began participating in the IMO in 1982.

In 2010, the editor of the *Arab Journal of Mathematics and Mathematical Sciences* was from Jordan. The Cyprus Mathematical Society was founded in 1983 and hosts activities like the Cyprus Mathematical Olympiad. Cyprus began participating in the IMO in 1984, Bahrain in 1990, the United Arab Emirates in 2008, and Syria in 2009. Saudi Arabia first participated in the IMO in 2004. It is also a member of the IMU, and mathematicians gather through the Saudi Association for Mathematical Sciences. Oman is an associate member of the IMU. Countries such as Qatar have developed mathematics standards for grades 1–9. Some countries in western Asia continue to be affected by the area's ongoing sociopolitical volatility. Georgia declared its independence from the Soviet Union in 1991 and is redeveloping many aspects of its national identity. It began participating in the IMO in 1993 and is a member of the IMU through the Georgian National Mathematical Committee. Iraq is also rebuilding itself after the turmoil of the late twentieth century and early twenty-first-century wars.

Some countries in the region participated in the Trends in International Mathematics and Science Study (TIMSS). In 2003, the study included fourth graders from the Republic of Yemen; eighth graders from Bahrain, Israel, Jordan, Lebanon, the Palestinian National Authority, the Syrian Arab Republic, and Saudi Arabia; and both fourth and eighth graders from Armenia and Cyprus. In 2007, even more countries from this region participated, including Armenia, Bahrain, Cyprus, Georgia, Israel, Jordan, Kuwait, Lebanon, Oman, the Palestinian National Authority, Qatar, Saudi Arabia, the Syrian Arab Republic, Turkey, and Yemen. In 2011, Armenia, Azerbaijan, Bahrain, Georgia, Israel, Jordan, Kuwait, Lebanon, Oman, the Palestinian National Authority, Qatar, Saudi Arabia, the Syrian Arab Republic, Turkey, the United Arab Emirates, and Yemen are included with benchmarking participants from this region listed as including Abu Dhabi, UAE, and Dubai, UAE.

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Dogan Comez
Sarah J. Greenwald
Jill E. Thomley

See Also: Africa, North; Arabic/Islamic Mathematics; Babylonian Mathematics; Mathematics and Religion.

Astronomy

Category: Space, Time, and Distance.
Fields of Study: Geometry; Measurement; Number and Operations; Representations.
Summary: Mathematics is used in astronomy to measure and model celestial bodies.

Astronomy is the science that deals with celestial objects. It is divided into two disciplines: positional astronomy (or "astrometry"), which deals with the positions and movements of celestial objects; and astrophysics, which deals with their chemical and physical properties.

Positional astronomy began as a practical science. The first astronomers, before the invention of writing, dealt with such questions as the proper time of the year to plant crops and the proper dates for religious festivals. As their understanding improved over the ages, astronomers tackled other practical problems such as how to predict eclipses, how to tell time within a day, and how to navigate at sea.

Ancient people could take simple observations of the sun and moon and observe the patterns they made. From there it was a short leap to predicting future patterns. They would first record (or, before writing, memorize) the observations, and then perform a mathematical analysis—even if the analysis were nothing more than counting (for example, discovering there were about 365 days between winter solstices).

A much more sophisticated accomplishment was working out the complicated cycles on which lunar and solar eclipses occurred. An eclipse can be terrifying for a people who are not expecting it. If astronomers (many of whom doubled as priests) could predict eclipses, they could warn people in advance and reduce the collective fear.

A number of ancient peoples, including Mayans, Chinese, and Babylonians, developed elaborate calendar systems and tracked the movements of the planets. The Chinese constructed star charts, kept records starting possibly as early as 4000 B.C.E., and developed astronomical instruments. The Babylonians mapped constellations and introduced 60-minute hours and 60-second minutes. Both the Chinese and the Babylonians were able to predict eclipses. By 2500 B.C.E, Egyptians had measured star positions well enough to orient the pyramids to face celestial north. Polynesians traveled throughout the Pacific Ocean using stars as navigational aids.

The Greeks

The ancient Greeks effectively applied mathematics to astronomy. Eratosthenes (c. 200 B.C.E.) used geometry to calculate the size of Earth. Hipparchus (c. 161–126 B.C.E.) discovered the precession of the equinoxes and created the most accurate Greek tables of lunar motion.

Lunar Calendars

Some cultures found it easier to use the moon instead of the sun to tell time. A quick glance shows the phase of the moon, while observing the sun takes careful measurement. There is not an even number of moon cycles in a solar year—making the exclusive use of moon calendars difficult—yet the Babylonians (and others) discovered that there was a 19-year cycle.

In 19 solar years (assuming 365.25 days per year), there are 6939.75 days. The moon takes 29.5306 days to cycle from one new moon to the next. In 235 lunar months, there are 6939.69 days. A lunar calendar based on this cycle has 12 months (six with 29 days and six with 30 days—figuring a lunar month of 29.5 days), adding up to 354 days. Seven times in the 19-year cycle, there is a leap year, with an extra month of 30 days added in—making the leap year 384 days.

$$(12 \times 354) + (7 \times 384) = 6936 \text{ days.}$$

Since the lunar month is slightly more than 29.5 days, a total of four days have to be added during the 19-year cycle, giving 6940 days. This lunar calendar, originated by the Babylonians and refined over the centuries, is still in use today—it sets the dates of Easter and of all Jewish holidays.

Like some other Greek astronomers, he held that Earth revolved around the sun. Claudius Ptolemy (c. 120–150 C.E.) combined observations from Hipparchus and others with his own observations to propose a model of how the solar system worked—assuming Earth was at the center. By using epicycles (circles revolving on circles), he produced what was by far the best model of the heavens until Nicolaus Copernicus.

The Greeks did not only conduct astronomical calculations by hand but used a computer as well. Though not much is currently known about it, a mechanical analog computer was built somewhere in the Greek world about 100 B.C.E., called the "Antikythera mechanism" after the place it was found. This remarkably sophisticated computer was able to show both solar and lunar calendars, track the complicated path of the moon using Hipparchus's results, and predict eclipses for years into the future.

The Renaissance

During the Middle Ages, Arabs, Persians, and Jews, as well as European Christians (after c. 1000 C.E.), continued the work of the Greeks, including making new tables of planetary positions to update Ptolemy's, and keeping track of the precession of the equinoxes. In 1543, Copernicus's book on the solar system was published. Through a mathematical analysis of Ptolemy's work and later observations, Copernicus showed that a system in which the sun was the center of the solar system led to simpler and more accurate analysis than Ptolemy's.

Johannes Kepler used Tycho Brahe's careful naked eye observations of the planets to show that Mars went around the sun in an ellipse, not a circle as the Greeks had assumed. Kepler stated his three laws, which relate the speed of a planet to the shape of its orbit, but he could not explain why these laws worked. Isaac Newton was the first to explain Kepler's laws. He was able to show that any object affected by gravity would move in one of the conic sections: Kepler's ellipse, a line, a circle, a parabola, or a hyperbola. The one exception was the planet Uranus, which did not follow its Newtonian orbit.

It was not until the 1800s that Urbain Leverrier, in France, and John Couch Adams, in England, (unknown to each other) made the assumption that the discrepancies were because of the gravitational pull of an unknown planet. The planet Neptune was discovered in 1846 using Leverrier's prediction. Neptune was found by the consideration of the three components, P_x, P_y, and P_z, of Neptune's position and the three components, V_x, V_y, and V_z, of Neptune's velocity.

Until 1821, Uranus was moving faster in its orbit than expected—more than 4 planetary diameters ahead of its predicted position. After 1821, Uranus moved slower than expected. Obviously, Uranus moved past Neptune around 1821. If one adjusted the coordinate system so that $P_x = 0$ was Uranus's position in its orbit in 1821 and examined how far Uranus was pulled above or below its expected orbit, then one can tell whether Neptune was above or below Uranus in 1821, which gives P_y, and also whether Neptune was moving up or down, which gives V_y. If we have P_z, which represents Neptune's distance from the sun in 1821, then Kepler's laws can be used to find the two remaining parameters: V_x and V_z. Leverrier and Adams used a shortcut to find P_z. Both used Titius-Bode's law, an empirical formula, to predict the next planet beyond Uranus to be 38.8 times Earth's distance from the sun. These predictions were good enough to find Neptune, although Neptune is only 30.1 times Earth's distance from the sun.

Leverrier later examined the orbit of Mercury and found a discrepancy of 43 seconds of arc (which sounds small but is twice the discrepancies used to find Neptune). He computed the orbit of a hypothetical planet, called "Vulcan," which would explain this 43-second variation. Vulcan has never been found, and Einstein's general theory of relativity also explains this discrepancy.

Parallax

The ancient Greeks made attempts using parallax (the difference in the angle to a distant body measured from two different locations, also called triangulation) to find the size of the solar system. Being restricted to naked-eye observations, their results were inaccurate. Using telescopes, a much more accurate measurement was made in 1761 in which observers scattered across Earth found the parallax of Venus when it passed in front of the sun. The observations gave a value of 95.25 million miles from Earth to the sun (the modern estimate is just under 93 million miles). A much more difficult problem was to find the distances of stars by their parallax when viewed from opposite sides of Earth's orbit, first accomplished by Friedrich Bessel in the 1830s. Is space Euclidean or non-Euclidean? If measurably non-Euclidean, this would show up in stellar parallax measurements. No such effect has yet been observed, so one can say—except for relativistic considerations—that space is Euclidean for hundreds of light-years from Earth.

Astrophysics

Astrophysical questions date to the ancient Chinese, who discovered sunspots, and Hipparchus (c. 190–120 B.C.E), who worked on the magnitude (or brightness) of stars. His magnitudes, much refined, are still in use today. However, astrophysics as a discipline can be said to have started with Joseph von Fraunhofer, who in 1815 devised a spectroscope and catalogued the various lines (known as the Fraunhofer lines) that can be seen in the solar spectrum. In the 1850s, Gustav Kirchhoff and Robert Bunsen determined that these lines belonged to different chemical elements. Thus, by examining the spectrum of a star, its chemical composition can be determined. In addition, it was discovered that magnetic fields caused broadening and splitting of Fraunhofer lines, allowing the magnetic fields of stars to also be investigated.

Over the course of the twentieth century, astrophysicists went from studying the spectrum of visible light to studying every frequency of electromagnetic waves—from gamma rays to radio waves. There is now no known radiation from a star that is not being used to help find answers to the questions of what stars are, and how they operate.

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James Landau

See Also: Babylonian Mathematics; Chinese Mathematics; Conic Sections; Egyptian Mathematics; Geometry of the Universe; Greek Mathematics; Incan and Mayan Mathematics; Renaissance.

Atomic Bomb (Manhattan Project)

Category: Government, Politics, and History.
Fields of Study: Algebra; Data Analysis and Probability; Number and Operations; Representations.
Summary: The atomic bomb was made possible by Einstein's discovery of energy-mass equivalence.

Influenced by a letter from the famous German-American theoretical physicist Albert Einstein and other prominent scientists, U.S. President Franklin D. Roosevelt authorized the establishment of the Manhattan Project (the code name given to an elaborate effort to design, construct, and detonate an atomic bomb) in mid-1942. The project was directed by physicist J. Robert Oppenheimer, and his group of scientists, mathematicians, and engineers conducted secretive, pioneering research that led to the development of the first nuclear weapons.

Among the scientists who worked on the Manhattan Project were Italian physicist Enrico Fermi, American theoretical physicist Richard Feynman, Hungarian-American mathematician John von Neumann, Hungarian-American theoretical physicist Edward Teller, and Polish-American mathematician Stanislaw Ulam (Einstein also worked as a consultant throughout the project). Notably, several of these scientists, including Einstein and Ulam, were of Jewish decent and eventually resided in America because of Nazi persecution.

Through much trial and tribulation, the first nuclear bomb detonation test titled "Trinity" was successfully conducted on July 16, 1945, in Alamogordo, New Mexico. The Manhattan Project ultimately produced two types of atomic bombs; the plutonium implosion device (the plutonium or implosion bomb), and the uranium bomb (the uranium "gun" bomb). The plutonium bomb was the more difficult of the two to construct and required testing, whereas the uranium bomb was comparatively more simplistic and remained untested until the war.

Following Trinity, the U.S. government attempted to end World War II by detonating its uranium bomb nicknamed "Little Boy" over Hiroshima, Japan, on August 6, 1945. The blast destroyed approximately one-third of the city and caused about 140,000 causalities. Japan's reluctance to surrender prompted the United States to drop its plutonium bomb nicknamed "Fat Man" over Nagasaki, Japan, three days later. This blast killed about 70,000 people, destroyed about one-third of the city, and subsequently ended the war.

The revolutionary science of the Manhattan Project—namely the process of creating atomic explosions—was seemingly insurmountable, and paved the way for significant advancements in physics, chemistry, and mathematics. However, the historical impact of the atomic bombs dropped on Japan, as well as the philosophical and ethical ramifications, is an issue still debated today. In this regard Oppenheimer said, "It is a profound and necessary truth that the deep things in science are not found because they are useful; they are found because it was possible to find them."

The First Nuclear Reactions

Nuclear fission is the splitting of the nucleus of a heavy atom into smaller pieces, which releases a gigantic amount of energy. When this type of reaction is self-sustaining (it stimulates further reactions), it is called a "chain reaction." A critical mass is the minimum mass of fissionable material needed to ensure that a nuclear reaction sustains a chain reaction. Achieving a critical mass and, ultimately, a chain reaction was the essential challenge in developing both Little Boy and Fat Man.

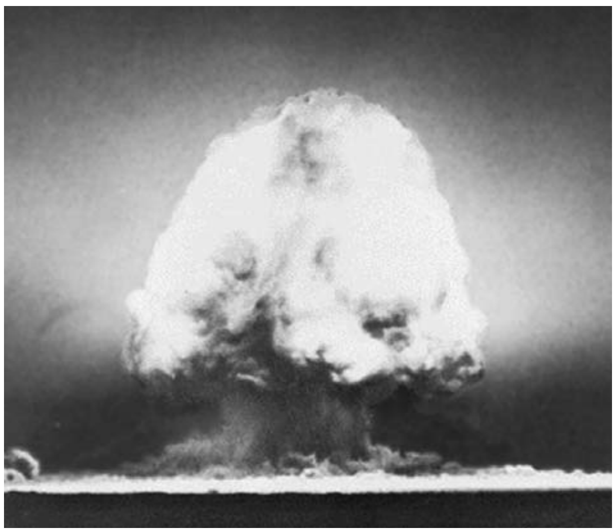
The Little Boy design utilized the "gun method," which was detonated by firing a mass of uranium-235 down a cylinder into another mass of uranium-235 to produce a chain reaction. Fat Man was an implosion-type device that used plutonium-239. In this design, plutonium was placed in the center of a hollowed-out sphere of high explosives, and a number of detonators located on the high explosive's surface were simultaneously fired pressurizing the core and increasing its density—creating an implosion that resulted in a chain reaction. The Trinity test bomb was similar and was nicknamed "The Gadget." Little Boy produced a blast of approximately 12,500 tons of trinitrotoluene (TNT). Fat Man had the explosive power of about 22,000 tons of TNT and The Gadget had a blast yield of around 15–20 tons of TNT.

A tremendous amount of engineering, chemistry, physics, and mathematics was involved in the development and deployment of the atomic bombs. Among these fields was a branch of theoretical physics called "quantum mechanics" (the set of scientific principles that describe the behavior of matter and energy predominating at both the atomic and subatomic levels), which at the time was in its infancy. Quantum mechanics was developed under the assumption that energy is not infinitely divisible but rather composed of quanta (small increments).

Unlike classical or Newtonian mechanics, which describes the motion of objects we encounter every day at the macrocosmic level, quantum mechanics deals with uncertainty in many of its results and is statistical and probabilistic in nature. Although initially this branch of physics was not readily accepted, it nonetheless proved an essential tool in the development of the atomic bomb as it provided many of the insights necessary for its construction. In regard to the science and mathematics utilized in the development of the bomb Stanislaw Ulam said, "It is still an unending source of surprise for me to see how a few scribbles on a blackboard or on a sheet of paper could change the course of human affairs."

The Energy-Mass Equivalence

One of the most imperative concepts in the development of the atomic bomb was the mathematical formulation of the energy-mass equivalence, which was derived by Einstein. He established that mass and energy are, in fact, both different manifestations of the same thing. This idea was a counterintuitive and revolutionary result that spawned from his 1905 special



Explosion resulting from the "Trinity" test of the plutonium bomb.

theory of relativity. Einstein's formulation implied that very minute amounts of mass can be converted into excessively large amounts of energy. For example, this very encyclopedia is, in actuality, a form of energy in storage, which could equivalently be called rest energy or mass. If this encyclopedia could be completely converted into energy, it would yield a gigantic amount of energy indeed. This energy-mass equivalence concept is depicted symbolically through one of the world's most famous equations:

$$E = mc^2.$$

This equation is interpreted as the rest energy E of an object being equivalent to the mass m of the object multiplied by the square of the speed of light c in a vacuum. Alternatively, $E = mc^2$ can be construed as the equation that allows one to determine the amount of mass needed to produce a certain amount of energy—assuming all of the mass can be converted completely into pure energy.

To better understand how this famous simple equation was crucial in the development of the atomic bomb, one needs to understand its nature. First, $E = mc^2$ is a "direct proportion" (E is directly proportional to m), and is symbolically expressed as $E \propto m$. In general, a direct proportion has the form of

$$x \propto y \text{ or equivalently } x = ay$$

where a is the proportionality constant. As a simple example:

$$4 \propto 2 \text{ or equivalently } 4 = 2a.$$

In this case, the proportionality constant is $a = 2$, whereas in the energy-mass equivalence, the proportionality constant is c^2. According to the International Bureau of Weights and Measures, the value of c is 299,792,458 meters per second (m/s), or about 186,282.4 miles per second (mi/s). For computational simplicity, c is often rounded to 300,000,000 m/s (186,000 mi/s), except when performing experiments that require exact values for light speed. Now, taking $c = 300{,}000{,}000 \text{ m/s} = 3 \times 10^8 \text{ m/s}$ one can compute that 1 kilogram of plutonium could theoretically turn into

$$E = 90{,}000{,}000{,}000{,}000{,}000 \text{ kg} \cdot \text{m}^2/\text{s}^2 = 9 \times 10^{16} \text{ kg} \cdot \text{m}^2/\text{s}^2$$

Therefore, one can intuitively understand why a small amount of uranium or plutonium can generate explosions as massive as the ones produced by Little Boy and Fat Man.

It is interesting to note that for the Trinity test, the mushroom cloud expanded to nearly 300 meters (about 984 feet) in .053 seconds.

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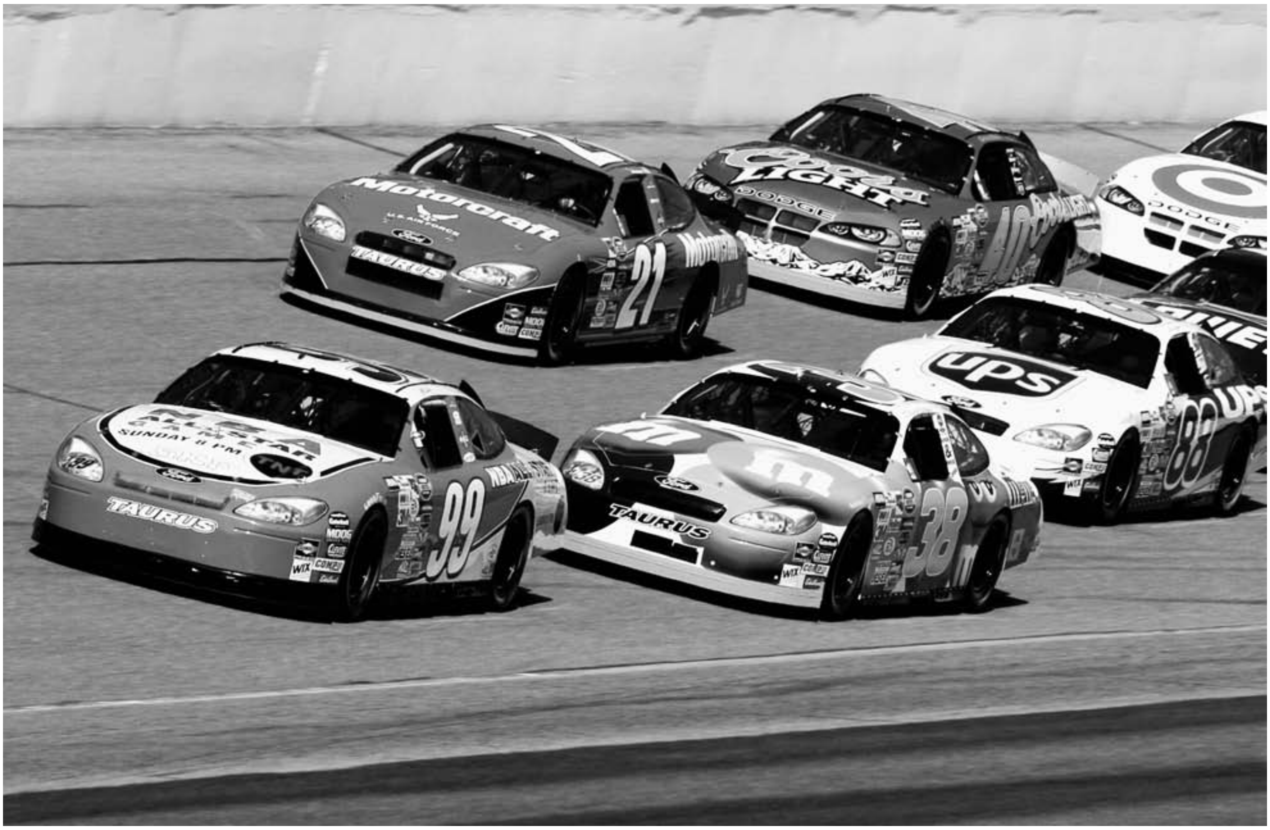
See Also: Cold War; Einstein, Albert; Radiation; World War II.

Auto Racing

Category: Games, Sport, and Recreation.
Fields of Study: Data Analysis and Probability; Geometry; Measurement.
Summary: Mathematics is essential in the design of race cars and racetracks, and the formulation of race strategy.

Auto racing has taken place for as long as cars have existed. While the early days of racing were related to



Drivers preparing during a practice run for the 2004 Daytona 500 race. An understanding of geometry is critical in determining how to set up cars to handle banking and high speeds.

fairly simple vehicles, it is now a very technical sport that has multiple branches with fans worldwide. Auto racing includes not only cars that are similar to those driven by the average citizen but also cars that are very sophisticated. The different branches of auto racing differ in the specifics of the car but all share a strong relationship to mathematical principles. The design of the car, its tires, the track, and the drivetrain require very careful measurement. The optimal path for a given track and weather condition requires a deep understanding of angles and geometry. Analysis of data to create probability information enables drivers and their teams to make wise decisions for a given set of conditions during a race.

Overview

Auto racing began as soon as the automobile was invented in the late 1800s. Auto racing is a broad term that includes single-seat cars or open wheel cars, which the Indianapolis 500 has made famous. Formula 1 racing is another type of open wheel racing but involves racing around courses that are not oval shaped. The National Association for Stock Car Auto Racing (NASCAR) utilizes cars that are modified from cars that can be bought by the general public. Many successful professional race car drivers began their racing careers with Kart racing, which involves vehicles that look like sophisticated go-karts.

Race Track Design

The racing surface and the track design are significant factors that affect both car design and driving strategy. Race surfaces can include asphalt, concrete, dirt, sand, and (sometimes) ice. Some tracks consist of a very short distance (1/4 mile) and are straight. These tracks are typically used for drag racing, which involves cars trying to go as fast as possible over a short distance. Many track designs have drivers travel in an oval, or near-oval, shape with some banking to help make high-speed turns easier. An understanding of geometry is imperative when determining how best to set up the car to handle the banking and the speeds. Track designs also include road courses in which racers turn both left and right and require a completely different car design to handle banking in both directions. The radius of a turn influences how fast the car can go without losing grip and crashing into the outer wall. The speed affects the size of the down force on the car (caused by spoilers), and, as such, different tracks require car designs.

Car Design

Race car designs evolve in response to technology changes and safety concerns, often as a result of mathematical or statistical analysis. Each branch of auto racing has very strict rules on car design, which are tested before—and sometimes after—each race. The testing includes very careful measurements of various components of the car from the size of various components of the engine, to the car's width, height, and weight. The tests focus on items that affect the car's power (the engine), response to the environment (temperature, air resistance, and gravity), and its influence on forces that are made on the car (width, height, and weight). Because they are such an important part of car performance, tires are supplied to the teams. A large amount of testing by tire companies goes into determining which type of tires will be provided for a particular track. The air and track temperatures often change drastically during a race and can affect how the tires interact with the track surface—providing more or less grip. Likewise, the gas that is put into the car is also provided to drivers. These standardizations provide a more even playing field for the teams so that the driver who wins is, presumably, the one with the greatest skill. Teams can alter the cars slightly during races to modify how the car receives forces from the track and from the air. These modifications include taking out or adding small wedges that alter the angle that the car sits on the track. The impact these small changes make on force is understood using trigonometry.

Race Strategy

Once teams have prepared their car and driver for the race, the issue of strategy plays an important role. Teams use probabilities to determine if and when to stop in the pits to change tires or to add gas. Gas mileage is estimated by using regression involving the number of laps, the speed of the car during the laps run, and the temperature. This estimation is not absolutely exact, and it is not uncommon for drivers to run out of gas near the end of some races because of an error in the team's regression model. Some teams alter the usual pit stop, which involves replacing all of the tires and adding gas, by replacing just some of the tires or just adding gas.

Technology and Safety

Technology is playing a bigger role in auto racing in both car development and car testing. Car teams now use technology to measure a large number of factors that influence their car's performance. For some branches of auto racing, these measurements are made during races. For other branches, the rules prohibit this during races but allow the measurements to occur during practice and research design. Because testing can be so expensive, some tests are done with a few drivers and then shared with all the teams. The use of computer simulation based on mathematical modeling is becoming more prevalent in all branches. It is not unusual for teams to use wind tunnels to test car design, and fluid dynamic modeling has been used to improve the aerodynamic properties of race cars. Off-season drivers use sophisticated driving simulators to hone their skills.

Technology has also been used to make racing safer. Race uniforms, helmets, and car interiors have become much less dangerous because of technological improvements. Additionally, track walls now include what is called a Steel and Foam Energy Reduction (SAFER) barrier, which dissipates the collision energy from a crash so that the impact force felt by the car and driver is smaller and less dangerous.

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See Also: Diagnostic Testing; Extreme Sports; Highways.

Axiomatic Systems

Category: History and Development of Curricular Concepts.
Fields of Study: Communication; Connections; Geometry; Reasoning and Proof.
Summary: An axiom is a statement that is assumed to be true, and axiomatic systems have a rich and interesting mathematical history.

Axiomatic systems provide a deductive framework for mathematicians to combine related definitions and theorems that make mathematical knowledge systematic and structural. Mathematical theories including number systems, set theory, probability, algebra, and many others are built by using axiomatic systems.

Axiomatic Method and Axiomatic System in Mathematics

To build a deductive mathematical system, one needs to observe two intrinsic limitations in this process.

Limitation 1: Not every mathematical term can be defined. The reason can be seen by the following considerations: To define a term *A*, one needs a term *B*, and possibly some other terms. To define the term *B*, one needs another term, *C*, and so on. One may eventually come back to the term *A*; in which case the definition would be circular as there are a finite number of words. This means that *A* is used to define *A*, which is undesirable.

If the definitions are not to become circular, some terms are needed to start with. The solution is that there will be some terms that will not be defined. These will be called "undefined terms," and will be used to define all the other terms to be considered. One may think that it is strange that this solution can work. How can undefined terms give meaning? This puzzle is partially answered upon consideration of the next limitation.

Limitation 2: Not every mathematical statement can be deduced or proven. The reason is similar to the one in Limitation 1; some statements are needed to start a chain of deduction: if *R*, then *S*; if *S*, then *T*; if *T*, then *U*; and so on. To deal with this limitation, certain statements must be accepted without proof. These statements are called "axioms," and they are the statements that we used to deduce other statements. Actually, the axioms are often statements about the undefined terms. In other words, the axioms often tell us certain properties or restrictions of the undefined terms. Thus, the axioms help provide meaning to the undefined terms. Starting with undefined terms, axioms, and definitions, and by using deductive reasoning to establish important mathematics facts in the form of theorems, the mathematics system so obtained is said to be built by using the "axiomatic method." Such a system that consists of undefined terms, axioms *A*, definitions *D*, statements of the form If *P* then *Q* and proof of such statements is called an "axiomatic system." In an axiomatic system, one does not talk about the validity of *A* or *P*, one talks only about the validity of the proof based on *A* and *D*.

Historical Developments

Historically, Euclidean geometry was the best-known model of an axiomatic system. Around 300 B.C.E., Euclid wrote his 13-volume *Elements*, which contained an axiomatic treatment of geometry. It starts with 23 definitions; Euclid stated 10 axioms. The first five axioms are geometric assumptions, which he called postulates. The last five are more general, which he called common notions. There, Euclid did not use undefined terms.

The most important and fundamental property of an axiomatic system is "consistency" (it is impossible to deduce from these axioms a theorem that contradicts any axiom or previously proved theorem). The Euclidean geometry provides such a consistent axiomatic system. An individual axiom is "independent" if it cannot be logically deduced from the other axioms in the system. The entire set of axioms is said to be independent if each of its axioms is independent. Mathematicians prior to the nineteenth century doubted very much about the independence of the fifth postulate (the parallel postulate). They tried to deduce such a postulate by using the first four postulates. Despite considerable effort devoted to the task, no significant result could be obtained.

Euclid's *Elements* indeed became the most influential book on geometry, as well as the model of logical reasoning and axiomatic system, until the nineteenth century when two fundamental developments took place. First, it was realized that Euclid's logical system was not rigorous enough. A rigorous axiomatic treatment of Euclidean geometry was given by David Hilbert (1862–1943) in his 1899 book *Grundlagen der Geometrie* (*The Foundation of Geometry*). Here, Hilbert used the undefined terms of point, line, lie on, between, and congruent for the geometry system. Second, research results of C. F. Gauss (1777–1855), J. Bolyai (1802–1860), and N. I. Lobachevsky (1793–1856) asserted that the parallel postulate was actually an independent axiom. Non-Euclidean geometry could be developed by replacing the fifth postulate with another independent axiom. The lesson from Euclidean and non-Euclidean geometry is that both are valid axiomatic systems. When studying Euclidean or non-Euclidean geometry, no claims are made on the truth of the axioms about the physical world. One merely claims that if the axioms are valid, then the theorems deduced therein are also valid. Whether the logical system describes the real world is another question.

Current Issues

There are still many issues regarding the axiomatic systems. The set of axioms in an axiomatic system is "complete" if the axioms are sufficient in number to prove or disprove any statement that arises concerning our collection of undefined terms. To determine whether an axiomatic system is complete is by no means an easy question to answer. A great surprise was discovered by Kurt Gödel (1906–1978) in 1931. He proved that in a formal mathematics system that included the integers, there exist statements that are impossible to prove or disprove. This result is called Gödel's incompleteness theorem. Also, to determine whether a given proposition is an axiom has been a very important issue in computer science and is important when one tries to use a computer to do proofs. If the computer cannot recognize the axioms, the computer will also not be able to recognize whether a proof is valid.

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Ka-Luen Cheung

See Also: Geometry and Geometry Education; Greek Mathematics; Mathematical Certainty; Parallel Postulate.

B

Babylonian Mathematics

Category: Government, Politics, and History.
Fields of Study: Algebra; Connections; Geometry; Measurement; Representations.
Summary: Babylon had an advanced utilitarian mathematics from which we inherited sexagesimal timekeeping.

Our knowledge of Babylonian mathematics (2100–200 B.C.E.) is based on extensive mathematical calculations found on clay tablets in the area of Mesopotamia (now Iraq), surrounding the ancient city of Babylon between the Tigris and Euphrates rivers. Because only a fraction of the tablets have survived—and only a small fraction of those have been translated—our knowledge of the depth and breadth of Babylonian mathematics is limited. Mathematics historian Otto Neugebauer likens the situation to tearing a few random pages out of a few textbooks and then trying to reconstruct a representation of modern mathematics. Nonetheless, Babylonian mathematics did involve complicated mathematics, and was used primarily to solve practical problems. These mathematical problems ranged from arithmetic calculations, to algebraic rules, to geometrical formulas, to numerical ideas.

Babylonian Number System

The Babylonian number system was sexagesimal, using both a place value notation based on powers of 60 and a base-10 grouping system for numbers between 1 and 59 within each place value.

Traces of their sexagesimal notation remain today in the recording of time (hours, minutes, seconds) and the measurement of angles (degrees, minutes, seconds). Their numbers were written in cuneiform, or the use of a triangular stylus to make wedges on a clay tablet. A vertical line represented unity and a horizontal wedge mark represented a 10.

For example, within each place value, the number 57 would be represented by 5 horizontal wedges and 7 vertical lines. Expanding the example, a cuneiform number represented in modern form as "3, 4, 57" was equivalent to

$$3\left(60^2\right)+4\left(60^1\right)+57\left(60^0\right)$$
$$=3\left(3600\right)+4\left(60\right)+57\left(1\right)=11097.$$

The Babylonians had neither a symbol for zero as a placeholder nor a symbol to designate the "decimal" point in their sexagesimal fractions. Writing and reading numbers required the Babylonian mathematician to understand the problem's context and the use of a space to represent either an "empty" place value or shift to fractional place values. Thus, the previous number, "3,4,57," possibly was equivalent to:

$$3(60^4)+0(60^3)+4(60^1)+57(60^0)$$
$$\text{or } 3(60^4)+4(60^3)+57(60^2)$$
$$\text{or } 3(60^1)+4(60^0)+57(60^{-1})$$
$$\text{or even } 3(60^{-1})+4(60^{-2})+57(60^{-3}).$$

To avoid ambiguity, modern translations of these numbers would be first "3,0,4,57" or "3,4,57,0,0" or "3,4;57" or "0;3,4,57" respectively, where the semicolon separates whole numbers from fractional numbers. Tablets from the Seleucid period (300 B.C.E.) did include a special symbol that played the double role of a placeholder (zero) and the separator between two sentences.

Babylonian Arithmetic

Using the sexagesimal system, the Babylonians were able to add, subtract, multiply, and divide numbers. Their computations were complemented by the use of extensive tables. Their multiplication tables had products ranging from 1 × 1 through 59 × 59, and seeming somewhat unusual, they had access to multiplications tables for "1,20" (or 80), "1,30" (or 90), "1,40" (or 100), "3,20" (or 200), "3,45" (225), and even "44,26,40" (or 160,000). Some of this can be explained by looking at their tables of reciprocals for working with fractions. For example, one table includes the deceptive notation 1 ÷ 1,21 = 44,26,40, with the latter value actually being "0;0,44,26,40."

The Babylonians produced extensive tables of squares and cubes, tables of square sides and cube sides (square and cube roots), and sums of squares and cubes. When a table side-value was not available, the Babylonians approximated roots using an interpolation process based on averaging and division; this process was quite fast, producing 26-decimal accuracy in five iterations.

Babylonian Algebra

Though without an algebraic notation, the Babylonians solved numerous types of algebraic equations. Each solution involved the replication of a formulaic prescription represented by a step-by-step list of rules. In effect, their prescriptions invoked algorithms, which were usually specific to a stated problem and not generalized to a class of problems.

For example, consider this Babylonian problem: the area and two-thirds of the side of my square have I added and it is 0;35. In modern notation, their step-by-step solution was: 1, the unit, you take; two-thirds of 1, the unit, is 0;40: Its half is 0;20 and 0;20 you multiply 0;6,40, you add 0;35 to it and 0;41,40 has 0;50 for its square root. 0;20 that you multiplied with itself, from 0;50 you subtract and 0;30 is the side of the square.

In modern mathematics, this same problem would involve solving the quadratic equation:

$$x^2+\frac{2}{3}x=\frac{35}{60}.$$

The steps in this problem also can be interpreted using geometrical algebra, where the square is "completed" in a manner similar to the derivation of our general quadratic formula.

In their solution of special types of algebraic equations, the Babylonians made extensive use of their tables of the sums of squares and cubes, especially if the equation was of the third or fourth degree. Some of their solutions to algebraic problems were quite sophisticated. For example, one problem involved a system of equations of the form

$$xy=n \quad \text{and} \quad \frac{mx^2}{y}+\frac{py^2}{x}+q=0 .$$

Its solution using substitution would normally lead to a single-variable equation involving x^6, but the Babylonians solved it by viewing it as a quadratic equation in x^3.

Babylonian Geometry

Dominated by their work with algebraic ideas, the Babylonians' geometry focused on practical measurements such as the calculation of lengths, areas, and volumes. Again, the Babylonians used prescriptive formulas. For example, to calculate a circle's circumference, they multiplied the diameter by 3, implying their value of π was 3. For the circle's area, they squared the circumference and divided by 12, which is equivalent to our modern formula $A = \pi r^2$ if the correct value of π had been used.

Mathematics historians credit the Babylonians with the division of a circle into 360 degrees. Neugebauer suggests it is related to their Babylonian mile, a measure of long distance equal to about 7 miles. This measure evolved into a time unit, being the time it took to travel this distance. After noting that 12 of these time units equaled a full day or one revolution of the sky, the Babylonians subdivided their mile into 30 equal parts for simplicity, leading to $12 \times 30 = 360$ units in a full circle.

The Babylonians computed areas of right triangles, isosceles triangles, and isosceles trapezoids, as well as the volumes of both rectangular parallelepipeds and some prisms. They had difficulties with certain three-dimensional shapes, being unable to compute correctly the length of the frustum of a pyramid (they claimed it was the product of the altitude by the average of the bases).

The Babylonians did know some general geometric relationships. For example, they knew that perpendiculars dropped from the vertex of an isosceles triangle bisected the base, that corresponding sides of similar triangles were proportional, and that angles inscribed in a semicircle are right angles. The Babylonians used this knowledge to solve difficult geometrical problems, such as their determination of the radius of a circle circumscribing an isosceles triangle.

Evidence suggests that they knew a precursor of the Pythagorean formula. One cuneiform tablet (c. 1700 B.C.E.) includes sexagesimal numbers written along a square's side (30) and diagonal ("42,25,35" and "1; 24, 51, 10").

The latter number is both the product of the other two numbers and a good approximation of the square root of 2 (1.414214). Also, in the Plimpton 322 collection, some of the tablets contain tables of Pythagorean triples ($a^2 + b^2 = c^2$), arranged with increasing acute angle of the associated right triangle.

Signs of Advanced Mathematical Thinking

For the most part, Babylonian mathematics was utilitarian, being tied to solving practical problems. Nonetheless, interpretations of some of the tables on the clay tablets suggest that the Babylonians occasionally explored theoretical aspects of mathematics. Examples include their tables of Pythagorean triples and tables of exponential functions (which perhaps were used to compute compound interest in business transactions). Also, the Louvre tablet (300 B.C.E.) includes two series problems

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^8 + 2^9 = 2^9 + 2^9 - 1$$

and

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots 9^2 + 10^2$$
$$= \left(1\frac{1}{3} + 10\frac{2}{3}\right)(55) = 588$$

but historians do not suggest the Babylonians knew general series formulas such as

$$\sum_{k=0}^{n} r^k = \frac{r^{n+1} - 1}{r - 1}.$$

Specific to number theory, mathematics historians point to the cumbersome nature of the Babylonians' sexagesimal system, making it difficult to explore ideas such as factors, powers, and reciprocals. Some suggest that this is symptomatic of the Babylonian's reasonable choice of 3 for π, rather than the fraction

$$\frac{22}{7}$$

equal to the more complicated repeating expression "3; 8, 34, 17, 8, 34, 17, ..."

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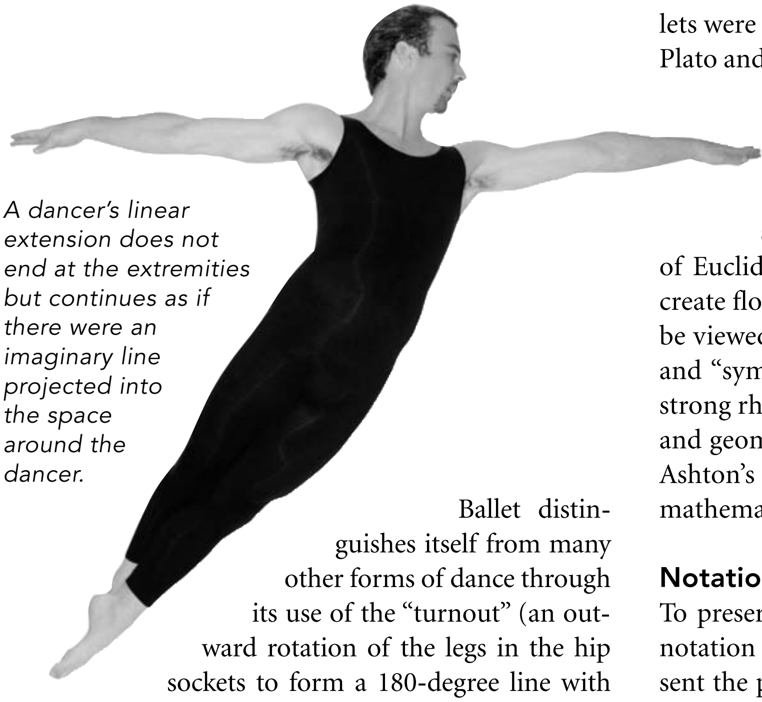
See Also: Arabic/Islamic Mathematics; Chinese Mathematics; Egyptian Mathematics; Greek Mathematics.

Ballet

Category: Arts, Music, and Entertainment.
Fields of Study: Communication; Geometry; Representations.
Summary: Ballet uses geometry to create captivating moving art.

Ballet can be considered mathematics in motion from basic counting (keeping time with music, and doing *demi-pliés* in childhood dance classes); making lines, angles, and geometric shapes in space via basic positions and choreographed routines of principal dancers and the *corps de ballet*; communicating stories in ballet productions (like the classic *Swan Lake*, or a seasonal favorite *The Nutcracker*); conversing visually among dancers (as in a *pas de deux* with Margot Fonteyn and Rudolf Nureyev); and by representing general emotions, moods, and abstract themes (as in George Balanchine's *Serenade*). Words from the French language may be common in ballet terminology, but concepts from mathematics abound as well. These representations, communications, and geometric creations can all be achieved and evidenced through the dance figures and ballet movements.



A dancer's linear extension does not end at the extremities but continues as if there were an imaginary line projected into the space around the dancer.

Ballet distinguishes itself from many other forms of dance through its use of the "turnout" (an outward rotation of the legs in the hip sockets to form a 180-degree line with the feet in first position). This turnout gives the dancer a strong base and the ability to move in any direction while allowing a more open body presentation to the audience, yet holding the graceful curves and shapes of the dancer's body to preserve a svelte "line." Other standard positions of the feet, carriage of the arms, or basic movements of the body produce angles such as a 135-degree arabesque, a 90-degree attitude, or a 45-degree *battement tendu.* The *rond de jambe à terre* or *en l'air* utilizes circular movements of the leg to trace semicircles or arcs, on or off the ground. These geometric lines, circles, and angles continue when basic steps become building blocks to more complicated movements. Meanwhile, dancing on the tips of the toes (*en pointe*), another distinctive ballet feature, heightens the dancer's lines in a vertical fashion. The linear extension, from head to toe, fingertip to fingertip, does not end at the extremities but continues as if through an imaginary line into the space around the dancer.

Ballet as Geometry

One of the earliest ballet performances was the sixteenth-century *Le Balet Comique de la Reine* by Balthazar Beaujoyeulx, commissioned by the court of France. During that elaborate production, the dancers performed dozens of geometric figures involving triangles, circles, and squares for their geometric proportions and spatial configurations. These beginning ballets were influenced by the writings of Pythagoras and Plato and represented the cosmic and heavenly significance of numbers and geometry. A twentieth-century choreographer, Frederick Ashton, however, was inspired by mathematics for its sheer beauty in his creation, *Scènes de Ballet*. Working from a book of Euclid theorems, he specifically used geometry to create floor patterns and dance movements that could be viewed from any angle to see the geometric figures and "symmetrical asymmetries." Combined with the strong rhythms and counts of Igor Stravinsky's music, and geometrically patterned costumes and set details, Ashton's work was said to have beautifully combined mathematics and ballet for its visual imagery.

Notation Systems

To preserve these choreographed works of art, dance notation systems were created to symbolically represent the positions, steps, and movements of the dancers. Early seventeenth- and eighteenth-century systems, such as Feuillet notation, recorded mainly floor patterns and feet positions, whereas the twentieth-century notation systems, Labanotation and Benesh Movement Notation (written on vertical and horizontal staffs, respectively), corresponded to the scores of accompanying music. These notation systems detailed the entire body movements from head to toe of every dancer. Even with the advent of video recording, it is these symbolic notations showing graphical representations of the step details that best preserve ballets for future generations.

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See Also: Ballroom Dancing; Contra and Square Dancing; Musical Theater.

Ballroom Dancing

Category: Arts, Music, and Entertainment.
Fields of Study: Communication; Geometry; Representations.
Summary: Ballroom dancing allows students to approach mathematics in a variety of ways.

Ballroom dancing, considered sophisticated for its elegance, is a style of choreographed dance showcasing not only the dancers' technical skill but also their poise and style. Originally danced primarily at balls for the social elite, ballroom dancing has become a competitive sport. Dancing allows students to approach mathematics in a variety of ways, from the basic arithmetic of the beats per minute (bpm) to the geometric spatial relationship with respect to the other dancers. Choreographers Erik Stern and Karl Schaffer have created a dance called a "math dance." The purpose is twofold: to use mathematics to create dance, and to help students learn mathematics concepts through the movements of the dance. Some of the topics explored in math dances are the mathematics of rhythm, polyhedra, symmetry, and dissection puzzles.

History

The phrase "ballroom dancing" derives from the Latin word *ballare* meaning to sweep or to dance. Now considered historical dances, the original forms of ballroom dancing included the minuet and quadrille. Some steps performed in the quadrille, such as the *entrechats* (crossing the legs one in front of the other multiple times) and the *ronds de jambes* (circular movement of the leg while it is extended, toe pointing to the floor), have disappeared from the modern ballroom yet still exist in the ballet world.

In the early 1800s, the waltz made its appearance; the distance between dancing partners was considered scandalous at the time since the waltz required the partners to dance in close proximity. The early 1900s brought the birth of jazz and new dance styles as dancers moved together yet independently of each other. In addition, lively dances such as the Foxtrot, otherwise known as the one-step or two-step, moved away from the traditional placement of feet being turned out and instead called for dancers to have their feet parallel to each other. While many people are unfamiliar with any ballroom dances besides the waltz, competitive ballroom dancing has gained notoriety; it has been showcased on the ABC television show *Dancing with the Stars* and has become an Olympic sport as well.

Beats

Ballroom dancing consists of a series of dance moves, where more complicated dance steps are called "figures" or "dance figures." Each of the formally named dances has a variety of dance moves that can be put together to form a personalized performance. Determining the dance moves to use involves more than merely counting the beats. One can calculate the total number of beats that will occur in a song and then determine how many different dance moves would be necessary. For example, if one hears 12 beats in a five-second segment of the song, it can be calculated that the song has 144 bpm. If the song is exactly two minutes long, one can calculate there are 288 beats to work with for the whole song ($2 \times 144 = 288$). Since each dance move is typically 8 beats, dividing 288 by 8 beats indicates one needs 36 dance moves. The moves can be repeated, using, for example, 9 moves 4 times each or 11 moves 3 times each (the second option gives the dancer three fewer moves than needed, requiring a dramatic flourish to end the dance). The total number of beats combined with the thematic moves of a particular dance and an individual's personal signature steps form a composite whole.

Rhythm

One rhythm option for the American-style Foxtrot consists of *Slow, Quick, Quick*, or half, quarter, quarter in 4/4 time; this approach to the dance gives teachers the opportunity to teach fractions to students using dancing. By creating a dance of successive moves in which two basic steps make one whole move, students will use fractions—adding and subtracting in 4/4 time and introducing the family of fractions

$$\frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \text{ and } \frac{1}{2}.$$

This also can be done in 6/8 time with $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$, and so on.

Geometry

As the lead dancer gauges the couple's location within the coordinate plane of the dance floor, he or she keeps them spatially equidistant from other couples. In addition to the symmetry involved in the various dance moves on the dance floor, symmetry is considered within each dancer's pose and posture (the form created by the two partners together—symmetrical or asymmetrical). This symmetry can lead to an understanding of angles and curves when various dance poses are examined, and allows students the opportunity to solve problems kinesthetically when they attempt to form a mirror image of their partner while executing the dance moves.

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See Also: Ballet; Contra and Square Dancing; Geometry of Music; Step Dancing.

Bankruptcy, Business

Category: Business, Economics, and Marketing.
Fields of Study: Data Analysis and Probability; Number and Operations; Problem Solving.
Summary: The value of a business entering bankruptcy is determined by the asset, income, or market approach and creditors are repaid according to their risk.

Bankruptcy of a business occurs when the business is legally declared insolvent (its assets are less than its liabilities). If the debtor files a bankruptcy petition, it is called a voluntary bankruptcy. However, if creditors force the debtor into bankruptcy, then it is called an involuntary bankruptcy. Most bankruptcies are voluntary. In either case, the value of the business needs to be determined for legal purposes. The standard of the value used in the valuation is the fair market value (the value of the price of the firm that a rational buyer is willing to pay to a willing seller in a free market). There are three basic approaches for valuating the business: the asset approach, the income approach, and the market approach. The hierarchy of the creditor in a bankruptcy is determined by the amount of risk the creditor bears: the creditor who bears least amount of risk will have priority to receive payment after liquidation.

Asset Approach

The asset approach determines the value of a company by adjusting its book value of assets to the current market value. It is based on the economic principles of substitution: a rational investor will not pay more for a business asset than the price of a different asset that provides similar utility. There are two methods associated with the asset approach: the adjusted book value method and the replacement cost method.

In the adjusted book value method, the assets and liabilities on the balance sheet are examined item by item by professionals to determine the business's current market value. Once the assets and liabilities have been adjusted to the current market value, the value of the company is calculated as the difference.

In the replacement cost method, the value of each asset and liability on the balance sheet is first determined as the cost to replace it. Then, the value the company is determined as the difference of its assets and liabilities.

The asset approach is not reliable for companies with significant intangible assets because the approach involves professional judgment. It is more suitable for companies that have many tangible assets and few intangible assets.

Income Approach

The income approach determines the fair market value of a firm by discounting its expected cash flows at an appropriate discount rate assuming the firm will continue to operate without liquidation. The discount rate is often chosen to be the firm's weighted average cost of capital (WACC). The procedure is completely analogous to that of determining the net present value of a firm in corporate finance theory. Mathematically, the fair market value under the income approach can be written as

$$\mathrm{FMV} = \frac{E(C)}{\mathrm{WACC}}$$

where FMV is the fair market value, $E(C)$ is the expected cash flows under the assumption that the firm will continue to operate, and WACC is the weighted capital of cost.

In corporate finance theory, WACC is often calculated as the weighted average of the cost of debt of the firm and the cost of equity of the firm

$$\mathrm{WACC} = (1 - T_c) r_D \frac{B}{B+S} + r_E \frac{S}{B+S}$$

where T_c is the corporate tax rate, r_D is the cost of debt, r_E is the cost of equity, B is the market value of the firm's bonds, and S is the market value of the firm's stocks.

WACC takes into consideration the facts of leverage and taxes and thus is the appropriate discount rate used for income approach. The income approach assesses the value of the debtor to the creditors. However, it fails to take account of the value inherent in the flexibility of decision making, which is often valued using a mathematical tool called "decision tree."

Market Approach

The market approach assesses a company's value by comparing it with similar companies in the market. The rationale behind this approach is that the price of the subject company should be very close to the values of the similar companies in the market. There are two methods associated with the market approach: the guideline public company method, and the comparable transaction method. In the guideline public company method, a peer group of public companies with similar sizes, natures, operations, and financial characteristics is first selected. Next, the enterprise value of each company in the group is calculated as

$$\mathrm{EV} = P_S \times N_S + D - C_E$$

where EV is the enterprise value, P_S is the stock price per share, N_S is the number of outstanding shares, D is total debt, and C_E is excess cash.

Then market multiples, such as enterprise value/revenue and enterprise value/earning before interest and tax, will be calculated using the enterprise value. Finally, the value of the subject company is determined by applying the calculated market multiples. For example, if the enterprise value/revenue is used, then the value of the subject company can be calculated as

$$V = \mathrm{EV} \times R$$

where V is the value of the subject company and R is the revenue of the subject company.

In the comparable method, the value of the subject company is determined in a similar fashion as in the guideline public company method. In other words, market multiples are derived, and then they are applied to the subject company to determine its value. However, in the comparable method, public data of comparable transactions are used to calculate the market multiples.

Thus, the comparable method also consists of three steps: selecting a group of comparable transactions, calculating market multiples, and applying the market multiples.

The biggest drawback to the guideline public company method is that it is not applicable for nonpublic companies. The challenge with the comparable method is finding appropriate and reliable comparable transactions.

Paying Creditors

When a company declares bankruptcy, its creditors must be paid, but the creditors receive only some of the money they are owed. For example, if a bankrupt company is ordered to pay 10 cents on the dollar, this means for every dollar the company owes a creditor, it will pay only 10 cents. This is a proportional solution that is easy to arrive at using simple algebra. However, this is not the only payout strategy. There are several mathematical methods that can be used to determine how much money each creditor should receive. In the total equality method, available capital is simply divided equally among debtors, regardless of how much they are owed. A variation, traced back to medieval philosopher Moses Maimonides, proposes giving every debtor as equal a share as possible but never more than they are owed. In modern terms, this is a constrained optimization problem that can be solved using methods such as linear programming. Other decision methods are logically and analytically more complex, like the Shapely value, which considers paying a sequence of creditors their full amounts owed, to the extent of available funds, for all possible orderings. This game-theory approach is named for American mathematician and economist Lloyd Shapely.

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See Also: Accounting; Bankruptcy, Personal; Budgeting.

Bankruptcy, Personal

Category: Business, Economics, and Marketing.
Fields of Study: Data Analysis and Probability; Number and Operations; Problem Solving.
Summary: Personal bankruptcy can be caused by exponentially increasing debt, and mathematics is used to calculate payments or to divide assets among creditors.

Personal bankruptcy is a legal proceeding intended to provide relief for the debtor. Personal bankruptcy essentially results from huge debts, which can be caused many factors including unexpected medical bills, huge credit card debts, poorly managed loans, unemployment, and divorce. The fundamental formula that lies behind most large debts is exponential growth.

Legal Procedure

Personal bankruptcy in the United States is usually a court-supervised procedure that provides the debtor with the opportunity for a fresh financial start. The earliest personal bankruptcy law in the United States can be traced to 1800. The most recent personal bankruptcy law passed by the U.S. Congress is the Bankruptcy Code of 1978. Under this law, an individual may file a voluntary petition under either Chapter 7 (liquidation) or Chapter 13 (Reorganization).

If a personal bankruptcy case is filed under Chapter 7, a court-supervised procedure begins. The debtor's assets will be classified as either exempt or nonexempt according to the state law. A trustee will then collect the nonexempt assets of the debtor. The debtor is allowed to keep all the exempt assets provided such an asset is not secured by any property. For example, a mortgage is secured by the house. Thus, debtors can still lose their houses if mortgaged payments fall behind. The debts of the debtor will be wiped out except certain non-dischargeable debts including alimony, child support, student loans, taxes, and any fines resulting from criminal conviction. The record of personal bankruptcy could stay on the debtor's credit history for up to 10 years. In summary, under Chapter 7, the debtor is discharged most of the debts and surrenders all possessions except those necessary for living. However, not everyone is qualified for Chapter 7 bankruptcy. To qualify, the debtor must complete Official Form 22A (Chapter 7) to pass the means test. The personal bankruptcy involves balancing the conflicting interests of the creditors and the debtor. While a qualified debtor can wipe out most debts under Chapter 7, some creditors will not receive any payment. The 2005 Bankruptcy Abuse Prevention and Consumer Protection Act was enacted to prevent the abuse of Chapter 7 and makes it more difficult for a debtor to file under Chapter 7.

If a personal bankruptcy petition is filed under Chapter 13, then the debtor is required to propose a repayment plan that will pay the debts during a specified period of time (typically three to five years). The plan must be reasonable and meet certain requirements. It must be approved by the court. Although the debts of the debtor cannot be written off immediately under Chapter 13, the debtor is protected from debt-collecting actions from the creditors while the repayment plan is in effect. Thus, the Chapter 13 bankruptcy is often chosen by those who have a stable income.

Exponential Growth

Personal bankruptcy results from unmanageably large debts that can be caused by many factors such as medical costs because of under-insurance and uninsured status, compulsive buying habits, loss of job, or irresponsible loans. The fundamental formula that leads to a large debt is the law of exponential growth, which occurs when the growth rate of a quantity is proportional to the current value. In mathematics, exponential functions generally involve the constant *e*. Mathematically equivalent forms with different bases may be used in order to more intuitively correspond to the parameters of a real-life problem, such as interest calculations. The traditional way of calculating interest on a loan is called compound interest, under which the interest earned during each interest measurement period (month, quarter, or year) will automatically be added to the principal to earn additional interest during the next interest measurement period. Mathematically, this can be expressed as

$$B_t = B_0\left(1+r\right)^t$$

where B_0 is the principal amount, B_t is the balance of the loan at the end of t-th interest measurement period, and r is the interest effective per interest measurement period.

The loan balance under the compounding interest grows rapidly over a relatively long period, even if the interest rate is not high. For example, consider a person who takes a loan of $10,000 from a bank at a monthly interest rate of 1.5%. The loan balance after one, five, and 10 years will be $11,956, $24,432, and $59,693, respectively.

Mathematical Division of Assets

The ideas of dividing and choosing have existed as long as mankind. The mathematical theory of fair division dates back to World War II, to Polish mathematicians Hugo Steinhaus, Bronisław Knaster, and Stefan Banach. The classic bankruptcy problem in game theory addresses fairness in one way. It involves allocating some amount of resources among two or more individuals who have a claim on them, assuming that any division of the assets is allowable and that there are not enough resources to satisfy all claims. Real-life examples include someone who has declared Chapter 13 bankruptcy and therefore must repay some creditors, or dividing a deceased person's estate among several heirs—especially when the estate cannot satisfy all the deceased's commitments.

Assets may be divided equally (with or without ensuring no claimant receives more than his or her claim), proportionally according to the relative size of the claim, or by other more complex strategies. The cake-cutting problem also tackles the issue of fair allocation but includes more subjective measures of valuation that must be modeled mathematically, and sometimes an asset pool with constraints on the ways in which it may be divided. Cake-cutting problems typically require iterative algorithms to solve.

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Liang Hong

See Also: Bankruptcy, Business; Budgeting; Credit Cards.

Bar Codes

Category: Business, Economics, and Marketing.
Fields of Study: Algebra; Number and Operations; Representations.
Summary: Bar codes encode numerical data visually for product identification and other purposes.

A bar code is a visual representation of information intended to be decoded by an optical scanner called a bar code reader. The reader illuminates the bar code, thus allowing its light sensor to detect the patterns of dark and light bars. The sequence and width of dark and light bars represents a unique sequence of numbers and letters.

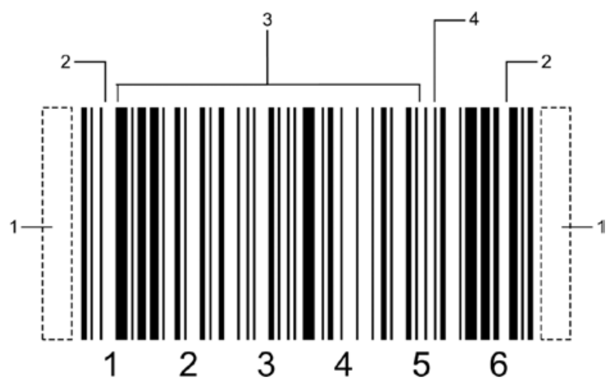
Origins

It took 26 years for the idea of bar codes to be successfully implemented in the retail industry. In 1948, two graduate students at Drexel University, Norman J. Woodland and Bernard Silver, overheard a conversation in which the president of a local supermarket chain in Philadelphia wished to automate the checkout process. At that time, a cashier in a supermarket would have to type into a cash register the price of all items in a purchase—a time-consuming and error-prone task. Woodland and Silver filed a patent application in 1949, obtaining the patent in 1952, for an optical device that would read information automatically. The first prototype was produced by IBM but was impractical because of both its size and the heat generated by the 500-watt light bulb used by the bar code scanner. The patent was sold in 1952 to the Philadelphia Storage Battery Company (Philco), which was also unable to produce a viable prototype, and sold the patent the same year to the Radio Corporation of America (RCA). Bernard Silver died in a 1963 car accident, before the bar code system was implemented in practical settings. The invention of lasers and integrated circuits in the 1960s allowed the manufacture of small, low-energy bar code readers. RCA developed a modern version of bar codes in 1972 in a Kroger store in Cincinnati, but the code was printed in small stripes that were easily erased or blurred by employees who had to attach them manually to each item. Norman J. Woodland was an employee at IBM at the time and led a team that produced bar codes according to a standard known as Universal Product Code (UPC) still in use today. Bar codes are used in nearly all retail products worldwide. The applications of bar codes have also reached far beyond the retail industry; they are now used in such disparate applications as patient identification, airline luggage management, and document management, as well as purchase receipts.

The Mathematics of Bar Codes

The most ubiquitous form of bar codes consists of a visual pattern of long lines (hence the "bar" in "bar code"), which has four well-defined zones (see Figure 1): (1) quiet zone, or empty zone, located in the left and right zones of the code; (2) initial character (right) and final character (left) are standard bars that appear on all bar codes, and indicate where the information begins and ends; (3) variable-length character chain, which contains as many characters as needed to encode the message; and (4) checksum, which is a number that is computed algebraically from the other characters

Figure 1. Zones of a bar code.



using modular arithmetic, and is used to verify that the characters have been correctly transmitted and interpreted. The digits are either simply added or are weighted. For example, the 10-digit International Standard Book Number (ISBN-10) uses weights based on digit position and modulus 11 arithmetic.

Each digit is encoded by two white and two black bars. The bars have widths of 1 to 4 units, and the total width for each digit is always seven units. Bar code readers are designed to read bar codes irrespective of their size; a magnified bar code encodes the same information as a reduced-size bar code. This property is mathematically known as scale invariance.

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See Also: Coding and Encryption; Comparison Shopping; Inventory Models.

Baseball

Category: Games, Sport, and Recreation.
Fields of Study: Data Analysis and Probability; Measurement.
Summary: Baseball is a mathematically rich sport, especially with regard to its array of statistics.

Though America's favorite sport for more than a century, the game of baseball has undergone many changes, many in response to statistics gathered regarding all parts of the game. At first, the statistics were limited to scorecard data but have expanded to include every action and detail of the game. More so, this gathering and analysis of data has expanded beyond the realm of statistical analysis, as mathematics is now used to examine all aspects of baseball—the physical characteristics and performance of its players, the analysis and modeling of each element (hitting, fielding, pitching, strategies), and the combined geometry and physics surrounding the game.

Although some fans object to this intrusion of mathematics into a competitive sport, most accept or even depend on the mathematical aspects as enriching their enjoyment of the game itself. That is, mathematics has become the arbiter in arguments, the stimulus for "hot stove league" discussions, a tool to help identify either patterns of team strengths and weaknesses or optimal strategies, and a decision-making tool for gamblers and fantasy league participants.

Sabermetrics

Bill James, a baseball writer, historian, and statistician, gave authenticity to the use of statistics in analyzing all aspects of baseball through his pioneering mathematics and statistics work. Multiple editions of his *Baseball Abstract* in the 1980s changed not only the play of the game itself but also how it is viewed by fans, and are the predecessor to many modern Web sites dedicated to analysis of the sport. James revolutionized the way mathematics is used to analyze sports to determine why some teams win and others lose. He coined the term "sabermetrics," which is derived from the Society for American Baseball Research acronym SABR, for his analytical and modeling methods. In 2006, *Time* magazine named him one of the most influential people in the world.

Mathematical statistics provide perspectives that explain game occurrences, provide comparative rankings of teams and players, and assist in managerial decision making. The primary example is the simple use of ratios, means, and medians as both descriptive and inferential statistics for a player, position, game, season, or career. Some examples include the following:

- Batting average, slugging percentage, on-base percentage, and batter's run average
- Effect of artificial turf on numbers of ground ball hits or base stealers' performances
- Performance of hitters and pitchers in different environments (outdoor versus dome stadiums; night games versus day games)

- Expected strike zones for umpires, given a pitcher or batter is right- or left-handed

Going beyond these descriptive statistics, the game of baseball can be analyzed using very sophisticated techniques. Some examples include the following:

- Connections between a player's characteristics and training regimens relative to game performance, or even to document the effects of steroid use
- Trend analysis, based on either a player's or team's performance (hitting, pitching, fielding) over the past five, 10, and 15 games
- Importance of pitcher throwing a "first strike"
- Effects of bringing in the infield when the bases are loaded with less than two outs
- Team winning tendencies based on run differential in innings seven, eight, and nine
- Impact of rule changes on pitching and hitting, such as the effects of elevating the pitching mound or changing foul-line distances to outfield fences
- Determination of coaching strategies such as sacrifice bunts, pitch-outs, stealing home, intentional walks, shifts of fielders for certain hitters, or use of pinch hitters and relief pitchers
- Determining the "best" all-time player in a particular position (for example, centerfielder, hitter, relief pitcher, base-stealer)
- Selection of players by professional teams during annual drafts, using both historical data for each player's performance and physical data
- Use of statistical data during contract negotiations between a player and management, or even the release or trading of players based on team needs

Mathematical probabilities, odds, and expected values can help examine the chances of particular events happening within a game or across games:

- Probability that the World Series will go four, five, six, or seven games
- Use of odds to determine personal or professional betting strategies

Although some fans object to using mathematics in baseball, some enjoy it as much as the game itself.

- Use of conditional probabilities to determine lineups or use of pinch hitters, reflecting the probability of a batter getting a hit given that the pitcher is right- or left-handed
- Correlations between a team's wins per season and player payrolls, or pitcher salaries and their ERAs
- Probability of a record being broken, either by a team or player, such as Joe DiMaggio's 56-game hitting streak

Though difficult to implement practically, geometry, trigonometry, and calculus can shed light on other important ideas:

- Length of a home run
- Actions of different pitches such as a curve ball, slider, fastball with movement, or forkball

- Determination or alteration of a hitter's batting stance or position in the batter's box
- Use of angles in fielding balls off outfield walls

Game theory also is used as part of the decision-making process within a baseball environment, leading to choices of optimal tactics. Some specific decisions are as follows:

- A manager's choice of batting lineups and pitching moves, relative to the opposing manager's choices
- A manager's calling for shifts of fielders, pitch-outs, or steals at times within a game
- A manager trying to argue, influence, or reverse decisions by umpires
- A manager's use of techniques to motivate specific players
- A team's selection of players during a draft, dependent on the player's apparent abilities, the inferred needs of other teams, and the specific draft round
- Contract negotiations involving players, agents, and team management

Finally, using all of these statistical data and mathematical modeling techniques, one can create realistic simulations of baseball games or end-of-year series, possibly using computer animations.

At the collegiate and professional levels, managers are increasingly using mathematics to remain competitive, even hiring mathematical statisticians as important parts of their staff. However, some authors and fans suggest that the team with the best players and managers will usually win, despite any use of sophisticated mathematics.

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Jerry Johnson

See Also: Basketball; Football; Hitting a Home Run; Hockey; Soccer.

Basketball

Category: Games, Sport, and Recreation.
Fields of Study: Data Analysis and Probability; Geometry; Measurement.
Summary: Play can be analyzed geometrically and probabilistically to inform strategy or construct simulations.

Basketball is an international sport that can be enjoyed either as a participant or as a spectator, regardless of one's sex or one's age. A growing number of coaches, reporters, and ardent fans are using mathematics to examine all aspects of basketball—the physical aspects and performance of its players, the analysis of each element (shooting, defense, strategies) of the game, and the combined geometry and physics surrounding the game. Perhaps as expected, this mathematical analysis can have opposite effects, either enriching or ruining the sports experience itself.

Basketball was intended to be a dynamic, fair competition between two teams; however, mathematical concepts and techniques can be used in a basketball environment to identify patterns of strengths and weaknesses, suggest optimal strategies for coaches and players, stimulate discussions, and resolve arguments. Statistician Dean Oliver is a well-known contributor to the statistical evaluation of basketball, which is called APBRmetrics. The name comes in part from the Association for Professional Basketball Research (APBR). This methodology is a very similar to the analysis of professional baseball using sabermetrics. Though difficult to implement practically, geometry, trigonometry, and calculus can shed light on these important ideas:

- Given a player's height, the best angle and velocity for shooting a basketball, assuming the intent is to have the basketball's parabolic arc pass through the basket (often called the "Shaq phenomena")
- The connection between the angle of shooting a ball and the event known as an "all-net" basket
- The connection between a player's height and where a player should aim a shot—at the center of basket, the front of the rim, or the back of the rim
- Use of angles in making bounce passes
- Determining defensive positions that maximize centers of gravity
- The connection between a player's position on the court and decisions to bank the basketball off the backboard as the best shot
- Comparison of the merits of shooting a free-throw underhand versus overhand

By gathering and analyzing the myriad of available data provided by a game experience, mathematical probabilities can help examine the chances of particular events happening within a game, including the following:

- The likelihood of a player making 0, 1, or 2 points in a 1-and-1 free throw opportunity

Mathematics can be used to examine the physical aspects and performance of players, the analysis of each element of the game, and the combined geometry and physics surrounding basketball.

- The reality of a player having a "hot-hand," based on his or her making successive shots
- The decision as to which player should be purposely fouled at the end of a close game
- The evaluation of a player's performance in terms of "per-possession efficiency"
- The probability of a record being broken, either by a team or a player

Similarly, the collection and organization of mathematical statistics can provide perspectives that explain game occurrences, provide comparative rankings of teams and players, and assist in future decision making by coaches and team management. The usual sources of statistics are data regarding shooting, rebounding, free throws, turnovers, defensive gains, and time management. Some specific examples include the following:

- The simple use of ratios, means, and medians as descriptive statistics for a player, a position, a game, or a season
- Connections between a player's characteristics and training regimens relative to game performance
- Trend analysis, based on either a player's or a team's performance in specific ways over the past five, 10, and 15 games
- Winning tendencies based on connections to lead changes during a game or knowledge of the team leading at the end of the third quarter
- The impact of rules changes on scoring and defenses within the sport itself, such as the observed effects of expanding either the three-point arc or the free-throw lane
- Determining the "best" all-time player in a particular position (for example, center), at a particular time in a game (for example, last-second shot), or in an era
- The seeding and selection of teams in a bracketed tournament, possibly as part of a betting pool with stated odds
- Selection of players by professional teams during the annual draft, using historical data for each player's performance in conjunction with physical data
- The use of statistical data as part of contract negotiation between players and management
- The release or trading of players based on team needs

The ideas of mathematical game theory have been applied to the decision-making process within a basketball environment, leading to choices of optimal tactics. The specific decisions range considerably:

- A coach's choice of designed offenses and defense strategies, relative to the opposing coach's choices
- A coach's calling of time-outs at opportune times within a game
- A coach trying to influence or reverse decisions by game officials
- A coach's use of techniques to motivate specific players
- A team's selection of players during a draft, dependent on the player's apparent abilities, the inferred needs of other teams, and the specific draft round
- Contract negotiations involving players, agents, and team management

Finally, using all of these available statistical data and mathematical modeling techniques, one can create realistic simulations of basketball events, full games, or even tournament series. At the collegiate and professional levels, coaches are increasingly using mathematics to remain competitive, even hiring mathematical statisticians as important parts of their staffs. Some mathematicians are even found on the court.

Retired San Antonio Spurs player Michael Robinson earned a bachelor's degree in mathematics from the U.S. Naval Academy, and is considered by many to be the best basketball player that school has ever seen. However, there are still some authors and fans who suggest the team with the best players and coaches will usually win, despite the use of sophisticated mathematics.

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Jerry Johnson

See Also: Baseball; Football; Hockey; Soccer.

Basketry

Category: Arts, Music, and Entertainment.
Fields of Study: Algebra; Geometry; Number and Operations.
Summary: Basket shapes and patterns are created by varying the weave.

Baskets are woven containers made of plant or artificial strips, such as palm fronds, willow branches, or fabric. People were already making baskets at least 10,000 years ago. Historians conjecture that basketry played a major role in the development of pattern, structure, and number in human cultures. Early humans could have observed birds and animals that wove to first learn the craft. Tracing basket-weaving patterns through cultures assists in creating models of human migration. "Underwater basket weaving," which is a technique in wicker, is a humorous idiom describing academic courses with low education standards or very narrow specializations. Mathematicians study and model patterns found in baskets from around the world, including those created by the Hopi people of the southwestern United States, various African peoples, and Pacific Islanders.

Weaves

There are several types of basket weaves, each with infinitely many possible patterns. Coiled baskets are made with two types of fiber—one thick, and one soft and pliable. The thick cord or vine forms the coil. Flat, pliable strips of materials such as grass or fabric are wound a number of times around the cord, then a number of times around its previous row or several rows in the coil, connecting the rows. Craftspeople can change patterns and shapes of baskets by varying these weave numbers. Wicker is a type of basket weave consisting of relatively stiff fibers of two types. One material, the foundation, is completely rigid, and the other, the weft, is more pliable. The pattern of individual weft fibers going over and under the foundation spokes determines the look of the basket's surface. Such patterns can become very complex. Weft fibers are often soaked to make them soft during weaving.

Twining also requires rigid foundation fibers and pliable weft. Several strands of pliable fiber—usually two—go around a foundation spike on either side, cross or twist in the middle, then go around the next spike. Twining patterns are created by changing the number of wefts or formulas of skipping spikes, and introducing braiding between spikes.

Plaited baskets consist of pliable fibers woven over and under one another, typically at right angles. This weave is very similar to how woven textiles are made, and some historians believe that textiles originated from this type of basket. Formulas—whose variables are the number of fibers that go over and under in each row—determine the pattern.

The physical properties of baskets are determined by the weave, the materials, and the pattern. Wicker baskets can be very sturdy, and wicker has been used in making fences, houses, and furniture like baby cradles.

Basket Patterns

Patterns found in baskets come up in many areas of mathematics. Frieze patterns have translational symmetry along lines, and there are seven types of them, all of which appear in traditional basket making. They are a part of more general wallpaper groups, of which there are seventeen types.

Some mathematics historians observed differences in patterns that involve six-fold symmetry, such as honeycombs, and more complex five-fold symmetry that comes up in basket weaving. For example, the traditional woven Malaysian ball is similar to the modern soccer ball (also known as "football") in that it contains pentagons.

Shapes

Baskets take a variety of three-dimensional shapes, such as cylinders, cubes, and prisms. Properties of weaving often determine the shape. For example, the stiff foundation fibers of twined or wicker baskets are usually straight lines, which only allows so-called ruled surfaces. By definition in analytic geometry, ruled surfaces are generated by straight lines. Cylinders, prisms, and cones are ruled surfaces and can be made by wicker. Spheres cannot be made out of straight lines, but spherical baskets are made by coiling, plaiting, or using bendable foundations in wicker and twining. Mathematicians and mathematical artists who use basket weaving to create striking sculptured models of complex surfaces have to select appropriate weaving techniques for their projects.

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Maria Droujkova

See Also: Sculpture; Surfaces; Symmetry; Textiles; Transformations.

Bees

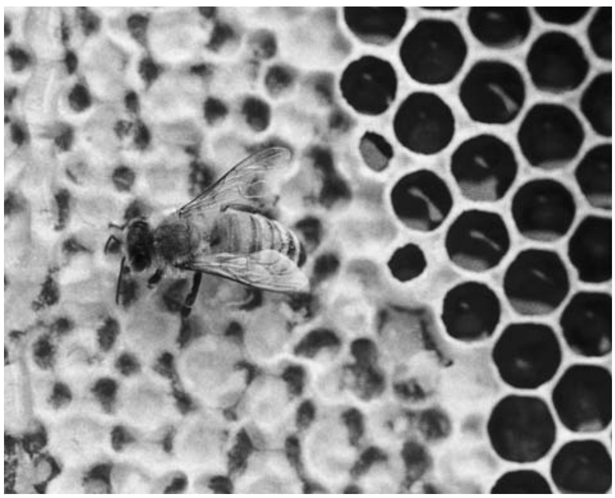
Category: Weather, Nature, and Environment.
Fields of Study: Algebra, Geometry; Representations.
Summary: Geometry explains why honeycombs are made of hexagonal cells, while bee movement patterns communicate information visually.

Honeycombs are remarkable for their beauty, precision, and symmetry. The honeycomb corresponds to a mathematical concept known as a "tiling of the plane." That bees use regular hexagons for this tiling (built to a remarkable level of precision) has fascinated human beings throughout history. At the end of the twentieth century, mathematician Thomas Hales rigorously proved a long-standing conjecture that fully justifies to humans what the bees have apparently known all along: the most efficient way to repeatedly enclose a fixed amount of storage space is to use regular hexagons to form the boundaries.

Honeycomb: How to Choose a Cell

Bees use honeycomb cells for storage. It takes work and material (wax) to create the boundary of each cell, so the bees want cells with as little boundary (perimeter) as possible, given that each cell should enclose a certain amount of storage (area). If a bee only needed to make one cell to store honey, it would likely use a shape other than a regular hexagon. For instance, a regular octagon holding the same area has less perimeter; a regular decagon will have less perimeter still. The more sides a polygon has, the smaller the perimeter will be, with the circle having the smallest perimeter-to-area ratio. That a circle is the least-perimeter shape to enclose a given area is a famous problem that goes back to the wonderful tale of Queen Dido of Tyre.

For example, suppose a bee wanted to enclose one square unit of area. The square that accomplishes this has a perimeter of 4. If the bee used an equilateral triangle instead, the necessary perimeter is larger, about 4.56. But the regular hexagon's perimeter is smaller, at



The most efficient way to repeatedly enclose a fixed amount of storage space is to use regular hexagons.

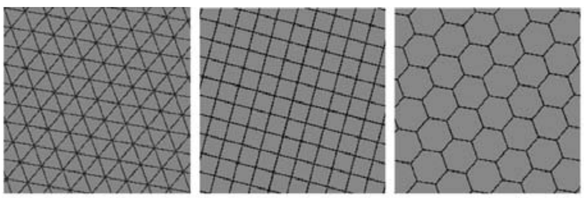
just over 3.72. The pattern of increasing the number of sides leading to a lower perimeter holds for all whole numbers $n > 2$, and every such regular polygon enclosing one unit of area has greater perimeter than a circle holding the same area. The circle that encloses one unit of area has a perimeter of approximately 3.54.

Tilings: Fitting the Cells Together

Bees do not need just one cell; they need many consecutive cells in which to place their honey, and therefore essentially have to create a "tiling" (a pattern involving polygons that will completely cover their work space without overlapping while leaving no space unused). Circular cells simply don't fit together as well because there are gaps between consecutive circles.

Many different kinds of floors and ceilings are tiled—usually with congruent squares or rectangles. Why don't bees use square cells in their honeycomb, rather than hexagons? Or equilateral triangles? It turns out that equilateral triangles, squares, and regular hexagons can all be used to tile the plane, as shown in the figures below. Bees choose hexagons from among these three options since a regular hexagon of unit area uses less perimeter (wax) than does a square or equilateral triangle; the hexagon is a more efficient choice (see Figure 2).

Figure 2.



So why not use regular octagons? Here it is not the efficiency of the individual cell that governs the choice but rather the overall packing of them: regular octagons cannot be used to tile the plane.

To understand why triangles, squares, and hexagons tile the plane, but octagons do not, observe that in a regular polygon with n sides, the sum of its interior angles is $180(n-2)$ degrees, and each of its n individual interior angles has the measure

$$\frac{180(n-2)}{n}.$$

For instance, with the square, each interior angle has the measure

$$\frac{180(4-2)}{4} = 90 \text{ degrees.}$$

Four squares arranged at a single vertex fit together perfectly, creating a full 360 degrees around the shared corner. Likewise, six equilateral triangles (each having 60-degree angles) can fit together perfectly for a full 360 degrees, as can three regular hexagons with their 120-degree interior angles.

But for the octagon with $n = 8$, each interior angle has the measure of 135 degrees. Three octagons put together at a shared vertex would have $135 \times 3 = 405$ degrees, which is simply impossible—as would be attempting to only have two octagons meet at a single vertex. Regardless of the number of sides of the regular polygon, the measure of the polygon's interior angle will need to divide evenly into 360 degrees. This forces

$$\frac{2n}{n-2}$$

to be an integer, and the only values of n for which that is true are $n = 3, 4,$ and 6: triangle, square, and hexagon! That the only ways to tile a flat surface using congruent regular polygons are with triangles, squares, or hexagons is a result often taught in high school geometry courses.

Irregular and Non-Polygonal Tilings

Since the time of the ancient Greeks, mathematicians conjectured that among all the ways to tile the plane so that each tile encloses just one unit of area, the way that uses the least perimeter is the tiling that uses all regular hexagons. This conjecture is much harder than it sounds to prove: one must consider irregular polygons (with sides of different lengths), as well as the possibility that the sides of some tiles might be curved. The first possibility is not too difficult to eliminate. For instance, it is straightforward to show that a regular hexagon with all sides of equal length will use less perimeter than any other hexagon to enclose the same area.

But the second possibility—using non-polygonal shapes—proved to be much, much more challenging. In this situation, one must consider the possibility of a shape that bows out on one side and, to fit into a tiling, bows in on another. Obviously, the part that bows out picks up area, while the part that bows in loses area. In 1999, mathematician Thomas Hales proved that any advantage that comes from a side of the tile bowing out is more than cancelled out by the disadvantage that follows from another side having to bow in. Thus, the ideal tile is one that has no bulges: a polygon!

What Professor Hales proved is essentially what the bees knew all along: of all possible tilings, the one using regular hexagons is the most efficient way to enclose cells of the same area.

Other Mathematical Aspects of Bees

Another way that mathematics relates to bees is when mathematicians work with bee researchers to solve problems such as those related to viral disease infection and pollination. Mathematics is also used to model the ways in which bees communicate locations. When a bee finds a source of food, it returns to the hive and performs an elaborate dance that conveys the direction and distance from the hive. Ethologist Karl von Frisch was one of the first to explore the meaning of the honeybee dance, and he won a Nobel Prize for his work. The angle that the bee dances expresses the direction. For example, if a bee dances in a straight line toward the upper part of the hive, then the flowers are located in the direction of the sun. The bee also takes into account the fact that the sun moves; the angle it describes inside the hive changes as the sun does. The duration of the dance and the number of vibrations give the exact distance. Other features of the dance remained unexplained until Barbara Shipman theorized that the honeybee's complex choreography is a projection of a six-dimensional space, and she was able to use this representation to reproduce the entire bee dance in all its parts and variations. To her, this implies that bees can sense the quantum world, although some researchers dispute her conclusions.

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Matt Boelkins

See Also: Animals; Farming; Polygons.

Betting and Fairness

Category: Games, Sport, and Recreation.
Fields of Study: Algebra; Data Analysis and Probability.
Summary: Mathematics is used to analyze betting and probabilities for games of chance and for investing in the stock market.

A pivotal moment in the early development of probability occurred in 1654, as the French mathematicians Blaise Pascal and Pierre de Fermat exchanged a series of letters. Pascal and Fermat were wrestling with questions involving the fair payoff for a gambler who is forced to quit in the middle of a game. In modern language, they were calculating the "expected value" of the game's payoff (the average payoff under the various possible outcomes, weighted according to the likelihood of those outcomes). A bet is said to be "fair" if the price of placing it is equal to the expected value of the payoff. Betting plays an integral part in our modern society. People place bets in casinos and at sporting events, as well as by buying lottery tickets. They are also placing bets when purchasing insurance or investing in the stock market. Some of these bets are fair, some are unfair, and some cannot be objectively categorized.

The primary problem that Pascal and Fermat solved (each employing a different method) can be used to illustrate some important ideas on fairness. In the problem, two gamblers are playing a game in which a coin is repeatedly tossed. The game is interrupted at a point where 2 more heads are required for Player A to win and 3 more tails are required for Player B to win (whichever occurs first). How should the potential winnings be divided at this stage of the game?

Fermat solved the problem by observing that at most 4 tosses remain in order to identify the winner, and that there are 16 equally likely ways in which 4 tosses could occur:

HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, HTTH, THHT, THTH, TTHH, HTTT, THTT, TTHT, TTTH, and TTTT.

In 11 of these possibilities (the first 11 items on the list), Player A would win, because 2 heads occur before 3 tails; in the other 5 possibilities, Player B would win, because 3 tails occur first. Therefore, Fermat reasoned that Player A should receive 11/16 of the winnings, and Player B should receive 5/16 of the winnings. In modern language, Player A would win the game with probability 11/16 and Player B would win with probability 5/16; Fermat was calculating the "expected value" of the winnings for each player.

Suppose that up to this point in their game, neither Player A nor B has paid any money for the opportunity to play, but that they are now required to pay a total of $1, altogether, and that this dollar will constitute the winnings. Fermat's solution to the previous problem allows for a fair method of dividing the payment: Player A should pay 11/16 of the dollar and Player B should pay 5/16, so that the payments match the expected winnings. In other words, if the game is being played for a $1 payoff, then the price for a fair bet is 11/16 of a dollar for Player A and 5/16 of a dollar for Player B.

Lotteries and Casinos

State-run lotteries are unfair to the player who purchases a ticket, because some of the revenue goes to the state and is not redistributed to the winner(s). Of course, even if all of the ticket revenue were paid to the winner(s)—so that the bets were fair—a lottery would be unfavorable to almost every player. Nonetheless, lotteries attract large numbers of players because people are willing to pay a small amount for the minuscule chance of winning a fortune.

A similar motivation attracts bettors to casinos, where almost all games are unfair. This casino advantage is known as the "house edge." On average, the house edge at a casino is 2% to 3%, which means that for each dollar that is bet, the house makes a profit of 2 or 3 cents. Over thousands of bets, this adds up to a significant profit. Some games, like slot machines, can have a house edge of up to 15%. Typically (in roulette, slot machines, and craps, for instance), the odds for each bet are slightly in favor of the house. Blackjack is a rare example of a casino game in which a player might be able to place bets that are better than fair from the player's perspective. In blackjack, two initial cards are dealt to each player as well as to the dealer. Certain strict rules dictate whether additional cards are dealt to the dealer, while each player has the choice of whether to receive additional cards. The objective of each player is to hold a total card value closer to 21 than the dealer holds, without going over. Each player knows which cards he or she holds, as well as some of the cards held by the dealer and other players, since some cards are dealt face up. An adept player can also keep track of cards that have been used in previous games following the last shuffle—though casinos often dissuade such card counting by combining several decks and shuffling regularly. By using this information, it is possible for a player to calculate the probability of drawing a particular card and, therefore, the expected value of the payoffs under the options of either receiving an additional card or not; often, one of these expected values is greater than the amount of the bet.

Subjective Probabilities

Early in the twentieth century, mathematicians realized the need to define probability in a rigorous way, if it were to be a formal part of mathematics. In problems involving tossing fair dice or coins, or counting card hands, it was obvious what should constitute the probabilities of the various occurrences, but in many other situations it was unclear. Usually people thought of probabilities as idealized frequencies: if a fair coin is tossed many times, for example, then the fraction of tosses which land heads should be approximately 1/2; so a fair bet for a $1 payoff on heads should cost $0.50. But there is not an obvious analogy for two boxers, for example, about to fight a match. Also, probability was becoming an increasingly important tool for the physical sciences, and mathematical theorems were required. As such, an axiomatic system was necessary. The Russian mathematician A. N. Kolmogorov and the Italian philosopher and mathematician Bruno de Finetti independently provided such a framework in the 1930s. Although different in appearance, their definitions are equivalent in most situations.

De Finetti's concept of a probability stems from gambling: the probability of an event is the price for a $1 payoff bet on that event. These prices may be assigned in whatever way one wants (hence the label "subjective probabilities"), provided certain consistency conditions are met. For example, suppose even money is coming into a betting house on two teams preparing to play a baseball game. This indicates that the bettors collectively value the two teams as equally likely to win the game. Ignoring the house fees, the price for a $1 payoff bet on either team is $0.50, because after the game, the entire pool of money will be redistributed to those who bet on the winning team.

Suppose, however, a particular bettor favors the home team, believing that team to have a 3/4 probability of winning the game. Then this bettor would price a $1 payoff bet on that team at $0.75; for this bettor, the $0.50 price generated by the betting pool is a bargain. From this bettor's perspective, a bet on the home team is better than fair: the price for a $1 payoff bet is $0.50, but the expected value of the payoff is $0.75. Such situations occur beyond sporting events, perhaps most prominently in the stock market. The fact that individuals' valuations often differ from those of the collective public is the driving force behind the trading of stocks. Individuals buy stocks that they believe to be undervalued and sell stocks that they believe to be overvalued. Because they are predicting the future performance of these stocks, they are essentially placing bets that they believe to be better than fair. In 1956, John Larry Kelly, Jr., a physicist who worked at Bell Labs, formulated and described the Kelly criterion. This algorithm for determining an optimal series of investments (or bets) is based on probability and economic utility theory, which tries to mathematically quantify satisfaction. In recent years, the Kelly criterion has been incorporated into many mainstream investment theories and betting strategies.

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John Beam

See Also: Dice Games; Expected Values; Probability.

Bicycles

Category: Travel and Transportation.
Fields of Study: Algebra, Geometry.
Summary: Bicycle geometry impacts performance, aerodynamics, efficiency, and stability.

The first bicycles of the early nineteenth century were simple designs of wooden frames and metal hoops for wheels. Though these early bicycles were propelled by feet pushing along the ground, soon pedals were added to the front axle allowing the rider to drive the front wheel for locomotion. It was not until the late 1880s when the first chain-driven bicycle was introduced, thereby separating the axles from the primary point of locomotion and overcoming problems with handling, steering, and weight distribution. This explosive decade of development also saw the first pneumatic tires, gearing, and coaster brakes, the latter allowing the rider to brake by pedaling backwards. Another series of innovations a century later was spurred by an explosion in frame design and fabrication techniques including the use of better materials such as aluminum, titanium, and, eventually, carbon fiber.

Bicycles serve as the primary means of transportation in several cultures, especially in southeast Asia. European communities are also known for embracing the bicycle as a legitimate form of transportation.

Mechanics

Bicycles have two in-line wheels and are driven by pedaling. The wheels each spin on axles rotating on bearing surfaces and most commonly support the rims via tension spokes. Pneumatic tires are secured to the outer surface of the rims to provide the primary contact with the ground. The centrally located bottom bracket is the rotating connection point of the pedals. Power is transferred to the rear wheel via a chain. Brakes are usually found on both wheels; most bicycles' brakes squeeze braking pads on the rim surface to create friction and slow the wheel and, as a result, the bicycle. Many newer mountain bicycles use disc brakes for increased stopping power. The rider sits on a saddle atop the bicycle and leans forward on handle bars, which provide support and the ability to steer. Many bicycles, especially mountain bicycles, have shock absorbers built into the front fork to provide cushioning over rough terrain. Some bicycles also feature rear suspension, which allows the rear triangle of the frame to rotate and further absorb the impacts of uneven terrain.

Gears (chain rings on the bottom bracket, a cassette on the rear axle) allow the rider to alter the ratio of pedal rotation to wheel rotation in order to go faster or slower. The gear ratio is determined by the diameter of the chain ring divided by the diameter of the rear cog. Since the number of teeth is proportional to diameter, tooth count is more typically used. For example, a 39-tooth chain ring used with a 15-tooth cog produces a gear ratio of

$$\frac{39}{15} = 2.6$$

that is, one revolution of the pedals produces 2.6 revolutions of the rear wheel. A standard 700C wheel (70 centimeters in diameter) will travel $0.7\pi = 2.2$ meters (7.2 feet) along the road with each revolution. Thus, a single rotation of the pedals produces

$$\frac{39}{15}(0.7)\pi = 5.7 \text{ meters (18.7 feet) of travel.}$$

Speed and distance traveled can then be calculated based upon the rider's revolutions per minute.

More than 10,000 riders have participated in the Tour de France since it began in 1903. Out of that group, only around 6,000 have been able to complete the grueling race that averages about 2200 miles.

Types of Bicycles

A bike with square wheels demonstrated at Macalester College.

Reflecting their wide versatility, bicycles come in a multitude of different styles. One of the most common is the road bicycle, which is distinguished by thin tires; a drop-style handlebar; and a stiff, light frame. Road bicycles are designed for fast travel over smoother road surfaces. The other most common bicycle is the mountain bicycle, which features wide, knobby tires designed for increased traction in the dirt; flat handlebars for a more upright position; and a wide range of gears, including very low gears for steep climbing. Most mountain bicycles have a front suspension fork and many feature a rear suspension as well.

Cyclocross bikes are closely related to road bikes but have slightly wider tires and lower gears for racing on cyclocross race courses or for exploring gravel roads. Comfort bicycles, commuters, and hybrids are usually compromises between the stiffness of a road bicycle and the comfort of a mountain bicycle; these bicycles' lower prices are often aimed at entry-level riders who are seeking practicality over high performance. Bicycle motocross (BMX) bicycles are single speed (no gears) with smaller, wider tires designed for racing on BMX courses. There are additional niche bicycles for special purposes such as time trialing, track racing, snow riding, and touring. Though most people cannot imagine a bicycle having anything but circular wheels, since that shape travels smoothly on flat roads, mathematicians have modeled as well as built wheels with other shapes, such as squares, three-leaf clovers, star-like shapes, and triangles. They found that a square-wheeled bike will travel smoothly on a road made of inverted catenaries, and each of the other types has at least one solution as well. A differential equation can be used to generally solve the problem of noncircular wheels.

Racing and Performance

Bicycle racing is a popular sport with a surprisingly active history. Near the end of the nineteenth century, bicycle racing was one of the most popular sports, drawing huge crowds of spectators across Europe and the United States. Today, bicycle racing is popular worldwide but has a stronger European following. Why certain cyclists are more successful than others can be analyzed in part using mathematics. Average riding speed, efficiency, and power are all calculated metrics useful for assessing performance. Seven-time Tour de France winner Lance Armstrong has been studied and modeled extensively throughout his career. American cyclist Greg LeMond overcame a 58-second deficit and won the 1989 Tour de France by 8 seconds over French favorite Laurent Fignon, which is generally attributed by most to the innovative aerodynamic handlebars he used in the last stage. Companies now routinely use mathematical modeling for cycling equipment, as well as to test aerodynamics and other essential properties, and teams use optimization strategies to construct bicycles within the sport's guidelines, since seconds can make the difference between victory and second place.

For the average rider as well as for professionals, the geometry of a bicycle plays a large role in its overall performance and stability. For example, the distance between the axles and the angle the front fork makes with respect to the ground are both important, according to bicycle makers. Some mathematicians have explored stability issues. In study released in 2007, researchers investigated and dynamically modeled 25 parameters believed to be important, with the goal of being able to construct bicycles targeted toward riders' specific needs.

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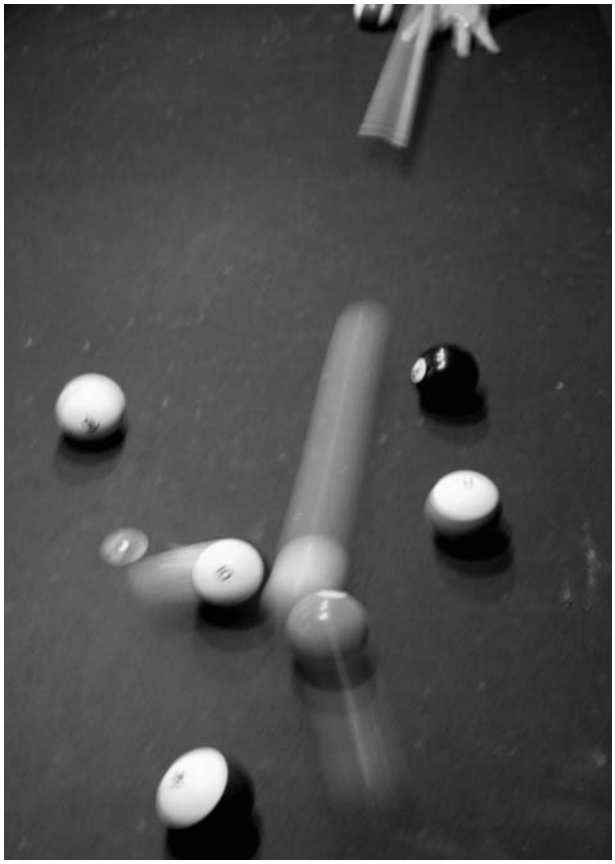
See Also: Extreme Sports; Perimeter and Circumference; Pi; Wheel.

Billiards

Category: Games, Sport, and Recreation.
Fields of Study: Geometry; Number and Operations.
Summary: Playing billiards depends on an understanding of spin, momentum, and angles.

Billiards is a cue sport game that involves the use of a rectangular table, billiard balls, and a stick called a "cue." Mathematics and physics are two important components of playing the game well. There are many different games within the cue sports that Americans typically name "billiards." Billiard tables with pockets comprise games that are termed as "pool" or pocket billiards. The rectangular table has two long sides (twice the short side) and two short sides with six pockets—one at each corner, and one midway along the longer two sides of the table. The object of the game is to hit the billiard balls into the pockets using a cue ball (the lone white ball in the set). Gaspard Coriolis, known today for the Coriolis effect, wrote a work on the mathematics and physics of billiards in 1835. He stated that the curved path followed by the cue ball after striking another ball is always parabolic because of top or bottom spin. Further, the maximum side spin on a cue ball is achieved by striking it half a radius off-center with the tip of the cue.

The game of billiards is also a source for interesting mathematical problems, which are connected to dynamical systems, ergodic theory, geometry, physics, and optics. In mathematical billiards, the angle of incidence is the same as the angle of reflection for a point mass on a frictionless domain with a boundary. The dynamics depend on the starting position, angle, and geometry of the boundary and the table. Mathematicians investigate the motion and the path of the ball on a variety of differently shaped flat and curved tables, like triangular or elliptical boundaries or hyperbolic tables. In 1890, mathematician Charles Dodgson, better known as *Alice in Wonderland* author Lewis Carroll, published rules for circular billiards and may have also had a table built. In 2007, mathematician Alex Eskin won the Research Prize from the Clay Mathematics Institute for his work on rational billiards and geometric group theory.



Billiard players can use transformational geometry to try to hit the ball so that it will return to a pocket.

Eight Ball

Eight ball is the pool game most commonly played in the United States, and it involves 16 billiard balls. To begin the game of Eight Ball, the numbered balls are placed in a triangular rack that sets the 8-ball in the middle position of the third row of balls with a single lead ball opposite the cue ball. The cue ball is placed on the midpoint of the line parallel to the short side at one-quarter of the long side known as the head spot. The point of the triangular-shaped racked set of billiard balls is placed on the opposite short end at one-quarter of the length of the long side from the other short side and is known as the foot spot. After one player "breaks" by hitting the cue ball from the head spot into the racked set of balls, the player then hits a set of balls into the pockets. A shot that does not cause a ball of his or her set to go into the pocket results in the next shot going to the other player.

Billiards Geometry and Physics

Shooting the balls into the pockets requires an understanding of angles and momentum, as well as placement of the cue so that the correct spin is achieved to place the cue ball where it can achieve the target ball going into a pocket. Coriolis investigated 90-degree and 30-degree rules of various shots and measured the largest deflection angle the cue ball can experience. Both skill and geometric understanding contribute to successful shots. Some shots require straight shooting; some shots need to be "banked" in by using the table sides. Players can use transformational geometry to approximate where on the table to hit the ball for it to return to a pocket. By measuring the angle from the ball to the side being used to bank off and reflecting the same angle with the cue stick, one can see the most viable spot to aim for so that the path of the caromed ball ends in a pocket. Using the diamonds found on the sides of most tables is one way of measuring these angles, and some systems for pool and billiards play use the diamonds. Using the diamond system for a different billiard game, Three Cushion Billiards is demonstrated on the 1959 Donald Duck Disney cartoon *Donald in Mathmagic Land*. The demonstration shows that it is possible to use subtraction to know where to aim the ball in relation to a diamond to make sure that all three balls are hit.

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See Also: Geometry in Society; Mathematical Modeling; Movies, Mathematics in; Surfaces; Transformations.

Binomial Theorem

Category: History and Development of Curricular Concepts.

Fields of Study: Algebra; Communication; Connections; Number and Operations.

Summary: The binomial theorem is the basis of Pascal's Triangle and is used to solve a variety of problems.

A binomial is an algebraic expression with two terms, like $x+y$. When binomials are multiplied together, they produce higher powers of the individual terms that are called "binomial coefficients." The binomial theorem states that for any real numbers x and y, and whole number n:

$$(x+y)^n = c_0x^ny^0 + c_1x^{n-1}y^1 + c_2x^{n-2}y^2 + \cdots + c_nx^0y^n$$

where c_k is the binomial coefficient

$$\frac{n!}{k!(n-k)!}$$

and $n!$ is the product of the numbers 1 through n. These coefficients are the entries in what is referred to as Pascal's Triangle, named for mathematician Blaise Pascal. Students typically encounter this theorem in middle school or high school algebra, and in high school or college calculus. It also has uses in other areas of mathematics, such as in combinatorics, where it helps in calculations for certain counting problems.

The binomial theorem is found across ages and cultures. It appears in the ancient world in the work of Greek mathematician Euclid of Alexandria. His formula was for the square $(n=2)$ of a binomial, but it was described geometrically rather than algebraically. There is also evidence that the Hindu scholar Aryabhata knew the theorem for cubes in the sixth century. At least as early as the eleventh century, Chinese mathematicians such as Jia Xian and later Zhu Shijie knew the binomial coefficients in the form of Pascal's Triangle. They used the binomial theorem to find square and cube roots, and evidence suggests they knew of the binomial theorem for large values of n. Around the fifteenth century, the binomial theorem and binomial coefficients to at least the seventh power were found in the writings of Islamic scholars including Omar Khayyam, Abu Bekr ibn Muhammad ibn al-Husayn Al-Karaji, Abu Ali al-Hasan ibn al-Haytham (Alhazen), and Ibn Yahya al-Maghribi al-Samaw'al.

In the sixteenth century, European mathematicians began using the binomial theorem and binomial coefficients. For example, mathematician Michael Stifel's 1544 work *Arithmetica integra* contained the binomial coefficients. Other contributors include François Viète, Blaise Pascal, James Gregory, Sir Isaac Newton, and Niels Abel. John Wallace's seventeenth century book *De Algebra Tractatus* is cited as the first published account of Newton's binomial work. While Pascal was not the first to study the binomial coefficients, he is credited with linking algebraic and combinatorial interpretations of the coefficients.

Pascal's Triangle is a triangular representation of the binomial coefficients, which may be attributed to him because of his 1653 work *Traité du Triangle Arithmétique* in which he compiled and expounded on much of what was known about binomial coefficients. The related Pascal matrix is a symmetric, positive definite matrix with the Pascal triangle represented on its antidiagonals.

Generalizations and Extensions

James Gregory and Sir Isaac Newton generalized the binomial theorem to allow first fractional, and then real powers, which requires replacing the finite sum with an infinite series and extending the definition of the binomial coefficients. When generalizing to an infinite series, another issue that must be considered is convergence, which imposes restrictions on the numbers x and y for which the series converges to the binomial. Newton came to this generalization indirectly while trying to calculate areas under certain curves.

Another way to generalize the binomial theorem is to broaden the types of values that x and y can take. One such generalization allows x, y, and n to be complex numbers. The definition of the binomial coefficients has to be generalized to complex numbers, and certain restrictions on the variables are required for convergence of the resulting infinite series. Alternatively, one can allow x and y to be commuting elements of a Banach algebra, a normed algebra studied in such fields as complex analysis, real analysis, and functional analysis. Banach algebra is named for twentieth century mathematician Stefan Banach.

One more type of generalization considers not just the sum of two numbers x and y, but sums with more terms. Such a sum would be called a multinomial, and the multinomial coefficients would be appropriate generalizations of the binomial coefficients. Pascal's Pyramid or Pascal's Simplex are extensions of Pascal's Triangle for three or more dimensions.

Applications

The binomial theorem gives a quick way of expanding a power of the form $(x+y)^n$, making the formula useful for basic algebraic calculations. The binomial theorem, along with De Moivre's formula, can be used to prove the trigonometric double-angle identities, as well as more general formulas for $\cos(nx)$ and $\sin(nx)$. The mathematical constant e also can be written as the infinite limit of

$$\left(1+\frac{1}{n}\right)^n.$$

Mathematical induction and the binomial theorem, or the multinomial theorem, can be used to prove what is known as "Fermat's little theorem," named for mathematician Pierre de Fermat. This result in number theory states that if p is a prime number and n is an integer not divisible by p, then $n^p - n$ is divisible

Probability Theory

In probability theory, the binomial distribution uses binomial coefficients in the computation of probabilities. A binomial distribution is used to model a situation or process in which a series of independent trials occurs. Each trial may have only one of two possible outcomes, traditionally labeled "success" and "failure." In each trial, the chance of success or failure is constant, such as flipping a fair coin and getting a head, or rolling a fair die and getting a 6.

In this context, the binomial coefficient indicates the number of permutations there may be of a specific number of successes in a given number of trials; for example, the orderings of 2 heads and 8 tails in a series of 10 coin tosses. Mathematicians such as Jacob Bernoulli, Abraham de Moivre, Pierre de Laplace, Simeon Poisson, and Pascal worked on the binomial distribution and extensions, such as the limiting Poisson distribution. Mathematician and statistician Samuel Wilks, as well as others, developed the multinomial distribution to extend the binomial to cases with more than two possible outcomes on each trial.

by p. Fermat's little theorem is itself used in cryptography, providing an indirect application of the binomial theorem. One theorem in graph theory states that a graph with n vertices and adjacency matrix A is connected if and only if all the entries in the matrix $(1+A)^{n-1}$ are positive. This theorem is proved using the binomial theorem, generalized to certain matrices, and some basic graph theory results. Certain colorings of Pascal's Triangle produce fractal figures like Sierpinski's Triangle, named for mathematician Waclaw Sierpinski. In set theory, the regions of a Venn diagram for n distinct sets are in one-to-one correspondence with the binomial coefficients c_k for k ranging from 0 to n. Venn diagrams are named for John Venn.

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See Also: Arabic/Islamic Mathematics; Chinese Mathematics; Cubes and Cube Roots; Permutations and Combinations; Sequences and Series.

Birthday Problem

Category: Friendship, Romance, and Religion.
Fields of Study: Algebra; Data Analysis and Probability; Measurement; Number and Operations.

Summary: The Birthday Problem is a classic example of how probability can reveal counterintuitive truths.

The Birthday Problem is a classic probability problem first presented by mathematician and scientist Richard von Mises in 1939, though the fundamental combinatorial concepts involved can be traced back as far as India in the sixth century B.C.E. Today, it is one of the most-explored problems in classrooms, often introduced as early as the middle grades. The problem asks: Given that there is some number of people (n) in a room, what is the probability that at least two of them share the same birthday? One of the aspects that makes this problem so intriguing is that the answer is much different than people intuitively expect.

Solving the Birthday Problem

The extreme cases of the problem are easy to determine logically. If there are fewer than two people, then it is impossible to have two who share the same birthday, making the probability 0. If there are more people than days in a year, then at least two people must share the same birthday, making the probability 1. Von Mises assumed a fixed 365 days per year, ignoring February 29 as a possible birthday, so the probability is always 1 if there are 366 or more people. More interesting and challenging are the cases for which there are anywhere from 2 to 364 people. For the purposes of modeling and computation, it is assumed that it is equally likely that someone will be born on one day of the year versus another, so there is a

$$\frac{1}{365} \text{ chance}$$

that a person will be born on any particular day.

The Birthday Problem is solved using the mathematical ideas of permutations and combinations, and it is more easily approached if one asks a slightly different but complementary question: What is the probability that everyone in the room has a unique birthday? That is, that no one shares. If there are two people in the room, the first can be born on any of 365 days of the year and the second must be born on any of the remaining 364 days. If there are three people in the room where the first is born on a particular day, the second must be born on one of the remaining 364 days of the year and the third on one of the remaining 363 days. The probability is

$$\frac{365}{365} \times \frac{364}{365} \times \frac{363}{365}.$$

If there are four people, the probability is

$$1\times\frac{364}{365}\times\frac{363}{365}\times\frac{362}{365}.$$

This pattern can be generalized as

$$P_{(\text{no match})}(n)=\frac{{}_{365}P_n}{365^n}.$$

The probability that at least two people in a room of n people share a birthday is 1 minus the probability that there is no match, which can be used to generate the following probabilities.

Number of people in the room	Probability that two of these people share a birthday
2	.00274
10	.117
20	.411
23	.507
30	.706
50	.970
60	.995

When there are 23 people in the room, there is slightly more than a 50% chance that two people will share the same birthday, which answers the original question. The probability of at least one match increases quickly and nonlinearly with the number of people, so that when the number reaches 60 (well below the certainty value of 366 people), there is a 99.5% chance that there will be a match—*almost* certain. For example, as of 2010, there are six pairs of men who share a birthday among the 74 unique winners of the Academy Award for Best Actor. For women, there are three pairs among the 69 unique Best Actress winners.

Applications of the Birthday Problem

Applications of the Birthday Problem exist in many fields. One is called Class Phenotype Probability. Given six characteristics (blood type, RH positive/negative, sex, mid-digital hair positive/negative, earlobes attached/unattached, and PTC taste receptor), it is possible to determine the probability that a particular combination exists and also the probability that two people share the same combination. This possibility is quite valuable in medicine when considering the likelihood of finding matches between donors and recipients. In computer security, a birthday attack is a computationally intensive strategy used to break encrypted digital signatures. A "collision" occurs when different sets of data yield the same cryptographic hash value, which is a function of the input data. The attack repeatedly evaluates a hash-generating function using random inputs until the output creates a collision with the true hash value it seeks to duplicate. On average, $1.2 \times \sqrt{k}$ trials are needed to get a match, where k is the number of possible outputs (for example, a 64-bit hash value has about 1.8×10^{19} outputs). The birthday attack strategy becomes much less efficient as the hash length increases.

There are interesting extensions of the Birthday Problem based on slightly altering the question or assumptions. The first comes from considering the chance that *three* or more people share a birthday (or four, or five, and so forth). The Almost-Birthday Problem expands the problem to finding at least two people whose birthdays are within one day of each other. The Movie Line Problem states that the first person in a line for a movie whose birthday matches someone in front of them wins free tickets, and it seeks to find where someone should stand to have the best chance of winning. The Goldberg Extension computes the expected number of different birthdays in a group, while the Tuesday Birthday Problem is given as, "I have two children, one of whom is a boy born on a Tuesday. What is the probability that my other child is a boy?" Other variations assume unequal distributions of birthdays throughout the year. As with the original problem, solutions usually run contrary to most people's intuition. The ideas provide the basis for many applied investigations, such as the photon behavior modeling done by mathematical physicist Satyendra Nath Bose, after whom the subatomic particle "boson" is named.

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Lidia Gonzalez

See Also: Bar Codes; Coding and Encryption; Permutations and Combinations; Probability; Statistics Education.

Black Holes

Category: Space, Time, and Distance.
Fields of Study: Geometry; Measurement; Number and Operations; Representations.
Summary: Black holes were implied by Einstein's general relativity and have challenged physicists' theories since.

A black hole is a finite region of space during a period of time (called space-time) subject to a singularity caused by a large concentration of mass in its interior. This massive object generates a gravitational field so powerful that atoms are compacted in super-high densities, which in turn increases the gravitational pull. A singularity is created in space because no particle of matter, not even light photons, can escape from that region. Hence the name: a black hole is an invisible region because it does not reflect any light (all light is absorbed). Many aspects of black holes can be described and studied using algebraic and geometric concepts, but the existence of black holes is still under debate. For example, Australian mathematician Stephen Crothers argues that black holes are inconsistent with general relativity and critiques the mathematics used by others to demonstrate their existence. It is believed that black holes originate when stars runs out of gas needed to maintain their temperature, causing a decrease in volume. As volume decreases, the proximity of particles increases the gravitational pull in a positive feedback loop; as particles get closer, the gravitational force keeps increasing. This compaction process continues until a singularity, called the "event horizon," is created. The event horizon is defined as a boundary in space and time beyond which events cannot affect an outside observer. The event horizon separates the black hole region from the rest of the universe and is the boundary of space from which no particle can leave, including light.

The singularity caused by a black hole is considered as a curvature in space-time. This curvature is explored by Albert Einstein's general relativity theory, which predicted the existence of black holes—though Einstein himself did not believe in them. In the 1970s, Stephen Hawking, George Ellis, and Roger Penrose proved several important theorems on the occurrence and geometry of black holes. Previously, in 1963, Roy Kerr had shown that black holes in a space-time have an almost-spherical geometry determined by three parameters: their mass, their total electric charge, and angular momentum.

It is believed that at the center of most galaxies, including the Milky Way, there are supermassive black holes. The existence of black holes is supported by astronomical observations, in particular through the emission of X-rays. Some black hole candidates have been identified experimentally using observations and data. There are different types of black holes, such as rotating black holes and stationary black holes, and these are described by using various metrics in physics and differential geometry.

Origins of Human Awareness of Black Holes

The concept of a body so dense that even light could not escape was described in a paper submitted in 1783 to the Royal Society by an English geologist named John Michell. By then, Isaac Newton's theory of gravitation and the concept of escape velocity were well known. Michell computed that a body with a radius 500 times that of the sun and the same density, would, on its surface, have an escape velocity equal to that of light and would therefore be invisible.

In 1796, the French mathematician Pierre-Simon Laplace explained in the first two editions of his book *Exposition du Système du Monde* the same idea; however, the concept that light was a wave without mass and therefore unaffected by gravitation was prevalent in the nineteenth century, and Laplace discarded the idea in later editions.

In 1915, Einstein developed his general relativity theory, and showed that light was influenced by the gravitational interaction. A few months later, Karl Schwarzschild found a solution to Einstein's equations, where a heavy body would absorb the light. We now know that the Schwarzschild radius is the radius of the event horizon of a black hole that will not turn, but this was not well understood at the time. Schwarzschild himself thought it was just a mathematical solution, not physical. In 1930, Subrahmanyan Chandrasekhar showed that any star with a critical mass (now known as the Chandrasekhar limit) and that does not emit radiation would collapse under its own gravity. However, Arthur Eddington opposed the idea that the star would reach a size zero, implying a naked singularity of matter; instead, the black hole should have something that will inevitably put a stop to collapse, an idea adopted by most scientists.

In 1939, Robert Oppenheimer predicted that a massive star could suffer a gravitational collapse and therefore black holes might be formed in nature. This theory did not receive much attention until the 1960s because after World War II he was more interested in what was happening at the atomic scale.

In 1967, Stephen Hawking and Roger Penrose proved that black holes are solutions to Einstein's equations and that in certain cases the creation of a black hole is the inevitable consequence of a star aging. The black hole idea gained force with the scientific and experimental advances that led to the discovery of pulsars. Soon after, in 1969, John Wheeler coined the term "black hole" during a meeting of cosmologists in New York, to designate what was formerly called "star in gravitational collapse completely."

The Entropy of Black Holes

The mathematical tools used to model black holes use fundamental laws of physics, particularly relativity and thermodynamics. According to initial theories by Stephen Hawking, black holes violate the second law of thermodynamics (the entropy, or disorder, of isolated systems tend to increase over time), which led to speculations about travel in space-time wormholes (tunnels that would allow time travel or fast travel over very long distances). Hawking has recanted his original theory and has admitted that the entropy of the matter is kept inside a black hole. According to Hawking, despite the physical impossibility of escape from a black hole, it may end up evaporating by constant leakage of X-ray energy that escapes the event horizon, called Hawking radiation.

An artist's concept chronicles a time lapse from left to right of an intact sun-like star (left) coming too close to a black hole (right) and its self-gravity becoming overwhelmed by a black hole's gravity.

According to this model, black holes have intrinsic gravitational entropy, which implies that gravity introduces an additional level of unpredictability over the quantum uncertainty. It appears, based on the current theoretical and experimental capacity, as if nature took decisions by chance or, more generally, far from precise laws.

The hypothesis that a black hole contains entropy and, furthermore, it is finite, required to be consistent with such holes emitting thermal radiation, at first seems contradictory. The explanation is that the radiation escapes the black hole in such a way that an external observer knows only the mass, angular momentum, and electric charge. This means that all combinations or configurations of radiation of particles having energy, angular momentum, and electric charge are equally likely. Physicists such as Jacob D. Bekenstein have been linked to black hole entropy and information theory.

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Juan B. Gutierrez

See Also: Astronomy; Einstein, Albert; Temperature.

Blackmun, Harry A.

Category: Government, Politics, and History.
Field of Study: Connections.
Summary: Harry A. Blackmun was a U.S. Supreme Court justice who applied mathematical logic in his judicial career.

Harry A. Blackmun (1908–1999) is best known as the author of the majority opinion in *Roe v. Wade*, the U.S. Supreme Court case that recognized a constitutional right to abortion. Blackmun, however, was perhaps the only Supreme Court justice to hold a degree in mathematics (A.B., Harvard, 1929), and one of the very few who has carefully applied mathematical concepts in judicial opinions.

Blackmun graduated from Harvard Law School in 1932 and served as law clerk to Judge John B. Sanborn of the U.S. Court of Appeals for the Eighth Circuit. His law practice, with a prominent Minneapolis law firm and the Mayo Clinic, focused primarily on tax law and estate planning. He succeeded Judge Sanborn on the Eighth Circuit in 1959 and was appointed to the U.S. Supreme Court by President Richard M. Nixon in 1970 following the Senate's rejection of two previous nominees. Both of Blackmun's judicial appointments were promoted by his childhood friend, Chief Justice Warren E. Burger. So close was this connection that for their first decade on the Supreme Court, the two men were referred to as the "Minnesota Twins," but thereafter they went separate jurisprudential ways.

Blackmun and Mathematical Reasoning

A particularly striking example of Blackmun's use of mathematical reasoning involves his application of the binomial distribution as a method for assessing discrimination claims. In *Castaneda v. Partida* (1977), a Texas county was accused of systematically discriminating against Mexican Americans in the selection of grand jurors. Mexican Americans constituted nearly 80% of the population but only 39% of the grand jurors during the 11-year period at issue. Blackmun's opinion for the Court noted the substantial absolute disparity but used the binomial distribution to explain the unlikelihood that such a disparity would have arisen by chance. Eschewing mathematical symbols, he verbally explained the calculations involved and the formula for making them. The difference between the observed and the expected number of Mexican-American grand jurors during this period was approximately 29 standard deviations, a result that would occur by chance less than once in 10^{149} times. The use of statistical evidence has since become standard practice in discrimination cases.

Justice Blackmun also relied heavily on empirical evidence in *Ballew v. Georgia* (1978), a case that established the minimum constitutionally acceptable size for juries. American juries traditionally had 12 members, but—in a case considered shortly before Blackmun took his seat—the Supreme Court held that six-member juries were permissible. *Ballew* held that a five-person jury was too small to satisfy the requirements of due process. Blackmun's opinion on this issue spoke for all of his colleagues, some of whom wrote separately on other issues. Engaging with a substantial body of statistical and experimental research, he observed that smaller juries are less likely to engage in effective deliberations, take diverse viewpoints seriously, and reflect a fair cross-section of the community than 12-person juries and that smaller juries are more likely to reach inaccurate judgments. Blackmun conceded that the differences between five- and six-person juries might be difficult to discern empirically, but he concluded that a line had to be drawn somewhere to avoid further reductions in jury size. The data actually raised troubling questions about the earlier decision upholding six-person juries, but the Court was unwilling to overrule that precedent.

Finally, Blackmun addressed statistical issues relating to the imposition of the death penalty, both on the Supreme Court and on the court of appeals. On the Eighth Circuit, in *Maxwell v. Bishop* (1968), Blackmun rejected a statistical study purporting to show that African Americans were much more likely than whites to get the death penalty in rape cases. The study was limited in scope, did not relate to the county where the case arose, failed to account for relevant variables, and did not show that racism affected the verdict or sentence in that case. The Supreme Court set aside this decision because of an intervening ruling on a different issue.

Two decades later, in *McCleskey v. Kemp* (1987), Blackmun dissented from a decision that rejected a claim of racial discrimination in a Georgia death penalty case. Blackmun explained that this claim was supported by a multiple regression analysis of every homicide case in that state during the relevant time period. The study, which included 230 variables, found that persons accused of killing whites were 4.3 times as likely to receive a death sentence as persons accused of killing African Americans and that African-American defendants were significantly more likely to be sentenced to death than white defendants. Blackmun emphasized the "sophistication and detail" of this study and concluded that it showed an unacceptable risk that racism had affected the decision to impose the death sentence.

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Jonathan L. Entin

See Also: Data Analysis and Probability in Society; Number and Operations in Society.

Blackwell, David

Category: Mathematics Culture and Identity.
Fields of Study: Algebra; Data Analysis and Probability; Number and Operations.
Summary: Statistician and game theorist David Blackwell (1919–2010) became one of the most esteemed African-American mathematicians.

David Harold Blackwell was one of the most famous American mathematicians. He was Professor of Statistics at the University of California at Berkley.

Early History

David Blackwell was born on April 24, 1919, in Centralia, Illinois, and was an African American. David was the oldest of four children in his family. His father worked for the Illinois Central Railroad looking after the locomotives, and his mother looked after the family. David attended the integrated elementary school rather than the existing segregated school for African Americans in Centralia. He said, "I had no sense of being discriminated against. My parents protected us from it and I didn't encounter enough of it in the schools to notice it."

Blackwell enjoyed geometry very much. In high school, he applied his mathematical skills to games. His interest in mathematics continued to grow after entering the University of Illinois, in 1935, at age 16. In a course on real analysis, he was especially interested in calculus, and he was excited by Newton's method for solving equations. The course on real analysis turned him on to a career in mathematics. He remarked upon the course, saying, "That's the first time I knew that serious mathematics was for me. It became clear that it was not simply a few things that I liked. The whole subject was just beautiful."

During his study at the university, his father had to borrow money to finance his education. David took jobs such as dishwashing to help earn money and, at the same time, he took courses over the summers and was able to graduate with a B.A. in 1938. After graduating, Blackwell continued to study at the University of Illinois for his master's degree, which he was awarded in 1939, and then for his doctorate, supervised by Joseph Doob. This was awarded in 1941 when Blackwell was only 22 years old. The dissertation title was "Some Properties of Markoff Chains." After that, Blackwell received a one-year appointment as a Rosenwald Postdoctoral Fellow at the Institute for Advanced Study in Princeton. This appointment caused some turmoil, of which he was not fully aware, because he was an African American.

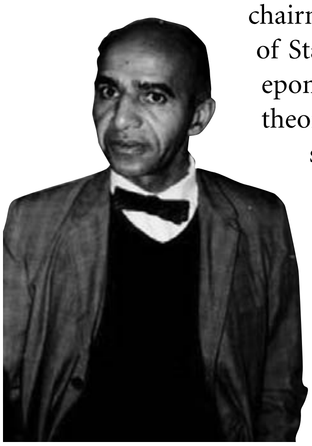
Personal and Professional Life

From 1942 to 1943, he had a post at the Southern University at Baton Rouge, followed by a year as an instructor at Clark College in Atlanta. Blackwell was appointed as an instructor in 1944 at Howard University and, after only three years, he was promoted to full professor and head of the Department of Mathematics.

In 1944 he married Ann Madison, with whom he had eight children.

He left Howard University in 1954 to take up a professorship at the University of California at Berkeley, where he taught students up to his retirement, and, after that, as Professor Emeritus.

During three summers between 1948 and 1950, he worked at the Research and Development Corporation (RAND). At about the time of his arrival and the beginning of his work at Berkeley, Blackwell's interests turned toward statistics, and he became a theoretical statistician. In 1956, he became chairman of the Department of Statistics. He was one of the eponyms of the Rao–Blackwell theorem, a famous theorem in statistical theory. Thanks to him, the areas of game theory and topology were connected as well.



David Blackwell photographed in Seattle in 1967.

Awards and Honors

Blackwell received honorary Doctorate of Science degrees from 12 universities. He was selected to be president of the Institute of Mathematics Statistics, the International Association for Statistics in Physical Sciences, and the Bernoulli Society. He was vice president of the International Statistical Institute, the American Statistical Association, and the American Mathematical Society. He was elected to the National Academy of Science, the first African-American mathematician to do so. He was also elected to the American Academy of Arts and Sciences. He held memberships in numerous professional organizations, including being a life member of National Association of Mathematicians. He was an Honorary Fellow of the Royal Statistical Society.

David Blackwell lived in Berkeley until his death in July 2010.

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Biljana Popovic

See Also: Game Theory; Probability; Statistics Education.

Board Games

Category: Games, Sport, and Recreation.
Fields of Study: Algebra; Data Analysis and Probability; Geometry.
Summary: While some games are explicitly mathematical, others are implicitly governed by math.

Humans have been playing games for as long as they have been around. Johan Huizinga was the first to call the attention to the fact that play precedes culture.

Board games, a very organized form of play, are part of human social nature. Human communities may differ in many ways, but they all play games. From the ancient Mancala, practiced for millennia in Africa, to our Monopoly, we find board games in many societies. Besides their cultural relevance—they are studied by anthropologists, historians, and others—board games are characterized by their sets of rules, which show mathematical structures and connections that are at times very surprising.

Game Classifications

Chess and Go come to mind as examples of traditional board games, and Monopoly and Scrabble are examples of proprietary games. The distinction between the two types of games is not always easy to identify. In chess, the movements of the pieces and the other rules are the main considerations. Chess is an abstract game, not considering the fact that it originally emulated a battle between two armies. Chess does have similarities with other games. When playing representational games like Monopoly or Diplomacy, players find themselves focusing on the possibilities and strategic choices, forgetting the particular settings. Accoring to David Parlett, positional games refer to games where pieces are played in a board or any other set of markings, as chess, checkers, and Go, and "theme" games are generally representational and commercial, like Monopoly and Diplomacy.

Board game classification has been inspired in the fact, first noted by H. J. R. Murray, that games are typical of early activities of man—the battle, the siege, the race, the hunt, alignment, arrangement, and counting. Parlett's classification, which evolved from Murray's and others, is as follows. In race games, the board is a linear track where each player tries to be the first to reach a particular cell or remove a set of pieces from the board. Most of the games under this category use dice or other randomizing devices, like Chutes & Ladders, Ludo, and Backgammon, but not all, such as Hare & Tortoise.

Space games, typically two-dimensional and free placing, comprise the alignment games, as Nine Men Morris; connection games, as Hex and Twixt; traversal games, in which a player tries to have one or several pieces cross the board, as Breakthrough, Halma, and Chinese Checkers; configuration games, where players try to achieve certain displays with their pieces, as Agon; restriction games, where the aim is to try to block the adversary, like Pentominoes; and occupation games, in which the winner is the player who achieves more space in the board, as in Go and Othello. Chase games are asymmetrical, one player having several pieces while the other has only one or two. Their goals are also distinct, as in Fox & Geese. "Displace games" include chess and checkers, where a player aims at capturing most of his opponent's pieces (as in checkers) or a particular one (as in chess), and other war games; the family of Mancala games belongs also to this class.

History

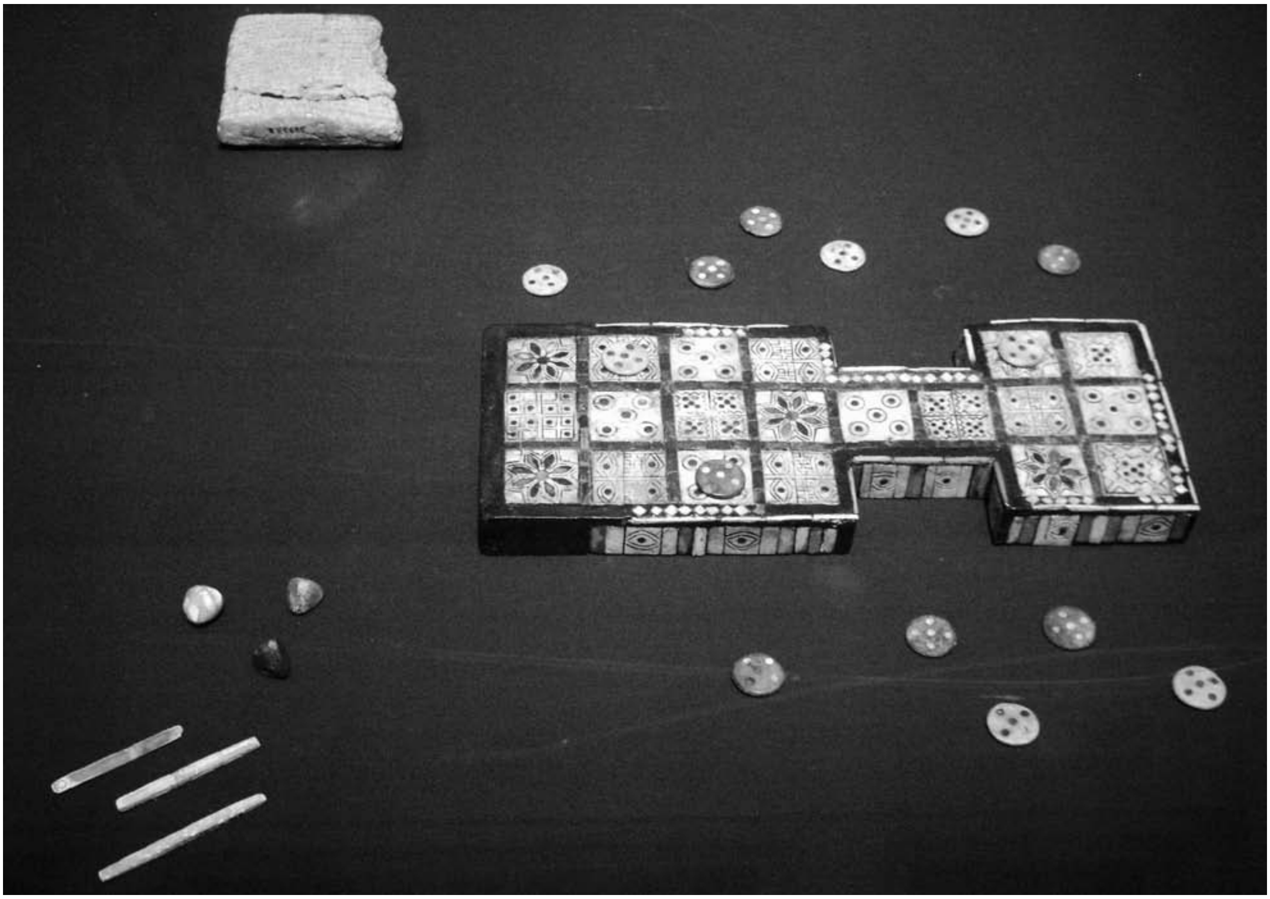
The Royal Game of Ur, also known as the Game of Twenty Squares, was found in the south of Iraq in the 1920s and is about 4500 years old. The board shows twenty squares, 12 in a three-by-four rectangular array, six in two rows of three, and two connecting cells. The reverse of the board corresponding to the 12 cells showed a zodiac, illustrating that in the past, the same object could be a board game and a divinatory device. Two cuneiform clay tablets give the exact rules for this game. Each player had seven pieces, which moved across the board according to the toss of three tetrahedral dice.

A similar game is found in Ancient Egypt, Senet or the Game of Thirty Squares. It was a race game as well, but it was more than a simple toy. In funerary monuments that date from 4000 years ago, images are shown of the deceased playing Senet against an invisible adversary. Osiris, which is present but not shown, decides on matters of life after death.

The Royal Game of Ur and Senet can be viewed as the oldest relatives of the modern Backgammon, a game in which the moves are decided by the players upon tossing two cubic dice. The player who better understands the probability laws that rule the dice is most often the winner.

The Chinese game Go is four millennia old. Nowadays, it remains one of the most complex games, despite the simplicity of its rules. Go is played on the intersections of a 19-by-19 grid, and each player fights to control the largest area.

Pure strategy games could also be found in Ancient Greece, like Petteia. This game, and the Roman Ludus Latrunculorum, shared the shape of the board, checkered, and the orthogonal movement of the pieces.



Two ancient copies of the Royal Game of Ur exist; one is exhibited in the British Museum in London (above). The game dates back to Iraq in 2600 B.C.E. and is thought to be oldest board game set ever found.

Chess, which originated in India about 1400 years ago, traveled to the West with the Arabs, and saw its rules evolve in the process. It was originally created as a war game between two armies, and its pieces represented the actors of the battle. However, the abstract shapes that reached Europe gave way to the symbolic representation of the European medieval society.

The Arabs introduced several other games in Europe. One game they introduced, Alquerque, was played on the intersections of a five-by-five lined board. The adaptation of this game to the chessboard originated the game of Checkers.

Board Games and Mathematics

The oldest known pedagogical game is Rithmomachia, also known as Philosopher's Game. It was invented in the eleventh century as a didactical device to teach mathematics. It was practiced wherever Boethius's arithmetic was taught. Pythagorean in nature, this tradition of mathematics dominated teaching at churches and universities for more than 500 years. In an eight-by-16 board, two armies fought each other. Pieces carried numbers and could have one of three shapes: circular, triangular, or square.

The movements depended on the shape of the piece played; the captures depended on the numbers and on arithmetical calculations. Victory was attained by means of a configuration of pieces holding numbers in progression (arithmetic, geometric, harmonic, or combinations of the three). This game spread throughout Europe, and only when the mathematical curriculum at universities changed in the sixteenth century did it vanish. Losing its pedagogical goal turned out to be fatal, as Rithmomachia lacked the qualities to survive as a purely recreational activity. Chinese scholars of the eleventh century also published work on permutations based on the Go board. John H. Conway's twentieth-century research on the game contributed to the invention of surreal numbers and the development of combinatorial game theory.

Ludus Astronomorum was a board game for seven players based on Ptolemaic astrological principles. In the sixteenth century, William Fulke, a professor at Cambridge who had written a manual of the Philosopher's Game, created two other games. One, intended to improve on the astronomy game, was Ouranomachia, the other, created to teach geometry, was Metromachia. Fulke published one book on each.

In the eighteenth century, George Berkeley invented a game to help teach algebra, a subject Berkeley had in very high consideration. The game was Ludus Algebraicus and essentially functioned as a randomizing device to generate algebraic equations.

Charles Dodgson invented a game in the nineteenth century to practice logical deduction and wrote a book about it, *The Game of Logic*, under his pen name, Lewis Carroll.

In Ireland, mathematician William Hamilton created in 1857 the Icosian Game and soon after Traveller's Dodecahedron. This comprised a dodecahedron and a piece of thread that should touch every vertex according to some rules. It was this game that gave rise to the concept of Hamiltonean Graph.

The familiar game of Nim in which a move consists of choosing from one of a pile of beans and reducing its cardinality, was first solved mathematically at the beginning of the previous century. In its normal form, where the winner is the one who takes the last bean, is the paradigm of a class of games studied in Combinatorial Game Theory. The familiar children's game Dots & Boxes was also treated mathematically with the same techniques. Some traditional games, like Konane, can be approached the same way.

The game Hex was invented independently by both Piet Hein and John Nash in the 1940s. It is a connection game played on a diamond-shaped board of hexagonal cells. David Gale noted that a game of Hex can never end in a tie, and that this fact is logically equivalent to a deep theorem in topology.

Abstract games with complete information and no chance devices are also called mathematical games. The mental processes present in their practice and in a typical mathematical activity, like problem solving, are far from disjointed.

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Jorge Nuno Silva

See Also: Dice Games; Mathematical Puzzles; Puzzles; Video Games.

Body Mass Index

Category: Medicine and Health.
Fields of Study: Algebra; Number and Operation.
Summary: Body Mass Index is a statistically useful index of a person's relative weight.

Body mass index (BMI) is an index of the relative weight of a person. In other words, it is an estimate of a person's weight, adjusted for height. The formula for calculating BMI is

$$\text{Weight in kilograms}/(\text{Height in Meters})^2.$$

The equation implies that, holding other factors constant, weight is proportional to height squared, at each level of height. This equation is counterintuitive because of the common assumption that to calculate mass or volume, a cubic function is necessary. In fact, dividing mass by height cubed was historically a popular method, called the "Ponderal index." However, Adolphe Quetelet (1796–1874) observed that, for an "average man," a squared function was a better fit to the data. With increasing age, humans' height increases at a faster rate than width. Taller adults tend to be slimmer—waistlines do not usually grow in proportion to increasing height. Quetelet observed that people do not expand equally in all three dimensions. If they did, then the Ponderal index would still be valid. In reality, he noted that, "weight increases nearly as the square of the height"—particularly between puberty and age 25. Before puberty, the Ponderal index does increase more proportionally to width. Quetelet stressed that there was considerable variance in body shape and size, which are determined by biological, psychological, and social factors. For example, he noted that "young persons who apply themselves to study, and persons in the affluent classes generally, are taller than others."

The Quetelet Index

The popularity of the Quetelet index increased following World War II, when epidemiological evidence began to accumulate that excess weight was a risk factor for premature mortality and morbidity. Historically, and in several cultures today, excess weight (corpulence) was considered healthy and desirable. Given this new evidence, actuaries needed a quick and reliable way to predict who might be most at risk, so that insurance premiums could be loaded against those with excess weight. They creating height-weight charts, based on Quetelet's data, which provided the typical weights expected at various levels of height for the average person—assuming that they were age 25. The index was later termed "body mass index."

Insurance companies, clinicians, and researchers began using BMI as a proxy variable for measuring excess weight at all ages, not simply as an index of relative weight at age 25, as originally intended. The popularity of BMI remains today. For example, the World Health Organization uses BMI in its definition of obesity, where "overweight" is defined as BMI equal to or greater than 25, and obesity is defined as a BMI more than 30. There are no agreed BMI cutoffs for childhood obesity in the same way that there are for adults. The categorization of continuous data (for example, overweight, obese) is controversial because it results in a loss of information. BMI is indeed a risk factor for chronic diseases, despite its usage deviating from the original intended purpose. In clinical settings, BMI is usually supplemented with other information regarding disease risk, such as blood pressure or lifestyle factors including cigarette smoking. Additionally, it may be necessary to take into account whether the person has an ectomorphic, mesomorphic, or endomorphic body type.

The Quetelet index was first formally evaluated by epidemiologists working on data from a large cohort study, called the Framingham Heart Study. They noticed that Quetelet's index was being widely used as an indicator of excess weight, not simply weight adjusted for height as it was originally intended. The epidemiologists wanted to evaluate the validity of this assumption, by comparing different methods for measuring relative weight against three criteria:

1. The proxy should not correlate with height.
2. The proxy should correlate highly with skinfold thickness measurements, since these are valid proxies for the thickness of the subcutaneous fat layers in different parts of the body, in turn.
3. The proxy should be easy to calculate.

After analyzing the data, they concluded that Quetelet's index was indeed the best available measure, and renamed it the Body Mass Index. However, it should be noted that correlations between BMI and skinfold thickness measurements varied considerably, and the highest was 0.8. The researchers noted that if height and weight are the only data available, excess body fat is unlikely to be measured in a satisfactory way. The lower a correlation between a proxy variable and the variable it is intended to measure, the less well that proxy will correlate with health outcomes. It should also be noted that the third criterion (the ease of calculation) is not statistical—the validity of an index or test should be based on how well it performs against a gold standard, not simply because it is easy to use.

Criticisms

Researchers have since argued that valid proxies for excess body fat should take into account its distribution in the body. Excess fat in the abdominal region (visceral fat) is a risk factor for metabolic diseases, regardless of total fat volume in the body. Waist circumference correlates highly with visceral fat, leading some researchers to suggest that waist circumference is a better proxy for excess weight than BMI. Similar alternatives include the ratio of waist circumference to hip circumference (waist-to-hip ratio). Because waist circumference is associated with increased morbidity and mortality risk, holding BMI constant, it provides additional information that is not captured by BMI. Both are considered independent risk factors, such that it may be necessary to measure both BMI and waist circumference. In fact, a consensus statement from Shaping America's Health concluded that waist circumference predicted cardiometabolic outcomes, and should therefore be measured in clinical settings as a matter of routine. However, waist circumference is difficult to measure reliably. BMI remains a useful index for many purposes.

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Gareth Hagger-Johnson

See Also: Cubes and Cube Roots; Growth Charts; Measurement in Society; Normal Distribution; Squares and Square Roots.

Brain

Category: Medicine and Health.
Fields of study: All.
Summary: The brain is studied through models and through algorithm-dependent medical technology. The neurology of mathematical thought is a vibrant field.

The applications of mathematics to the study and understanding of the brain have been varied and widespread. They include models to predict the start of seizures using dynamical systems; maps of the brain using projective, hyperbolic, or other geometries, as well as graph theory; applications of morphometrics, which is the statistical study of shapes, to schizophrenic brains; and dynamic simulations and visualizations of electrochemical activity in neurons. Other models are used to study how electrical signals propagate along nerve cells and the way in which electrical discharges in nerve cells tend to synchronize and form waves. Medical technology used in brain treatment and studies uses mathematical algorithms; for example, to create and process computer-generated images of brain cells, as well as to measure functions like blood flow, glucose consumption, and electrical activity. Mathematics is also important in modern medical devices that involve nerve fibers within or leading to the brain, such as cochlear implants. How mathematical thought arises in the brain—from arithmetic to abstract thinking—is also of great interest. Mathematics and the Brain was the theme of Mathematics Awareness Month in 2007.

Brain Composition and Structure

Before proceeding with some applications of mathematics in the study of the brain, it is important to have an idea of brain composition and structure. In humans, this complex organ consists of perhaps 100 billion nerve cells (or neurons), with roughly a total of 100 trillion connections between neurons (or synapses). Although some nerve cells do regenerate, and new connections between nerve cells are made, overall these numbers tend to decline after birth. Even with advances in computer processing and storage, the sheer number of neurons and connections hints at the enormous scope of the problem inherent in understanding the brain. By comparison, the nematode *Caenorhabditis elegans* has 959 cells in the entire organism, 302 being nerve cells, which result in over 5000 connections between neurons. Even for something of this vastly smaller scale, the nematode's neural connections were initially mapped after more than 10 years of effort by the mid-1980s, and earned a Nobel Prize for Sydney Brenner—who famously called *C. elegans* "nature's gift to science." Those results have since been updated.

A single neuron generally consists of: a main cell body (or soma); many filamentous dendrites, which are where signals from other neurons are usually received; and a single axon, which typically communicates to the dendrites of other neurons. The electrical voltage across the neuron's cell membrane varies as the concentrations of calcium, sodium, potassium, and chloride ions fluctuate, producing a fluctuating electrical signal. The electrical signal is transferred from one neuron's axon to another's dendrite across a gap known as a "synapse." Some synapses, known as "electrical synapses," involve a direct channel that connects the two cells' cytoplasm and allows for very fast electrical transmission. By contrast, chemical synapses involve molecules known as "neurotransmitters," which mediate signal transmission. The human brain utilizes more than 100 types of neurotransmitters. However, just two of these types arise at the vast majority of synapses; namely, glutamate and gamma aminobutyric acid. In addition to neurons, glial cells serve various support functions for neurons. One important function of special kinds of glial cells—namely, special kinds of oligodendrocytes—is the myelinization of axons. Myelin, a fatty substance, essentially electrically insulates neurons. Because they are pinkish white, and white when stored in formaldehyde, bundles of myelinized axons make up what is known as "white matter" in the brain. On the other hand, "grey matter," as seen on the surface of the cerebral cortex in a typical brain slice image, comprises of the soma, dendrites, and other kinds of glial cells, such as astrocytes. While *C. Elegans* has fewer than 60 glial cells, the human brain likely has at least as many glial cells as neurons, although the ratio varies widely in different brain regions.

Applications of Neural Networks

How neurons collectively convey information is also of much interest to researchers. Interestingly, attempts to model so-called "artificial neural networks" have led to highly useful algorithms used in many areas of mathematics, science, and engineering in their own right, having nothing to do with the study of the brain. Computers are often "trained" with data sets using such neural networks to help process data. Neural networks can be found in software used in fields as varied as financial analysis and fraud detection, robotics, handwriting analysis, and voice recognition. As another example, much mathematics is used in processing and analyzing the enormous amount of neuron image data, and neural network algorithms are now being used to help automate that processing to help computers track neural connectivity.

Brain Mapping and Study

Mathematics also has been used to help in producing accurate maps of the cortex of various parts of the brain. The extensive folding in the human brain in the cerebral cortex, which produces peaks or ridges (or *gyri*) and valleys or furrows (or *sulchi*) makes it difficult to compare two different brain surfaces. A calculus-based geometry is used to find effective maps. As another example of an application, mathematics is used extensively in devices such as cochlear implants, useful to deaf individuals who still have a functioning auditory nerve. In humans, as many as 30,000 individual nerve cells in the inner ear pass through the auditory nerve to the brain. Different sound frequencies innervate different nerve cells; roughly speaking, lower frequencies innervate nerve cells in the basilar membrane closer to the beginning of the cochlea, as opposed to higher frequencies innervating cells further along. But the precise mapping of which cells are affected by which frequencies follows a logarithmic mathematical pattern, as a function of distance in the cochlea. Using various radio signal technologies, external sounds are transmitted to a receiver in the inner ear, which connects to implanted electrodes for nerve innervation. There, mathematics is used in the computational processing to convert the received frequencies into the appropriate electrical innervations, so that only certain nerve cells are stimulated for certain frequencies.

Examples of mathematics applied to the study of the brain abound in the five-year, National Institutes of Health–funded Human Connectome Project. This project, somewhat analogous to the Human Genome Project, was funded in 2010 for approximately $40 million. Mapping all the connections between neurons in the human brain in a meaningful way is the goal of the Connectome project. One component involves constructing connection data from 1200 individuals, including numerous twins. Developing effective ways to collect the data set, as well as analyze the results, involves several areas of mathematics in crucial ways. First, instruments must be able to create high-resolution images of the brain tissue of living humans in a completely noninvasive way. Next, the enormous image data must yield to automated computer analysis that can determine the actual neural connections within the brain. Finally, the connection data set must be amenable to meaningful analysis by researchers interested in understanding normal brain processing as well as diseases. At each stage, mathematics plays a crucial role.

Brain Imaging Technologies

Magnetic resonance imaging (MRI) is commonly used today for noninvasive imaging of the internal structure of the human body; for example, to help determine if knee surgery or back surgery is warranted. In standard MRIs, a powerful magnetic field changes rapidly, and, by doing so, it manipulates the minute magnetic fields produced by protons in water molecules inside the body—a weak signal can be detected externally from the protons being flipped around by the strong magnetic fields. From these weak and indirect measurements, solving the inverse problem using mathematics related to calculus is used to create what appear as two-dimensional slices through the body. In the case of brain studies, the resolution of standard MRIs is adequate to see tumors but is too crude to see individual neurons or even to effectively track bundles of neurons. Since myelin is a fatty substance, water outside neurons will generally not diffuse into axons; rather, this water will tend to diffuse along the length of axons—the water percolates along axons or white matter.

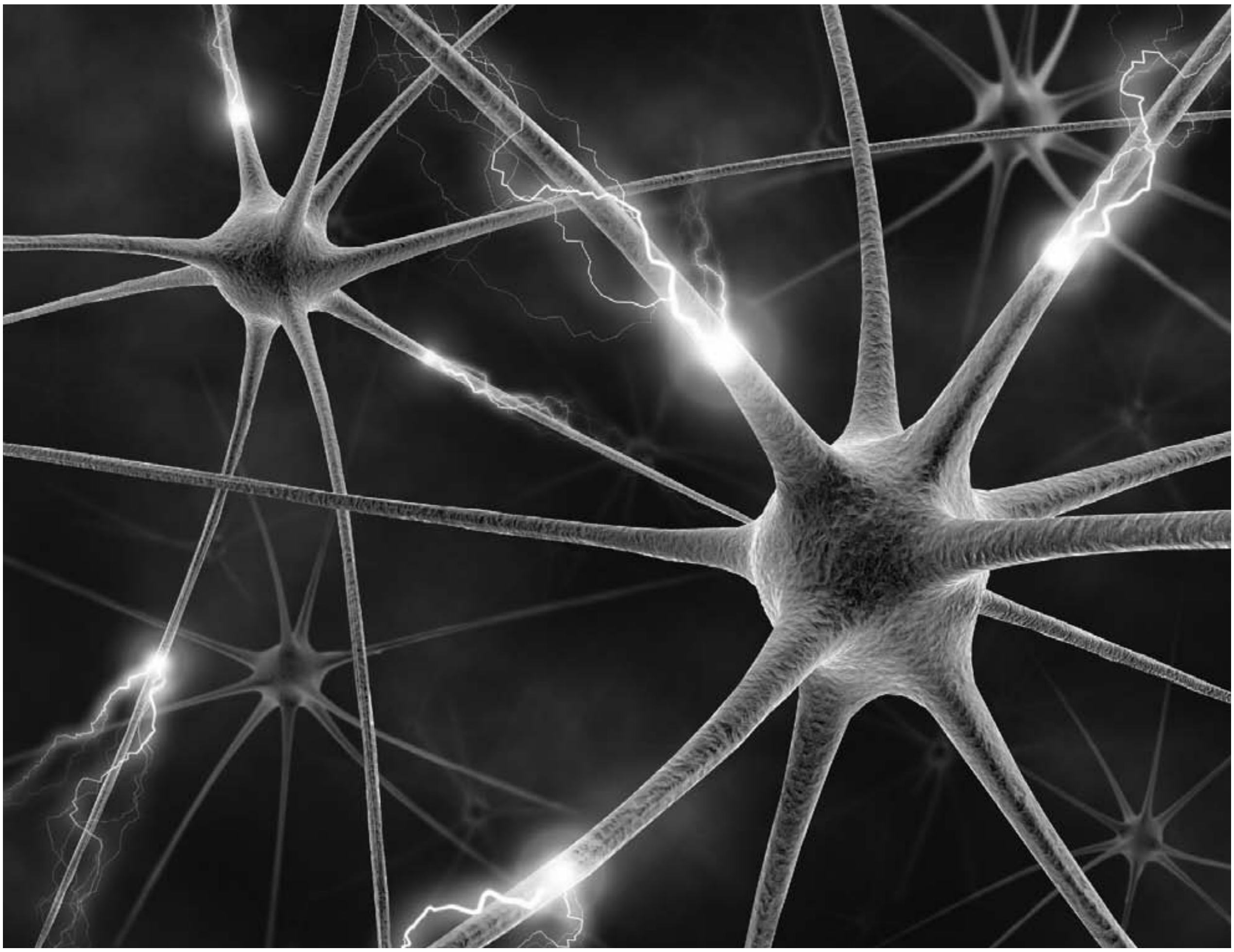
Diffusion tensor imaging (DTI) uses a variation of a standard MRI to determine the diffusion direction, and hence determine bulk nerve fibers. While DTI can produce high-resolution images of nerve fibers, difficulties arise when fibers cross. Water diffusion in this case can now take multiple paths at the crossing points, and it is thus difficult to track nerve fibers at these crossing points. Diffusion spectrum imaging (DSI) involves more mathematics that determines more precisely how water diffuses and is not limited to thinking that water diffuses in only one direction. Roughly speaking, the mathematics is a mixture of calculus and statistical ideas, and it is interesting that two-dimensional ellipses and three-dimensional ellipsoids play a role in the mathematics of DTIs and DSIs. The resulting images of nerve fibers are visually striking.

Not all techniques for imaging neurons rely on such indirect approaches as conventional MRI, or the MRI-based DSI and DTI. Recall that those techniques are used primarily for imaging nerve fibers, not individual neurons. Techniques for higher-resolution imaging of actual neurons are somewhat direct. Jeff Lichtman and others developed the use of genes encoding three proteins that fluoresce in, essentially, the colors red, blue, and green. Genetically modifying mice with these genes, as well as an enzyme that randomly arranges the genes amongst neurons, allows for mice neurons to appear in one of now approximately 150 colors, creating what Lichtman has termed a "brainbow." Another approach relies on a genetically modified version of the rabies virus, which ordinarily is well suited for traveling from neuron to neuron on its journey to the brain. By tagging the modified virus with a fluorescent molecule, one obtains bright images of neurons connecting to just one other neuron to further aid in understanding neural connectivity. All in all, exceptionally striking images are displayed in many places including on the Internet. While the imaging of individual neurons is often more direct and makes less use of mathematics than the MRI approaches to imaging nerve fibers, much mathematics is subsequently used in automating the process of tracking neurons and nerve fibers and, ultimately, the connections found might be described and analyzed by an area of mathematics known as "graph theory."

The approaches to imaging nerve fibers discussed above, such as DTI and DSI, rely on MRI instruments; that is, they rely on indirect methods involving minute variations in very small magnetic fields from protons that are assailed by powerful externally generated magnetic fields. They use indirect information coming from throughout the local environment inside the brain, and mathematics is crucial to inverting the recorded data to recover what is going on at a particular location in the brain. But not all imaging inside the brain focuses just on the connections between individual neurons or bundles of neurons. Other areas of interest include determining which parts of the brain are stimulated at which times by which activities. Here, other inversion processes are used to see what is happening in the brain, including functional MRI (fMRI), positron-emission tomography (PET), and electroencephalograms (EEG). Blood oxygen level dependence (BOLD) uses MRI technology that takes into account the very slight differences in the magnetic fields from water molecules in blood, depending on whether the blood is carrying oxygen. Hemoglobin bound to oxygen is diamagnetic, essentially repelled by a magnetic field. Hemoglobin without bound oxygen is paramagnetic, or attracted to a magnetic field. In either case, it affects the overall magnetic fields from the water molecules. This effect leads to functional MRI (fMRI), which is used to examine oxygenated blood in the brain. The principle is that high neural activity is probably associated with increased blood flow.

PET, another imaging approach, uses an analog of glucose with a radioactive fluorine atom attached. When it decays, it produces a particle known as a positron that is quickly annihilated upon encountering an electron, and two photons stream out in opposite directions. Photomultiplier detectors essentially notice the two photons, and mathematics is used to invert this problem and determine where the annihilation occurred, which presumably is near where the brain was consuming the glucose-like food. A single-photon to PETs, Single Photon Emission Computed Tomography (SPECT), is also utilized.

As the final example of imaging approaches, EEGs focus on using electrical activity recorded on the scalp to see what voltages are created by bulk neurons extending over somewhat larger regions of the brain, as neurons synchronize their electrical signaling. EEGs thus have less spatial resolution than some of the other imaging approaches. Magnetoencephalography (MEG) is a magnetic analog of the EEG in that it is also a noninvasive procedure. Rather than using electrodes attached to a person's scalp for measurements, as in an EEG, very precise superconducting quantum interference devices (SQUIDs) detect weak magnetic fields directly arising



An illustration of hardwired neurons, transferring pulses and generating information. The electrical signal is transferred from one neuron's axon to another's dendrite across a gap known as a synapse.

from electrical brain activity. A mathematical inverse process makes the externally obtained magnetic data usable and converts it to internal electrical activity. MEG typically offers greater resolution, so it can localize the electrical activity more precisely, than EEGs.

Much mathematics is used to model the flow of electrical impulses in the brain. Wave phenomena in the brain arise in varied contexts, from the propagation of signals down a neuron, to collective behavior of many neurons resulting in rhythmic activity. More specifically, the Nernst equation, named for Walther Nernst, and its generalization, the Goldman equation, named for David Goldman, help relate ion concentrations to voltages. How those voltages change in a neuron as it is stimulated by other neurons is modeled by the Hodgkin–Huxley set of equations, which are a calculus-based set of differential equations that resulted in a Nobel Prize for Alan Hodgkin and Andrew Huxley. Next, an area of mathematics known as "dynamical systems" helps model how the firing of individual neurons can naturally become synchronized and produce wave behavior at different frequencies, which are ultimately recorded on EEGs. Normal rhythmic activity is important in activities such as sleeping, breathing, or walking. Abnormal rhythmic activity is manifested in various diseases; for example, forms of schizophrenia, Parkinson's disease, and epilepsy demonstrate deviations from what is considered typical rhythmical behavior.

Neurology of Learning Mathematic

How the brain learns mathematics is another area of interest to researchers. Psychology and other social sciences bring light to bear on this subject but so too does the study of various neural pathologies. As an example, dyscalculia, which has been called a form of "number blindness" (by analogy to color blindness), is a pathology wherein individuals cannot acquire arithmetic skills. For instance, individuals fully capable of language communication who cannot tell if one whole number is larger than another or are unable to do 2-digit computations are considered "number blind." For other examples, there are cases of individuals with increasing difficulties with speech—primarily because of atrophy in the temporal lobes leading to dementia—having highly reduced vocabulary including an inability to name common objects, yet whose arithmetic abilities remained virtually flawless. Similarly, there are instances of autistic individuals essentially unable to speak or understand speech, who nevertheless can perform computations. Infants can notice when the number of objects in a display changes or when a number of objects are hidden behind a screen.

Finally, there are instances of stroke victims who have fully intact language but lack numerical skills, such as not being able to count past 4, or say how many days are in a week. These examples indicate that language is not crucial for arithmetic computations, and, further, language may not be necessary for learning to calculate. Generally speaking, computations seem to be localized to the parietal lobe at the top of the brain, whereas key language areas, such as Broca's (frontal lobe), named for Paul Broca, and Wernicke's (temporal lobe), named for Carl Wernicke, reside elsewhere. However, there are ongoing debates among neuroscientists regarding what the highlighted areas on images mean with regard to brain functionality.

A related issue is how mathematical thinking beyond the level of simple arithmetic evolved in humans, including its relationship with the development of language and increasingly abstract reasoning. There are different and intriguing hypotheses regarding why language evolved roughly 200,000 years ago, whereas various forms of numerical and algorithmic abstraction evolved within the past few thousand years.

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Rick Kreminski
Dedicated to Wanda Kreminski (1925–2010)

See Also: EEG/EKG; Medical Imaging; Nervous System; Optical Illusions; Visualization.

Bridges

Category: Architecture and Engineering.
Fields of Study: Algebra; Geometry.
Summary: Bridges are subject to various complex forces, the distribution of which are determined by their structures.

Bridges are structures built to span a gap or a physical obstacle such as a road or body of water. The many forces acting on bridges make different designs variously suited to different conditions, uses, and building materials. The earliest manmade bridges emulated naturally occurring bridges, like fallen trees that spanned rivers, and were improved upon by lashing logs into place, cutting planks to form a more even travel surface, and eventually building bridges out of stone. The mathematics of bridges was not well understood and most improvements were achieved through trial and error, one of the most significant being the advent of the arch bridge, introduced in Greece in 1300 B.C.E. and used extensively by the Romans. Arch bridges use arch-shaped abutments, sometimes in a series, to distribute much of the bridge's load into horizontal thrust the abutments can restrain—not only a major improvement over earlier designs, but a design well-suited to the simple building materials of the time as stone is strong in compression but weakly resists tension. As applied mathematics became more sophisticated, bridges were often objects of study.

Most bridges are built for functional purposes, but some of them are works of art, like the Golden Gate Bridge or the London Bridge. Mathematicians have long worked on various aspects related to the design and construction of bridges. For instance, Charles Hutton worked on equilibrium principles and Claude-Louis Navier developed a theory for suspension bridges. Applied mathematician P. Joseph McKenna analyzed bridge oscillations and differential equation models of the collapse of the Tacoma Narrows Bridge. The configuration of bridges in Konigsberg served as the subject of mathematical study for Leonhard Euler and is sometimes noted as the beginning of graph theory.

The Tacoma Narrows Bridge

The Mathematics of Bridges

A bridge has to support various forms of forces: tension, compression, bending, torsion, and shear. It has to carry its own weight (or "dead load"), the weight of the traffic for which it was intended (or "dynamic load"), and it should resist various natural forces, such as wind or earthquakes. The Tacoma Narrows Bridge is often presented in engineering, physics, or mathematics classes as an application of oscillation problems or differential equations. It was a 1.1 mile (1.9 kilometer) long suspension bridge and collapsed in 1940—four months after being opened—because a 35–46 mile per hour wind produced an oscillation, which ultimately broke the entire construction.

Types of Bridges

There are various types of bridges. Beam bridges consist of a horizontal beam with two supports called "piers" at the ends. Arch bridges are one of the oldest types of bridges and distribute the load of the bridge outward along the curve of the arch to the supports at the ends. Suspension bridges are light and strong and can span longer distances than any other type of bridge, but they are expensive to build. Large bundles of cables suspend the roadway from one end of the bridge to the other. Early Asian suspension bridges were suspended with bamboo cables. Cable-stayed bridges look like suspension bridges, but their cables are secured to towers that bear the load of the bridge. They cost less and their construction is faster than suspension bridges, since they need fewer cables and builders can use pre-cast concrete sections. Movable bridges can be occasionally levered for making way for ships or other kinds of traffic. Double-decked bridges have two levels and are used for multiple forms of traffic—subway, pedestrian, automobile, or bicycle.

The Seven Bridges of Konigsberg

Mathematician Leonhard Euler posed the problem of the seven bridges of Konigsberg in a 1736 paper. The town of Konigsberg contained an island with two branches of a river flowing around it. There were seven bridges spanning the river, and the question was whether a person could start at some point and follow a path that would cross each bridge exactly once and return to the starting point. Euler proved that there was no such path.

Famous Bridges

Millau Bridge, France, is 1125 feet high—higher than the Eiffel Tower. Hangzhou Bay Bridge, China, is 22 miles long. The Rolling Bridge, England, is 39 feet long and rolls itself up until the two ends meet, using a hydraulic press. Tower Bridge, England, is a landmark of London and opens in the center, allowing ships to sail through. Ponte Vecchio, Italy, is considered by some to be the oldest stone arch bridge in Europe. Lake Pontchartrain Causeway, Louisiana, is 24 miles long. Vasco da Gama Bridge, Portugal, is 10.5 miles long. Confederation Bridge, Canada, is 8 miles long. Golden Gate Bridge, California, is one of the most famous symbols of San Francisco. Evergreen Point Floating Bridge, Washington, is a 1.5-mile-long floating bridge.

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Simone Gyorfi

See Also: Engineering Design; Graphs; Highways; Levers; Tunnels.

Budgeting

Category: Business, Economics, and Marketing.
Fields of Study: Measurement; Number and Operations.
Summary: Creating a viable budget requires mathematical analysis.

The word "budget" originally meant a small pouch. By the end of the sixteenth century, people used the word to refer to the contents as well as the bag. The connection with finance dates back to at least 1733. In general, a budget is a balanced plan for spending and saving that includes expected incomes and expenditures. Individuals or families use budgets to manage earnings; pay bills; save for events like retirement, college, or vacations; and to plan for large purchases like a car or a home.

Businesses manage revenues and expenses for materials, taxes, advertising, and payroll using budgets. They may also have smaller budgets for individual projects. City, state, and national governments use budgets to distribute incomes from taxes and other sources among expenditures like infrastructure, social programs, national defense, and debt. Mathematicians play a large role in developing mathematically sound budgets at all levels, especially accountants and actuaries. In the past, budgeting in classroom settings was confined largely to home economics classes, but now budgeting activities are often used to teach various mathematical principles in context.

Some budgets are created using known amounts. Other times, the values are forecasts of income or expenditures based on data or mathematical models. Budgets themselves can also be used for modeling and production. For example, a static budget is a fixed budget created before any input and output values are

The 10-10-80 principle is to give 10% to charity, save 10%, and live on the remaining 80% of your earnings.

known, while a flexible budget can be adjusted based on information about actual activity. A metric called "flexible budget variance" compares flexible budgets to actual results to determine the effects of economic variables on business operations. Sales volume variance compares flexible budgets to static budgets to determine the effect a company's activity had on its operations. Budget accuracy ratios also quantify differences between various budgets or actual production. These can be used to create more accurate future budgets and to plan operations. Budgeting concepts can also apply to resources other than money. Lisa Sullivan, a senior budget analyst working for the U.S. Department of the Navy, regularly uses algebra, statistics, mathematical modeling, and operations research to explore resource allocation problems that affect budgeting. She often works on unique mathematical problems that do not occur in private industry, such as determining the optimal number of Navy surgeons needed in wartime.

Budgeting Basics

Creating a spending plan can be complicated; however, the easier the plan, the more likely it is to be followed. One of the simplest budgets used is the 10-10-80 principle. John D. Rockefeller, the first person in the world to amass a fortune of $1 billion and the wealthiest American ever when adjusted for inflation, is reported to have used the 10-10-80 principle. The crux of the principle is simple: give 10% of your earnings to charity, save 10% of your earnings, and live on the remaining 80%.

Anytime you receive income (for example, paycheck, gift, or prize), first multiply that amount by 10%. Finding 10% of an amount is a relatively easy process: move the decimal point to the left one place value. For example, if you received earnings of $342.57, multiplying by 10% would yield $34.257 (rounded to $34.26). Based on the 10-10-80 principle, you should first give $34.26 away to charity. Many people donate this money to religious institutions or charities such as the Red Cross or the United Way. One argument for giving, besides being altruistic, is to show ourselves that we have control of our money. By freely and willingly giving some of it to others instead of tightly holding onto it, we gain confidence that we have enough and can therefore live on what we are given.

The next 10%, or $34.26 in this example, is given to yourself into some sort of savings vehicle like a savings account or a money market fund. Ideally, this money is never needed as it becomes part of your long-term savings. This money may go toward retirement or an emergency fund in case of job loss or major disaster. Many people are tempted to use this long-term savings for expenses like taking vacations, buying a car, or replacing an appliance. However, these foreseeable expenses should be budgeted as part of the remaining 80%.

Once you have given 20% of your income away (10% to others and 10% to yourself), the remaining 80% can be used for living expenses (including short-term savings). How that 80% is spent can vary depending on many factors including how many people are being supported (for example, you do not need to buy as much food for a single adult as you do for a family of five). Usually the biggest expenditures are for housing and transportation.

Combined, these two categories should not account for more than 50%, or half, of your income. Of course, the less you spend on these the more you have to spend on other areas. Housing, by itself, should account for less than 35% of your income. In the previous example, 35% of $342.57 is $119.91. Set aside this $119.91 to cover any housing expenses you have.

Housing expenses include not only the obvious rent or mortgage but also utilities (heat, electric, plumbing, sanitation, telephone, and Internet), insurance, property taxes, and property maintenance (property maintenance is usually about 5% of the property value each year).

If housing and transportation together should be 50% (or less) of your income, then 15% should be used on transportation. In the example, 15% of $342.57 is $51.39. This amount becomes earmarked to cover all transportation expenses. These expenses include car payments, insurance, license, gasoline, parking, and maintenance (car maintenance is usually about 10% of your transportation costs).

If you spend 50% of your income on housing and transportation, this leaves a mere 30% for everything else. If you have been spending more than you earn, you probably have credit card debt or other personal debt. Ideally, your debts (not including housing or transportation debts) do not account for more than 5% to 10% of your income. What remains should be used to pay for food, life insurance, medical insurance, medical and dental expenses, clothing, entertainment, short-term savings (for vacations and replacement costs), and other miscellaneous spending.

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Chad T. Lower

See Also: Comparison Shopping; Coupons and Rebates; Home Buying; Money.

Burns, Ursula

Category: Business, Economics, and Marketing.
Field of Study: Connections.
Summary: The CEO of the Xerox Corporation and *Forbes*' 20th most powerful woman in the world, Ursula Burns (1958–) is an accomplished mathematics education advocate.

Ursula Burns was the first African-American woman to be named chief executive officer (CEO) (2009) and ultimately chairman (in 2010) of a Fortune 100 company. Burns has a bachelor of science degree in mechanical engineering from Polytechnic Institute of NYU and a master's of science degree in mechanical engineering from Columbia University. In addition to her work at the Xerox Corporation, she has been passionate about mathematics, science, and engineering education.

Career

Burns has dedicated her entire professional life to the Xerox Corporation. She began her career with the company in 1980 as a mechanical engineering summer intern. After completing her master's degree, she joined the company as a full-time employee. In her early career, she worked in product development and planning, and later in manufacturing and supply chain operations. She noted, "This company was my family." In 2007, she was appointed president of the company, and in 2009, at the age of 50, she succeeded another female CEO, Ann Mulcahy, to become the first African-American woman named CEO of a Fortune 100 company. She advises students, "Find something that you love to do, and find a place that you really like to do it in. . . . I'm a mechanical engineer by training, and I loved it. I still do. . . . I got to work on these great problems."

Commitment to Mathematics and Science

In addition to her business successes, Burns advocates for stronger educational efforts in science, technology, engineering, and mathematics. In a 2010 interview with *Fortune* magazine, she reflected, "If you get kids when they're young from just about any background, you can create people who are capable of utilizing science, technology, math, and engineering to solve problems. If you look at the list of the top nations and try to find out where we are in reading, math, and any science, it is stunning. I don't look at the list anymore because it's an embarrassment. We are the best nation in the world. We created the Internet and little iPods and copying and printing machines and MRI devices and artificial hearts. That's all science and engineering. Who's going to create those things?" Her concern is supported by data from sources such as the Trends in International Mathematics and Science Study (TIMSS).

In response to her strong dedication to mathematics and science learning, in 2009, President Barack Obama appointed Mrs. Burns a member of Educate to Innovate, a White House Initiative on Science, Technology, Engineering, and Mathematics Education (STEM) to help lead a national program to improve student learning in these fields. This committee was also charged with creating public-private partnerships to foster innovation and creativity in the STEM fields.

This program was expanded in 2010 to include Change the Equation, a CEO-led initiative. Funded in part by the Bill and Melinda Gates Foundation and the Carnegie Corporation of New York, the group provides financial support to assist high school students in passing advanced placement tests in science and mathematics, as well as promoting the professional development of STEM teachers. Burns has been quoted as saying, "If we inspire young people today, we secure our ability to innovate tomorrow…[because] [i]nnovation is central to our nation's overall growth."

Accomplishments

In 2010, Burns was named the 20th most powerful woman in the world by *Forbes* magazine. The award, in part, was based on her commitment to mathematics and science education. Similarly, Fortune magazine has ranked her as number nine on its list of the 50 most powerful women for 2010. In addition to her position at Xerox, she is on the board of directors for several organizations including the American Express Corporation, FIRST Robotics, the National Academy Foundation, MIT, and the University of Rochester. Burns credits her success in part to the lessons in life she learned from her mother. She is often quoted as saying, "Don't ever do anything that won't make your mom proud." Clearly, her accomplishments would please any mother.

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Konnie G. Kustron

See Also: Careers; Curriculum, K–12; Fax Machines.

Bus Scheduling

Category: Travel and Transportation.
Fields of Study: Algebra; Data Analysis and Probability; Measurement; Number and Operations.
Summary: Mathematical modeling can be used study and create optimal bus schedules.

Public transportation systems, like buses, are the primary mode of transportation for millions of people worldwide. Many people advocate for the increased use of buses to alleviate problems such as pollution and roadway congestion. Most public bus systems use fixed routes and schedules that specify the times and places at which the bus will stop so that people can plan their travel. However, most bus riders have had an occasion when their bus arrived late or have seen several buses arrive in quick succession. At peak times, buses may also be too full to admit new riders. Operations research is a subdiscipline of mathematics that focuses on these sorts of scheduling problems and mathematicians in a wide variety of areas work on related theories, problems, and applications.

Since buses usually travel several circuits in the same closed loops, and since there may be several buses following the same path, scheduling buses is similar to the problem of people waiting in a line or queue at the grocery store or the movies. Queuing theory uses mathematical techniques and concepts such as Markov chains, boundary models, series and cycles, numerical methods, simulation, and stochastic modeling to optimize scheduling. These problems can be challenging because of the need to quantify human behavior. Of particular interest to some mathematicians is the amount of slack that must be allowed in the schedule to allow buses to complete their routes in a timely and efficient fashion while accounting for natural variability and unexpected events. A related phenomenon is "bunching," which happens when buses traveling the same route get too close together. Both result in delays, lack of reliability, and customer dissatisfaction. In 2006, engineers Maged Dessouky, Jiamin Zhao, and T. S. Bukkapatnam published a mathematical model that created curves to correlate average delay times and slack time ratios with passenger waiting times. The curves were used to estimate optimal slack as a function of total round-trip travel time. They found an exact solution for the simplified case of a single bus on a closed loop with a known distribution of travel delays, with approximate extensions for more buses. In contrast, physicists Petr Seba and Milan Krbalek studied unscheduled, privately owned buses in Mexico. Passengers waited at known stops, and the drivers competed for passengers rather than assigning specific pickup times. While this system may appear to be chaotic, it has been shown in some studies to be more efficient than scheduled stops, and it can be modeled with a mathematical concept known as random matrices. Theoretical physicists have used these matrices since the 1970s to model complex quantum systems. They also have applications such as describing the distribution of prime numbers, and the possible arrangements of shuffled playing cards.

Queues

Both the problems of scheduling and queuing have commonality and are studied under the title "Queues" or "Queuing Theory." To think of the simplest problem is to consider a single customer service counter where the server takes a random amount of time serving each customer, and customers come one by one to the counter. A customer arriving at the counter is served straight away if the counter is idle when the customer arrives. If a customer is being served when additional customers arrive, then these new customers have to wait in a queue for their turn to be served. This method by which a queue forms leads to several interesting questions. How does one model the arrival pattern of customers? What is the expected time of service for different customers in the queue? What is the expected length of the queue as a function of time? How many customers will be served in a day given a model for the random service times?

This queuing problem can be translated to scheduling a bus to run in a city. Various specific questions arise in this scenario. When should the bus start? Which route should it take so that the service is available to the maximum number of commuters? How much time should the bus wait in intermediate bus stops? How often should the bus repeatedly go in the same route? Here, the objectives may be to maximize the utility of people who commute using the bus, minimize the fuel costs for running the bus, and optimize the use of the available buses. The problem of finding the optimal routes is called a "routing problem" or "bus scheduling problem." Given information about the number of buses available, the layout of the city, and the number of commuters who are likely to use the bus facility in the city, the scheduling problem can be posed as an optimization problem. To find out the number of commuters who may use the facility, one can perform a pilot study to ascertain the views of the people who may be interested in using a bus for their transport.

Modeling bus schedules is necessary to predict the arrival time of a bus at a particular station. Stochastic modeling must be employed since many random factors are involved like possible delay in the starting station because of commuter rush, and unexpected hurdles in the route because of weather. Modeling also helps in avoiding the clustering of buses at some points in a route. Another application of modeling is to track the buses and monitor the speed of buses on the routes. Once the bus scheduling is completed, service reliability has to be studied so as to make adjustments in the bus scheduling for improving the service. Efficient bus scheduling also helps in increasing the profitability of running bus service.

Scheduling Factors and Models

Bus scheduling involves a lot of random factors. Some of the factors are the number of people who will use the service, the amount of time the bus takes to cover a particular route, the delay caused by traffic jams, the number of commuters getting on and off at a particular bus station, the monthly income generated by the bus service along a particular route, and the maintenance costs for the bus. This necessitates stochastic modeling for the bus schedules. Models can be proposed based on historical data, pilot studies, and experiments. One of the important parameters considered in bus scheduling is the waiting time of commuters at a particular station. The objective of scheduling should be to minimize the waiting times of commuters at several points along a route, and for this, it is necessary to provide the most accurate bus schedules possible so that commuters get the maximum benefit. Queuing theory addresses most of these problems discussed and is a good source for solutions to problems in bus scheduling. Data mining techniques can also be used to look at patterns of commuter behavior across routes and this may be helpful in improved bus scheduling.

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See Also: Animation and CGI; Data Mining; Probability.

C

Calculators in Classrooms

Category: History and Development of Curricular Concepts.
Fields of Study: Communication; Connections; Number and Operations.
Summary: Calculators can be used in classrooms to augment rather than replace learning mathematical calculations.

Calculators have a long history. They can be traced back to the ninth century when the original compact calculator, the abacus, was developed in China. Nowadays, calculators are small (often handheld), electronic, digital, and inexpensive devices to perform various operations of mathematics. There are many kinds of calculators. Simple calculators just perform the basic operations of arithmetic.

Advanced calculators include scientific calculators and graphing calculators. Scientific calculators can work on complex operations such as trigonometric, logarithmic, and statistical calculations; some are even able to perform computer algebra. Graphing calculators usually have similar abilities to scientific calculators; however, they can be used to graph functions defined on the real line or higher dimensional Euclidean space. Since the advent of handheld calculators in the 1970s, the daily lives of people have been changed profoundly. Almost each business office and every high school student has at least one calculator. People can solve tedious computations in a few seconds by calculator, which was impossible before 1970.

Calculators in Primary Mathematics Classrooms

The availability of inexpensive calculators in primary classrooms has been increasing rapidly. However, the debate on their effectiveness in student learning and their role in mathematics instruction continues. Whether calculators should be used in primary classroom remains a controversial issue. On one hand, some people worry that calculators may hinder students' learning and obstruct the development of basic arithmetic operations such as addition, subtraction, multiplication, and division. On the other hand, research has shown that calculators can promote problem solving in students with a strong foundation in basic skills.

In primary classrooms, the use of calculators aims at facilitating the learning of mathematics rather than replacing mental arithmetic and written calculation. Pupils can use calculators to explore number patterns, construct concepts, and check different methods and results of problem solving. With the help of calculators, children can strengthen their abilities in mental arithmetic and estimation, as well as judge the sensibleness of the results of calculation. For instance, pupils may be asked to estimate the sum of $9 + 99 + 999$ and explain how they get the answer. One method of estimating the sum is the calculation $10 + 100 + 1000 - 3$.

After they have done the estimate mentally, they can check their estimation by calculator. Depending on their abilities, pupils may be asked to estimate the sum of more complicated operations such as $999 + 9999 + 99{,}999 + 999{,}999 + 9{,}999{,}999$ and then check the answer by calculator. This kind of activity facilitates the development of inquiry mind and higher order thinking in children. When pupils are allowed to use calculators to check the answers they have come up with by themselves, they have immediate feedback, have more time for solving additional problems, and make fewer errors. Calculators help pupils concentrate on thinking rather than on computation.

Pupils cannot benefit much if they are requested to compute traditional calculations such as $2 + 7$ or 3×5 by using a calculator. However, when they are asked to explore what calculations would give an answer of 10 with the aid of calculator, the effect can be very positive. Pupils may find patterns such as $1+9,\ 2+8,\ 3+7, \ldots$; $11-1,\ 12-2,\ 13-3, \ldots$; and $10 \div 1,\ 20 \div 2,\ 30 \div 3, \ldots$; and so on. Such open-ended tasks provide opportunities for children to explore basic arithmetic operations, natural numbers, fractions, and decimals. Through these exploratory activities, children can develop number sense and strengthen inquiry mind by making and testing conjectures. Calculators help children quickly detect and correct their misunderstandings experientially.

There are many good calculator activities that will enrich the learning experience for pupils. Different activities may be suitable for different classrooms; however, the focus should be on the mathematics rather than the calculator.

Calculators in Secondary Mathematics Classrooms

The use of scientific and graphing calculators in secondary school causes much less controversy than the use of simpler calculators in elementary schools. In fact, many countries allow their secondary students to bring in approved calculators for their university-entrance examinations.

Over the past 10 years, many innovative methods of teaching secondary mathematics have been developed with the advancement of handheld calculators and the needs of society. Many of the ideas require only basic calculators, but scientific and graphing calculators open up more possibilities—particularly for the learning of complicated functions, shapes, and graphs.

A graphing calculator typically refers to a class of handheld calculators that are capable of plotting graphs, solving systems of equations, and performing numerous other tasks with variables. For instance, graphing calculators allow students to explore the effect of varying the coefficients in the quadratic equation $y = ax^2 + bx + c$ by plotting graphs for different set of values of a, b and c in seconds. Plotting quadratic graphs by paper and pencil would consume a lot of time and effort, which would slow down the learning pace and reduce learning interest in the topic.

Graphing calculators save students from laborious work and provide opportunities to facilitate independent learning and give scope for open-ended exploration. If students go further in their investigation, they may generalize the conditions under which only one solution is obtained for the quadratic equation $0 = ax^2 + bx + c$.

The power of calculators advances rapidly. Some people worry about the use of symbolic calculators that can perform symbolic computations. They have argued that the use of symbolic calculators can cause core mathematical skills to wither, or that such use can prevent understanding of advanced concepts. It is not unusual that students use a symbolic calculator to find

$$_{x}\lim_{\infty}\left(\frac{1}{1+x}\right)$$

without realizing the mathematical principle or skills involved.

Concerns on Usage

From time to time, calculators are accused of making children lazy and replacing the need for them to use or remember number facts. They provide a means for getting answers without understanding mathematical processes. Some people worry that the extensive use of calculators in mathematics instruction interferes with students' mastery of basic mathematical skills and the understanding they need for more advanced mathematics.

In reality, the calculator is a tool that, if used in the right way, can support and encourage children's mathematical thinking. It is not calculators themselves that matter but when and how they are used that is important. To avoid overemphasis on the use of calculators, students should be guided to recognize the functions and limitations of calculators, so as to strengthen their abilities in exploring and solving mathematical problems. For instance, in a classroom activity, all pupils are given the same set of calculation questions, such as $789 + 0$, 25×4, 17×8, and 299×10. Pupils work in pairs; one is requested to find the answers by mental computation while the other uses a calculator.

At the end, they have to record the time needed and the number of correct answers. Pupils have to discuss and identify which calculations can be easily done mentally and which cannot. This activity can facilitate pupils' communication in mathematics and understanding that mental calculation sometimes is more powerful than the calculator.

Research also suggests that inadequate guidance in the use of calculating tools can restrict the kind of mathematical thinking that students engage in. Therefore, it is important that schools implement a balanced program that develops students' understanding of the appropriate use of the calculator.

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Eddie Leung

See Also: Calculators in Society; Software, Mathematics.

Calculators in Society

Category: Communication and Computers.
Fields of Study: Algebra; Measurement; Number and Operations.
Summary: Advancements in calculator technology have profoundly changed society and mathematics education.

In the decades since the invention of a truly handheld calculator, these devices have evolved from four-function curiosities costing hundreds of dollars to sophisticated machines capable of performing a wide range of mathematical and statistical functions at the same cost as that "four-banger" from the early 1970s. The effect on society has been considerable, as the laborious arithmetic involved in routine scientific or financial calculations can be done by nearly anyone with minimal effort and accuracy that was unthinkable in the 1950s. A variety of technological advances and a new market for calculating power during the 1970s led to the "calculator wars" among a variety of manufacturers, and frequent major advances in the power of a calculator were marketed to a willing society. These powerful calculators have changed the school mathematics curriculum in a variety of ways and brought a new focus to the Advanced Placement (AP) calculus exams.

Early History of Calculators

In their 1951 textbook *Mathematics of Investment*, Paul Rider and Carl Fischer made occasional reference to the ability of "computing machines" to facilitate involved calculations in financial mathematics. However, since such machines were by no means common in the 1950s, the book includes 123 pages of numerical tables, roughly one-third of the book's total length. These references were essential to actuarial calculations for many years, and their analogous tables of values of trigonometric functions, exponentials, and logarithms were a staple of mathematics textbooks for a comparable time period.

The rapid rise of low-cost electronic calculators—a generation beyond the electric computing machines to which Rider and Fischer referred—reduced those tables to a mere historical curiosity within a generation. In 1958, Texas Instruments (TI) engineer Jack Kilby invented the integrated circuit, which became known as the "calculator on a chip," that revolutionized the world of calculating devices. Large electromechanical desktop calculators soon gave way to more compact electronic machines, which culminated with the development of the Cal-Tech handheld calculator in 1965 at Texas Instruments. The Cal-Tech was a simple four-function calculator that used a paper tape for output. With a new standard for what was possible, the rush to advance calculating machines, both handheld and desktop, was on.

Engineers at Hewlett-Packard (HP) merged the old with the new in 1968 with the development of the HP-9100A, the first fully electronic desktop calculator. The 9100A was considerably larger than the Cal-Tech but was much more versatile. Its function set included all of the functions found on a modern scientific calculator—trigonometric functions, logarithms, reciprocals, and others—and it was fully programmable. On viewing the 9100A, company founder Bill Hewlett included among his words of praise for the developers the challenge that the world needed a similar machine that would fit into a shirt pocket.

In 1972, Texas Instruments introduced the Datamath, a four-function calculator released under the TI name. This was a departure for the company, which until then had confined its calculator work to manufacturing parts for other companies' machines. Indeed, the Cal-Tech was built primarily to show other manufacturers what the company's parts could do, not as an eventual consumer product. In that same year, Hewlett-Packard engineers developed the HP-35, a fully scientific calculator that could fit into a shirt-pocket. With these two companies at the forefront of a rapidly advancing technology, and with many other manufacturers in close competition, the "calculator wars" began. The rapid evolution of affordable competing calculators from a variety of manufacturers went on throughout the 1970s and into the early 1980s.

A major innovation was TI's introduction of the TI-30 scientific calculator, which sold for under $30 beginning in 1976. The full scientific function set of the TI-30 on a low-priced machine was a huge advance over the $395 price tag of the original HP-35, and the TI-30 was regarded for many years as the best-selling calculator of all time.

HP introduced the first handheld programmable calculator, the HP-65, in 1974 (fewer than two years after its first scientific calculator), and followed it up in 1977 with the HP-67. TI countered with the SR-52 in 1975, which was succeeded by the TI-58 and TI-59 in 1977. Each of these milestone calculators allowed the user to specify a sequence of steps into a special memory. These steps could then be repeatedly executed as many times as desired. The HP models and the 52 and 59 provided the option of recording programs onto small magnetic cards for permanent storage, while the 58 and 59 came equipped with a slot for read-only memory (ROM) cartridges with space for dozens of specialized prewritten programs that were stored on the chip and could be run as needed without the need for repeated keying.

Special-Purpose Calculators

Special-purpose calculators are preprogrammed with functions and formulas that are specific to a particular profession or interest. Among the earliest were calculators designed for financial mathematics, with keys and

Hewlett-Packard developed the world's first pocket scientific calculator, the HP-35, in 1972.

routines for solving the time value of money problems and automating interest calculations—and here was where Rider and Fischer's prediction was exceeded. These business calculators were considerably more sophisticated than could have been imagined in 1951.

By far the most successful business calculator is Hewlett-Packard's HP-12C, which was introduced in 1981 and is still in production 30 years later. In most senses, the 12C is the industry standard financial calculator, and it has been the key to HP's successful focus on the business calculator market. In 2003, the 12C got a facelift—and a faster processor—as the HP 12C Platinum Edition.

Unit conversion calculators inspired by the push in the 1970s to introduce the metric system in the United States live on in a variety of construction calculators, many of which have been produced by a small company, Calculated Industries (CI). CI was founded in the 1978, and its first product was a real estate calculator dubbed "The Loan Arranger." Future financial calculators from CI would expand in capability to accommodate more sophisticated calculations, and a separate line of CI financial calculators is specific to Canadian interest calculations. Later product lines from CI included the Construction Master and Measure Master lines—which were specialized for the building industry. CI also produces a series of electrical engineering calculators and a pair of professional plumbing calculators.

CI also manufactures special-purpose calculators for a variety of niche markets. Do-it-yourselfers can find the calculations they need preprogrammed into the ProjectCalc series. Several of these have been rebranded by Sears under the Craftsman line. The KitchenCalc Pro is preset to convert cooking measurements and includes a built-in timer. The Quilters' FabriCalc is one of the company's most successful hobbyist calculators and automates the considerable mathematics involved in quilting. Most recently, the Mr. Gasket Hot Rod Calc was developed to serve performance automotive enthusiasts with a collection of functions for use in assessing an automobile's performance.

Calculators in the Classroom

In 1976, Texas Instruments released the Abstract Linking Electronically (ABLE) calculator system, which represented the first attempt to manufacture a calculator specifically designed for elementary school classrooms beginning in the earliest grades. The ABLE system consisted of a standard four-function calculator with six interchangeable faceplates. These faceplates blocked access to some of the calculator's functions and could be switched out to allow a richer selection of options as a child's mathematical sophistication grew.

There was then, and continues to be, considerable tension over the question of calculator use in school mathematics. The conflict is generated by the ability of inexpensive calculators to automate routine arithmetic problems, which had led one side of the debate to suggest that there is no need to require computational automaticity, such as memorizing multiplication tables, which a calculator can handle. These advocates then assert that calculators free up room in the curriculum for what are called "higher-order" mathematical thinking skills. Those opposed to this view assert that higher-order skills are not useful without a sound foundation based on mastery of routine calculations. Sensible middle ground exists between these two viewpoints, and a variety of combinations of these approaches are advocated in textbooks and available to teachers.

In 2000, TI expanded the Explorer line to include the TI-15 Explorer calculator, which was designed for use in grades 3–6. This calculator contains specialized keys for computations like place value calculations and fraction operations without cluttering the keyboard with higher-level computations, like trigonometric functions, that are not studied in elementary school. Additionally, the TI-15 Explorer includes two keys that can be programmed to repeat simple operations, a randomized arithmetic tutor, and tools for exploring inequalities. A simpler companion calculator, the TI-10, was introduced in 2002 and is aimed at kindergarten through third grade classrooms.

At higher grade levels, one effect was far less controversial. With the advent of inexpensive powerful scientific calculators, there was no longer a need for extensive tables of functions in precalculus textbooks.

Graphing Calculators

In 1985, Casio introduced the first graphing calculator, the fx-7000G. In addition to serving as a fully functional scientific calculator, the fx-7000G had a large (1.4-by-2-inch) LCD screen on which graphs of functions could be displayed. This allowed students to work with functions from both numerical and graphical perspectives, and set the stage for a revolution in mathematics teaching. Graphing calculators soon came to be seen as one of the primary components of this shift in teaching and learning.

Hewlett-Packard advanced handheld capacity further with the HP-28C, introduced in 1987. In addition to numerical and graphical approaches to functions, the 28C was able to perform symbolic algebra and calculus, working with variables directly without the need for numbers. Texas Instruments released its first graphing calculator, the TI-81, in 1990, and the TI-85 soon after. The TI-82, 83, 84+, 86, and 89 have extended this successful product line, while the TI-80 and 73 have reached downward into middle schools.

As graphing calculators and computer algebra systems, such as Derive and Mathematica, competed for space in calculus classrooms around the world, it became clear that standardized testing would have to accommodate these new devices. Beginning in 1995, the Advanced Placement calculus exams have required the use of a graphing calculator on part of the exam, one that can plot graphs of functions, solve equations numerically, compute numerical derivatives, and evaluate definite integrals numerically. The College Board, which administers the AP exams, draws the line at calculators with a typewriter-style QWERTY keyboard, such as the TI-92 (introduced in 1996) and Voyage 200 (introduced in 2002) from Texas Instruments. The concern here is for the security of the tests, as the typewriter keyboard and text-processing capability are thought to make it too easy to collect confidential test questions and remove them from the testing site.

The Future of Calculators

It is unclear what new ground remains to be broken in future calculators. Three-dimensional graphing is available on a variety of TI and HP machines, but the size of the screen and the challenge from computer algebra systems, such as Mathematica, have limited the reach of this feature. Calculating power is finding its way into a variety of other handheld devices. Just as many people no longer wear watches because they can get the time from their cell phones, calculator applications for cell phone platforms may render the cell phone an attractive alternative to a specialized calculator. While there are cost and durability issues to be considered in this comparison, CI has recognized this alternate platform by marketing its Construction Master Pro software for the iPhone.

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Mark Bollman

See Also: Calculators in Classrooms; Calculus and Calculus Education.

Calculus and Calculus Education

Category: History and Development of Curricular Concepts.
Fields of Study: Calculus; Communication; Connections.
Summary: Once reserved for upper-level majors, the study of derivatives and integrals of functions has been mainstreamed by its applications.

Calculus, which takes its name from the Latin word for "pebble," is one of the most important branches of mathematics and one of the cornerstones of mathematics education. In ancient history, pebbles were used for counting, and "calculus" initially referred to that. The word now represents the method of calculation linked often with the study of change attempting to investigate motions and rates of change. From its mathematical development to the philosophy of calculus education, calculus has been fraught with rigorous debate and change.

Its appreciation and deeper understanding is a fundamental requirement in order to proceed toward a more advanced mathematical education and be involved with topics such as mathematical analysis. The topic finds its use and application in a vast number of different applied disciplines, such as biology, engineering, physics, population dynamics, statistics, and, in general, any scientific area that involves the study of instantaneous change.

Calculus education has a rich and varied history. Takakazu Seki is remembered as an influential teacher who passed his form of calculus on to his students. However, during the seventeenth century, secrecy surrounded rival schools in Japan, so it is difficult to determine his exact contributions. Successful calculus textbooks date back to at least Maria Gaetana Agnesi in the eighteenth century. She wrote *Analytical Institutions*, probably as a textbook for her brothers. She mastered many languages, which were useful when she integrated the knowledge of the time. She also added her own examples and expositions. Her book was widely translated and used all over the world, making the concepts of calculus more accessible.

Calculus education underwent many changes in the twentieth century. Early on, calculus was often an upper-division college course in North America while it was a pre-college course in France. U.S. President John F. Kennedy's race to the moon impacted calculus education in the United States. More engineering students were recruited, and as a result, calculus shifted earlier in the college curriculum. Another change was an emphasis on set theory in such texts as Tom Apostol's *Calculus*. Beginning in the latter half of the twentieth century, high schools offered AP Calculus. The shift of calculus to lower-level students also occurred in other countries, such as in Japan.

However, students who did not have the aptitude to succeed in competitive programs were filtered out in lower-level college courses, and educators debated this problem internationally. A calculus reform movement in the United States originated in the late 1980s, epitomized by the slogan "Calculus should be a pump, not a filter." Teachers debated the roles of lectures, technology, and rigor. With minimal theoretical support for the choice of teaching strategies, mathematicians relied on empirical studies to determine what would help calculus students succeed. Educators tested many different approaches, such as those emphasizing active learning, graphing calculators, computer algebra software, historical sources, writing, humanistic perspectives, real-life applications, distance education, or calculus as a laboratory science. New teaching approaches were met with widespread acceptance on some campuses and rejection and backlash on others. Some schools reported a decline in the number of students failing calculus. In the early twenty-first century, mathematicians continue to discuss and refine the calculus course.

Calculus—A Journey Through History

Even though counting as a process appears from the very first stages of humanity and its traces are lost in history of various civilizations, calculus was officially introduced as a realization of the deeper need to set rules and construct generally approved techniques that would assist toward quantification of any kind of change in time or space.

It could also perform modeling of systems that continuously evolve, and hence aid the interpretation and deduction of the consequences of the existence of such systems. Basic ideas of calculus involve limits, continuity, derivatives, and integrals.

Archimedes is one of the main scholars of ancient history who is linked with the ideas of calculus (c. second century B.C.E.). However, two important scholars of the seventeenth century made significant contributions to the introduction and the establishment of calculus as a quantitative language. Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716) are both recognized as fathers of modern calculus. Even though they worked independently and were influenced by two different areas—Newton by physics and Leibniz by geometry—they both reached into discovering the same fundamental ideas of calculus.

Differential Calculus

If x is a variable that changes with time (for example, $x = x(t)$ is a function of time *t)* then one denotes

$$\dot{x} = \frac{dx}{dt}$$

the first derivative of the function x, which represents the rate of change of $x(t)$ with respect to time t. Newton used the notation $\dot{x}$ while Leibniz used

$$\frac{dx}{dt}.$$

In case of a moving object in one dimension, $x(t)$ represents the position of the object and $\dot{x}(t)$ its velocity.

The term "function" was first introduced by Leibniz and is one of the fundamental terms of mathematics. In practice, a quantity y is defined as a function of another quantity t if there is a rule (method or process) in which a unique y will be assigned to any t. Leonhard Euler (1707–1783) introduced the notation $y = f(t)$ to identify a function f.

"Method of fluxions" is the term Newton used for his set of techniques to study the continuous flow of change. The process of finding a formula for the function x, given the formula for the function of x, is known as "differentiation" and the methods used for this belong to the field of differential calculus.

Rate of Change

There is a particular interest in studying the change of a quantity and by extension the rate of change of a quantity as another quantity changes in a very small amount. As Newton and Leibniz were developing calculus, they both used "infinitesimals" in order to emphasize the idea of such a small quantity that is not zero and that cannot be measured ("infinitesimal calculus" or "calculus of infinitesimals"). Hence, the infinitesimal number dx was considered to be different from zero and less than any positive real number.

Their approach raised criticism among other well-known scholars such as George Berkeley (1685–1783), and the idea of using infinitesimals became gradually unpopular. The introduction of "limits" from Augustin Louis Cauchy (1789–1857), Karl Theodor Wilhelm Weierstrass (1815–1897), and Georg Friedrich Bernhard Riemann (1826–1866) led in a better realization of the fundamental ideas of calculus and reestablished the topic within a more sound framework. However, nonstandard analysis (Abraham Robinson, 1960) and smooth infinitesimal analysis as introduced in the twentieth century have brought back into use the idea of infinitesimals.

Limit

The definition of limit is a cornerstone for advanced mathematics and especially for mathematical analysis. Limit is what distinguishes calculus from other areas of mathematics, such as algebra, geometry, and number theory. Even though mathematics has a history of more than 3000 years, limits were treated as a special area of mathematics only from 1823 C.E. when the concept was published for the first time in Cauchy's book *Résumé of Lessons of Infinitesimal Calculus*. The first appearance of the term belongs to the Greek mathematician and philosopher Zeno of Elea (495–435 C.E.). However, the definition that was finally accepted and used by the mathematical community is the (ε, δ) definition as stated by Weierstrass.

Weierstrass Definition

Assume that L is a real number and that $f(x)$ is defined in an open interval where x_0 belongs. Then the limit of $f(x)$ as x tends to x_0 is equal to L and is denoted with

$$\lim_{x \to x_0} f(x) = L$$

if the following is true: for any real number ε, there exists another real number δ such that for all x in the interval $x_0 - \delta \le x \le x_0 + \delta$ the value $f(x)$ of f lies within the range from $L - \varepsilon$ to $L + \varepsilon$.

In terms of infinitesimals, the limit is defined as follows: L is the limit of $f(x)$ as x tends to x_0 and is denoted as

$$\lim_{x \to x_0} f(x) = L$$

if the following is true: for any infinitesimal number dx, the value of $f(a + dx)$ is finite, and the standard part of $f(a + dx)$ equals to L.

Equation of Tangent

The term "derivative" as introduced from Newton and Leibniz signified a new era in mathematics. The term assisted mathematicians in finally solving rigorously the problem of constructing a unique tangent passing from a point of a curve. Historically, mathematicians since Archimedes' period were constantly trying to solve the prob lem of a unique tangent on a point of a curve. Ancient Greeks believed first that the tangent at a point of a circle should be the line that passes from the particular point and is vertical to the radius of the circle. Archimedes devoted a significant part of one of his books to this specific problem, which is known as "Archimedean spiral." However, it was because of the introduction of the first derivative that the researchers could actually provide the equation for the tangent line of a curve $C: y = f(x)$ at a point $(x_0, f(x_0))$ as

$$y = f(x_0) + f'(x_0)(x - x_0) \text{ where}$$

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

is the rate of change of the function at that point

$$(x_0, f(x_0))$$

corresponds and defines the slope of the line tangent to the curve C at point

$$(x_0, f(x_0))$$

if and only if the limit exists.

Higher Order Derivatives

Thinking of the example of a moving object in time, it can be easily identified that there is a need for estimating the acceleration of the object. Acceleration is the rate at which an object changes its velocity. Therefore, acceleration in mathematical terminology is nothing else but the derivative of the derivative of x denoted

$$\ddot{x} = \frac{d^2x}{dt^2}$$

and called "second derivative." Since the first derivative provides information on the rate of change of a function, the second derivative refers to the rate of change of the rate of change. In general, a higher order derivative is denoted as

$$\frac{d^n x}{dt^n}.$$

In a more geometric framework, the first and second derivatives can be used to determine the concavity; in other words, the way that the slopes of tangent lines of a curve $y = f(x)$ change as x changes in an interval (a,b).

If f is a differentiable function and f' is increasing on (a,b), then f is concave up on (a,b). The slopes of the tangent lines of the graph of f increase as x increases over (a,b); a concave up graph looks like a right-side up bowl.

If f' is decreasing on(a,b), then f is concave down on (a,b). The slopes of the tangent lines of the graph of f decrease as x increases over (a,b); a concave down graph looks like an upside-down bowl.

Points where the concavity changes are known as inflection points of f. Given that a function is increasing throughout an interval, if its first derivative is positive throughout the interval and vice versa, and given that f' is differentiable, then the following can be obtained: If

$$(f')'(x) > 0$$

for all x in (a,b), then f' is increasing on (a,b) and therefore f is concave up on (a,b). If

$$(f')'(x) < 0$$

for all x in (a,b), then f' is decreasing on (a,b) and therefore f is concave down on (a,b). A natural application of this concept is to find the maximum or the minimum of a function in a case in which the function is concave down or concave up throughout the whole domain respectively. This can be used further to solve problems where an optimal solution is requested.

According to Hans Hahn (1879–1934), the fundamental problem of differentiation can be expressed by two problems: (1) if the path of a moving object is known, estimate its velocity, and (2) given the existence of a curve, estimate its slope. Therefore, the inverse of these problems are (1) if the velocity of a moving object at every instance is known, estimate its path, and (2) if the slope of a curve is known, find the curve.

Integral Calculus

Generally, the process to find a formula for a function of x given the formula for the derivative of the function of x is known as "integration" and the methods used to find the formula belong to the field of integral calculus.

Historically, integral calculus was motivated by the geometric problem of estimating the area of a region in xy-plane bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$. The solution of this problem came as a realization for the need of integral calculus and is linked with

$$\int_a^b f(x)\,dx$$

which is known as the "definite integral."

It is not known exactly for how long the aforementioned problem troubled the mathematical world. In

1858, Alexander Henry Rind (1833–1863), an Egyptologist from Scotland, discovered parts of a handwritten papyrus document that is considered to have been written in 1650 B.C.E. *The Rind Papyrus*, as it is known today, consisted of 85 problems by the Egyptian scribe Ahmes, who claimed that he had copied these problems from an older document. Problem number 50 indicates that before 4000 C.E., Egyptians knew how to compute the area of a circle by using the formula Area = 3.16 × radius^2.

Eminent interest toward computations of areas of regions bounded by different kinds of curves is also seen in ancient Greece. Archimedes, whom several scholars consider as the "father of integral calculus" because of his method to estimate that the area bounded from the parabola $y = x^2$ and the rectangular lines $x = 1$ and $y = 0$ would be equal to 1/3. His method, which is known as the "method of exhaustion," was an attempt to approximate the area of a curve by inscribing first in it a sequence of polygons and computing afterward their area, which must converge to the area of the containing curve. However, this method was first developed by Eudoxus; Archimedes just applied this method in order to establish the said area. This method was later generalized in what is known now as "integral calculus."

The fundamental problem of integration focuses on finding the actual function (or, equivalently, its indefinite integral) if the derivative of the function is known.

Assume that function f exists. If there is a function $F : y = F(x)$ such that

$$F'(x) = \frac{dF}{dx} = f(x)$$

then F is called the indefinite integral or antiderivative of f and it is denoted as

$$F(x) = \int f(x)\,dx = \int f = I$$

where I stands for the first letter of the word "integral."

Cauchy was most probably the first mathematician who provided a rigorous definition for the integral by using the limit of a sum. Riemann, later on, influenced by the theory of trigonometric series of the form

$$\sum \left(a_n \cos(nx) + b_n \sin(nx)\right)$$

continued Cauchy's work and defined the integral in a similar way, with the only difference that he studied the whole family of functions that can be integrated—functions for which the integral exists. During Cauchy and Riemann's period, mathematicians were mainly concerned with integrating bounded functions. However, the need for integrating functions that cannot be bounded was soon apparent. Carl Gustav Axel Harnack (1883) and Charles De La Vallée-Poussin (1894) were among the first mathematicians to be occupied with such a problem. However, Henri Léon Lebesgue (1875–1941) is the one who, with his Ph.D. thesis titled "Integral, Length and Area" published in 1902, brought integral calculus into a new level. He defined the Lebesgue integral, which is a generalization of the Riemann integral, and defined a new measure known today as the Lebesgue Measure, which extends the idea of length from intervals to a large class of sets.

Other important scholars whose names are tightly linked with the development of modern calculus are Frigyes Riesz (1880–1956), Johann Radon (1887–1956), Kazimierz Kuratowski (1896–1980), and Constantin Caratheodori (1873–1950). They succeeded in generalizing and extending even further Lebesgue's work.

The symbol $\int$, which is used for integration, is a big *S* (the first letter of the German word *summe,* meaning "sum") and was used for the first time by Leibniz. There are several theories regarding the origin of the symbol. F. M. Turrell has supported the theory that almost every botanist knows that if an apple is peeled by hand, and, with the help of a knife, starting from the stem and continuing in circles around the central axis without cutting off the apple skin until the opposite end is reached, then a spiral is produced that creates an extended S once placed on the top of a horizontal surface with the inner part of the skin facing upward. This observation, according to Turrell, could possibly explain the symbol of integration. Finally, the Greek letter Σ is strongly linked with $\int$ as Euler used it to denote a sum.

The fundamental theorem of calculus asserts that differentiation and integration are inverse problems. If a function f is continuous on the interval $[a,b]$ and if F is a function whose derivative is f on the interval (a,b), then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This realization has proved to be a very useful technique to estimate definite integrals in an algebraic way. Isaac Barrow, Newton, Leibniz, and Cauchy worked on the concepts and early proofs, and Riemann and Vito Volterra explored what conditions on functions were necessary in the theorem. Lebesgue's definition of integrals avoided some of the previous problems.

Probability theory and statistics are disciplines that use calculus. A valuable application is to determine the probability of a continuous random variable from an assumed density function and define the average of the variable and a range of variation around it. The basic method used to approach the underpinning problems is to find the area under the corresponding curves (compute an integral).

For the study of joint distributions of several random variables (multivariate distributions), students and researchers need to be familiar with the fundamental ideas of multidimensional calculus. Optimization in statistics is another area where calculus is significant; when, for instance, there is a demand to find an estimator of an unknown parameter that satisfies an optimality criterion, such as minimum variance.

Other Types of Calculus

Other calculi that are linked strongly with the undergraduate and postgraduate curriculum, indicating the broadness of the topic, are vector calculus and calculus of variations.

- Stochastic calculus is tightly linked with financial calculus. It is mostly found in higher levels of mathematical education as it requires knowledge of measure theory, functional analysis, and theory of stochastic processes.
- Malliavin calculus or stochastic calculus of variations was initiated by Norbert Wiener (1894–1964) in an attempt to provide a probabilistic proof of Hörmander's "sum of squares" theorem. It is an infinite-dimensional differential calculus on a Gaussian space with features that can be applied in a wide variety of advanced topics of stochastic analysis. Its development has enormously facilitated the study of stochastic differential equations where the solution is not adapted to the Brownian filtration.
- Quantum and quantum stochastic calculus, which have gained the interest of quantum mechanics specialists, use infinitesimals rather than limits.
- π-calculus and λ-calculus offer a simpler syntax, which is highly appreciated by those in computing, offering an easier development of the theory of programming languages: network calculus and operational calculus.

Calculus, with its all variations, can be characterized as the mathematical language that unifies science by linking different disciplines together; this is why it plays a central role in the mathematical curriculum with students exposed to its basic ideas from the high school level.

The appreciation of the influence of calculus upon the vast majority of disciplines promotes a simultaneous intuitive approach by providing sufficient examples that illustrate the applicability of the topic. Modern technology in the form of computers and graphical calculators provides the tools that can assist not only in applying the mathematical techniques but also in a smooth transmission of the scientific ideas and basic mathematical concepts.

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See Also: Archimedes; Calculus in Society; Curriculum, College; Curves; Functions; Software, Mathematics.

Calculus in Society

Category: School and Society.
Fields of Study: Calculus; Connections.
Summary: Since its introduction in the seventeenth century, calculus has been applied to more and more practical endeavors, from engineering and manufacturing to finance.

Since its initial development in the seventeenth century, calculus has emerged as a principal tool for solving problems in the physical sciences, engineering, and technologies. Applications of calculus have expanded to architecture, aeronautics, life sciences, statistics, economics, commerce, and medicine. Contemporary society is impacted continually by the applications of calculus. Many bridges, high-rise buildings, airlines, ships, televisions, cellular phones, cars, computers, and numerous other amenities of life were designed using calculus.

Since the 1970s, calculus in conjunction with computer technology has resulted in the emergence of new areas of study such as dynamical systems and chaos theory. Such vast applications have established the study of calculus as essential in preparation for numerous careers. Indeed, calculus is considered among the greatest achievements of humankind, making it worthy of study in its own right in a society that places rational thought and innovation in highest esteem. Recent curricular and pedagogical reforms in calculus have made it more academically accessible to the school population.

What Is Calculus?

Calculus originated from studying the physical motions of the universe, such as the movement of planets in the solar system and physical forces on Earth. It involves both algebra and geometry, in combination with the concepts of infinity and limits. In contrast to algebra and geometry, which focus on properties of static structures, calculus centers on objects in motion. There are two principal forms of calculus, differential calculus and integral calculus, which are inversely related. At its most basic level, differential calculus is used in determining instantaneous rates of change of a dependent variable with respect to one or more independent variables; integral calculus is used for computing areas and volumes of nonstandard shapes.

Who Invented Calculus?

In the late seventeenth century, Isaac Newton (1646–1727) of England and Gottfried Wilhelm Leibniz (1646–1716) of Germany independently invented calculus. Isaac Newton began his development of calculus in 1666 but did not arrange for its publication. He presented his derivations of calculus in his book, *The Method of Fluxions*, written in 1671. This book remained unpublished until 1736, nine years after his death. Gottfried Leibniz began his work in calculus in 1674. His first paper on the subject was published in 1684, 50 years earlier than Newton's publication. Because of these circumstances and fueled by the eighteenth-century nationalism of England and Germany, a bitter controversy erupted over who first invented calculus. Was it Isaac Newton or Gottfried Leibniz?

Investigators found that Leibniz had made a brief visit to London in 1676. Supporters of Newton argued that during that trip, Leibniz may have gained access to some of Newton's unpublished work on the subject from mutual acquaintances within the mathematics community. However, these two prominent and outstanding mathematicians used their own unique derivations and symbolic notations for calculus, with Newton developing differential calculus first and Leibniz developing integral calculus first. For many decades, the calculus feud divided British mathematicians and continental mathematicians, and it remains a historical mystery into the twenty-first century. It was an unusual controversy in that it erupted rather late in the development of calculus and was ignited by the respective followers of Newton and Leibniz. In the twenty-first century, the general consensus is that both Newton and Leibniz invented calculus, simultaneously and independently.

Isaac Newton (1646–1727): The Man

Isaac Newton was revered in England during his lifetime and is recognized as one of the foremost mathematicians and physicists of all time. In addition to his invention of calculus, Newton is famous for designing and building the first reflecting telescope, formulating the laws of motion, and discovering the white light spectrum. He held many prestigious positions, including Fellow of Trinity College, Lucasian Professor of Mathematics, Member of Parliament for the University of Cambridge, Master of the Royal Mint of England, and many others. Even though Newton was extremely productive and admired universally for his work, on a personal level he was humble, cautious of others, and angered by criticism. His modest nature is embodied in his famous statement, "If I have seen farther than others, it is because I stood on the shoulders of giants." His works in mathematics and physics were recognized throughout Europe when he was honored as Fellow of the Royal Society of London in 1672. He subsequently served as the Society's president from 1705 until his death. In 1705, Newton was knighted in Cambridge by Queen Anne of England for his contributions to the Royal Mint. In 1727, Newton's name was immortalized in English history by his burial in London's Westminster Abbey and by the accompanying monument honoring his contributions to mathematics and science.

Gottfried Leibniz (1646–1716): The Man

Gottfried Leibniz is recognized as one of Germany's greatest scholars of philosophy, history, and mathematics. He was the son of a philosophy professor and a leader in the philosophy of metaphysics. His optimism is reflected in his words, "We live in the best of all possible worlds." On a personal level, Leibniz was considered likeable, friendly, and somewhat boisterous. Professionally, Leibniz was employed by a succession of German princes in the capacities of diplomat and librarian. He planned and founded several academies throughout Europe. For his knowledge of law, he was appointed Councilor of Justice for the Germanic regions of Brandenburg and Hanover. Similarly, Russian Tsar Peter the Great appointed Leibniz as Court Councilor of Justice for the Habsburgs. For his work in mathematics (derivations in calculus and invention of the binary number system), in 1673, Leibniz was appointed Fellow of the Royal Society of London, a society honoring outstanding mathematicians and scientists throughout Europe. By 1706, however, Leibniz's stellar reputation had begun to disintegrate. Accusations of plagiarism regarding the invention of calculus were unrelenting until Leibniz's death in 1716. In contrast to Newton, the only mourner at Leibniz's funeral was his secretary. Eventually, more than a century after his death, Leibniz's outstanding contributions to mathematics were recognized in Germany when a statue was erected in his honor at Leipzig, one of Germany's major centers of learning and culture.

Interestingly, it is Leibniz's symbolic notations for calculus, namely dy/dx and $\int y\, dx$, that have stood the test of time. These notations are most prevalent in calculus classrooms in the twenty-first century because of their consistency with the operations of differential equations and dimensional analysis. The most significant contribution to mathematics by Newton and Leibniz was their derivations of the Fundamental Theorem of Calculus, a theorem that unites both differential and integral calculus.

Building on Newton's and Leibniz's Work

Following the invention of calculus, additional contributions to calculus were made by John Wallis (1616–1703), Michel Rolle (1652–1719), Jacob Bernoulli (1654–1705), Guillaume de l'Hôpital (1661–1704), Brook Taylor (1685–1731), Colin Maclaurin (1698–1746), Joseph-Louis Lagrange (1736–1813), Bernard Bolzano (1781–1848), Augustin-Louis Cauchy (1789–1857), Karl Weinerstrasse (1815–1897), and Bernhard Riemann (1826–1866).

The Power of Calculus

The power of calculus in contemporary society rests primarily in its applications in the physical sciences, engineering, optimization theory, economics, geometrical measurement, probability, and mathematical modeling.

The following is a sampling of basic applications using the two major branches of calculus.

Applications of Differential Calculus

- *Environmental science*: An oil tanker runs aground and begins to leak oil into the ocean and surrounding land areas, resulting in potentially devastating consequences. Differential calculus can be used to supply information essential for assessing the leakage and resolving the problem. For example, the rate and volume at which the oil is leaking can by determined using calculus.
- *Business and economics*: Important applications of calculus in business and economics involve marginal analysis (known as the first derivative). Marginal costs, revenues, and profits represent rates of change that result from a unit increase in product production. This information is valuable in developing production levels and pricing strategies for maximizing profits.

- *Medicine*: Calculus can be used for evaluating the effectiveness of medications and dosage levels. For example, calculus can be used in determining the time required for a specific drug in a patient's bloodstream to reach its maximum concentration and effectiveness.
- *Biology and chemistry*: Assessments of chemical treatments for reducing concentration levels of biological contaminants (such as insects or bacteria) can be determined by calculus. For instance, calculus can be used in measuring the concentration levels, effectiveness, and time necessary for a chemical treatment supplied to a body of water to reduce its bacterial count to desired minimal levels.
- *Physics* (*velocity and acceleration*): For moving objects (such as rolling balls or hot-air balloons), their maximum velocities, accelerations, and elevations can be determined using calculus.
- *Politics*: The number of years required in a city for the rate of increase in its voting population to reach its maximum can be determined using calculus.
- *Manufacturing*: The design of containers, meeting specific constraints, can be determined using calculus. For example, calculus will supply the dimensions of a container that will maximize its volume or minimize its surface area.

Applications of Integral Calculus

- *Inverse of differential calculus*: In mathematics, most operations have inverse operations. In calculus, the inverse of differentiation is integration. Therefore, a fundamental application of integral calculus is to find functions that produce the answers to a problem in differential calculus.
- *Measurement, area, and volume*: Integral calculus can be used to find (1) the areas between the graphs of functions over specified intervals, (2) the surface areas of three-dimensional objects, and (3) the volumes of three-dimensional objects.
- *Centroids*: The centroid (or center of mass) of an object can be found using integral calculus. For two- and three-dimensional objects, the centroid is the balancing point of the object. Calculus can be used to locate the position of the centroid on the object.
- *Fluid pressure*: Integral calculus is essential in the design of ships, dams, submarines, and other submerged objects. It is used in determining the fluid pressure on the submerged object at various depths from the water's surface. This information is essential in the design of submerged objects so they will not collapse.
- *Physics* (*work*): When a constant force is applied to an object that moves in the direction of the force, the work produced by the force is found by multiplying the force by the distance moved by the object. However, when the applied force is not constant or is variable, calculus can be used in determining the work produced by the variable force (for example, the variable force needed to pull a metal spring, or the force exerted by expanding gases on the piston in an engine).

The aforementioned applications are examples of the most elementary applications of calculus. In the technological world of the twenty-first century, applications of calculus continue to evolve. The consequences of calculus are ubiquitous in contemporary society and impact every walk of life.

Recommendations for Mathematics Curriculum Reform

In 1983, following a harsh report from the National Commission on Excellence in Education, U.S. society began to question seriously the effectiveness of its educational systems. The report, titled *A Nation at Risk: the Imperative for Educational Reform*, was commissioned by U.S. President Ronald Reagan. The report cited U.S. students for their poor academic performance in every subject area at every grade level and their underachievement on national and international scales. The Commission warned the United States that its education system was "being eroded by a rising tide of mediocrity." In the years that followed, the Commission's explicit call for educational reform in U.S. schools served to generate numerous curricular reform efforts at the pre-college and college levels.

In response to this call for reform, in 1987, the Mathematical Association of America (MAA) and the National Research Council (NRC) co-sponsored a conference held in Washington, D.C., titled Calculus for a New Century. The conference was attended by more than 600 college and pre-college calculus teachers. The conference focused on the nature and need for calculus reform in college and pre-college institutions throughout the nation. During that conference, the phrase "Calculus should be a pump, not a filter in the pipeline of American education" became a national mantra for calculus reform.

National educational assessments conducted in 1989 further supported initiatives for calculus reform. During the 1980s, approximately 300,000 U.S. college students were enrolled annually in science-based calculus courses. Of that number, only 140,000 students earned grades of D or higher. Thus, more than 50% of U.S. college students were failing the calculus courses required for their majors, which included mathematics, all of the natural and physical sciences, and computer science. These bleak statistics served to motivate concerned calculus teachers to examine the traditional calculus curriculum, as well as their own teaching methodologies, with the intention of increasing course enrollments, student achievement, and enthusiasm for the subject.

Their efforts resulted in major calculus reform initiatives as early as 1989. The first set of recommendations for reform in school mathematics (grades prekindergarten–12) came from the National Council of Teachers of Mathematics (NCTM). These recommendations were delineated in NCTM's publication, *Principles and Standards for School Mathematics* (also known as NCTM Standards).

Four overarching standards (called Process Standards) were identified for improving mathematics instruction at all levels. These standards identified problem solving, reasoning and proof, connections, and communications as the four primary foci for mathematics instruction. During the 1990s, most U.S. states adopted this document as their curriculum framework for school mathematics. Decisions regarding the mathematics curriculum, textbook selections, and instructional strategies were revised in accordance with the recommendations of the NCTM Standards. Interestingly, the same document served to inspire pedagogical reform in mathematics at the college level, especially in calculus.

Traditional Calculus Versus Reformed Calculus

Until 1990, the calculus curriculum had remained basically the same for decades. The traditional calculus curriculum reflected formal mathematical language, mathematical rigor, and symbolic precision. Computations with limits, mathematical proofs, and elaborate mathematical computations were common practice in calculus classrooms. Students took careful notes, asked clarifying questions, and completed voluminous amounts of homework in preparation for test questions similar to those completed for homework. Instruction was teacher-centered and delivered through a lecture approach. Relevant applications were seldom considered, and graphing calculators and computers were rarely used in calculus instruction, and students were not allowed to use them for computations, graphing, or problem solving. Mathematics educators attributed the dismal performance of the majority of students in the nation's calculus classes to this traditional calculus curriculum. Consequently, by the mid-1990s, calculus reform movements had been initiated in many of the colleges and pre-college classrooms throughout the nation.

Calculus reform efforts at the college level in the 1990s often applied the pedagogical recommendations found in NCTM Standards. These pedagogical recommendations were also reflected in the revised Advanced Placement Calculus (AP Calculus) and International Baccalaureate Calculus (IB Calculus) courses offered in the nation's high schools. A measure of the subsequent success of the calculus reform movement at the pre-college level can be seen in the dramatic increase in numbers of students who took these courses from the 1980s into the twenty-first century. Specifically, the National Center for Education Statistics reported that the percentage of students completing calculus in high school had risen from 6% to 14% in the years from 1982 to 2004. The number of students completing calculus in high school continues to grow exponentially, at an estimated rate of 6.5% per year.

Several reform calculus curricula originated in the 1990s, and continue into the twenty-first century. The following examples are prominent reform calculus projects: Calculus, Concepts, Computers and Cooperative Learning (C^4L) conducted at Purdue University; the Calculus Consortium at Harvard (CCH) conducted at Harvard University; and Calculus and Mathematica (C&M) conducted at the University of Illinois at Urbana-Champaign and at Ohio State University.

While these three reform calculus projects differ from each other in significant ways, they share the following characteristics:

- They use graphing calculators, computers, and computer algebra systems (CAS) extensively for instruction, exploration, and visual representations. Supporters argue that technology serves to alleviate the huge burden of algebraic computation so characteristic of traditional calculus. The rationale for this reform is that technology facilitates instructional processes that focus on the principles of calculus rather than on computational procedures. Moreover, the graphical and visual representations provided by these technologies offer alternative modalities for learning that accommodate students' different learning styles. The curricula for CCH and C^4L focus heavily on graphing calculators, whereas the curriculum for C&M relies heavily on the computer software, Mathematica.
- The teacher serves as a facilitator of learning rather than the main conveyor of knowledge. While the teacher continues to initiate instruction and answer questions, mathematical situations are often explored by groups of students, using cooperative learning strategies. Using the principles of constructivist learning, students are guided to discover mathematical properties for themselves in a laboratory setting.
- A major focus is placed on real applications from multiple disciplines. The intention is to raise students' interest in the subject and motivate them with relevant applications.
- Mathematical rigor and formal language are de-emphasized. The abstractions of mathematical proof and rigor are postponed for several semesters to provide sufficient time for students to gain practical and intuitive knowledge of the subject.
- Assessment focuses heavily on students' writing, explanations of problem solutions, and open-ended projects. Sometimes students' explanations are valued as highly as the accuracy of their answers.

Whereas all of the above instructional practices have shown varying degrees of success in reform calculus classrooms, some areas of concern have been identified by those involved in the projects. Specifically:

- Focusing heavily on relevant applications sometimes results in the omission of important calculus content that cannot always be motivated by applications.
- The use of everyday language sometimes results in imprecise and incorrect mathematical definitions.
- Overuse of technologies for computation and graphing can weaken the development of students' quantitative reasoning and computational skills in calculus.
- Real-world problems are sometimes too complex and frustrating to students because of the extraneous and irrelevant information they usually contain.
- Short-answer problems for assessment are often easier for students than describing their problem-solving procedures in writing.
- Constructivist approaches are often too time consuming, allowing insufficient time for covering the entire calculus curriculum during class time.

Resolution of these concerns will surely be addressed in future curriculum revisions, and changes or modifications will be made accordingly. However, these accommodations are consistent with the historical evolution of calculus, which is the study of change and systems in perpetual motion.

Summary

In the past, calculus was taught in ways that made it accessible to only a small proportion of the population. However, recent curricular and pedagogical reforms in calculus, both at the college and pre-college levels, have served to increase student success, include twenty-first-century-technologies, and triple course enrollments. Statistics indicate that calculus enrollments will continue to increase exponentially. These findings suggest that calculus instruction in the United States is responding positively to the academic needs of society.

Indeed, by combining the power of technology with calculus, new areas of mathematics are emerging (for example, fractals, dynamical systems, and chaos theory). These new branches of mathematics have allowed humans to mimic nature's designs of mountain ranges, oceans, and plant growth patterns—which once were considered random acts of nature. In conclusion, calculus as a subject is still growing, and its applications are continually expanding to meet the needs of a dynamic, diverse, and technologically driven society.

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Sharon Whitton

See Also: Algebra and Algebra Education; Archimedes; Calculus and Calculus Education; Function Rate of Change; Functions; Geometry and Geometry Education.

Cameras

See *Digital Cameras*

Calendars

Category: Space, Time, and Distance.
Fields of Study: Measurement; Number and Operations; Representations.
Summary: Various calendars use different methods of resolving the need for "leap" days, months, or years.

Even the earliest human beings must have noticed the astronomical cycles: the alternation of day and night, the pattern of the changes in the moon's shape and position, and the cycle of the seasons through the solar year. It must have been frightening every autumn as the days became shorter, causing concern that the night might become permanent. This led to celebrations of light in many areas as the days began to lengthen again. Once the repetitions of the patterns were recognized, people could count them to keep track of time. Longer cycles helped avoid difficulties in keeping track of large numbers—once approximately 30 days had been counted, people could, instead, start counting "moons."

This same technique of grouping also occurred in the development of counting systems in general—leading to place-value structures in numeration systems.

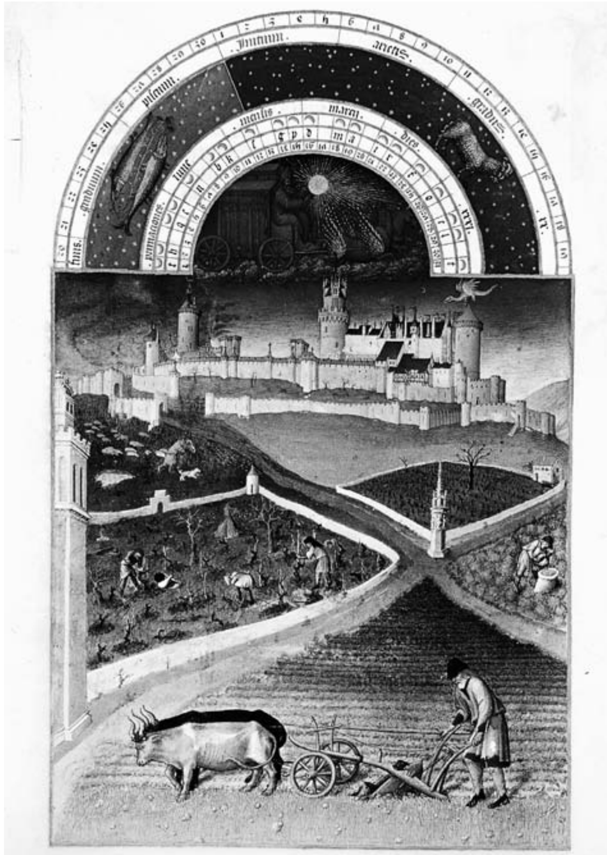
The problem was that the shorter cycles did not fit evenly into the longer cycles. Trying to fit the awkward-length cycles together actually led to some mathematical developments: two different cycles would come together at the least common multiple of the lengths of their cycles; modular arithmetic and linear congruences were methods of handling leftover periods beyond the regular cycle periods.

The Julian and Gregorian Calendars

The Romans developed the Julian calendar (named for Julius Caesar), recognizing that the exact number of 365 days in one year was slightly too short and would soon throw the calendar off the actual cycle of the solar year. They found a remedy by assuming the solar year to be 365.25 days. To handle the one-quarter day, they added one full day every four years—the day that we call "leap-year day" on February 29 of years whose number is a multiple of four. This gives $3(365)+366=1461$ days in four years, or an average of 365.25 days per year as desired. However, the actual solar year is 365.2422 days long (to four decimal places), about 11 minutes less than the Romans' value. Even in a human lifetime, this is negligible. Over centuries, however, the extra time builds up so that by the 1500s, the calendar was 10 days off from the solar cycle (for example, the vernal equinox seemed to be coming too late).

In 1582, Pope Gregory XIII assembled a group of scholars who devised a new system to fit better. It kept the Roman pattern except that century years (1600, 1700), which should have been leap years in the Roman calendar, would not have a February 29 unless they were multiples of 400. For example, 1900 was not a leap, year but 2000 was. In the full 400-year cycle, there are (400×365) regular days + 97 leap-year days = 146,097 days, making an average of 365.2425 days per year. This cycle is only .0003 days (about 26 seconds) too much; in 10,000 years, we would gain three extra days. This system was called the Gregorian calendar. Since the longer Julian calendar had fallen behind the solar year by about 10 days, the changeover to the Gregorian required jumping 10 days.

Various countries in Europe changed at different times, with each switch causing local controversy as people felt they were being "cheated" out of the skipped days. The effects of the change are noticed in history. When Isaac Newton was born, the calendar said it was December 25, 1642; but later England changed the calendar, so some historians today give Newton's birthday as January 4, 1643. The Russians did not change their calendar until after the 1917 October Revolution, which happened in November by the Gregorian calendar.



A 1412–1416 illumination depicting the month of March with the constellations of the zodiac on top.

The Lunar Calendar

The other incongruity of calendar systems is that the moon cycle of 29.53 days does not fit neatly in the 365.2422 days of the year. Twelve moon periods is 11 days shorter than a year, and 13 "moons" is 18 days too long. It is interesting to note that of the three major religious groups of the Middle East—the Christians, the Muslims, and the Jews—each chose a different way to handle "moons/months." The Christians (actually, originally, the Romans) ignored the moon cycle and simply created months of 30 and 31 (and 28 or 29) days. The Muslims considered their year to be 12 moon cycles and ignored the solar year. This means that dates in the Muslim calendar are shifted back approximately 11 days each year from the solar calendar, and Muslim festivals move backward through the seasons.

People in the Jewish faith chose to keep both the solar and lunar cycles. After 12 lunar months, a new year begins—as in the Muslim calendar—11 days "too early." However, after the calendar slips for two or three years—falling behind the solar calendar by 22 or 33 days—an extra month is inserted to compensate for the loss. There is a 19-year pattern of the insertion of extra months, which keeps the year aligned with the solar year. Interestingly, the traditional east Asian calendar follows a pattern very similar to the Jewish calendar.

The Mayan Calendar

The Mayans of Central America had a very complex pattern of cycles leading to a 260-day year for religious purposes, and a regular solar year that was used for farming and other climate-related activities. Their base-20 numeration system, which should have had place-value columns of 1-20-400-8000, was adjusted to 1-20-360 to fit into the 365+ days of the year. They were also notable for developing massive cycles of years lasting several millennia, including one ending in late 2012 of the Gregorian calendar.

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Lawrence H. Shirley

See Also: Astronomy; Incan and Mayan Mathematics; Measuring Time.

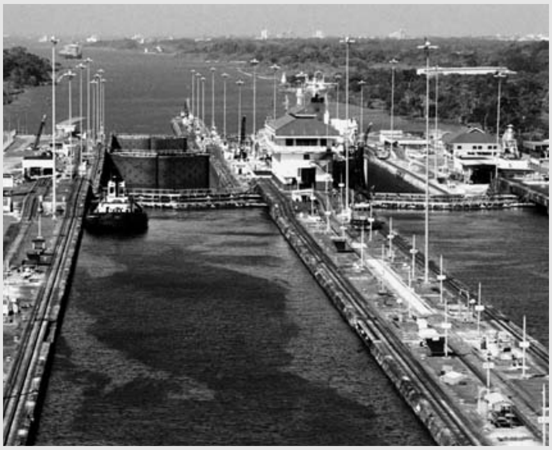
Canals

Category: Architecture and Engineering.
Fields of Study: Algebra; Geometry; Number and Operations; Problem Solving.
Summary: Modern canal design, particularly the challenges of a lock system, depends on partial differential equations and other mathematics.

Canals are human-made channels for water, including both waterways big enough to be traversed by ship (built for transportation), and aqueducts (built for water supply and irrigation). The building of canals was critical to the formation of many ancient civilizations, which needed to manipulate water access in order to enable an early urban lifestyle. Many ancient mathematics texts address such large-scale ancient engineering projects.

A number of the surviving Babylonian tablets dealing with geometry were composed for canal projects: they calculated the number of workers necessary to build the canal in a given number of days, the dimensions of the canal, and the total wage expenses so that the ruler for whom they were built would know how much the project would cost. Mathematical problems related to the construction of canals can also be found in the fifth chapter of *The Jiuzhang Suanshu* (*Nine Chapters on the Mathematical Art*), one of the earliest surviving ancient Chinese mathematics texts. Mathematicians and engineers have long investigated canals.

For instance, Jacopo Riccati worked on hydraulics and constructed dikes in Venice, and Barnabé Brisson employed descriptive geometry in the design and construction of ship canals. Mathematicians like George Green and Joseph Boussinesq analyzed and modeled wave motion in canals. John Russell tested and studied steam-powered canal transportation and wave creation for the Union Canal Company. Mikhail Lavrentev created a theoretical foundation for large projects on the Volga, Dnieper, and Don rivers. Mathematics theories and techniques are critical when engineers, mathematicians, and software programmers model the



The Gatun (above) and Miraflores Locks of the Panama Canal can be viewed from a webcam.

Major Canals

Significant canals include the Erie Canal in the United States, the Suez Canal in Egypt, the Panama Canal in Panama, and the Grand Canal in China, each of which was constructed as a major operation for the sake of hastening trade and transport. Judge Benjamin Wright, who some call the father of American civil engineering, was appointed the chief engineer of the Erie Canal. Astronomer and mathematician Guo Shoujing (also known as Kuo Shou-ching) was the head of the Water Works Bureau in the thirteenth century. He made improvements to control the water level in existing canals and built new ones.

The Suez Canal was imagined long before it was completed, and the Egyptians were masters of large-scale engineering projects. Napoleon Bonaparte, during the French invasion of Egypt, reportedly discovered ruins of an ancient canal, which inspired him to order a preliminary survey exploring the possibility of a north–south canal joining the Mediterranean and the Red Sea (the ancient canal had been east–west and was intended to link the Red Sea and the Nile). The project was abandoned—possibly because of the belief that the Red Sea was higher than the Mediterranean—and so the canal remained unbuilt for 70 years.

changing flow rates and levels of a canal. They rely on mathematics like the Saint-Venant equations (partial differential equations that are named after mechanic and mathematician Jean Claude Saint-Venant).

The simplest canals are merely trenches through which water runs, usually lined with some kind of construction material. Canals need to be level in order to be navigable (a ship cannot move "uphill"). When the land itself is not level, a lock system must be used. Locks are systems for raising and lowering boats from one stretch of water to a stretch of water at a different level. The most common type of canal lock—used in ancient China, and most likely in the ancient West, and still common today—is the pound lock, which consists of a watertight chamber with gates at either end to control the water level in the chamber.

Engineer Chiao Wei-Yo is credited with the design of the lock system, which he used on the Grand Canal in the tenth century. In the pound lock system, a ship enters the chamber (the "pound") from one length of canal; water is raised or lowered to bring the ship to the level of the next length of canal; and the ship exits the chamber. The necessity of locks added much complexity, time, and room for error to the construction of canals, which would have been sufficient to discourage Napoleon's aims. In 2010, the Panama Canal commemorated its one-millionth transit, and engineers plan to expand the canal by adding more locks. It has been referred to as one of the seven wonders of the industrial world.

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Bill Kte'pi

See Also: Floods; Tides and Waves; Tunnels; Water Distribution.

Carbon Dating

Category: Weather, Nature, and Environment.
Fields of Study: Algebra; Data Analysis and Probability.
Summary: Exponential and logarithmic functions are used in carbon dating—a method of determining the age of plant and animal fossils.

As is demonstrated throughout this encyclopedia, mathematics provides explanations for many interesting physical phenomena, and enables humankind to better understand its surrounding world. One of our ongoing intellectual projects is simply to make sense of the world we inhabit, based on the evidence that surrounds us. As anthropologists, archaeologists, and geologists have worked to determine the age of the earth and to track the evolution of species, radioactive isotopes have played a prominent role in efforts to create a timeline that charts a wide range of historical developments. In particular, carbon-14 dating has provided a fundamental test enabling scientists to accurately date certain plant and animal fossils that are approximately 60,000 years old or less. Willard Libby was one of the first to research radiocarbon dating, and he won a Nobel Prize in chemistry. Carbon dating is not an exact science, and statistical methods are used to enhance the reliability of the methods.

The Mathematics of Carbon Dating

Left alone, a radioactive quantity will decay at a rate proportionate to the amount of the quantity present at a given time. More specifically, a radioactive chemical element (such as uranium) is one that is unstable; as it decays, it emits energy and its fundamental makeup changes as the mass of the element is changed to an element of a different type. Because such an element is losing mass at a rate proportionate to the available mass at time t, an exponential function may be used to model the amount of the isotope that is present.

Letting $M(t)$ represent the mass of the element at time t, it turns out that $M(t) = M_0 e^{-kt}$, where M_0 is the mass at initial measurement (at time $t = 0$), and k is a constant that is connected to the rate at which the element decays. Furthermore, k is tied to the isotope's half-life (the amount of time it takes for 50% of the mass present to decay). In the given model, if h represents the half-life, then when $t = h$, it follows that

$$M(h) = \frac{M_0}{2}.$$

That is, the equation

$$\frac{M_0}{2} = M_0 e^{-kh}$$

must hold. Dividing both sides by M_0, yields

$$\frac{1}{2} = e^{-kh}$$

and using the natural logarithm function, one may solve for k and thus rewrite the most recent equation as

$$-kh = \ln\left(\frac{1}{2}\right).$$

This can be rewritten as

$$k = \frac{-\ln\left(\frac{1}{2}\right)}{h}.$$

A property of the natural logarithm is that

$$-\ln\frac{1}{2} = \ln(2)$$

so that in slightly simpler terms,

$$k = \frac{\ln(2)}{h}.$$

Therefore, the model for radioactive decay of an element having half-life h is

$$M(t) = M_0 e^{-(\ln(2)/h)t}.$$

With this background in place, one is now ready to understand how carbon dating works.

All living things contain carbon, and the preponderance of the carbon present in plants and animals is its stable isotope, carbon-12. At the same time, every living being takes in radioactive carbon-14, and this carbon-14 becomes part of our organic makeup. While carbon-14 is constantly decaying simply by doing the normal things that come with being alive, each living organism continuously replenishes its supply of carbon-14 in such a way that the ratio of carbon-12 to carbon-14 in its body is constant.

When no longer living, a plant or animal lacks the ability to ingest carbon-14, and thus the ratio of carbon-12 to carbon-14 starts to change, and this ratio changes at the rate that carbon-14 decays. Chemists have long known that carbon-14 has a half-life of approximately $h = 5700$ years, and this knowledge, together with the exponential model

$$M(t) = M_0 e^{-(\ln(2)/h)t}$$

enables people to determine the age of certain fossils. Consider, for example, the situation where a bone is found that contains 40% of the carbon-14 it would be expected to have in a living animal. With less than half the original amount present, but more than 25%, it can be determined that the bone is somewhere between one and two half-lives old; that is, the animal lived between 5700 and 11,400 years ago.

Through our understanding of exponential functions and logarithms, this estimate can be made much more precise.

Specifically, let $t = 0$ be the year the animal died. The present year *t* satisfies the equation $M(t) = 0.4M_0$, since 40% of the initial amount of carbon-14 remains. From the model, it is known that *t* must be the solution to the equation

$$0.4\,M_0 = M_0 e^{-(\ln(2)/5700)t}.$$

First, divide both sides by M_0 to get $0.4 = e^{-(\ln(2)/5700)t}$ and then, taking the natural logarithm of both sides ofthe equation, it follows that

$$\ln(0.4) = \frac{-\ln(2)}{5700}t.$$

Thus, solving for *t* yields

$$t = \frac{-5700\ln(0.4)}{\ln(2)} \approx 7500 \approx \text{years}$$

and the skeletal remains have been dated according to their carbon content.

Limitations of Carbon Dating

Carbon dating does have some reasonable limitations. One of these involves the complications of measuring only trace amounts of carbon-14, and emphasizes the behavior of functions that model exponential decay. For each half-life that passes, half of the most recent quantity of the element remains. That is, after one half-life,

$\frac{M_0}{2}$ remains; after two, half of that amount,

or $\frac{M_0}{4}$ is left;

after three, $\frac{M_0}{8}$ is present.

The quantity rapidly diminishes from there. For instance, after 10 half-lives have elapsed, there is

$$\frac{M_0}{2^{10}} = \frac{M_0}{1024} \text{ or approximately}$$

$0.0009766M_0$ left. Because each living organism only contains trace amounts of carbon-14 to begin with (of all carbon atoms, only about one-trillionth are carbon-14), after 10 half-lives elapse, the remaining amount of carbon-14 is so small that it is not only difficult to measure accurately, but it is difficult to ensure that the measured carbon-14 actually remains from the organism of interest and was not somehow contributed from another source. Ten half-lives is approximately 60,000 years, so any organism deemed older than that needs to be dated in another manner, typically using other radioactive isotopes that have considerably longer half-lives.

Finally, because radiocarbon dating depends on naturally occurring radioactive decay, its accuracy depends on such decay not being accelerated by unnatural causes. In the 1940s, the Manhattan Project resulted in humankind's development of synthetic nuclear energy and weapons; subsequent nuclear testing and accidents have released radiation into the atmosphere that makes the accuracy of carbon-14 dating more suspect for organisms that die after 1940.

New Developments

The exponential model $M(t) = M_0 e^{-kt}$ of radioactive isotope decay has enabled humans to better understand our surrounding world, and to know with confidence key information about the history of the existence of plant and animal life on Earth. Even today, there are new developments in the science of radiocarbon dating as experts work to understand how subtle changes in Earth's magnetic field and solar activity affect the amounts of carbon-14 present in the atmosphere. In addition to continuing to help analyze fossil histories, carbon-14 dating may prove an important tool in ongoing research in climate change.

The Accelerator Mass Spectrometry method of dating directly measures the number of carbon atoms rather than their radioactivity, which allows for the dating of small samples. Other methods under development include nondestructive carbon dating, which eliminates the need for samples. A group of Russian mathematicians have proposed a new chronology of history based on other methods for dating; however, many have dismissed their work as pseudoscience. Physicist Claus Rolfs explores methods to accelerate radioactive decay in the hope of reducing the amount of radioactive material.

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Matt Boelkins

See Also: Algebra and Algebra Education; Calculus and Calculus Education; Exponentials and Logarithms; Functions.

Carbon Footprint

Category: Weather, Nature, and Environment.
Fields of Study: Algebra; Data Analysis and Probability; Measurement; Representations.

Summary: A carbon footprint is a mathematical calculation of a person's or a community's total emission of greenhouse gases per year.

Carbon footprint is intended to be a measure of the ecological impact of people or events. It is a calculation of total emission of greenhouse gases, typically carbon dioxide, and is often stated in units of tons per year. There is no universal mathematical method or agreed-upon set of variables that are used to calculate carbon footprint, though scientists and mathematicians estimate carbon footprints for individuals, companies, and nations. Many calculators are available on the Internet that take into account factors like the number of miles a person drives or flies, whether or not he or she uses energy efficient light bulbs, whether he or she shops for food at local stores, and what sort of technology he or she uses for electrical power. Some variables are direct, such as the carbon dioxide released by a person driving a car, while others are indirect and focus on the entire life cycle of products, such as the fuel used to produce the vegetables that a person buys at the grocery store and disposal of packaging waste.

The notion of a carbon footprint is being considered in a wide range of areas, including the construction of low-impact homes, offices, and other buildings. The design must take into account not only the future impact of the building in terms of carbon emissions, but carbon-related production costs for the materials, labor, and energy used to build it. Mathematical modeling and optimization helps engineers and architects create efficient, useful, and sometimes even beautiful structures while reducing the overall carbon footprint. Mathematicians are also involved in the design of technology that is more energy efficient, as well as methods that allow individuals and businesses to convert to electronic documents and transactions rather than using paper. These methods include using improved communication technology, faster computer networks, improved methods for digital file sharing and online collaboration, and security protocols for digital signatures and financial transactions. Manufacturers are increasingly being urged and even required to examine their practices, since manufacturing processes produce both greenhouse gasses from factory smokestacks and waste heat. Mathematicians and scientists are working on ways to recycle much of this heat for power generation. One proposed device combines a loop heat pipe, which is a passive system for moving heat from a source to another system, often over long distances, with a Tesla turbine. Patented by scientist and inventor Nikola Tesla, a Tesla turbine is driven by the boundary layer effect rather than fluid passing over blades as in conventional turbines. It is sometimes called a Prandtl layer turbine after Ludwig Prandtl, a scientist who worked extensively in developing the mathematics of aerodynamics and is credited with identifying the boundary layer.

These are in turn related to the Navier–Stokes equations describing the motion of fluid substances, named for mathematicians Claude-Louis Navier and George Stokes. The Navier–Stokes equations are also of interest to pure mathematics, since many of their mathematical properties remain unproven at the beginning of the twenty-first century.

Carbon Footprints of People

A calculation of the carbon footprints of different aspects of people's lives, and then the aggregate for a year, is always an estimate. For example, different towns use different methods for generating electricity. Entering data for an electric bill allows for a rough estimate of the household's carbon footprint, but not exact numbers, which would depend on the electricity generating methods. Houses contribute to carbon footprints through their building costs, heating and cooling, water filtration, repair, and maintenance—all of which use products with carbon footprints.

Travel is another major contributor to peoples' carbon footprints. Daily commutes and longer trips with any motorized transportation contribute to carbon dioxide emissions. When computing carbon footprints, fuel production and storage costs have to be taken into consideration.

The food that people eat contributes to the carbon footprint if it is transported by motorized vehicles before being eaten. The movement of locavores (people who eat locally grown foods) aims to minimize the carbon footprint of food. Also, different farming practices may contribute more or less to the carbon footprint of food.

The objects people use contribute to their carbon footprints. Recycling and reusing reduces the need for landfills, waste processing, and waste removal, all of which have carbon footprints. There are individuals and communities who avoid waste entirely; several countries, such as Japan, have plans to mandate zero-waste practices within the next few decades.

Economy and Policy

There are two main strategies for addressing carbon footprints. The first strategy is to lower the carbon footprint by modifying individual behaviors, such as traveling by bike, eating locally, and recycling. The second strategy is to perform activities with negative carbon footprints, such as planting trees, to match carbon footprints of other activities.

Some companies incorporate activities that offset the carbon footprint of their main production into their business plans, either lowering their profit margins or passing the cost to their customers. There are economic laws and proposals that attempt to integrate carbon footprint considerations into the economy, usually through taxes on use of fuel, energy, or emissions. Carbon dioxide emissions, in economic terms, are a negative externality (a negative effect on a party not directly involved in the economic transaction). Money collected through carbon taxes is generally used to offset the cost to the environment.

Emissions trading is another mathematics-rich area of dealing with carbon footprints economically. Governments can sell emission permits to the highest-bidding companies, matching their carbon footprints, and capping the total emission permits sold. This method allows prices of permits to fluctuate with demand, in contrast with carbon taxes in which prices are fixed and the quantities of emissions can change. Economists model the resulting behaviors, and advise policymakers based on the models' outcomes.

Marginal Abatement Cost Curve

"Marginal cost" is an economic term that means the change of cost that happens when one more unit of product is made, or unit of service performed. For physical objects, the curve is often U-shaped. The first units produced are very costly because their cost production involves setting up the necessary infrastructure. As more units are produced, and the infrastructure is reused, the price goes down until the quantities of production reach such levels that the logistic difficulties drive the price per additional units higher again.

A marginal abatement curve shows the cost of reducing emissions by one more unit. These curves are usually graphed in percents. For example, such a curve can be a straight line, with the cost of eliminating the first few percent of emission being zero or even negative. This happens because it can be done by changing practices within existing economic infrastructures, such as cheap smart switches into the residential sector's lighting grids. Additional lowering of the carbon footprint, however, requires deeper and costlier changes to the way of life. For example, there are relatively high costs involved in switching to wind and solar power, or switching to the use of crop rotations that do not require high-carbon fertilizers.

Country by Country

The average carbon footprint of citizens varies by country. For example, in late 2000s, the average annual carbon footprint of a U.S. citizen was about 30 metric tons per year, and a Japanese citizen about 10 metric tons per year. However, these calculations are extremely complicated because of global trade. For example, many developed countries "export" or "outsource" their carbon emissions to developing countries. Products imported from developing countries account for anywhere from a tenth to a half of the carbon footprints of developed nations.

International calculations indicate a strong correlation between the average carbon footprint of a country's citizen and the average per capita consumption. The higher the consumption rates, the higher the average carbon footprint.

The categories used for calculation for countries are similar to those used for individuals and include construction, shelter, food, clothing, manufactured products, services, transportation, and trade. The ratios of these items to one another in the carbon footprints vary by country. For example, the greatest item in the U.S. carbon footprint is shelter (25%), with mobility

Carbon footprints are calculated to include travel, fuel production, transportation, and storage. In Canada, mobility is the highest contributor to the national carbon footprint.

being second (21%). In contrast, Canada's greatest item affecting carbon footprint is mobility (30%), and its second greatest is shared between shelter and service (18% each).

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Maria Droujkova

See Also: City Planning; Climate Change; Electricity; Energy; Farming; Fuel Consumption; Green Design; Green Mathematics; Recycling; Traffic; Wind and Wind Power.

Careers

Category: Mathematics Culture and Identity.
Fields of Study: All.
Summary: There are a wide variety of careers in many disciplines available to those with a mathematics background.

"What can one do with a mathematics degree other than teaching?" It is a question asked by many aspiring mathematicians. In fact, a more accurate question to ask should be "What can't one do with a math degree?" Actually, the study of mathematics extends far beyond mere number crunching and doing fast mental arithmetic in grocery stores. The fact is that studying mathematics can prepare one for numerous careers.

In general, companies believe that studying mathematics develops analytical skills and the ability to work in a problem-solving environment. These are the skills and experiences that are essential assets to one's success in the workplace. Precisely, mathematics is often the quintessential element to fluently communicate with people of various backgrounds. It is the ability to efficiently process a manifold of information and deliver the technical details to a general audience that makes mathematicians valuable. Having a mathematics background not only helps people broaden their pool of career options, it also helps to land some of the best jobs available.

According to an article published in the *Wall Street Journal* on January 26, 2009, a "mathematician" is considered to be the best occupation in the United States. This ranking was determined based on five criteria inherent to every job: environment, income, employment outlook, physical demands, and stress. In fact, five out of the six "best jobs" in terms of low stress, high compensation, autonomy, and hiring demand in the *Job Related Almanac* by Les Krantz are all mathematics related: (1) mathematician, (2) actuary, (3) statistician, (4) biologist, (5) software engineer, and (6) computer systems analyst. In this entry, a collection of possible career opportunities appropriate for someone with a mathematics background is provided, and a list of resources is given on how to find a job with different levels of academic degrees. The lists are by no means exhaustive and should only be used as a reference.

Analytical Thinking

Why is mathematics a required subject in school curricula at all levels? Why is mathematics so essential for the proper functioning of everyday tasks in society? Why do most people who excel in their field credit their success to their formal training in mathematics? One possible reason is that a proper training in mathematics provides people with abilities to think and solve problems critically in novel settings.

A Web site sponsored by the Department of Mathematics at Brigham Young University provides a list of possible career options for someone with a background in mathematics. Some of the more common professions include actuary, architect, chemical engineer, college professor, computer scientist, cryptanalyst, economist, mechanical engineer, quantitative financial market analyst, and statistician; some less well-known career options include air traffic controller, animator, astronaut, epidemiologist, geologist, hydrologist, lawyer, market research analyst, composer, physician, technical writer, and urban planner. Certainly, a fixed set of mathematics curriculum will not prepare one for all the jobs listed here. What will be consistent is gaining the ability to solve problems analytically and critically.

Not many people know that the San Antonio Spurs Basketball Hall of Famer David Robinson had a B.S.

in mathematics from the U.S. Naval Academy. Even Michael Jordan toyed with the idea of being a mathematics major in his early college years. It is perhaps not surprising that one of the world's most influential bankers and financiers, J. P. Morgan, majored in mathematics, but not many would think that mathematics would find its way into entertainment. For example, American actress Danica McKellar, who had a leading role in a television comedy-drama *The Wonder Years*, is a well-known mathematics author and education advocate. The popular television drama, *Numb3rs* featured a mathematician who helped his brother in the FBI to solve crimes with his mathematical genius. A few popular movies that successfully portray mathematicians in society include *Good Will Hunting* (1997), *A Beautiful Mind* (2001), and *Proof* (2005).

Although it is seemingly impossible to categorize every branch of mathematics in society, career options available for those who study under common branches of mathematics include the areas of applied mathematics, actuarial mathematics, financial mathematics, and other emerging fields.

Applied Mathematics

Applied mathematicians often solve problems that originate in physics, chemistry, geology, biology, or various disciplines of engineering. Mathematics is used to model physical phenomena, to answer questions derived from observations, to learn characteristics of large quantities, and to make predictions and improvements for future events. A representative mathematical training includes coursework in numerical analysis and methods, computer programming, computer languages, applied and experimental statistics, and probability theory, as well as a few courses in another field of interest.

Often, a typical applied or computational mathematics problem is interdisciplinary in nature and derived from realistic demands in industry. People who wish to gain a general sense of what these types of problems entail are encouraged to attend mathematics-in-industry workshops that are available in Europe and some parts of the United States and Asia. Mathematics in Industry and International Study Groups maintains a Web site that provides updated information for future study groups and meetings. The Society for Industrial and Applied Mathematics (SIAM) maintains a list of example organizations, corporations, and research institutions that hire mathematicians and computational scientists with an applied mathematics training. These organizations, corporations, and research institutions include the following:

- Aerospace and transportation equipment manufacturers such as Aerospace Corp., Boeing, Ford Motor Co., General Motors, Lockheed Martin, and United Technologies
- Chemical and pharmaceutical manufacturers such as DuPont, GlaxoSmithKline, Kodak, Merck & Co., Pfizer, and Wyeth
- Communications service providers such as Clear Channel Communications, Qwest Communications and Verizon
- Electronics and computer manufacturers such as Bell Laboratories, Alcatel-Lucent, Hewlett-Packard, Honeywell, IBM Corporation, Motorola, Philips Research, and SGI
- Energy systems firms such as Lockheed-Martin Energy Research Corporation and the Schatz Energy Research Center (SERC)
- Engineering research organizations such as AT&T Laboratories—Research, Exxon Research and Engineering, and NEC Laboratories America
- Federally funded contractors such as the Mitre Corporation and RAND
- Medical device companies such as Baxter Healthcare, Boston Scientific, and Medtronic
- U.S. government agencies such as the Institute for Defense Analyses, NASA's Institute for Computer Applications in Science and Engineering, National Institute of Standards and Technology, Naval Surface Warfare Center, Supercomputing Research Center, and the U.S. Department of Energy
- U.S. government labs and research offices such as the Air Force Office of Scientific Research, Lawrence Berkeley National Laboratory, Los Alamos National Laboratory, Oak Ridge National Laboratory, Pacific Northwest National Laboratory, and Sandia National Laboratories
- Producers of petroleum and petroleum products such as Amoco, Exxon Research and Engineering, and Petróleo Brasileiro S/A, Petrobras

Actuarial Mathematics

An actuary is a risk management professional who helps design insurance plans by recommending premium rates and making sure companies are designating enough funds to pay out on claims. Actuaries may also help create new investment tools for financial institutions. The main type of mathematics an actuary uses on a daily basis is applied statistics, which involves arithmetic, basic algebra, and practical applications such as using numbers and math to generate tables and graphs. Actuaries should also have a general understanding of business, economics, and corporate finance, all of which have mathematical components.

Most actuaries have at least a four-year degree in mathematics, business, economics, statistics, or, in some cases, a specific degree in actuarial science. As computer modeling replaces traditional graphs and tables, computer and programming skills have become increasingly important as well. The last step to becoming a licensed actuary is to get certified by passing a series of exams sponsored by either the Society of Actuaries or the Casualty Actuarial Society. The list of possible job choices for someone with an actuarial background is relatively small compared to that of the applied mathematicians. These include the following:

- Consulting firms such as Daniel H. Wagner Associates, Deloitte Touche Tohmatsu, Ernst & Young, Hewitt, McKinsey & Company, and KPMG
- Banks or related financial institutions such as AIG, ING, Capital Management, Chase Manhattan Bank, CitiGroup, Fidelity Investments, Goldman Sachs & Co, HSBC, JP Morgan Securities, Lehman Brothers, Mercer Investment Consulting, Merrill Lynch, Morgan Stanley & Co, Standard and Poor's, TD Ameritrade, and Wachovia Securities
- Brokers such as Acordia, Benfield, Cooper Gay, Heath Lambert, HLF Group, March & McLennan, and Willis Group
- Actuarial software development companies such as Actuarial Resources Corp. (ARCVal, HealthVAL, STAR, UltraVAL, CARVM), BLAZE SSI Corp., EMB America, Integrated Actuarial Services (Total Solution, RAAPID), TAG, and WySTAR Global Retirement Solutions (DBVAL, DCVAL, OPEVS)
- Miscellaneous jobs in large companies and government agencies such as ACTEX Publications, Casualty Actuarial Society, Coca-Cola, Ford Motor Co., International Actuarial Association, National Association of Insurance Commissioners, and the Society of Actuaries
- Insurance companies including both property and liability insurance, and life and health insurance such as AFLAC, AAA of CA, Allstate, Blue Cross and Blue Shield, Safeco, Sun Life, Universal Care, and WellPoint

Financial Mathematics

Financial mathematics is the development of mathematical tools and computational models used in the financial industry and on Wall Street. People in this profession are referred to as "quantitative analysts," or "quants." As new quantitative techniques have transformed the financial industry, banks, insurance companies, investment and securities firms, energy companies and utilities, multinationals, government regulatory institutions, and other industries have all come to rely on applied mathematics and computational science.

Sophisticated mathematics models and the computational methods and skills needed to implement them are used to support investment decisions, to develop and price new securities, and to manage risk, as well as for portfolio selection, management, and optimization. For example, modern hedge funds depend on these sophisticated techniques, as do pricing of bonds and commodity futures. Typically, someone who is interested in working in financial service and investment firms such Citibank, Moody's Corporation, Morgan Stanley, or Prudential will need to have a solid background in mathematical modeling, numerical and computational mathematics, applied statistics, business, economics, and finance.

Emerging Fields

Biomathematics and Bioinformatics. This emerging field can be thought of as a computer science/mathematics/biology hybrid that integrates mathematics and computer technology in the study of biological sciences. Broadly speaking, bioinformatics is the recording, annotation, storage, analysis, and searching/retrieval of nucleic acid sequence (genes, RNAs, and DNAs), protein sequence, and structural information. Mathematicians in this area contribute to the development of new algorithms with which to detect patterns and assess relationships among members of large data sets.

Computer Visions and Computer Graphics. Mathematicians in the field of computer vision work on developing theoretical machine learning algorithms to extract meaningful information from images. The images take on various forms such as waveforms from voice recorders or three-dimensional images from a magnetic resonance imaging (MRI) device. Its example applications include (1) artificial intelligence and controlling processes (for example, industrial robots and autonomous vehicles), (2) pattern recognition and verification (for example, public surveillance and biometric identification), (3) modeling and processing (for example, medical image analysis and terrain modeling), and (4) communication (for example, brain-computer interface for people with disability). Mathematicians in the field of computer graphics develop ways to represent and manipulate image data to be used by computers. The most well-known applications under this category are the video game and computer animation industries, where various transformation matrices and interpolation techniques are used to create smooth and believable subjects in successive frames. Companies such as Pixar and DreamWorks hire mathematicians in their research divisions to come up with innovative ways to enhance visual effects to be more aligned to reality. Other companies that hire mathematicians with backgrounds in computer vision and graphics include Siemens, Hewlett-Packard (HP), Honeywell, Flash Foto, GeoEye, Nokia, Microsoft, Apple Inc., Amazon.com, and Google.

Operations Research. This is a highly interdisciplinary branch of applied mathematics that uses methods such as mathematical modeling and optimization to solve problems that require a complex decision-making process. Mathematical areas such as game theory and graph theory have become useful tools in solving problems under the umbrella of operations research (OR). Examples of disciplines that use OR are financial engineering, environmental engineering, manufacturing and service sciences, policy-making and public sector work, revenue management, and transportation. Almost all companies hire operations research analysts to use mathematics and computers to develop software and other tools that managers can use to make decisions such as how many people to hire and retain in order to maximize productivity and minimize costs.

It is worth reemphasizing that having a mathematics degree or a mathematics-related degree increases one's chance of securing a position in nearly any company. Even areas that are traditionally viewed as pure mathematics such as combinatorics, number theory, topology, algebraic and differential geometry, analysis, and algebra often turn out to have real-world applications; for example, number theory in cryptography, Fourier analysis in speech recognition, and differential geometry in face recognition. Some additional career choices are as follows:

- Nonprofit organizations such as the American Institute of Mathematics (AIM), and SIAM
- Publishers and online products such as Birkhauser, Springer, and Elsevier Science
- University-based research organizations such as the Institute for Advanced Study, the Institute for Mathematics and Its Applications (IMA) and the Mathematical Sciences Research Institute (MSRI)
- Government agencies such as the National Security Agency (NSA) and the U.S. Department of Defense (DoD)
- Teaching at academic institutions. To teach at the high school level, one needs a bachelor's degree in mathematics and a teaching credential; to teach at the community college level, one needs a Master of Science or Master of Art degree in mathematics; to teach at the college level, one needs a Ph.D. in mathematics, mathematics education, applied mathematics, or statistics

Online Mathematics Jobs Listings

The American Mathematical Society (AMS) has an extensive set of resources to help someone in the market for academic positions and is the premier source for information on careers in mathematics. This includes a list of job postings organized by country and state. It has useful features such as an e-mail service that notifies applicants of all new job listings and an online storage of curriculum vitae (academic resume) and transcripts that can be used repeatedly for different applications. In addition, it allows one to register for the job fairs at the annual AMS meetings and has a list of graduate programs for students.

The Math-Jobs Web site lists international and national job openings for mathematicians in both industry and academics.

The Mathematical Association of America (MAA) has a comprehensive set of resources for students, faculties, professional mathematicians, and all who are interested in the mathematical sciences. In particular, MAA Math Classifieds helps people to find career in the diverse field of mathematics.

The *Chronicle of Higher Education* has academic and nonacademic job advertisements. Use the searchable index to find mathematics jobs.

The Mathematical Sciences Career Information by AMS-SIAM has information on nonacademic jobs, profiles of mathematicians in industry, job search tips, and links to many online job-posting services.

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Jen-Mei Chang

See Also: Accounting; Mathematics, Applied; Problem Solving in Society; Professional Associations.

Caribbean America

Category: Mathematics Around the World.
Fields of Study: All.
Summary: The diverse islands of Caribbean America have produced notable mathematicians.

The Arawaks, Caribs, and other pre-Columbian peoples lived in the area of the Caribbean Sea before Spanish, French, English, or Scottish sea traders settled there. Linguists explore the different languages that were spoken in the Caribbean, traces of which can be found in the twenty-first century. Along with these languages, there were possibly different numerical systems. Sea merchants needed bookkeepers and accountants to keep track of their business, and although Port Royal and other places in the seventeenth-century Caribbean were notorious for piracy and lawlessness, there were also many counting houses and legitimate business operations.

The development of schools and universities led to more mathematical opportunities. According to the United Nations, the Caribbean America region encompasses Anguilla, Antigua and Barbuda, Aruba, the Bahamas, Barbados, the British Virgin Islands, Cayman Islands, Cuba, Dominica, the Dominican Republic, Grenada, Guadeloupe, Haiti, Jamaica, Martinique, Montserrat, Netherlands Antilles, Puerto Rico, Saint-Barthélemy, Saint Kitts and Nevis, Saint Lucia, Saint Martin (French part), Saint Vincent and the Grenadines, Trinidad and Tobago, Turks and Caicos Islands, and the U.S. Virgin Islands. By the end of the twentieth century, there were numerous Caribbean mathematicians, and *The Caribbean Journal of Mathematical and Computing Sciences* has published volumes of research articles.

Mathematicians in the Caribbean, and around the world, have also worked on mathematics history and research that is specifically related to the Caribbean area, like C. Allen Butler, who investigated optimal search techniques for smugglers in the Caribbean. Mathematicians have also discussed the high numbers of university graduates who have left the Caribbean, and they have created educational initiatives and mathematical texts designed for Caribbean children. The Caribbean and Central America areas combine for a joint Mathematical Olympiad. The most well-known mathematician in the region is perhaps Keith Michell from Grenada who, after completing his doctoral thesis from the American University, was a professor at Howard University, and then returned to Grenada, becoming prime minister in 1995, a position he held until 2008.

Barbados

On the island of Barbados, although education was an important facet of colonial life from the late nineteenth century on, few students were able to continue with mathematics. One exception was Merville O'Neale Campbell, who had become fascinated with mathematics at an early age and won a scholarship to study at Cambridge University in England. He then went to teach at the Gold Coast (now Ghana), completing his doctoral thesis, "Classification of Countable Torsion-

Free Abelian Groups," from the University of London, and is noted as the first Barbadian to have a Ph.D. in mathematics. His daughter, Lucy Jean Campbell, also completed her doctoral thesis in mathematics, and specializes in geophysical fluid dynamics, nonlinear waves, and a variety of numerical and analytical methods at Carleton University in Ottawa. Other prominent Barbadian mathematicians include Charles C. Cadogan, who has edited the *Caribbean Journal of Mathematical and Computing Sciences* and has contributed papers in journals around the world; and Hugh G. R. Millington, who completed his doctorate, "Cylinder Measures," from the University of British Columbia and then worked at the University of the West Indies, Cave Hill, Barbados.

British Caribbean

Well-known mathematicians from the British Caribbean include those from Jamaica. Earl Brown, who was the head of the Department of Science & Mathematics at University of Technology (Jamaica) from 1997 to 2000, completed his doctoral thesis at the Massachusetts Institute of Technology. Joshua Leslie completed his doctoral thesis from the Sorbonne in Paris, and was the chair of the Mathematics Department at Howard University; and Kweku-Muata Agyei Osei-Bryson from Kingston completed his doctoral thesis, "Multi-objective and Large-Scale Linear Programming," at the University of Maryland—College Park in 1988, and from 1993 until 1997 was the Faculty Fellow (Information Systems) for the U.S. Army, The Pentagon. Other prominent mathematicians from Jamaica, or whose ancestors were from Jamaica, include Garth A. Baker, Charles Gladstone Costley, Leighton Henry, Fern Hunt, Lancelot F. James, Clement McCalla, Bernard Mair, Claude Packer, Paul Peart, Donald St. P. Richards, and Karl Robinson.

Elsewhere in the British Caribbean, there have also been a number of mathematicians who held senior positions in the region and in the United States including Ron Buckmire from Grenada, who has specialized in computational aerodynamics; Edward Farrell from Trinidad, who has published extensively on polynomials; and Velmer Headley from Barbados, who has concentrated on the study of differential equations.

Cuba

One notable Cuban mathematician is Argelia Velez-Rodriguez, who was born in Havana and won her first mathematics competition when she was 9. She was the first Afro-Cuban to complete a doctorate from the University of Havana but left Cuba two years later to live in the United States. Since the 1959 Revolution, there has been an increased emphasis on the education system in Cuba, and Cuban students have long shown a high aptitude for mathematics.

French Caribbean

French Caribbean mathematicians include those from Haiti, with a desperately poor education system, and Guadeloupe. Louis Beaugris completed his doctoral thesis, "Some Results Related to the Generators of Cyclic Codes Over Zm," at the University of Iowa. Serge A. Bernard completed his doctoral thesis, "A Multivariate EWMA Approach to Monitor Process Dispersion," at the University of Maryland—College Park; and Jean-Michelet Jean-Michel completed his doctorate at Brown University. Alex Meril from Guadeloupe completed his thesis at the University of Bordeaux and worked at the University of Guadeloupe.

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Justin Corfield

See Also: Central America; North America; South America.

Carpentry

Category: Architecture and Engineering.
Fields of Study: Algebra; Geometry; Measurement.
Summary: Precise measurement is the foundation of the building trades.

While the word "carpentry" originally comes from the Latin root for chariot maker, today, the term refers to a number of trades that use wood for the construction of buildings and other articles. As there is a wide range of activities involved in carpentry tasks, carpenters must possess many different manual and intellectual skills to function in the profession.

Types of Carpenters

Carpenters who work on houses often fall into one of two broad categories: framing carpenters who work on the rough frame of a building, and finish carpenters who complete trim, stairs, railings, shelves, and other detail work. However, in practice, many carpenters end up doing some of each type of work, and carpenters who specialize in remodeling may not only do framing and finish carpentry but also tasks that are not strictly carpentry at all, such as plumbing, wiring, sheetrock finishing, and painting. There are also carpenters who specialize more narrowly, such as cabinet makers or carpenters who work on the specialized joinery between large posts and beams required in timber frame and log cabin construction.

Carpenters need to be able to read blueprints, and measure, cut, fasten, and finish a variety of materials.

Tasks of the Carpenter

Carpentry requires a variety of skills, including reading blueprints, measuring, cutting, fastening, and finishing. In addition, a carpenter must have knowledge of materials, including a variety of wood products and fasteners; and tools, including measuring devices, saws, drills, hammers, planes, and sanders. Carpenters who work on their own or as subcontractors on larger jobs must also have skills in cost-estimation and billing.

Consider, for instance, a carpenter who has been hired to add a covered deck onto a house. This carpenter might begin by working with the homeowner to determine the size and shape of the deck, possibly using a Computer Assisted Design (CAD) program to generate three-dimensional representations of how the finished project will look. After deciding on a design, the carpenter will need to use structural engineering tables to assess structural issues related to the design, such as the dimensions required for posts, the placement and size of cross-bracing, and the sizes of timbers that will be needed to span the distance between posts. From the calculations, the carpenter will then generate a price estimate, based on a materials list and an estimate of labor. The actual construction will include pouring concrete footers for the posts, measuring and cutting posts and joists with a circular saw, fastening materials to one another and to the house, screwing decking materials to the framing, framing a roof, installing roofing materials, constructing a railing, and building and finishing a set of stairs from the yard to the deck.

A Carpenter's Calculations

In the process of creating a simple covered deck, this carpenter will be making many measurements, calculations, and decisions regarding:

Layout: The initial position of the deck must be laid out so it is square to the house. To do this, the carpenter will construct a set of batter boards that are set outside the corners of the proposed deck and allow strings to be pulled to mark the edges of the deck. Employing the rule that the diagonals in a rectangle are equal to one another, the carpenter adjusts the strings to bring the corners to 90 degrees. Corner square may also be established and checked using the Pythagorean theorem.

Footers: Each post will be anchored to a concrete footer that will prevent it from moving or sinking into the ground. The bottom of the holes for these footers must be dug below the freeze level for the geographic area where the deck is being built so that the footers will not be heaved out of place by the freezing and resulting expansion of the soil. By consulting the building code, the carpenter will determine the appropriate area for the footer in square feet; multiplying by the height will give the cubic feet. If this is a large project, where the concrete will be delivered, the carpenter will have to convert cubic feet to cubic yards, as this is the unit in which concrete is ordered.

Raising the Posts: After pouring the footers, the carpenter will raise the posts for the deck being built. Since these posts will also support the roof in this example, they must be cut carefully to take into account any variation in the height of the footers. This measurement will be done by using a transit, a laser level, or a water level to assess the difference in the height of the footers. The carpenter will then add or subtract length to the height of each post to compensate. Once the posts are cut, they can be raised into position, ensuring each is plumb (perfectly vertical) using a level.

Joists and Decking: The sizing for all the wooden parts of the deck is determined by calculating how long a distance must be spanned and the weight the span will carry. The timber that is parallel to the house and runs between the posts must be sized to be strong enough to carry all the weight between each pair of posts; the longer the span between posts, the larger this timber must be. Similarly, the floor joists that butt into this timber will need to be large enough to carry the weight over their length, and the decking will be sized so that it does not sag between the joists.

Fasteners: In our example, the deck will be fastened to the building using bolts, and held together using nails, while the decking itself will be screwed on. The carpenter has many fasteners to choose from with many different finishes. Each type of fastener has special characteristics that make it useful for certain tasks. Nails are typically sold by the pound and come in sizes from large 20d framing nails (often called 20 penny nails) to small 6d finish nails. Screws are also sold by the pound but are sized by length and by a number that can be converted, using a chart, to their diameter. Bolts are sold by diameter and length; as is the case with all fasteners, there are many different types among them, lag bolts, carriage bolts, and through bolts.

The Roof: The roof over the deck will be set at an angle so water runs off it and away from the house. The pitch of a roof is typically measured in "rise over run," with the denominator of this fraction always given as 12. Thus, a roof that goes up four feet over a run of 12 feet is said to be a "4:12 roof." The carpenter will use a special tool called a "speed square" that allows the direct conversion of roof pitch to angles and mark rafters for cutting.

Stairs: While stairs can be constructed to be more or less steep, a carpenter must keep in mind a basic mathematical relationship between tread length and riser height that will make a set of stairs comfortable to ascend. It turns out that because of the characteristic of the human gait, the steeper a stair, the less wide each tread should be. The formula that carpenters use is that for each stair, twice the rise plus the run should equal 24–26 inches.

Of course, once the carpenter is done with the project, there are still numerous other tasks to complete, including building railings and benches, as well as finishing and waterproofing the surfaces. If the homeowner were to want an outdoor grill area, with built-in cabinets, the carpenter would have a whole new set of challenges worthy of a cabinet maker and finish carpenter.

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Jeff Goodman

See Also: Geometry in Society; Measurement, Systems of; Pythagorean Theorem.

Castillo-Chávez, Carlos

Category: Mathematics Culture and Identity.
Fields of Study: Algebra; Connections; Data Analysis and Probability; Measurement.

Summary: Carlos Castillo-Chávez works in the field of mathematical epidemiology, which deals with the spread, treatment, and eradication of diseases.

Carlos Castillo-Chávez (1952–) is a Mexican-American applied mathematician, eminent in the field of mathematical epidemiology. His research and writing has advanced human understanding of the mechanisms by which diseases spread and by which they can be contained. The specific diseases that he has worked with extensively include human immunodeficiency virus and acquired immune deficiency syndrome (HIV/AIDS), tuberculosis, influenza, and many others.

He grew up in Mexico, where he excelled academically. Motivated in part by the Tlatelolco massacre in which hundreds of Mexican students were killed, he emigrated to Wisconsin in 1974. In 1984, he was awarded a Ph.D. in mathematics from the University of Wisconsin—Madison. He spent 18 years as a professor at Cornell University before coming to Arizona State University, where he is both professor of mathematical biology and executive director of the Mathematical and Theoretical Biology Institute, as well as the Institute for Strengthening the Understanding of Mathematics. He is considered an important voice of the mathematical biology community and has served on many influential committees and panels, including the National Science Foundation, the National Institutes of Health, the Society for Industrial and Applied Mathematics, and the American Mathematical Society.

Mathematics and Biology

In the past decades, mathematics and biology have enjoyed an increasingly symbiotic relationship. Mathematical biology is a wide area of applied mathematics, focusing principally on modeling. A "model" of a biological process or phenomenon is a mathematical system that obeys certain rules and properties abstracted from what we know (or suspect) about the biology in the real world. Two primary mathematical tools for mathematical biology are the study of dynamical systems and differential equations, since we are often most interested in how certain quantities change in response to other quantities.

A distinguishing feature of mathematical biology is the mutual feedback between the mathematicians and the scientists involved. A model based on today's understanding of a certain disease (or of the action of neurons, or of cellular growth) may make certain predictions, suggesting that certain experiments be performed. The results of these experiments can improve, correct, and refine scientists' understanding of the underlying biology. Mathematicians can then incorporate this new knowledge into more sophisticated, more accurate models.

Carlos Castillo-Chávez is a leader in the area of mathematical epidemiology, the branch of mathematical biology dealing with the spread, treatment, and eradication of diseases. Mathematical epidemiologists can use mathematical modeling techniques to predict how certain diseases might affect the population. More sophisticated models can incorporate the effects of various proposed treatment and control options. Properly applied, these techniques can enable epidemiologists to effectively predict the effects of methods of prevention, allowing for a more effective allocation of resources in responding to disease threats.

Minorities in Mathematics and Science

Carlos Castillo-Chávez is an outspoken advocate of minorities, women, and other underrepresented groups in mathematics and the sciences. He has expressed the belief that people from different backgrounds may bring different perspectives to mathematics and science, leading them to directions of research that may have gone unnoticed or uninvestigated. Since mathematics and the sciences are driven by the questions that participants pursue, asking a richer set of questions leads to a fuller body of knowledge; supporting students from underrepresented groups minorities is therefore a matter both of social justice and of enhancing the discipline.

Dr. Castillo-Chávez has supported these beliefs with his actions at all stages of his career. As a Ph.D. student in Milwaukee, he spent his summers teaching mathematics to Latino students in the area. He has served as a mentor to numerous female and minority students, helping and encouraging them at all stages of education. He is also an active member of the Society for the Advancement of Chicanos and Native Americans in the Sciences (SACNAS); during his time at Cornell, he was the founding president of a northeast chapter of SACNAS and was instrumental in initiating a special summer program intended to provide Latino, Chicano, and Native American students with mentorship, encouragement, and training in the sciences. Carlos

Castillo-Chávez is highly acclaimed for his work in this regard.

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Michael "Cap" Khoury

See Also: Disease Survival Rates; Diseases, Tracking Infectious; Mathematics, Applied; Viruses.

Castles

Category: Architecture and Engineering.
Fields of Study: Algebra; Geometry; Measurement.
Summary: Mathematics has been used to both construct and study castles.

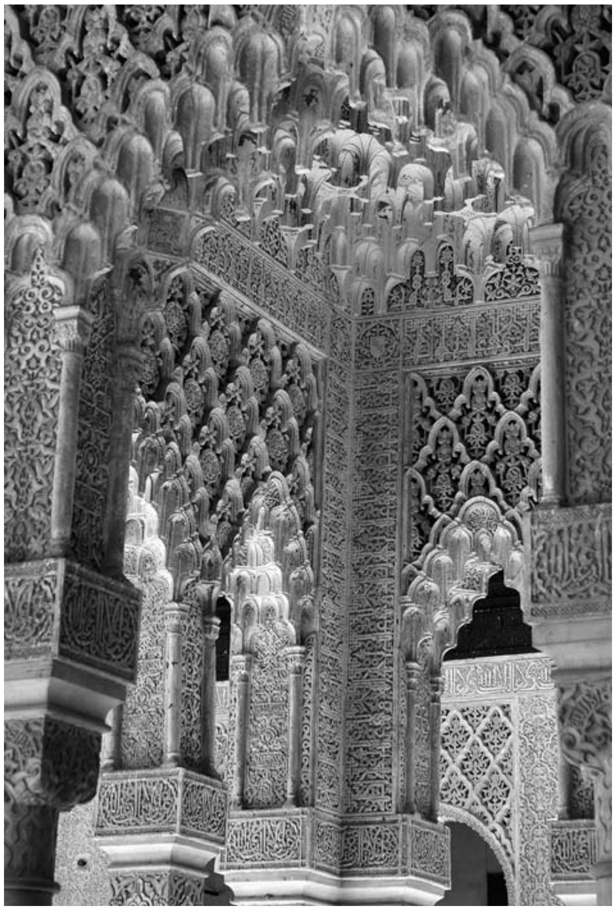
Castle are fortified structures, used as residences by European nobles in the Middle Ages. Early castles were often made of wood, but with the development of better attack methods, castle builders switched to stone as the main building material. With the extensive use of artillery, residential castles became indefensible. They were replaced by purely military forts (not used for administrative and residential purposes) and decorative residences resembling castles (not used during wars). The geometry of a castle was often dictated by defense considerations. Architect Benjamin Bramer fortified castles and published a work on the calculation of sines.

The Alhambra, a fourteenth-century palace and fortress, is well known for its mathematical tiles. In the early twenty-first century, the American Institute of Mathematics proposed a headquarters in California that would be modeled after the Alhambra, popularly referred to as a "math castle."

Castles are frequently found in fantasy and horror literature. One common image is that of Dracula's castle. Dracula author Bram Stoker earned a degree in mathematics. Some mathematics teachers use castles like Cinderella's castle or sand castles to explore concepts such as ratios, fractions, volume, statistics, and geometric shapes. Scientists, including physicist Mario Scheel, explore the physical properties of sand-like material, and researchers in experimental archaeology model and design castles.

Geometry of Castle Defense

Both the layouts of castles and the shapes of their parts were dictated by defense needs. For example, concentric



The Alhambra, in Granada, Spain, is known for its Moorish use of mathematical symmetry groups.

castles consisted of several concentric walls. The barbican (the outer wall) had relatively many entrances, while the inner wall had few, making the attacking army crowd between walls and, thus, become vulnerable to defenders.

Keeps and towers were mostly round to allow for a larger arc of shooting coverage from each arrowslit. In addition, the isoperimetric theorem states that for a given area, the circle has the least perimeter among all shapes, thus minimizing the amount vulnerable walls (not to mention reducing the costs of building materials). Each corner introduced blind spots where enemies could avoid arrows, and circles have no corners. Also, corners are more vulnerable for mining.

Cylindrical towers led to the invention of spiral staircases. Most castle staircases were built so attackers would ascend clockwise, making the central shaft of the staircase interfere with their right hands—often the hand that held the sword.

Stonemasons building castles used simple tools, such as compasses, dividers, and straightedges. Their manuals included descriptions for creating a variety of shapes with these tools. For example, pointed and rounded arches, including Tudor, lancet, and horseshoe arches, could be traced with compasses and straightedges.

Shooting from high towers allowed for better view, and also used gravity to add acceleration to arrows and other projectiles. When glass windows were installed in circular towers, they were made by blowing glass inside a cylinder, cutting it, and then connecting multiple pieces with lead to match the curvature of the castle wall.

Castle builders used terrain geometry to support defense. In addition to the height advantage of the castle walls and towers, castles were frequently situated on hills (either natural or artificial) or on earthen mounds called "mottes." Defensive ditches around castles, called "moats," prevented siege towers from coming close. When moats were filled with water, they could also make digging tunnels for mining the walls more difficult.

The construction of moats led to the invention of drawbridges and the mechanisms of raising and lowering them. The drawbridge mechanisms involved levers and pulleys.

Logistics and Finance

Building a large castle was a major financial undertaking spanning many years, and occasionally bankrupted the ruler attempting it, such as King Edward I. Supplying the castle, especially with enough supplies to withstand lengthy sieges, presented another organizational problem. A siege was a common method of castle attack in which the attackers surround the castle grounds and waited for the defendants to starve. The siege process could sometimes last for months or even years.

Experimental archaeology is a new field of study that combines archaeological research, computer modeling, and actual building. Observations in building experiments allow for conclusive results of how models can be made to work. For example, Project Gueledon is a real-size castle built recently to help give people a deeper understanding of how castles were constructed in medieval times. The researchers used building methods and materials similar to those used by thirteenth-century castle builders, with a team of 50 workers from various professions.

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Maria Droujkova

See Also: Archery; Artillery; Geometry in Society; Middle Ages; Tunnels.

Caves and Caverns

Category: Weather, Nature, and Environment.
Fields of Study: Algebra; Geometry; Representations.
Summary: Several metrics are used to describe caves while mathematical measurements can detect them.

Caves are underground spaces large enough for a human to enter. The science of studying caves is called speleology and the practice of exploring caves is spelunking. Caves can be formed through a variety of ways, such as solutional caves (made by rocks dissolving in acids in water) or littoral caves (made by waves pounding cliffs). They are also categorized by the passage patterns, such as angular networks or ramiform caves. Mathematical techniques are used to model and understand the structures and ages of caves and caverns. For instance, the topology of the cave highlights the number of tunnels and how they are connected while the geometry shows accurate distances, curvatures, and steepnesses. Statistical methods as well as fractal concepts of self-similarity have been used to estimate the number of entranceless caves. Archaeology has revealed that caves are among the oldest known human habitations.

Some researchers analyze ancient cave paintings for mathematical, astronomical, or geographical interpretations. Mathematical objects and mathematicians have also been connected to caves and caverns. The Lebombo bone was discovered in a Swaziland cave in the 1970s. It dates to approximately 35,000 B.C.E and is thought to be the oldest known mathematical artifact. The bone holds 29 tally notches and it has been compared to calendar sticks that are still in use in Namibia. In France, numerous mathematicians trained at École des Mines including Henri Poincare, who was employed as a mine engineer and was eventually promoted to inspector general.

Visitors today can enter Pythagoras' cave in Samos, where he apparently lived and worked on mathematics. In *The Republic*, Plato imagines chained prisoners in a cave who can only see shadows of the movement behind them. Similar metaphors continue to be explored in order to explain higher dimensional realities and other concepts in mathematics, physics, and philosophy, including investigations of quantum caves.

Cave Measurement and Records

There are several metrics used to measure caves, including total length of passages, depth from the highest entrance to the lowest point; total volume, or height, depth, length, area; and volume of individual passages, shafts, and rooms. The deepest cave is 2191 meters meters (7188 feet) deep and the greatest total length cave is 591 kilometers (67 miles) long. These numbers are updated as more parts of caves are explored and new caves are discovered.

Geophysical Detection of Caves

The mapping of hidden caves and smaller karst formations is done for scientific and recreational explorations, as well as to ensure the stability of constructions, such as houses and bridges. Geophysical detection methods use contrasts in a physical property, such as electric resistance or density, between different parts of the underground medium. To detect variations, scientists measure microscopic changes in gravity caused by empty spaces, or transmit electromagnetic waves into the ground and measure their reflections. Another method is to transmit an electric current and measure changes in ground resistance. Seismic tomography depends on collecting massive amounts of data from inducing stress through boring holes, but it can be very accurate. All these methods depend on mathematical models of changes in physical properties between different surfaces.

All geophysical techniques require contrasts of some physical property (density, electrical resistivity, magnetic susceptibility, seismic velocity) between subsurface structures.

Cave Patterns

The geometry of a cave depends on many geological factors, such as the structures dominant in the rock and the sources of water for solution caves. Spongework caves consisting of large, connecting chambers formed in porous rocks. If the rock also fractures easily, large chambers will be interspersed with long passages formed by fracturing in a pattern called "ramiform" (branchlike). Nonporous rock that fractures will produce a distinct pattern called rectilinear branchwork, with straight passages at angles to one another. Lava tubes are round in cross-section, long, and relatively even; they are formed by a lava flow that develops a hard crust.

Cave Meteorology and the Geothermal Gradient

Heat in caves comes from water or air entering the cave, or from overlying and underlying rock. Overlying rock does not transmit the surface heat well. For example, a difference of 30 degrees Celsius between day and night on the surface translates into 0.5 degrees Celsius difference one meter (3.28 feet) deep into limestone. Seasonal fluctuations penetrate deeper but still become negligible at depths of 10 or so meters (32.8 feet).

In most parts of the world, the temperature increases by about 25 degrees Celsius for every kilometer of depth, because of the molten interior of Earth, the rate called "geothermal gradient." As one goes deeper into a cave that starts at a sea level, the temperature first drops because of insulation from the surface but then increases because of the geothermal gradient. In areas of high volcanic activity near the surface, caves can be very hot, or even contain molten lava. Some of the deepest caves in the world are cold, because their entrances are high in the mountains.

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Maria Droujkova

See Also: Geothermal Energy; Measurements, Area; Measurements, Volume; Stalactites and Stalagmites; Temperature; Tides and Waves; Tunnels.

Cell Phone Networks

Category: Communication and Computers.
Fields of Study: Algebra; Geometry; Measurement.
Summary: Mathematics is involved in the design of the cell network and the assignment of calls to frequencies, as well as in data compression and error compression.

Cell phones have grown from a novelty, to a luxury, to a virtual necessity since the 1990s, with the number of cell phone subscribers in the United States growing from about 91,000 in 1985 to 276 million in 2009. Part of the reason that cell phones have become so reliable, cheap, and secure has to do with mathematics. Mathematics is involved in the design of the cell network and the assignment of calls to frequencies (or channels), as well as data compression and error compression that allow a large number of clear calls to be carried over a small bandwidth. The concept of a tree from graph theory can be used to understand cell phone networks, which are challenging because of the large amount of data and links. Mathematicians like Vincent Blondel analyze millions of users and months of communication.

Cellular Radio Networks

Cell phones work by communicating via radio signal with a nearby cell phone tower. In a cellular radio network, the type of system used for cell phone coverage, the land area to be supplied with coverage is divided into regular shaped regions (or "cells"), each of which has a corresponding radio base station or cell tower. Phones within a particular cell connect via radio signal to the tower for that cell, which then connects to the public telephone network through a switch. The range of a tower may be about one-half mile in urban areas up to about five miles in flat rural areas.

Because of this relatively short transmission range, cell phones and towers can use low power transmitters. In addition to allowing phones to be small and use smaller batteries, the low power also means the radio frequencies can be reused by towers not too far away from each other without any interference between the transmissions. This function allows cell phone networks to carry a larger number of calls in a smaller bandwidth. Typically, cell companies will divide their coverage area into regularly shaped cells or regions with each one covered by a single tower. In fairly flat areas, these regions are usually hexagonal in shape—an idea developed by Bell Labs engineers W. Rae Young and Douglas Ring in the middle of the twentieth century.

The frequencies used by a particular tower for transmissions in its region cannot be used by any of the six regions with which it shares a boundary. The Four Color Theorem from graph theory indicates that only four frequencies are needed to ensure that regions that share a boundary do not use the same frequency. However, companies usually want to further buffer the distance between reuse of the same frequency, so they divide the frequencies up into seven bundles and use a different one on each of the six cells sharing a boundary with a given cell.

A cell phone tower disguised as an evergreen tree. Cell signals are sent through towers via radio signals.

Cell Phone Channels

During the twentieth century, there were many discussions among professionals at the Federal Communications Commission regarding the possibility of opening up frequencies for phone use. Cellular networks began to appear around the world. For instance, Japan offered a 1G system in 1979, and, in 1983, AT&T and Ameritech tested a commercial cellular system in Chicago. Much of the advancement in cell network technology has been focused on the frequency band within a cell, which must be divided up to carry several calls at the same time. In first-generation cell technology, calls were transmitted in analog, which allowed only one call per frequency. Typically, a cell phone carrier was assigned 832 radio frequencies to use in a city. Each call was full duplex, meaning that it used two frequencies: one to transmit and one to receive.

Thus, typically there were 390 voice channels with the remaining 42 radio frequencies used for control channels that were used to locate and communicate with phones but not to carry calls. If the 395 voice channels were divided into seven frequency bundles, that made 56 voice channels per region. So if more than 56 calls were in progress in a given region at a given time, then one of the calls would be disconnected or dropped. Fortunately, first-generation technology is no longer in use. With second-generation (2G) cell technology, calls were no longer analog signals but were converted to a digital (0 and 1) format. This shift is similar to the change from cassette tapes to compact discs in the recording industry.

The greatest advantage to digital technology is that it allows for sophisticated data compression techniques to be used without losing acceptable call quality. Data compression allows for between three and 10 digital calls to be carried in the bandwidth necessary for a single analog call. Further advancements in compression have allowed for even newer third-generation (3G) technology. 3G networks have much faster transmission speeds and allow the use of smartphones that can transmit data fast enough to surf the Internet, send and receive e-mail, and even instant message with a cell phone. Newer 4G technology adds even more speed and capacity to cell phone networks.

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Mark Ginn

See Also: Coding and Encryption; Digital Storage; Telephones; Wireless Communication.

Census

Category: Government, Politics, and History.
Fields of Study: Data Analysis and Probability; Measurement; Representations.
Summary: Conducting a valid and reliable census depends on mathematical and statistical methods.

The term "census" comes from the Latin word *censere*, meaning "to assess." A census is a systematic collection of data about an entire population of interest. Usually the population is people but historically it has also been done for land, livestock, and trade goods. Sometimes a census is a one-time event, or it may be repeated periodically, like the decennial census in the United States.

There are also two primary philosophies of data collection that can affect the outcome of a census: *de jure* and *de facto*. *De jure* counts people at their usual place of residence, while *de facto* counts people where they are on the day of the census.

For example, one biblical account of the birth of Jesus involves a census in which individuals were required to return to their town of origin rather than being counted where they lived, as opposed to the U.S. census, which is centered about people's permanent residences. Archaeological records indicate that many ancient civilizations conducted censuses, the purpose of which was often taxation or military recruitment. The constitutionally stated purpose of the U.S. census is to determine each state's congressional representation, though it has grown to include additional descriptive and predictive activities. The U.S. Census Bureau is one of the largest employers of mathematicians and statisticians, who not only collect and analyze data but also lead the way in developing new data collection and analysis methods.

Statisticians work internationally as well. For example, in 1949, British statistician Frank Yates was appointed to the United Nations Commission on Statistical Sampling and published *Sampling Methods for Censuses and Surveys*, which is widely acknowledged to have been influential in establishing sound principles and technical terminology. Overall, mathematical and statistical procedures improve the quality, reliability, and representation of census data, and the methods used by census-takers are constantly evolving.

Census Controversies

While the aim of collecting census data is to provide complete record of data on a population, there can be many difficulties in obtaining such comprehensive data. Past problems have included members of the population objecting to the potentially intrusive nature of such a full-scale inquiry, which has the potential to be misused, and difficulties reaching the entire population.

This second problem was especially problematic in the 1990 decennial census and spurred a great deal of developmental activity with regard to statistical survey methods. Even further in the past, there was a heated debate among the U.S. Founding Fathers about how to account for slaves in the U.S. census, as these counts had the potential to dramatically shift the balance of representative power between the northern and southern colonies. Even now, evolving social constructs and definitions of significant demographic variables, like race, can be a controversial topic.

The History of the Census

The practice of completing a census for an entire population occurred in many ancient civilizations. Records suggest that the Babylonians conducted a census in about 3800 B.C.E., and that Egyptians did so in the second millennium B.C.E. Male Roman citizens had to register for a census every five years and declare both family and property. Elected censors oversaw and coordinated the census process. The censors would then summon every tribe in the country to appear before them so they could record the relevant details. In ancient Rome, the census recorded the names of the family members, along with details of any property or land they owned. This provided the leaders of the country at the time the ability to tax their citizens according to the amount they owned. William the Conqueror carried out a census in Britain in 1086 C.E. for taxation purposes. This census took years to complete and attempted to compile a comprehensive list of all land and property in Norman Britain. Such a comprehensive exercise was previously unheard of in Europe, and it preceded an early example of a modern census by nearly 600 years.

Instructed by King Louis XIV in 1666, Jean Talon, a French colonial administrator, conducted a census in order to expand the colony in New France, North America. Talon used the *de jure* method and visited many of the colonial settlers personally, compiling data on settlers' names, age, sex, and occupation. The aim of this census was to help the colony settle by using the statistics to decide how best to develop agriculture, trade, and manufacturing industry. In all, Talon managed to compile details of 3215 inhabitants and paved the way for the development of a number of further censuses in the New World. In Britain, a 1798 paper written by demographer Thomas Malthus discussed the possibility that not knowing the population size and growth rate (demographics), of a country could lead to food shortages and overuse of other resources, resulting in famine and disease as the population is unable to sustain itself. These revolutionary modeling ideas led the British government to pass through parliament the Census Act of 1800. The first modern British census took place in 1801; the process has been repeated decennially since then, except for in 1941 during World War II.

The Modern-Day Census—Data Collection

The U.S. census is required by constitutional law to take place every 10 years and involves sending forms to every residence in the United States and Puerto Rico. The data are then analyzed in order to determine how each state is represented in the U.S. House of Representatives and to provide the correct resource allocation for the current population, that is, how much of the federal fund is given to hospitals, schools, and other public services. Individual responses to the U.S. census are kept confidential for 72 years.

A similar process is used in the United Kingdom, although the census details are kept confidential for 100 years. A key difference in the census forms in the United Kingdom (UK) and the United States is that the U.S. form has just 10 questions and is two pages long. The UK census form for 2011 contained 43 questions in a 32-page booklet.

In Canada, a national census is taken every five years. Each household receives a census questionnaire, to either be filled out online or returned in the post.

Practical Problems With Census Taking

A number of problems can arise when attempting to take a census of an entire population. For the data to be useful, the characteristics of the whole population need to be reflected. This requirement means that any nonresponse could jeopardize the quality of the data. Nonresponses can happen, for example, when an address list is not comprehensive, or people fail to fill in their forms fully and return them. There are a number of measures used to prevent this, including following up with nonrespondents in a face-to-face interview, and setting fines for nonrespondents.

A statistical technique called "imputation" was used by the U.S. Census Bureau in its 2000 census to create data using the nearest neighbor "hot deck" method. Where a gap in the counting (for example, an entire household's data was missing) was identified, data from the geographically closest neighbor were used instead. Where a household had not completed every question fully, the missing data were imputed from a nearest neighbor record where the households are of the same size. Where a respondent within a household gave incomplete data, the characteristics were imputed from the characteristics of other household members. This technique enabled the U.S. Census Bureau to produce a more complete set of data on the U.S. population.

In order to overcome the obstacle of an incomplete address list, a number of different address lists can be combined to get a more complete list—thus ensuring a wider population is reached, and improving the reliability of the data. Alternatively, another statistical method, called "sampling," can be used to estimate features of the population. A forward-thinking statistical sampling plan was proposed by many mathematicians and statisticians after the 1990 census turned out to be particularly problematic in terms of issues such as undercoverage of certain subpopulations. The U.S. Supreme Court refused to permit sampling to completely substitute for counting. These mathematical methods are used, however, for other types of estimation and to gauge how much undercoverage or other biases might exist.

Analysis of Census Data

A number of mathematical and statistical techniques can be used to draw the most descriptive and predictive information possible from raw census data. For example, to identify resource need, data can be ranked in such a way to identify areas where there are more children, thus enabling the government to plan where to locate schools. Alternatively, areas with a high percentage of elderly people could be identified and provided with more social care. Since the 1990s, census data have become a major resource for both amateur and professional genealogists now that older records are being digitized. Census data are also used to find ways to make future collections efforts better.

Edna Lee Paisano grew up on a Nez Perce Indian Reservation in Sweetwater, Idaho. Talented in both mathematics and science, she attended the University of Washington and earned a graduate degree in social work, studying statistics in the process. In 1976, she was hired by the U.S. Census Bureau to work on issues regarding Americans and Alaskan Natives, and was the Bureau's first full-time Native American employee.

Using data from both the 1980 census and a survey she developed, Paisano discovered that Native Americans in some locations were undercounted. This was a serious issue, as allocation of federal funds to tribal units is based on census figures. She used statistical methods to improve the accuracy of the census and encouraged others in the Native American community to become educated in mathematics-related fields such as computer science, demography, and statistics. The 1990 census showed a 38% increase in U.S. residents counted as American Indians.

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Amy Everton

See Also: Congressional Representation; Data Analysis and Probability in Society; Probability; Sample Surveys.

Central America

Category: Mathematics Around the World.
Fields of Study: All.
Summary: Mesoamericans were sophisticated mathematicians, and mathematics continues to be important in the area.

Central America is defined as the southern part of the North American continent, reaching from Mexico to Panama. The portion of the region in which corn, beans, and squash were reliable crops during the pre-Columbian era is referred to as "Mesoamerica," reaching from the mountains of Mexico to Guatemala and down the Pacific coast into Nicaragua. Teotihuacan, Olmec, Maya, and Aztec were among the many cultures sharing the same prehistoric land and cultural legacy.

The development of the area and its perspective on mathematics were shaped in part by the origins of civilization isolated from the other large centers of civilization in the Eastern Hemisphere. Spanish colonization in the sixteenth century brought the first introduction to European cultures. Efforts at spreading Christianity resulted in the loss of much of their rich, ancient heritage. The area had gained independence by the mid-nineteenth century, variously structured as separate nations and unified groups. Struggles to achieve stability continue into the twenty-first century in many parts of the region. Education and mathematics are highly valued as keys to further progress.

Ancient Mesoamerica

Without the benefit of influence from other cultures, the ancient Mesoamericans built large city-states sometimes supporting several hundred thousand people, and extensive empires, with no domesticated large mammals and with no use of the wheel, other than in children's toys. They mastered basic arithmetic, with a concept of zero evident a millennium before European civilizations. They shared a counting system based on 20 rather than 10. Numeral representations included dots for units, bars for five, and a circle or seashell for zero. Ancient ruins show evidence of meticulous accounting of trade and personal lives. The construction of imposing pyramids and other structures aligned to astronomical features and adorned with harmonic geometric design reveal an advanced level of engineering, architecture, and astronomy to rival that found in Europe at the same time. From as early as 2000 B.C.E., the people of the area had sophisticated calendars, which were used in tandem to mark time reflecting both human and solar cycles. Ethnomathematicians continue to study ancient and modern Central America, and many teachers use Mesoamerican mathematics concepts as the basis of lesson plans and assignments.

Modern Central America

Central America is defined by the United Nations to include the modern countries of Belize, Costa Rica, El Salvador, Guatemala, Honduras, Mexico, Nicaragua, and Panama. The countries share ethnic, economic, and geological features. The peoples are primarily Spanish, Amerindian, or Mestizo (a mixture of the two). The climate ranges from mountainous to tropical coastline. While significant portions of the population are centered in large urban areas, much of the population of the region is located in small villages, sometimes isolated by rugged terrain.

Education

Central Americans are continuously improving their education systems, and efforts at reform often include careful inclusion of children from both urban and rural areas with the use of radio, television, and computer technologies. Teacher salaries and the contrasts of management of schools by local or federal administrators are recent areas of research. United Nations data report high participation in formal schooling. Private schools usually are more prestigious than public schools in most areas.

As calendars held power in ancient Mesoamerica, knowledge of mathematics is held to be essential for the people in modern Central America. High school graduates receive extensive content instruction in mathematics and science but historically with little emphasis on mathematics applications. Teachers are encouraged to teach mathematics in context rather than as an isolated, esoteric discipline both for the better understanding and for the application of learning to solve problems and promote progress. Recent research in mathematics from the region includes a diverse range of areas like topology, noncommutative geometry, and applied mathematics.

Mathematics researchers gather for conferences, research seminars, educational forums, and social events. For example, the Sociedad Matemática Mexicana (Mexican Mathematical Society) was founded in 1943. The society's goals include encouraging mathematical research, including cooperation with related scientific disciplines; improving mathematics education at primary, secondary, and college levels; and providing various forums for discussion and dissemination, including journals and conferences.

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See Also: Calendars; Castillo-Chávez, Carlos; Curricula, International; Incan and Mayan Mathematics.

Cerf, Vinton

Category: Communication and Computers.
Fields of Study: Connections.
Summary: Computer Scientist Vinton Cerf helped create the Internet and continues to be a leader in Internet innovation.

Vinton Gray Cerf is an American computer scientist and is one of the creators of the Internet. He worked on Internet architecture and the design of TCP/IP protocols in the 1960s and 1970s, eventually moving from academia, to government, and to corporations like MCI and Yahoo. He continues to work in advancing Internet applications and policies, such as laws regarding "net neutrality." He has won many prestigious awards in conjunction with collaborator Robert E. Kahn, including the U.S. National Medal of Technology, the Association for Computing Machinery's Alan M. Turing Award (sometimes called the "Nobel Prize of Computer Science"), and the Presidential Medal of Freedom, which is the highest civilian award given in the United States. In December 1994, he was also listed as one of *People* magazine's "25 Most Intriguing People." Cerf and his wife, Sigrid, have been married since 1966, and he has spoken of her support regarding his education and career. They have two sons, and, at times, his family's needs have influenced where he has decided to work. Since 2005, he has been a vice president at Google, and continues to be a leader in Internet innovation.

Reflecting on his own education, Cerf traced his interest in mathematics to primary school. He cited his fifth grade mathematics teacher as being an influential force. When Cerf complained of boredom with the standard curriculum, the teacher introduced him to more advanced mathematics. Cerf said, "I fell in love with algebra. It was wonderful.…Frankly, I liked the word problems the best because they were like little mystery stories.…I still love word problems. To this day, give me an algebra word problem, and I'll have a great old time with it." Outside the classroom, Cerf also enjoyed the camaraderie and challenge of his high school math club and mathematics competitions led by a young teacher named Florence Reese. Of the experience, he noted positively, "It would be weeks and weeks of just working problems, and then the morning of the event we'd all get up and have a big steak and egg breakfast at 7:00 in the morning.…You didn't want to dull your brain with a lunch of any kind." He went on to earn a B.S. in mathematics from Stanford University, then a M.S. and a Ph.D. in computer science from the University of California, Los Angeles, along with multiple honorary doctoral degrees from universities around the world. Regarding his change of field from undergraduate to

Vinton Cerf playing a game on the Computer History Museum's PDP-1 computer from 1959.

graduate school, Cerf said, "I had already figured out that I wasn't going to be a world-class mathematician. I sort of broke my pick on Riemannian geometry...." At the same time, he credited his education in geometry with developing his thinking skills, saying, "...I enjoyed the reasoning part of it, which is probably one of the reasons why I've enjoyed being a programmer, because you have to go through the same line of thinking."

As a graduate student, Cerf was part of Professor Leonard Kleinrock's data packet networking group when they conducted the first connection tests and demonstrations of the Advanced Research Projects Agency Network (ARPANet), one predecessor to today's Internet. ARPANet was created as a joint project between MIT and the Defense Department's Defense Advanced Research Projects Agency (DARPA). After earning his doctorate, Cerf returned to Stanford as an assistant professor from 1972 to 1976, where he continued to work on packet networking and worked with Robert E. Kahn—who was instrumental in ARPANet's hardware design—to develop the TCP/IP protocol for the Department of Defense. Various protocols had to be developed to enable computers to communicate with one another. TCP/IP was a suite of two such protocols: the Transmission Control Protocol, used to exchange data; and the Internet Protocol, which handles routing and addressing. The early version of TCP/IP was introduced in Cerf and Kahn's 1974 paper "A Protocol for Packet Network Interconnection," published by the Institute of Electrical and Electronic Engineers (IEEE). In the twenty-first century, it continues to be the protocol that most Internet applications rely on, including e-mail, file transfer, and the World Wide Web.

Cerf left Stanford in 1976, to work for DARPA directly until 1982, roughly the dawn of the personal computer era, when he was hired as vice president of MCI Digital Information Services (which has since been acquired by Verizon Communications). Cerf oversaw the development of MCI Mail, the first commercial e-mail service, which was officially in service from 1983 to 2003. Messages over MCI Mail were sent over any standard telephone landline with the use of a modem and could be delivered to any other MCI Mail user, a telex, or an MCI Mail print site—an important option in days when access to a personal computer was often limited. Eventually, messages could be sent to any e-mail user regardless of his or her service, as well as to FAX dispatchers. He also led teams at MCI that developed Internet solutions for data, voice, and video transmissions. As the Internet became more widespread, he continued to be an advocate for its use and development. For example, from 1999 to 2000, he served on the board of the Internet Corporation for Assigned Names and Numbers, and some attribute the group's survival to Cerf's business prowess, technical knowledge, and ability to work with players at all levels of Internet governance. He has consulted with NASA's Jet Propulsion Laboratory to develop an Internet standard for planet-to-planet communication and testified before the U.S. Senate in favor of "net neutrality" as that has become an increasing concern of the twenty-first century.

In 2005, Google hired Cerf as vice president and "Chief Internet Evangelist," which has given him a prominent platform from which to address issues from environmentalism, to artificial intelligence, to the imminent transformation of the television industry's delivery model. When asked about the process of innovation and where innovators like him get their ideas, Cerf said, "Part of it is being willing to think literally, out of the box....The people I find most creative are also the ones who really know a lot about what they're doing. They either know a lot of physics, or a lot of math." In addition, he noted that "depth of understanding" means not only knowing the terms of a formula but being able to convey the intuitive meaning of the mathematics.

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Bill Kte'pi

See Also: Data Mining; Internet; Personal Computers.

Cheerleading

Category: Games, Sport, and Recreation.
Fields of Study: Geometry; Number and Operations.
Summary: Cheerleading demonstrates and depends on an understanding of gravity and other forces.

Cheerleading is an activity that can be considered both recreation and a competitive sport, depending on the context. It typically consists of choreographed routines that require energy, discipline, and stamina, and may include chants, dance, tumbling, and other physical stunts. Cheerleaders make what they do look easy when, in reality, the underlying mathematics, such as symmetry, sequences, and physics, helps them to conquer gravity and fly. In 2008, the show *Time Warp* on the Discovery Channel analyzed the physics of cheerleading and gymnastics using slow-motion cameras.

History

In 1898, University of Minnesota football student Jack "Johnny" Campbell became the first person to lead football fans in cheers, using a megaphone, which had been invented by Thomas Edison in 1878, in order to spur his school's football team to victory. This cheering gave rise to organized cheerleading. Women joined the sport in the 1920s, bringing an opportunity to add gymnastics and throws to the cheerleading repertoire. Showmanship and pom poms were later added to the sport. The Dallas Cowboys cheerleaders' skimpy outfits in the 1970s changed the outward appearance of cheerleaders, while the 1980s brought the pursuit of more technical stunt sequences. In the new millennium, the *Bring It On* movies highlight the sport's challenges as well as its technical aspects. Although college squads are currently about 50% male, youth cheerleading is predominantly female. Cheerleaders are now found all around the world.

Focusing on symmetry helps to maintain an equal distribution of weight when acting as a base.

The Physics of Cheerleading

Cheerleaders are focused on center of mass and axes of rotation in order to maintain balance and complete pivots, jumps, and flips. Focusing on symmetry not only helps both their formations and individual poses have a more appealing look but also keeps them focused on maintaining an equal distribution of weight when they act as "bases" for a "climber" or "flyer. "

Cheerleaders need a firm grasp of gravity and the physics involved in their work, including Newton's Third Law, which states that for every action there is an equal and opposite reaction. For example, in a "full extension," the climber pushes off the two bases' shoulders and pulls up with his or her own shoulders to bear some the weight. The two bases move into a "chest prep" with their knees locked, their arms extended and locked, holding the climber's feet at chest level; the climber is now referred to as a flyer. The back person, or "spotter," will often be used as an additional holder to both hold some of the flyer's weight as well as to solidify the overall hold.

As the bases bend their knees, preparing to exert upward force in order to toss the flyer, each base's arms hold half the flyer's weight—uneven distribution of weight is seen when the bases' hips are uneven, exhibiting a loss of symmetry. The bases will extend their knees, letting go of the flyer's feet, to give the flyer upward force; the flyer lands exerting greater force on the way down, so the bases bend their knees and lock hands to cushion the catch. If the bases have not evenly distributed the weight, or have exerted unequal amounts of force, the flyer will not go straight up and the bases will need to move to catch the flyer.

In preparing to execute a flip, the cheerleader bends his or her knees to exert the upward force. To execute, the cheerleader needs to stay tight, keep the axis of rotation steady, point the feet, and land lightly, snapping together to a final pose to stop his or her momentum. The cheerleader's angular speed can change by changing the distance of mass to the axis of rotation; the cheerleader gets momentum from the push off as well as from reducing the distance from mass to axis of rotation by tucking the body in as he or she rises from the ground.

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See Also: Gymnastics; Knots; Skating, Figure.

Chemotherapy

Category: Medicine and Health.
Fields of Study: Algebra; Measurement.
Summary: Mathematical modeling has improved chemotherapy protocols and saved patients' lives.

Chemotherapy is the use of chemical drugs to kill cancerous cells in the body. Although cancerous cells are the target of chemotherapy, traditional chemotherapies do not distinguish between "good" and "bad" cells. Hence, chemotherapy often results in side effects, such as hair loss and toxicity damage to body organs. Because of these chemotherapy side effects, chemotherapy protocols attempt to kill as much of the tumor as possible while incurring as little damage to the patient as can be managed. Thus, chemotherapy regimens are managed according to different variables, including how much drug is given in a treatment, how frequently treatments are given, and the total number of treatments given. Historically, chemotherapy protocols were designed only through experimental data from clinical trials and practice. However, such experiments can be costly or even pose ethical dilemmas. Chemotherapy variables are quantitative—each lends itself to a mathematical understanding and description that can be used to model and simulate treatment experiments, adding to the information gained in clinical settings.

Mathematics in Cancer Chemotherapy News

Mathematics is becoming an increasingly powerful tool in cancer chemotherapy treatments, especially in the dosing and management of chemotherapy protocols. For example, in 2004, Dr. Larry Norton received the American Society of Clinical Oncology's David A. Karnofsky Award, which is given for an outstanding contribution to progression in cancer treatment. Norton's award is notable because of his quantitative contribution to the field of chemotherapy dosing. The National Cancer Institute has a Center for Bioinformatics that addresses the issue of systematically studying the vast amounts of data associated with cancer growth and treatment response.

Cancer Geometry and Treatment

Cancer cells appear visibly different in shape and structure than normal healthy cells. This fact helps practitioners identify unhealthy cells. Quantitative measurements are associated with the geometry and the complexity of cancer cells. These measurements are related to fractal geometry. Tumor fractal dimensions reflect more complex structures generally because of the arrangement of blood vessels in the tumor. Abnormal blood vessel arrangements inhibit the tumor's uptake of therapeutic drugs. This understanding has led to the use of anti-angiogenic drugs that inhibit the production of new blood vessels and lower the measurement of the tumor's complexity. These drugs can now be used in concert with other cancer treatments in order to create a more effective cancer-fighting regimen.

Cancer Growth and Chemotherapy Treatment

Historically, it was believed that cancer cells grew in an exponential manner over the entire period of a tumor's growth. In exponential growth, the doubling time of a population is constant. This belief affected the way that chemotherapy was delivered, since chemotherapy works by attacking rapidly dividing cells. If a tumor's growth rate were constant, there would be no difference in how many cells were killed during any chemotherapy treatment, regardless of the size of the tumor. This is the "log-kill" model of tumor growth.

However, in the mid-twentieth century, it was experimentally discovered that many tumors exhibited a different kind of population growth: "Gompertzian growth," named for Benjamin Gompertz. When populations grow in a Gompertzian fashion, they grow very rapidly at first—when the population is small. As the population size increases, the growth rate of the population slows. Thus, many tumors would have a smaller doubling time when smaller, and a larger doubling time when larger. Because chemotherapy attacks the most rapidly dividing cells, smaller tumors would be more susceptible to chemotherapy treatments. Thus, if a tumor has been reduced in size by one chemotherapy treatment, it would be better to give a second chemotherapy treatment as soon as possible without costing the patient in terms of healthy cell function. This Norton–Simon hypothesis, named for Larry Normal and Richard Simon, has led to a change in the frequency of standard chemotherapy regimens—the time between treatments was decreased in order to take advantage of the more rapid growth rate in the smaller tumor that had resulted from the previous treatment. This change in treatment timing has increased the survival time of patients undergoing chemotherapy treatments.

Looking Ahead

Although the Norton–Simon hypothesis is a prominent example of how mathematics has helped improve cancer chemotherapy treatments, there are ongoing studies by mathematicians to further improve treatment of cancer. Using a field of mathematics known as optimal control, some mathematicians study how to make chemotherapy treatments as ideal as possible. Although practitioners can make (and have made) use of the Norton–Simon hypothesis, the increase of chemotherapy treatments for a patient, while better, is not necessarily best. Using optimal control theory on mathematical models of cancer and cancer treatment, researchers can investigate the best timing and dosing strategies for chemotherapy based on the variables mentioned above. This work may even lead to determining cancer treatment plans based on a particular individual or a particular kind of cancer in the future.

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Angela Gallegos

See Also: Mathematical Modeling; Mathematics Research, Interdisciplinary; Medical Simulations.

Chinese Mathematics

Category: Government, Politics, and History.
Fields of Study: Algebra; Connections; Geometry; Representations.
Summary: Chinese mathematicians have a long history of investigation and discovery, sometimes predating similar findings in other cultures.

Chinese mathematics has a very long history, and its development is quite independent of other civilizations before the thirteenth century. Roughly speaking, it has four periods of developments before the middle of the Qing dynasty, namely

- *The early development period*: from ancient times to the Qin dynasty (2700–200 B.C.E.)

- *The foundation period*: from the Han dynasty to the Tang dynasty (200 B.C.E–1000 C.E.)
- *The golden period*: from the Sung dynasty to the Yuan dynasty (1000–1367)
- *The east meets west period:* from the Ming dynasty to the middle of the Qing dynasty (1367–1840)

Early Chinese mathematics is problem based and is motivated by various practical problems, including astronomy, trade, land measurement, architecture, and taxation.

The Early Development Period

It is written in *Yi Jing* (*I-ching* or *Book of Changes*) that, "In early antiquity, knotted cords were used to govern with. Later, our saints replaced them with written characters and tallies." In other words, the ancient Chinese used knotted cords to record numbers. Later, written symbols and tallies were used instead. In the Shang dynasty (1600–1050 B.C.E.), numerals were invented and inscribed on oracle-bones or tortoiseshells for recording numbers. It was a decimal system and was widely used at the time.

In the Zhou dynasty (1050–256 B.C.E.), mathematics was one of the *Six Arts* (*Liu Yi*), which were taught by teachers at schools. The other five arts were rites, music, archery, charioteering, and calligraphy. From this dynasty onward, the ideas of *Taichi*, *Ying Yang*, *Trigrams,* and *Hexagrams* largely influenced the developments of sciences, mathematics, philosophy, arts, architecture, and many other areas in Chinese culture. For example, *Luoshu* (3×3 magic square) is closely related to the eight *Trigrams*. It has both ceremonial and metaphysical importance, which plays a significant role in Chinese philosophy for several thousands of years.

From the Kingdoms of Spring and Autumn (720–480 B.C.E.) to the period of Warring States (480–221 B.C.E.), the Chinese used counting rods to do calculations. Numbers were expressed by nine symbols, and blanks were used to denote zeros. The numeration system was already a decimal place-valued system.

The first definitive work on geometry in ancient China was the *Mo Jing*, which was compiled after the death of Mozi (470–390 B.C.E.). Many basic concepts of geometry can be found in this book. For example, the *Mo Jing* defines a point to be the smallest unit that cannot be divided, and points on a circle to be equidistant from the center. The book also mentions the definitions of endpoints, straight lines, parallel lines, diameter, and radius.

In the Qin dynasty, the famous Great Wall and many huge statues, tombs, temples, and shrines were built, which required sophisticated skills and mathematical knowledge for calculating proportions, areas, and volumes. Unfortunately, not much is known about the actual mathematical development in the Qin dynasty now, because of the burning of books and burying of scholars ordered by Emperor Qin Shi Huang.

The Foundation Period

In 1984, a Chinese mathematics text called *Suan Shu Shu*, completed at about 200 B.C.E., was discovered in a tomb at Zhangjiashan of the Hubei Province. It is about 7000 characters in length, and is written on 190 bamboo strips. Its content is mainly concerned with basic arithmetic, proportions, and formulas of areas and volumes. The next complete surviving text is the *Zhou Bi Suan Jing*, written between 100 B.C.E. and 100 C.E. Although it is a book on astronomy, it contains a clear description of the *Gougu Theorem* (the Chinese version of the Pythagorean theorem), which is very useful in solving problems in surveying and astronomy. This work is perhaps the earliest recorded proof of the Pythagorean theorem.

After the book burning in 212 B.C.E., the Han dynasty (202 B.C.E.–220 C.E.) began to edit and compile the mathematical works lost in the Qin dynasty. The most important one is the *Nine Chapters on the Mathematical Art* (*Jiuzhang Suan Shu*), completed at around 179 C.E. Although the editor is unknown now, this book had a great impact on the mathematical developments in China and its neighboring countries, such as Japan and Korea. It contains a collection of 246 mathematical problems on agriculture, engineering, surveying, partnerships, ratio and proportion, excess and deficit (the method of double false positions), simultaneous linear equations, and right-angled triangles.

The general method of solutions is provided, but no proof is given in the Greek sense. Most of the methods are of computational nature, and they can be applied to solve problems algorithmically. For instance, square roots, or cubic roots, can be found in a finite number of steps by using a procedure called *Kai Fang Shu*. For skillful users of this method, the answers can be computed efficiently by manipulating the counting rods.

For circular measurements, the approximated value of π is taken as 3. Some problems are expressed in terms of a system of linear equations and then solved by algebraic techniques. For instance, a problem in Chapter Eight leads to the system

$$3x + 2y + z = 39$$

$$2x + 3y + z = 34$$

$$x + 2y + 3z = 26$$

which can be solved by a method like the matrix approach described in modern textbooks.

In the third century, Liu Hui wrote his famous commentary on the *Nine Chapters on the Mathematical Art.* He also wrote a book called *The Sea Island Mathematical Manual* (*Haidao Suan Jing*), to demonstrate how to apply the Gougu Theorem. He was the first Chinese mathematician to deduce that the value of π lies between 3.1410 and 3.1427, by repeatedly doubling the number of sides of a regular polygon inscribed in a circle. It is called the method of dissection of a circle. Liu Hui also discovered the Cavalieri's Principle and used it to find the volume of a cylinder. About two centuries later, Zu Chongzhi (430–501) and his son, Zu Geng, found that the value of π lies between 3.1415926 and 3.1415927, based on the pioneer works of Liu Hui. He also obtained the remarkable rational approximation $355/113$ for π, which is correct to six decimal places. Working with Zu Geng, he successfully applied the Cavalieri's Principle to deduce the correct formula for the volume of the sphere by computing the volume of a special solid called *Mouhe Fanggai* (the double vault) as proposed earlier by Liu Hui.

Unfortunately, his own work called *Zhui Shu* was discarded from the syllabus of mathematics in the Song dynasty and was finally lost in the literature. Many believed that *Zhui Shu* probably describes the method of interpolation and the major mathematical contributions by Zu Chongzhi and Zu Geng.

At the beginning of the Tang dynasty, Wang Xiaotong (580–640) wrote the *Jigu Suanjing* (*Continuation of Ancient Mathematics*), a text with only 20 problems that illustrate how to solve cubic equations. His method was a first step toward the *Tian Yuan Shu* (the method of coefficient array), which was then further developed by other mathematicians in the Sung and Yuan dynasties.

In the sixth century, mathematics was a subject being included in the civil service examinations. Li Chunfeng (602–670) was appointed by the Chinese emperor as the chief editor for a collection of mathematical treatises for both teachers and students. The collection is called the *Ten Classics* or the *Ten Computational Canons*, which include the *Zhou Bi Suan Jing*, the *Jiuzhang Suan Shu*, the *Haidao Suan Jing*, the *Sunzi Suan Jing*, the *Wucao Suan Jing*, the *Wujing Suan Shu*, the *Shushu Jiyi*, the *Xiahou Yang Suan Jing*, the *Zhang Qiujian Suan Jing*, and the *Jigu Suan Jing*. The book *Zhui Shu* by Zu Chongzhi had been included in the *Ten Classics* at the beginning, but it was later replaced by the *Shushu Jiyi* because of it being lost in the Sung dynasty.

The Golden Period

No significant advances in mathematics were made between the tenth century and the eleventh century. However, Jia Xian (1023–1050) improved the methods for finding square roots and cube roots, and also extended them to compute the numerical solutions of polynomial equations by means of the *Jia Xian Triangle* (the Chinese version of the Pascal Triangle).

The golden period of mathematical development in China occurs in the twelfth and the thirteenth centuries, which is called the "Renaissance of Chinese mathematics" by some authors. Four outstanding mathematicians appeared in the Sung dynasty and the Yuan dynasty, namely Yang Hui (1238–1298), Qin Jiushao (1202–1261), Li Zhi (also called Li Yeh, 1192–1279), and Zhu Shijie (1260–1320). Yang Hui, Qin Jiushao, and Zhu Shijie all used the Horner–Ruffini method to solve quadratic, cubic, and quartic equations. Li Zhi, on the other hand, revolutionized the method for solving problems on inscribing a circle inside a triangle, which could be formulated as algebraic equations, and solved by using the Pythagorean theorem. Another mathematician, Guo Shoujing (1231–1316), worked on spherical trigonometry for astronomical calculations. Therefore, much of the modern mathematics in the West had already been studied by Chinese mathematicians in this period.

Qin Jiushao (1202–1261) invented the symbol for "zero" in Chinese mathematics. Before this invention, blank spaces were used to denote zeros. Qin Jiushao also studied indeterminate problems and generalized the method of Sunzi to become the now-called "Chinese Remainder Theorem." He wrote the *Shushu Jiu-*

zhang (*Mathematical Treatise in Nine Sections*), which marks the highest point in indeterminate analysis in ancient China.

Yang Hui was an expert in designing magic squares. He discovered elegant methods for constructing magic squares with an order greater than three. Some of the orders are as high as 10. He was also the first in China to give the earliest clear presentation of the *Jia Xian Triangle* in his book *Xiangjie Jiuzhang Suanfa*.

The famous work of Li Zhi is the *Sea Mirror of the Circle Measurements* (*Ce Yuan Hai Jing*). It is a collection of some 170 problems. He used *Tian Yuan Shu* or the Method of Coefficient Array to solve polynomial equations of degree as high as six. He also wrote the book *Yi Gu Yan Duan* (New Steps in Computation) in 1259, which is an elementary book related to solution of geometric problems by using algebra.

The most important text in the thirteenth century is the *Precious Mirror of the Four Elements* (*Si Yuan Yujian*), written by Zhu Shijie in 1303. This book marks the peak of the development of algebra in China. The unknowns that appeared in equations are called the four elements, namely heaven, Earth, man, and matter. This book describes how to solve algebraic equations of degrees as high as 14. The method is the same as the Horner–Ruffini method. Zhu Shijie also used the matrix methods to solve systems of equations. He was also an expert in summation of series. Many formulas on summation of series can be found in the *Precious Mirror of the Four Elements*. He also wrote an elementary mathematics text called the *Introduction to Computational Studies* (*Suanxue Qimeng*) in 1299, which had a significant impact on the development of Japanese mathematics later.

The East Meets West Period

In the Ming dynasty, not much original mathematics work emerged in China. Even the famous work *Suanfa Tongzong* (*General Source of Computational Methods*) by Cheng Dawei (1533–1606) was an arithmetic book for the abacus only. Its style and content were still influenced very much by the *Nine Chapters on the Mathematical Art*. It was only when the Italian Jesuit Matteo Ricci (1552–1610) came to China in 1581 that the development of mathematics in China was influenced by the West from this time onwards. For instance, Xu Guangqi (1562–1633) and Matteo Ricci translated a number of Western books on sciences and mathematics into Chinese, including the famous *Euclid's Elements*, the influence of the Western culture on China became more apparent.

However, the Chinese mathematicians also did an excellent job in editing and recording their traditional mathematics and science works in the early Qing dynasty, so that much of them can come down to us now. For example, Mei Juecheng (1681–1763) edited the famous mathematical encyclopedia *Shuli Jingyun* in 1723, and Ruan Yuan (1764–1849) edited the *Chouren Zhuan* (*Biographies of Astronomers and Mathematicians*) in 1799. Both of these works are very valuable and useful references for historians to study the mathematical developments in China before the middle of the Qing dynasty.

Achievements in Chinese Mathematics

After the decline of Greek mathematics in the sixth century, Western Europe was undergoing the period of Dark Ages. On the other hand, many of the achievements of Chinese mathematics predated the same achievements before and shortly after the Renaissance. For instance, before the fifteenth century, China was able to (1) adopt a decimal placed-value numeral system, (2) acknowledge and use negative numbers, (3) obtain precise approximations for π, (4) discover and use the Horner–Ruffini method to solve algebraic equations, (5) discover the *Jia Xian Triangle*, (6) adopt a matrix approach to solve systems of linear equations, (7) discover the Chinese Remainder Theorem, (8) discover the method of double false position, and (9) handle summation of series with higher order. It was only after the fourteenth century that the development of Chinese mathematics began to decline and lag behind the Western mathematics in the Ming and Qing eras. However, it is worthy to note that the traditional Chinese mathematics still can find its contribution in mechanized geometry theorem proving in the twentieth century, because of its algorithmic characteristics.

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Yiu-Kwong Man

See Also: Asia, Eastern; Binomial Theorem; Greek Mathematics; Pi; Pythagorean Theorem.

Circumference

See *Perimeter and Circumference*

City Planning

Category: Architecture and Engineering.
Fields of Study: Geometry; Number and Operations; Problem Solving.
Summary: Mathematics is used to model optimal city designs and reduce problems of traffic congestion, sanitation, and water distribution.

Also called "urban planning" and "town planning," city planning is a discipline that focuses on the various economic, environmental, historical, physical, political, and social characteristics of the urban environment and their harmonious organization. It encompasses a variety of projects, processes, and goals that involve multiple disciplines and fields of expertise, such as physical design, and quantitative and qualitative research, as well as analysis, forecasting, strategic planning, negotiation, and public mediation. Since the late nineteenth century—and especially during the second half of the twentieth century—the profession has increased its reliance on statistics and mathematics.

Early History

The early origins of urban planning can be traced in the physical design and purposeful spatial organization of some ancient cities in Mesopotamia, Egypt, the Mediterranean Basin, South and Central America, the Yellow River Basin, and along the Indus Valley. Many of these settlements present a hierarchical system of paved streets, often following a rectilinear grid, with water supply and drainage systems. The Middle Ages was not a propitious era for urban planning. It became popular again during the Italian Renaissance with the design of ideal cities. Influenced by the belief that a perfect form was the image of a perfect society, designers opted for radial or centrally planned cities frequently uniting the perfect geometric figures of the square and circle into a star-shape layout. In the seventeenth century, the rise of nation states and absolutism was conducive to the development of the monumental baroque city with its straight and endless avenues, unbroken horizontal rooflines, and repetition of uniform elements, which glorified the ruling power. Simultaneously, the advances of warfare techniques led to the disappearance of the old city walls and the adoption of new complicated systems of fortification with considerable outworks and bastions in spearhead forms.

The Industrial Era

The modern origins of city planning have their roots in the industrial city of the mid- and late nineteenth century. In both Europe and the United States, rapid technical progress, tremendous industrial development, and massive displacements of rural population to urban areas created considerable problems that threatened to disrupt the existing social order. The dreadful conditions experienced by masses of people living in abject poverty and misery in overcrowded slums sprawling around wealthier districts became a source of concern for the general public health. In 1854, Dr. John Snow—the father of modern epidemiology—identified the source of a cholera outbreak in London by studying the patterns of the disease and using statistics and a spot map illustrating the clustered death cases of cholera around the Broad Street pump. The fears of major epidemics resulted in the rise of a social movement for urban reform and planning, which first focused on water supply and sanitation improvement, and later on housing provision.

In the 1880s, the basic lack of information regarding the extent and distribution of poverty in London led English philanthropist Charles Booth to develop a comprehensive and scientific social survey investigating the incidence of pauperism first in East London, and

Lithograph by Currier and Ives of the 1893 World's Columbian Exposition in Chicago. The "White City" in the exhibition was the first large-scale planning execution of the City Beautiful movement in the United States.

later in the entire city. His quantitative statistical analyses and qualitative research presented in 17 volumes with accompanying colored maps indicating the levels of poverty and wealth by street received considerable attention. They were also influential in demonstrating the importance of social surveys for public policy, demographics, and sociology as well as in improving census data collection.

Similar problems affected Paris and, after visiting London, Napoléon III placed considerable emphasis on urban planning to modernize the medieval capital into the capital of light. The large-scale restructuring program under the direction of Baron Haussmann affected not only the center of Paris but also its surrounding suburbs. At the time, it was the largest urban renewal project ever implemented. The plan created a network of large, easily accessible avenues and boulevards with radiating vistas terminated by prestigious public edifices and monuments. In addition to the building of 71 miles of new roads, the layout of 400 miles of pavement, and the doubling of the number of trees lining the streets, the city's infrastructure was entirely renovated. The construction of more than 340 miles of sewers and hundreds of miles of aqueducts increased the water supply by 400%. Haussmann also created two major urban parks and two large natural preserves on the periphery. This urban metamorphosis influenced the design of numerous cities worldwide and in particular the "White City" of the World's Columbian Exposition of 1893 in Chicago, which was the first large-scale project of the City Beautiful movement in the United States. The aim of expanding civic consciousness and raising the standards of civic design culminated in the publication of the famous Plan of Chicago in 1909, which coincided with the first university course in city planning at Harvard and the first National Conference on City Planning. In 1917, the American City Planning Institute was founded.

The Twentieth Century

Nevertheless, with the growth of the automobile as a favorite mode of transportation, it was the "Garden City" concept invented by Ebenezer Howard that became the leading model for the development of U.S. suburban residential communities. As the old city centers became increasingly congested, transportation planning became increasingly important to ensure an efficient balance between land-use activities and the potential communications between them. Transportation planners regularly collect data, which they analyze and process, to forecast future traffic using various techniques such as land-use ratio methods, multiple regression models, category analysis, growth-factor methods, synthetic models, modal split analysis, diversion curves, and geographic information systems (GIS).

After the U.S. Department of Commerce published "A Standard State Zoning Enabling Act" and "A Standard City Planning Enabling Act" in the 1920s and the U.S. Supreme Court upheld the constitutionality of zoning in 1926, most U.S. cities established planning departments to adopt master plans and zoning regulations that allowed them to control land-use development, protect property values, and segregate uses. Cities also started implementing subdivision controls and regulations. These new tools contributed to the belief in part of the planning community of the possible rational and scientific management of cities. On the other hand, idealists such as Frank Lloyd Wright and Lewis Mumford criticized the new pragmatic and technological approach, preferring a philosophy of city development for humanistic and social ends as epitomized in the design of Radburn, New Jersey. Over time, zoning regulations revealed some drawbacks. They often increased traffic congestion, and sometimes prevented the construction of affordable homes. Some courts struck them down as exclusionary.

City planning in the post–World War II era was dramatically affected by four significant federally funded programs: public housing, urban renewal, home mortgage insurance, and highway building. The miserable failure of urban renewal—and the urban crisis of the 1960s that ensued—required new approaches to urban planning. During the second half of the twentieth century, city planning became increasingly defined as a cyclical process attempting to balance conflicting social, economic, environmental, and aesthetic demands while implementing selected objectives and goals. Therefore, regular monitoring became necessary to test, evaluate, and review the strategies and policies adopted on a continuous basis. City planners regularly use a wide range of models ranging from basic descriptive statistics to more complex mathematical models that allow them to understand the nature of various urban components and forecast the consequence of change.

Because of the tremendous complexity of urban systems, models can provide only a simplified representation of the studied phenomena. Consequently, there is considerable attention and controversy regarding the choice of variables, and their level of aggregation and categorization, as well as the handling of time, specification, and calibration. Although deterministic models are the dominant type of predictive models used by urban planners, there has been some attempt at developing stochastic models. Urban planners are also concerned with the accuracy, validity, and constancy of the models they use. Most models tend to be topic specific, focusing, for example, on population, housing, employment, shopping, transport, or recreation, but integrated forecasting systems have become more common as there has been an increasing recognition of the interdependence of the various subfields of a city.

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See Also: Engineering Design; Highways; Water Distribution; Wright, Frank Lloyd.

Civil War, U.S.

Category: Government, Politics, and History.
Fields of Study: Algebra; Geometry; Measurement; Number and Operations; Problem Solving.
Summary: The U.S. Civil War saw numerous advances in firearms, cryptography, and strategy.

The U.S. Civil War, also sometimes known as the War Between the States, was a conflict fought from 1861 to 1865 between 11 southern U.S. states that seceded from the Union to form the Confederate States of America and the remaining United States. Precipitating causes of the war centered on economic issues and states' rights versus federal power, often symbolized by the central dividing issue of slavery. More than 600,000 men on both sides died, which is greater than the combined U.S. losses in all subsequent wars and military conflicts through the beginning of the twenty-first century, though World War II exceeds this count if the metric is combat deaths versus deaths from all causes. Some also refer to the Civil War as the first "industrialized war" because of the extensive use of the telegraph, railroads, and mass-manufactured goods and weaponry.

Mathematics was instrumental in this war in many ways. Introduction of the Spencer repeating rifle has been cited by many as the turning point toward the eventual Union victory. Ciphers and code-breaking efforts were important in communicating military strategies and plans. The U.S. Army Signal Corps, founded in 1860, used both telegraphy and line-of-sight methods, such as the wig-wag signaling system in which the left, right, or upward positions of a single flag represented the numbers 1–3 and specific number combinations corresponded to letters. The ranks of leaders on both sides were filled with mathematically educated graduates of schools like the Unites States Military Academy at West Point. Mathematics education and research were also impacted by the war.

Weaponry

Changes in small arms and artillery that were occurring in Europe and the United States at this time had a tremendous impact on the war. Many different types of smoothbore or rifled artillery were used during the Civil War. One way in which they were distinguished was by their bore size, which was the diameter of their barrels, usually expressed in inches. Another differentiating feature was the weight of the projectiles they fired, in pounds. Different classes of weapons also had different trajectories. Cannons known as "guns" had relatively flat trajectories. Mortar rounds followed a steeply arcing path. Howitzers fell between the other two because the possible angles of inclination and powder charges could be varied more than the other two types.

The most common artillery piece was the Napoleon, a howitzer named after Napoleon Bonaparte, an avid student of mathematics who had revolutionized infantry and artillery warfare. Artillery ammunition included solid shot or balls, grape, canister, shell, and chain shot. Canister and grape shot could be particularly devastating to humans, since they disintegrated into smaller, scattering projectiles along a number of trajectories when fired. At the start of the war, both sides relied primarily on muzzle-loading rifled muskets such as the 58 Springfield, though some smoothbore muskets were also in use. Rifled weapons had greater range—nearly half a mile—versus the 100-yard range for smoothbores, which affected infantry tactics. The breech-loading Spencer repeating rifle was a major innovation, and was considered the most advanced weapon of its time. It used metal cartridge ammunition and it could hold seven cartridges at a time, which greatly increased rate of fire and accuracy, though the Union was initially concerned about the corresponding increase in the demand for ammunition.

Revolvers also replaced muzzle-loading pistols, with similar effects. Minié ball ammunition, named for French military officer Claude Minié, was used extensively in the Civil War. Previously, rifles had been difficult to load because the bullets fit tightly in the bore of the weapon, which was necessary for them to be propelled effectively by explosive powder charges. Despite being called a "ball," the Minié projectile was conical. It was smaller in diameter than older ammunition, and also had grooves that allowed it to fall smoothly and quickly into the barrel of the rifle. A hollow indentation extending from the base caused it to expand to the size of the gun's bore when fired, optimizing the combination of loading time and accuracy.

Cryptography

One problem addressed by mathematical problem-solving approaches was the terrible problem of "hacking." Increasingly, communications were being relayed by the newest technology—the telegraph and Morse code. It was relatively easy for someone to climb the telegraph pole, connect a telegraph key to the wires, and intercept messages that were being relayed back and forth from the front lines to the base camp. Messages needed to be coded so that interceptors could not interpret them, which was not a new problem. The problem of encoding military messages can be dated to at least Julius Caesar and earlier to the Spartan military.

The governors of Ohio, Indiana, and Illinois were close enough to the Confederate border that they felt the need to have their messages encoded. Governor William Dennison of Ohio asked Anton Stager to prepare a cipher (a code) that could be used by these three governors. Stager adapted a transposition coding system that had been used in Great Britain years earlier. Words were rearranged into a grid. The first word of the message was the "key" to indicate how many columns were to be formed and in what order they were to be read. Instructions for these codes were printed on cards about the size of a standard index card, which were the precursors of codebooks. Included on the cards were the route, the keys, the code words, and words used to check the cipher.

This system underwent a number of modifications, and Stager's route cipher was eventually adopted as the Union's official cipher. Increasingly sophisticated ciphers were created during the war and, as a result, the instructions could no longer fit on cards. Some of the resulting codebooks were 48 pages long. The messages were intercepted by the Confederacy and sent to Richmond, Virginia, the capital of the south. By twenty-first-century coding standards, they appear to have been relatively easy codes to break, but evidence suggests that the south did not have either sufficient manpower or mathematical knowledge to decode them.

By contrast, the Union forces had a team to work in breaking the southern codes, including many times President Abraham Lincoln. The codes used by the south at the beginning of the war were not standardized, resulting in many messages that were unreadable. The Confederacy eventually settled on a code from 1587, the Vigenère cipher, named for Blaise de Vigenère—although others before him, such as Giovan

A 12-pdr. howitzer gun captured by Butterfield's Brigade of the 12th Maine Infantry in May 1862.

Bellaso, are also noted as having invented it. This code consisted of a tableau of staggered alphabets. The ease of this code was that the code did not have to change if a coded message was captured, only the code phrase. The problem with the Vigenère code came in errors in transmission over the telegraph. Even though the code was harder to break than the Union cipher, it was more difficult to implement because a missed letter would result in an incoherent message. For instance, General Edmund Smith reportedly spent 12 hours trying to decode a message from General Joseph Johnston during the Vicksburg campaign. The message requested reinforcements, but Smith was unable to read it. He eventually sent a courier, but it was too late for reinforcements; Johnston's army was already cut off. Revisions to the code to avoid this problem in the future made deciphering easier as well.

Mathematically Educated Leaders

Many of the military leaders for both the north and the south were graduates of the U.S. Military Academy at West Point, which was the United States' only engineering school for an extended period of time. The Civil War was fought with 359 generals who graduated from West Point. They served on both sides, 217 for the Union and 142 for the Confederacy. This list of elite officers and leaders includes many well-known officers. Ulysses S. Grant had plans to return to West Point to teach mathematics, and these plans were changed by the outbreak of the Mexican War. Robert E. Lee graduated second in his class in 1829 and served as an assistant professor for mathematics for his first two years at West Point. Edmund Kirby Smith (1845) taught mathematics at the University of the South after the war, where he joined another West Point graduate, Josiah Gorgas. Other well-known graduates who served both sides during the Civil War included Confederate President Jefferson Davis, Braxton Bragg (1837), John Bell Hood (1853), Thomas Jonathan "Stonewall" Jackson (1846), Albert Sidney Johnston (1826), James Longstreet (1842), George E. Pickett (1846), J. E. B. Stuart (1854), William Tecumseh Sherman (1840), George G. Meade (1835), George McClellan (1846), Joseph Hooker (1837), Abner Doubleday (1842), George Armstrong Custer (1861), and Don Carlos Buell (1841).

All these men were mathematically educated, which was unique for that point in U.S. history and likely played a role in many aspects of the war. For example, the maps and messages of the military in the Civil War show the influence of what is now referred to as "descriptive geometry," which was created by Gaspard Monge and was incorporated into the curriculum after engineer Claudius Crozet brought it to the Military Academy. Mathematics textbooks used by most of the leaders on both sides of the Civil War include those written by mathematicians Charles Davies and Albert Church, some of which were adaptations of earlier French works. This education of the leaders of both sides of the conflict may have had a great deal to do with the long length of the conflict and allows historians the opportunity to study other differences in the two sides.

It was not just the military leaders during the Civil War who made use of mathematics. Lincoln purportedly had a great reverence for Euclid of Alexandria and geometry. Some historians assert that he kept a copy of Euclid's *Elements* in his saddlebag and studied it by lamplight to develop his logic and reasoning skills. Phrasing in Lincoln's well-known 1863 address at the Gettysburg battlefield has sometimes been compared to Euclid.

Education

While mathematics undoubtedly influenced the course and outcome of the war, the Civil War also affected mathematics education and research. Antebellum college curriculum in schools such as The Citadel or Harvard consisted of classes in mathematics that were filled with "practical applications," such as mercantile transactions, navigation, surveying, civil engineering, mechanics, architecture, fortifications, gunnery, optics, astronomy, geography, history, and "the concerns of Government." These topics were all expanded in one of the common textbooks of the day, *An Introduction to Algebra*, by Jeremiah Day. Geometry and trigonometry were also commonly taught, and analytic geometry, conic sections, and calculus were often optional classes.

The problems discussed and worked in these classes, both in surveying and in navigation, were carefully chosen and adapted to make them easily done but not extremely realistic. Thus, the navigators and the surveyors being prepared for the Army were ultimately ill prepared to handle the realistic situations of making measurements under fire or in harsh seas. According to the work of Andrew Fiss, the Union army regulations required that surveyors plot the best course for the army to take. The topographical engineers worked so slowly that many of the generals took to asking local citizens for the best directions. By two years into the war the topographical engineers were incorporated into the Army Corps of Engineers. Likewise, the Navy found that the U.S. Naval Academy, founded near the middle of the nineteenth century, could prepare navigators better than a mathematics department. However, the academy was negatively impacted by the temporarily relocation from Maryland to Rhode Island during the war.

After the war, many universities started offering higher level mathematics courses, and some increasingly focused on research. Harvard and John Hopkins University graduated doctorates in mathematics within the next decade.

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See Also: Artillery; Coding and Encryption; Firearms; Revolutionary War, U.S.; Strategy and Tactics.

Climate Change

Category: Weather, Nature, and Environment.
Fields of Study: Algebra; Calculus; Data Analysis and Probability; Problem Solving; Representations.
Summary: Mathematicians and scientists use sophisticated models to track and predict global climate change.

The term "climate change" refers to the changing distribution of weather patterns. Climate is considered to be the average of 30 years of weather. In other words, climate is the distribution from which weather is drawn. Global warming refers to the change in climate in such a way that warmer weather is increasingly likely. In fact, it is not just the warming itself that is of concern but also the rate of change of the warming process since ecological systems typically cannot adapt to a rapidly changing climate. According to the *2007 Synthesis Report* by the Intergovernmental Panel on Climate Change (IPCC), "Warming of the climate system is unequivocal, as is now evident from observations of increases in global average air and ocean temperatures, widespread melting of snow and ice and rising global average sea level." The main cause of changing climate is the increasing atmospheric concentrations of greenhouse gases (carbon dioxide, methane, and nitrous oxide), which effectively act as a blanket over the atmosphere.

The IPCC report noted, "There is very high confidence that the net effect of human activities since 1750 has been one of warming." The evidence for the warming of the climate includes more than the measurement of global average temperatures, as physical evidence such as glacier melt also exists. Most predictions of global warming are based on data models, and mathematics is used extensively to measure and quantify atmospheric carbon dioxide and aerosols, which are believed to add to the problem. The National Oceanic and Atmospheric Administration (NOAA) and the U.S. National Aeronautics and Space Administration (NASA) are two large federal agencies that are involved in the collection, analysis, and dissemination of climate data. They employ a diverse range of mathematicians, statisticians, scientists, and others, and they have partnerships with many academic institutions, government agencies, and businesses around the world. Researchers do agree, however, that the current and future consequences of climate change disproportionately impact the world's poor.

Climate as a Distribution

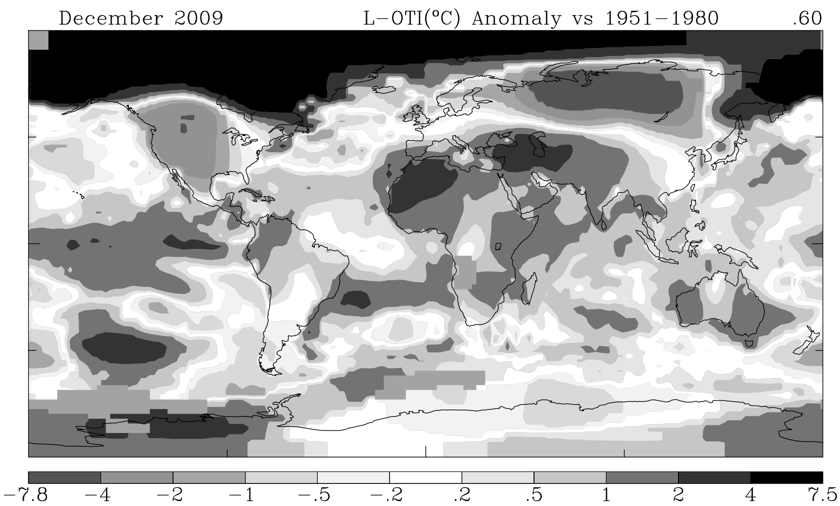
In order to understand global warming, it is important to understand that the term refers to a distribution. It is easy to dismiss the notion of global warming on a cold winter's day, a mild summer day, or any day where the weather is cooler than expected. In fact, an unusually hot or cold event is not evidence for or against global warming. To aid in understanding climate as a distribution, consider a set of 20 cards numbered 1–20. One card will be drawn at a time with replacement. The value 10.5 is the average of these cards; a card selected below 10.5 will represent a below-average temperature for the day, and one selected above 10.5 will represent an above-average for the day. Further, the farther away the value of a card is from 10.5 will represent a larger deviation from average temperature. This example represents a stable climate. Some days are colder than average, and others are hotter than average. But, over time, roughly an equal number of colder days and hotter days occur. Moreover, if the value of the cards were unknown, basic statistical sampling ideas could be used to estimate the average.

To represent a changing climate, start with the same set of cards and consider values below or above 10.5 as a colder or hotter day. But this time, every time a card is drawn and replaced, the next higher card will be added to the set. For example, after the first card is drawn, a 21 will be added to the set, then a 22 will be added after the second draw, then a 23 after the third, and so on. At first, this change would be barely noticed if at all since the cards drawn will be roughly equal above and below 10.5. After some time, however, one would start to question the assumption that 10.5 is the average. In this case, if the values of the cards were unknown, basic statistical sampling techniques could not be used to estimate the average since the average is in fact changing. In this example, if 10.5 is taken as the average, then values below 10.5 still occur but are becoming less likely. In other words, record lows can still occur—and will still happen—even though climate is warming.

To complicate this example further, consider this same experiment being performed simultaneously by 2000 people to represent different locations around Earth. When the set of cards have values from 1 to 100, one individual would have only a 10% chance of drawing a card below 10.5, but it is expected that approximately 200 of the 2000 experiments will draw a card below 10.5. In other words, even though climate is warming, there will still be places that have colder than average days.

In terms of actual weather, consider Figure 1, which provides the average monthly temperature anomalies in degrees Celsius for December 2009 compared to the average from 1951 to 1980. The month of December was slightly colder for most of the United States. The overall average for the world was 0.60 degrees Celsius

Figure 1. Global temperature anomalies for December 2009.



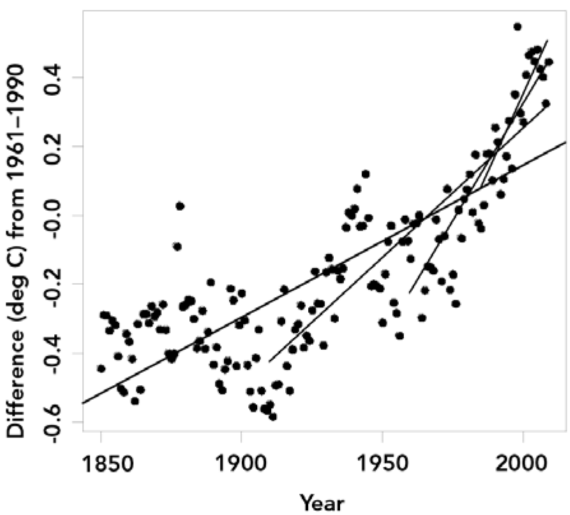
higher than the baseline years. One month, or even one year or a few years, of above average temperatures does not provide conclusive evidence for or against global warming as these abnormalities could be explained as normal variations in weather.

Evidence of Warming

Calculating global mean temperatures each year provides one form of evidence for global warming. For example, Figure 2 displays mean global temperature anomalies dating to 1850. Even though the overall trend is upward, variation from one year to the next can go in either direction. Gerald Meehl, who has a Ph.D. in climate dynamics and works at the National Center for Atmospheric Research (NCAR), collected information from 1800 weather stations across the United States that have been operating since 1950. He and his colleagues looked at the ratio of record highs to record lows and grouped the ratios by decade. From the 1950s through the 2000s, the ratio of record highs to lows was 1.09:1, 0.77:1, 0.78:1, 1.14:1, 1.36:1, 2.04:1. From the 1950s through the 1980s, the ratios might be considered to be in the range of normal variation for a stable climate. On the other hand, by the 2000s, it certainly appears that the observations no longer represent normal variation, and that the climate distribution is getting warmer.

In Figure 2, the baseline is the average from 1961 to 1990, with regression lines for different time periods from 1850 to 2009, 1910 to 2009, 1960 to 2009, and 1985 to 2009. The data are the HadCRUT3 data set provided by the University of East Anglia Climatic Research Unit.

Figure 2. Global mean temperature anomolies.



Beyond data, a warming climate should present physical evidence in the form of melting ice. Figure 3 is one of a number of glacier image pairs, which are pictures of glaciers taken from the same vantage but 40–100 years apart. The change is striking. Where there was once ice, there is now ocean water with the glacier retreating about seven miles. In the foreground, thick vegetation exists where there was once rock. This change is because of microclimate changes since the ice is no longer cooling that area. Along with melting glaciers, Arctic sea ice is decreasing rapidly and permafrost is melting. In fact, the entire village of Newtok, Alaska, must be relocated because the loss of permafrost has allowed the banks of the Ninglick River to erode.

Melting ice is just one source of evidence of a changing climate. During most of the twentieth century, sea level was rising at a rate of 0.07 inches per year, but by the 1990s that rate increased to 0.12 inches per year. In 2006, the National Arbor Day Foundation updated its plant hardiness zone maps, and most of the zones shifted northward. In other words, many plants can now be grown where they could not before because of their cold hardiness. There have already been observed shifts in species ranges, a northward shift, as well as shifts in phenology (seasonal biological timing) toward events such as early blooming. In fact, many species have seasonal behavior that is occurring 15–20 days earlier than the behavior occurred in the mid-twentieth century.

The general trend of warming is only part of the story. If the planet warmed a degree or two over millions of years, then ecological processes could adapt and societies could migrate. Figure 2 has least squares regression lines calculated over the time periods of 1850–2009, 1910–2009, 1960–2009, and 1985–2009. The four regression lines are as follows:

$$y_{159}(t) = 0.0041\,t - 8.67281$$

$$y_{100}(t) = 0.00750\,t - 14.75315$$

$$y_{50}(t) = 0.01364\,t - 26.96187$$

$$y_{25}(t) = 0.01801\,t - 34.67615.$$

In each case, the slope of the line, with units of degrees Celsius per year, is increasing as the time periods are shortened toward more recent years. More importantly, the 95% confidence intervals for the slopes are (0.00387, 0.00495), (0.00657, 0.00844), (0.01151, 0.01577), (0.01254, 0.02347), respectively. The first three intervals do not overlap, and so the slopes of the lines are significantly different. This provides evidence not only for overall warming but also that the rate of warming is increasing. Some species, trees for example, will simply not be able to adjust their ranges quickly enough to adapt to the warming climate.

Climate Science

Climate models, which incorporate mathematical topics such as dynamical systems, statistics, differential equations, and applied probability, are used to predict future global average temperature.

Mathematician Ka-Kit Tung, in his book *Topics in Mathematical Modeling,* provides a simple climate model. The model is

$$R\frac{\partial}{\partial t}T = Qs(y)(1-\alpha(y)) - I(y) + D(y).$$

The left-hand side represents change in temperature. There are three basic terms on the right-hand side that contribute to temperature change. The first term has incoming solar radiation at the top of Earth's atmosphere,

$$Qs(y),$$

where the $s(y)$ term distributes the radiation differently depending on the latitude $y = \sin\theta$ with θ representing latitude. The term also takes into consideration how much radiation is absorbed

$$(1-\alpha(y))$$

where $\alpha(y)$ is the fraction reflected or albedo. The next term,

$$I(y)$$

represents outward radiation, and the last term,

$$D(y)$$

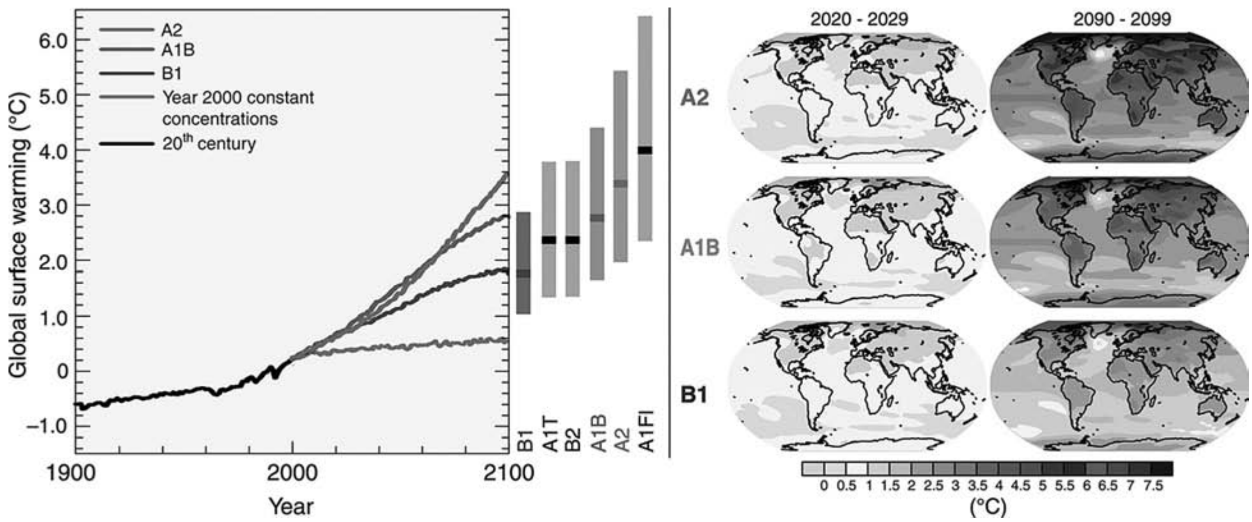
represents heat transportation from warmer latitudes to colder latitudes. In Tung's textbook, this simplified model is analyzed to gain understanding of possible locations in ice lines.

The more complex computer simulations that model climate are built with assumptions related to population growth and societal choices, such as energy use or technological change. These assumptions are then used to predict how greenhouse gases will increase. The effect of increased greenhouse gases in trapping heat is well understood, and in terms of the simple climate model above, the increase in greenhouse gases decreases outward radiation. Beyond that, the increase in carbon levels itself is a problem as oceans work to absorb some of this carbon in the form of carbonic acid. The increase in carbonic acid in the oceans increases the acidity levels, which damages coral, crustaceans, sea urchins, and mollusks.

For each scenario, many different models are considered, and the predictions are averaged to produce the graph on the left side of Figure 4. The three higher curves illustrate the average warming. On the right side of the graph is a range based on the various models. A distribution has been created, and based on the graph, one could say that, by 2100, global mean temperatures will increase between approximately 1.5 degrees Celsius and 3.5 degrees Celsius, but the distribution around the three scenarios presented is from approximately 1 degree Celsius to 6 degrees Celsius. The right side of Figure 4 presents the predicted temperature changes as a distribution across Earth, and it is

Figure 3. Images of the Muir glacier taken from the same vantage on August 13, 1941 (left), and August 31, 2004 (right).

Figure 4. Climate model predictions of future average global temperature and distributional changes of temperature.



predicted that the Arctic region will warm more than the equatorial region.

A key complication in climate modeling is the existence of feedback loops. A feedback loop is created when a change in one factor causes a change in a second factor that then either reinforces or diminishes the change in the first factor. While each scenario sets out greenhouse gas levels, the models must then attempt to take into account how warming may, in fact, increase warming or decrease warming. For example, one positive feedback loop involves melting ice. As ice melts, the Earth's albedo (reflectivity) changes so that less solar radiation is reflected out to space. In the climate model above, the $\alpha(y)$ term is decreased so that more solar radiation is absorbed. In other words, as the planet warms, ice melts. However, there are now fewer reflective white surfaces and more dark surfaces, which will then absorb even more solar radiation and increase the planet's warming. Another potential positive feedback loop arises from melting permafrost. As the permafrost melts, partially decomposed organic matter will decompose more fully and release carbon into the atmosphere. Even more uncertainty arises with the effect of clouds. Low clouds tend to cool by reflecting more energy than they trap, while the reverse is true for high clouds. As surface temperature increases, there is increased evaporation from the oceans, creating more water vapor and hence clouds. But the type of clouds that arise will depend on whether this is a positive or negative feedback loop.

Of course, to many people, an increase of a few degrees Celsius does not seem to be drastic enough to impact life on Earth significantly. But consider that during the twentieth century, global average temperatures increased by less than 1 degree Celsius. Nevertheless, there has already been observed disappearing glaciers, loss of Arctic sea ice, changing species habitat and phenology, and a new plant hardiness map. In fact, a difference of approximately 0.2 degrees Celsius was the difference between the Medieval Warm Period (c. 950–1250) and the cooling period (c. 1400–1700). The warm period led to the Norse migrating to Greenland and bountiful harvests and population increases in Europe. This period was followed by a cooling period that led to the collapse of the Norse Greenland society and starvation in Europe.

Impacts of Climate Change

The general consensus in the scientific community in 2010 is that warming has occurred and will continue to take place even with changes. Debate continues on precisely how much warming will occur and the exact nature of the ramifications. The questions are by how much, and what should people expect to happen? Species ranges are already changing, and, in some cases, species ranges are disappearing as appears to be the case for polar bears. Unfortunately, the speed of warming will lead to some species not being able to change their range quickly enough, resulting in extinction. The changing phenology is already causing ecological disruption. Some plants are blooming earlier, but the species that feed on them are not arriving earlier, leading to decreased food supply.

As Figure 3 shows, ice is melting and more of that is expected. The loss of Arctic ice will decrease polar bear populations. The melting glaciers of the Tibetan Plateau are of particular importance. These glaciers are responsible for supplying water to about 2 billion people, and data suggests that the Tibetan Plateau is warming twice as fast as the global average. Once these glaciers are gone, so is the water supply. The melting of glaciers and land ice, along with the thermal expansion of water, will raise sea levels. One example is Bangladesh, which faces severe threats from sea level increases since millions of people live along a coastline that may be underwater in the future.

There are additional predictions as of 2010, based on models and scientific expertise. An increase of 2 degrees Celsius from pre-industrial levels would lead to a fall in agricultural yields in the developed world, a 97% loss of coral reefs, and 16% of global ecosystems transformed. With an increase of 3 degrees Celsius, few ecosystems could adapt and an additional 25–40 million people would be displaced from the coasts because of sea level rise. If global average temperatures rose to 4 degrees Celsius above pre-industrial levels, entire regions would be out of agricultural production, including Australia.

Climate Change and Societies

The joint science academies' statement on sustainability, energy efficiency, and climate protection issued in 2007 by the G8 nations and Brazil, China, India, Mexico, and South Africa, said that, "Many of the world's poorest people, who lack the resources to respond to the impacts of climate change, are likely to suffer the most." The warming of the planet will have some advantages and disadvantages, although there will be more disadvantages. Some warmer climate species will have expanded ranges and be able to thrive, while arctic species may lose their entire ecosystem. Some countries will be impacted more than others, and the wealthier countries will have a better ability to adapt. The examples that have been given here of societies that already have been or will likely be impacted are all examples of poorer societies.

The people of Newtok, Alaska, are poor; in 2010, Bangladesh ranked 183 in the world in terms of GDP per capita; and there is considerable poverty in regions in and around the Tibetan Plateau. Part of the tragedy is that these are not the people who are largely responsible for increasing greenhouse gases. China and the United States are the largest emitters of carbon dioxide, but on a per capita basis, the United States far exceeds China. In general, it is the industrialized nations that contribute the most to greenhouse gases. Figure 4 provides different models for future climate change, and these are primarily based on the models that predict future greenhouse gas emissions, and it is the more industrialized nations that have the resources to make reductions in these emissions.

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See Also: Coral Reefs; Forecasting; Function Rate of Change; Mathematical Modeling; Weather Forecasting; Weather Scales.

Climbing

Category: Games, Sport, and Recreation.
Fields of Study: Algebra; Data Analysis and Probability; Geometry; Problem Solving.
Summary: Effective climbing relies on mathematical principles, and there are connections between climbing and mathematical problem solving.

Climbing is the use of the human body and assisting equipment to ascend or descend steep surfaces. Climbing can be done professionally, such as for construction or in the military, for exercise or competition, or for performance—in the case of parkour. There are different styles of climbing depending on the object, such as bouldering, ice, tree, and rope climbing. If the weight of the climber is supported by equipment, it is called aid climbing; when the weight is supported only by the climber's muscles, it is called free climbing. Mathematics plays a role in successful climbing and in analyzing various aspects of the discipline. Mathematician Skip Garibaldi said, "Climbing has a lot of puzzles that have to be solved. It's not just strength or skill."

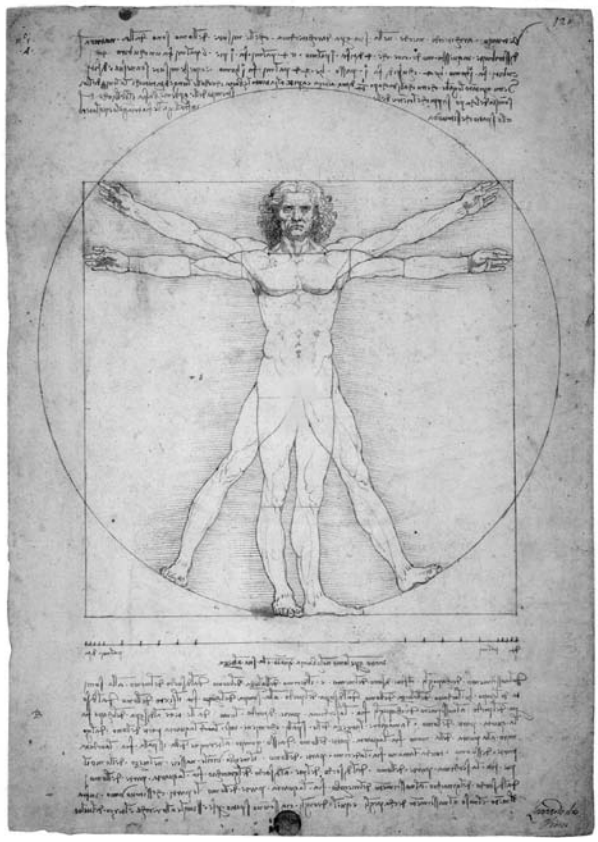
Anthropometry in Climbing

Anthropometry is the mathematical study of body measurements in order to understand human variability. For example, studies show that elite climbers, on average, tend to have small stature, low body mass, and a high handgrip-to-mass ratio compared to the population as a whole. Compared to nonclimber athletes with similar physical conditioning, they are frequently linear, with narrow shoulders relative to hips. Ape index is the ratio of a climber's arm span to height. In adults, it is usually close to one, as illustrated in Leonardo da Vinci's "Vitruvian Man." An ape index greater than one is reputedly advantageous for climbing, and some researchers have found ape index to be a statistically significant predictor of climbing success.

Fall Factor and Impact

Fall factor quantifies how hurtful a fall may be to a roped climber. Mathematicians such as Dan Curtis have derived the fall factor (F_{max}) using differential equations. It is a function of the ratio of the total distance the climber falls (D_T) to the length of the unstretched rope (L) between the climber and belayer or anchor at the rope's other end. It is also a function of the climber's mass (m), the elasticity or "stretchiness" of the rope (k), and gravity (g). Algebraically, it is represented as

$$F_{max} = \sqrt{2mgk \frac{D_T}{L}}.$$



The arm span to height ratio for an adult is generally near one, as shown in da Vinci's "Vitruvian Man."

Climbing ropes must pass a statistically designed drop test to be certified for sale and use. Other critical safety equipment is also designed using mathematics. One example is the curve of cams used in the "friend" devices that secure ropes to crevices in rock walls, which may be optimized using systems differential equations, sometimes with polar coordinates. The devices themselves are an application of logarithmic spirals.

Climbing Theories and Modeling

Many people have drawn parallels between climbing mountains and solving mathematical problems, especially great challenges like summiting Mount Everest and solving a problem like the Riemann hypothesis, first proposed by mathematician Bernhard Riemann. Analyses have shown that Everest climbers engage in multistep problem solving with altitude changes, rates, percentages, conversions, approximations, and division of large numbers. Mathematician-climber John Gill said that problems in both mathematics and climbing are often solved by "quantum jumps of intuition." Patterns found in the natural features of some popular climbing locations can very mathematical. The Navajo Sandstone formation includes rounded domes and saddle shapes with remarkably precise-looking contour lines.

At the same time, the geometric diversity and complexities of climbing surfaces and the variety of techniques used by climbers have made developing a single theory of optimal climbing strategy difficult. However, several methods are used to quantify characteristics of different climbs and probabilistic models can be used to make decisions. Competitive climbers assign climbing grades to climbing routes, using objective and subjective criteria, to describe their difficulty. Other systems assess the technical difficulty of required moves, the stamina necessary, exposure to the elements, or the frequency of difficult moves. Mathematician Alan Tucker demonstrated using graph theory that the classic Parallel Climbers mathematical puzzle has a solution for any mountain range.

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Maria Droujkova

See Also: Algebra in Society; Elevation; Extreme Sports; Puzzles.

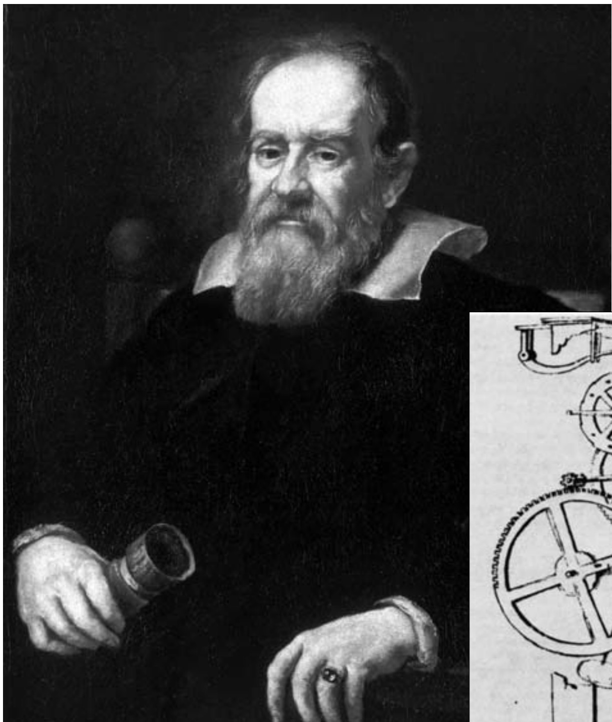
Clocks

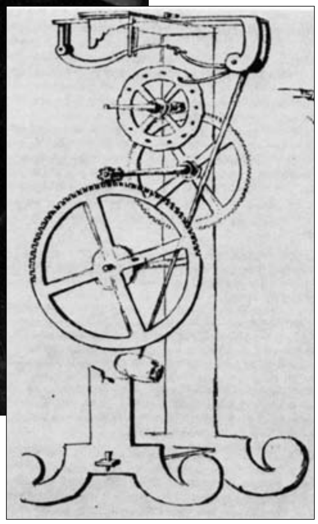
Category: Space, Time, and Distance.
Fields of Study: Measurement; Number and Operations; Representations.
Summary: Clocks are devices for timekeeping and are used for a variety of mathematical calculations, including finding one's longitude.

The term "clock" in a generic sense is applicable to a broad range of devices for timekeeping usually concerning fractions of the natural unit of time—the day. Modern clocks operate through various physical processes. It does not matter what kind of periodic signals a clock produces—ringing a bell, firing a cannon, flashing a light, moving a hand, displaying a number, or generating electric impulses. Mathematics has been fundamental both in the design of clocks and in the measurement of their accuracy. Modular arithmetic, an algebraic concept involving cycles, is sometimes informally known as "clock arithmetic." In the realm of biology, mathematicians have also worked on theories related to the operation of humans' internal biological clocks and bacterial genetic clocks.

History of Clocks

In everyday English language, watches and other timepieces that can be carried individually sometimes continue to be distinguished from clocks. Via Dutch, Northern French, and Medieval Latin, the word "clock" is derived from the Celtic *clagan* and *clocca* meaning "bell." Those old clocks had a striking mechanism for announcing intervals of time acoustically. The history of clocks is much deeper, however. It started in early prehistoric times with sundials (often a vertical post or pillar on horizontal ground exposed to the sun or a post parallel to the Earth's axis) that were the first and





Galileo Galilei, with his design for a pendulum clock around 1641, drawn by Vincenzo Viviani in 1659.

oldest scientific instruments of archaic humankind. They worked only in the daytime. In the terminology of ancient Greece, such a device was called a *gnomon*, and the entire branch of science on sundials is *gnomonics*. Famous Egyptian obelisks—now reerected in some European capitals—were also sundials.

Timekeeping devices of different types were called *horologium* by the Romans. In its corrupted forms, this term later on entered many languages of the world. A noticeable step in the history of timekeeping was the invention of a "water clock" (the specific Greek name is the *clepsydra*). Water clocks could be used at night. Some of the water clocks in China and the Near East were quite large. Another type of simple clock was the "sandglass."

The modern era of clock-art started with the invention of weight-driven mechanical clocks (sometimes known as "chimes"). The inventor of such a novelty is unknown. Because daily prayer and work schedules in medieval times were strictly regulated, religious institutions required clocks, and it is certain that the earliest medieval European clockmakers were Christian clerics. Mechanical clocks were designed en masse in the thirteenth century in Western Europe. They were bulky and appeared on cathedral towers in many countries. Some of them have survived up to now and are among the great artifacts of the medieval epoch.

After the invention of tower clocks, efforts were made to design smaller pieces for tabletops and personal "pocket" clocks (watches) for individuals. Peter Henlein (c. 1480–1542), a locksmith from Nuremberg, Germany, is often credited as the forerunner of the first portable timekeeper, but this claim is disputed. His drum-shaped *Taschenuhr* was too big for a pocket. The first individual clocks were usually worn on the neck or beneath the knee. Timepieces of this type were often known as "Nuremberg eggs." The earliest clocks are very expensive now and are subjects for collectors.

Clocks for Navigation

A great chapter in clock-making began in conjunction with the rapid development of seafaring after the European discovery of the Americas. In order to determine one's position at sea, it is necessary to calculate two geographical coordinates: latitude and longitude. Latitude is easily computed directly from trivial astronomical considerations (the latitude of a locale is equal to the altitude of the celestial pole). As for longitude, it is equal to the difference between local time and the time of a prime meridian chosen specifically for cartographic purposes; navigators used different prime meridians in different countries in different epochs. To discover one's longitude, an observer must know the time at the prime meridian, which requires the art of "transporting" accurate time.

The search for accurate and convenient timekeeping became one of the most impressive scientific and technological challenges of the seventeenth century. Numerous mathematical and astronomical methods were proposed, such as observations of the moon. However, the computations would have been difficult for the typical sailor and the mathematical methods were not yet well-developed enough to provide an accurate prediction. This problem was among the foci of scientific activities of Galileo Galilei of Italy (1564–

1642), who discovered the key property of pendulums that makes them useful for timekeeping: isochronism, which means that the period of swing of a pendulum is approximately the same for different sized swings. Galileo developed the idea for a pendulum clock in 1637, but did not have enough time to complete the design.

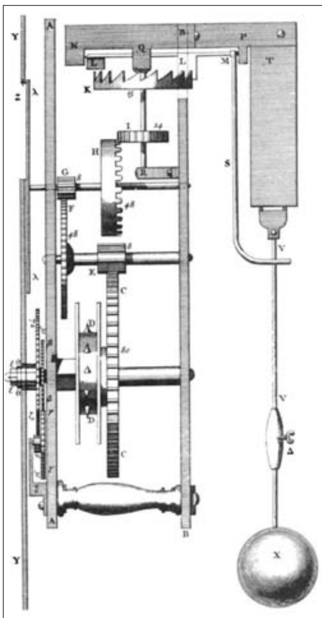
Dutch scholar Christian Huygens (1629–1695) successfully built a pendulum clock in 1656 and patented it the following year. Its design incorporated concepts derived from mathematical work on cycloids. The introduction of the pendulum—the first harmonic oscillator for timekeeping—increased the accuracy of clocks enormously, from about 15 minutes per day to 15 seconds per day. In addition to building a clock, Huygens investigated the properties of synchronization of identical pendulum clocks. Researchers have been interested in the subject of synchronization of clocks and oscillators since that time.

The design of the first marine chronometer was performed by the self-educated English carpenter and clockmaker John Harrison (1693–1776). This device dramatically revolutionized and extended the possibility of safe long-distance sea travel. At the time, the problem was considered so intractable that the British Parliament offered a prize of 20,000 British pounds sterling (comparable to about $4.72 million in modern currency) for the solution. Sailors and astronomers continued to be the principal consumers of accurate timekeeping. Precise clocks became essential equipment for each and every astronomical observatory.

Modern Clocks

The problem of "transportation" of accurate time to determine longitudes lost its actuality with the invention of the telegraph and, later on, with utilization of radio signals. But with the advancement of the twentieth century, new scientific and applied challenges demanded increasingly accurate time reckoning. As a result, new clocks were created based on newly discovered physical principles that were operationalized using mathematics. The crucial step in this direction was the invention of so-called quartz clocks. A quartz crystal has the unusual property of piezoelectricity—when stimulated with voltage and pressure, it oscillates at a constant frequency.

The vibration of a quartz crystal regulates the clock very precisely. Quartz crystal clocks were designed in 1927 by two engineers at Bell Telephone Laboratories: the Canadian-born telecommunications engineer Warren Marrison (1896–1980) and an electrical engineer from the Massachusetts Institute of Technology (MIT), Joseph Warren Horton (1889–1967). Since the 1970s, quartz clocks have become the most widely used timekeeping technology. Atomic clocks followed quartz clocks toward the end of the century. The U.S. National Bureau of Standards (now the National Institute of Standards and Technology) based the time standard of the land on quartz clocks between the 1930s and the 1960s. Eventually, it changed to atomic clocks, the best of which are accurate to 5×10^{-15} seconds per day. Researchers are now developing optical clocks that can be up to 100 times more accurate than the best atomic clocks. Further, satellite-based global



Christian Huygens (right) claimed his 1656 pendulum clock (above) had an accuracy rate of 10 seconds per day.

positioning systems are now a primary source of time for some scientists and people in everyday life. This system provides almost unlimited transportation of time using variety of mobile devices in space and on Earth.

Today, the reckoning and keeping of precise and super-precise time continues to be requisite for numerous scientific and applied problems. Astronomers are still important users of this data. It is important, for instance, in cosmic navigation, in the measurement of variations of the rotation of Earth, and in the implementation of a particular technology into everyday life, such as radio interferometry with a hyperlong base. Every developed country now has a specialized national service for addressing questions regarding precise timekeeping and time reckoning. For a long time in the Paris Observatory, there was the Bureau International de l'Heure (The International Time Bureau), which played an important role in the research of timekeeping. In 1987, the responsibilities of the Bureau were taken over by the International Bureau of Weights and Measures (BIPM) and the International Earth Rotation and Reference Systems Service (IERS).

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See Also: Calendars; GPS; Measuring Time.

Closed-Box Collecting

Category: Arts, Music, and Entertainment.
Fields of Study: Data Analysis and Probability; Number and Operations.

Summary: Collecting objects in sets is a popular pastime that can require a great deal of effort and such collections have inspired mathematical investigations.

A set of closed-box collectibles is a set of similar objects that are sold interchangeably. The objects might be anything; for example, cards, figurines, or trinkets. The term "closed-box" means that the consumer purchases each item without knowing exactly which thing the consumer will get—the package will contain a random item from the collection. Because some collectibles may be more rare or more valuable than others and because individual preferences vary, side markets for these collectibles may emerge, with an identified item selling for many multiples of the price of a random one. Baseball cards, collectible card games, and other trading cards give a familiar example of closed-box collectibles. Toy prizes in cereal boxes and Kinder Surprise eggs are other examples. This problem is one of the classics of probability theory. It has many extensions and can be solved by many methods, including combinatorics and generating functions. It is also known as the Coupon Collector Problem.

Promotional Contests

There have been many contests based on closed-box collecting used for promotional purposes by various businesses and products. Two well-known examples include McDonald's annual Monopoly game and Subway's Scrabble game. In these cases, certain purchases come with one or more random game pieces, which will be one of a large number of types. The game pieces come in various groups; a complete group of collectibles can be exchanged for a contest prize.

From the perspective of the business running such a promotion, the contest design creates certain mathematical problems. What proportion of the game pieces should be manufactured of each type? A main goal is to minimize unpredictability. If too many grand prizes are collected, the company may have to pay out a substantial amount of money; this might be too great a risk to tolerate even if it is very rare. On the other hand, if too few major prizes are awarded, the public may become dissatisfied, negating the public relations goals of the promotion. The usual solution for the significant prizes is to make one type of piece in each group extremely rare, manufacturing only as many as they intend to pay out prizes. The other types can then be made relatively common without risk. This system generally has the effect of maintaining public interest by giving a large number of people the "feeling" of getting closer to winning a big prize as they accumulate common tokens in the group, without risking a huge payout.

Expectations in Closed-Box Collecting

Suppose that a consumer is interested in one particular collectible from a set, and the consumer decides to purchase collectibles one at a time until getting the desired one. Assume each collectible purchased will be the desired kind with probability p, independently of the others. (In real life, this assumption will not be strictly valid, but the discrepancy is negligible if the number of collectibles purchased by an individual is small compared to the total number in existence.) The chance that it is not the kind desired is then $1-p$, and this scenario is modeled by a geometric random variable. The probability of getting the desired item on the first try is p, on the second try is $p(1-p)$, on the third try is $p(1-p)^2$, and so on. Then the expectation is

$$\sum_{n-1}^{\infty} p(1-p)^{n-1}.$$

Standard techniques of basic analysis now show that the expected number of purchases needed is $1/p$.

It should be emphasized that this is the expectation in the sense of probability theory and that there are some common misconceptions about what it means. If the probability of getting the desired item is 1/100, this does not mean that 100 is the most likely number of purchases, nor that the 100th item is any more likely to be the desired type than any other. It means that on average—in the long run—it will take 100 tries to get the desired item. This also means, for example, that when rolling a fair die, it will take an average of six tries to roll a 1, squaring well with intuition.

Another important issue in understanding the dynamics of closed-box collecting is the expected number of purchases to collect a complete set. Suppose that there is a set of 100 collectibles, each item purchased being equally likely to be any of the hundred types. If a consumer purchases collectibles one at a time until obtaining a complete set, how many purchases will be made? It will take one purchase to get one item. With one item type, each purchase will add to the collection with probability 0.99, so the consumer expects to purchase 100/99 more items to get the second item. With two item types, each purchase will add to the collection with probability 0.98, so the consumer expects to purchase 100/98 more items to get the third item. This process continues until the consumer has all the items but one; then each purchase will complete the collection with probability .01, so the consumer expects to purchase 100/1 more items to get the last type, completing the collection. This process indicates a total of

$$\frac{100}{100} + \frac{100}{99} + \frac{100}{98} + \frac{100}{97} + \cdots + \frac{100}{1}$$

purchases, about 519. In general, if there are n types, then the expectation is

$$n\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}\right).$$

For large n a good approximation is

$$n\ln(n) + \gamma n + \frac{1}{2} + o(1) \text{ purchases}$$

to collect all n objects. The constant $\gamma \approx 0.5772156649$ is the Euler–Mascheroni constant, named for Leonhard Euler and Lorenzo Mascheroni, while $o(1)$ is a constant used in computer science meaning a function that converges to zero for very large inputs, such that the value is effectively zero for very large n.

This illustration gives insight into why it seems harder and harder to make further progress in collecting, the further you get. In the example of collecting a complete set of 100 collectibles, with each purchase equally likely to be any of the hundred types, the expected purchases needed is about 519. Suppose now that one has accumulated a collection of 50 different items; is that really halfway to a complete collection? By a similar analysis, the expected number of additional purchases to collect the remaining 50 items is about 450. So there is a meaningful sense in which 450/519 of the collecting task is still undone; a more accurate description of the progress is that the collection is 13.3% completed. In the sense of expectation, one is not really halfway through collecting 100 items until obtaining the 93rd item. While the assumption that all types are equally likely does not usually hold in practice (some types are rarer, some more common), the qualitative conclusion applies in general, unless a few of the items are extremely rare.

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Michael "Cap" Khoury

See Also: Coupons and Rebates; Expected Values; Probability.

Clouds

Category: Weather, Nature, and Environment.
Fields of Study: Data Analysis and Probability; Geometry.
Summary: The formation and behavior of clouds can be mathematically modeled and studied.

Mathematics has been called "the science of patterns." In clouds and the atmosphere, generally there is no end to the patterns that may be observed, quantified, and more clearly understood using mathematics. Mathematicians have long modeled the behavior and structure of clouds.

Applied mathematicians continue to develop ways to detect clouds and quantify motion, composition, density, top altitude, and the distance between clouds, among other characteristics. In 1999, the U.S. National Aeronautics and Space Administration (NASA) launched the Multi-Angle Imaging SpectroRadiometer to measure environmental and climate data from nine different angles, including cloud data.

The U.S. National Oceanic and Atmospheric Administration (NOAA) is one of the largest organizations specializing in the study of the environment. In 2010, a NOAA team led by physicist Graham Feingold reported its findings that clouds form synchronous patterns, meaning that individual clouds in a group respond to signals from other clouds, an effect also observed in chirping crickets or flashing fireflies. This research has implications for interpreting climate change data. There are also mathematical objects such as point clouds that are of interest in geometry, imaging, and efficient distribution mining. Fractal clouds are appreciated for their mathematical properties and their artistic qualities.

Water in the Air

Air is composed primarily of nitrogen (78%) and oxygen (21%). Argon comprises nearly 1%, leaving little room for the remaining gasses, including carbon dioxide, ozone, and neon. This recitation, however, is for dry air. Water vapor, the invisible gas from which clouds are constructed, can account for 0% to 4% of any given parcel of air. In order to form a cloud, water vapor must change phase to either liquid water droplets or ice crystals.

The Transformation of Water into Clouds

The amount of water vapor that can be held in a parcel of air is determined primarily by the temperature of the air; warm air can hold more and cold air less. The amount of water vapor held in a parcel of air is identified by the mixing ratio:

$$w = \frac{\text{grams of water vapor in a parcel}}{\text{kgs of dry air in the same parcel}}.$$

The amount of water vapor a parcel of air *can* hold is called the "saturation mixing ratio":

$$w_1 = \frac{\text{grams of water vapor in a saturated parcel}}{\text{kgs of dry air in the same parcel}}.$$

Relative humidity is a measure of how much vapor a parcel of air is holding compared to how much it could possibly hold and is expressed algebraically as

$$\text{RH} = 100\frac{w}{w_1}.$$

The dew point is the temperature at which a parcel of air becomes saturated. At this point, the saturation mixing ratio and the actual mixing ratio are equal to one another, and the relative humidity is therefore 100%. A further drop in temperature should produce condensation as water changes phase from vapor to liquid cloud droplets or solid ice crystals—a cloud is born.

The Unstable Atmosphere

Clouds are often the result of lifting in the atmosphere. When a parcel of air rises, it generally cools, and this cooling produces condensation. The way in which the lifting is accomplished can lead to dramatic differences in the appearance of the cloud. When whole layers of air are gently lifted in an atmosphere that is stable, stratus clouds are formed, whereas the more dramatic vertical structure of a cumulus cloud comes from runaway convection, a self-perpetuating process that can build clouds more than 12 kilometers (km) or 40,000 feet tall.

What is a stable atmosphere? Temperatures generally decrease with height. The rate of change is, of course, variable but it is referred to as the "lapse rate" (Γ) of the atmosphere. A parcel of air, distinct from the air that surrounds it, may be forced to rise or descend and will cool or warm as a result. Pressure generally decreases with height, and a parcel that rises into a zone of lower pressure will expand, doing work on the environment and therefore cooling. The rate at which a parcel of air cools as a result of this sort of ascension is known as the "dry adiabatic lapse rate" (Γ_d) which is approximately 10 degrees Celsius per km. When the dew point is reached in the parcel and condensation occurs, latent heat is released as a result of the phase change and the parcel is warmed.

The result is a lower lapse rate, the saturated adiabatic lapse rate (Γ_s). The saturated lapse rate depends on the amount of moisture being condensed but 6 degrees Celsius per km may be used as a rough estimate.

Now if $\Gamma < \Gamma_d$, the atmosphere is stable because unsaturated air that is made to rise will cool at approximately 10 degrees Celsius per km and will find itself in air that is increasingly warmer than itself. The greater the difference $\Gamma_d - \Gamma$, the greater the force restoring the parcel to its previous altitude. The force may be quantified as

$$\frac{g\left(\Gamma_d - \Gamma\right)\delta z}{T}$$

where g is the gravitational constant, T is temperature, and δz is a small upward displacement of the parcel from its equilibrium level. Consider the implications of a temperature inversion in which temperature actually increases with height and Γ is a negative quantity. Now consider a situation in which the atmosphere cools strongly with height, that is $\Gamma > \Gamma_d$. Then, the restoring force becomes negative. Air that rises becomes warmer than its surroundings and so continues to rise. This leads to the runaway convection that builds the towering cumulonimbus clouds that can produce thunderstorms, lightning, and hail.

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Mark Roddy

See Also: Energy; Forecasting; Hurricanes and Tornadoes; Wind and Wind Power.

Clubs and Honor Societies

Category: Mathematics Culture and Identity.
Fields of Study: Communication; Connections.
Summary: Various clubs and honor societies add a social dimension to the enjoyment of doing mathematics and can provide networking and scholarship opportunities for talented students.

Mathematics clubs are often designed to provide a fun atmosphere outside of the classroom environment in order to promote mathematics and create a sense of community and camaraderie. Clubs exists for students of all ages as well as adults. Many undergraduate mathematics clubs are affiliated with organizations like the Mathematical Association of America, the Society for Industrial and Applied Mathematics, or the Association for Women in Mathematics. These clubs are open to all students, regardless of gender, race, color, religion, age, national origin, sexual orientation, or disability. However, other student extracurricular activities like semester or summer programs or camps may use a variety of selection criteria for membership.

Mathematical honor societies like Pi Mu Epsilon often consider both mathematical GPA and overall scholarship in their selection of candidates. Participants in mathematics clubs, programs, and honor societies seem to be attracted to the social aspects and related food opportunities; pizza or donuts are often a component of such activities. Researchers have investigated the impact of participation on achievement. There is some evidence that participation may be correlated with an increase in retention, positive attitudes about mathematics, and higher grade point averages.

Activities and Participation

People in mathematics clubs enjoy a wide range of activities together. Clubs may participate in mathematical contests, homecoming activities, mathematical murder mysteries, π-day celebrations, mathematics *Jeopardy*, or recreational mathematics. Some clubs bring in outside speakers, work on problems together, or write and perform mathematical plays or poetry. Other clubs perform service activities, like volunteering to be tutors. Faculty and teacher advisors run or help student members organize clubs at all levels, from primary school through graduate school, although some clubs may be completely run by student members. Adults also organize mathematical clubs for themselves. For instance, in Los Angeles, California, Math-Club's catchphrase is "Be there and be square." Listed members are employed in a wide variety of careers and include professional Hollywood writers, actors, designers, journalists, and musicians.

Clubs may be funded from schools and private donations or they may raise funds from activities like the sale of mathematical T-shirts with slogans like "Know your limits—don't drink and derive," "Math club is as sweet as π," or "Nerds now, rich later." In fact, some journalists have noticed that members of mathematics clubs may enjoy embracing their status as "nerds" or "intellectuals." This may be connected to the same occurrence in popular culture, where nerds are sometimes hip. Clubs are often open to anyone who wishes to join. Specific clubs also exist for members with more specialized interests, like for prospective mathematics teachers or mathematical knitters.

Honor Societies

Members of honor societies are recognized for their successful pursuit of mathematical understanding. The most well-known mathematics honor society in the United States may be Pi Mu Epsilon. As of April 2010, there were 343 chapters. The organization promotes student scholarly activity through a student research journal as well as grants for contests, conferences, and speakers.

Another college society is Kappa Mu Epsilon, which listed 144 chapters and more than 75,000 members in 35 states as of March 2009. The organization focuses on the power and beauty of mathematics and the connections between mathematics and society through a journal and regional conventions. Mu Alpha Theta is an honor society for high schools and two-year colleges, which listed more than 75,000 members in more than 1,500 schools as of October 2010. There are also mathematics honor societies for the homeschool community as well as for some states, schools, and colleges.

Other Programs

There are many other opportunities for students to engage in club-like or honor society–like activities. The Budapest Semesters in Mathematics (BSM) study-abroad program holds courses in English and is seen by many as a prestigious program for students bound for graduate school. In the summer, students may participate in a variety of mathematics camps, workshops, or research programs, such as the U.S. Space & Rocket Center's mathematics camp, Clarkson University's Roller Coaster Camp, or Research Experiences for Undergraduates. Some programs charge money for such activities, and others are funded by grants.

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See Also: Careers; Competitions and Contests; Professional Associations; Social Networks; Succeeding in Mathematics.

Cochlear Implants

Category: Medicine and Health.
Fields of Study: Algebra; Data Analysis and Probability.
Summary: Cochlear implants use signal processing and algorithms to transmit electrical impulses to the brain to simulate hearing.

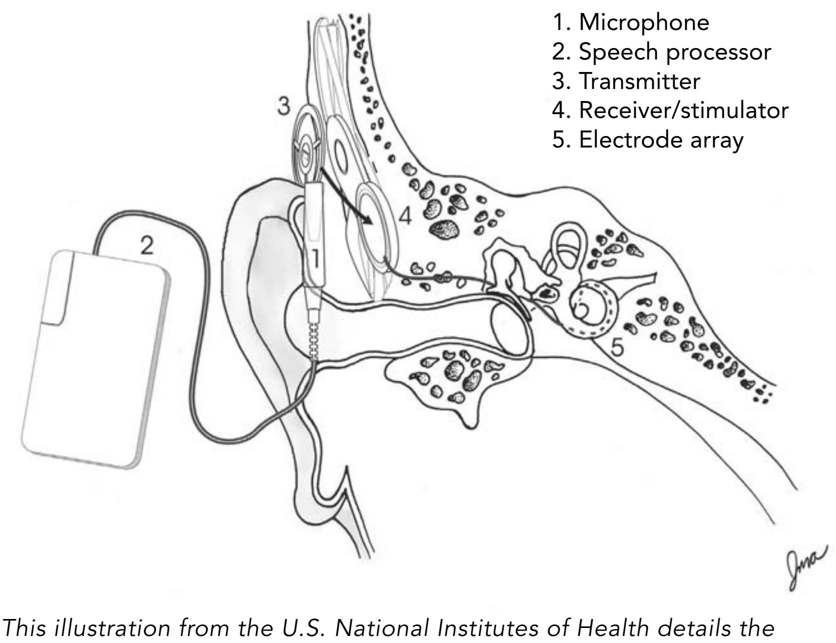
A cochlear implant is an electrical device that can help provide a representation of sound to a deaf or severely hard-of-hearing person. Unlike a hearing aid, a cochlear implant does not amplify sound; instead, it directly stimulates the auditory nerve, which sends these signals to the brain, where they can be interpreted as sound. Development of the cochlear implant relied in part on discoveries by French mathematician Joseph Fourier (1768–1830), whose studies in heat transfer led to the development of mathematics that can also be used to describe sound. Fourier analysis allows mathematicians to describe complex wave patterns, including the pressure waves that produce sound, as the combination of a number of component waves. Cochlear implants also draw on the discovery by the Italian physicist Alessandro Volta (1745–1827) that electrical current could be used to stimulate the auditory system and produce the sensation of sound. Practical work on cochlear implants dates back to the mid-twentieth century, and cochlear transplants were first approved by the U.S. Food and Drug Administration in 1984 for adults and in 1990 for children age 2 years and older (a limit since lowered to 12 months for one type of implant).

Sound Waves and Hearing

In a person with normal hearing, sound waves are collected by the outer (visible) ear (pinna) and sent down the ear canal to the eardrum (tympanic membrane). Movement of the eardrum is amplified by three small bones in the middle ear, commonly referred to as the "hammer," the "anvil," and the "stirrup," before being passed on to the cochlea in the inner ear. In the cochlea, this information is converted into electrical impulses by the hair cells of the organ of corti, and these impulses are sent on to the brain, where they are interpreted as sound. The cochlea has a spiral shape (sometimes likened to that of a snail shell) and scientists have recently discovered that the shape itself is significant in the cochlea's function. The spiral shape produces a "whispering gallery" effect as the energy of the frequency waves accumulate against the outer edge of the chamber, increasing humans' ability to detect low-frequency sounds.

Signal Processing

Most sound, including speech, is complex, meaning that it consists of multiple sound waves with different frequencies. In a person with normal hearing, the ear acts as a kind of Fourier analyzer, which decomposes sound into components. A cochlear implant attempts to mimic this activity, translating sound waves into electrical impulses and transmitting them directly to the brain. Two basic signal-processing strategies have been used in designing cochlear implants: filter bank strategies, which use Fast Fourier Transforms to divide sound into different frequency bands and represent



This illustration from the U.S. National Institutes of Health details the components of a cochlear implant.

this information as an analog or pulsatile waveform; and feature-extraction strategies, which use algorithms to recognize and emphasize the spectral features of different speech sounds.

A cochlear implant somewhat simulates normal hearing rather than restoring it, and individuals who receive an implant require special training in order to learn to recognize the signals as sound. In addition, cochlear implants are not advisable for every type of hearing loss, and a number of factors should be considered by the individual and his or her physician before committing to an implant. These factors include current age, age at which the person became deaf, how long the person has been deaf, the availability of support (including financing) to see him or her through the training period, and the health and structure of the individual's cochlea.

Although cochlear implants are growing in popularity and being used for younger and younger children, they are also controversial for several reasons, some of which were discussed in the 2000 documentary film *Sound and Fury*. One is based on the cost of the operation and follow-up therapy necessary to help the recipient learn to process the electrical impulses as sound. Another is that the surgery requires destroying whatever hearing may remain in the ear where the implant will be placed; for this reason, it is common to have the implant in one ear only. In addition, the surgery is done on children as young as 1 year in order to take advantage of peak language learning periods, so parents must make this decision for their children. Finally, many members of the deaf community feel that they should not be regarded as being defective or handicapped, that they can function successfully in the world using sign language and lip reading. They fear that widespread adoption of cochlear implants will ultimately destroy a distinctive and flourishing deaf culture.

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Sarah Boslaugh

See Also: Brain; Functions; Harmonics; Mathematical Modeling; Nervous System.

Cocktail Party Problem

Category: Friendship, Romance, and Religion.
Fields of Study: Data Analysis and Probability; Measurement.
Summary: A metaphorical cocktail party is the setting for a source separation problem and other challenges.

The eponymous Cocktail Party Problem is a source separation problem in digital signal processing, wherein digital systems have difficulty separating out one signal among many—the metaphorical conversation in a noisy cocktail party, which is comparatively easily handled by the human brain. More broadly, distinguishing signal from noise is a data analysis challenge with many specific applications. The metaphor of the cocktail party also lends itself to a number of other problems in combinatorics, graph theory, probability, and functional analysis.

Conversations and Background Noise

With all the noise at a party, it can be difficult to focus on one conversation, although many people are able to do so. Telecommunication professor Colin Cherry conducted experiments in this area, and he is considered by some to be a pioneer in cognitive science. Many people can even recognize the sound of their name from across a noisy room. However, this is not as easy when heard on a recording. One cocktail party problem arises from concerns about separating each individual's voice characteristics in a recording from the other voices and background noise. People in surveillance and intelligence are inherently interested in such a problem, and scientists and engineers have worked on solutions since at least the 1950s. One common method is mathematical signal processing. Mathematicians and engineers digitize a signal using a Fourier transform, named for Joseph Fourier. They process it using a variety of methods to remove noise and other extraneous information, and then reconstruct the signal using the inverse transform.

While the process may result in an improved recording of one person's voice, early twenty-first-century technology and methods do not provide perfect separation, so the recording still includes at least some distracting background noise. However, engineers have conjectured that the signal should be able to be reconstructed without the noisy phase. Mathematicians Radu Balan, Peter Casazza, and Dan Edidin made progress on the problem in 2006, when they showed—using a neural net—that it is mathematically possible to retain the voice characteristics without the noise. Scientists continue to work on developing algorithms for practical use. Casazza made another fundamental mathematical discovery during his work on the cocktail party problem. He and his wife, Janet Tremain, also a mathematician, showed that the Kadison–Singer problem, named for mathematicians Richard Kadison and Isadore "Iz" Singer, is equivalent to other unsolved problems in areas of pure and applied mathematics and engineering, such as operator theory, harmonic analysis, and signal processing.

Mathematicians also investigate other party problems, like the probability that when people at a party are chosen to be partners for a card game—like bridge—no randomly chosen partners will contain spouses or members of the same family. The solutions require finding specific combinations or permutations of the guests. Under certain constraints, the maximal probability for some problems may be bounded at less than certainty as the number of people at the party grows. There are also connections between this question and the card game War, as well as with a related set of problems that focus on orders and arrangements of guests around a single dinner table or in various groupings, with applications in areas like queuing theory and assignment problems. The classic dining philosophers problem is yet another variation that has applications in resource sharing and task allocation in computer science.

Another party problem asks how many people must be present at a party in order to ensure that there will be a group of three people who share the characteristic of being acquaintances or strangers. There is no guarantee that three people will all know each other or will be strangers in parties of five or less people since counterexamples exist. The Java game HEXI, named so because the game is played on the vertices of a hexagon, is modeled on this question. The six vertices are connected by edges and each player takes a turn coloring an edge his or her color. One color represents acquaintances, and the other represents strangers. The goal of the game is to avoid making a triangle of the same color. Mathematicians model this question using graph theory, and show that in any group of at least six people, it is possible to find a group of three people satisfying one of the mutually exclusive relationships. Hence HEXI will always have a loser. Instead of people at a party or vertices of a polygon, one could explore other objects like nations embroiled in a conflict, sequences of randomly generated numbers, or stars. Mathematicians investigate problems like these concerning the existence of regular patterns in sets of objects in Ramsey theory, named for Frank Ramsey.

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See Also: Daubechies, Ingrid; Graham, Fan Chung; Intelligence and Counterintelligence; Probability; Wireless Communication.

Coding and Encryption

Category: Communication and Computers.
Fields of Study: Algebra; Data Analysis and Probability; Number and Operations; Representations.
Summary: Mathematical algorithms are used in modern encryption and decryption.

Human beings have a propensity to preserve and share secret information. Cryptography, from the Greek *kryptos* (hidden) and *graphein* (to write), is the art and science of coding and decoding messages containing secret information. Encryption is the algorithmic process that converts plain-text into cipher-text (looks like a collection of unintelligible symbols), while decryption is the reverse process that converts the cipher-text back to the original plain-text. A cipher algorithm and its associated key control both directions of the sequence, with the code's security level directly related to the algorithm's complexity. The two fundamental types of cryptography are symmetric (or secret keys) or asymmetric (or public-key), with multiple variations. Claude Shannon, an American mathematician and electronic engineer, is known as the father of information theory and cryptography. Some claim that his master's thesis, which demonstrates that electrical applications of Boolean algebra can construct and resolve any logical numerical relationship, is the most important master's thesis of all time.

Around 2000 B.C.E., Egyptian scribes included non-standard hieroglyphs in carved inscriptions. During war campaigns, Julius Caesar sent coded information to Roman generals. Paul Revere's signal from a Boston bell tower in 1775 is even a simple example of a coded message. Success of the Allies in both World Wars depended on their breaking of the German's Enigma code. With the world-wide need for more sophisticated coding algorithms to transmit secure messages for military forces, businesses, and governments, people began capitalizing on the combined powers of mathematics, computer technology, and engineering.

The simplest examples of ciphers involve either transpositions or substitutions. In 450 B.C.E., the Spartans used transposition ciphers when they wound a narrow belt spirally around a thick staff and wrote a plain-text (or message) along the length of the rod. Once unwound, the belt appeared to be a meaningless sequence of symbols. To decipher the cipher-text, the receiver wound the belt around a similar staff. Variations of transposition ciphers are the route cipher and the Cardan grill.

Julius Caesar used substitution ciphers, where each letter of the plain-text is replaced by some other letter or symbol, using a substitution dictionary. For example, suppose:

Original Alphabet:
A B C D E F G H I J K L M N O P Q R S T U V W X Y Z
Key Dictionary:
K L M N O P Q R S T U V W X Y Z A B C D E F G H I J

where the key dictionary is made by starting with "code" letter K and then writing the alphabet as if on a loop. To encode the plain-text, "The World Is Round," each letter is substituted by its companion letter, producing the cipher-text "CRO FXAUN SB AXDWN." To disguise word lengths and to add complexity, the cipher-text was sometimes blocked into fixed-length groups of letters such as "CROF XAUN SBAX DWN." To decipher the cipher-text, one needed to know only the "code" letter. Though simple and initially confusing, substitution ciphers now are easily broken using frequency patterns of letters and words. Variations of the substitution cipher involve the suppression of letter frequencies, syllabic substitutions, or polyalphabetic substitutions such as the Vigenère or Beaufort ciphers.

The Playfair Square cipher used by Great Britain in World War I is a substitution cipher, but its encryption of letter pairs in place of single letters is more powerful yet easy to use. The cipher-key is a 5 × 5 table initiated by a key word, such as "mathematics."

M	A	T	H	E
I	C	S	B	D
F	G	K	L	N
O	P	Q	R	U
V	W	X	Y	Z

The table is built by moving left to right and from top to bottom (or other visual pattern as in a spiral) by first filling in the table's cells with the keyword's letters—avoiding duplicate letters. Then, the subsequent cells are filled with the remaining letters of the alphabet, using the "I" to represent the "J" to reduce the alphabet to 25 letters (instead of 26). Both the coder and the decoder need to know the both the keyword and the conventions used to construct the common cipher-key.

The coder first breaks the plain-text into two-letter pairs and uses the cipher-key via a system of rules:

- If double letters occur in the plain-text, insert an X between them.
- Rewrite the plain-text as a sequence of two-letter pairs, using an X as a final filler for last letter-pair.
- If the two letters lie in the same row, replace each letter by the letter to its right (for example, CS becomes SB).
- If the two letters lie in the same column, replace each letter by the letter below it (TS becomes SK and PW becomes WA).
- If the two letters lie at corners of a rectangle embedded in the square, replace them by their counterpart in the same rectangle (TB becomes SH and CR becomes PB).

Using this cipher-key, the plain-text "The World Is Round" becomes first

TH EW OR LD IS RO UN DX

which when encoded, becomes

HE ZA PU BN CB UP ZU ZS.

The same cipher-key is used to decode this message, but the rules are interpreted in reverse. It is quite difficult to decode this cipher-text without access to both the keyword and the conventions to construct the common cipher-key, though very possible.

The problem with all substitution and transposition encryption systems is their dependence on shared secrecy between the coders and the intended decoders. To transmit plain-text via cipher-text and then decode it back to public-text successfully, both parties would have to know and use common systems, common keywords, and common visual arrangements. In turn, privacy is required, since these systems are of no value if the user learns the key-word or is able to use frequency techniques of word/letter patterns to break the code. A more complicated and secure encryption process was needed, but it was not invented until the 1970s.

The revolutionary idea in encryption was the idea of a public key system, where the encryption key is known by everyone (that is, the public). However, the twist was that this knowledge was not useful in figuring out the decryption key, which was not made public. The RSA public-key cipher, invented in 1977 by Ronald Rivest, Adi Shamir, and Leonard Adleman ("RSA" stands for the names of the inventors), all of whom have bachelor's degrees in mathematics and advanced degrees in computer science, is still used today thanks to powerful mathematics and powerful computer systems.

In a RSA system, the "receiver" of the intended message is the driver of the process. In lieu of the "sender," the receiver chooses both the encryption key and the matching decryption key. In fact, the "receiver" can make the encryption key public in a directory so any "sender" can use it to send secure messages, which only the "receiver" knows how to decrypt. Again, the latter decryption process is not even known by the "sender."

Because the problem is quite complex and uses both congruence relationships and modular arithmetic, only a sense of the process can be described as follows:

- As the "receiver," start with the product n equal to two very large prime numbers p and q.
- Choose a number e relatively prime to $(p-1)(q-1)$.
- The published encryption key is the pair (n,e).
- Change plain-text letters to equivalent number forms using a conversion such as $A=2, B=3, C=4, \ldots, Z=27$.
- Using the published encryption key, the "sender" encrypts each number z using the formula $m \equiv z^e \bmod (n)$, with the new number sequence being the cipher-text.
- To decode the text, the "receiver" not only knows both e and the factors of n but also the large primes p and q as prime factors of n.
- Then, the decryption key d is private but can be computed by the "receiver" using an inverse relationship $ed \equiv 1 \bmod (p-1)(q-1)$, which allows the decoding of the encrypted number into a set of numbers that can be converted back into the plain-text.

The RSA public key system works well, but the required primes p and q have to be very large and often involve more than 300 digits. If they are not large, powerful computers can determine the decryption key d from the given encryption key (n,e) by factoring the number n. This decryption is possible because of the fact that, while computers can easily multiply large numbers, it is much more difficult to factor large numbers on a computer.

Regardless of its type, a cryptographic system must meet multiple characteristics. First, it must reflect the user's abilities and physical context, avoiding extreme complexity and extraneous physical apparatus. Second, it must include some form of error checking, so that small errors in composition or transmission do not render the message into meaningless gibberish. Third, it must ensure that the decoder of the cipher-text will produce a single, meaningful plain-text. There are many mathematicians working for government agencies like the National Security Agency (NSA), as well as for private companies that are developing improved security for storage and transmission of digital information using these principles. In fact, the NSA is the largest employer of mathematicians in the United States.

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See Also: Bar Codes; Intelligence and Counterintelligence; Mathematics, Applied.

Cold War

Category: Government, Politics, and History.
Fields of Study: All.
Summary: The Cold War had a broad influence on mathematics, including education, coding theory, game theory, and many applied fields.

The Cold War was a 45-year-long period of bitter competition between two large groups of nations. It lasted from the end of World War II in 1945 to the collapse of the Soviet Union in 1991. The two groups never came to direct combat—hence the term "cold"—but it was a war in every other way, fought with deadly ferocity in the political, economic, ideological, and technological arenas. It was also a time of unprecedented investment in new mathematical ideas, driven in part by the desire of each side to dominate the other through nuclear intimidation, economic strength, espionage, and political control. The Cold War had a great impact on mathematics education, on the study of codes and code-breaking algorithms, and the development of new fields such as game theory. More generally, the term "cold war" can be applied to any fight-to-the-death competition between nations in which the two sides avoid direct military combat.

In the original Cold War (1945–1991), the two groups of nations divided along ideological lines. One side, the Soviet bloc, adhered to the communist political and economic philosophy of Karl Marx and Vladimir Lenin. The other side, the Western bloc, adhered to the older free-market capitalist philosophy originated by Adam Smith.

The two sides of the Cold War were essentially forced to avoid military conflict by the recent invention of the atomic bomb, because neither side wanted to risk combat that might give rise to an unstoppable military escalation. The inevitable result of such an escalation would have been worldwide nuclear war, with most large cities destroyed in an instant by nuclear warheads, followed by massive clouds of radioactive ash circling the globe and causing the death of hundreds of millions of innocent people.

Mutually Assured Destruction (MAD)

Prior to the development of the atomic bomb, there were no weapons capable of destroying the population of an entire city in a single blow. Wars were fought as purely military conflicts, without risking the life of civilization itself. This nature of conflict changed forever with the advent of nuclear weapons.

Prior to the Cold War, the dominant mathematical model of warfare was a simple predator–prey model invented by Frederick Lanchester during World War I. Perhaps not surprisingly, given the slow, grinding progress of World War I, the Lanchester model places primary emphasis on the rates of attrition of the military forces. The side that survives this deadly attrition process wins the battle. In a nuclear-armed battle, however, it is survival rather than attrition that is the vital concern, and the Lanchester equations are irrelevant.

In the earliest years of the Cold War, only the United States possessed the theory and technology to construct an atomic bomb. The presence of the bomb in the arsenal of one side but not the other made possible a strategy known as "nuclear blackmail." The owners of the atomic bomb could threaten to use the bomb if their adversaries did not comply with their demands. For example, newly declassified documents have revealed that in 1961, Great Britain threatened China with nuclear retaliation if China were to attempt a military invasion of the British Crown Colony of Hong Kong. The United States backed up this threat, and China refrained from invading Hong Kong.

Clearly, the ownership of the secret of the atomic bomb by just one nation in 1945 had destabilized the military balance of power among the victors of World War II. Great Britain and France allied themselves with the United States and were given access to atomic secrets. The Soviet Union chose to develop its own versions of the atomic bomb, or to steal the secrets through espionage. Thus arose the great division of the Cold War, between the respective allies of the Soviet Union—known as the Warsaw Pact—on one side, and the United States—through the North Atlantic Treaty Organization (NATO)—on the other.

The Soviet Union tested its first nuclear weapon in 1949, ending its four-year period of vulnerability to nuclear blackmail. The military doctrine that took its place was known as mutually assured destruction (MAD). As long as each side in the Cold War could assure the other that it would be utterly destroyed in a nuclear exchange, then—so it was hoped—military conflict could be prevented. MAD did indeed prevent the two nuclear powers from directly attacking each other, but it had two unfortunate consequences: the people of both sides lived in terror of nuclear annihilation, and both superpowers engaged in so-called proxy wars, using much smaller nations as their proxies in localized military conflicts.

Albert Wohlstetter was an influential and controversial strategist who was a major force behind efforts to deter nuclear war and avoid nuclear proliferation. He worked as a consultant to the RAND Corporation's mathematics division starting in 1951. Initially, he collaborated on problems related to modeling logistics, but then he was asked to turn his skills to a problem posed by the U.S. Air Force regarding the assignment and location of bases for Strategic Air Command (SAC). On the surface, it was a common logistics problem, but ultimately SAC's method of basing its medium-range manned bombers, which were one of the country's major deterrents against a Soviet invasion of western Europe, had far greater implications. This work drew him into global strategy. He and his wife. Roberta Wohlstetter, a historian and intelligence expert, received the Presidential Medal of Freedom in 1985. Wohlstetter was also reputedly one of the inspirations for the film *Dr. Strangelove*.

The Arms Race as a Nash Equilibrium

From its very outset, neither of the Cold War's two superpowers—the United States and the Soviet Union—believed that they could stop developing new and ever more powerful nuclear weapons. The MAD

In accordance with a post–Cold War agreement, the Titan II Missile silo doors can only open halfway.

doctrine applied only as long as the forces of each side could pose a credible nuclear threat to the other. Therefore, each side worked to create new weapons as fast as possible. Throughout the 1950s and well into the 1960s, both nations tested ever more powerful nuclear weapons. This became known as the "arms race."

In the mathematical theory of games, the military arms race brought about by the MAD doctrine is an example of a Nash equilibrium, named for mathematician John Nash. In the decades-long arms race between the two Cold War competitors, each side could be seen as playing a simple noncooperative game. Each player in this game has a choice: to construct new and more terrible nuclear weapons, or not. If either player chose not to develop further weapons, while the other did, then the first player would face the very real risk of eventually facing nuclear blackmail. Each player understood the other's dilemma all too well, and so both continued to develop new weapons as fast as possible.

The persistence of this behavior comes from the fact that neither player can benefit by changing strategy unilaterally. When this occurs in a game, then it is in a form of equilibrium whose existence was first proved in the general case by Nash in 1950.

Game theory itself was a child of the Cold War, having been created in 1944 by John von Neumann, a mathematician who also played a key role in the development of the first atomic bomb, and Oskar Morgenstern, an economist. Throughout the Cold War, the theory of games was studied and elaborated, both by the military and by economists, as a means for better understanding the fundamental nature of competition, cooperation, negotiation, and war.

The fundamental irrationality of the nuclear arms race, in which each side became able to kill every single person on the planet many times over, was apparent to almost everyone. This realization did little to stop the arms race, because of the power of the Nash equi-

Old Soviet anti-aircraft missile rockets, first deployed in 1957. Generations of children grew up with the ever-present threat of war during the Cold War (1945–1991) and were taught bomb threat procedures.

librium to trap the players of the game into modes of behavior that, individually, they deplored.

In some critical respects, an arms race resembles a famous game known as the Chicken: two cars race toward each other down a narrow road, with the driver who first swerves away to avoid a crash being the loser. The key to winning a game of chicken is to act in such a way that your opponent comes to believe that you are so irrational as to be willing to die before swerving. In other words, the "rational" solution to the game is to be utterly and convincingly irrational. The same principle holds in a nuclear arms race.

The game of Chicken and other apparent paradoxes of rationality within the theory of games led to the development in the 1970s of meta-game analysis. This and other mathematical forms of strategic analysis played an important role in the eventual winding down of the arms race with a series of strategic arms agreements between the major powers of the Cold War.

Political Competition in the Cold War

The bitter competition of the Cold War was at least as much political and economic as military, and new mathematical ideas contributed mightily to this competition. In the economic arena, the Cold War was fought between the proponents of multiparty, free-market economies on the Western side, and the proponents of single-party, command economies on the Soviet side.

Both sides claimed to be democratic in the Cold War, but they used different meanings for the word. In the West, the word "democracy" retained its historic meaning, a political system in which leaders are chosen in free elections. In the Soviet system, "democracy" meant a "dictatorship of the proletariat" in which all political power rested in a hierarchy of labor councils, known as *soviets*, and the supreme soviet could dictate any aspect of public affairs. Soon after the Russian Revolution, however, the Communist Party seized control of the soviets, and after that, no election in the Soviet system was free.

The intense political competition between these two systems of government led to great interest in the West in how to conduct elections in the fairest possible way. A large body of mathematical theory of elections emerged, much of it devoted to the study of election systems that come the closest to meeting a measure of fairness known as the Condorcet criterion. In an election, the Condorcet winner is the candidate who can beat any of the other candidates in a two-person run-off election. Many forms of preference balloting, in which voters rank the candidates, come quite close to the Condorcet criterion, but none is without problems. Arrow's Paradox, discovered and proved by Kenneth Arrow in 1950, states that when voters have three or more choices, then no voting system can convert the ranked preferences of the voters into a community-wide ranking that meets a particular beneficial set of criteria. This Cold War mathematical discovery is the starting point of the modern theory of social choice, the foundation of the mathematical theory of political science.

Economic Competition in the Cold War

There are many forms of socialism known in economic theory, but the form practiced by the Soviet bloc of nations was particularly severe. In its purest form, Soviet socialism entailed state ownership of all means of economic production: all industrial plants, all commercial businesses, all farms, and all financial institutions. Soviet socialism was a command economy, meaning that the state had to tell every plant, business, and institution how much to produce, and at what price they should sell their goods and services.

In order to come up with the enormous number of production and price commands that had to be sent out every month and year, the Soviet system of government employed a vast bureaucracy. The system used by these bureaucrats was developed in the 1920s, during the early years of the Soviet Union, without the benefit of mathematics. Known as the "method of balances," this system attempted to function so that the total output of each kind of goods would match the quantity that its users were supposed to receive.

In practice, the Soviet "method of balances" functioned very much like the U.S. War Production Board during World War II, and by its counterparts in the war economies of Great Britain and Germany. The first production decisions were made with respect to the highest priority items (ships, tanks, airplanes) and those were balanced with the available amounts of strategic resources (iron, coal, electricity), and so on down to the lowest-priority items. The command system was thought to be crude and error-prone, and its mistakes and imbalances were widely noticed.

In the West, the response of mathematicians to these failures of the wartime command economy was the development of the field of engineering known as operations research. The mathematical technique known as "linear programming"—originally a little-known Russian discovery—was successfully developed by George Dantzig and John von Neumann in 1947 to optimize production quantities under linear constraints on supplies. The Soviet economy was very slow to adopt these ideas, preferring for ideological reasons to stick with the inefficient and error-prone method of balances until very late in the Cold War.

After World War II, the nations of the West ended their wartime command economic systems and reverted back to using the free market to make price and production decisions. The Soviet Union and its allies, however, continued to rely on a large army of bureaucrats to make all economic decisions without the aid of good operational theory.

Wassily Leontief, a Russian economist working in the United States, solved one of the fundamental problems of a command economy in 1949 with his method of input–output analysis. This method required the creation of a very large matrix showing the contribution of each component or sector of the economy to every other component. When properly constructed, the required inputs of raw materials to the economy can be calculated from the desired outputs by matrix inversion. The Soviet Union failed to quickly see the significance of Leontief's achievement, and did not incorporate his ideas into its planning system for many decades.

It is one of the great ironies of the Cold War that the mathematical theories that were required to make a command economy function properly were perfected in the West, where they are now universally employed within industrial corporations—some now larger than the entire economy of the old Soviet Union—to run their operations in the most efficient way possible. In the end, the economy of the Soviet Union and its satellites was not able to keep pace with the West, and in 1991, it suffered catastrophic political and economic collapse.

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See Also: Atomic Bomb (Manhattan Project); Game Theory; Predator–Prey Models; World War II.

Combinations

See *Permutations and Combinations*

Comic Strips

Category: Arts, Music, and Entertainment.
Fields of Study: Communications; Connections.
Summary: Mathematics plays a role in comic strip formats and is sometimes even the subject of comics.

The comic strip is a combination of word and picture in a narrative structure, unique from forms of communication based solely on the one or the other. The standard comic strip format presents its creator with two unique mathematical puzzles: to tell a story that fits into the pattern unconsciously expected by the reader, and to organize the illustrations of a Sunday strip into a format of exacting geometric and narrative demands.

Origins

The term "comic strip" entered the English language in 1922, via a poem by Carl Sandburg, describing the single-strip, black-and-white cartoons published in daily newspapers. Scholars disagree on the origins of the comic strip. Daily strips first appeared circa 1903, as part of the racing tips section of the newspaper. This was some 20 years after the appearance of the first full- or half-page color comics in supplemental sections, a reaction to protests against publishing on Sundays.

Color cartoons were a continuation of the European tradition of sociopolitically inspired prints that date back to the widely circulated wood block broadsheets of fifteenth-century Germany. Some scholars go further, tracing ancestry as far back as La Tapisserie de La Reine Mathilde, narrative scrolls of China and Japan, Trajan's Column, Bronze Age logographs, or even ancient petroglyphs and cave paintings. Regardless of the exact origins, the current format presents the cartoonist with two challenges of special interest to the mathematician.

Story and Art

The original supplements carried full- or half-page features with detailed drawings and developed stories. Older strips would not be legible if published at current smaller sizes. As a result, the expansive serial strip has been replaced almost entirely by the gag strip.

Theorists suggest that humor is based on pattern recognition. If the audience recognizes a pattern, it begins to anticipate what will come next. A deviation from the pattern, if done correctly, is perceived as humorous. In gag strips, between two-thirds and three-quarters of the space is spent establishing the pattern. The deviation happens next, sometimes followed by a character reaction to the deviation, sympathetically reinforcing the audience reaction, or providing additional deviation.

Not only have comic strips become smaller, all color comics must fit into an extremely limited template. Syndicates require a minimum of six panels, but some newspapers elect not to publish one or both of the first panels. Character poses and scene layouts in the two-dimensional plane are designed to lead the reader's gaze from one point of interest to another, driving the story forward. Deciding on the orientation of visual images and their relationships can be difficult when the artist does not know where one panel will be in relation to the next.

Mathematics in Comic Strips

Mathematics is used not only to decide the layout and flow of comic strips but can be used within comic strips as an element of humor. Many comics reveal or satirize widely held societal attitudes and beliefs about mathematics. Bill Amend, creator of the widely circulated *Foxtrot* comic strip, has a degree in physics, and his strip frequently features mathematically based humor. The same is true of Randall Munroe's Web comic *xkcd*, which is subtitled "A webcomic of romance, sarcasm, math, and language." Comic strips may be used in classrooms as motivators for serious discussions about mathematics concepts and analysis of peoples' attitudes about mathematics. There are also entire comic books and graphic novels intended to teach mathematics. The work *Logicomix* dramatizes the life story of philosopher and mathematician Bertrand Russell, who spent his life trying to establish an indisputable logical foundation for mathematics. In the course of the novel, he encounters many mathematicians of note, including Gottlob Frege, David Hilbert, Kurt Gödel, and Ludwig Wittgenstein.

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John N. A. Brown

See Also: Communication in Society; Connections in Society; Representations in Society; Sequences and Series.

Communication in Society

Category: School and Society.
Fields of Study: Communication; Connections.
Summary: Communication helps mathematicians and others be informed of past and current research and to formulate and organize their own ideas.

Communication is fundamental to mathematics as a discipline, the mathematics community, mathematics education, and society as a whole, since communication is an essential part of everyday life and any social interaction. Effective communication is inherent in validating mathematics. Using a common language and a set of notions and drawing upon a shared body of knowledge, mathematicians communicate with each other—both orally and in writing—about their mathematical ideas, perceptions, or methods.

For example, mathematicians exchange ideas with their colleagues, write technical reports, publish original research papers and expository articles in professional journals, or give oral presentations. Some associate good mathematics communication with beautiful expository lectures or clear writing, while others focus on the quality of the interactions between people, such as those working in a group on mathematics. A peer review process is frequently part of mathematics communication and dissemination, ensuring some degree of consensus on what constitutes *appropriate* or *valid* mathematics. In this way, the standards of mathematics are socially developed. In addition to interacting with their colleagues, mathematicians need to communicate with the rest of the society using a language and terminology that are more familiar to the general public. For instance, mathematicians explain to the public how the discipline of mathematics contributes to society or demonstrate the various applications of mathematics in fields such as engineering, medicine, and communication technologies.

The role of communication in the education of mathematics is similar to the vital role communication plays in the discipline of mathematics. Drawing upon mathematical language and notation, teachers and students talk about mathematics; share, explain, and justify mathematical ideas; or analyze, discuss, and interpret mathematical concepts. Communication about mathematics and communication using mathematical language do not occur only in the mathematics community or in mathematics classrooms. Regardless of one's profession, wise decision making in personal lives and participation in civic and democratic life increasingly demand mathematical communication skills. For example, people need to communicate with mortgage companies when buying a house and interpret various mathematical concepts (such as percentage and rate) presented in the media. Thus, communication with mathematics and about mathematics is an essential part of daily life.

Communication Media

In the twenty-first century, there are a wide variety of electronic and print venues for communicating mathematics, and the evolution of electronic media and databases has vastly changed the way people access mathematics. Historically, mathematicians communicated by letters, during visits, or by reading each other's published articles or books once such means became available. Some mathematical concepts were developed in parallel by mathematicians working in different areas of the world, such as German Karl Friedrich Gauss and American Robert Adrain, who both made advances in the theory of the Normal distribution in the early nineteenth century. Some mathematicians were not aware of each other's progress because they did not have the venues of communication that are available in the twenty-first century. In an effort to increase the accessibility of mathematics research articles, reviews began appearing in print journals like *Zentralblatt für Mathematik*, which originated in 1931, and *Mathematical Reviews*, which originated in 1940. Since the 1980s, electronic versions of these reviews have allowed researchers to search for publications on a specific topic. In 2010, MathSciNet, the electronic version of *Mathematical Reviews*, listed more than 2 million items and more than 1 million links to original articles. In 2011, the database Zentralblatt MATH listed more than 3 million items from approximately 3500 journals and 1100 serials. Both contain work dating back to the early 1800s. There are also thousands of mathematics journals that are not listed in these collective databases, such as most mathematics education research.

Some mathematicians publish open access drafts of their papers on their personal Web pages before official publication in peer-reviewed and other journals, or in other online settings such as the ArXiv.org e-print archive. Co-authors from around the world can work together using e-mail or other Web-based collaborative tools. Mathematics students, teachers, and researchers often discuss mathematics ideas and share resources on blogs, through online chats, or using other forums. For instance, what began in 1992 as the Geometry Forum was extended in 1996 to become the Math Forum. There are many additional resources for sharing and teaching mathematics content, both in print and in electronic media. Some electronic examples include the National Council of Teachers of Mathematics Illuminations Web site; Wolfram MathWorld, which was developed by Eric Weisstein; and Math Fun Facts, developed by Francis Su. Social and historical context is also often addressed in sites such as The MacTutor History of Mathematics archive, developed by John O'Connor and Edmund Robertson, or Mathematicians of the African Diaspora, created by Scott Williams.

One important question related to online communication is how to represent and display mathematical notation, which is an important part of mathematical validity and understanding. Some Web pages contain fixed images for each equation or graph. Others use Java applets for dynamic display. The Mathematical Markup Language (MathML) is one way to encode mathematics. TeX was created by Donald Knuth in order to typeset scientific and mathematical research. TeX-based software such as LaTeX has become the standard in printing mathematics. Another issue is the validation of online resources, which may be created or published without peer review. On one level, this issue is an extension of the existing issue of peer review for print media, as mathematics journals already employ varying degrees of rigor when reviewing and publishing papers. At the same time, there is in increasing trend of creating printed works from electronic sources or using electronic sources as references, which creates an added difficulty in ensuring the collective accuracy of the body of mathematics communication.

With so many options available, the specific nature of mathematics communication depends in large part on the purpose and intended audience. There are some mathematics publications and communications aimed at a general audience, others aimed at students, and yet others intended for researchers. Mathematicians, educators, and other communications specialists work to match the form and venue of the mathematics communication to the need. Some careers that are regularly involved in communicating mathematics include technical writers or publication editors. The Society for Technical Communication and the Council of Science Editors are two professional associations that address this need. In 2007, Ivars Peterson became the director of Publications and Communications at the Mathematical Association of America, which, like other professional associations, publishes items for both the specialist and the nonspecialist. He previously wrote MathTrek for *Science News*. In 1991, he received a Joint Policy Board for Mathematics (JPBM) Communications Award for his "exceptional ability and sustained effort in communicating mathematics to a general audience." He also served as East Tennessee State University's Basler Chair of Excellence for the Integration of the Arts, Rhetoric, and Science in 2008 and taught a course there called Communicating Mathematics. In a talk on the topic of communication in mathematics, he noted:

> The importance of communicating mathematics clearly and effectively is evident in the many ways in which mathematicians must write, whether to produce technical reports, expository articles, book reviews, essays, referee's reports, grant proposals, research papers, evaluations, or slides for oral presentations.

National Council of Teachers of Mathematics

The National Council of Teachers of Mathematics (NCTM) emphasizes clear and coherent communication of mathematical ideas and thinking as a skill that students need to learn from pre-kindergarten through grade 12. Given the essential role of communication in teaching and learning of mathematics, NCTM has set forth process standards for communication for primary and secondary mathematics curricula. The Principles and Standards for School Mathematics (2000) states that instructional programs from pre-kindergarten through grade 12 should enable all students to

- Organize and consolidate mathematical thinking through communication
- Communicate their mathematical thinking coherently and clearly to peers, teachers, and others
- Analyze and evaluate the mathematical thinking and strategies of others
- Use the language of mathematics to express mathematical ideas precisely

Communication in Schools

Communication, both oral and written, is an essential part of mathematics education. The act of communication allows students to systematize and incorporate their mathematics thinking and understanding, both for learning mathematical theory and mathematical problem solving. For example, when students communicate their own mathematical thinking and understanding, they are required to rationalize and organize their reasoning and also formulate puzzling or complex questions well enough to present them as clearly as possible to a reader. As a result, the process guides students toward greater insights to their own thinking and learning. Focused reflection, which is conceptually intertwined with communication, helps students to increase the benefits of communicating their ideas with peers, teachers, and others. Written or oral reflections in which ideas are shared among peers, teachers, and others provide students multiple perspectives that sharpen ideas explored. The American Society of Mathematics (ASM), which is also known as the American Society for the Communication of Mathematics, sponsors problem-solving contests and the U.S. National Collegiate Mathematics Championship.

Proofs

One topic that illustrates the importance and the diverse nature of mathematics communication is the notion of proof. Researchers have proposed a wide variety of roles for proof in mathematics, such as establishing the truth of a statement, communicating mathematical knowledge, opening the way for further understandings and discoveries in mathematics, providing new techniques for doing mathematics, and organizing statements into systems of axioms and theorems. Throughout history, proofs and communication via proof have been incorporated in many different ways in mathematics education in the United States. The National Council of Teachers of Mathematics' (NCTM) 2000 Principles and Standards for School Mathematics emphasized the role of proof in mathematics learning for all students and helped to formalize its curricular importance and place in prekindergarten though high school education. Further, as proof became more systematized in K–12 education, some mathematics education researchers began to more deeply explore students' understanding of the definition or nature of proof, the role of proof as a mode of communication, and peer acceptance of the validity of a given proof, as well as how proof is taught in classrooms.

As the concept of proof came under investigation, an important issue was the conceptualization and the roles of proof in school mathematics. The NCTM defined proof in *Principles and Standards for School Mathematics* as "arguments consisting of logically rigorous deductions of conclusions from hypotheses." One element in the definition of proof is the acceptability of an argument as proof, which is referred to as "logically rigorous." An important question that NCTM's definition entails is who decides if a proof is logically rigorous enough to be accepted. To conceptualize the definition and identify the roles of proof in school mathematics, mathematics education researchers have referred to the qualifications and function of proof in the discipline of mathematics and investigated how it is implemented in mathematics classrooms.

Research has demonstrated the social nature of argumentation and justification in the classroom and beyond, and communication and validation by peers plays an important role in proof within and outside the classroom. This social dimension of proof is grounded in sociocultural theories of mathematical learning and is believed to reflect the process of becoming a mathematician. Yu Manin argued that within mathematics community, "A proof becomes a proof after the social act of 'accepting it as a proof.'" Erna Yackel and Paul Cobb concluded that acceptable justifications in mathematics education are interactively constituted by individual teachers and students in each classroom, where the teacher is the representative of the mathematical community. Mathematical justifications and argumentations are regulated by the general expectations and the regulations of the classroom community.

Thus, they are a part of the classroom norms and, more specifically, the sociomathematical norms, which are the extension of general classroom social norms to specifically focus on the normative aspects of mathematical discussions as students participate in mathematical activities. Yackel and Cobb argued that: "Normative understandings of what counts as mathematically different, mathematically sophisticated, mathematically efficient, and mathematically elegant in a classroom are sociomathematical norms.…The understanding that students are expected to explain

A student at a science, technology, engineering, and math (STEM) summer camp examines her robot before releasing it for a test. The camp encouraged K–12 students to pursue education and careers in the STEM fields.

their solutions and their ways of thinking is a social norm, whereas the understanding of what counts as an acceptable mathematical explanation is a sociomathematical norm." This idea plays a role in mathematics educator Andreas Stylianides's conceptualization of proof. He proposed four aspects that are required to consider an argument a proof: foundation, formulation, representation, and social dimension. He presented an example in which an elementary school student constructed a mathematical argument that was founded on definitions of mathematical constructs, formulated using deductive reasoning from these definitions, and then represented verbally. Regarding the social dimension of the proof, although the student's argumentation was logically rigorous and would have been accepted as a proof in the wider mathematical community, it generated counterarguments among her classroom peers and her argument was not accepted as a proof by the classroom community.

Indeed, the conceptualization of mathematics, in particular the social dimension that is appropriate for school mathematics, requires more research to develop. Mathematical discourse is an important factor in the development of shared understanding of mathematically valid justifications. However, students at various levels, particularly younger elementary school students, may have different levels of understand regarding the rules and norms of mathematical discourse, and understanding is not necessarily shared by all. Thus, as was the case in Stylianides' study, a valid mathematical argument was not accepted as valid by all students. In such cases, the teacher, acting as an authoritative representative of the mathematical community, could intervene and explain why the argument is indeed valid by broader standards. However, in some ways this action would negate the social dimension aspect that is used to evaluate mathematical acceptability, at least with respect to the classroom environment. Thus, the subtleties in what constitutes a valid argument within a mathematics classrooms and the relation to a teacher's role as the communicator of other mathematical norms as they acculturate students in the processes of proving need to be explored. It is important to note that teachers need to know when, how, and how much to intervene so as to not play an authoritarian role, thereby creating a learning environment in which students are forced into authoritarian schemes and communication is essentially unidirectional, from teacher to student.

Mathematical Applications in Communication Technologies

In the increasingly digital world of the twenty-first century, the safe communication of information has become a major issue for discussion and research in mathematics and science, in large part because of theft and fraud often perpetrated using new technologies. Mathematics plays an important role in making communication as safe as possible. Cryptology is a technique used to ensure that messages or data are transmitted safely to the receiver. Dating to the substitution ciphers used in ancient Rome and other civilizations, this field has always drawn heavily from mathematics. Research in mathematics and other disciplines, such as computer sciences and engineering, has resulted in an increasingly sophisticated array of coding techniques and technologies, as well as code-breaking methods. Some of the most common and known applications of cryptography include encryption of credit card numbers or passwords for electronic commerce and encryption of e-mail messages for secure communication. Confidentiality, authenticity, and integrity in electronic commerce or communication have become an apparent and sensitive issue for people who engage in online transactions such as buying or selling items online, online banking, and online communications, as well as for applications like medical records. If proper action is not taken for data transmission, information sent over an open network can be stolen by hackers. Such an action can reveal secret information or messages containing personal information, like a credit card number, a password, or online banking information, facilitating crimes like identity theft. A hacker can use digital data to clone a person's identity and use a victim's resources for the hacker's own good. Even worse, this information could be a national secret, and it may cause more serious problems. For that reason, the National Security Agency (NSA) uses its cryptologic heritage in the midst of challenging times to protect national security systems, and the NSA is one of the leading employers of mathematicians in the United States at the start of the twenty-first century.

Along with digital security, mathematics also plays a fundamental role in both the hardware and software that make the increasingly wireless, globally connected world possible. The *Advances in Mathematics of Communications* journal publishes research articles related to mathematics in communication technologies. Mathematicians and mathematical methods contribute to many aspects, including the Internet's computer server backbone and communications protocols; vast cell phone networks; and smartphones that act as mobile platforms for an array of communications methods, such as voice, text, photo, e-mail, and Internet. Music, movies, dance, art, theater, and many other methods people use to convey ideas to one another involve mathematics as part of the creative endeavor. Humans can communicate with neighbors next door, with people on the opposite side of the world, with satellites orbiting the planet, or even with probes that have been sent into the far reaches of the solar system thanks to mathematics. Some would in fact argue that mathematics is itself a universal language or method of communication.

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See Also: Coding and Encryption; Professional Associations; Reasoning and Proof in Society; Universal Language; Wireless Communication.

Comparison Shopping

Category: Business, Economics, and Marketing.
Fields of Study: Geometry; Measurement; Number and Operations.
Summary: Both simple and complex algorithms are used to compare consumer prices and contextualize mathematics instruction.

The globalization of the marketplace has resulted in a plethora of choices for any given item at both the local store and via the Internet. People comparison shop for both very expensive items like a car or a plane ticket and fairly inexpensive purchases like a box of cereal. Comparison shopping is perhaps one of the most widely used applied mathematics lessons, both in K–12 and lower-level college courses.

Mathematics is at the forefront of comparison shopping through unit pricing, which makes use of division and fractions. Geometric methods can be used to compare volume or weight. Notions from pre-algebra and algebra model financial decisions such as purchasing a cell phone plan or taking out a car loan. Students explore parameters in order to make balanced and informed choices. Mathematics educators not only use these examples in classrooms, but they also study their effectiveness. Researchers and online shopping agents take advantage of mathematical methods to extract, compare, and mine huge amounts of data. Comparison techniques also include data envelopment analysis and multiple regression.

Unit Pricing

One method of comparing differently priced items in different sized containers is through unit pricing. Dividing the price by the quantity or amount of items, such as how many ounces, will yield a cost per unit term that can be used for comparison purposes. For example, an 11.5 oz box of cereal might cost \$4.49, while a 24 oz box of cereal costs \$4.99. The unit price of the first box is \$4.49/11.5 ≈ \$0.39 per ounce, while the second box is \$4.99/24≈ \$0.21 per ounce. Some items are already priced by their weight, like meats, fruits, vegetables, or coffee, and others are priced according to their volume, that is, by the container size. For those items that are not priced by weight or volume, unit pricing is listed on the shelf tag in many stores. However, the unit price is not the only important feature in comparison shopping. Personal preferences and other important factors must also be taken into consideration, like whether one will be able to use up a larger quantity before the expiration date. Unit pricing examples proliferate in lessons on fractions and in classes like pre-algebra and developmental mathematics. Students also compare scenarios in which sales occur or other discounts are applied.

Debt and Interest

Another common classroom scenario is found in comparing house and car purchases in financial mathematics segments. For instance, students can use the loan payment formula to calculate the monthly payment R in terms of the monthly interest rate r, the loan amount P, and the number of months, n

$$R = \frac{rP}{1-\left(1+r\right)^{-n}}.$$

Then they can calculate the total interest by multiplying the monthly payment and the number of months and subtracting the loan amount. One comparison scenario is determining how the monthly payment and total interest change as the price of the car or house changes or the interest rate fluctuates. Another is determining whether one should take out a smaller loan versus paying loan points to buy down the interest rate. Students also compare car prices to income level using the debt-to-income ratio. The debt-to-income ratio is the debt divided by the income, which is the percentage of debt. Banks use the debt-to-income ratio in making decisions about mortgage or car loans. From the Great Depression in the 1930s until the deregulation of banking restrictions in the 1970s, an upper limit of 25% was typical. However, that level rose after deregulation and with the increase in consumer credit card debt. In the twenty-first century, it is common for an upper limit to range between 33% and 36%. Given a monthly car payment, house payment, and other monthly debts, students can add up the total debt and solve for the necessary income level in order to stay below 36%. They can also compare the way that debt and the needed income change as the interest rates vary.

Contextualizing Instruction

Mathematics educators use purchasing scenarios in the classroom and study and debate their effectiveness. Some studies have found that the contextualization of mathematics using examples from shopping helps students. Terezinha Nunes, Analucia Schliemann, and David Carraher compared the mathematical abilities of children who were selling items in Brazilian street markets to questions in school. They found that the closer to the real-life situation, the more successful the student. Other studies have also found that there can be a disconnect between performance in the supermarket and performance in school. Some researchers assert that the contextualization may disguise the mathematics and be problematic in elucidating the underlying mathematical processes.

Mathematical Models for Comparison Shopping

Businesses and researchers employ a variety of mathematical techniques in order to compare large shopping data sets. Online shopping agents use mathematical methods in situations such as a Web search for airplane ticket prices or hotel rooms. Historically, dating back to at least the nineteenth century, travel agents sold vacations to consumers on behalf of suppliers. Travel agencies grew in popularity with the increase in commercial aviation after World War I. At the end of the twentieth century, the Internet vastly changed the way in which consumers compared and purchased vacation travel. Airlines, hotels, and other vacation companies offered online services directly to consumers, bypassing travel agents. In response, some travel agencies created travel Web sites that would compare options. Their computer programs extracted comparative price data from Web sites in order to build comparison shopping engines. Researchers continue to develop advanced comparison shopping techniques including methods in data mining, data envelopment analysis, and multiple regression. They create sophisticated algorithms to analyze data and find patterns. In data envelopment analysis, networks can be viewed as decision-making units, and efficient configurations are selected. In multiple regression, several variables are combined in an attempt to create a meaningful predictor or measure. Mathematical methods are also important in predicting shopping preferences and consumer behavior.

Marketplace globalization has resulted in increased choices at both local stores and via the Internet.

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See Also: Coupons and Rebates; Data Mining; Inventory Models; Market Research; Predicting Preferences.

Competitions and Contests

Category: Mathematics Culture and Identity.
Fields of Study: Communication; Connections; Problem Solving.
Summary: Mathematics competitions and contests help encourage students to practice and study mathematics and develop problem-solving abilities.

Well-designed mathematics contests provide excellent vehicles for students to hone their skills, expand their knowledge, develop their ability to focus, practice creative problem solving, and join a community of peers who love mathematical challenges. Mathematicians and educators organize competitions, help students prepare for them, participate on committees to grade the results, and assess contests' long-term impact. Some mathematics competitors are known as "mathletes."

MATHCOUNTS and USAMTS

There are a number of well-known mathematics competitions in the United States for middle school and high school students. MATHCOUNTS, a mathematics competition for sixth, seventh, and eighth graders, emphasizes problems from geometry, combinatorics, and algebra. The competition includes written and oral rounds with both individual and team competitions, and students advance from school, to chapter, to state, and to national levels. The USA Mathematical Talent Search (USAMTS) is an open mathematics competition for U.S. middle and high school students. USAMTS consists of two rounds of six problems per round and operates on the honor system, since participants are given a full month to work on the problems. The goal of USAMTS is to help students develop their proof writing ability, improve their technical writing abilities, and mature mathematically while having fun. The organizers strive to foster insight, ingenuity, creativity, and perseverance. The American Mathematics Competitions (AMCs) provide three levels of competitions. Students who perform well on the AMC 10 or AMC 12 exams, for students in grades 10 or 12 and below, respectively, are invited to participate in the American Invitational Mathematics Examination (AIME). Approximately the top 270 performers on the AIME and the AMC 12 advance to the United States of America Mathematical Olympiad (USAMO), which is the final round of the AMC series of contests. The top 230 AIME and AMC 10 only participants take part in the USA Junior Mathematical Olympiad (USAJMO). The top 30–40 performers on the USAMO, along with a dozen or so others from the USAJMO, attend the Mathematical Olympiad Summer Program, a training program from which the six members of the U.S. International Mathematical Olympiad (IMO) team are selected. Students who do well on the AIME typically receive scholarship offers from prestigious colleges and universities.

IMO

The International Mathematical Olympiad (IMO) is an annual two-day, six-problem, mathematical competition for pre-collegiate students that began in 1959. Approximately 100 countries send teams of up to six students. The problems are extremely difficult and involve ideas that are not usually encountered in high schools or colleges. Many IMO participants have become world-class research mathematicians, such as Noam Elkies, who eventually became the youngest full professor in Harvard University's history at the time of his promotion. Filmmaker George Csicsery documented the 2006 U.S. IMO team in Slovenia. The documentary also included segments on families and schooling, girls, and the Olympiad, as well as the problems and their solutions. Melanie Wood, who was the first female to represent the United States in the IMO, noted: "Math competitions are great. They introduce all these new ideas and in particular give students who are at school the first chance to see how you can be creative in solving a problem." She went on to obtain her Ph.D. in mathematics in 2009.

The Benet Academy Math Team poses with their trophy after a state math competition awards ceremony at the University of Illinois at Urbana-Champaign in 2007. Mathematics competitors are sometimes called "mathletes."

William Lowell Putnam Mathematical Competition

College students also participate in mathematical contests. The William Lowell Putnam Mathematical Competition is an annual mathematics competition for mathematically talented undergraduate college and university students in the United States and Canada administered by the Mathematical Association of America. The competition, in which both individuals and teams compete, consists of morning and afternoon three-hour exams, each with six problems. Although the problems are extraordinarily difficult and require highly creative thinking, they can typically be solved with only knowledge of college-level mathematics. The problems are so challenging that a median score for the 120-point exam is often 0 or 1. Of the more than 120,000 times the exams have been taken since the competition's inception in 1938, there have been only three perfect scores as of 2010.

In recent years, about 4000 students and 400 teams have participated. The top five teams and individual scorers receive thousands of dollars in prize money. Many top five scorers, named as Putnam Fellows, have become distinguished researchers in mathematics and other fields, including Fields Medalists (the highest award in mathematics for people younger than 40) John Milnor, David Mumford, and Daniel Quillen, and Nobel laureates Richard Feynman and Kenneth G. Wilson. Several Putnam Fellows have been elected to the National Academy of Science. In 2010, Putman Fellow David Mumford received the National Medal of Science, bestowed by President Barack Obama.

MCM

Unlike other competitions, which place a premium on speed and individual performance, the Mathematical Contest in Modeling (MCM) contest rewards teamwork, research skills, programming skills, organizing ability, writing ability, and stamina. The MCM is a 96-hour mathematics competition held annually since 1985 by the Consortium for Mathematics and Its Applications and sponsored by the Society of Industrial and Applied Mathematicians, the National Security Agency, and the Institute for Operations Research and the Management Sciences. Approximately 1000 international teams of three undergraduates each produce original mathematical papers in response to one of two open-ended modeling problems. The students may use any references and the Internet but are not permitted to discuss their problem with anyone not on their team. Approximately 1% to 2% of the teams are designated Outstanding Winners. The skills required in the modeling contest are those typically most valued by employers. Many students who do not excel in problem-solving contests excel in the modeling competition.

Value and Benefits

The value of mathematics competitions is that they pique interest in mathematics and encourage students to pursue intellectual activities. The benefits of participating in mathematics competitions are very much like the benefits derived from athletic contests or becoming accomplished in playing a musical instrument. The intention is that those engaged in such activities develop a sense of accomplishment and a positive self-image. On the other hand, some object to mathematics being presented as a competition. While some students may thrive in a competitive environment, others may be discouraged. For some, the competitive environment highlights mutual interests, which can help create lasting bonds and friendships.

Like sports, participants in mathematics contests may learn to set goals and work toward them, be highly motivated, be able to focus, have self-discipline, perform under pressure, cope with success and failure, and have a competitive spirit. As in music, participants in mathematics contests must learn self-discipline, develop the ability to concentrate, pay attention to detail, and practice many hours. Perhaps the important lesson learned from participating in mathematics contests is that success is the fruit of effort.

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See Also: Careers; Clubs and Honor Societies; Professional Associations; Succeeding in Mathematics.

Composing

Category: Arts, Music, and Entertainment.
Fields of Study: Algebra; Number and Operations; Representations.
Summary: Mathematics and music developed in tandem and composition is firmly grounded in mathematics.

Throughout the history of Western music, composers have utilized mathematical techniques in creating musical works. From Pythagoras, Plato, and Ptolemy in ancient Greece to the sixth-century music theorist Boethius, music was thought to be a corollary of arithmetic. With the widespread development of modern standardized musical notation thought to have begun in the Renaissance, compositional craft became more highly developed. Compositions intertwined with mathematical patterns were particularly highly regarded.

The eighteenth-century composer and theorist Jean-Philippe Rameau was unequivocal in his views on the connection between mathematics and music in his 1722 *Treatise on Harmony*, writing, "Music is a science which should have definite rules; these rules should be drawn from an evident principle; and this principle cannot really be known to us without the aid of mathematics." Fugal composition techniques in the high Baroque period were highly mathematical. The classical and romantic eras, characterized by a movement away from polyphonic music, produced less obvious mathematically oriented composition technique. In the twentieth century, however, mathematical formalisms were fundamental as replacements for the tonal structures of the romantic era. There are even subgenres of rock music (started in the 1980s) called "math rock" and "math-core" (after metalcore, a fusion of heavy metal and hard-core punk), which uses complex and atypical rhythmic structures, angular melodies, unusual time signatures, and changing meters. Metalcore, in particular, also uses harmonic dissonance. In another example, Robert Schneider composed a mathematical score for a play in 2009. He said:

> I wrote a composition called 'Reverie in Prime Time Signatures,' that is obviously written in prime time signatures, that is, only prime numbers of beats per measure. Also the piece has kind of a sophisticated middle section that encodes some ancient Greek mathematics related to prime numbers in musical form, that I am proud of.

The Renaissance Canon

During the Renaissance, mathematical devices were developed to a considerable degree by Northern European composers. In the canons of Johannes Ockeghem, a single melodic voice provides the basis by which one or more additional voices are composed according to various mathematical transformations of the original: mirror reflection of musical intervals (inversion), time translation, mirror reflection in time (retrograde), or a non-unit time scaling (mensuration canon). Composers of this period understood the word "canon" to mean a rule by which secondary voices could be derived from a given melody, in contrast to our modern usage of the word, which means a simple duplication with later onset time, as in the nursery rhyme round "Row, Row, Row Your Boat."

Mathematical Transformations in Composition

In addition to standard musical notation, music can be represented mathematically as a sequence of points in an algebraic structure. A musical composition can be represented as a sequence of points from the module M over the cyclic groups of integers Z_p

$$M = Z_{p1} \times Z_{p2} \times Z_{p3} \times Z_{p4},$$

with the coordinates representing (respectively) onset time, pitch, duration, and loudness. For example, the 12

Bach: The Canon Master

Johann Sebastian Bach was a master of canonic composition. Bach's canons challenged performers to solve puzzles he set before them. Examples abound in *A Musical Offering* (BWV 1079), written in 1747. The first of two *Canon a 2* (canon for two voices) from *Musical Offering* appears to have two different clef symbols: one at the beginning of the first measure, and one at the end of the last. The first singer had to read from beginning to end, and the second had to start at the same time and read in the opposite direction. In this small piece, Bach provides an example of retrograde or cancrizan (crab) canon. The puzzle in the second *Canon a 2* is even more cleverly concealed: a single line with two clef signs in the first measure, one upside down. The cryptic instruction *Quaerendo invenietis* ("Seek and ye shall find") is inscribed at the top of the manuscript.

The second, inverted clef sign indicates that the second voice of the canon is to proceed in inversion, and the performer is left to "seek" the appropriate time translation at which the second voice should begin. Another example of Bach's masterful canonic treatment is BWV 1074: *Kanon zu vier Stimmen*, which with its numerous key signatures, clefs, and repeat signs can be played from any viewing angle.

notes of the chromatic scale would be represented in the second coordinate by Z_{12}. In this schematic, if a point (x_1, x_2, x_3, x_4) in a musical motif were repeated later at a different volume level, the repetition would differ in the first and last coordinate and would be represented as $(x+\alpha, x_1, x_2, x_3, x_4+\beta)$, where α is the time shift and β is the amount of the volume difference.

Inversion takes the form $(x_1, 2\alpha - x_2, x_3, x_4)$. Mensuration, as in the canons of Ockegham, is written $(x_1, x_2, \alpha \cdot x_3, x_4)$. Transformations of this form were used extensively in the Renaissance and Baroque eras and played a fundamental role in post-tonal era of the twentieth century.

Mathematical Structure in Atonal Music

At the turn of the twentieth century, music theorists and composers looked for new organizing principles on which atonal music could be structured. Groundbreaking composer Arnold Schoenberg turned to the idea of "serialism," in which a given permutation of the 12 chromatic pitches constitutes the basis for a composition. The new organizing principle called for the 12 pitches of this "tone row" to be used—singly, or as chords, at the discretion of the composer, always in the order specified by the row. When the notes of the row have been used, the process repeats from the beginning of the row.

Composers like Anton Webern, Pierre Boulez, and Karlheinz Stockhausen consciously used geometric transformations of onset time, pitch, duration, and loudness as mechanisms for applying the tone row in compositions. In the latter half of the twentieth century, set theoretic methods on "pitch class sets" dominated the theoretical discussion.

Predicated on the notions of octave equivalence and the equally tempered scale, Howard Hanson and Allen Forte developed mathematical analysis tools that brought a sense of theoretical cohesion to seemingly intractable modern compositions. Another mathematical approach to composition without tonality is known as aleatoric music, or chance music. This technique encompasses a wide range of spontaneous influences in both composition and performance. One notable exploration of aleatoric music can be seen in the stochastic compositions of Iannis Xenakis from the 1950s. Xenakis's stochastic composition technique, in which musical scores are produced by following various probability models, was realized in the orchestral works *Metastasis* and *Pithoprakta*, which were subsequently performed as ballet music in a work by George Ballanchine.

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Eric Barth

See Also: Geometry of Music; Harmonics; Scales; Time Signatures.

Computer-Generated Imagery (CGI)

See *Animation and CGI*

Congressional Representation

Category: Government, Politics, and History.
Fields of Study: Algebra; Data Analysis and Probability; Number and Operations.
Summary: Though the Constitution dictates proportional representation by state, there are multiple methods for attempting to achieve fair apportionment.

Apportionment is the process of distributing a fixed resource on a proportional basis, particularly associated with government. The legislative branch of the

U.S. federal government—and most U.S. states—is bicameral, meaning that two separate bodies deliberate on laws. Reflecting a great political compromise of American government, these bodies are formulated on two distinct representative principles. The U.S. Senate has equal representation from each state to ensure that states have equal voices. For the House of Representatives, the U.S. Constitution requires that "Representatives…shall be apportioned among the several States which may be included within this Union, according to their respective Numbers."

This requirement ensures that larger states have a voice that fairly represents their greater constituencies. The primary mathematical challenge in most systems of representation is that typically not all representatives will represent the same number of citizens, and calculations rarely result in integers. Deciding a fair system of rounding for representative numbers for fractional constituencies has proven surprisingly challenging, and Congressional apportionment has generated substantial controversy throughout the history of the United States.

Numerous serious apportionment methods have been proposed. Most have names associated with the people who proposed them, such as third U.S. president Thomas Jefferson, and are generally classified as "divisor methods" or "quota methods." Many systems have been used in the United States, and mathematicians have long investigated fair apportionment. In 1948, at the request of the National Academy of Sciences, mathematicians Luther Eisenhart, Marston Morse, and John von Neumann recommended the Huntington–Hill method, proposed by mathematician Edward Huntington and statistician Joseph Hill. Apportionment is a prominent aspect of social choice theory, extensively studied by mathematicians such as Peyton Young and Michel Balinski. There have also been innovative links between apportionment and other areas of mathematics, like just-in-time sequencing and scheduling problems for manufacturing.

Apportionment Methods

A state's proportion of the total population of a country can be found by dividing the state's population by the total population. The state's fair share of the total seats in the nation's legislature, called its "standard quota," is the product of this proportion and the total seats. Alternatively, the standard quota can be found by using the standard divisor, which measures the average number of people per seat on a national basis, and is found by dividing the total population by the number of available seats.

For example, suppose that a small country consists of four states (A, B, C, D), with populations given as

State	Population
A	791
B	892
C	6987
D	530

The total population of this country is 9200, and State A has 791/9200 or approximately 8.6% of the population. If there are 25 seats in the country's legislature, then State A's standard quota is

$$\frac{791}{9200}\left(25\right)\approx 2.149\,.$$

seats. State A's population therefore warrants slightly more than two seats but less than three. The standard divisor in this case is 9200/25 = 368 people per seat, and State A's standard quota can also be represented as 791/368. Similarly, the standard quotas for States B, C, and D are calculated as 2.424, 18.986, and 1.440, respectively.

The requirement that each state be assigned an integer number of representatives forces a country to impose a systematic method for rounding standard quotas to whole numbers. It is reasonable to expect that any reasonable method will assign each state either its lower quota or its upper quota—its standard quota rounded down or up, respectively. This requirement is known as the Quota Rule. One method that arises naturally is to round up those standard quotas that are closest to the next number of seats. Specifically, one may choose to initially apportion each state its lower quota, which will always yield leftover seats. These surplus seats are distributed to the states whose standard quotas have the largest fractional part. This method is known as Hamilton's Method, Vinton's Method, or the Method of Largest Remainders, named for the first Secretary of the Treasury Alexander Hamilton and Congressman Samuel Vinton. In the above example, after assigning each state its lower quota, only 23 of the 25 seats have been apportioned. The first surplus seat is

Table 1: Distribution of Congressional Seats

State	Population	Standard Quota	Lower Quota	Apportionment
A	791	2.149	2	2
B	892	2.424	2	2
C	6987	18.986	18	19
D	530	1.440	1	2

assigned to State C, whose standard quota is very close to 19, while the other surplus seat is assigned to State D (See Table 1).

Some apportionment methods solve the problem of apportionment by using a specific rounding rule and modifying the standard divisor if necessary. In Jefferson's Method, for example, all quotas are rounded down to the integer part of the quota. As with Hamilton's Method, this yields unassigned seats in an initial apportionment. Rather than distributing those surplus seats as Hamilton's Method does, Jefferson's Method instead modifies the divisor by making it smaller. This method makes it easier for the states to obtain a seat and allows the states' quotas to grow larger. In a successful Jefferson apportionment, a modified divisor is found so that when the modified quotas are rounded down, the total number of seats apportioned is the desired number.

Adams' Method, named for sixth U.S. president John Quincy Adams, is similar to Jefferson's Method. Rather than rounding down, however, quotas are instead rounded up to the next largest integer. In this case, the initial attempt at apportionment results in too many seats being distributed and a modified divisor must be chosen that is larger than the standard divisor, reflecting a need to make it more difficult to obtain a seat.

Other divisor methods differ from the methods of Jefferson and Adams only in how the rounding is conducted. In Webster's Method, named for Senator and Secretary of State Daniel Webster, for example, all quotas are rounded conventionally—to the nearest whole number. If a state's quota has a fractional part that is 0.5 or greater, the quota is rounded up. Otherwise, it is rounded down. In other words, one can think of the tipping point for rounding in Webster's method as being the arithmetic mean of a state's lower and upper quota. In the Method of Huntington–Hill, a state's cutoff for rounding is the geometric mean of the state's lower and upper quotas. For this method, if a state's lower quota is L and its upper quota is U, then the cutoff for rounding is

$$\sqrt{LU}\,.$$

If the quota is less than the cutoff, then it is rounded down, otherwise it is rounded up. In Dean's Method, named for mathematician and astronomer James Dean, the cutoff for rounding is the harmonic mean of the lower and upper quota, expressed algebraically as

$$\frac{2}{\frac{1}{L}+\frac{1}{U}}\,.$$

Applying any of these methods requires searching for a modified divisor so that when the modified quotas are calculated and rounded according to the given rule, the number of seats distributed is the correct total.

Applying these divisor methods to the sample situation given above results in the apportionments seen in Table 2. A number of important aspects of apportionment can be seen in the table. First, the apportionment method makes a difference; different methods can yield different apportionments. Jefferson's Method has a substantial bias toward larger states. Adams's Method, on the other hand, is biased toward smaller states and can cause lower quota violations. Quota rule violations can occur with Webster's Method as well, though they are relatively rare. Webster's Method demonstrates little bias overall. The HuntingtonHill Method and, to a greater degree, Dean's Method, have biases toward smaller states.

History of U.S. Apportionment

The U.S. Constitution mandates a decennial census. Congressional representatives are reapportioned every 10 years based on the results. Though the Constitution provided for an initial distribution of U.S. Congressional representatives, it specified no particular apportioning method. Following Constitutional ratification in 1787 and the census of 1790, the first apportionment was carried out. In 1792, Congress passed a bill instituting Hamilton's Method, which George Washington vetoed. Congress then approved Jefferson's Method, which was used through 1832 when a Quota Rule violation was observed. New York State had a standard quota of 38.59 seats, so New York should have received either 38 or 39 seats. However, Jefferson's Method assigned New York 40 seats. John Quincy Adams and Daniel Webster immediately put forth separate bills calling for the adoption of the apportionment methods that carry their names. Though both bills failed, this was the last apportionment in which Jefferson's Method was used.

Webster's Method was used for the apportionment of 1842, but in 1852, Hamilton's Method was adopted as "permanent" by Congress. In 1872, Hamilton's Method was not applied correctly. In 1882, additional difficulties arose with the method itself. While considering different sizes for the House of Representatives, observers noted that with a House size of 299, Alabama would receive eight seats under Hamilton's Method, but with a House size of 300, Alabama would receive only seven seats. This flaw, whereby increasing the number of seats to apportion can, in and of itself, cause a state to lose a seat, became known as the "Alabama paradox." Congress sidestepped this issue in 1882 by increasing the size of the House to 325 seats, but the flaw led to their discarding Hamilton's Method in 1901. Webster's Method was used in the apportionments of 1901, 1911, and 1931, though no apportionment was completed after the 1920 census. In 1941, Congress adopted the Huntington–Hill Method as "permanent," with the House size of 435 seats, which is the method still in use at the start of the twenty-first century, though controversy continues.

Impossibility

Many mathematicians and others have asked whether there is an "ideal" apportionment method that solves the apportionment problem in a reasonable way and is free of flaws such as the Alabama paradox and quota rule violations. In the 1970s, Balinski and Young proved that no such method exists. Every apportionment method will either potentially violate the quota rule or cause either the Alabama paradox or another problematic paradox called the "Population paradox," whereby one state whose population is growing at a faster rate can lose a seat to a state with a slower growth rate. The search for perfection in apportionment is an inherently impossible task, but mathematicians continue to study the problems and paradoxes and seek new approaches to reduce bias.

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Table 2. Dean's Method.

State	Standard Quota	Jefferson	Adams	Webster	Huntington–Hill	Dean
A	2.149	2	2	2	2	2
B	2.424	2	3	2	2	3
C	18.986	20	18	20	19	18
D	1.440	1	2	1	2	2
Valid Range of Divisors		333–349	396–411	357–358	365–374	396–411

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Stephen Szydlik
Matt Kretchmar

See Also: Census; Gerrymandering; Government and State Legislation; Voting Methods.

Conic Sections

Category: History and Development of Curricular Concepts.
Fields of Study: Algebra; Communication; Connections; Geometry.
Summary: Conic sections have many interesting mathematical properties and real-world applications.

Conic sections, or simply "conics," are the simplest plane curves other than straight lines. Students in the twenty-first century begin to study these curves in middle school. In coordinate geometry, they can be expressed as polynomials of degree 2 in two variables while straight lines are polynomials of degree 1 in two variables. Conic sections can further be divided into three types: ellipse, parabola, and hyperbola. Conics were named and systematically studied by Apollonius of Perga (262–190 B.C.E.). At that time, the study of conics was not merely to explore the intrinsic beauty of the curves but to develop useful tools necessary for applications to the solution of geometric problems. Today, the theory of conics has numerous applications in our daily lives including the designs of many machines, optical tools, telecommunication devices, and even the tracks of roller coasters.

Representations of Conics and Their Applications

One can generate a two-sheet circular cone by fixing a straight line as the axis of the cone in the space first. Choose a fixed point on it as the vertex of the cone. Rotating another straight line through the vertex that makes a fixed angle with the axis, we obtain the desired cone as the trace of the rotating line. Any straight line on the trace is called a "generating line" of the cone. Conic sections are obtained by intersecting the two-sheet cone with planes not passing through its vertex as shown in Figures 1A–C.

The three types of conic sections are generated according to the positions of the intersection.

Ellipse

When the intersecting plane cuts only one sheet of the cone and the intersection is a closed curve, an ellipse is created. A circle is obtained when the intersecting plane is perpendicular to the axis of the cone; an ellipse is obtained when the intersecting plane is not perpendicular to the axis of the cone. A circle, as such, can be considered as a particular case of an ellipse (Figure 1A). As illustrated in Figure 2, an ellipse is the collection of points in a plane that the sum of distances from two fixed points F_1 and F_2, the foci, to every point in the collection is constant.

On the coordinate plane, if the foci are located on the x-axis at the points $(-c, 0)$ and $(c, 0)$ and the constant distance between F_1 and F_2 is $2a$, the equation of an ellipse can be derived as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ with } b^2 = a^2 - c^2 < a^2.$$

Figure 1A. Figure 1B. Figure 1C.

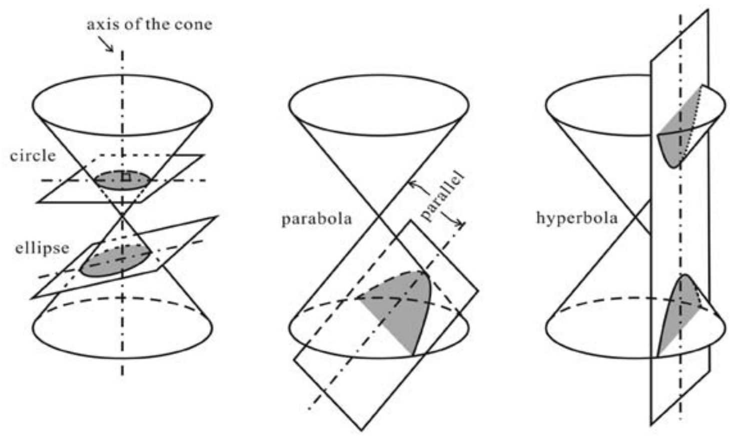


Figure 2.

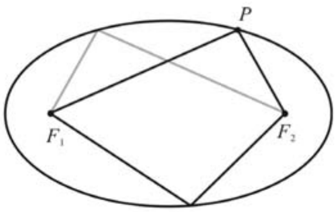


Figure 3A.

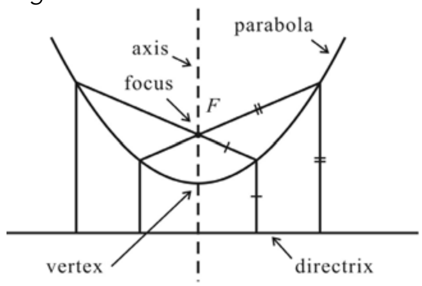
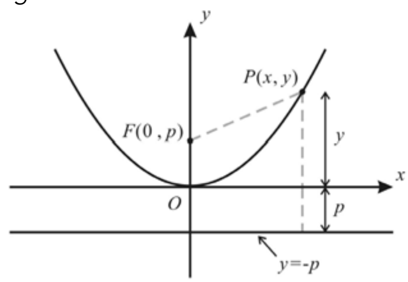


Figure 3B.



Parabola

When the intersecting plane cuts only one sheet of the cone and is parallel to exactly one generating line of the cone, the intersection is a non-closed curve—a parabola (Figure 1B). A parabola is the collection of points in a plane that are equidistant from a fixed point F (called "focus") and a fixed line (called "directrix"). The graph of the parabola is illustrated in Figure 3A. The graph is symmetric with respect to the line through the focus and perpendicular to the directrix. This line of symmetry is called the "axis" of the parabola. The intersection of the graph with the axis is called the "vertex" of the parabola. On the coordinate plane, if the vertex is located at the origin *O*, and the focus at the point $(0, p)$, then its directrix will be on the line $y = -p$ (Figure 3B), and the equation of the parabola can be derived as $x^2 = 4py$.

Hyperbola

When the intersecting plane meets both sheets of the cone, the intersection is a hyperbola, which consists of two identical non-closed parts, each located in one of the two sheets of the cone (Figure 1C). A hyperbola is the collection of all points in a plane that the difference of distances from two fixed points F_1 and F_2, the foci, to every point in the collection is constant. The graph of a hyperbola is drawn as shown in Figure 4A.

On the coordinate plane, if the foci $(-c, 0)$ $(c, 0)$ are located on the *x*-axis and the differences of distance is $\pm 2a$, then the equation of the hyperbola can be derived as

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ with } c^2 = a^2 + b^2 \text{ (See Figure 4B)}$$

A Brief History of Conic Sections

Between 460 B.C.E. and 420 B.C.E., three famous geometry problems were posed by the ancient Greeks. These problems were (1) the trisection of an angle, (2) the squaring of the circle, and (3) the duplication of the cube. The last problem merely asks that given any cube of side length *a*, can one construct another cube with exactly twice the volume, $2a^3$. Hippocrates of Chios (circa 470–410 B.C.E.) had the idea of reducing that problem by finding two quantities *x* and *y* such that

$$\frac{a}{x} = \frac{x}{y} = \frac{y}{2a}.$$

Then, $x^2 = ay$, $y^2 = ax$, and $xy = 2a^2$.

As such, *x* is the required solution for the problem. This solution is equivalent to solving simultaneously any two of the three equations ($x^2 = ay$, $y^2 = 2ax$, and $xy = 2a^2$) that represent parabolas in the first two and a hyperbola in the third. However, no explicit construction of the conic sections was given. Menaechmus (380–320 B.C.E.) is believed to be the first mathematician to work with conic sections systematically, which is theorized to have arisen because of curves traced out by sundials. At his time, the conic sections were formed by cutting a right circular cone with a plane perpendicular to a side.

The sections were named according to whether the vertex angle was acute, right, or obtuse (Figure 5). Menaechmus constructed conic sections that satisfied the required algebraic properties suggested by Hippocrates and thus obtained the points of intersection of these conic sections that would lead to the solution of the problem of the duplication of the cube.

Figure 4A.

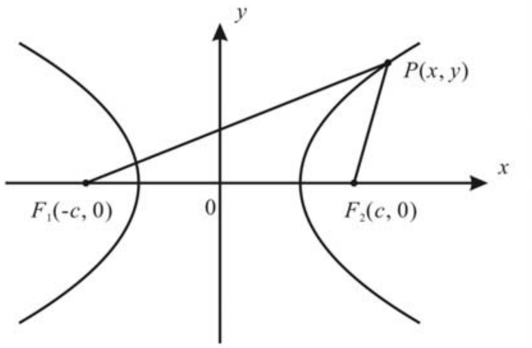
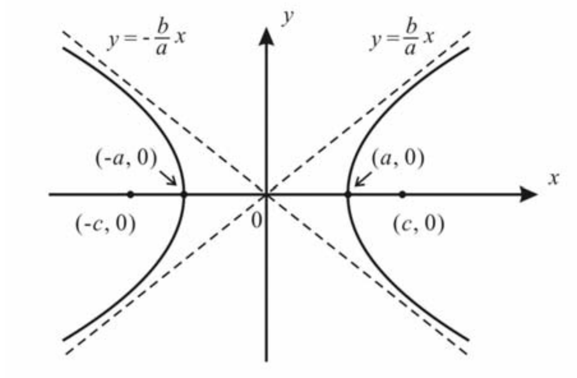


Figure 4B.



The breakthrough in the study of conics by the ancient Greeks was attributed to Apollonius of Perga. His eight-volume masterpiece *Conic Sections* greatly extended the existing knowledge at the time (one of the eight books has been lost to history). Apollonius' major contribution was to treat the conic sections as plane curves and use their intrinsic properties to characterize them. This method allowed conic sections to be analyzed in great detail by the ancient Greeks.

Abu Ali al-Hasan ibn al-Haytham studied optics using conic sections in the tenth and eleventh centuries. Omar Al-Khayyami (Omar Khayyam) authored *Treatise on Demonstration of Problems of Algebra* in the eleventh century. This work showed that all cubic equations could be classified using geometric solutions that involve conic sections. Later, in the seventeenth century, Gerard Desargues (1591–1662) and Blaise Pascal (1623–1662) connected the study of conic sections to developments from projective geometry. At the same time, René Descartes (1596–1650) and Pierre de Fermat (1601–1665) also connected it with the developments from coordinate geometry. Eventually, problems of conics in geometry could be reduced to problems in algebra.

Johan Kepler (1571–1630) revolutionized astronomy by introducing the notion of elliptical orbits. According to Isaac Newton's later law of universal gravitation, the orbits of two massive objects that interact are conic sections.

If they are bound together, they will both trace out ellipses; if they move apart, they will both follow parabolic or hyperbolic trajectories.

The Applications of Conic Sections

Besides applications in astronomy, conics have many other applications.

In an ellipse, any light or radiation that begins at one focus will be reflected to the other focus (Figure 6 on following page). This property can be used in theater designs. In an elliptical theater, the speech from one focus can be heard clearly across the theater at the other focus by the audience. It can also be applied in lithotripsy, a medical procedure for treating kidney stones. The patient is placed in an elliptical tank of water, with the kidney stone fixed at one focus. High-energy shock waves emitted at the other focus can be directed to pulverize the stone. Also, elliptical gears can be used for many machine tools.

Figure 5.

Figure 6.

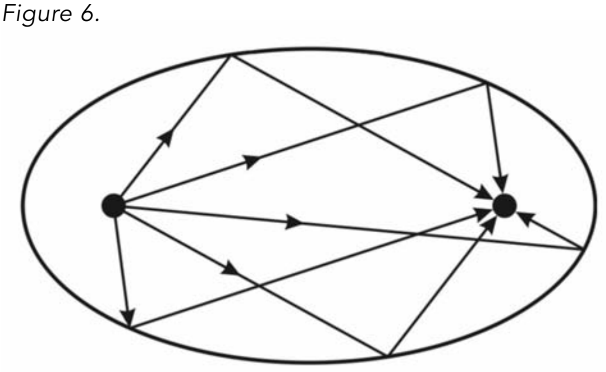
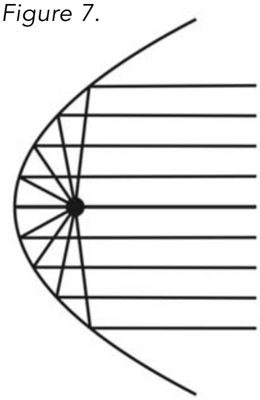


Figure 7.



In a parabola, parallel light beams will converge to its focus (see Figure 7 on following page). Parabolic mirrors are used to converge light beams or heat radiations, and parabolic microphones are used to perform a similar function with sound waves.

In reverse, if a light source is placed at the focus of a parabolic mirror, the light will be reflected in rays parallel to said axis. This property is used in the design of car headlights and in spotlights because it aids in concentrating the parallel light beam. Hyperbolas are used in a navigation system known as Long Range Navigation (LORAN). Hyperbolic—as well as parabolic—mirrors and lenses are also used in systems of telescopes.

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Ka-Luen Cheung

See Also: Curves; Geometry of the Universe; Greek Mathematics.

Connections in Society

Category: School and Society.
Fields of Study: Connections.
Summary: An integrated approach to mathematics stresses the importance of making connections among various perspectives and applications.

While mathematics in educational settings is often separated out into differing subjects, it is important to understand that mathematics is an interconnected field of study. While most individuals are aware that they must be familiar with basic addition and subtraction to ensure the proper handling of money, very few individuals give any thought to the multitude of deeper mathematical connections they experience daily. In fact, both the National Science Foundation and the National Council of Teachers of Mathematics have recently begun to strongly advocate for the use of an interconnected curriculum in K–12 mathematics education. An integrated approach to mathematics education stresses the importance of making connections among mathematical perspectives, as in algebra and geometry, making connections to other fields, as in physics or religion, and connecting mathematical concepts to society as a whole, as in applications and usefulness in daily living.

The purpose of an interconnected curriculum is to help students better understand how the various branches of mathematics are connected and how mathematics is connected to the real world. By teaching mathematics as a unified whole, rather than multiple discrete subjects, students may better understand that mathematics is not a set of indiscriminate rules and isolated skills; rather, it involves a rich interplay between mathematical concepts, as well as complex interactions with other academic subjects. It is this integrated approach to mathematics that seeks to answer that question, "When are we ever going to use this in *real life?*" When this objective is met, students often show an increased appreciation and enthusiasm for mathematical principles.

People use many different interrelated approaches to process ideas, analyze objects, make decisions, or solve problems. For example, one might calculate the optimal viewing distance of a painting in order to see the depth that the artist intended, examine the surface of the painting to appreciate the finer details and glazes, or stand back to appreciate the overall effect and balance of colors. Real-life situations are not divided the way they are in textbooks by their applicability to a certain topic or technique, like exponential models. In fact, throughout the twentieth century, employers, such as engineering firms, complained about the lack of connections made in school between different subjects. Mathematician Eliakim Moore discussed this problem a century ago in his 1902 address as the president of the American Mathematical Society. In 1989, the National Council of Teachers of Mathematics published a set of national standards for mathematics that included connections as a significant component.

Whereas traditional mathematics curricula in the twentieth century separated subject areas like algebra and geometry, an integrated approach involves presenting mathematical subjects as one interrelated whole that also connects to other subjects and real-world experiences. In antiquity, the square of a number was defined as the area of a square with the same side length. People with interdisciplinary interests were perhaps more common at that time—Greek mathematicians were also astronomers, inventors, engineers, and philosophers. Throughout history, mathematicians such as Carl Friedrich Gauss contributed to so many areas of mathematics and to other fields, like geodesy; but in the twenty-first century, researchers who specialize in a subdiscipline are more common. However, connections among multiple mathematical perspectives are still important in the development of mathemat-

Mathematics and the FIFA World Cup Finals

Mathematics is also used in one of the most popular sporting events in the world: the FIFA World Cup Finals. Thirty-two teams qualify for the World Cup Finals, and they are assigned to eight groups of four teams. The top seven teams in the world and the host country's team are seeded based on the FIFA World Rankings and recent World Cup performances and put into the eight different groups. The rest of the teams are put into different pots based on their geographical location, and then teams from each pot are randomly assigned to the eight groups. In the group stage of the World Cup Finals, each team plays every other team in its group.

A team earns three points for a win, one point for a tie, and zero points for a loss. In each group, the two teams with the most points advance to the knockout stage. If teams are equal in points, the greatest goal difference, the greatest number of goals scored, and other various statistics can be used to determine the top two teams to advance to the next stage. Sixteen teams advance to the knockout stage, which is a single-elimination tournament. At the end of the tournament, FIFA crowns one World Cup champion, as well as several individual awards, such as the Golden Ball for the best player and the Golden Boot for the top goal scorer. The winner of the Golden Ball award is based on a vote of media members. The Golden Boot award is given to the player with the greatest number of goals scored, as well as with the greatest number of assists. Mathematics is used in important calculations of sports statistics, college and world rankings, tournament rankings, and awards for individual performances.

ics. Algebra and geometry remain linked and the field of algebraic geometry is active today. Many researchers use techniques from a variety of different mathematical fields. Geometers may heavily rely on concepts from analysis, linear algebra, number theory, or statistics, for example. Other researchers work in the intersection of fields like statistical analysis.

Mathematics can easily be connected to other scientific disciplines, like physics or biology. Mathematics is sometimes referred to as the "foundation" or "language" of science. However, there are many other types of links between mathematics and the sciences. Some researchers work on problems at the interface of mathematics and a scientific field, while others translate ideas from the sciences to solve problems in mathematics and vice versa. Scientific disciplines like physics are often referred to as partner disciplines for mathematics. Researchers have met for a conference named "Connections in Geometry and Physics" that explores the interdisciplinary facets. In geometry and physics there is a concept called a "connection," which is an operator that allows for comparison at different points in a space via parallel transport. Mathematics has been interwoven with physics since antiquity. There have also been historical linkages between mathematics and biology, but the interdisciplinary field of mathematical biology has grown rapidly in the early twenty-first century.

Students may have difficulty appreciating the importance of mathematics in nonscientific disciplines, but the connections between mathematics and subjects like business, art, music, or religion are multilayered and multifaceted. For example, mathematics has played a part in religious life since the earliest documented cultures. The ancient Mesopotamians, embracing a polytheistic faith, developed the time system we use today with bases of 60 (60 seconds make up a minute, and 60 minutes comprise an hour). Adherents of Christianity, Judaism, and Islam have all embraced elements of mathematics in the conceptualization of sacred time. Given the importance of religion today, this time is still of great value for humankind. Mathematics plays a key role in the calculation of religious celebrations around which many faiths flourish. The week and solar day provide a delineation of sacred days that are different from the others—Sunday for Christians, Saturday for Jews, and Friday for Muslims. In other ways, numeric or geometric symbolism plays a significant part of religious practice.

There are countless examples that highlight the importance of mathematics in daily tasks. In the twenty-first century, it is almost impossible to find a task that does not connect to mathematics, either directly or through the tools and technologies in which mathematics plays an important role. In turn, mathematicians formulate new theories and concepts in order to meet the needs of society.

Mathematics as a Universal Language

Many people consider mathematics as the only truly universal language, regardless of gender, culture, or religion. For example, while the precise number of digits that are used in applications may differ, the ratio of the circumference to the diameter of a circle is still π, irrespective of the cultural context. Calculating the cost of groceries involves the same mathematical processes whether one is paying for those groceries in dollars, pesos, or pounds. With the universal language of mathematics, regardless of the unit of exchange, humans are likely to arrive at similar mathematical results. In fact, there are many examples of researchers in different areas of the world who independently arrived at the same theorems. Thus, mathematics as a universal language provides a common ground, creating the capacity for human beings to connect to one another across continents and across time.

Nutrition Labeling

An important way that mathematics can be found in our everyday life is on nutrition facts panels, which are mandated by the Nutrition Labeling and Education Act of 1990 to be placed on nearly all multiple-ingredient foods. The nutrition facts label on foods must list the fat, saturated fat, trans fat, cholesterol, sodium, total carbohydrate, fiber, sugar, protein, Vitamin A, Vitamin C, calcium, and iron content of the food. Other nutrients may be listed voluntarily. These labels also include a column that lists the percent Daily Value (% DV) to help consumers decide whether the nutrient content of a serving of the food product is a lot or a little. Mathematics is used to calculate the calories per serving and the % DV of a serving listed on the nutrition facts label.

As shown in Figure 1, at the top of the nutrition facts label, the serving size, as well as the number of servings per container, is listed directly underneath "Nutrition Facts." In this case, a serving size is ½ cup and there are eight servings per container. This means that there are four cups (½ cup × eight servings = four cups) of food in this package. If a person consumed half the container, or two cups of food, he or she would have had four servings (the amount of food consumed divided by a serving size, or two cups divided by ½ cup per serving = four servings).

Next, the calories per serving and the calories from fat per serving are listed. In this food, there are 200 calories per serving and 130 calories from fat in one serving. If the person consumed four servings and there are 200 calories per serving, then he or she consumed 800 calories (four servings × 200 calories/serving = 800 calories). Similarly, this person consumed 520 calories from fat (four servings × 130 calories from fat/serving = 520 calories from fat).

Figure 1. A Common Nutrition Facts Label.

Nutrition Facts

Serving Size 1/2 cup (about 82g)
Servings Per Container 8

Amount Per Serving	
Calories 200	Calories from Fat 130
	% Daily Value*
Total Fat 14g	**22%**
Saturated Fat 9g	**45%**
Trans Fat 0g	
Cholesterol 55mg	**18%**
Sodium 40mg	**2%**
Total Carbohydrate 17g	**6%**
Dietary Fiber 1g	**4%**
Sugars 14g	
Protein 3g	

Vitamin A 10%	• Vitamin C 0%
Calcium 10%	• Iron 6%

*Percent Daily Values are based on a 2,000 calorie diet. Your daily values may be higher or lower depending on your calorie needs:

	Calories:	2,000	2,500
Total Fat	Less than	65g	80g
Saturated Fat	Less than	20g	25g
Cholesterol	Less than	300mg	300 mg
Sodium	Less than	2,400mg	2,400mg
Total Carbohydrate		300g	375g
Dietary Fiber		25g	30g

Calories per gram:
Fat 9 • Carbohydrate 4 • Protein 4

Following the calorie content, the nutrition facts label also lists the number of grams of total fat, total carbohydrate, and protein, which are calorie-yielding nutrients. A gram of fat contains nine calories, which is listed at the very bottom of the label. In this food, a single serving contains 14 grams of fat, which yields 126 calories (14 grams of fat × 9 calories/gram of fat = 126 calories from fat). This calculation was done to create the number of calories from fat listed on the panel (they rounded up to 130). As previously mentioned, if a person ate four servings, he or she consumed about 520 calories from fat.

The number of calories from carbohydrates and proteins can also be calculated. Both carbohydrates and protein yield four calories per gram, which is also listed at the very bottom of the nutrition label. In this food, there are 17 grams of carbohydrates, which provides 68 calories (17 grams × 4 calories/gram = 68 calories). In four servings, a person would ingest about 272 calories from carbohydrates (68 calories/serving × 4 servings = 272 calories from carbohydrates). There are three grams of protein in one serving, which means there are 12 calories from protein in one serving (3 grams × 4 calories/gram = 12 calories) and 48 calories from protein in four servings (12 calories/serving × four servings = 48 calories).

On the right side of the nutrition facts panel, the % DV is also listed. These daily values are based on a 2000-calorie diet, which is stated on the label next to the asterisk. Near the bottom of the label, it lists the maximum number of grams or milligrams of total fat, saturated fat, cholesterol, or sodium that a person should consume per day if on a 2000-calorie diet. It also lists the number of grams of total carbohydrate and fiber a person should eat if on a 2000-calorie diet.

If there are 14 grams of fat in one serving of this food and a person on 2000-calorie diet should consume no more than 65 grams of fat per day, then one serving of this food yields 22% of a person's DV of fat (14 grams of fat/65 grams of fat = about 22%). If this person has consumed four servings, then he or she has eaten 88% of his or her DV of fat (22%/serving × four servings = 88%). The same calculations can be made for the saturated fat, cholesterol, sodium, total carbohydrate, and fiber. Similar calculations are also made for the vitamins listed on a nutrition facts panel.

As demonstrated, mathematics is used in the calculations surrounding calorie content and % DV on nutrition labels. The mathematics used can affect a person's choice of foods and, in turn, a person's health.

Sports

Mathematics is used in numerous other everyday activities, such as sports. It is common in popular sports to calculate statistics to measure performance. In baseball, a common statistic is a batting average. A batting average is a simple calculation: the number of "hits" divided by the number of "at bats." This statistic is used to estimate an individual's batting skills. In professional baseball, a batting average of .300 is considered an excellent batting average.

A similar statistic to the batting average is in volleyball, which is called a hitting percentage. However, it is slightly different because it tries to measure an individual's hitting or attacking skills and takes errors into accounts. It is calculated by taking the number of kills, subtracting the number of errors, and then dividing the difference by the number of attempts. A "kill" is when a hitter's attack results directly in a point (the ball falling into the opponent's area of the court, an opponent not being able to return the ball, or the opponent making a blocking error as a result of the attack). An "error" is when a player hits the ball and it goes into the net (does not cross to the opponent's side) or out of bounds. An "attempt" is anytime the player tries to attack the ball. For example, if a player had 10 kills, 3 errors, and 17 attempts, the player's hitting percentage would be about .412 ($(10-3)/17 = 0.412$), which would also be considered a good hitting percentage, similar to the guidelines to the batting average.

Mathematics is important in the calculation of college football Bowl Championship Series (BCS) rankings as well. A mathematical formula is used to calculate these rankings, which order the top 25 NCAA Division I-A football teams based on their performance during the prior week. At the end of the season, the top two teams play each other in the national championship bowl. Mathematical formulas are also used to calculate which teams will play in the other bowls, taking into consideration the conference the team comes from and how many fans and advertising dollars the team is likely to bring in as well.

More specifically, the main factors that go into these rankings are subjective polls, computer rankings, the difficulty of a team's schedule, and the number of losses. The subjective poll numbers come from the average of two rankings from the Associated Press (AP) and the *USA Today*/ESPN Coaches Poll Ratings. Sports writers and broadcasters vote in the AP poll and a select group of football coaches vote in the *USA Today*/ESPN Coaches poll on which football teams they think are the best, and then these two rankings are averaged. The computer rankings are based on eight different computer rankings that are calculated based on a team's statistics for that week (strength of the opponent, final score, win-loss record, and so forth). The strength of a team's schedule is based on a cumulative win-loss record of its opponents, as well as their opponent's opponents. The calculation of the number of losses is straightforward. Each loss that a team suffers corresponds to one point, which is added to its final score. Points from each category are assigned to the team, and then these values are added to create a team's final score. The team with the lowest point total is ranked "number one" in the rankings.

Speedometers

Mathematics is also used in cars. All cars have a speedometer, which is a device used to calculate an instantaneous speed of a vehicle. It is important for a driver to know the speed of the vehicle at all times to ensure the safety of passengers and pedestrians and to abide by local traffic laws. In the United States, speedometers are read in terms of miles per hour. The calculation of the speed of the vehicle requires significant mathematics.

In many vehicles, an eddy current or mechanical speedometer is used, which is the speedometer with a needle that points to the speed that the vehicle is travelling. In these cars, there is a drive cable that runs from the speedometer to the transmission, which has a gear that tracks the rotational speed of the wheels. In other words, the gear tracks the number of revolutions the wheel makes within a certain time frame. Digital speedometers calculate miles per hour slightly differently, using a vehicle speed sensor. The vehicle speed sensor is in the transmission and also tracks the rotations of the wheels. From this information, the vehicle's speed is calculated and displayed on either a digital screen or a traditional needle-and-dial display.

The calculation of a vehicle's speed is dependent on the size of the tire as well. For example, if the tire rotates x times per minute, then the vehicle's speed can be calculated in miles per hour. Knowing the diameter of the tire, the circumference of the tire can be calculated (diameter $\times$ π). Therefore, the vehicle travels the distance of the number of revolutions times the circumference of the tire, within a certain time frame. This ratio can then be converted to miles per hour by converting the units. Because all of these calculations are based on an assumed tire diameter and circumference, it is very important for drivers to ensure that the correct size tires are on their vehicle. If a car's wheels are too large or too small, the speedometer will read slower or faster than the vehicle's actual speed, which may lead to accidents, speeding tickets, or just slower driving.

Conclusion

Mathematics can be found in everyday situations that have a real and important effect on our lives. All areas of one's life are in some way connected to mathematical principles. Only a small number of examples have been presented here—the list can be expanded infinitely. In fact, one would be hard pressed, in today's technologically advanced world, to present even a handful of activities that do not involve some mathematical concepts, if even at the unconscious level. By bridging the disconnect between "school mathematics" and "real-life mathematics," individuals gain a greater appreciation for—and curiosity of—mathematical applications.

By viewing mathematics as an integrated whole and understanding its connectedness to society, individuals become active participants, rather than passive recipients, of information. When one becomes aware of mathematical connectedness, rather than viewing math as a series of isolated and disconnected concepts to be learned though rote memorization, an individual develops the understanding of mathematics as a crucial and meaningful tool that can aid in the understanding, predicting, and quantifying of the world around us.

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See Also: Mathematics, Applied; Mathematics, Utility of; Mathematics and Religion; Mathematics Research, Interdisciplinary; Painting.

Continuity

See *Limits and Continuity*

Contra and Square Dancing

Category: Arts, Music, and Entertainment.
Fields of Study: Communication; Geometry; Representations.
Summary: Square and contra dancing employ many mathematical principles, including symmetries and permutations.

Square dance is geometry and combinatorial mathematics in motion. A caller directs the dancers through a set of choreographed dance movements unique to each type of square dancing. The dancers are sorted and shuffled in a myriad of ways by the caller and then returned to their original positions. Not only do the participants create mathematical forms as they move, mathematics is used to analyze different aspects of square dancing and its related form, contra dancing. For example, graph theory, matrix theory, and group theory can be used to represent the various structures and symmetries. Mathematics has also been used to analyze optimal calling patterns depending on the specific combinations of movements in the dance. Square dancing is a popular pastime for many people with an interest in mathematics. Several colleges have square dancing clubs, such as the Square Roots at North Central College in Illinois. That college has also offered a course called "The Mathematics of Square Dancing," which combined advanced dance patterns with discussions of mathematics theory, including parallelogram or hexagon dancing.

The Basic Square

The basic square consists of four couples. A square is symmetric under rotations of 90, 180, 270, and 360 degrees. Some or all of the dancers in the basic square can rotate in a circular movement according to these symmetries. Including the mirror reflections about each of the two lines of symmetry passing through the

Decomposing Squares Into Columns and Lines

Besides arranging dancers in squares and circles, the caller can also arrange them into columns and lines. A column arrangement occurs when all the couples are aligned one behind the other. A caller can shuffle this arrangement into any of 24 possibilities. A column of dancers can then be bisected longitudinally into two lines or crosswise into two smaller squares. There are two kinds of lines: one in which all dancers face the same direction, and another (a wave) in which they alternate the direction they are facing.

center of the square and parallel to an edge, there are six different targets of movement for the dancers. Further, in respect to each male (m)-female (f) pair, there are 10 possible movements. Thus, f_1 could be directed to replace either f_2 or m_2, m_1 could replace either f_2 or m_2, or both f_1 and m_1 could replace f_2 and m_2. Since there are four pairs, there are 240 possible movements among the dancers ($6 \times 10 \times 4 = 240$). Dance is about movement and not positions; thus, dance movements are not transitive. A movement of f_1 to f_2 is not the same as a movement of f_2 to f_1, although the outcome is the same arrangement. The two cases differ in respect to who initiates the action and who must react to the other's actions.

Secondary Squares

Besides the basic square, several other squares are part of square dancing. First, each m-f pair is a square. Several calls direct the movements of these dancers relative to one another. Thus, in a Do-Si-Do, the two members dance a square around one another and return to their initial positions. Alternatively, the basic square can be divided into a square within which a circle is inscribed. Four of the dancers constitute the square, while the remaining dancers move inward so that they are contained by the larger square. These can then be instructed to move according to the four symmetries. This arrangement can be inverted. The pairs can move toward a center point and form the radii of a circle, while the square that contains the circle is implicit. Again, the four symmetries constrain these movements. Instead of being expanded, the square can be constricted. The larger square can be divided into two smaller squares, each with four dancers. The dancers can be instructed to form smaller squares with the pair on the right, the pair opposite, or the pair on the left.

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See Also: Ballet; Permutations and Combinations; Polygons; Similarity; Symmetry.

Cooking

Category: Arts, Music, and Entertainment.
Fields of Study: Number and Operations; Measurement.
Summary: A good cook must be able to compute conversions, costs, and measurements.

In his Renaissance play, *The Staple of News,* Ben Jonson likens a master cook to—among other things—a mathematician. Although many people would think this comparison is an exaggeration, the mathematical requirements placed on the modern cook are significant.

In the past, cooking skills were passed on orally and through apprenticeship from generation to generation; today, inexperienced cooks are expected to learn to cook from recipes, which consist of a list of measured ingredients followed by instructions that refer to temperatures, times, and possibly more esoteric measurements. In addition to being able to scale recipes, the cook in our global world encounters many interesting recipes from diverse cultural traditions, which use a variety of systems of measurement.

Cooks must also be able to plan healthy and cost-effective menus.

Measurement of Ingredients

In recipes written in the United States, quantities for both liquid and dry ingredients are often specified by volume, and are measured in terms of teaspoons, tablespoons, or cups, in which there are 3 teaspoons to 1 tablespoon, 16 tablespoons to 1 cup, and 2 cups to 1 pint. Special measuring cups are made that permit the leveling of dry ingredients to ensure precise measurement. For measuring liquid ingredients, different cups are used that have graduation marks down the side and a convenient pouring spout. Measuring spoons are used for smaller quantities of both liquid and dry ingredients. For an experienced cook, the quantities given in recipes serve as general indications; however, in baking, when certain chemical reactions are expected to be balanced, precision is needed.

For more consistent outcomes, quantities are specified by weight. Ingredient densities vary. For example, a cup of water weighs 8 ounces, whereas a cup of flour—depending on how it was scooped—weighs about 5 ounces. Tables to assist in conversion between weight and volume can be found on the Internet. There can be confusion with the word "ounces," which can refer to either weight or volume. Ounces used for dry ingredients refer to one-sixteenth of a pound. Ounces measuring liquid ingredients refer to either one-sixteenth of a pint or to one-twentieth of a pint, depending on what is being measured.

Modern recipes written outside the United States provide measurements in the metric system. Liquid ingredients are specified in liters (volume) while dry ingredients are specified in grams (mass). Since kitchen scales actually measure weight, most cooks view grams as measuring weight. One liter of water weighs approximately 1000 grams. A liter is 1000 cubic centimeters, or about 1.057 quarts. A kilogram, 1000 grams, is approximately 2.205 pounds. A deciliter is one-tenth of a liter and is often used for recipes designed for home use. The metric system—based on multiples of 10—is designed to simplify calculations and scaling of measurements and is becoming the preferred system for cooks.

Scaling a Recipe

Recipes often specify the number of portions that they produce. To alter the number of portions generated, the recipe is scaled. This involves multiplying the quantity of each ingredient by a scale factor. To double a recipe, the scale factor is 2, while to halve a recipe, the scale factor is 1/2. At times, a more complex scaling is required. For example, imagine a baker is following a recipe that calls for 125 grams of pre-fermented dough. The recipe to make pre-fermented dough calls for 1000 grams of flour, 10 grams of yeast, and 0.6 liters of water and results in 1610 grams of dough. Since only 125 grams of pre-fermented dough are needed, the required scale factor is 125/1610 = 0.078.

A naïve scaling results in 78 grams of flour, 47 grams of water (.047 liters), and the absurdly small amount (0.78 grams) of yeast. An experienced cook would add more yeast. Most recipes written for home use can only be scaled up or down by less than a factor of 4. Additionally, some ingredients, like spices, gelatin, and leavening agents, should not be scaled proportionately. Most good general cookbooks will give advice on scaling recipes. A good collection of professional recipes for large numbers of portions is available from the Armed Forces Recipe Service.

Measuring Temperature

Controlling temperatures on most modern stovetops is easier than doing so on wood-burning stoves. However, techniques vary significantly among gas, electric, and induction cookers and are best described by the manufacturer. In some instances, such as deep fat frying or candy making, temperature on the stovetop is measured by a thermometer. In making candy sugar syrup, temperature can also be measured by "feel" or by the way a drop of the syrup interacts with cold water. Books on making candy describe the relationships among these methods. The temperature of an oven is accurately monitored by a thermostat, which can be set. Often, an oven thermometer is also used to check the oven thermostat. Most recipes give the required temperature in either Fahrenheit or Celsius (previously called centigrade). The formula for converting from Fahrenheit to Celsius is given by

$$C = \frac{5}{9}(F - 32)$$

and from Celsius to Fahrenheit by

$$F = \frac{9}{5}(C + 32).$$

Thus, an oven temperature of 350 degrees Fahrenheit is about 177 degrees Celsius. Temperatures in some older British recipes are given in gas mark settings, in some older French recipes in numbered settings, in some older German recipes as Stufe settings, and in some much older recipes as verbal descriptions such as Very Slow or *Doux*. Tables showing conversions among these various approaches to measuring temperature can be found in general cookbooks and on the Internet.

Other Important Measurements

Other important quantities that need to be measured when cooking include time, acidity, and density. Time measured in seconds, minutes, and hours—a system based on 60—is now probably universal. Because estimating the passage of time is fraught with error, early recipes specified important times "as measured by the clock." Acidity is measured on the pH scale. Water, which is neutral, has a pH of 7. An acidic solution, like orange juice, might have a pH of 3, while a basic solution of baking soda in water might have a pH of 9. In home cheese making, the conversion of lactose to lactic acid is tracked by monitoring pH levels of the milk; however, traditional cheese makers will use the Dornic scale.

Measuring the density of a solution is important in wine and beer making, and in candying fruits. For example, the density of fresh grape juice indicates the ripeness of the grapes and the alcohol content of the finished wine. Candying fruit in sugar water can take many days. The daily gradual increase of sugar in the syrup where the fruit is steeping maximizes the amount of sugar absorbed by the fruit. The density of the syrup is carefully checked to ensure the correct increase of sweetness. Density of syrups is measured with a hydrometer, and a variety of scales, including Brix, Baumé, and specific gravity, have been used in

Molecular gastronomy is a new trend in cooking with a scientific slant. A chef plates a dish called strawberry ravioli created using reverse spherification, and places so-called caviar spheres of sauce with chop-sticks.

recipes. Although older French recipes will refer to the Baumé scale, since the 1960s, most recipes have used specific gravity. For syrups that are denser than water, a simple approximate conversion from Baumé to specific gravity (sg) is given by:

$$sg = \frac{145}{145 - ^{\circ}B}.$$

Menu Planning and Budgeting

Cost and nutrition are also important factors for cooks. Many modern recipes, in addition to giving calories per serving, will give grams of carbohydrates, protein, fat, cholesterol, sodium, and calcium. This information, along with labels on prepared food, helps guide the cook in making nutritional choices. A cook might also be interested in knowing the cost of a portion size. For example, consider a portion of boneless chicken breast. The cost as purchased is what the chicken breast with bone costs per pound. Once the breast has been boned, what remains weighs less and results in a higher cost per pound of the edible portion. During cooking, the breast will shrink, resulting in an even higher cost per pound of the breast as served. Being aware of these costs, along with labor costs and inventory costs, helps the cook determine the cost of each item served. Although the home cook probably does not go through all these computations, a good home cook will have an idea of monthly food expenditures and how these costs are distributed among the various kinds of food served.

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See Also: Connections in Society; Measurement, Systems of; Number and Operations in Society.

Coordinate Geometry

Category: History and Development of Curricular Concepts.
Fields of Study: Algebra; Communication; Connections; Geometry.

Summary: The development of coordinate geometry revolutionized mathematics, has a wide variety of applications, and is now widely used in many areas of mathematics.

The discovery that plane geometric configurations could be entirely described by real number pairs and two-variable equations revolutionized geometry and many other important fields of mathematics that emerged later, including real analysis, vectors, calculus, linear algebra, and matrix theory. Also referred to as "analytic geometry" or "Cartesian geometry," named for the great philosopher and mathematician René Descartes, the subject of coordinate geometry is the study of geometry using the Cartesian coordinate system with algebraic operations. In twenty-first century classrooms, children in primary school begin to examine coordinate systems and create plots on graph paper.

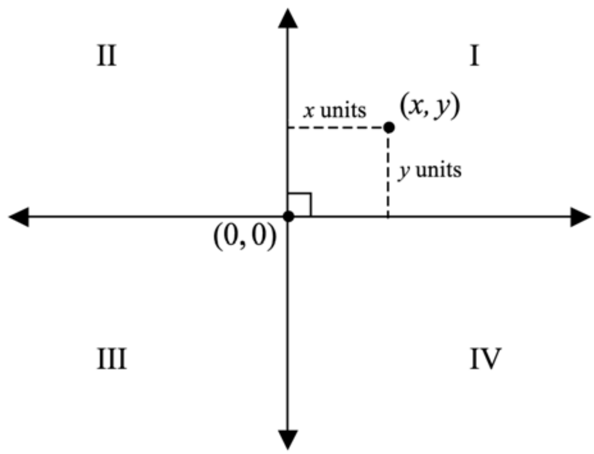
The level of sophistication of knowledge builds through high school and college through the use of various coordinate systems including Cartesian, polar, and spherical systems and by representations in two- and three-dimensional geometry. Some calculus courses are titled "Calculus and Analytic Geometry." Various coordinate system standards are in use in physics or mathematics, for surveyors, or at the state or company level. High school and college students learn to convert between some of these representations. Coordinate geometry has many applications and is used in every conceivable area of mathematics, science, and engineering to calculate precise locations and boundaries, distances and bearings from reference points, and to define graphs and curves using a point location, radius, and arc-lengths.

The fundamental building block of coordinate geometry is the Cartesian coordinate system, which includes an infinite collection of points on a plane determined by an ordered pair of numerical coordinates (x, y). The x-coordinate (called "abscissa") represents the horizontal position, and the y-coordinate (called "ordinate") represents the vertical position. These positions can be expressed as signed distances from the origin $(0, 0)$, a point that is at the intersection of two perpendicular reference lines called the "coordinate axis" (see Figure 1).

Once points are determined by ordered pairs (x, y) on the coordinate plane, one can then obtain analytic formulas for various geometric quantities on the plane. For example, an application of the Pythagorean theorem then yields the distance between any two points (x_1, y_1) and (x_2, y_2) given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Figure 1.



Early Variations

Coordinate-like types of systems arose in cartography well before Descartes. Maps with grids date back to ancient times, including those by Dicaearchus of Messana and Eratosthenes of Cyrene. Claudius Ptolemy attempted to create coordinates of well-known places in the world, essentially their latitude and longitude, from spherical projections, although the astronomical and mathematical methods to accurately calculate these would not be completely developed until much later. Islamic Mathematicians in the medieval Islamic world, such as Abu Arrayhan Muhammad ibn Ahmad al-Biruni, who compared the work of Ptolemy and Abu Ja'far Muhammad ibn Musa Al-Khwarizmi, provided coordinates for more than 600 geographical locations. Al-Biruni also used rectangular coordinates to represent three-dimensional space as well as ideas that some consider as a precursor to polar coordinates. In the twenty-first century, the global positioning system calculates the coordinates of a user from a system of satellites.

Other aspects of coordinate geometry can also be found in various early contexts. Some have noted that the mathematical work of ancient Greek mathematician Menaechmus could be interpreted as one that used coordinates. However, there was no algebra in ancient Greece, and others have highlighted the challenge that mathematics historians face in judging historical works. Coordinate geometry is a natural leap for the historians but probably not for Menaechmus, critics assert. Graphing techniques were developed in the fourteenth century in publications of Nicole d'Oresme and a work titled *De latitudinibus formarum* (The Latitudes of Forms), which some attribute to d'Oresme. Others assert that this attribution is an error and that the author is unknown. These works may have influenced coordinate geometers.

Transformations of coordinate-like systems developed along with perspective drawing techniques of curves and shapes, like in the works of Leone Battista Alberti and Piero della Francesca. Polar coordinates were motivated through the work of mathematicians such as Bonaventura Cavalieri on spiral curves like the Archimedean spiral, named for Archimedes of Syracuse.

Development

Descartes and Pierre de Fermat are both credited with independently introducing coordinate geometry. They each introduced a type of single-axis system or ordinate geometry. Distances could be measured at a fixed angle to the reference line. In Fermat's work, curves are generated as loci rather than by plotting points. Historian of science Michael Sean Mahoney noted: "There is connected with the system an intuitive sense of motion or flow wholly in keeping with the intuition which underlies the notion of an algebraic variable." Descartes' published work on coordinate geometry dates to 1637 in the appendix (*La Géométrie*) of a short book entitled *Discourse on the Method*. Descartes defined the five algebraic operations of addition, subtraction, multiplication, division, and extraction of square roots as geometric constructions on line segments and showed how these operations could be performed in the Euclidean plane by straightedge-and-compass constructions. He also developed geometric techniques for solving polynomial equations by intersecting curves, such as conic sections, with each other or with lines to obtain solutions algebraically. Coordinate geometry helps to classify conic sections, which are curves corresponding to the general quadratic equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

where a, b, c, d, e, and f are constants and a, b, and c are not all zero. Coordinate geometry became useful in a wide variety of mathematical and physical situations. Sir Isaac Newton and others investigated various coordinate systems as well as how to convert between them. In the nineteenth century, Christof Gudermann investigated the sphere, and Julius Plücker published numerous volumes on analytic geometry.

Variations

In situations where there is no obvious origin or reference axes, mathematicians developed local coordinates or coordinate-free approaches. For instance, the Frenet–Serret frame is named for Jean Frédéric Frenet and Joseph Serret. It is a type of coordinate axis system for a curve in three-dimensional space and represents the twists and turns of a curve as three vectors that move along the curve. Jean-Gaston Darboux explored the analog for a surface.

Another example is "isothermal coordinates" on surfaces in the work of mathematicians, like Carl Friedrich Gauss. Engineer, mathematician, and physicist Gabriel Lamé is noted as the first to use the term in his 1833 work on heat transfer. August Möbius introduced barycentric coordinates, which utilizes notions related to the center of mass and the centroid of a triangle, and these coordinates can be found in computer graphics. Möbius' work used both the position and magnitude.

Other mathematicians developed similar systems, including vectors, which allowed for compact notation. Hermann Grassmann and William Hamilton created the algebra of vectors. The development of vectors was especially useful when extending the geometry or physics to higher dimensions. A point (x, y) in the plane can also be represented by a vector as $r = x\hat{\imath} + y\hat{\jmath}$ where $\hat{\imath}$ and $\hat{\jmath}$ are unit vectors. Mathematicians including Jean-Victor Poncelet and Michel Chasles developed synthetic projective geometry, which focused on axioms instead of coordinates. Gregorio Ricci-Curbastro and Tullio Levi-Civita explored a coordinate-independent calculus, which led to the development of tensor analysis that later became important in general relativity. Bernhard Riemann's work on geodesics and Riemannian geometry led to geodesic coordinates, which also became important in relativity.

Education

Coordinate geometry took on an increased prominence in schools in the nineteenth and twentieth centuries. One reason was the development and curricular use of graph paper. A patent for printed graph paper dates back to Dr. Buston in the late eighteenth century. Graph paper makes it easier to plot points and create curves, and it was found to be useful in surveying and civil engineering projects.

Mathematicians in the nineteenth century, like E. H. Moore, advocated the use of paper with "squared lines" in algebra classes. Coordinate geometry topics were also included in algebra textbooks and in textbooks devoted to the subject.

One notable textbook was published by Scottish mathematician Robert J. T. Bell in 1910. His treatise on coordinate geometry in three dimensions became a very successful textbook on the subject and was translated into numerous languages.

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Padmanabhan Seshaiyer

See Also: Algebra and Algebra Education; Equations, Polar; Geometry and Geometry Education; Geometry in Society; GPS; Graphs; Maps; Relativity.

Coral Reefs

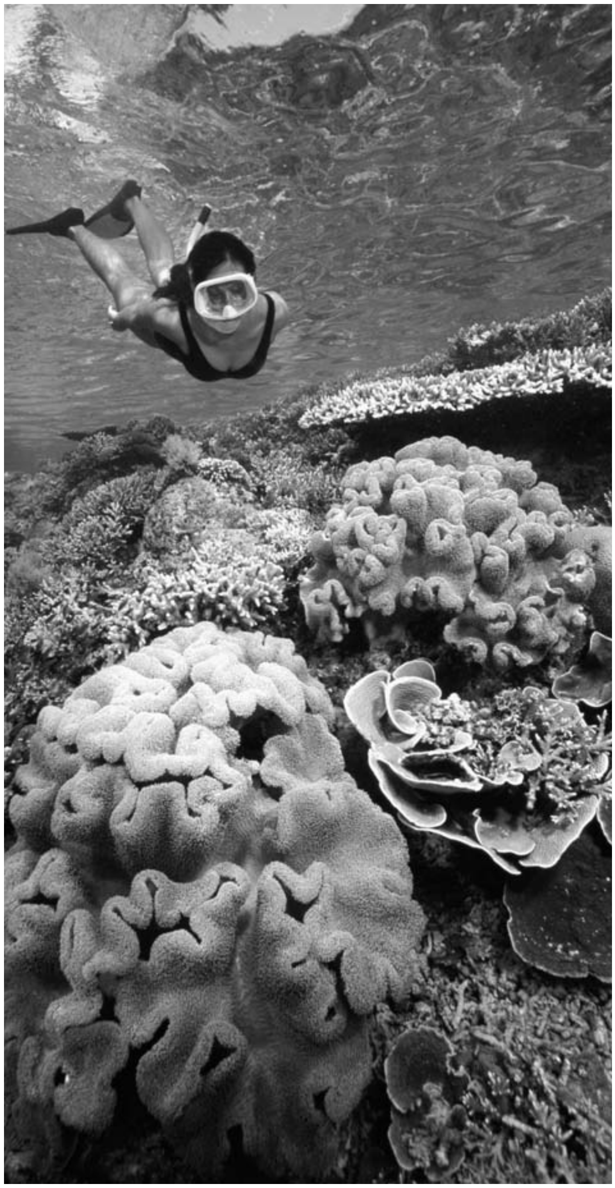
Category: Weather, Nature, and Environment.
Fields of Study: Connections; Data Analysis and Probability; Geometry.
Summary: Mathematics helps describe and explain the formation of coral reefs.

Coral reefs are complex stony structures made of exoskeletons of coral polyps. Colonies of polyps form corals, with their stony parts consisting of calcium carbonate. All polyps in a single coral are genetically identical. Polyps get their energy from photosynthesis of their internal symbionts, one-cell algae living in the polyps. Some corals also have stinging tentacles for catching plankton, and can be painful for people to touch. The development and growth of coral reefs and atolls was fiercely debated in the nineteenth and early twentieth centuries. Charles Darwin argued in his 1842 publication *Structure and Distribution of Coral Reefs*, based on his personal observations, that the geometry of coral reefs resulted from the natural geological subsidence of oceanic islands.

In other words, coral reefs formed around islands, growing as the islands sank away. Darwin's chief opponent in this debate was Alexander Agassiz, who advocated the theory that coral reefs were not wholly dependent on subsistence for their formation but rather arose from a variety of geological and biological factors. Agassiz collected data from nearly every coral reef on Earth before his death in 1910, but none of his research had been published at that time. Contemporaries of both Darwin and Agassiz were inhibited by the inability to collect data other than observations and relatively shallow rock samples. In the 1950s, geologist Harry Ladd conducted tests in conjunction with the U.S. War Department, including boring thousands of holes in the coral of Eniwetok Atoll. Ladd's drill went to a depth of nearly 5,000 feet before finally passing completely through the coral into the soil below, confirming in many scientists' minds that the atoll had been built up as the land had sunk away. Ladd purportedly erected a sign on Eniwetok that read, "Darwin was right!"

Measurements and Variables

The shape of a coral reef is determined by the sea floor and the historical changes in sea levels. Reef scientists recognize three main shape types: fringing reefs, barrier reefs, and atolls. Fringing reefs stay close to shores, and their shape is determined by the shore they circle. Barrier reefs start as fringing reefs, but as the water levels rise relative to the shore, there are deep, large lagoons separating the shore and the reef. When volcanic islands completely subside underwater, their fringing or barrier reefs can stay near the surface, forming a circular lagoon. Such reefs are called "atolls."



Reefs need clear waters for photosynthesis and can be modeled as interesting hyperbolic structures.

In most places, sea levels rise over the land. The speed of reef growth depends on multiple variables, including temperature, water salinity, water clarity necessary for photosynthesis, and wave action. Reefs can grow up to 25 centimeters (about 10 inches) per year in height. Reefs cannot grow faster than sea levels rise, because the polyps can survive out of water only for a short time—for example, during the low tide. When the speed of reef growth matches the rise of the sea level, they are called "keep-up reefs." When the speed of reef growth is slower than the rise of the sea level temporarily, reefs may become either "catch-up reefs" when the speeds eventually match, or "drowned out" reefs that die as they are submerged too deeply. Global warming threatens to increase the rate at which sea levels rise beyond the speed of reef growth.

Because reefs need clear waters for photosynthesis, they grow in the parts of the ocean that are relatively nutrient-poor. However, reefs themselves support rich and diverse ecosystems—the contradiction called "Darwin's paradox." Reefs underlie less than 1% of the world's ocean beds but host about 25% of the marine species. They are called "underwater rainforests" because of their active biomass production, measured in weight per area per day.

Coral reefs have high fractal dimensions; in other words, their surface is rough, wrinkled, and uneven. This characteristic explains why corals thrive in moving waters. The fractal-like coral surfaces break the still water barrier surrounding them, with any agitation of water creating and amplifying turbulence. This turbulence means more water moves through the polyps, delivering nutrients to them and removing sediments that could prevent photosynthesis.

Mathematical Models

Coral reefs are vulnerable to storms, tsunamis, and other strong natural events. By modeling reef damage, it is possible to intervene, and to preserve some reefs that would otherwise be destroyed. Existing models include equations that measure the forces applied to reefs, and the forces reefs can withstand.

The ratio between the area of attachment of a reef and its total surface area plays a role in the models. The higher the surface area of the reef, the higher the pressure storms apply to it. On the other hand, the higher the area of attachment, the more force it takes to detach the reef. By modifying these variables, as well as the force of the storm, oceanologists can predict what happens to particular reefs. Moreover, with more computation power comes the opportunity to model detailed shapes of reefs, individual currents, and other local variables, making predictions more precise.

Dynamic systems of differential equations are the area of mathematics applicable to complex ecosystems such as coral reefs. More deterministic models such as algebraic or simple differential equations do not capture the reality as well.

Hyperbolic Crochet Coral Reef Project

The crocheted coral reef is a collaborative project with hundreds of contributors and several exhibits worldwide, and is coordinated by the Institute for Figuring. It demonstrates hyperbolic geometry, which is a non-Euclidian geometry discovered about 200 years ago and found in nature—including corals. "Hyperbolic crocheting," the process for modeling corals, was first described in the late 1990s. It involves a simple repeating algorithm with introduced "mutations" that produce varied forms.

The models explore mathematical entities that can be found in coral reefs, such as the hyperbolic radius of curvature, pseudospheres, hyperbolic planes, and geodesics.

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Maria Droujkova

See Also: Animals; Crochet and Knitting; Geometry of the Universe; Surfaces; Tides and Waves; Transformations.

Counterintelligence

See *Intelligence and Counterintelligence*

Coupons and Rebates

Category: Business, Economics, and Marketing.
Fields of Study: Algebra; Measurement.
Summary: Mathematical differences between coupons and rebates provide different rewards to consumers.

Offering price reductions through coupons and rebates is a popular means of increasing the number of sales of a product, attracting customers to retail stores (both physical and online), and promoting public awareness of a brand name or product. One of the first known instances of a coupon was in 1894, when the Coca-Cola Company gave out handwritten tickets for samples of its new soft drink. The next year, Charles Post, of Post Cereal, started issuing coupons to help sell groceries. By the 1930s, these coupons were increasingly popular for saving money during the Great Depression. Some researchers claim that by the mid-1960s, half of American households used coupons.

In the twenty-first century, coupons are available on the Internet, or as permanent discount cards, in addition to their traditional paper form. In the age of online shopping, coupons for free shipping are cited by some as one of the most important factors in determining where to shop. For the customer, coupons and rebates bring savings on regularly purchased items and provide an incentive to try new products or services. In general, coupons are small discounts (a few dollars or cents) redeemed at the time of payment. The term "rebate" generally refers to larger reimbursements or discounts where the price reduction is either applied at the time of sale (an instant rebate) or reimbursed after required documents are mailed in by the customer. The specific type of coupon or rebate affects the calculation of both the discount and any applicable sales tax.

Coupons

Coupons may be issued by a product manufacturer or a store, and the redemption is somewhat different for the two types. When a customer presents a manufacturer coupon to a retailer, the customer pays any applicable tax on the full price of the item before the coupon is applied. For example, if a retailer charges $50 for a product with an 8% sales tax and a customer presents a manufacturer coupon for $10, then the cost to the customer at the time of purchase would be $44; that is, the original $50, plus sales tax of $4 ($\$50 \times 0.08$), minus the $10 coupon. Typically, the manufacturer reimburses the retailer for the amount of the coupon plus handling.

Sometimes a retailer like a grocery store, pizza restaurant, or automobile detailer will offer its own store coupons or rebates on its products or services. When a customer uses a store coupon, tax is computed on the balance after the coupon is deducted. If the $10 manufacturer's coupon is replaced by a $10 store coupon, the cost to the customer would be less: $43.20 versus $44.00.

Many retailers issue plastic cards that customers present to take advantage of weekly card specials or to receive a certain percentage discount on purchases made with the card. These cards not only allow shoppers to save money, but also the data collected when these cards and the associated purchases are scanned allow stores to better track their sales and inventory, and sometimes offer additional discounts tailored to a specific buyer's purchasing patterns. Sometimes these cards are free, but other times they require an initial or annual fee.

A retailer may offer a card at a cost of $10 that can be used for a 10% discount on all purchases at that store for one year. If a first-time customer checks out with a balance of $110 before tax, the customer can determine whether to purchase the card and take the 10% reduction. Although the card costs $10, the customer would save $11 on the initial balance (10% of the $110 total), resulting in a final cost of $109. The tax would be marginally less as well, since the total was reduced. Thus, the card would pay for itself at the first purchase, even before any other savings occur.

Another form of coupon is a card that is stamped each time the customer purchases a specified type of product, until a certain number of stamps are accrued. The customer then receives the next purchase of the specified product type free of charge, except for—possibly—sales tax. This form of coupon may be offered by certain restaurants, food markets, coffee shops, or bookstores.

Rebates

For a manufacturer rebate, an electronics retailer may sell a computer for $1,500, together with a free $100 printer after a mail-in rebate. The customer pays the tax on both the computer and the printer, and the manufacturer reimburses the customer $100 after the rebate is processed. With 8% sales tax, the cost to the customer after the rebate would be $1,628 (where the sales tax was 8% of $1,600, or $128).

Historically, economists have viewed consumer spending as a function of income. Politicians often cite this principle when pushing for tax rebates, believing they will increase consumption. However, there is little empirical evidence to support this notion, and in some cases there is contrary evidence. In 2001, the U.S. Congress enacted a tax rebate, giving $300 to anyone who had paid income taxes the previous year ($600 for couples). Economic indicators showed no associated increase in spending but rather a spike in saving. A survey of a sample of households that received a rebate reported that roughly one in five of those asked said they would spend the money. The *Wall Street Journal* ran the headline "Rebates Boost Incomes, But Not Spending." A study of the 2008 rebate found similar results.

Coupon Collector's Problem

There is a classic probability problem known as the Coupon Collector's Problem, which has been explored by a number of mathematicians, including the prolific Paul Erdos. The problem supposes that there is some number of different coupons (n) a person needs to collect to win a prize and asks how many coupons will he or she have to acquire, one at a time, to get a complete set. Usually, the coupons are equally likely to be drawn, and getting one of the n coupons does not prevent another of the same type from being drawn. Solutions to the problem can be found in a number of ways, including harmonic numbers, probability generating functions, and simulation. Extensions of the Coupon Collector's Problem are very useful in manufacturing quality control, for situations in which a number of product types must be sampled.

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See Also: Budgeting; Comparison Shopping; Market Research.

Credit Cards

Category: Business, Economics, and Marketing.
Fields of Study: Algebra; Number and Operations; Data Analysis and Probability.
Summary: Credit card issuers use mathematical models to determine credit lines and interest rates, as well as to detect fraud and analyze offers.

Credit card issuers use statistical analysis in a wide variety of ways. Statistical models of risk help the banks decide whom to approve for card membership and what interest rate to charge. Models also help issuers manage the risks of their existing customers and detect fraudulent transactions. Credit card issuers use designed experiments to help decide which offers have the largest potential to be profitable. Typically, the bank tries out the new offer on a sample of people (while leaving others in a control group) before deciding whether the new offer will be successful if given to the entire customer base. Data mining techniques help banks look at customers' past transactions in order to model future uses of the card and to help decide which customers are most likely to want which other products and services that the bank offers.

History

The first credit card was born when businessman Frank McNamara realized that he had forgotten his wallet at a New York City restaurant. After his wife rescued him by bringing cash to the restaurant, he vowed he would never face that embarrassment again. The Diners Club card was born a few months later in 1950 and became the first widespread alternative to cash. The first businesses honoring Diners Club purchases were charged 7% of each transaction (typical costs are now 2% to 5%), and subscribers were charged $3 per year.

Bank of America pioneered its BankAmericard program in Fresno, California, in 1958, and American Express issued the first plastic card in 1959. Carte Blanche was another early card. The idea of a credit "card" really gained momentum when a group of banks formed a joint venture to create a centralized system of payment. National BankAmericard, Inc. (NBI) took ownership of the credit card network in 1970 and for simplicity and marketability changed its name to Visa in 1976. (One reason for the name "Visa" is that it is pronounced nearly the same way in every language.)

Fraud Detection

Credit card banks use statistical algorithms to detect fraudulent use of credit cards. During the few seconds that it takes to approve or deny a credit card transaction at a merchant's site, information about the card is sent to a processing center. Typically at this point, only cards that are known to be stolen, fraudulent, delinquent, or other states that can be looked up will cause a denial. After the transaction has been approved, algorithms examine transactions to see if the pattern is suspicious. The cardholder may be contacted, usually by telephone, to verify that the transaction was made by the cardholder. The algorithms that identify a suspicious transaction can be quite sophisticated and are based on the past behavior of the cardholder.

That year, Visa processed 679,000 transactions—a volume that is processed on average every four minutes today. The Visa system is currently able to handle a load of about 6800 transactions per second, a capacity nearly exceeded on December 23, 2005, during the height of the Christmas shopping season. Visa is the largest merchant network, although MasterCard, American Express, and others process many transactions as well.

The Fair Isaacs Company (FICO) has grown in parallel with the credit card industry. It was founded in 1956 by mathematician Earl Isaac and engineer Bill Fair with the idea that data, used intelligently, can be used to make better business decisions. The next year, Conrad Hilton hired FICO to design and implement a complete billing system for his Carte Blanche card. FICO next developed the methodology to "score" the credit rating of customers but was unable to sell the idea to credit card banks until the 1970s. By the early 1990s, nearly every credit card bank was using some form of credit card scoring to help decide which customers to approve for credit and at what price. In 1995, both Fannie Mae and Freddie Mac, the two largest mortgage brokers in the United States, recommended using FICO scores for use in evaluating U.S. home mortgages. Today, U.S. citizens can access their various credit scores through online credit bureaus and, in fact, the U.S. government developed a policy allowing consumers to find out their scores once a year for free.

Credit Scoring

Credit bureaus use statistical analysis on past transactions, as well as income and other demographic information, to generate a credit score, usually referred to as a FICO score. This number is on an arbitrary scale that generally runs from 350 to 850 (with slight variations). The three main credit bureaus are Experian, TransUnion, and Equifax. Credit scores on the same individual may differ among the credit bureaus because of slight variations in the statistical model used to generate the number and slightly different data reported to the various bureaus. In all cases, the credit score is a prediction of how likely a borrower is to pay back the loan. For credit card companies, the score is used to decide both whether to issue the card, and what price (annual percentage rate) to charge on a balance that's carried over from month to month.

Data Mining

Credit card transactions, while vital to the running of the credit card bank, also contain information on the cardholder's spending patterns. These databases are very large, containing the records of tens of millions of customers, and dozens to hundreds of transactions per record. Using statistical models (often logistic regression models), banks can use these vast data repositories to identify the customers who are predicted to have the highest probability of enrolling for a new product or service. These offers may be made via a number of different channels. The offer may be given while the cardholder is calling a call center (800 number) with an issue concerning his or her card (in which case, the statistical algorithm will notify the operator that this customer should get the specific offer), by e-mail, by an outbound telemarketing call, by a targeted ad that pops up while the customer is visiting the issuer's Web site, or as direct marketing (so-called junk mail).

Experimental Design

To evaluate whether a new type of offer (the so-called "challenger") will be more effective (as measured by higher enrollment, revenue, profit, or other criteria) than the current offer (the "champion"), banks often use statistically designed experiments. The simplest such experiment is randomized at two levels, also known as a champion/challenger design. In this design, a sample is selected at random from the entire customer database. A proportion of those are chosen as the control group. They receive the current offer (the champion), and the rest are chosen to receive the challenger. The data are then collected, and the differences in response between the two groups are evaluated. The design can be complicated by blocking (stratification) on card type, region, income, or other demographic variables. Designs can be complicated by adding more factors, more levels, and by asymmetries introduced by infeasible treatment combinations. In the credit card industry, analysis is also complicated by the fact that one cardholder may be getting more than one experimental treatment (offer) simultaneously from different groups within the same organization and from different organizations. Capital One Bank claims to run upward of 40,000 such experiments a year on its cardholders.

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Richard De Veaux

See Also: Accounting; Budgeting; Data Mining; FICO Score; Money.

Crime Scene Investigation

Category: Government, Politics, and History.
Fields of Study: Data Analysis and Probability; Geometry; Measurement; Number and Operations; Problem Solving.
Summary: Crime scene investigation uses sophisticated mathematical models to determine what events took place at a crime scene, based on the available physical evidence.

Crime scene investigation (CSI) is the rigorous preservation and documentation of physical evidence at a specific location related to a criminal event. Investigators meticulously collect and measure crime-related evidence for scientific and mathematical analysis, reconstruction, and courtroom presentation. Overall, crime scene investigation and reconstruction involve the application of basic mathematical formulas and equations, as well as physics, geometry, and analytical thinking. Applied mathematical procedures based on well-collected data produce accurate results that generate reliable evidence for presentation in a criminal trial. Analysis of bullets, blood patterns, fingerprints, vehicle skid marks, chemical traces, and other data yield quantitative results that are invaluable in finding, arresting, and convicting suspects.

According to mathematician Chris Budd, "Many of the mathematical techniques used by forensic scientists are similar to those used in medical imaging for brain tumors, oil prospecting and remote sensing by satellites.…It is remarkable how often ideas which might be thought of as pure mathematics often find very real and important applications."

Mathematical Modeling of Projectiles

An automated ballistic identification system (ABIS) is a computer system designed to capture, store, and compare digital images of bullets and cartridge casings. A scanner captures images of bullets and cartridges so that a mathematical algorithm can extract their unique shapes, marks, and striation patterns (signatures), which are compared to a vast database of stored images. Both wavelets and statistical correlation techniques play a role in these analyses. Forensic ballistics involves the study of a projectile in motion, from the time of shooting to the time of impact with the target. Mathematics is used to analyze and describe a projectile's path though both the air and any obstructions, such as a body, as well as the mechanical characteristics of the weapon that fired the projectile.

Unobstructed projectile motion through air is typically parabolic, but a bullet may trace a complex path if deflected or stopped by an object, which requires more advanced mathematics, such as fractional differential equations, to describe. Blood droplets are another sort of projectile found at crime scenes, and the blood spray patterns are analyzed with geometric and trigonometric methods to determine the point of origin and other crucial characteristics. Along with ballistics and blood spatter, precise wound descriptions, which are closely related to fields like surveying and topography, can be mathematically modeled to suggest the type of weapon or bullet most likely to have made the wound.

Locations and Relationships

The locations and relative relationships among the various pieces of evidence are also important in making sense of a crime scene. Precise measurements allow investigators to place every item of evidence in its original location with some degree of certainty. These may be represented in a two-dimensional diagram, or in a computer reconstruction that uses two- or three-dimensional representation. Newer laser technology can record distance very quickly and precisely, as well as compute height using trigonometry. Mathematical computer algorithms can then combine data from multiple measures of a single object, taken from many angles, to produce three-dimensional models with minimal error. Another example of imaging used to solve a famous ancient "murder mystery" is the case of King Tutankhamen. X-rays from the 1960s, which could only provide two-dimensional images, were inconclusive. However, using CAT scans, which can mathematically construct three-dimensional images, scientists concluded that the king probably died from an infection in a broken leg.

Probability

Though the phrase "innocent until proven guilty" is often heard in connection with criminal investigations, in many cases the available evidence allows only a statement of what probably happened versus absolute certainty. Homicide investigators must logically infer or deduce what transpired at the crime scene by using evidence to reconstruct events and by matching a crime scene's characteristics to other examples. They may hypothesize a timeline or scenario and then apply scientific analysis to verify or refute the sequence of events to a high degree of probability.

This process requires critical scientific thinking and logical analysis. Investigators may use controlled experimentation, such as firing several bullets from the same weapon to look for variations in the pattern. Increasingly, they can also use computerized reconstructions of crime scenes and data to manipulate critical variables and conduct multiple "what if" simulations to eliminate unlikely scenarios and narrow the set of possible suspects and causes. Probability also comes into play in DNA analysis, where results are given the form of probability matches, and scientific tests such as gunshot residue, which are not 100% accurate and may occasionally result in a false outcome.

Digital images of evidence like bullets and cartridge casings are processed in an automated ballistic identification system (ABIS).

Conclusion

In summary, crime scene investigation requires investigators to apply many scientific and mathematical analyses to determine an accurate sequence of events and reconstruct what actually happened at a crime scene. Physical evidence helps investigators focus on a suspect and the manner in which the crime was committed. Successful crime scene investigations, reconstructions, and interpretations are the result of sound hypothesis formulation, experimentation, laboratory examination, and logical analysis. Applied mathematics provides the logic and rational simulations for scientific reasoning and assumptions.

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Thomas E. Baker

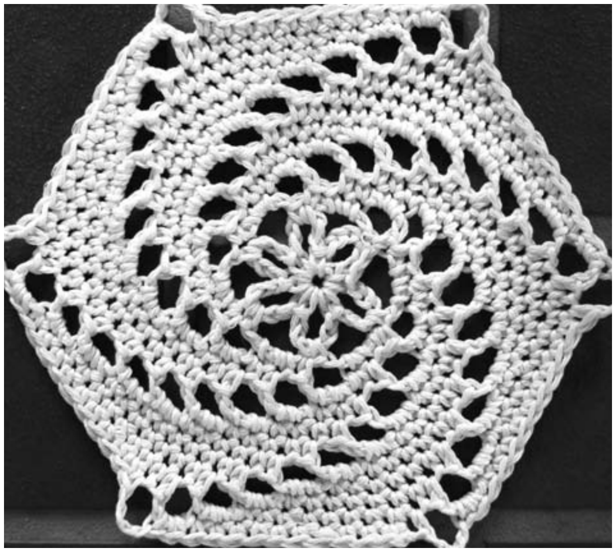
See Also: Fingerprints; Intelligence and Counterintelligence; Mathematics, Applied; Measuring Time; Medical Imaging; Probability.

Crochet and Knitting

Category: Arts, Music, and Entertainment.
Fields of Study: Geometry; Measurement; Representations.
Summary: Crochet and knitting can be used to create models of mathematical surfaces.

Crochet and knitting are techniques for turning one-dimensional yarn or thread into two-dimensional fabric by knotting it in a regular pattern. Both produce flexible, elastic fabric, although crochet is firmer than knitting. Historically, crochet and knitting were used to produce both functional and ornamental textiles by hand, but both are now hobby pursuits.

Since both techniques produce regular arrays of stitches, they can be used to display a wide variety of symmetric patterns. Furthermore, both can be used to make intrinsically curved fabrics. This allows mathematicians and others to approximate or replicate the geometry of hard-to-visualize objects, including models of two-dimensional mathematical curved surfaces, such as spheres, tori, or sections of the hyperbolic plane. Crocheting and knitting circles have been held at professional mathematics conferences for both recreation and serious discussion of mathematical concepts. Mathematician Carolyn Yackel has noted, "Knitting and crocheting are helping us think about math we already know in a different light."

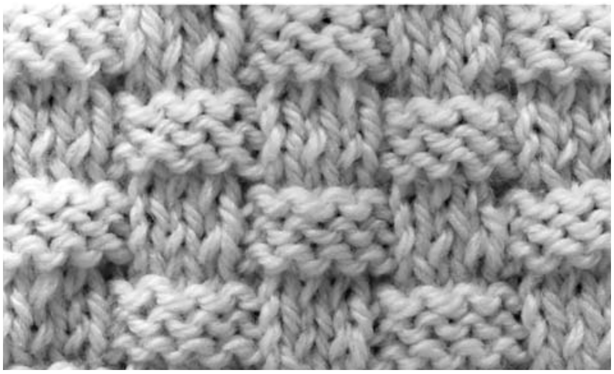


A hexagonal medallion made by crocheting in rounds. Crochet can be also worked in rows.

Crochet

In crochet, stitches are made by pulling loops of yarn through each other with a hook. One stitch is worked at a time. Every crochet stitch is attached at its base to an earlier stitch. Varying the type of stitch and the way new stitches are worked into earlier stitches can produce many different patterns. Crochet can be worked back and forth in rows or in circular rounds. Working two stitches into one base stitch increases the number of stitches and makes the fabric wider; decreasing the number of stitches reduces the width of the fabric. Placing increases or decreases at the edges of the work makes flat fabric with curved edges. Placing increases or decreases in the middle of the fabric makes it intrinsically curved.

The origins of crochet are not well understood. Few—if any—samples are known from before the nineteenth century. At that time, it was generally worked in fine cotton or linen thread and used for lace edgings,



Knit and purl stitches combined to create a basketweave pattern.

doilies, and other household textiles. From the middle of the twentieth century on, crochet has generally been worked in thicker yarn. It is often used to make blankets known as "afghans." The hobby of crocheting stuffed animals, known as "amigurumi," has spread around the world in recent years; because of the curved shape that these toys are crocheted in, they have few seams.

Several mathematicians have designed crocheted models of mathematical curved surfaces. As mathematician Daina Taimina has pointed out, it is especially simple to crochet negatively curved surfaces, such as a hyperbolic plane; the crocheter simply works an increase (an extra stitch) once every two or three (or *n*) stitches in every row. These increases cause the fabric to fold back on itself rather than lie flat. The closer together the increases are, the more ruffled the fabric.

The Hyperbolic Crochet Coral Reef, a project by the Institute for Figuring in Los Angeles, is intended to increase awareness of global warming issues by bringing together mathematicians, marine biologists, and community crafters in a highly visible way. The project asks volunteers to crochet models of coral reef life forms using Taimina's patterns. This effort and other mathematical crochet or knitting projects have been used successfully by mathematics educators in their classrooms.

Knitting

In knitting, as in crochet, stitches are made by pulling loops through each other. Knitting can also be worked in either rows or rounds. Two (or more) needles are used and many stitches are held on the needles simultaneously. The most basic stitches are "knit" and "purl" and there are techniques for increasing, decreasing, and making textural elements such as holes, cables, or bobbles. Knitting produces a flatter, stretchier fabric than crochet. (Indeed, most elastic fabric produced today is machine knitted.) As with crochet, increases and decreases allow the knitter to change the shape and curvature of the fabric. The shaping and elasticity make knitting ideal for garments such as socks, hats, gloves, and sweaters where both fit and comfort are important.

Hand knitting was once an important industry in Europe. Medieval guilds produced stunning garments for the wealthy in the Middle Ages, and a large cottage industry knitted stockings in the eighteenth and nineteenth centuries. Written patterns become available in the nineteenth century, and ornate knitting in fine thread became a popular pastime for ladies.

Hand knitting resurged in popularity in the first decade of the twenty-first century. Many current designers of garments and home textiles take their inspiration from mathematics, using symmetry and geometry to create attractive garments and household items.

Like crochet, knitting can be used to produce curved mathematical surfaces. Wide, soft, knitted Mobius bands are often knitted for use as scarves.

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Elizabeth L. Wilmer

See Also: Crystallography; Escher, M.C.; Origami; Sculpture; Textiles.

Crosswords

See *Acrostics, Word Squares, and Crosswords*

Crystallography

Category: Weather, Nature, and Environment.
Fields of Study: Geometry; Measurement; Number and Operations; Representations.
Summary: Various mathematical principles are inherent in the structure of crystals and are used to study and classify them.

Crystallography is the study of the periodic structural arrangements of particles in solids. The first discoveries of the crystallographic structure of materials were made in the early twentieth century with the X-ray diffraction technique pioneered by Max van Laue. Solids that have crystal structures have a sharp melting point, which distinguishes them from amorphous substances, such as glass, which has neither a sharp melting point nor a crystal structure.

All matter tends to crystallize, since a crystal form is the lowest energy state. In reality, most physical crystals will have flaws rather than a perfect geometric structure. The chemical composition of a substance does not determine its crystal form. Calcareous spar, for example, has at least three distinct crystal types. Although crystals exist in three dimensions, some substances, such as graphite, form strong bonds between molecules in a plane, and only weak bonds between parallel planes. Mathematics is inherently connected to crystallography, as mathematicians describe and classify crystal structures and also use crystallographic methods to solve mathematical questions, such a packing problems. Despite almost a century of the existence of the modern science of crystallography, scientists do not have a good understanding of how local ordering principles produce large-scale order.

Lattices

The first consideration in crystal structure is the lattice, also known as the Bravais lattice, after August Bravais. There are 14 types of lattices. In a crystal structure, a translation is a motion in space in a certain direction through some distance. The arrangement of atoms, ions, and molecules must be periodic, and there must be three nonunique axes of translation. An axis of translation specifies a direction in which the structure repeats. If the whole structure is moved the proper distance in the direction of an axis, it will exactly cover itself. The lattice can be considered to be all the points to which any given particle can be translated by a translation, which also moves the entire crystal structure onto itself. Thus, the lattice consists of all the points that a given point or particle is moved to by a translation. From every point in the lattice, the view of the rest of the crystal is exactly the same. The portion of the crystal obtained by starting with a particle and moving it the smallest possible distance in each of the three translation directions is known as the unit cell.

Symmetries in Crystals

The geometry of a crystal structure is characterized by its symmetries. Besides translations, other symmetries include reflections in a plane, rotations through an angle about an axis, glide reflections (translation combined with a reflection), and screw translations (translation with a rotation). A crystal structure can only have rotations that are one-half, one-third, one-fourth, or one-sixth of a complete revolution. Mathematically speaking, two crystallographic structures are the same if their symmetries are the same. A collection of symmetries for an object is called a "symmetry group." Yevgraf Federov and Arthur Schoenflies, in the late 1800s, independently discovered that there are 230 distinct crystallographic symmetry groups in three-dimensional space.

Other Crystals

Wilson Bentley provided a wealth of insight into the structure of snow crystals using a photographic microscope, taking thousands of photographs of individual snowflakes over the course of 50 years. His photographs show that although snowflakes always have a basic hexagonal symmetry, they exhibit an endless variety of detail and seem to have a limitless number of forms. The simpler snowflakes grow slowly at high altitudes in low temperatures, and the more complex ones form at higher temperatures at greater humidity. Besides direct examination, information about the structure of snowflakes has been deduced by the forms of halos that they cause around the sun and moon.

In recent years, substances such as various aluminum alloys have been discovered to have regularity of structure but no translational symmetry. These substances are called "quasicrystals," and unlike true crystals, they can have 5-fold, 8-fold, 10-fold, or 12-fold rotational symmetry.

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See Also: Molecular Structure; Nanotechnology; Polyhedra; Symmetry.

Cubes and Cube Roots

Category: History and Development of Curricular Concepts.
Fields of Study: Algebra; Communication; Connections; Geometry.
Summary: Cubes and cube roots have been the subject of classical problems in mathematics, some of which were not solved for centuries.

Cubes and cube roots of numbers have played an important part in the development of mathematics. Middle school students are taught cubes and cube roots in order to solve equations and to calculate volumes of solids. In calculus, the cube root function is a common example of a function that is continuous everywhere but has an infinite derivative at one of its points. In addition, the cube function is an example of a function that is strictly increasing everywhere but has a point where the derivative is zero. The cube rule relates the percentage of popular vote in an election with the expected percentage of seats won in a two-party election. The power needed to overcome wind resistance is directly proportional to the cube of the wind speed. One model shows the heart rate in mammals is inversely proportional to the cube root of the weight of the animal. People in many different cultures have studied cubes and cube roots, and numerous interesting stories are found in its history. These simple objects have also generated many new ideas and new fields of mathematics.

Definition

To cube a particular number x, multiply it by itself 3 times—this is denoted x^3. If x is a number such that $x^3 = y$ for some other number y, then x is a cube root of y, written as $x = \sqrt[3]{y}$. Since $(-5)^3 = -5 \times -5 \times -5 = -125$, -5 is the cube root of -125, and the notation is

$$\sqrt[3]{-125} = -5.$$

The cube of any real number is unique; however, every real number has exactly one cube root that is a real number and two cube roots that are complex numbers.

Early History

As with squares, the earliest uses of cubes of numbers involved common geometric objects, specifically the cube, which is a three-dimensional object with six sides, all of which are congruent squares. The volume of a cube is the cube of the length of one of its sides. The volume of a sphere is directly proportional to the cube of its radius. One of the classical problems in Greek mathematics was the problem called "Duplication of the Cube." The problem was to find the length of an edge of a cube that has double the volume of a given cube using the tools of the time, the ruler and compass. It is now known that if x is the length of the side of the given cube, then $\sqrt[3]{2} \times x$ is the length of the cube with twice the volume. One possible origin of this problem is that, in 430 B.C.E., it was proclaimed through the oracle at Delos that the cubical altar to Apollo was to be doubled in volume in order to alleviate a plague that had befallen the people. Another possibility is that the Pythagoreans successfully doubled the square and doubling the cube was a natural extension. In any event, many great mathematicians throughout history worked on this problem, and, in the nineteenth century, it was proven that a solution was impossible.

Cube roots can be exact numbers if the cube root is an integer or a fraction. However, the cube root of most numbers is irrational (it has a infinite non-repeating decimal expansion) and its value can only be approximated. The easiest method to approximate the real cube root of a real number is to raise the number to the 1/3 power on a calculator. Obviously, the calculator is a recent invention, and other methods have been developed for approximating a cube root. Some of the earliest known methods are found in the Chinese text *Nine Chapters on the Mathematical Art* (c. first century C.E.) and in the book *Aryabhatiya* by the Indian mathematician Aryabhata (b. 476 C.E.). Both methods use the formula $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ repeatedly to generate the successive digits of the cube root. Approximations to cube roots can also be computed with the Chinese abacus, the *suànpán*, which dates to 200 B.C.E. In many cases, scribes would create tables of cube roots, which people would use to look up values for use. Barlow's Tables, named for mathematician Peter Barlow, who originally published in 1814, give the value of cubes and cube roots to nine decimal places and are still in print in the twenty-first century. Recreational mathematicians have found it fun to devise ways to compute cubes and approximations to cube roots in their head without outside assistance.

Cubic Equations

Cubic equations are equations that involve positive integer powers of x where the highest power is 3. Mathematicians have been trying to solve these equations from the earliest times. The Babylonian text *BM 85200* (c. 2000 B.C.E.) contains many problems that compute the volume of an excavated rectangular cellar by setting up and solving a cubic equation. Another Babylonian tablet contains, among other things, a table of integers and the sum of each integer's square and cube, and it was presumably used to solve cubic equations.

Archimedes of Syracuse (c. third century B.C.E.) considered the problem of passing a plane through a sphere such that the volumes of the two pieces had a certain ratio. This problem gives rise to what would be a cubic equation. A manuscript, thought to have been written by Archimedes, was found centuries later that gave a detailed solution to the problem that involved finding the intersection of a parabola with a hyperbola. Omar Khayyam (c. eleventh century C.E.) was the first to find a positive root of every cubic equation having one. Before this time, numbers were thought of as specific quantities of objects, so very little was done with negative numbers—and certainly not complex numbers. As with Archimedes, Khayyam's solutions involved intersecting conic sections.

By the fifteenth and sixteenth centuries, negative numbers and zero were accepted, and many of the Greek mathematical texts were translated to Latin. The field of algebra had been developed, and people could study equations as expressions with variables that can be manipulated (as is done in the twenty-first century). As the solution of the general quadratic equation had been discovered, Italian mathematicians focused their attention to the solution of the general cubic equation $ax^3 + bx^2 + cx + d = 0$. During this period, academic reputations and employment were based on public problem-solving challenges, and discoveries were kept secret so they could be used to win one of these challenges.

The solution of certain cubic equations provided the backdrop to one of the more entertaining chapters in the history of mathematics. On his deathbed in 1526, Italian mathematician Scipione del Ferro told one of his students, Antonio Maria Fior, how to solve a specific type of cubic equation. Nine years later, Fior submitted 30 cubic equations of this type to mathematician Niccolo Tartaglia in a public challenge. During the contest, Tartaglia himself discovered the solution and won the contest. After hearing of the contest, Girolamo Cardano contacted Tartaglia to inquire about his method. Tartaglia told him his solution, only after Cardano agreed to keep it secret as Tartaglia indicated he was going to publish it (thinking he was the first to discover it). Years later, Cardano found out that del Ferro actually discovered the formula and published it as del Ferro's method in addition to solutions to the cubic equation in all cases that he and his assistant, Lodovico Ferrari, discovered. Tartaglia was extremely angry and felt Cardano had broken his promise. In the twenty-first century, the formula for the solution of the cubic is known as the Cardan(o)–Tartaglia formula.

Uses and Applications

The cubic equation also played an essential role in the formulation of complex numbers. In his 1572 text, *Algebra,* Rafael Bombelli considered the equation $x^3 = 15x + 4$.

Applying the formula of Tartaglia and Cardano, one obtains a solution

$$x = \sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}}.$$

However, Bombelli knew the solution was actually 4 and that, somehow, the square root of −121 could be manipulated in a way to reduce this expression to 4. He developed an algebra for working with these roots of negative numbers (thought to be of no use to earlier mathematicians), and complex numbers and the field of complex analysis was born. With complex numbers, one can show that any cubic equation has exactly three solutions, two of which must be complex and one real.

In 1670, it was discovered that French mathematician Pierre De Fermat claimed that for all natural numbers $n > 2$ there are no nontrivial solutions of positive integers *a*, *b*, *c* such that $a^n + b^n = c^n$. Andrew Wiles proved this theorem in 1994 using objects called "elliptic curves." Elliptic curves are defined by a cubic equation of the form $y^2 = x^3 + ax + b$, whose graph has no cusps or self-intersections. These curves are studied in the twenty-first century and used in both number theory and cryptography (the study of coding information). Even though Fermat's equation has no positive integer solutions for $n = 3$, other problems involving sums of cubes have been studied.

In 1770, Edward Waring proposed the following question: for every positive natural number *k*, does there exist a natural number *s* such that every natural number *N* can be written as the sum of at most *s* numbers which are *k*th powers? If $k = 3$, the question becomes: can every positive number be written as a sum of at most *s* cubes? Some examples are $5 = 1^3 + 1^3 + 1^3 + 1^3 + 1^3$ and $23 = 2^3 + 2^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3$.

As 23 shows, one requires at least 9 cubes. In 1909, David Hilbert proved that 9 is the maximum number of cubes that are required for any positive natural number. The Waring-Goldbach problem asks a similar question, except it requires at most *s* cubes of prime numbers. Some progress has been made, but this question remains unsolved as of 2010.

One of the more interesting recent mathematicians is Srinivasa Ramanujan from India (1887–1920). He was mostly self-educated and was able to prove theorems in number theory that shocked one of the eminent mathematicians of the time, G. H. Hardy. Once when Hardy visited Ramanujan, he mentioned he arrived in a cab numbered 1729, which did not seem very interesting. Ramanujan responded that 1729 is a very interesting number in that it is the smallest positive integer that can be represented by a sum of two cubes in two different ways, $1729 = 1^3 + 12^3 = 9^3 + 10^3$, which is correct. The taxicab numbers are generalizations of this idea. The *n*th taxicab number, denoted $Ta(n)$, is the smallest positive integer that can be written as two different cubes in *n* different ways. By Ramanujan's comment, $Ta(2) = 1729$. It is also true that $Ta(1) = 2$, since $2 = 1^3 + 1^3$ and $Ta(3) = 87{,}539{,}319$, since

$$87{,}539{,}319 = 167^3 + 436^3 = 228^3 + 423^3 = 255^3 + 414^3.$$

The first 6 taxicab numbers are known, but $Ta(7)$ and beyond are all unknown as of 2010.

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See Also: Algebra and Algebra Education; Measurements, Volume; Numbers, Complex; Squares and Square Roots; Units of Volume.

Currency Exchange

Category: Business, Economics, and Marketing.
Fields of Study: Algebra; Measurement; Representations.
Summary: Mathematical models seek to price financial products in the foreign exchange market.

The term "currency exchange" refers to the business transaction that trades one currency for another. Such a transaction happens in the foreign exchange (FX) market and is measured by foreign exchange rates, which are often called exchange rates. Exchange rates fluctuate all the time. There are many factors that influence the movements of exchange rates. After all, foreign exchange rates are largely determined by the supply and demand in the FX market. Numerous mathematical models have been proposed by financial mathematicians and financial engineers to price different financial products in the FX market. Some of them have been used successfully by practitioners.

Exchange Rate Definition

There are many different currencies in the world. A measurement of the value of one currency in terms of another is called a (foreign) "exchange rate" or a "currency rate." In simple terms, an exchange rate of *K* currency *X* to currency *Y* means the value of *K* units of currency *X* is equivalent to the value of 1 unit of currency *Y*. It is often quoted as the price of currency X divided by currency Y is K. For example, the price of "euros/U.S. dollars is 1.3578" denotes an exchange rate of 1.3578 U.S. dollars to euros. In other words, it means the value of 1 euro is the same as that of 1.3578 U.S. dollars.

Types of Exchange Rates

A fixed exchange rate (also known as "pegged rate") means one currency is pegged to a major currency such as the U.S. dollar. Usually, the government or the central bank of a country will intervene in the market to peg its currency to a major currency to maintain a fixed exchange rate.

In contrast, a floating exchange rate is determined by the market forces of demand and supply.

Exchange Rate Fluctuation

Fluctuation of exchange rates, like fluctuation of stock prices, interest rates, and many other economic indices, is a ubiquitous phenomenon. Many factors drive the exchange rates up and down. These factors include but are not limited to capital flows, international trades, speculation, political factors, government or central bank intervention, and interest rates. However, the fundamental driving force is the invisible hand—the demand and supply—of the market.

Besides those quantifiable drivers of the FX market, there are other nonquantifiable ones such as the expectation of the investors. Attempts have been made by economists to account for those driving forces as well. Some economists have put the theory of exchange rate into a behavioral finance framework. Others used information theory and game theory.

FX Markets and FX Financial Products

The FX market is where the currency exchange happens, and is one of the largest financial markets in

Having exchange rates for national currencies allows us to consistently express the value of an item across borders of countries and cultures.

the world. Its major participants include commercial banks, investment banks, companies, investors, hedgers, speculators, traders, governments, and central banks. A variety of financial instruments are traded in the FX market, including currencies, currency forward contracts (also known as "FX forward contracts"), currency futures contracts (also known as "FX futures contracts"), currency options (also known as "FX options") and currency swaps (also known as "FX swaps"). Thus, the FX market has several important submarkets: the FX spot market, the FX forward market, the FX futures market, the FX options market, and the FX swaps market.

Although hundreds of financial products exist in the FX market, the basic ones are currencies, currency forward contracts, currency futures contracts, currency options, and currency swaps. Currencies are priced by the exchange rates. Both currency forward contracts and currency futures contracts are agreements made between two parties to exchange a specified amount of currency for a specified price at a specific future date. The main difference is that a currency forward contract is traded over the counter, whereas a currency futures contract is traded on an exchange. They both are financial derivatives. Their prices can be determined using simple algebra and are expressed in terms of exponential functions. Currency options and currency swaps are also financial derivatives. A currency call/put option gives one party the right—but not the obligation—to buy or sell a specific amount of the currency at a price (called "strike price") at a specific time in the future.

A European option can be exercised only at maturity, whereas an American option can be exercised at any time up to maturity. The cash flows of currency options are more complicated than those of the currency forward and currency futures contracts. The pricing requires sophisticated mathematical tools from stochastic calculus. Fisher Black, Myron Scholes, and Robert C. Merton made fundamental contributions in option pricing by giving the basic pricing formulas of European options. Scholes and Merton were awarded the Nobel Prize in Economics for this accomplishment in 1997 (Black was not awarded the prize because he had passed away).

A currency swap is an agreement between two parties to exchange the principal and interests of one currency at an interest rate for the principal and interests of another currency at another interest rate for a certain period of time. For example, suppose party A enters into a currency swap contract with party B today. For the next five years, party A will pay party B the interest of a principal of $1 million at an annual interest rate of 5%. In return, party B will pay party A the interest of a principal of 95 million Japanese yen at an annual interest rate of 4.5%. The two parties will also exchange the principals at the end of the fifth year. Like currency forward and currency futures contracts, the currency swap can also be priced using simple algebra.

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See Also: Connections in Society; Money; Risk Management.

Curricula, International

Category: Mathematics Around the World.
Fields of Study: All.
Summary: Comparisons of mathematics curricula worldwide help facilitate growth and development.

A long history exists of comparisons between undergraduate mathematics curricula in other countries and the United States, and in recent decades, similar comparisons are being made at the primary and secondary levels. A recent movement in mathematics education has shifted the focus of how mathematics at all grade levels is taught. This movement was in large part spurred by the results of international testing. Since 1995, the Trends in International Mathematics and Science Study (TIMSS) has collected data on student achievement for fourth-, eighth-, and twelfth-grade students around the world.

The TIMSS was designed to allow for international comparisons, and has motivated educators to examine more closely those countries that consistently show success in educating students. Another international assessment, the Programme for International Student Assessment (PISA), focuses on measuring the mathematical literacy of 15-year-olds. The results of the PISA reflected those of the TIMSS, prompting educators in less successful nations to explore how some countries, such as Singapore, Japan, and Korea, educate students in mathematics. One area that has been explored as a result of the TIMSS and PISA is that of curricula. Liping Ma's 1999 book *Knowing and Teaching Elementary Mathematics*, which compared teaching methods in the United States and China, has also spurred numerous discussions about curricula and teaching methods, including teacher education and preparedness of teachers for presenting mathematical concepts at all levels.

It is important, first of all, that a distinction be made between curriculum and instructional programs. "Curriculum" is generally defined as a set of standards or objectives that guides what is taught at a particular age or grade level. "Instructional programs," on the other hand, are resources that are available to teach the curriculum, such as textbooks. On the international stage, a variety of instructional programs exist and are in use, but mathematics curricula across nations remain surprisingly similar.

An analysis of 16 countries' curricula conducted by Graham Ruddock demonstrated that different nations used the same basic mathematical principles as a foun-

Since 1995, the Trends in International Mathematics and Science Study (TIMSS) has collected international data on student achievement in mathematics for fourth-, eighth-, and twelfth-grade students.

dation for building mathematics curricula: number, algebra, geometry, measures, probability, and statistics. While some of the principles may be combined together into a single topic (for example, probability and statistics), these basic principles existed in the curricula of all nations that were studied. However, Ruddock pointed out that it is important to realize that just because nations use the same label, it does not mean that the content included in the principles is consistent across nations, nor does it mean that each nation explores each of the principles with equal rigor.

Nations also generally agree which principles of mathematics should be taught in the lower grades. Number is the primary focus for younger students, with a shift in focus toward algebra as students move into the middle grades. Nations vary widely in their mathematics curricula for upper grades, because of the nature of the different educational systems. For example, Japan uses an integrated approach to mathematics through the upper grades, where all principles are taught in varying degrees at all grade levels, while the United States utilizes a traditional division of mathematics topics (for example, algebra, geometry, calculus as separate courses).

Recent Pedagogical Changes

Interestingly, most nations at the beginning of the twenty-first century incorporate what is known as a "spiral curriculum," which is designed so that students revisit topics that were previously learned. This form of curriculum represents a shift in thinking in mathematics education that occurred during the 1990s. The purpose of the spiral curriculum is to assist students in making connections between mathematical ideas as well as ensure that students retain the knowledge that has been previously taught. A well-designed spiral curriculum is designed to encourage students to view mathematics as an integrated whole, rather than as discrete, unrelated topics.

An additional pedagogical shift has come as mathematics educators consider the value of conceptual understanding versus procedural understanding. Curricula in various nations have been adapted to include a stronger focus on the conceptual understanding of mathematics, rather than rote memorization and mastery of basic math skills. For example, curricula in Japan, Korea, and Singapore, all of which have consistently performed well on the TIMSS and PISA, have shifted from the learning of basic skills through rote memorization to an emphasis on problem solving and critical thinking. Curricula in other nations have followed this example.

National Mathematics Curricula

Some nations, such as England, France, Italy, and Japan, have required national mathematics curricula. Other nations, such as the United States, Australia, Canada, and Germany, view education as a local responsibility; therefore, a national mathematics curriculum does not exist. However, organizations such as the National Council of Teachers of Mathematics have developed national standards as suggested guidelines for what mathematics should be taught at different grade levels.

The greatest difference between nations regarding curricula is that of implementation. Curricula implementation varies widely among different nations, with some nations, like Hungary and Spain, placing a focus on local implementation while Japan has national guidelines for how teachers are to implement the curricula into their classrooms. From this variety of approaches comes the question of intended versus enacted curriculum. In other words, are teachers implementing the mathematics curriculum as it was designed? While the intended curricula across nations appear to have some strong similarities, especially at the lower grades, the enacted curricula may be quite different, thus resulting in substantial differences in student learning.

Current Trends in Curriculum Approaches

In recent years, the Singapore mathematics curriculum has garnered a great deal of attention because of the impressive performance of Singapore students on the TIMSS. The Singapore curriculum focuses on developing concept mastery through an in-depth exploration of a few mathematical topics each year. Also emphasized are the use of visual strategies in problem solving and establishing connections between mathematical topics. The Singapore mathematics curriculum has undergone a variety of changes since it was first developed in 1981, with the latest version including the introduction of calculators at a younger age and a reduction in emphasis on mental mathematics. Several countries, including the United States and Canada, have begun to implement curricula that mirror the Singapore mathematics curricula in the hopes of acquiring similar levels of student achievement on national and international assessments.

The International Baccalaureate (IB) Programme has also gained in popularity in recent years. The IB is designed to be a broad-based international curriculum, and is offered at three different levels: the Primary Years Programme (PYP), the Middle Years Programme (MYP), and the Diploma Programme (DP). While the IB does not focus specifically on mathematics, all three levels include mathematics as an integral part of the IB experience, as "mathematics is a universal language with diverse applications." Mathematics in the IB is viewed as a key connection to students' understanding of culture and history, and as a primary method of developing students' logic and critical thinking skills.

Since World War II, a growing number of foreign-educated students in mathematics and other related fields have chosen to attend graduate school or seek postdoctoral positions at American universities, with the largest growth occurring in the 1990s. For example, studies show that in 2002, nearly one-third of all graduate students enrolled at U.S. universities came from abroad. Many reasons are cited for this effect, including the quality of research universities, the availability of funding, and the existence of desirable job opportunities. A phenomenon colloquially known as "brain drain" reflects the significant migration of students with mathematical and technical skills away from their native countries, diminishing these countries' ability to compete in the global marketplace. In response, countries are beginning to expand their efforts to retain these students. For example, China has reorganized some current universities and built new ones, as well as engaged in significant curriculum reform. This reform includes new partnerships, such as a new Danish-Chinese University Centre for collaborative technology research, which was formalized in 2010.

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Calli A. Holaway

See Also: Connections in Society; Curriculum, K–12; Succeeding in Mathematics.

Curriculum, College

Category: School and Society.
Fields of Study: All.
Summary: Collegiate mathematics education is determined by the student's choices within the constraints of graduation and department requirements.

For thousands of years, mathematics has been considered an important part of a liberal arts education. Examples of this idea abound, including schools and scholars in ancient Greece, China, and the medieval Islamic world, as well as in the rise of North American colleges in the seventeenth century.

Debate has existed for decades about which topics should be a part of the college curriculum and how best to teach them. Common curricula, such as geometry, or educational tools, like the abacus, have been replaced by other focuses as societies' needs have changed and technology has advanced.

New discoveries in mathematics and emerging disciplines also result in curriculum changes. In the twenty-first century, the mathematics curriculum at the university level varies depending on the educational goals of the student. In the United States, the types and the number of mathematics courses required in the curriculum are typically based on a student's major subject of concentration. In this regard, there tend to be three broad categories into which a typical college student may be classified: a student who needs to fulfill a general education requirement in the mathematical sciences; a student majoring in a partner discipline, such as the physical sciences, the life sciences, computer science, engineering, economics, business, education, and the social sciences; and a student whose major is in the mathematical sciences, including pure (theoretical) or applied mathematics, statistics, actuarial sciences, and mathematics education. At most colleges and universities, curriculum is approved by both internal governing bodies, such as curriculum committees, and external accrediting agencies. Local, national, and specialized accrediting agencies may approve programs at the department or college level.

History

There is a rich history of mathematics in higher education contexts. From the schools of ancient Greece to the universities of the Renaissance, mathematics was an important component of the seven liberal arts, and mathematics was seen as a way to understand reality. Three of the liberal arts, the Trivium, consisted of grammar, rhetoric, and logic. In the Quadrivium, the other four liberal arts, arithmetic was the study of numbers, geometry was the study of numbers in space, music was the study of numbers in time, and astronomy or cosmology was the study of numbers in space and time. The first college in the United States was Harvard University, founded in 1636. Harvard and other institutions of higher learning included mathematics in their curriculum. Around the time of the Revolutionary War, advanced mathematics included topics in surveying, algebra, geometry, trigonometry, and calculus. In 1776, Congress advised that disabled veterans, "[w]hen off duty, shall be obliged to attend a mathematical school, appointed for the purpose, to learn geometry, arithmetic, vulgar and decimal fractions, and the extractions of roots." This obligation led to the official founding of the United States Military Academy in 1802. After World War II and the beginnings of the Cold War, the growing emphasis on computer technology greatly impacted the mathematics curriculum in the United States.

Teachers have long explored different methods to help students succeed in mathematics. The philosopher Socrates is known for the Socratic Method, and in the early part of the twentieth century, topologist Robert Lee Moore developed a Socratic style of teaching that became widely known as the Moore Method. Versions of the Moore Method, or a modified Moore Method, continue to be used in twenty-first-century undergraduate and graduate mathematics classrooms. In some implementations, students work on problems and present proofs or solutions they develop on their own, with the class being responsible for corrections and the teacher acting as a guide. In the 1980s, a calculus reform movement that is often referred to as the "calculus wars" spurred debates among mathematicians regarding various aspects of teaching, including the use and balance of lectures, technology, and rigor in calculus classrooms.

Calculus education had already undergone many changes in the twentieth century, such as a shift to calculus being taken earlier in the college curriculum. Following the ethos of "calculus should be a pump, not a filter," educators explored many different approaches, often based on empirical studies. Some campuses embraced new approaches, while others soundly rejected them. In the early twenty-first century, mathematicians continue to discuss and refine the calculus course as well as other mathematics courses. There are also discussions at both the college and federal level of the possibility of standardized college mathematics assessments.

General Education Mathematics Requirement

For the college student majoring in a subject area that does not require specific mathematics courses, the extent of the mathematics curriculum may consist of mathematics courses that satisfy general education core requirements. At most colleges and universities, these courses enroll almost twice as many students as all other mathematics courses combined. These students represent a broad variety of majors, including students from the humanities, fine arts, elementary education, and several branches of the social sciences.

Courses that fall into this category may be termed or described as one of the following: quantitative literacy; liberal arts mathematics; finite mathematics; college algebra with modeling; or introductory statistics. These courses are designed to have students learn to think effectively, quantitatively, and logically, and may actually also be requirements for a student's major. Such courses often serve as students' final experience of college mathematics. While these courses may be terminal, such courses could also entice students to study mathematics further, and therefore, such course offerings may act as a springboard or gateway through which a student chooses to continue the study of the mathematical sciences.

There is a wide variety of topic options in these courses. Some professors incorporate topics directly from daily life, like financial mathematics, while others focus on algebraic or statistical techniques that might be important in future coursework. General education courses are also seen as the final place to impact students' perceptions about mathematics and its role in society. In the same way that a survey course on important literature might include works by William Shakespeare, some mathematicians select course topics from the masterpieces of mathematics, which might include great theorems, like Euler's theorem, named for Leonard Euler; interesting applications, like Chvátal's art gallery theorem, named for Václav Chvátal; interdisciplinary topics, like fractals, perspective drawing, or the philosophy of mathematics; or beautiful mathematical topics, like the golden mean. Some classes focus on the breadth of mathematics, while others try to cover a few topics in depth. There is also a wide variety of teaching methodologies and pedagogy. In some classrooms, the focus is on lectures, while in others it is on discussion or presentations. Technology may be a fundamental part of the class, or the class might focus on pencil-and-paper methods.

Mathematics and Partner Disciplines

During the second half of the twentieth century and into the twenty-first, there has been an enormous growth and development of scientific and technological disciplines, and, consequently, the role of mathematics is increasing in an expanding array of subject areas and professional programs. Students may be required to take specific mathematics courses that complement their major field of study. These partner or client disciplines include physics, chemistry, biology, computer science, engineering, business, finance, economics, nursing, psychology, and education. Partner and client discipline courses may impact mathematics as well as the respective discipline.

Some of these courses are taught in mathematics departments; others are taught as a quantitative course in the major, as in some psychology departments. This system provides numerous opportunities for faculty and students in mathematics departments to collaborate with their counterparts in other academic departments on campus. It is not uncommon for students who major in these partner disciplines to also study advanced mathematics, often resulting in dual majors or a minor in mathematics. One emerging area in the twenty-first century has been calculus for the life sciences. Cutting-edge pedagogies may come from mathematics or a client discipline. Faculty in either mathematics or a client discipline may lead efforts in interdisciplinary curricular development, or departments may resist changes because of staffing or philosophical considerations, sometimes leading to friction between departments.

Such courses need not be limited to calculus-based courses. For example, students in the sciences often benefit from the skills and techniques used in introductory statistics and discrete mathematics courses, which may not have a calculus prerequisite. The ability to visualize in three dimensions is also valued by partner disciplines, and courses that emphasize geometric and graphical reasoning, linear systems, and vector analysis may also be required.

Precise, logical thinking is an essential part of mathematics. While it remains a component of the mathematics courses taken by students who study the aforementioned partner disciplines, additional needs specific to such fields of study are also imbedded in the courses taken by these students. Logical and deductive reasoning skills may need to be developed in a specific context, and certain disciplines may or may not have a need for the use of formal proof found in the mathematics courses. Also, the level and type of logical reasoning may vary depending on discipline. For example, business majors may require more quantitative or statistical analysis, while engineering students need to engage in more formal analysis in a course like multivariable calculus. Students studying the natural sciences benefit from heuristic arguments and data analysis, while computer scientists and software engineering students need the ability to use logic to write simple proofs. The courses that bridge various other subject areas with mathematics attempt to balance the rigorous proof and deductive reasoning inherent to mathematics with the skills these partner disciplines require of their students.

Students who are preparing to teach elementary or middle school mathematics also fall into this category. The curriculum designed for future primary mathematics teachers varies state by state, with many requirements set by schools or teacher program accrediting agencies. They often aim to provide these students with a firm foundation in various mathematical topics, such as number and operation, algebra and functions, geometry and measurement, and data analysis, probability, and statistics. These topics are studied at a level above and beyond that which they will eventually teach. Courses are designed to provide students with an understanding of these broad areas as well as an ability to make connections among various mathematics topics and with other subjects taught in the elementary and middle school curriculum. The intent is that the future teachers will be able to guide their students in ways that instill mathematical breadth and depth and "plant the seeds" of ideas that will come later. From 2003 to 2009, the Mathematical Association of America ran a program that was funded by the National Science Foundation called Preparing Mathematicians to Educate Teachers (PMET). PMET strove to improve the mathematics education of teachers by targeting the development of faculty awareness and teaching as well as instructional materials.

Concentration in the Mathematical Sciences

For students who choose to major in the mathematical sciences, their college curriculum is centered on this goal of study. Major programs in the mathematical sciences include courses that focus on pure (theoretical) mathematics, applied mathematics, statistics, actuarial sciences, or secondary mathematics education. Depending on the college or university, the programs and faculty for statistics or applied mathematics, as well as actuarial sciences and mathematics education, may be housed in a department distinct from the traditional mathematics department.

The actual course of study for mathematics majors will differ depending on the specific college or university. In general, students in their first years of study will take a sequence of courses in calculus consisting of single- and multivariable calculus, which include the topics of differentiation and integration, sequences and series, vector analysis, and differential equations. Beyond calculus, mathematics students often take a transition course that includes an introduction to proof-writing techniques demonstrated by a study of various foundational topics in mathematics, such as logic, set theory, functions and relations, and cardinality.

Other commonly required courses for the mathematics major include linear algebra, abstract algebra, and real analysis (or advanced calculus). Several other advanced courses in mathematics that make up the major include ordinary differential equations, partial differential equations, discrete mathematics, probability and statistics, modern geometry (Euclidean and non-Euclidean geometry), complex analysis, topology, combinatorics, and number theory. Students who are interested in learning more applied mathematics may take courses in dynamical systems, numerical analysis, cryptanalysis, and operations research. A course in the history of mathematics may also be offered, especially for those students preparing to teach mathematics.

Because there are numerous topics that connect mathematics with other disciplines, various interdisciplinary courses may also be offered by mathematics departments in conjunction with other academic departments on campus. Some schools use common syllabi or exams for certain courses, and other schools allow more flexibility in what is taught and how it is taught. Regardless, there are at least some common expectations because mathematical definitions, ideas, and proofs build upon one another across courses, and so earlier courses in the major impact later ones. For example, a single-variable calculus class impacts multivariable calculus, and an analysis course impacts courses in complex analysis and topology.

The curriculum for students majoring in mathematics is designed so that there is a progression from the study and practice of computational methods and procedures toward an extensive understanding of the subject, which may include logical reasoning, generalization, abstraction, sophisticated applications, and formal proof. Students majoring in mathematics are also encouraged to demonstrate their mathematical knowledge in both written and oral formats. Students should also gain experience in the analysis of data, gaining the ability to move between context and abstraction—an especially important ability for students whose course of study focuses on applied areas of mathematics as well as for those becoming mathematics teachers. While mathematics students may prefer one area of mathematics over another, they are encouraged to gain a broad view of the subject, recognizing the complementary nature of the following concepts: theory versus application; discrete versus continuous; algebraic versus geometric; and deterministic versus probabilistic.

In addition to specific mathematics courses, students majoring in mathematics may take courses in computer science. On some college and university campuses, mathematics and computer science are classified in the same department or division. The natural affinity between the skills used by mathematicians and computer scientists makes this partnering possible, since the application of logical reasoning to the task of programming enhances the learning of both disciplines. For mathematics majors who are preparing to enter the nonacademic workforce, experience with teamwork, creativity, and problem synthesis skills is enhanced by computer programming coursework.

Undergraduate Research in Mathematics

Many mathematical science departments require their mathematics majors to engage in some form of research at the undergraduate level. This research can take many forms, such as a capstone course, a thesis, or some other form of a project during the senior year of college. The

Mathematics Departments

Mathematics departments need to serve all students well—not only those who major in the mathematical or physical sciences. The following steps will help departments reach this goal.

- Design undergraduate programs to address the broad array of problems in the diverse disciplines that are making increasing use of mathematics.
- Guide students to learn mathematics in a way that helps them to better understand its place in society: its meaning, its history, and its uses. Such understanding is often lacking even among students who major in mathematics.
- Employ a broad range of instructional techniques, and require students to confront, explore, and communicate important ideas of modern mathematics and the uses of mathematics in society. Students need more classroom experiences in which they learn to think, to do, to analyze—not just to memorize and reproduce theories or algorithms.
- Understand and respond to the impact of computer technology on course content and instructional techniques.
- Encourage and support faculty in this work—a task both for departments and for administrations.

The CUPM Guide 2004 presents six general recommendations to assist mathematics departments in the design and teaching of all of their courses and programs:

1. Understand the student population and evaluate courses and programs.
2. Develop mathematical thinking and communication skills.
3. Communicate the breadth and interconnections of the mathematical sciences.
4. Promote interdisciplinary cooperation.
5. Use computer technology to support problem solving and to promote understanding.
6. Provide faculty support for curricular and instructional improvement.

From the Introduction of "Undergraduate Programs and Courses in the Mathematical Sciences: CUPM Curriculum Guide 2004" by Committee on the Undergraduate Program in Mathematics (CUPM) of The Mathematical Association of America (MAA).

The Hutlee'/Umyuarchdelee program is funded by the National Science Foundation to help Natives and rural Alaskans succeed in college and pursue careers in science, technology, engineering, and mathematics.

area of study for such research may connect knowledge of previous courses in an advanced manner. Such research often culminates in both a written paper and an oral presentation. This presentation provides the opportunity for mathematics students to not only study the mathematics, but write and speak about their results in the fashion conventional to the discipline.

Separate from major program requirements, research in mathematics at the undergraduate level can also be performed at National Science Foundation (NSF) programs, such as Research Experiences for Undergraduates (REUs) held at various schools across the country, often during the summer months. These opportunities allow students to become actively involved in current mathematical research projects under the guidance of faculty, and thus demonstrate how mathematical research is done and how it differs from research done in other fields. Programs such as REUs demonstrate how the activities of a professional mathematician are performed, including the various stages: formulating and solving a problem, writing a mathematics paper, communicating the results in a talk or poster (perhaps at a local or national mathematics conference), and possibly publishing a research article. The topics of study in REUs go beyond the standard undergraduate curriculum and also draw upon previous coursework and experience. By conducting research before they graduate from college, students get a taste of what happens in graduate school programs in mathematics, specifically the research component of the dissertation requirement.

Two-Year Colleges

A significant percentage of students who receive a bachelor's degree in the mathematical sciences have taken some of their mathematics courses at two-year colleges. While many college students may fulfill their general education requirement in mathematics by taking such courses at a two-year college prior to attending a four-year college or university, many potential mathematics majors complete a variety of mathematics courses that satisfy requirements in the major program. Such courses include developmental mathematics, precalculus, introductory calculus, multivariable calculus, linear algebra, differential equations, discrete mathematics, and statistics. While an associate's degree in mathematics may not be obtainable from a two-year college, it is becoming more common that future mathematics majors are beginning their mathematics career at these schools, including students who are also preparing to become mathematics teachers at the various school levels.

Technology

With the advances made in science and technology during the latter half of the twentieth century, many new instructional techniques are being designed and utilized in the mathematics classroom at all levels, including the collegiate. With an emphasis on critical thinking and deductive reasoning and a movement away from rote memorization of mathematical theories and algorithms, there has been an increase in the use of technology for teaching and learning advanced mathematics. Accurate visualization of graphs and geometric objects and easy manipulation of algebraic constructs are some of the benefits of current technology available for mathematics education.

Computational technology changed rapidly during the latter part of the twentieth century. At the beginning of the twenty-first century, computer algebra systems (CAS), such as Mathematica, MATLAB, and Maple, are often helpful tools for both in-class demonstrations and independent student assignments. These software packages are commonly implemented in a variety of courses, such as calculus, linear algebra, differential equations, statistics, real analysis, and complex analysis.

Other software packages, such as Geometer's Sketchpad and Exploring Small Groups (ESG), are more course-specific to geometry and group theory, respectively. In addition to desktop or laptop computer technology, the development of handheld graphing calculators, such as the various models produced by Texas Instruments (TI-83+, TI-84, TI-86, and TI-89), has also influenced the use of this technological tool in the classroom. Computer programs and graphing calculators are also being used at the secondary school level, and the transition to using such technology in the mathematics classroom at the collegiate level is often a smooth experience for the mathematics student.

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See Also: Calculus and Calculus Education; Mathematics, Applied; Mathematics, Theoretical; Pythagorean School.

Curriculum, K–12

Category: School and Society.
Fields of Study: All.
Summary: Curricular standards of mathematics have undergone a series of changes in the twentieth and twenty-first centuries in response to various national concerns.

The term "curriculum" has been variously defined as a coherent program of study in a specific subject area consisting of a set of courses and learning experiences provided by an educational institution. Since the beginning of the twentieth century, the school curriculum (grades K–12) of the United States has undergone numerous changes, particularly in mathematics. Indeed, the U.S. school curriculum has been more profoundly influenced by the demands of society than any other country in the world.

Twenty-first-century debates over the school curriculum are essentially debates on how best to prepare students to live in a just, democratic society; be competitive in a global economy; and thrive in a technologically literate workforce. Employers in both explicitly technical and nontechnical fields seek candidates who can use mathematics content, logical reasoning, and other problem-solving skills that are ideally acquired by the end of high school. Both government and private employers have publicized the various requirements for different career paths. Differences of opinion often stem from opposing views about the nature of learning, the needs of society, and the purposes of schooling. The evolution of the school curriculum in the United States and the changing nature of its mathematics component are most easily understood from an historical context.

Change in the Mathematics Curriculum

Since the inception of the United States, the predominant view has been that education is necessary for the common good of society and the survival of democracy. Although mathematics has always been recognized as an essential component of the K–12 curriculum, the role of mathematics in public schools and the nature of its content have fluctuated over the years. Benjamin Franklin was one of the nation's first leaders to understand the need for mathematics instruction beyond basic arithmetic and measurement. For example, in 1751, he helped institute an academy in which geometry and algebra were among curricula designed to meet the practical needs of merchants, seamen, builders, and artisans.

Common School Movement

Throughout the 1800s, the curricular and educational trends of Europe influenced the mathematics curriculum of the United States. The focus of the mathematics curriculum was on basic arithmetic skills in the early grades and algebra and geometry in the upper grades. In 1837, the *Common School* movement was instituted by Horace Mann (1796–1859) from Massachusetts. He worked to develop a statewide common-school (public school) system. The philosophy was that education is a major "human equalizer" that balances the social structure of a country. For this reason, Horace Mann is often considered the "father of American public education." The curriculum of the common schools of the 1800s served to reflect the values and needs of a democratic society and instituted free education for all U.S. citizens, making mathematics education much more broadly available

The Progressive Movement

For the majority of the twentieth century, U.S. educators consistently promoted a "Progressive Education" agenda, spearheaded by John Dewey (1859–1952). Progressive educators believed that the school curriculum should be determined primarily by the needs and interests of children. Dewey advocated a school curriculum that encouraged students to be thinkers and problem solvers. He encouraged instructional methods that were experiential and child-centered, covering content arising naturally within the child's environment. This method is in contrast to traditional instruction which is usually classroom-based and teacher-centered, covering predetermined content. During the Progressive movement, mathematics instruction emerged primarily when needed within the real-life experiences of the child and was thus widely varying

By the 1940s, an alternate version of Progressive Education called "Life Adjustment" had gained popularity among some U.S. schools. The curriculum of Life Adjustment schools was designed to prepare many of the students for the working world and everyday living, though some opponents claim it was motivated by anti-intellectual philosophies. These students focused on practical concerns, such as home budgeting, consumerism, taxation, health, and citizenship, and mathematics courses, such as algebra, geometry, and trigonometry, were deemphasized.

By the end of World War II, rapid societal and technological changes abruptly came to the forefront. Public knowledge of the impact of atomic energy, radar, cryptography, and other scientific and technological advances underscored the need for a strong national curriculum in mathematics and science to maintain national security, to retain the nation's lead in technology, and to prepare students for jobs in the sciences. As a consequence, Progressive Education came under severe attack following World War II.

New Math

A momentous event occurred in 1957 that impacted the nation's mathematics and science curricula at all levels. U.S. society was stunned by the launching of the first space satellite, Sputnik, by the Soviet Union. Sputnik was considered a national embarrassment and a potential security threat. Its mere existence suggested that the Soviet Union was technologically superior and had a military capacity of launching offensive missiles at the United States. It also underscored an overall weakness in the U.S. educational system, particularly in mathematics, science, and technology.

The U.S. Congress responded to the nation's panic and the implications of a security threat by passing the 1958 National Defense Education Act, intended to increase the quantity and quality of mathematics and science professionals. That same year, the American Mathematical Society (AMS) established the School Mathematics Study Group (SMSG), headed by Edward G. Beg1e of Yale University, to develop a new mathematics curriculum for the nation's high schools. The aim was to produce the most highly capable mathematics students in the world, with a view toward regaining the nation's technological superiority and bolstering its defense system against the Soviet Union. This marked the beginning of the New Math movement.

Funded by the National Science Foundation, the SMSG created a new, more-rigorous high school mathematics curriculum for college-bound students and wrote textbooks supporting the new curriculum. The SMSG curriculum was developed by mathematics professionals consisting of working mathematicians, university professors, high school teachers, and school supervisors. The SMSG soon expanded its curriculum to include mathematics for grades K–12. Similar mathematics curricula emerged in the early 1960s, modeled after the original work of SMSG. These curricula were products of other federally funded projects such as the Ball State Project, Greater Cleveland Mathematics Program, the University of Maryland Mathematics

U.S. Mathematics Curriculum Evolution

The evolution of the mathematics curriculum in the United States has been unstable. Over the years, it has responded to the demands of society, professional educators, and national organizations, often at the expense of the needs of the country. While unpopular in some segments of society, the New Math movement responded favorably to the national panic following the Soviet Union's launch of Sputnik. The New Math movement was responsible in large measure for regaining the nation's international lead in technology and winning the race to the moon. Since the 1980s, the National Council of Teachers of Mathematics has contributed significantly to the nation's mathematics curriculum by developing a series of well-articulated standards that have informed the curriculum and assessment strategies for every state.

Student assessment has also gained a more prominent role in federal funding and curriculum development. Specifically, the success of states in meeting national education standards will be measured by students' performance on high-stakes tests. A positive contribution of the Common Core State Standards, initiated by the nation's state governors, will be the establishment of a national school curriculum. According to U.S. Secretary of Education Arne Duncan, "For the first time, a child in Mississippi and a child in Massachusetts will be judged by the same yardstick."

Project, and the Minnesota School Science and Mathematics Center. Each curriculum mirrored the rigorous mathematics content and educational philosophy of the New Math movement.

The New Math curriculum included advanced content that had never before been covered in public schools, such as set theory, Boolean algebra, base arithmetic, field axioms, algebraic structures, and formal math language and symbolism. The curriculum was designed to provide the theoretical foundations for studying calculus and abstract algebra in college, with the intent of producing as many mathematics, science, and engineering majors as possible.

Even though there were numerous successes in the New Math movement, after a decade of implementation, it was slowly removed from the nation's public schools. Some believed its downfall was a result of excessive rigor and mathematical formalism at the expense of basic skills and problem solving. Many parents and school administrators were confused by the unfamiliar mathematics content and advanced symbolism. More importantly, a large number of the nation's older teachers were unable to implement the New Math curriculum because they, themselves, were not academically prepared to deal with the content. Notwithstanding, 12 years following the launching of Sputnik, the United States succeeded in placing the first man on the moon with its 1969 Apollo 11 mission.

Back to Basics

By the early 1970s, the New Math movement was over. The National Science Foundation discontinued its funding of New Math programs, and the U.S. public called for a return of a "Back to Basics" curriculum, under a Progressive agenda. Mathematical rigor and advanced symbolism were discouraged; teachers experimented with child-centered instructional approaches, such as Individualized Instruction, Continuous Progress, and Open School environments. The failures of such practices were soon exposed as standardized mathematics test scores of U.S. students steadily declined throughout the decade.

Fueled by the failure of the Back to Basics movement, the lowering of college entrance requirements, and reduced enrollments in higher-level mathematics courses, by the end of the 1970s, the U.S. school curriculum was once again under severe public attack. Two publications by U.S. federal agencies had a significant impact on public perceptions of the U.S. educational system, *An Agenda for Action* and *A Nation at Risk*.

An Agenda for Action

Based on multiple national assessments, the National Council of Teachers of Mathematics (NCTM) produced the 1980 *An Agenda for Action*, which provided recommendations for reform in mathematics education. Its primary recommendation was that problem solving should be the primary focus of the mathematics curriculum, supported by the following instructional practices:

- Calculators and computers should be used in K–12 classrooms.
- Estimation and approximation should be an integral part of instruction.
- Team efforts in problem solving should be encouraged in the elementary classroom.
- Manipulatives should be used to develop new mathematical concepts and skills.
- Instructional strategies should provide for situations requiring student discovery and inquiry.
- Mathematics programs and student performance should be evaluated on a broader range of measures than conventional testing.

A Nation at Risk

Although *An Agenda for Action* provided innovative and lofty recommendations for reform in the school mathematics, it was overshadowed by the 1983 publication of *A Nation at Risk,* a report by the National Commission on Excellence in Education. In graphic terms, it warned Americans, "The educational foundations of our society are presently being eroded by a rising tide of mediocrity that threatens our very future as a nation and a people," and, "If an unfriendly foreign power had attempted to impose on America the mediocre educational performance that exists today, we might well have viewed it as an act of war."

A variety of educational issues and specific weaknesses in the mathematics curriculum were addressed. Specifically, the commission found that the textbooks used for instruction were void of rigorous content, the curriculum lacked continuity and depth, and high school teachers were typically underprepared in mathematics and academically weak. Despite out-dated information, *A Nation at Risk* is still often quoted in the twenty-first century and remains an influential publication.

National Standards-Based Curriculum

The publication of *A Nation at Risk* and similar reports of the dismal performance of U.S. students on international assessments have all served to provoke U.S. society and government to demand higher academic standards in public schools. International assessments provided strong evidence that mathematics teaching and its school curriculum must change if U.S. students are to be competitive in the global economy and able to deal with the complex decisions they will confront as responsible citizens and members of a technologically literate workforce.

In 1989, NCTM took a giant step in recommending a national agenda for curriculum reform, resulting in NCTM's 1989 publication of *Curriculum and Evaluation Standards for School Mathematics*. This document initiated a national standards-based curriculum movement, influenced by its earlier work reported in An Agenda for Action.

Within five years, NCTM also produced two supporting documents: the 1991 *Professional Standards for Teaching Mathematics* and the *1995 Assessment Standards for School Mathematics*. These documents recommended teaching standards, instructional methodologies, and an array of assessment strategies for accommodating the new standards-based curriculum.

An updated version of NCTM's original 1989 *Standards* was published in 2000, having a new title: *Principles and Standards for School Mathematics*. These three standards documents continue to be profoundly influential in the twenty-first century in matters of curriculum and assessment decisions for U.S. school mathematics. Specifically, Principles and Standards for School Mathematics provides six principles for school mathematics, five process standards, and five content standards.

The six principles for school mathematics are as follows:

- Equity—high expectations and strong support for all
- Curriculum—courses and learning experiences focused on important mathematics, well articulated across the grades
- Teaching—instruction that is challenging, supportive, and focused on what students know and need to learn
- Learning—develops understanding by building new knowledge on students' experiences and prior knowledge
- Assessment—provides useful information to teachers and students and supports the learning of important mathematics
- Technology—essential for teaching and learning mathematics, influences the mathematics that is taught, and enhances student learning

The five process standards considered essential for teaching all mathematics, are problem solving, reasoning and proof, communication, connections, and representation. These processes were expected to be integrated into the teaching of all of mathematics, regardless of the topic or the grade level.

The five content standards include each of the following: number and operations, algebra, geometry, measurement, and data analysis and probability. Each content area is expected to be covered to some degree of integrity at every grade level.

Overall, given the breadth of mathematics applications found in modern society, both in work and everyday life, schools were encouraged to widen their mathematics offerings. For example, one recommendation for the high school curriculum was that calculus should not always be the primary goal for the mathematics curriculum. Instead, discrete mathematics, probability, and statistics should also be considered valuable goals. The curriculum must prepare students for a variety of career paths that use mathematics; for example, actuarial science (probability), engineering and electronics (technical mathematics), economics and behavioral science (statistics and decision theory), theoretical or nuclear physicist (calculus), and numerous others.

In 2006, NCTM released another supporting document, *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence*, which articulated the specific topics that should be the focus for each grade level pre-K–8. The curriculum focal points acknowledged that NCTM's five content standards are not equally weighted and should have greater emphasis at different grade levels. These topics are identified in this document.

In 2009, NCTM released *Focus in High School Mathematics: Reasoning and Sense Making*. This publication was designed to provide teachers with curriculum guidance and content focal points for high school mathematics, modeled after NCTM's 2006 pre-K–8 document, *Curriculum Focal Points.* In this 2009 document, NCTM stresses that reasoning and sense making should be the focus of all high school mathematics, spanning all content areas, and evident in the teachers' instructional strategies and assessment practices. The goal is for mathematics to be viewed as a logical, problem-solving tool, rather than a set of meaningless procedures, disconnected from everyday life and decision making. It is stressed that students should have experiences with reasoning and sense making within a broad curriculum that may deviate from the textbook. Such experiences should be designed to meet students' future needs and prepare them for citizenship, the workplace, and future careers.

Twenty-First-Century Mathematics Curriculum

History has shown that as national needs and societal perceptions change, so does the school curriculum. Pervasive and radical changes have occurred throughout the world since the 1990s, such as genetic engineering, nanotechnologies, global economies, environmental disasters, global warming, depleting energy sources, and countless others. It is clear that U.S. citizens must be prepared to deal creatively and competently with a multitude of rapid changes and to tackle complex problem situations. The school curriculum must respond accordingly to provide students with the content knowledge, problem-solving skills, and learning experiences that are necessary for students to meet these immense challenges.

Unfortunately, the U.S. mathematics curriculum still has a long way to go in preparing students to meet these challenges. In fact, international assessments report serious deficiencies in the mathematical performance of U.S. students. In 2009, the Programme for International Student Assessment (PISA) reported that 15-year-olds from the United States ranked 18 among 33 developed nations in mathematical literacy and problem solving. In sum, compared to other developed nations, students from U.S. schools score in the lower 50th percentile in mathematics. Furthermore, the National Center for Education Statistics reported in 2003 significant racial achievement gaps in the United States. Societal concerns for economic stability, national security, and equity in instruction all demand immediate and substantial reforms in the U.S. mathematics curriculum and educational system as a whole.

No Child Left Behind

Research has shown that the school curriculum is closely tied to assessment. One governmental attempt to address the school curriculum and the lagging academic achievement of U.S. students is the No Child Left Behind (NCLB) Act of 2002. NCLB includes a number of mandates designed to promote significant gains in student achievement and to hold states and schools accountable for meeting curricular goals. NCLB supports a standards-based curriculum and was founded on the belief that setting high standards and measurable objectives would result in improved teaching and learning in the nation's schools.

As a provision for federal funding, the NCLB Act requires that states develop assessments in basic skills for students at specific grade levels and that each state set its own curriculum, content standards, and achievement benchmarks. The Act further mandates that 100% of the students in each school be proficient in reading and mathematics by the year 2014. As a consequence, any school not showing significant progress toward meeting these goals will be subject to sanctions, culminating in the closing of the school and termination of the faculty and staff.

The underlying theory is that schools will show significant improvement if children in grades 3–8 are held accountable for their academic achievement, as measured by their test scores every year. As of 2011, more than 10,000 schools have been labeled as "failures"; thousands of teachers have been fired; and numerous schools, heretofore considered "very good," are being forced to close. To meet NCLB goals, many schools have eliminated studies in art, history, science, foreign languages, physical education, and geography from their offerings. The majority of school time is now devoted to preparing students for high-stakes tests in the basic skills, the results of which will determine if the school remains open for the following year.

The consequences of students' test performance are so punitive that some districts have experienced record amounts of cheating. Some states have even lowered the passing score on their annual mathematics exams to increase the pass rates for their schools. Reactions such as these to the mandates of the NCLB Act underscore the fact that testing alone will not increase student achievement nor improve instruction. Regardless of how well the curriculum is constructed, meaningful instruction will be abandoned for the sake of test preparation.

Common Core State Standards

Several of the nations to which the United States is often compared academically do have national curricula, such as Great Britain, Germany, France, and Japan. Even though NCTM has provided national guidelines for mathematics education, until 2010, nearly every state had its own unique set of mathematics standards and curriculum for each grade level. In some cases, decisions about curricula were made by county and local school districts and boards. Consequently, state mathematics standards have varied considerably from state to state, and valid comparisons are difficult to make with respect to student performance. Because of the absence of a common set of standards among states, 48 of the nation's state governors and their chief school officers set forth to create the Common Core State Standards (CCSS), released in 2009.

The CCSS were developed in collaboration with content experts, college professors, public school teachers, school administrators, and parents. They are designed for a curriculum that includes rigorous content and applications; requires high-order thinking skills; and prepares students to succeed in a global economy. They are also aligned with the mathematics curricula of top-performing countries in the world. As of 2011, 41 of the 50 states have adopted CCSS.

Race to the Top

The rapid adoption of the CCSS by nearly every state in the nation was surely spurred by the Race to the Top (RTTT) program funded the Educational Recovery Act of 2009. RTTT is a $4.35 billion U.S. Department of Education program offering competitive grants designed to promote educational reforms in state education. The underlying federal agenda is to establish national standards, tests, and curricula. Even though the principle of states' rights ensure that individual states have total control over their educational systems, the promise of RTTT's discretionary funding of hundreds of millions of dollars is a huge incentive for states to adopt the CCSS, which is prerequisite to RTTT funding. When states receive RTTT discretionary federal funding, they must agree to implement the CCSS as well as comply with other stipulations.

Consistent with NCLB, state assessments for RTTT's funding are highly reliant on students' test scores as the sole measure of student achievement. Additionally, many states intend to use students' test scores to evaluate their teachers' performance and determine salaries and bonuses. As of 2011, there is also a rising movement among state governors to push for an end to teachers' unions, tenure, and rights to due process, many of which have existed since at least the early twentieth century. It appears that if these movements continue in the twenty-first century, teachers will soon have no organized voice for addressing teaching conditions, budgetary concerns, or program and curricular issues.

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See Also: Careers; Competitions and Contests; Curricula, International; Curriculum, College; Educational Manipulatives; Educational Testing; Government and State Legislation; Learning Models and Trajectories; Math Gene; Mathematicians, Amateur; Mathematics Literacy and Civil Rights; Professional Associations; Schools.

Curves

Category: History and Development of Curricular Concepts.
Field of Study: Algebra; Calculus; Communication; Connections; Geometry.
Summary: Curves have many different definitions and applications in various fields of mathematics.

Intuitively, a curve might be thought of as a path, like that of a curveball. A curve is viewed and defined in several ways depending on the branch of mathematics. A curve can be defined as the one-dimensional continuous trajectory of an object in space moving in time, the intersection of two surfaces in space, the image of the unit interval under a continuous function, or the graph of a solution of a polynomial equation. Each of these approaches captures the intuitive idea of a curve in their respective domains; the first is more physical, the second is geometric, the third is topological, and the last is an algebraic view of a curve. Curves can be used to create figures, model paths of motion, or express relationships between variables.

There are many types of curves that are the focus of classroom investigations, including yield curves, which are important to investors, and the normal distribution or bell-shaped curve. Felix Klein is noted to have said, "Everyone knows what a curve is, until he has studied enough mathematics to become confused through the countless number of possible exceptions." In education, a "learning curve" is a phrase that is meant to informally capture the notion of the change in knowledge over time. Algebraic and geometric curves are also important in school. Children study lines and circles in primary and middle school. They investigate their lengths and areas. By high school and college, they learn about parametric equations of curves and the area under a curve. In order to enrich classroom learning, mathematicians and mathematics historians created the National Curve Bank Web site.

Early History of the Study of Curves

The Greeks initiated the study of curves and discovered numerous interesting curves. Apollonius of Perga studied conic sections as the intersections of a plane and a cone by changing the angle of intersection. Diocles of Carystus invented the cissoid curve and used it in his attempts to solve the problem of doubling the cube. Nicomedes invented the conchoid curve and used it in his attempts to solve the problems of doubling the cube and trisecting an angle. Some have noted that aspects in the design of the columns of the Parthenon may resemble a conchoid of Nicomedes, although others present different curves as the model. Canon of Samos invented the spiral that was eventually called the "Archimedean spiral."

This curve was utilized by Archimedes of Syracuse as a method to attempt to trisect an angle and square the circle. The Greek view of curves was geometric, since Greek mathematics was, essentially, geometry-centered. Hence, their study of curves usually was through some elaborate and often ingenious methods of construction. Besides the lack of analytical tools, their insistence of having concrete or mechanical methods of construction, and—more importantly—their attempts to solve some important problems of antiquity that later were shown to be unsolvable by ruler and compass constructions are some of the factors contributing to the Greek concept of curves.

Mathematicians, philosophers, and others introduced and investigated the geometry of many interesting curves long after the ancient Greeks. For example, Nicholas of Cusa lived in the fifteenth century. He is noted as the first of many to explore the cycloid, which was eventually known as the path of a point on a wheel as the wheel rolls along a straight line.

Developments Since the Seventeenth Century

With the introduction of analytic geometry in the seventeenth century, the theory of curves received a new impetus—expressing curves by equations would make their study much easier compared to doing it via elaborate geometrical constructions. Analytic geometry enabled mathematicians to focus on the intrinsic features of curves; discover and investigate new curves; study curves in a more systematic way, leading to their classification into algebraic versus transcendental categories; and apply the results to various physical problems, such as the long-standing problem of determining the orbits of planets or solving the problem of a hanging chain, which was posed by Jacob Bernoulli. Gottfried Leibniz, Christiaan Huygens, and Jacob Bernoulli's brother Johann Bernoulli responded to the elder Bernoulli's challenge with the equation of the catenary. In the eighteenth century, Guido Grandi investigated rhodonea curves that resemble roses and what was later to be known as the Witch of Agnesi, named because of a mistranslation of the example in Maria Agnesi's famous calculus textbook.

Beginning with the seventeenth century, smooth curves have been an intense subject of investigation leading to determination of various features. Smooth curves, like lines, circles, parabolas, spirals, and helices, possess properties that make them amenable to numerous applications besides lacking any jagged behavior. For example, younger students learn that a straight line is the shortest path between two points in the plane, and mathematicians in the seventeenth century wondered about an analog for surfaces. A geodesic curve is locally a minimizing path; as a result, it is important in advanced mathematics and physics classes. Leonhard Euler published differential equations for geodesics in 1732. Mathematicians also investigated the classification of smooth curves. One invariant is the length of a curve. In general, length does not distinguish two different curves. It turns out that two other invariants, called the "curvature" and "torsion," work much better for this purpose. Broadly speaking, at any point on the curve, the curvature measures the deviance of the curve from being a straight line, and the torsion function measures the deviance of the curve from being a plane curve. Furthermore, the fundamental theorem of curves states that these invariants determine the curve, a result that is proved in twenty-first-century college differential geometry classrooms using the Frenet–Serret Formulas. These are named for Jean Frédéric Frenet and Joseph Serret, who independently discovered them in the nineteenth century.

With further investigations by prominent mathematicians, like Carl Friedrich Gauss, Gaspard Monge, Jean-Victor Poncelet, and their students, the theory of curves, particularly smooth curves, matured into an active field of research. The findings in the theory of curves not only enriched the realm of curve studies, they also contributed to the development of new ideas that ended up revolutionizing mathematics in the nineteenth century. Broadly speaking, the general definition of a curve is topological; namely, a curve is defined as a continuous map from an interval to a space. Curves can be algebraic (those defined via algebraic equations). For instance, a plane curve can also be expressed by an equation

$$F(x, y) = 0$$

and a space curve can be expressed by two equations

$$F(x, y, z) = 0 \text{ and } G(x, y, z) = 0.$$

A curve is algebraic when its defining equations are algebraic—a polynomial in x and y (and z). The cardioid, a heart-shaped curve whose Cartesian equation is

$$\left(x^2 + y^2 - 2ax\right)^2 = 4a\left(x^2 + y^2\right)$$

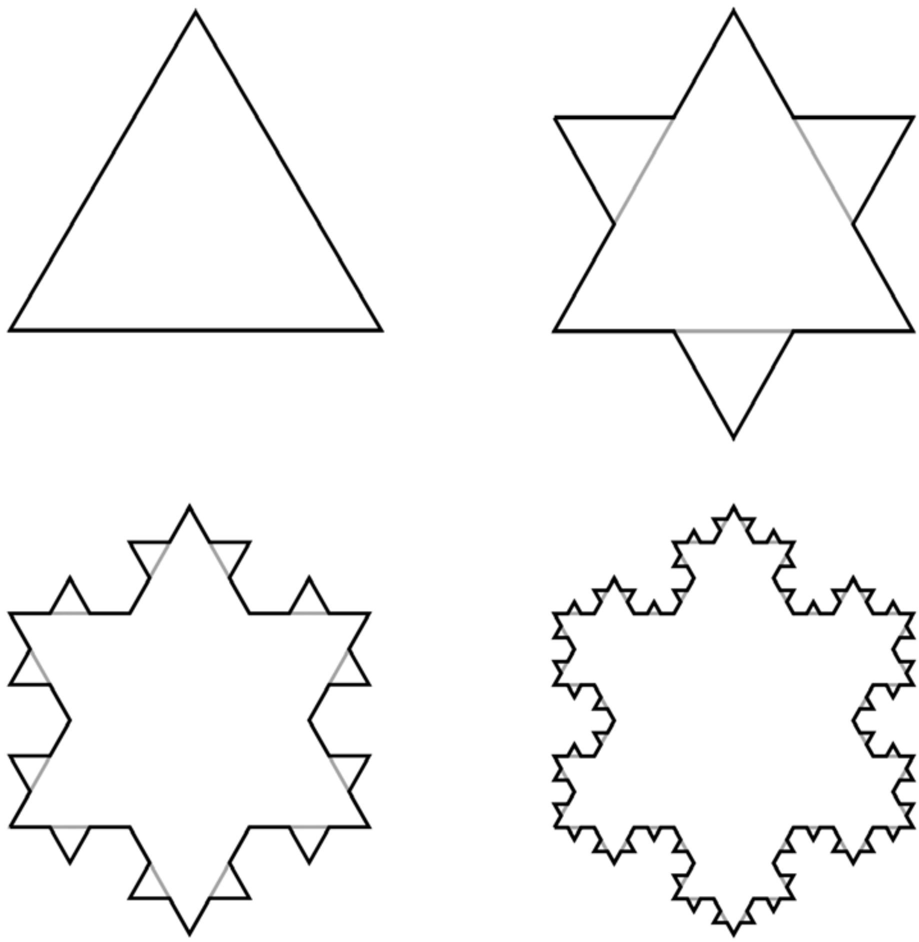
and the asteroid, whose Cartesian equation is

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

where a is a constant, are algebraic curves.

Before analytic geometry, each of these curves had been expressed using geometric investigations; for example, a circle turning around a circle that sweeps out the cardioid, or wheels turning within wheels that form the asteroid. Transcendental curves cannot be defined algebraically and include the brachistochrone curve, also known as the "curve of fastest descent"; very complicated looking fractal curves, such as the Koch snowflake, named for Helge von Koch, who explored the geometry in a 1904 paper; and paradoxical sounding space-filling curves, discovered by Giuseppe Peano in 1890. The last two types of curves can be extremely jagged curves with no smooth components.

An algebraic curve of the form $y^2 = x^3 + ax + b$, where a and b are real numbers, satisfying the relation $4a^3 + 27b^2 \neq 0$, is called an "elliptic curve." Geometrically, this condition ensures that the curve does not have any cusps, self-intersections, or isolated points. On the points of elliptic curves (including the point at infinity), one can define an operation by three



Curves can be extremely jagged curves with no smooth components, such as the complicated Koch snowflake fractal curve named for Helge von Koch who explored the geometry in a 1904 paper.

points sum to zero, if and only if they are collinear. This interesting feature of elliptic curves, besides being an important algebraic structure to be studied on its own, also has found some astonishing applications, such as in cryptography for developing elliptic curve-based public-key cryptosystems. Elliptic curves are also important in number theory; they are effective tools in integer factorization problems. They also turned up as an instrumental tool in the proof of Fermat's Last Theorem, named for Pierre de Fermat.

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See Also: Conic Sections; Limits and Continuity; Normal Distribution; Planetary Orbits; Polynomials.

D

Dams

Category: Architecture and Engineering.
Fields of Study: Algebra; Geometry; Measurement.
Summary: Mathematics is vital to the design, monitoring, maintenance, and safety of dams.

Dams are embankments across a waterway for control of water or for water storage; they have served many functions in societies throughout history. The earliest dams were primarily used for irrigation and as a water source for livestock. Today, smaller dams provide water for livestock, fish and wildlife habitat, and recreation. Larger dams can provide flood control in places below sea level, like New Orleans and the Netherlands; municipal and industrial water supply; irrigation for crops; hydroelectric power; commercial navigation; and recreation. They are typically earthen dams, concrete structures, or some combination. Older dams were sometimes made of timber, masonry, or steel. Mathematicians and engineers investigate many aspects of the construction and maintenance of dams using geometry, trigonometry, and stochastic and limit-state analyses. For instance, Boris Galerkin, who had degrees in applied mathematics and mechanics, studied stress in dams, and Pelageia Polubarinova, who had a degree in mathematics, contributed to the theory of seepage flow of groundwater through porous materials that included earth dams. Some well-known dams are the Itaipú Dam in Brazil and Paraguay, the Hoover Dam in the United States, the Aswan Dam in Egypt, and the Dneproges Dam in the Ukraine.

Considerations for building a dam must take into account both positive and negative impacts. There are a variety of benefits of a dam that are closely related to its uses—providing water supply, flood control, hydroelectric power, and navigation. Hydroelectric power provides an important source of electrical power around the world. Commercial navigation through river systems provides efficient and economical transportation of agricultural products and commercial goods. Many dams that control flood plains provide farmers with an increased crop yield because land that would once have been flooded is now controlled upstream by the dam. Negatively, some dams may hinder fish movement; for example, along some streams, salmon are not able to get back to their native spawning areas because of the dam. Additionally, dams affect the natural order of a stream—its sediment load and flooding characteristics.

Purposes and Design

Dams are constructed with a definite purpose in mind based on the function(s) they are to serve. Dams are built to control watershed areas (all the area upstream of the dam, which provides runoff to the structure).

Engineers use a variety of mathematics skills as they plan, design, construct, and operate a dam. During the planning stage, engineers work with sponsors to scope out the needs and develop a basic design for the structure including design issues such as location, height, and base flow of the structure. Base flow is calculated with the formula $Q = v \times A$ where Q is the base flow rate, v is the velocity of water, and A is the area. Another important part of the planning stage is determining the economic feasibility of building the dam by calculating a benefit-to-cost ratio. Using a mathematical model, both the benefits of the dam over its life and the total cost of building and maintaining the dam are calculated. Ideally, for the construction of a dam to be feasible, the benefit-to-cost ratio needs to be greater than 1.

As a part of the design process, engineers must create detailed blueprints for the structure and an accompanying cost sheet that includes items such as quantities or volumes of a variety of materials (for example, cubic yards of concrete) and the cost of the removal and placement of earthen materials, which can be millions of cubic yards in the case of large dams. During the construction of the dam, the blueprints must be followed with precision and detail to ensure the integrity of the dam. Once the dam is constructed, regular monitoring is important to ensure the most efficient use of the available storage. Engineers monitor the amount of water leaving the dam through its spillway, as well as the amount of water entering the watershed. These inflows and outflows must be balanced in order to maintain storage needs and prevent flooding or low flows in the river downstream.

Safety

A major consideration in the planning, design, construction, and maintenance of any dam is safety. Engineers determine a hazard rating for each dam, with the highest hazard rating dealing with potential loss of human life. A breach in a dam can be catastrophic. A breach in a dam can be caused by a flaw in the design of the structure, extreme rainfall, lack of or poor maintenance of the structure, or a geological occurrence. Regular inspection and maintenance are important to ensure the safety of those downstream from the dam.

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Juliana Utley

See Also: Engineering Design; Floods; Water Distribution.

Data Analysis and Probability in Society

Category: School and Society.
Fields of Study: Connections; Data Analysis and Probability.
Summary: Today, most industries depend on data analysis for some aspect of their work.

Data analysis can be thought of as the process of collecting, transforming, summarizing, and modeling data, usually with the goal of producing useful information that facilitates drawing logical conclusions or making decisions. Virtually any field that conducts experiments or makes observations is involved in data analysis.

There are many mathematical data analysis methods, including statistics, data mining, data presentation architecture, fuzzy logic, genetic algorithms, and Fourier analysis, named for mathematician Joseph Fourier. Probabilistic statistical methods are among the most widely applied tools, and they are what many people think of when they hear the term "data analysis." The use of probability, statistical analysis, and other mathematical data analysis methods is widespread, especially given technological advances and computer software that facilitate rapid, automated data collection and efficient, effective processing of massive data sets. According to forecasts included in the U.S. Bureau of Labor Statistics' 2010–2011 *Occupational Outlook Handbook*, the demand for statisticians and individuals with mathematical data analysis skills is expected to grow. Jobs that involve data collection, probabilistic modeling, statistical data analysis, data interpretation, and data dissemination are found in both the public and private sector, as well as in a diverse array of disciplines, including agriculture, biology, computer science, digital imaging, economics, engineering, education, forestry, geography, insurance, law, manufacturing, marketing, medicine, operations research, psychology, and pharmacology.

Many specialized data analysts are known by job titles or classifications other than statistician, such as actuary, biostatistician, demographer, econometrician, epidemiologist, or psychometrician. In the twenty-first century, both probability and data analysis are components of U.S. primary and secondary mathematics education, usually starting in the earliest grades and continuing through high school. This curriculum has been advocated by the National Council of Teachers of Mathematics in its *Principles and Standards for School Mathematics*, published in 2000, as well as by professional organizations such as the American Statistical Association and the Mathematical Association of America.

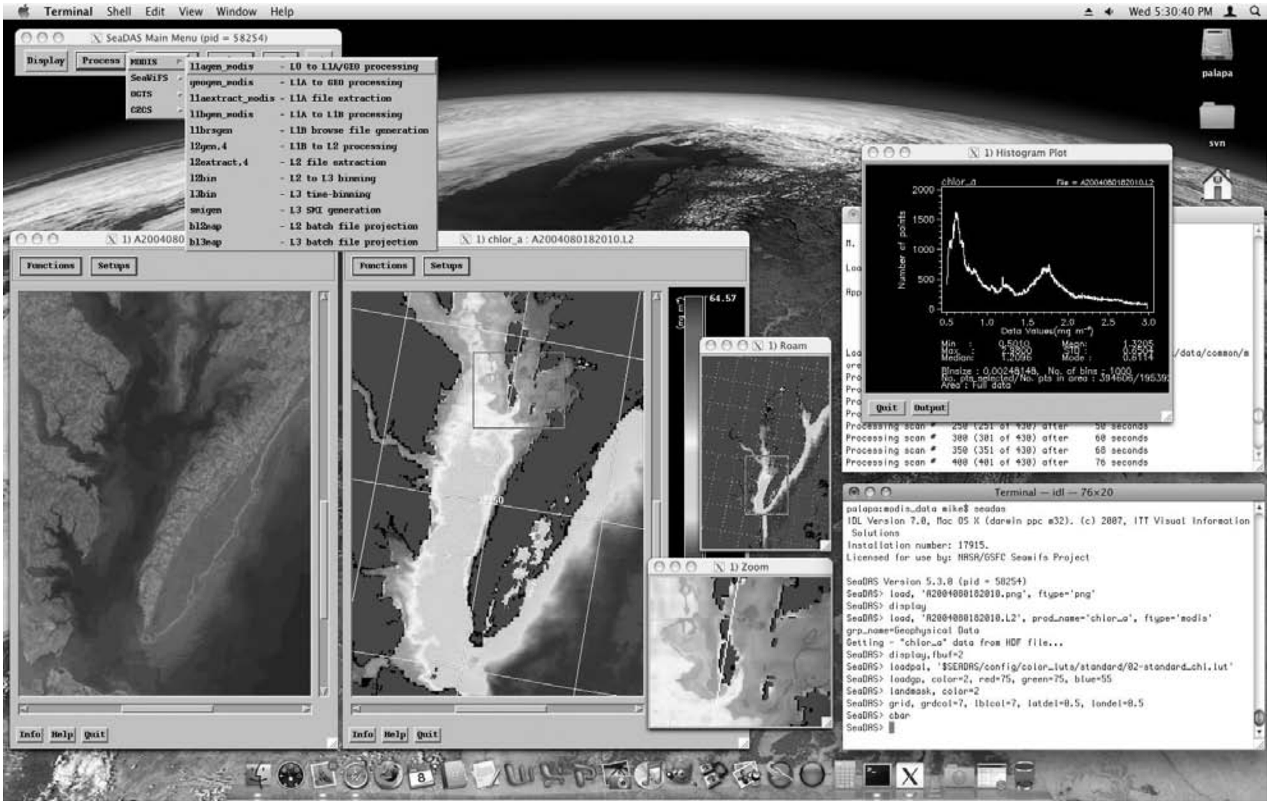
Professional Education

The first college statistics department was founded in 1911 at University College London. Other departments in universities around the world followed. In the twenty-first century, more than 200 colleges and universities in the United States offer undergraduate statistics degrees, and many more schools offer minors and courses in probability and statistics, data mining, and other mathematical data analysis methods. These courses may be taught either in mathematics and statistics departments or, often, in one of many partner disciplines, such as psychology, biology, or business. Graduate degrees in statistics do not necessarily require an undergraduate degree in statistics or mathematics, but most graduate degree programs prefer strong mathematical or statistical backgrounds with courses in areas like differential and integral calculus, mathematical modeling, probability theory, statistical methods, vector analysis, linear algebra, and mathematical statistics. Historically, computational methods were a primary focus of statistics education. With the evolution of technology and the growing role of statistics in everyday life, statistics education has shifted to focus on conceptual understanding, analysis of real data in context, survey sampling and experimental design methods, technology for analysis and presentation, communication of methodology and results to both technical and nontechnical audiences, and statistical thinking or literacy. People with bachelor's degrees in mathematics, statistics, or related mathematical fields, like operations research or decision sciences, can often find entry-level data analysis positions in government and industry, but research-related jobs and teaching at the community college level typically require master's degrees. Teaching or research-related jobs at four-year colleges and universities usually require doctoral degrees. Work experience or qualifying exams, such as those administered by the Society of Actuaries, are often necessary for employment in some industries. Training and certification programs like Six Sigma Black Belt also signify a certain level of data analysis skill and knowledge.

Government

Virtually all federal organizations have data analysis specialists or entire statistical subdivisions that use mathematical and statistical models. Since ancient times, governments have collected data and used mathematical methods to perform necessary functions. Archaeological evidence suggests that many ancient civilizations conducted censuses to enumerate their populations, often for taxation or military recruitment. Livestock, trade goods, and other property were sometimes counted in addition to people. Mathematics facilitated decisions regarding the distribution of resources like land, water, and food. The German word for this process of "state arithmetic" is cited as the origin of the English word "statistics," which first appeared in *Statistical Accounts of Scotland*, an eighteenth-century work by politician John Sinclair that included data about people, geography, and economics. In the United States, counting of the population is required by the U.S. Constitution, and congressional representation for the U.S. House of Representatives is determined by the decennial census population values.

Over the decades, many mathematicians and statisticians worked on planning and implementing the census, like Lemuel Shattuck, who also co-founded the American Statistical Association in 1839. Since its creation in 1902, the duties and activities of the U.S. Census Bureau have grown beyond the mandated 10-year census to include collecting and analyzing data on many social and economic issues, and the U.S. Census Bureau is one of the largest employers of mathematicians and statisticians in the country. At the start of



The SeaWiFS Data Analysis System at the National Aeronautics and Space Administration (NASA) is an image analysis package for the processing, display, analysis, and quality control of ocean color data.

the twenty-first century, various agencies of the U.S. government employed approximately 20% of the statisticians in the country. An additional 10% were employed by state and local governments, including state universities.

Statisticians and other mathematical data analysts working within many federal agencies are also responsible for developing new and innovative methods for gathering, validating, and analyzing data, especially the massive, messy, or incomplete data sets that are increasingly common in technological and industrialized societies. They also work to reduce bias and more accurately model issues that affect individuals and organizations. Many countries and governing entities around the world have agencies that perform similar functions. One major area of interest for most governments is the economic health of the country and the well-being of its workers. In the United States, the Bureau of Labor Statistics measures and forecasts factors such as labor market activity, productivity, price changes, spending, and working conditions. They began collecting data at the federal level in 1884. The *Current Population Survey*, implemented by the U.S. Census Bureau, is a monthly survey of about 50,000 households that has been conducted for more than 50 years, and the *Current Employment Statistics Survey* gathers data from about 410,000 worksites to summarize variables such as hours worked and earnings.

While the Bureau of Labor Statistics focuses mostly on manufacturing and services, the U.S. Department of Agriculture's Economic Research Service, established in 1961, is responsible for data about farming, natural resources, and rural development, addressing issues like food safety, climate, farm employment, and rural economies. Its online *Food Environment Atlas* includes indicators that describe the U.S. "food environment" and model concepts like people's geographic proximity to grocery stores or restaurants and food prices. The National Agricultural Statistics Service, also established in 1961, conducts the *Census of Agriculture*. It can be traced in part to a 1957 Congressional decision to approve probability survey methods for agriculture research. The U.S. Internal Revenue Service's Statistics Income Division, created in 1916, was among the first federal agencies to use stratified random sampling and machine summarization of data, both in the 1920s. In the twenty-first century, it assesses the tax impact of federal legislation.

Beyond their workforces, governments are also typically interested in the overall health, safety, and education of members of the broader society. The U.S. National Center for Health Statistics, established in 1960, compiles public health statistics, tracks federal health initiatives, and helps assess trends related to health care and health behaviors. For example, it has monitored efforts to reduce obesity and teen pregnancy. Other data include health care delivery and changes, such as the use of prescription medications and emergency rooms. The Bureau of Justice Statistics, founded in 1980, is primarily responsible for crime and criminal justice data collection, analysis, and dissemination in the United States. One of its principal reports is the annual *National Crime Victimization Survey*. The Federal Bureau of Investigation, founded in 1908, creates the annual *Uniform Crime Report*. The National Center for Education Statistics is mandated by the 2002 Educational Sciences Reform Act to collect and analyze "statistics and facts as shall show the condition and progress of education in the several states and territories" of the United States. The U.S. Congress uses data from this agency to plan education programs and to apportion federal funds among states.

In the twentieth century, issues like the energy crisis of the 1970s, climate change, and concerns over the future availability of oil focused more attention on U.S. energy resources and infrastructure. The Energy Information Administration (EIA) was established in 1977 to independently and impartially collect and analyze data to disseminate information about energy resources, uses, infrastructure, and flow, as well as their impacts on and responses to economic and environmental variables. The goals are to assist in creating policies and making energy decisions as well as educating the public about all aspects of energy. The EIA's "Energy Kids" Web site contains educational materials for primary and middle school students, and its "Energy Explained" Web site is aimed at the older students and the general public.

While government is one of the largest producers and users of statistics, not everyone agrees on their validity or utility. Many have criticized politicians for selectively using or deliberately misusing data and statistics, while others have suggested that the issue is insufficient training or understanding of mathematical data analysis—though statistical methods are increasingly part of political science degree programs. Former North Carolina Representative Lunsford Richardson Preyer once said: "Statistics do not always lie, but they seldom voluntarily tell the truth. We can argue any position on this bill on a set of statistics and some study or another." At the same time, some propose that effective democracy depends on citizens being able to access and understand current statistics. The burden and responsibility to produce credible information then rests with both the public, which has an obligation to provide valid data and seek to understand the outcomes, and the government, which must collect, analyze, and publicize information in a reliable, timely, and nonpartisan manner.

Industry and Manufacturing

The notion of interchangeable parts—pioneered by individuals like eighteenth-century army officer and engineer Jean-Baptiste Vaquette de Gribeauval and inventor Eli Whitney—followed by the mass production of goods during the Industrial Revolution, ushered in a new era of data collection and analysis to ensure the quality of manufactured products. In the early twentieth century, physicist and statistician Walter Shewhart pioneered data analysis methods in manufacturing that led some to call him the "father of statistical quality control." Among other accomplishments, he developed specialized charts using data and probability to sample and track the variability in processes to identify both natural, random process deviations and nonrandom deviations in order to eliminate the latter and thus improve consistency in the product.

W. Edwards Deming expanded on these notions to help develop the industrial management practice known as "continuous quality control" or "continuous quality improvement." Deming is credited with significant contributions to Japan's post–World War II reputation for high-quality products, and his data-based control methods have been widely adopted in the United States. For example, Motorola's Six Sigma program, founded in the 1980s, focused on training managers and employees at various levels in statistical methods and practices designed to identify and remove causes of product defects with the overall goal of minimizing process variability. The program name derives from statistical notation: sigma (σ) is commonly used to represent standard deviation, a measure of variability. Six standard deviations on either side of the mean in a bell-shaped or normal curve encompasses virtually all of the data values. If there are six standard deviations between the process mean and the nearest product specification limit, only three or four items per million produced will fail to meet those specifications. General Electric and other companies adapted and evolved the original Six Sigma ideas by merging them with other management strategies. For example, in the 1990s, concepts from a manufacturing optimization method known as "lean manufacturing" resulted in a hybrid program called "Lean Six Sigma."

Data analysis and probability are also used in advertising and market research. Many of the common market research practices used in the twenty-first century are traced to the work of engineer and pioneer television analyst Arthur Nielson. These practices include data analysis to quantify market share and determining sales patterns by combining consumer surveys with sales audits.

Medicine and Pharmacy

In the nineteenth century, some in the medical community began to investigate the idea of using data analysis for medical applications. Physician William Farr applied data analytic methods to model epidemic diseases. He is often credited as the founder of epidemiology. Physician John Snow gathered data to trace the source of an 1854 cholera outbreak in London. Along with his census work, Shattuck helped implement many public health measures based on data analyses. Florence Nightingale invented her own graphical data presentations in order to summarize data on the health impacts of poor hygiene in British military hospitals. In the twenty-first century, agencies like the U.S. Centers for Disease Control and Prevention and the World Health Organization collect, analyze, and model data in order to, among other goals, track the spread of infectious disease; assess the impact of preventive measures, like vaccinations; and test the virulence of infectious agents.

Clinical trials or experiments are also performed to determine the effectiveness and safety of new medical procedures and drugs. In the eighteenth century, physician James Lind tested remedies for scurvy aboard a British navy ship, which can be cited as one of the first recorded cases of a controlled medical trial. Statistician and epidemiologist Austin Bradford Hill helped pioneer randomized, controlled clinical trials in the twentieth century and also worked to develop the Bradford-Hill criteria, a set of logical and mathematical conditions that must be met to determine causal relationships. Approval and patenting of pharmaceuticals and medical devices by federal agencies like the Food and Drug Administration, part of the U.S. Department of Health and Human Services, require extensive experimentation and data analysis. For example, when a television commercial for a drug states that it is "clinically proven," this usually means that it has gone through experimental testing and that appropriate analyses of data have determined that it is very probably effective and safe, according to measures like the Bradford-Hill criteria.

Finance and Insurance

Probability is essential for quantifying risk, a concept that underlies most financial ventures and drives interest, credit, loan, and insurance rates. Data analysis can be used to derive probabilities and create financial models or indices like Fair Isaac Corporation (FICO) scores, the Dow Jones Industrial Average, and nations' gross domestic products. Engineer and economist William Playfair is considered to be one of the creators of graphical data analysis. Beginning in the eighteenth century, he researched trade deficits and other types of economic and financial data. Mathematician Louis Bachelier is known as the "father of financial mathematics" for his use of Brownian motion to model stock options at the turn of the twentieth century. Brownian motion, named for botanist Robert Brown, is a stochastic (probabilistic or random) process. The international Bachelier Financial Society is named for Louis Bachelier. Its goal is "the advancement of the discipline of finance under the application of the theory of stochastic processes, statistical and mathematical theory," and it is open to individuals in any discipline. Actuarial scientists or actuaries are also widely employed to develop models of the financial impact of risk. For example, they may use a combination of theoretical probability and data analysis to determine appropriate premiums for life or health insurance using variables such as life expectancy, which is adjusted for characteristics or behaviors that modify risk, like gender or smoking. Astronomer and mathematician Edmund Halley, for whom Halley's

Comet is named, is also often cited as the founder of actuarial science. He calculated mortality tables using data from the city of Breslau, Germany (now Wrocław, Poland). Published in 1693, these tables are the earliest known works to mathematically quantify the relationship between age and mortality.

Entertainment and Gambling

Archaeological evidence suggests that games of chance have existed since antiquity. Probability appears in different forms in written works throughout the centuries, like the body of Talmudic scholarship and the 1494 treatise of mathematician and friar Luca Pacioli known as *Summa de arithmetica, geometria, proportioni et proportionalita*. The mathematical study of probability as it is known in the twenty-first century is traditionally traced to seventeenth-century mathematicians Blaise Pascal and Pierre de Fermat, who were inspired to formulate their mathematical "doctrine of chances" by problems in gambling. In the twenty-first century, gambling is a multibillion dollar industry. In Las Vegas and other places, oddsmakers use probability to determine risks, point spreads, and payoff values for games of chance, sporting events, and lotteries. Players often use betting systems that are based on data analysis or probability to attempt to beat the odds and increase their chances of winning.

One example was a group of students from the Massachusetts Institute of Technology and other schools who used card counting techniques and mathematical optimization strategies in blackjack, which was the basis of the 2008 movie *21* and a television documentary *Breaking Vegas*. The television game show *Deal or No Deal*, which has aired versions in approximately 80 countries around the world, has been studied by mathematicians, statisticians, and economists as a case of decision making involving probability and data analysis concepts, like expected value. Probability-based random number generation is incorporated into many popular video games to increase realism and create multiple scenarios, while moviemakers are exploring probability-based artificial intelligence systems to generate realistic behavior in large, computer-generated battle scenes. The pioneering *Lord of the Rings* movies used a program developed by computer graphics software engineer Stephen Regelous and named Multiple Agent Simulation System in Virtual Environment (MASSIVE), which uses probabilistic methods like

World Statistics Day

The growing importance of statistical data analysis in global twenty-first century society was highlighted by the first World Statistics Day, which was held on October 20, 2010 ("20.10.2010" in common international date notation). In his letter to world leaders, United Nations Secretary General Ban Ki-moon emphasized the importance of data and statistical analysis to the current and future welfare of global society: "Let us make this historic World Statistics Day a success by acknowledging and celebrating the role of statistics in the social and economic development of our societies and by dedicating further efforts and resources to strengthening national statistical capacity." More than 130 countries and areas, as well as professional statistical organizations, universities, and other groups, held celebrations. Several international organizations also hosted a World Statistics Day conference in Geneva, Switzerland. That gathering brought together data analysis professionals from academia, government, and business, along with various end-users of statistics, to discuss the essential role of statistical data analysis in everyday life and in solving humanity's most pressing social, economic, and environmental issues. In the United States, President Barack Obama cited the importance of such methods: "Statistical data drives countless decisions which impact our nation. It guides representation in the United States Congress; informs our economic, social service, and national security outlook; and helps determine where infrastructure like schools, hospitals, and roads should be built.

fuzzy logic, derived from the fuzzy set theory of computer scientist and mathematician Lotfi Zadeh. Most sports collect a wide variety of data about their players, but in the latter twentieth century, advanced mathematical modeling, such as sabermetrics, developed by statistician George William "Bill" James, gained popularity for analyzing player and team performance and making predictions.

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See Also: Baseball; Census; Congressional Representation; Deming, W. Edwards; Diseases, Tracking Infectious; Energy; Expected Values; Forecasting; Gerrymandering; Industrial Revolution; Inventory Models; Life Expectancy; Market Research; Normal Distribution; Probability; Quality Control; Sample Surveys; Statistics Education; Stock Market Indices.

Data Mining

Category: Business, Economics, and Marketing.
Fields of Study: Data Analysis and Probability; Measurement; Number and Operations.
Summary: Data mining is the relatively recent practice of using algorithms to distill patterns, summaries, and other specific forms of information from databases.

Advances in technology in the latter half of the twentieth century led to the accumulation of massive data sets in government, business, industry, and various sciences. Extracting useful information from these large-scale data sets required new mathematical and statistical methods to model data, account for error, and handle issues like missing data values and different variable scales or measures. Data mining uses tools from statistics, machine learning, computer science, and mathematics to extract information from data, especially from large databases. The concepts involved in data mining are drawn from many mathematical fields such as fuzzy sets, developed by mathematician and computer scientist Lotfi Zadeh, and genetic algorithms, based on the work of mathematicians such as Nils Barricelli. Because of the massive amounts of data processed, data mining relies heavily on computers, and mathematicians contribute to the development of new algorithms and hardware systems. For example, the Gfarm Grid File System was developed in the early twenty-first century to facilitate high-performance petascale-level computing and data mining.

History

Data mining has roots in three areas: classical statistics, artificial intelligence, and machine learning. In the late 1980s and early 1990s, companies that owned large databases of customer information, in particular credit card banks, wanted to explore the potential for learning more about their customers through their transactions. The term "data mining" had been used by statisticians since the 1960s as a pejorative term to describe the undisciplined exploration of data. It was also called "data dredging" and "fishing." However, in the 1990s, researchers and practitioners from the field of machine learning began successfully applying their algorithms to these large databases in order to discover patterns that enable businesses to make better decisions and to develop hypotheses for future investigations.

Partly to avoid the negative connotations of the term "data mining," researchers coined the term "knowledge discovery in databases" (KDD) to describe the entire process of finding useful patterns in databases, from the collection and preparation of the data, to the end product of communicating the results of the analyses to others. This term gained popularity in the machine learning and AI fields, but the term "data mining" is still used by statisticians. Those who use the term "KDD" refer to data mining as only the specific part of the KDD process where algorithms are applied to the data. The broader interpretation will be used in this discussion.

Software programs to implement data mining emerged in the 1990s and continue to evolve today. There are open-source programs (such as WEKA, http://www.cs.waikato.ac.nz/ml/weka and packages in R, http://www.r-project.org) and many commercial programs that offer easy-to-use graphical user interfaces (GUIs), which can facilitate the spread of data mining practice throughout an organization.

Types of Problems

The specific types of tasks that data mining addresses are typically broken into four types:

1. Predictive Modeling (classification, regression)
2. Segmentation (data clustering)
3. Summarization
4. Visualization

Predictive modeling is the building of models for a response variable for the main purpose of predicting the value of that response under new—or future—values of the predictor variables. Predictive modeling problems, in turn, are further broken into classification problems or regression problems, depending on the nature of the response variable being predicted. If the response variable is categorical (for example, whether a customer will switch telephone providers at the end of a subscription period or will stay with his or her current company), the problem is called a "classification." If the response is quantitative (for example, the amount a customer will spend with the company in the next year), the problem is a "regression problem." The term "regression" is used for these problems even when techniques other than regression are used to produce the predictions. Because there is a clear response variable, predictive modeling problems are also called "supervised problems" in machine learning. Sometimes there is no response variable to predict, but an analyst may want to divide customers into segments based on a variety of variables. These segments may be meaningful to the analyst, but there is no response variable to predict in order to evaluate the accuracy of the segmentation. Such problems with no specified response variable are known as "unsupervised learning problems."

Summarization describes any numerical summaries of variables that are not necessarily used to model a response. For example, an analyst may want to examine the average age, income, and credit scores of a large batch of potential new customers without wanting to predict other behaviors. Any use of graphical displays for this purpose, especially those involving many variables at the same time, is called "visualization."

Algorithms

Data mining uses a variety of algorithms (computer code) based on mathematical equations to build models that describe the relationship between the response variable and a set of predictor variables. The algorithms are taken from statistics and machine learning literature, including such classical statistical techniques as linear regression and logistic regression and time series analysis, as well as more recently developed techniques like classification and regression trees (ID3 or C4.5 in machine learning), neural networks, naïve Bayes, K-nearest neighbor techniques, and support vector machines.

One of the challenges of data mining is to choose which algorithm to use in a particular application. Unlike the practice in classical statistics, the data miner often builds multiple models on the same data set, using a new set of data (called the "test set") to evaluate which model performs best.

Recent advances in data mining combine models into ensembles in an effort to collect the benefits of the constituent models. The two main ensemble methods are known as "bootstrap aggregation" (bagging) and "boosting." Both methods build many (possibly hundreds or even thousands of) models on resampled versions of the same data set and take a (usually weighted) average (in the case of regression) or a majority vote (in the case of classification) to combine the models. The claim is that ensemble methods produce models with both less variance and less bias than individual models in a wide variety of applications. This is a current area of research in data mining.

Applications

Data mining techniques are being applied everywhere there are large data sets. A number of important application areas include the following:

1. *Customer relationship management* (*CRM*). Credit card banks formed one of the first groups of companies to use large transactional databases in an attempt to predict and understand patterns of customer behavior. Models help banks understand acquisition, retention, and cross-selling opportunities.
2. *Risk and collection analytics*. Predicting both who is most likely to default on loans and which type of collection strategy is likely to be successful is crucial to banks.
3. *Direct marketing*. Knowing which customers are most likely to respond to direct marketing could save companies billions of dollars a year in junk mail and other related costs.
4. *Fraud detection*. Models to identify fraudulent transactions are used by banks and a variety of government agencies including state comptroller's offices and the Internal Revenue Service (IRS).
5. *Terrorist detection*. Data mining has been used by various government agencies in an attempt to help identify terrorist activity—although concerns of confidentiality have accompanied these uses.
6. *Genomics and proteomics*. Researchers use data mining techniques in an attempt to associate specific genes and proteins with diseases and other biological activity. This field is also known as "bioinformatics."
7. *Healthcare.* Data mining is increasingly used to study efficiencies in physician decisions, pharmaceutical prescriptions, diagnostic results, and other healthcare outcomes.

Concerns and Controversies

Privacy issues are some of the main concerns of the public with respect to data mining. In fact, some kinds of data mining and discovery are illegal. There are federal and state privacy laws that protect the information of individuals. Nearly every Web site, credit card company, and other information collecting organization has a publicly available privacy policy. Social networking sites, such as Facebook, have been criticized for sharing and selling information about subscribers for data mining purposes. In healthcare, the Health Insurance Portability and Accountability Act of 1996 (HIPAA) was enacted to help protect individuals' health information from being shared without their knowledge.

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Richard De Veaux

See Also: Data Analysis and Probability in Society; Forecasting; Neural Networks; Predicting Preferences; Statistics Education.

Daubechies, Ingrid

Category: Mathematics Culture and Identity.
Fields of Study: Algebra; Number and Operations; Representations.
Summary: The first female president of the International Mathematical Union, Belgian Ingrid Daubechies revolutionized work on wavelets.

Ingrid Daubechies is a physicist and mathematician widely known for her work with time frequency analysis, including wavelets, and their applications in engineering, science, and art. Some people even refer to her as the "mother of wavelets." In 1994, Daubechies became the first tenured woman professor in the Mathematics Department of Princeton University, and in 2004 she was named the William R. Kenan, Jr. Professor of Mathematics at Princeton. Daubechies has achieved many honors internationally and was the first woman to receive a National Academy of Sciences Award in Mathematics. In 2010, she became the first woman president of the International Mathematical Union.

Daubechies was born in Houthalen, Belgium, in 1954. As a child she enjoyed sewing clothes for dolls, saying about her experiences, "It was fascinating to me that by putting together flat pieces of fabric one could make something that was not flat at all but followed curved surfaces." She also computed powers of two in her head before sleeping, a childhood activity that coincidentally her future husband also engaged in. She had the support of her parents, which she appreciated. Her father, a coal mine engineer, answered her mathematical questions, and she tried to do the same with her own children. She attended a single-sex school and was not exposed to the idea that there might be gender differences in mathematics, saying, "So it didn't occur to me.…Later on, I did meet people who felt or even articulated very clearly that women were less 'suited' for mathematics or science, but by then I was confident enough to take this as a sign of their narrow-mindedness rather than let it influence me." She earned her bachelor's degree in 1975 in physics, and her Ph.D. in physics in 1980 from the Free University (*Vrije Universiteit)* in Brussels, Belgium. She held a research position at the Free University until 1987, when she accepted a position as a member of the technical staff of the Mathematics Research Center at AT&T Bell Laboratories in the United States. She remained at the Bell Labs until 1994, although she took two leaves of absence for research: one for six months at the University of Michigan, and another for two years at Rutgers University

Physicist and mathematician Ingrid Daubechies is the first woman president of the International Mathematical Union.

Wavelet Analysis

Daubechies is best known for her work in wavelet analysis, a cross-disciplinary field that allowed her to combine her interest in mathematics with her training in physics. She has stated that she now considers herself a mathematician rather than a physicist because her work in physics was always highly theoretical and mathematical, and because she is interested in applications outside physics, particularly in engineering. A wavelet is an oscillation that has an amplitude that moves from zero to some point and then decreases back to zero (similar to an oscillation on a heart monitor). A wavelet transform is a mathematical function, similar to a Fourier transform, which allows data to be divided into frequency components and may be used to analyze signals that contain discontinuities and spikes. Jean Morlet and Alex Grossman developed the continuous wavelet function in the 1980s, and Daubechies, working with Yves Meyer and Alex Grossman, developed a discrete approach that allowed the reconstruction of wavelets from discrete values.

Applications of Wavelet Analysis

Wavelet analysis has many practical applications, particularly in creating and storing digital images. For instance, the U.S. Federal Bureau of Investigation (FBI) has used wavelet analysis since 1993 to encode digitized fingerprint records. This application is due in large part to the fact that a wavelet transform of an image reduces the amount of computer memory required to store it by as much as 93% compared to conventional image storage methods. Another application of wavelet analysis is in medical imaging systems, such as magnetic resonance imaging and computerized tomography. These technologies use scanners to collect digital information that is then assembled by a computer into a two- or three-dimensional picture of some internal aspect of the patient's body. Data processing methods involving wavelet transforms "clean up" and smooth digital information to yield a sharper image. Using wavelet transforms in medical scanning also reduces the time used to take the scan (thus reducing the patient's exposure to radiation) and makes the process of acquiring usable images faster and cheaper.

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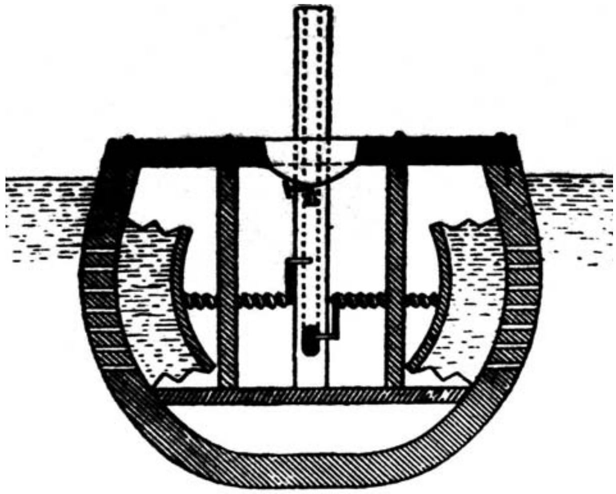
Sarah Boslaugh

See Also: Animation and CGI; Digital Images; Digital Storage; Women.



William Bourne's 1578 design was one of the first recorded plans for an underwater navigation vehicle.

Deep Submergence Vehicles

Category: Travel and Transportation.
Field of Study: Algebra; Measurement; Number and Operations.
Summary: Submergence vehicles must be carefully designed to take into account undersea conditions.

Deep submergence vehicles are primarily designed to aid researchers in exploring the depths of Earth's oceans. Much is unknown about the suboceanic environment, and exploration of these depths requires transport vehicles that can withstand tremendous pressures. Modern submergence vehicles can not only dive to great depths but can also stay submerged for hours at length, and are equipped with external lights and tele-operated robotic manipulators to gather deep sea samples for further research. Besides researching marine life, deep submergence vehicles also play vital roles in the oil exploration and the telecommunications industries where robotic submarine vehicles known as "autonomous underwater vehicles" detect faulty cables and help in oil field exploration. English mathematician William Bourne may have been the first to record a design for an underwater navigation vehicle in 1578. In addition to mathematics and mathematicians impacting deep submergence vehicles, submarines have also impacted the development of mathematics. Mathematicians examined the optimal way for airplanes to search for submarines, and the field of operations research was born.

Physical Characteristics of the Abyss

Pressure. At any given depth under the sea level, the pressure on a body can be calculated as

$$P = \rho \times g \times h$$

where P is pressure, ρ is the density of the seawater, g is the acceleration because of gravity, and h is the depth at which the measurement is being taken.

The atmospheric pressure at sea-level is about 100 kPa (~ 14.6 psi), the same amount of water pressure at about 10 meters (33 feet) below the surface, making the combined pressure experienced by a body at a 10 meter depth almost double of that at the surface.

Light. Most of the visible light entering the ocean is absorbed within 10 meters (33 feet) of the water's surface. Almost no light penetrates below 150 meters (490 feet). Solid particles, waves, and debris in the water affect light penetration. The longer wavelengths of light, red, yellow, and orange, penetrate to 15, 30, and 50 meters respectively, while the shorter wavelengths—violet, blue, and green—can penetrate further. The depth of water where sunlight penetrates sufficiently for photosynthesis to take place is called the Euphotic Zone and is normally around 200 meters (655 feet) in the ocean. The zone where filtered sunlight only suffuses in the water is known as the Disphotic Zone and extends from the end of the Euphotic Zone to about a depth of 1000 meters. Below that, no sunlight ever penetrates, and this is known as the Aphotic Zone.

Temperature. There is a significant difference in the temperatures between the Euphotic and Aphotic zones. However, in the Aphotic Zone, the temperature remains almost constant, hovering around 2 to 4 degrees Celsius. The only exception occurs when deep-sea volcanoes or hydrothermal vents exist, which cause significant warming of the waters.

History

The earliest deep-sea submersibles were known as "bathyspheres" (from *bathys*, Greek for "deep"). They were raised in and out of the water by a cable. They were fitted with oxygen cylinders inside to provide air to the divers, and had chemicals to absorb the expelled carbon dioxide. The early bathyspheres were not maneuverable—the only degree of freedom they had enabled them to go up and down.

The notable Swiss physicist Auguste Piccard (1884–1962) was influential in making the next design iteration to the bathysphere, called the "bathyscaph." The vessel was not suspended from a ship but instead attached to a free-floating tank filled with petroleum liquid. This tank made it buoyant (lighter than water). The bathyscaph had metal ballasts that, when released, allowed the vessel to surface. Auguste and his son Jacques designed the next generation bathyscaph, the Trieste. The Trieste set a new world record when it reached the lowest point on Earth, the Marianas Trench (35,800 feet).

Improvements in electronics and materials engineering have led to the design of Alvin, a deep-sea vessel capable of accommodating up to three people and diving for up to nine hours. Alvin sports two robotic arms that can be customized depending on the mission it is undertaking. Alvin's most notable contribution was its role in exploring the RMS Titanic.

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Ashwin Mudigonda

See Also: Coral Reefs; Marine Navigation; Robots; Tides and Waves.

Deforestation

Category: Weather, Nature, and Environment.
Fields of Study: Algebra; Measurement; Problem Solving.
Summary: Mathematicians study and model many aspects of deforestation.

Deforestation is the removal of forests by logging or burning. While some deforestation can occur accidentally as a result of wildfires, most is deliberate. Trees may be sold for lumber or charcoal, and land may be cleared for housing, farming, or pasturing livestock. Trees may also be removed for beneficial purposes, such as directing water flow or controlling future forest fires. Many people believe that deforestation is a significant factor in climate change and biodiversity loss, and research has shown that deforested regions are much more vulnerable to soil erosion and desertification.

While logging is linked to deforestation in the popular imagination, the United Nations Framework Convention on Climate Change actually found that in the early twenty-first century, logging actually accounted for less than 20% of deliberate deforestation. In contrast, commercial agriculture claimed about one-third of deforested lands and subsistence farming nearly one-half. This statistic indicates one reason why deforestation is increasing primarily in relatively poorer countries. However, within an industrialized country, like the United States, logging and clearing land for housing or other real estate development account for far more deforestation than subsistence farming, which few Americans have practiced since the dawn

Many people believe that deforestation is a significant factor in climate change and biodiversity loss, and research has shown that deforested regions are much more vulnerable to soil erosion and desertification.

of the twentieth century. Mathematicians study and model many aspects of deforestation, including possible causes and the biological, geological, social, and economic effects; uses of deforested land; patterns of regrowth and biodiversity in areas where the forest has been allowed to return; and spatial mapping and visualizations of geographical regions before, during, and after deforestation. Data collection, statistical analyses, and spatial dependency analyses, as well as stochastic spatial modeling, linear programming, geometry, and digital image analysis, are all mathematical methods that have played a role in such analyses.

Environmental Effects

Deforestation is implicated in numerous environmental problems. The relationship between the forest and atmospheric carbon dioxide, for instance, is complicated. While they are alive and actively growing, trees remove carbon dioxide from the atmosphere, store it as carbon, and release oxygen back into the atmosphere through respiration. This process reduces the amount of greenhouse gases in the atmosphere, and this basic dichotomy—plants breathing in carbon dioxide and releasing oxygen, while humans and animals do the opposite—has long been taught to schoolchildren as the critically interdependent relationship between flora and fauna on Earth. In the early twenty-first century, the world's forests store roughly three-quarters or greater of aboveground and soil carbon. When trees are cut down and burned, they release their stored carbon back into the atmosphere. When trees die and decay, they do the same, as fungi and bacteria break down the carbon products into carbon dioxide and methane. Their effect on the world's oxygen supply is actually very minor—the amount of oxygen they release is not as significant as the amount of carbon dioxide involved in a tree's lifespan.

But cutting trees down and turning them into long-lived products (using them to build houses, for instance) stores the carbon just as efficiently. For forests to continue to take carbon dioxide in from the atmosphere, the trees must be harvested regularly—with new trees planted—so that there are always actively growing trees. Left to their own devices, mature forests cycle through periods as carbon dioxide sources (when the carbon dioxide released by decaying or wildfire-burned trees exceeds that taken in by growing trees) and sinks (when the net carbon dioxide release is negative).

The greatest amount of carbon dioxide is taken in by deciduous trees when spring leaves are growing, which results in an observable dip in the Keeling Curve (a graph that tracks variation in the concentration of atmospheric carbon dioxide from 1958 onward). The dip is mirrored by a rise corresponding to the release of carbon dioxide back into the air every fall when these leaves fall and decay. The curve is named for Charles Keeling, a University of California, San Diego, oceanographer whose observations helped bring global attention to anthropogenic climate change. Measurements continue to be taken at Mauna Loa, in Hawaii, and those data have shown a roughly 20% to 25% increase in the amount of atmospheric carbon dioxide between 1958 and 2010. There have been no declining trends in that time, countering the pre-Keeley claim that an apparent rise in carbon dioxide atmospheric concentration was the result of random fluctuations. Periodic local decreases and increases of about 1% to 2% are associated with seasonal cycles.

Anti-Deforestation Efforts

Recent efforts to reduce greenhouse gas emissions, and international agreements binding countries to do so, have brought more focus to the task of accurately measuring those emissions. It came to light in 2010 that Australia's efforts to reduce emissions in order to comply with the Kyoto Protocol goals were hampered by their inaccurate measurement of deforestation emissions. Since 1990, Australia has had the highest rate of deforestation in the developed world, and thus is the only developed country targeting deforestation emissions as its primary way of reducing overall emissions. But its inability to generate an accurate figure of what those emissions currently are, to establish a baseline, or reliably measure them in the future, has thrown a wrench in its efforts.

Data Collection and Mathematical Modeling

The highly complex nature of forest ecosystems and even individual trees makes it virtually impossible to collect complete data on the system dynamics of natural forests. As a result, investigations of long-term dynamics rely heavily on scientific inference. One way of making any estimate, heavily relied on when considering the environmental costs of possible actions, is through ecosystem modeling, which constructs mathematical representations of ecosystems. The entire ecosystem need not be represented (though this leaves open the possibility of unforeseen consequences in parts of the ecosystem not modeled). Typically, models are constructed to examine the inventory of a specific chemical in the environment, like carbon, nitrogen, phosphorous, or a toxin. The ecosystem is reduced to a set of state variables that describe the state of a dynamic system, like the population of a specific species or the concentration level of a particular substance.

Mathematical functions define the relationships between those variables, such as the relationship between new leaf growth and carbon dioxide intake. A usable model typically requires many variables and much fine-tuning to affirm that the relationships have been defined accurately, and, in some cases, a model may be constructed simply to test a hypothesis about those relationships, by comparing the behavior of the model ecosystem to the real one. For example, mathematician and ecologist Nandi Leslie developed mathematical models using techniques such as spatial statistics, mean field and pair approximation, and the theory of interacting particle systems to investigate questions about forest fragmentation and degradation, ecology and biodiversity in lands reclaimed by forests, and landscape-level impact of land-use activities in Bolivia and Brazil. Leslie is included on a Web site called Mathematicians of the African Diaspora and is the daughter of mathematician Joshua Leslie, who has published widely in the fields of algebraic and differential geometry. The applications of modeling in deforestation are as broad as the types of models. Some mathematicians have used calculus to measure tree density, including the number of trees per acre and the quantity of foliage. Logistic functions have been used to estimate insect density or infestations. Many linear and nonlinear modeling techniques, like regression analysis, are widely employed to help reveal and explain associations between multiple variables, such as social choices and government policies; economic measures; environmental measures; geographic features, like altitude and slope; and human constructions, like roads. These models are then frequently used to forecast important quantities of interest, like deforestation rates and the overall proportion of deforested land. However, inappropriate extrapolations and generalizations can lead people to make inaccurate predictions or conclusions. For example, extrapolations from exponential models tend to lead to overestimation of future values. This has an impact on contentious and world-reaching scientific debates, such as global warming.

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Bill Kte'pi

See Also: Floods; Forest Fires; North America; South America.

Deming, W. Edwards

Category: Business, Economics, and Marketing.
Fields of Study: Connections.
Summary: W. Edwards Deming (1900–1993) was an American applied statistician who revolutionized Japanese management as part of the rebuilding effort after World War II.

W. Edwards Deming was an internationally renowned consultant whose work led Japanese industry into new principles of management and revolutionized quality and productivity in Japanese companies. He believed: "Innovation comes from freedom. It comes from those who are obligated to no one. It comes from people who are responsible only to themselves." He was born October 14, 1900, and died on December 20, 1993. His undergraduate degree in engineering was from the University of Wyoming in 1921. He earned a M.S. in physics and mathematics from the University of Colorado in 1925 and a Ph.D. in physics from Yale University in 1928. He had a wide and varied career, which included his first scientific paper on the nuclear packing of helium, mathematical and statistical work for the Department of Agriculture in Washington, D.C., and work on sampling issues for the U.S. Census Bureau. He noted that he also worked on many different studies, including: "application of statistical theory to problems that arise in industrial production, in tests of physical materials . . . motor freight, rail freight, accounting . . . average life of returnable bottles, comparison of medical treatments, comparison of methods of diagnosis, social, and demographic problems created by physical or mental handicaps. My part in any study is the design thereof, followed by evaluation of the statistical reliability of the results." He received many honors and awards and was also an active member in professional societies, such as the American Statistical Association and as president of the Institute of Mathematical Statistics in 1945.

In 1999, the *Los Angeles Times* recognized him as one of the 50 people who most influenced business in the twentieth century because of his work in rebuilding Japan after World War II. He urged Japanese companies to concentrate on constant improvement, improved efficiency, and doing things right the first time. The essence of his ideas was based on the concepts of statistical process control, a process originally developed by Walter A. Shewhart in the 1920s. It has since been expanded to include the total quality management approach.

The essence of Deming's process was to record the number of product defects, statistically analyze why those defects occurred, institute changes to correct the defects, record how much the quality then improved, and to continue to refine the production process until it was done correctly. He said: "If you don't have a method, you were goofing off. A system must be managed and must have an aim."

Deming first successfully applied his ideas in the United States during World War II in improving the manufacture of munitions and other strategically important products. As mentioned above, he brought those same ideas to Japan in the 1950s and early 1960s. During that time period, "Made in Japan" went from being a joke and a synonym for poor quality to a symbol of some of the highest quality products. The focus on quality that he emphasized was defined as the ratio of results of work efforts with total costs. If a company or manager focuses on quality, Deming's work demonstrated that, over time, quality will increase and costs will fall. On the other hand, if the focus is primarily on costs, then costs will rise and quality will decline.

Two major publications have outlined his theories and the processes he developed. In his 1982 book *Out of the Crisis*, Deming discusses his 14 key principles for management for transforming business executives. Deming felt that if his 14 points were applied in a meaningful way, they would lead to a process of continual improvement. *The New Economics*, published in 1993, emphasized that the solution to problems comes from cooperation, not competition. This concept is accomplished through a new type of management, which Deming identified as profound knowledge and which includes four parts: appreciation for a system, knowledge about variation, theory of knowledge, and psychology.

Deming also had an interest in music. He composed several pieces, mostly liturgical. He also composed a new rendition of the *Star Spangled Banner* with the same words set to a different tune. He had always felt that the "pub" music of the original version was not appropriate for a national anthem.

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Jim Austin

See Also: Data Analysis and Probability in Society; Quality Control; Scheduling; Statistics Education.

Diagnostic Testing

Category: Medicine and Health.
Fields of Study: Data Analysis and Probability; Measurement; Number and Operations.
Summary: Diagnostic tests rely on statistics from clinical research to predict the presence or severity of a disease in a specific patient.

The ability of humans to detect and treat diseases has advanced considerably in the past two centuries, with the discovery of underlying causes, such as microorganisms, and treatments, like antibiotics, as well as methods for diagnosing injury and disease. In medicine, a diagnostic test is in an instrument used to detect or predict the presence or absence of disease or the severity of disease.

The instrument used may take a variety of forms, including a patient inventory or a mechanical device. In clinical research, it is common practice to assess the quality of such instruments relative to established gold standards.

Here, the intention is often to replace a traditional method by a newer one that offers greater benefits to health providers or patients, including cost reduction and less physical or psychological discomfort.

It may be of interest to use the diagnostic tool to predict outcomes based on existing symptoms. In this case, the gold standard is used to confirm patient outcomes for comparison with test predictions based on surrogate measures.

Common measures of instrument quality include reliability, validity, sensitivity, specificity, positive predictive value (PPV), and negative predictive value (NPV). Strictly speaking, these measures apply specifically to the scores forthcoming from the instruments rather than the instruments themselves, as they are based on studies applied to a specific sample of patients. Mathematicians and statisticians are essential partners in creating many diagnostic tools, such as magnetic resonance imaging, as well as for developing and refining the measures that allow clinicians and researchers to determine the efficacy of diagnostic instruments. They also help design experiments in which new instruments are tested and compared.

Nursing and other healthcare education programs frequently require courses in mathematics or statistics, and the field of biostatistics is one of the fastest-growing occupations in the late twentieth and early twenty-first century.

Reliability represents the reproducibility of the test outcomes. A simple case involves estimation of the extent of chance-corrected agreement in the interpretation of categorical findings from medical images derived from patients. Here, agreement might be measured across different clinicians based on a single imaging procedure or alternatively, across different imaging procedures. In such cases, an appropriate choice of Kappa statistic or intra-class correlation coefficient may prove helpful. For continuous data, the Bland–Altman method has also proved particularly popular in measuring agreement across different methods. This is especially so within medicine, where for example, there may be a need to compare residual tumor sizes obtained using magnetic resonance imaging, and pathologic findings (the gold standard) in breast cancer patients who have undergone neoadjuvant (preoperative) chemotherapy.

The remaining measures above represent the accuracy of the test outcomes. Validity, which is a function of reliability, represents the extent to which the diagnostic test measures what is intended and is particularly relevant in psychological testing. Sensitivity (specificity) measures the proportion of genuine instances of disease (absence of disease, respectively), which are detected as such by the diagnostic test. By contrast, the PPV (NPV) measures the proportion of cases diagnosed by the test as instances of disease (absence of disease, respectively) which are, or will turn out to be, genuine. In assessing test accuracy, it can prove misleading to focus exclusively on sensitivity and specificity.

The PPV and NPV for a disease are influenced strongly by disease prevalence (the pre-test probability that a randomly chosen person from the study cohort has the disease). The PPV increases with increasing prevalence and where prevalence is particularly low (less than 5%), the PPV can be markedly improved by moderate increases in test specificity. In interpreting a published PPV, it is essential not only to consider the CI but also to verify whether disease prevalence for the published study is representative of that for the types of patient currently under consideration. This requirement is also particularly true of the NPV.

Further, it is typically the case that an initial stage has occurred whereby diagnostic test measurements in continuous form have been classified into categories. This categorization requires the derivation of a threshold value for differentiating between diseased and non-diseased patients. The clinician may be interested in finding the threshold value that offers an optimal combination of values for sensitivity and (1-specificity). Examples of scores that have been used in this way include

- The GRACE (Global Registry of Acute Coronary Events) score in predicting death and myocardial infarction for patients with Acute Coronary Syndrome
- The APACHE (Acute Physiology and Chronic Health Evaluation) II score and GS (Glasgow Severity) score in the prediction of each of onset of severe pancreatitis, MODS (multiorgan dysfunction syndrome), and death in patients presenting with acute pancreatitis
- The MELD (Model of End-Stage Liver Disease) and UKELD (United Kingdom MELD) scores in the assessment of risk of acute liver failure and hence the prediction of waiting list mortality in patients awaiting liver transplants

The underlying procedure for deriving the threshold value involves the segregation of the test instrument scores into two groups, as determined by the gold standard, namely those who do and those who do not have the condition of interest. The accuracy of the diagnostic test is in turn assessed on the basis of these two groups. This assessment involves generating a series of threshold values and corresponding values for sensitivity and 1-specificity. The ROC curve (Receiver Operating Characteristic) involves a plot of sensitivity versus 1-specificity. If the intention is to compare the performance of competing diagnostic tests, ROC curves for the different tests can be plotted on the same graph. For any one plot, the numerically optimal combination of sensitivity and specificity values is represented by the point on the curve that is closest to the top left-hand corner. However, the trade-off between sensitivity and specificity must also be carefully weighed.

For example, if the test is confirmatory, as might be the case in human immunodeficiency virus (HIV) testing, it may be preferable to choose a slightly different point, which further reduces the proportion of false positives (1-specificity) with a small cost to sensitivity. In comparing the accuracy of two tests by means of ROCs, it is common to use the area under the curve (AUC).

Where the diagnostic test identifies cases falling into the upper (lower) range of a test score, the AUC may be interpreted as a measure of the likelihood for a randomly chosen diseased patient and disease-free patient that the diseased patient will have a higher value (lower value, respectively) than the disease-free patient.

Where ROCs do not overlap, therefore, the greater the area under the curve, the more effective the diagnostic tool. Where they do overlap, the curve with the lower overall AUC may have a peak at an optimal combination of sensitivity and specificity values not attained by the other curve. It may therefore make sense to compare the partial areas under the curves within one or more ranges of specificity values.

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Margaret MacDougall

See Also: Chemotherapy; Data Analysis and Probability in Society; Diseases, Tracking Infectious; HIV/AIDS; Medical Imaging; Probability; Psychological Testing; Surgery; Transplantation.

Dice Games

Category: Games, Sport, and Recreation.
Fields of Study: Algebra; Data Analysis and Probability; Number and Operations.
Summary: Probability is the key factor for winning any dice game.

Dice games use one or more dice as central components of the activity, which excludes board games using dice solely as random devices to determine moves. The definition can be murky, as in the case of Backgammon, dice outcomes determine a player's moves and are integral parts of game strategies. Historically, dice games involving gambling led to the creation of probability.

History

Archaeological evidence from as early as 6000 B.C.E. shows that dice games were part of early cultures, where dice were cast to invoke personal divinations. The notion of "luck" was not involved, with the dice rolls controlled by the gods. Gamblers still refer to Fortuna, the Roman goddess and Jupiter's daughter, as their "Lady Luck."

The ancient die differed from the six-sided cube bearing pips, as the number of sides varied with the materials used, including fruit pits, nut shells, pebbles, and animal knucklebones. The latter, with four sides involving different probabilities, led to the phrase "rolling the bones."

Compulsive gambling and dice games have always been connected, being traced to Egyptian pharaohs, Chinese leaders, Roman emperors, Greek elite, European academics, and English kings. On the request of professional gamblers in the fifteenth and sixteenth centuries, mathematicians such as Fra Luca Bartolomeo de Pacioli and Girolamo Cardano began to study the probabilities of winning dice games. In the seventeenth century, correspondence between Blaise Pascal and Pierre Fermat ultimately solved the "problem of points" and established basic principles of probability.

The problem of points involves a dice game between two players; multiple rounds are played with each player having an equal chance of winning on each roll. If the game was interrupted before either player had won the necessary number of rounds, gamblers could not determine the "fair" division of stakes based on current scores. Fermat and Pascal's solution analyzed the probability of dice rolls and each player winning the pot.

Types of Dice Games

The simplest dice game involves a single die, where the winner is the person rolling the highest number. This can be extended to rolls of multiple dice, with the player's score being the sum or product of the numbers shown. Since these dice games involve only luck, gamblers prefer variations with elements of strategy.

The dice game craps involves strategy, as the "shooter" controls the number of dice rolls and betting options. Though craps is complex, key elements can be explained. Mathematically, each roll of two dice has 36 possible outcomes with shown totals ranging from "2" to "12". However, the probabilities of the totals vary, as the probability of a "2" (known as "snake eyes") or "12" (known as "boxcars") is 1/36, while the probability of a "7" is 6/36. Prior to the first "come out roll," players bet on the "Pass Line" or "Don't Pass Line." If the "shooter" then rolls a "7" or "11," the "Pass Line" bet wins double their amount and the "Don't Pass Line" bet is lost. However, if the initial roll is a "2," "3," or "12," the "Pass Line" bet is lost, while the "Don't Pass Line" bet is doubled if a "2" or "3" shows and is returned if a "12" ("push") shows. A sum of "4," "5," "6," "8," "9," or "10" becomes the "point" number, which the shooter tries to duplicate on the second roll. If the point number is made, the point bet is won and additional rolls can be made. But, if a "7" is rolled before the point number, the shooter "craps out" and a different shooter starts a new round. Craps games involve many other options, such as "Come/Don't Come Bets" and "Horn Bets."

Other dice games are used for gambling, each with their own multiple versions and strategies. For example, in the dice game Ship, Captain, and Crew, a players gets three rolls of five dice to gain a ship ("6"), a captain ("5"), and a crew ("4") in that order (or simultaneously). When those special numbers are rolled, that die is removed from play, with a successful player's score being the sum of a roll of the two remaining dice.

In Buck Dice, a player throws one die to determine the "point number." Another player then rolls three dice, continuing the rolls as long as one of the dice equals the point number. When this doesn't occur, the player's score for that round is the number of rolls.

The dice game craps is thought to have developed from a simplification of the Old English game Hazard.

A"big buck" occurs when all three dice equal the point number, and the player withdraws from the game. A "little buck" occurs if all three die do not equal the point number, which adds 5 points to the player's score. Any player with exactly 15 points withdraws from the game; any score forced higher than 15 nullifies a roll, and the player must reroll. The loser is the last person without reaching 15.

In Aces, a player starts with at least five dice, which he or she loses according to the numbers thrown. All rolled "1"s are placed in the table's center and eliminated. All rolled "2"s are passed to the player on the left, while all "5"s are passed to the player on the right. Turns continue with rolls of the remaining dice until players either do not throw a "1," "2," or "5," or have lost all of their dice. Play continues around the table until the last die rolled is a "1," and the player who threw it is the winner.

Farkle begins with a player rolling six dice. Each "1" adds 100 points, each "5" adds 50 points, and if three dice show the same number, the player adds 100 times that shown number. A player can stop after any roll and keep the current total. Alternately, a player can roll again to possibly increase his or her score. But, if the next dice do not produce a positive score, the player lose all accumulated points for that round. The winner is the first to reach 10,000 points. Some variations of Farkle give 1000 points for shown runs of "1–5" or "2–6."

In line with their history, multiple versions of dice games exist and will continue to be used by gamblers. Thus, the players who understand the probabilities involved will always have the advantage.

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Jerry Johnson

See Also: Betting and Fairness; Board Games; Game Theory.

Digital Book Readers

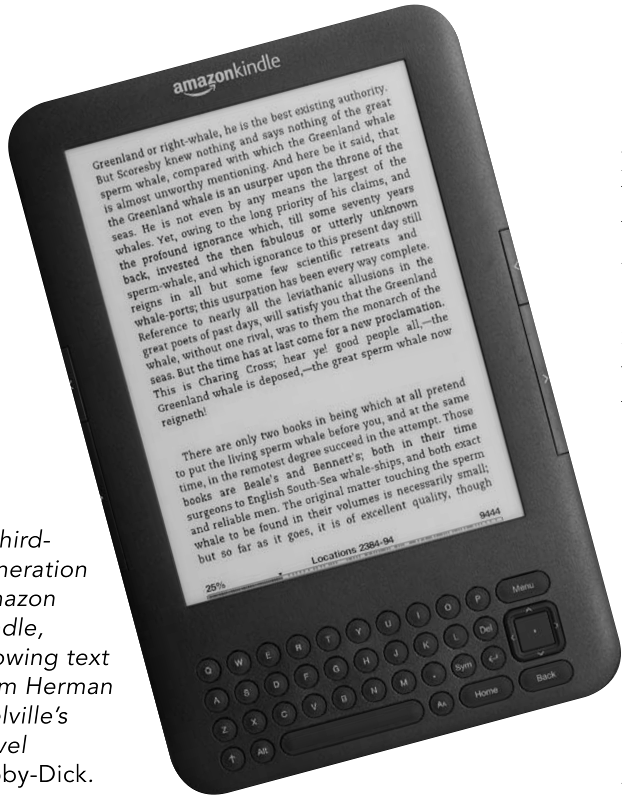
Category: Communication and Computers.
Fields of Study: Geometry; Measurement; Number and Operations; Representations.
Summary: The twenty-first-century surge in e-books began with the advent of "electronic ink" and future innovations include sketchpad-like functionality.

People have been reading digital content on computer screens since the 1970s, but the technology used for most computer screens at the end of the twentieth century made them somewhat less useful for replacing paper books, magazines, and newspapers. In 1971, volunteers started digitizing and archiving books for Project Gutenberg, whose goal was to encourage the development of electronic books. Research on electronic paper began in the 1970s. Many open and proprietary digital document formats were devised for potential use in e-books, like Adobe's Portable Document Format (PDF), created by mathematician and engineer John Warnock. However, most early attempts at digital books were unsuccessful or aimed at niche technical audiences.

In the early twenty-first century, the E Ink company introduced electronic ink technology, which revolutionized digital books. The company was co-founded by several individuals, including physicist Joseph Jacobson and Russell Wilcox, who has a degree in applied mathematics. The resulting "electronic paper" has a high contrast ratio similar to standard paper, and for most users it closely matches the experience of reading on standard paper. One early application was flexible, changeable store signs. The 2004 Sony Librié, released in Japan, was the first e-reader to make the technology widely available, while the Amazon Kindle is credited with popularizing it in the United States. As of 2010, there were many variations on e-readers with the ability to display multiple e-book formats. Some of the most popular included the Sony Reader, the Amazon Kindle, and the Barnes & Noble Nook. Motorola's FONE F3 was the first portable phone to include this technology.

Electronic Ink

Electronic ink technology is based on microcapsules, which were already in use for applications like scratch-and-sniff stickers and time-release medications. Rotating microcapsule spheres for electronic ink are filled with a clear liquid containing a mix of small, electrically



A third-generation Amazon Kindle, showing text from Herman Melville's novel Moby-Dick.

charged black and white particles. Some implementations contain on the order of 100,000 spheres per square inch. Electronic paper is a sheet of plastic coated with millions of microcapsules and equipped with an electronic device to draw the black and white particles into desired patterns of black and white dots.

When viewed from a distance, the patterns create words and pictures. The dots can also be mixtures of black and white, resulting in a range of grayscale tones. To change the image, computer programs in the reader send an electronic pulse to rearrange the pattern. Microcapsules are bistatic, which means they stay in place once they are arranged without drawing continuous electrical power. This factor contributes to long battery life. Electronic paper also has no backlighting like personal computer screens; it uses light reflection for viewing, just like ordinary paper. Scientists are investigating red, green, and blue filters to produce full-color electronic ink images. A version of the Barnes & Noble Nook released in 2010 uses a liquid crystal display (LCD) screen for color and touch-screen functionality. Some praise this, while others consider it to be a step backward in e-reader technology.

Early twenty-first-century digital book readers embody several other features that make them well-suited alternatives for leisure reading and textbooks in schools. One important aspect is their portability with high-capacity storage. Typical readers have the capability to store hundreds of books, so all required textbooks could be stored in single digital reader. Connectivity via wireless networking allows the downloading of a variety of books or teacher-created documents, including RSS feeds for blogs and Web content. RSS was developed by programmers like David Winer, who has degrees in mathematics and computer science. The reading experience is customizable; some e-readers have touch-screen navigation, adjustable font levels, the ability to take notes directly on screen or highlight text sections, built-in dictionaries, or search functions.

Readers that debuted in 2010 featured applications to allow users to write or draw, like a tablet PC, which would be important for many mathematical subjects, like geometry. Some mathematics educators have explored the use of electronic ink to support mathematics distance education. For example, electronic ink tools in a chat program allowed students and instructors to post and edit mathematical formulas, diagrams, and graphs while communicating in real time.

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Serkan Ozel
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See Also: Cell Phone Networks; File Downloading and Sharing; Personal Computers.

Digital Cameras

Category: Arts, Music, and Entertainment.
Fields of Study: Geometry; Measurement; Number and Operations; Representations.

Summary: Rapid advances in digital camera technology have led to their widespread use.

From the invention of modern photography in the 1800s to the rise of digital photography in the twenty-first century, the function of the camera has been the same: to record patterns of light. The word "photography," coined by Sir John Herschel in 1839, is from the Greek *phos* (light) and *gráphein* (to write). Simple pinhole cameras were described as early as the fourth and fifth centuries B.C.E. by Chinese philosopher Mo Ti and Greek mathematicians Aristotle and Euclid of Alexandria. Mathematician and physicist James Maxwell created the first color photograph in 1861. Not long after, American inventor and Kodak founder George Eastman developed inexpensive equipment and film that made photography practical for common use. Until recently, cameras recorded images on media coated in photosensitive compounds. Incoming light was registered as a chemical change that could be seen upon development in specialized photochemistry. Digital cameras use an electronic chip that is sensitive to light. The chip, either a charge coupled device (CCD) or complementary metal-oxide-semiconductor (CMOS), converts the light into an electrical signal, and a small computer in the camera then transforms that signal into the "ons" and "offs" (or "1"s and "0"s) of binary code for storage on a digital storage device. The digital information that represents an image can easily be copied onto a computer, manipulated, published electronically, and printed. Researchers also investigate mathematical questions like how many images one should shoot in order to be reasonably confident that no person in the photograph blinks. For groups under 20 people, the number of images is approximately equal to the number of people divided by one-half.

The Lens and the Shutter

Like most cameras, a digital camera begins the process of taking a photograph by letting light in through a lens (a curved piece of glass or plastic) that bends light through the principle of refraction and focuses the image. The light then passes through an opening called an "aperture" whose size can be adjusted to let more or less light pass. Apertures are described by an f-stop number, which is proportional to the focal length of the lens over the diameter of the entrance pupil. Since it is a ratio, larger f-stop numbers refer to smaller apertures. Each doubling or halving of the f-stop number translates to a change in amount of light let in by a factor of four. Thus, an f-stop of 11 lets in four times more light than an f-stop of 22. Finally, the light comes to a shutter that opens for a period of time when the shutter release is triggered, allowing light into the camera body. Usually, a shutter speed is a fraction of a second, though long shutter speeds can be used in low light, or for a variety of effects. With especially long shutter speeds, heat can build up in the CCD or CMOS, causing electrical interference that interferes with accurate, binary recording of the image, resulting in error or "noise," though camera manufacturers are developing a number of processes that have made this less of a problem over time.

The CCD or CMOS

In order to capture an image, the light that comes into the camera falls on the CCD or CMOS chip, which changes the image into electric current. A CCD is made up of tiny regions called "picture elements" or "pixels" that will correspond to the points in the photograph. A CMOS works similarly, though the specific underlying technology is a bit different. Some cameras have one CCD for all three primary colors of visible light, red, green, and blue; each pixel records only color of light from the scene. More advanced cameras use three different CCDs, one for each primary color, resulting in a more accurate image. Ultimately, the electric current from the CCD is encoded by a small computer in the camera into a stream of binary information in the form of "ons" and "offs" that ultimately will be stored on a flash memory card.

Sensitivity and ISO

In film cameras, different formulations of film are used for different light conditions, with more sensitive films employed in low light. In a digital camera, the signal from the CCD can be boosted to handle low light levels; however, doing so introduces noise in the signal. The setting for camera sensitivity is described as its "ISO number," an international standard for measuring the speed of color film. It uses both an arithmetic and logarithmic scale to combine two previous film standards. In the arithmetic scale, which is commonly the only one given, each doubling of the ISO representing a doubling of the sensitivity. Thus, a camera set to ISO 100 will be half as sensitive to light—and will require twice as long an exposure for a given scene to achieve the same result, given the same f-stop setting—as one with an ISO of 200.

Pixel Dimensions

One of the factors that determines the picture quality of an image produced by a digital camera is the number of pixels it records. This is especially relevant when images are blown up to large dimensions, as the individual pixels begin to become visible. Pixels are the individual binary units into which the image is "broken up" and stored during the electronic conversion process by the camera's chip. For example, the Droid Incredible phone, released in 2010, contains an 8 megapixel camera, which means its photographs are composed of about 8 million individual pixels, with each picture having a possible resolution of roughly 3264 pixels wide by 2468 pixels high. However, in practice, there are many factors that affect picture quality. The size of the electronic chip plays a large role. When the photosensitive regions of a camera's chip are packed too tightly together, they create electronic interference in their neighbors, potentially affecting the binary storage, and ultimately affecting the accuracy and quality of the stored image.

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Jeff Goodman

See Also: Digital Images; Digital Storage; Movies, Making of.

Cameras in Mathematics Classrooms

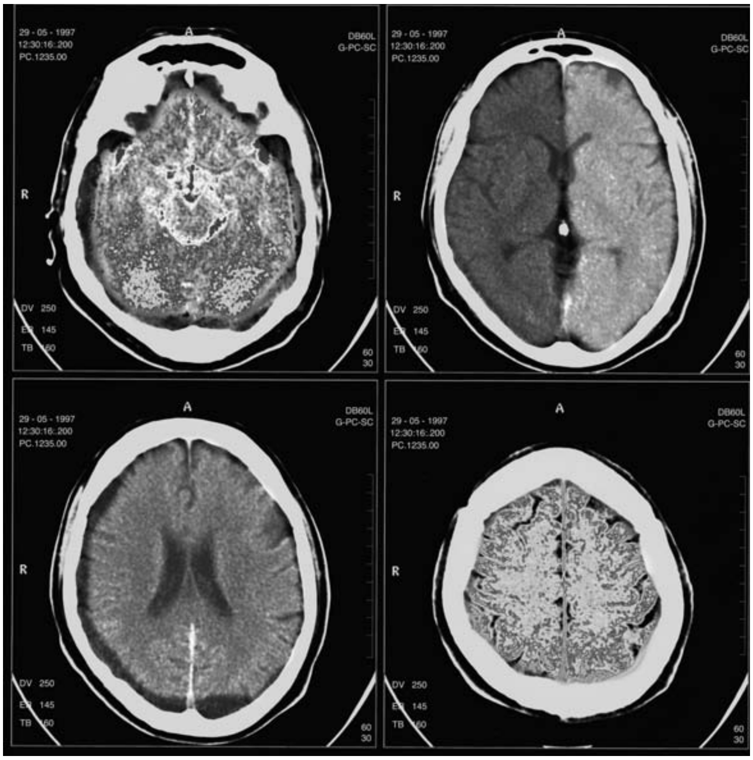
Cameras have grown in popularity since Eastman first made them readily available. Digital cameras are relatively inexpensive, and, in fact, are now standard features on many cell phones. Educators have seized on the digital camera as a very useful tool in the classroom for introducing concepts, making connections, and enriching educational experiences in a very hands-on way. For example, students in middle grades and above have been asked to use digital cameras to record their own examples of geometric concepts found in the world. They can then use the photographs, along with various mathematics concepts such as scaling and trigonometry, to answer questions like "Is the Houston Astrodome really round?" or "What is the slope of a roof?" In other cases, students use photos to record and measure themselves and their classmates—either once or repeatedly over time—to provide data for many interesting mathematics activities and discussions, such as variability and the importance of repeated sampling.

Digital Images

Category: Arts, Music, and Entertainment.
Fields of Study: Algebra; Geometry; Number and Operations; Representations.
Summary: Digital images are recorded as a binary account of pixels, which algorithms may compress.

Digital images are not images at all but rather are visual information encoded as binary data. Viewing a digital image requires a computer to decode binary information and display it on a screen in the form of an array of discrete lights called "picture elements" or "pixels." The first computer-generated digital images were produced in the early 1960s. The needs of the Cold War, medicine, and the space race drove many developments in digital imagery, some of which were achieved in the context of projects on satellite imagery, medical imaging, optical character recognition, and photo enhancement. The advent of microprocessors in the 1970s and advances in digital storage and display technologies made possible sophisticated imaging tools, like computerized axial tomography (CAT scanning).

The degree of mathematical sophistication that CAT scans introduced into medical imaging, such as integral geometry, optimal sampling, and transport equations, was unheard of at that time. It is reflected in further advances such as magnetic resonance imaging as well as developments in other fields that use similar imaging techniques, like seismic and electron microscopy. At the same time, scanners to digitize analog images began to be used in a diverse array of fields, such as archaeology and law enforcement. The first fully digital camera was released in 1995, and by the end of the twentieth century, charge-coupled devices (CCDs) largely displaced analog film and tape for photography and videography. Willard Boyle and George Smith shared the 2009 Nobel Prize in Physics for their invention of the CCD, an idea they first brainstormed at Bell Labs in 1969. Improved computing power also allowed for production of near-photorealistic images. All areas of digital imagery (creation, compression, restoration, recognition, and display) involve mathematics. In the twenty-first century, digital images are regularly used in both mathematics research and teaching.



CAT scanning of the head is typically used to detect tumors, calcifications, hemorrhage, and bone trauma.

Bitmap Graphics

In most digital images, each pixel has been defined numerically and this number has been converted into a string of "1"s or "0"s. This system is the approach of "bitmap graphics" (also known as "raster graphics"), and it is how digital cameras work. Depending on the number of bits used to represent each pixel, more or less color information is given. For example, a one-bit system would allow only a black or white pixel, as the only choices would be a "0" or a "1." A two-bit system would gives four choices per pixel, "00" (black), "01" (dark grey), "10" (light grey), and "11" (white). Typically, in photo editing programs of the early twenty-first century, each pixel is described by 24 bits of information, yielding more than 16 million possible colors.

Resolution

Bitmap images contain information for a given number of pixels. The larger the pixel number, the more information is in the image and the higher the resolution; typically, this also results in a bigger file. Screens are all made of pixels, whether they are on computers or cell phones; if an image is viewed at full size, each pixel in the image will show up as one pixel on the screen. However, if a viewer zooms in beyond this point, the pixels in the file are actually represented by big blocks of pixels on the screen, and the image is said to become "pixelated."

Thus, if an image is to be viewed on a screen, it will ideally have the same number of pixels as the size one wants it on the screen; any more than that is wasted file space, and any fewer will result in an image that appears pixelated. If images are going to be printed, however, more pixels will translate into sharper pictures, limited only by the resolution of the printer. Again, the larger the print, the more pixels you will need for a sharp print.

File Types and Compression

Bitmap graphics can be stored in a variety of file formats depending on how they will be used. Raw files, which store all the raw data for the light that hits each CCD pixel, are commonly used by photographers who wish to have maximum flexibility and are not worried about file size. In order to make files smaller, computers use mathematical algorithms to compress the files. For example, instead of recording values for each pixel, the values for some could be calculated by the difference between a pixel and its surroundings, thus yielding substantial file size savings where blocks of pixels are the same as their neighbors. Some kinds of compression are considered "lossless," because all the information from the original can be re-created when the file is decompressed. However, there are a number of compression schemes such as the popular jpeg format in which the mathematical approximations do not quite match the original. In these cases, accuracy is sacrificed in order to save file size, and these approaches are said to be "lossy." However, the algorithms used to compress and decompress files are generally so good as to be unnoticeable in many cases. The JPEG 2000 image compression standard for both lossless and lossy compression uses biorthogonal wavelets, which extends from the work of mathematician Ingrid Daubechies, known as the "mother of wavelets."

Vector Graphics

Certain kinds of images, especially those created in computer graphics programs, use a different method for describing the content of the image. Instead of denoting each pixel with a number, these vector graphics are described mathematically as a set of equations representing the lines and curves that make them up. When a viewer zooms in on a vector graphic, the image does not become pixelated, because the computer recalculates the curve or line based on the new image size. While vector graphics are not appropriate for photographs, photo editing programs may use them when overlaying text or graphics on a digital image.

Image Reconstruction

The basic problem of image reconstruction is to build a "best-guess" object out of averaged data and then estimate how close the reconstruction is to the actual object. For example, in a single-angle X-ray of a person, the amount of radiation going in and coming out the other side can be measured and visualized on X-ray film. The difference between the values is how much was absorbed, but there is limited information about the inner structures that blocked the radiation. This limitation can make diagnoses difficult. However, if the same person is X-rayed from several directions and angles, the resulting information can be compiled, averaged, or mathematically modeled to estimate what the internal structure looks like.

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Jeff Goodman

See Also: Daubechies, Ingrid; Digital Camera; Digital Storage; Medical Imaging; Televisions.

Digital Storage

Category: Communication and Computers.
Fields of Study: Algebra; Measurement; Number and Operations.
Summary: Information can be stored digitally—a process that requires information to be translated into binary code.

Digital information is information in binary code. In order to create, manipulate, and store this digital code, it must be created in physical form. This creation is done by using media that can exist in one of two distinct states and assigning one state to each of the two digits ("0" and "1") in binary code. Within a computer, the "1"s and "0"s are represented as "ons" and "offs"; on a magnetic hard disk, they are tiny magnets pointing one way or another; and on a CD, the two states are shiny and dull spots. Engineers used metal tape on reel-to-reel machinery to record audio signals in the early twentieth century. In 1952, IBM introduced a tape drive with iron oxide–coated plastic tape. Reel-to-reel tape drives were the standard for data storage by the mid-1970s. IBM also created magnetic hard disks in the late 1950s, but it took decades to overcome size and access speed issues to make hard disk drives (HDDs) feasible for applications like personal computers. Solid-state drive (SSD) technology, such as flash memory, was the necessary next step to overcoming the lagging mechanical speeds of HDDs. Mathematicians in many fields have been essential in all stages of development and continue to address emerging issues. Ingrid Daubechies, "the mother of wavelets," is perhaps best known for her work with wavelet-based algorithms for compressing digital images. Irving Reed and Gustave Solomon developed algebraic error-detecting and error-correcting codes. These Reed–Solomon codes are widely used in digital storage and communication, from satellites to CDs.

Bits and Bytes

The smallest unit of stored digital information, corresponding to a single "1" or "0," is called a "bit." The term "bit," a contraction of "binary digit," is commonly attributed to statistician John Tukey, working in conjunction with mathematician John von Neumann. Bits are collected into 8-unit chunks called "bytes," and these collections of 8 bits can represent various types of information. The lowercase letter "a," for instance, can be represented as 01100001, and "b" as 01100010. The music on a compact disc is encoded as a set of 44,100 reading (or samples) per second, with each reading represented by 2 bytes containing 16 bits.

Storage Size

Sizes of files, and the capacity of storage devices, are often referred to as multiples of the byte. A kilobyte (KB) is approximately 1000 bytes, enough information to store about 150 words, or about half a page of text from a paperback book. As larger units are used, the naming system employs other metric prefixes, with each step up representing a multiple of either 1000 or 1024, depending on the device. Thus, a megabyte (MB) is approximately 1000 KB, and a gigabyte (GB) is approximately 1000 MB. Units beyond the gigabyte include the terabyte (TB), petabyte (PB), and exabyte (EB).

Magnetic Storage

Since grains in a magnetic medium can be magnetized with the north pole pointed in either of two directions, magnetism is an ideal medium for representing binary information. In addition, since information stored in this way is relatively stable, it is useful for long-term storage. Finally, since this magnetism can be reset easily using an electromagnet, magnetic media are easy to erase and rewrite.

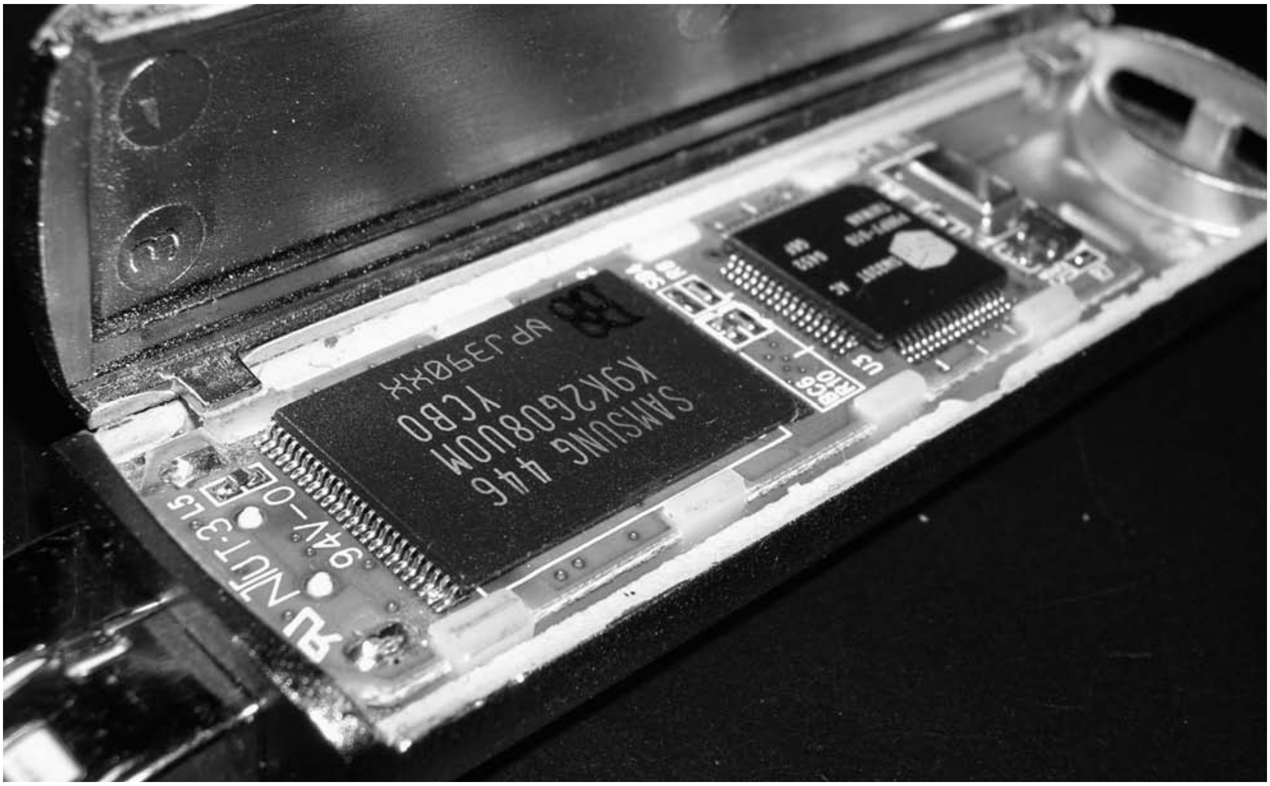
A magnetic hard disk employs one or more spinning platters coated in a magnetic medium. An arm with tiny electromagnetic heads floats over the surface of the disk and is used to magnetize regions of the disk corresponding to the "1"s and "0"s of binary code. To retrieve information, the disk spins past the heads, generating current that corresponds to the code stored on the disk. While the principle is straightforward, it has been a remarkable feat of engineering to create disks that spin up to 7200 revolutions per minute with arms that can travel across the surface of a platter 50 or more times per second as they seek and write information. Even so, writing and retrieval speeds have not increased over time at the same exponential rate as the amount of information that can be stored on such disks, resulting in undesirable lags.

Even in the early twenty-first century, long-term backup of computer information is often done on low-cost magnetic tape, with bits of information laid down as magnetic regions on moving tape. However, since the information is laid down on a long piece of tape, there can be no random access of information, limiting its usefulness in everyday applications. Until recently, digital camcorders used magnetic tape to record video; however, the desire to have random access of footage and recent advances in hard drive and other storage techniques have brought on a new generation of tapeless camcorders.

CDs, DVDs, and Flash Memory

Both CD and DVD players are optical devices that use lasers to read the shiny and dull spots encoded on a plastic disk. Information is recorded by burning nonreflective pits into the surface of the disk to represent "0"s and leaving the reflective surface to represent "1"s. When the disk is played, it spins past a laser. When the light encounters a pit, it is not reflected, and the player registers an "off" signal ("0"), and when the light bounces back off a shiny region, the player registers an "on" signal ("1"). This information is interpreted by a small computer in the player.

Many devices, including digital cameras, camcorders, video game consoles, and cell phones, use flash memory, which can store large amounts of information on small cards that have no moving parts. This technology employs an array of microscopic transistors



USB flash drives are smaller, faster, and have thousands of times more capacity than floppy disks or CD-ROMs. Flash memory stores and retrieves information accurately most of the time, but the devices are not problem-free.

through which current may pass. Whether this current passes through or not is controlled by what is called a "floating gate," and the path through the transistor can be electrically opened or closed. This method allows the transistor to have the two states needed for binary code. Sections of flash memory can easily be reset (erased) by flushing out the electrons trapped in the floating gate. One of the primary benefits of this technology is that information can be stored on a card with no moving parts, improving both access speed and portability.

Data Rot and Error Correction

Tape, hard disks, CDs, and flash memory store and retrieve information accurately most of the time, but they are not problem-free. Errors and noise can happen in an electromechanical recording system—"1"s that should have been "0"s, and vice versa—which diminish information accuracy. Mathematical methods are used to check for and correct errors. For example, cyclic redundancy check (CRC) coding algorithms calculate a fixed-length binary sequence (code) for each block of data using polynomial division in a finite field. The codes and data blocks are stored together, and they can be checked after transmission or retrieval. CRC was invented by mathematician W. Wesley Peterson, who also devised many error-correcting codes.

Even if the recording is perfect, the media that hold binary code can degrade in a variety of ways over time. For instance, magnetic media can lose their magnetic orientation, especially if they are subjected to a strong magnetic field. In addition, the substrates on which the magnetism is stored—the platters on hard drives and plastic backing on magnetic tape—will invariably degrade over time. Even the plastic on CDs and DVDs will begin to break down, and flash memory floating gates will ultimately leak the electrons that maintain data in their flash memory transistor states. Even if the storage media and binary information survive over time, there is a real chance that in the future there may not be hardware available to read information encoded in an outdated media.

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Jeff Goodman

See Also: Digital Images; MP3 Players; Personal Computers.

Disease Survival Rates

Category: Medicine and Health.
Fields of Study: Data Analysis and Probability; Number and Operations.
Summary: Sophisticated mathematics is used to calculate disease survival rates and to help doctors and patients make treatment decisions.

Disease survival rates indicate the seriousness of a certain disease, and the prognosis of a person with the disease based on the experience of others in the same situation (in terms of the stage of the disease, gender, and age). "Overall survival rate" is defined as the percentage of people who are alive after a specific period of time after diagnosis with the disease, which is computed using the following formula: Overall Survival Rate = 100 (Number alive at the end of a time period ÷ Number alive at the start of a time period).

Standard time periods such as one, five, and 10 years are often used. For instance, the five-year overall survival rate for stage-I breast cancer is said to be "95%" if 95% of all people who are diagnosed with stage-I breast cancer live for at least five years after diagnosis. Conversely, 5% of these people die within five years.

Survival rates depend on many factors, including both the type and stage of disease, as well as age, gender, health status, lifestyle, and treatment. Doctors and researchers use survival rates to evaluate the efficacy of a treatment, compare different treatments, and develop treatment plans. For example, the treatment having the highest survival rates over time is usually chosen. If treatments have similar survival rates but different numbers of side effects, the treatment with the fewest number of side effects is often selected.

Other Types of Survival Rates

Overall survival rates have some limitations. First, they do not distinguish causes of mortality within a given time period. For instance, a death may be caused by a car accident rather than by the disease. Second, they fail to indicate whether the disease is in remission or not at the end of the time period. Moreover, they do not directly provide the prognosis for a specific patient. For instance, the 95% five-year survival rate for stage-I breast cancer does not guarantee that every patient will survive more than five years. When considering only deaths caused by the disease, relative survival rate or cause-specific survival rate is often used. Relative survival rate is the ratio of the overall survival rate for people with the disease to that for a similar group of people in terms of age and gender without the disease.

One advantage is that relative survival rates do not depend on the accuracy of the reported causes of death. On the other hand, cause-specific survival rate is computed by treating deaths from causes other than the disease as withdrawals so that they do not deflate the survival rate due to the disease. When using this rate, there is no need to involve a similar group of people without the disease. Sometimes more detailed survival rates in terms of the status of a disease after a given period of time, such as disease-free survival rate and progression-free survival rate, are of interest. The computation for disease-free survival rate is similar to that of the overall survival rate except that the numerator is the number of patients who are cured at the end of the time period. Similar computation applies to the progression-free survival rate except that the numerator is the number of people who are alive and still have the disease, but the disease is not progressing at the end of the time period. As before, disease-free and progression-free survival rates can be adjusted by filtering out the effect of deaths from causes unrelated to the disease.

Survival Function

Related to survival rates, the survival function is a mathematical function that uniquely determines the probability distribution of a random variable. In survival analysis, the random variable of interest is survival time or time to a certain event, denoted by T. For instance, survival time could be time until recovery from a disease, or time to death. The survival function for T is a function of time point t defined as

$$S(t) = P(T > t)$$

which is the true probability that the survival time of a subject is beyond time t. The survival rates with an adjustment for deaths because of unrelated causes are estimates of the survival function at some t based on existing data. For a study with n patients, the survival function can be estimated by the empirical survival function:

$S_n(t)$ = Number of patients not experiencing the event up to t/n.

In follow-up studies, however, a patient with a certain disease may withdraw, die from other causes, or still be alive at the end of the study. In such cases, the survival time T of the patient is not exactly observed but only known to be greater than a certain time (withdrawal time, death time, or time at the end of the study) called "censoring time." Then T is said to be right-censored, and the resulting set of data is called right-censored data. Based on right-censored data, the survival function can be estimated by

$$S_{KM}(t),$$

the K-M estimator developed by statisticians Edward Kaplan and Paul Meier in 1958. As a special case,

$S_{KM}(t)$ coincides with $S_n(t)$ when there is no censoring.

When estimating survival probability,

$P(T > t)$ at a given time t,

$S_{KM}(t)$ or a cause-of-death-adjusted survival rate introduced earlier can be used. Taking a more sophisticated approach, $P(T > t)$ can be estimated using a confidence interval. For example, one may conclude that, with 95% confidence, $P(T > t)$ is between two numbers, say 0.80 and 0.90. Such a confidence interval can be constructed using $S_{KM}(t)$ and its variance estimate from statistician Major Greenwood's formula based a normal distribution.

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See Also: Data Analysis and Probability in Society; Functions; Statistics Education.

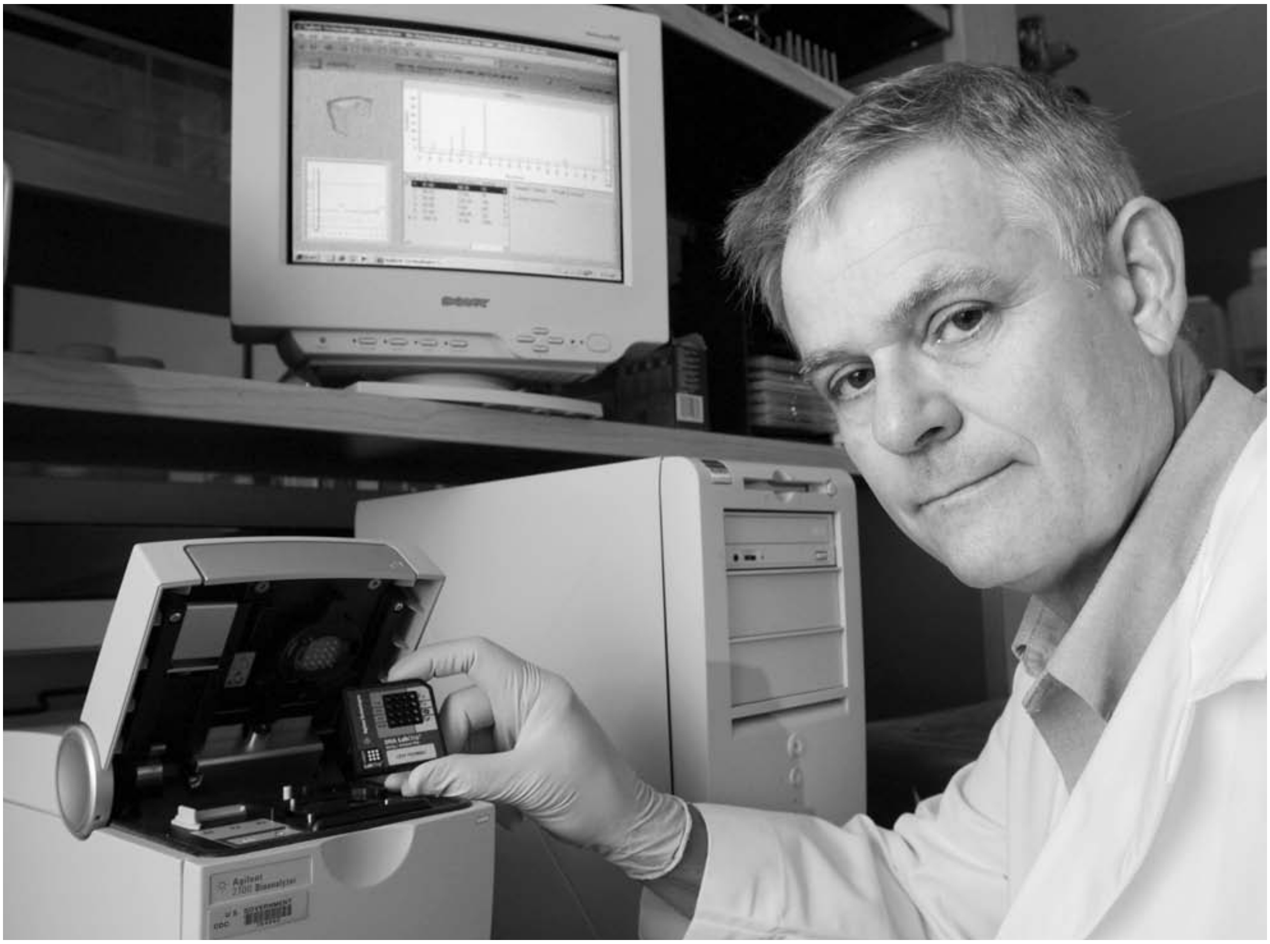
Diseases, Tracking Infectious

Category: Medicine and Health.
Fields of Study: Communication; Data Analysis and Probability.
Summary: Physicians and mathematicians have long worked together to develop and use models that track the spread of infectious diseases in order to develop appropriate countermeasures and responses to halt the disease spread.

The health of societies relies on quickly and correctly tracking and predicting the growth and spread of disease in populations. Epidemiology is a mathematically rich area. Exposure and infection are both probabilistic processes, and tracking infectious diseases is a dynamic application of mathematics. The World Health Organization (WHO) and other organizations concerned with public health use mathematical models in their decision-making, such as when WHO analyzed the risks and benefits of travel restrictions during the early twenty-first-century H1N1 (swine flu) epidemic. Epidemiology has a long history with important societal connections. Some trace one early use of mathematical modeling for disease to eighteenth-century mathematician Daniel Bernoulli. He presented an analysis of smallpox morbidity and mortality to demonstrate the efficacy of vaccination.

Nineteenth-century physician William Farr is often called the "father of epidemiology" and was responsible for the collection of official medical statistics in England and Wales. His most important contribution was to set up a system for routinely recording causes of death. Physician John Snow is frequently cited as using graphical methods to propose a mechanism of transmission and the source of a cholera epidemic in nineteenth century London. Epidemiologists using mathematical and statistical models have been influential in research, treatment, and some methods of prevention for potentially devastating diseases, like tuberculosis, smallpox, typhus, and malaria.

Infectious diseases are a leading cause of death for humans. In order to understand the dynamics of tracking infectious disease at the population level, it is important to understand the responsible mechanisms at the individual level. Infectious disease is caused by a pathogenic agent (for example, a virus, bacterium, or parasite) transmitted through one of many methods, such as air or body fluids. One method scientists have developed for investigating why outbreaks of disease take place and how to contain or end them is to design a system of surveillance and data collection from individual cases, which can then be used to model the infection's trajectory through a population. Other



This microfluidic labchip was used in a CDC bioanalyzer to evaluate Mycobacterium cosmeticum *strains. There are some 115 species of* Mycobacterium, *causing infectious diseases like tuberculosis and leprosy.*

times, they may use data from past similar situations to extrapolate possible solutions.

Surveillance of Infectious Disease

Central public health institutions have created computer systems to monitor emerging outbreaks of illnesses. Traditional notification has relied on disease reporting by laboratories and hospitals. However, the first indications of an outbreak usually occur before a formal diagnosis. People respond to illness with a variety of behaviors to illness that can often be tracked; for example, the number of visits to emergency rooms, or purchases of over-the-counter drugs. Other people's behaviors are more difficult to track, such as those people who continue their daily routines even though they feel sick. Systems of surveillance may compile data from many sources to look for unusual patterns or significant increases in activities like emergency room visits. Another approach, based on Internet search queries, collects disease-related searches. The searches are linked to geographic mapping tools and are used to identify clusters of symptoms. Further analysis and modeling using mathematical and statistical methods are needed to estimate the potential impact of a disease outbreak.

Modeling Infectious Disease

Quantitative analysis describes probable disease trajectories for predicting impact over time. The parameters may include the variables of time, geographic location, population density, contact rate, and saturation, as well as the personal characteristics of those who contract the disease. For example, eighteenth-century mathematician Daniel Bernoulli created mathematical models for smallpox to support the use of inoculations. At the turn of the twentieth century, British physician Ronald Ross began to develop mathematical models to help him understand malaria's trajectory, rate of progression, and probability of infection. He received the Nobel Prize in Physiology or Medicine in 1902, indicating the importance of his mathematical contributions to epidemic theory. Another early twentieth-century model is the Reed–Frost epidemic model, which was developed by scientists Lowell Reed and Wade Hampton Frost. It models disease transmission via person-to-person contact in a group and includes concepts like a fixed probability of any person coming into contact with any other individual in the group.

Quantitative research continued throughout the twentieth century and continues to be active in the twenty-first century. There are many large agencies that use epidemiological models, such as WHO and the U.S. Centers for Disease Control and Prevention (CDC). As medicine and technology advance, new variables become important in models; for example, global air travel, which brings previously isolated populations into greater contact with one another, along with new vaccinations and vaccination policies. Differential use of longtime practices, like quarantining sick and potentially exposed individuals, may also be a factor.

Other models incorporate seasonal information, such as varying contact rates, which can be affected by societal structures, such as school schedules. In the latter twentieth century, computer networking and the subsequent spread of computer viruses have led mathematicians and others to extend epidemiological models to research and model the spread of computers worms and viruses using mathematical techniques, such as directed graphs and simulation. In such an active field of research, new technologies and methods for quickening the pace of identifying patterns of disease are expected to be developed.

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Douglas Rugh

See Also: Disease Survival Rates; HIV/AIDS; Mathematics, Applied; Viruses.

Division

See *Multiplication and Division*

Domes

Category: Architecture and Engineering.
Fields of Study: Geometry; Measurement.
Summary: Domes have been used throughout history to cover open spaces.

In architecture, a dome is a hemispherical structure with a circular, polygonal, or elliptical base that is usually used to cover a large open space. It developed as a generalization of the full revolution of an arch around a vertical axis.

Early domes appeared in small buildings and tombs of the ancient Middle East, India, and the Mediterranean. Because these domes consist of horizontal layers of materials progressively cantilevered inward until they reach the top of the roof, they are considered "false domes" and called "corbel domes." True domes present the characteristic of having a continuously changing slope ranging from being vertical at the base to horizontal at the top, which requires adaptable roofing materials.

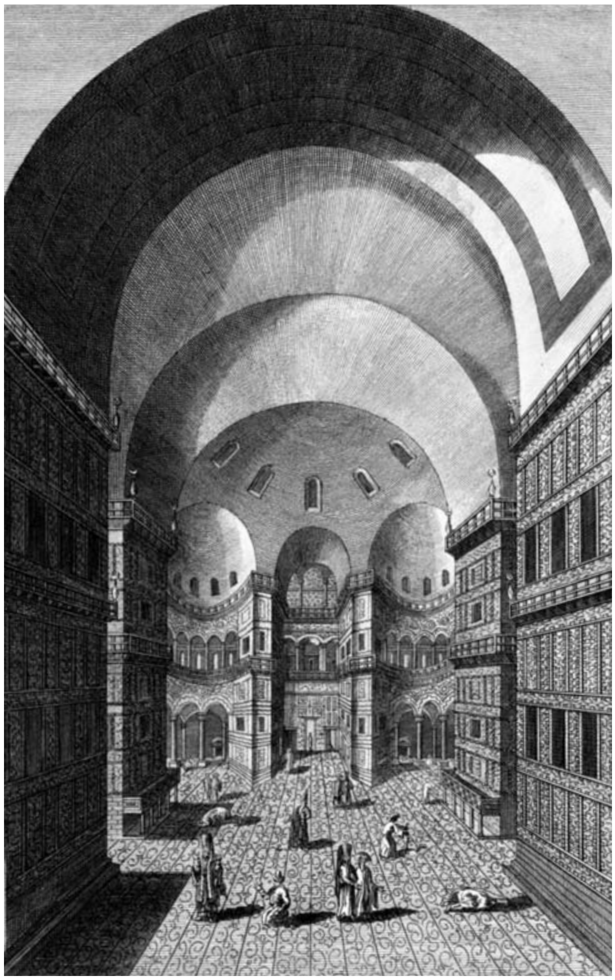
Large-scale masonry domes were first introduced by the Romans in public buildings, such as baths, temples, mausoleums, and basilicas. With an interior diameter of 142 feet, the Pantheon, built during the second century in Rome, remained the largest dome until 1881 and is still the world's largest unreinforced concrete dome. Built on a rotunda, it exerts tremendous thrusts on the perimeter walls. It is not only an engineering triumph but also a tremendous achievement in sacred geometry and cosmography. Its hemispherical ceiling has regularly been compared to the vault of heaven.

Carried on four pendentives, the 102-foot central dome that covers Hagia Sophia in Istanbul is a masterpiece of Byzantine architecture. Built in the sixth century, it exemplifies the full development of the pendentive (a triangular segment of a sphere) as a constructive solution allowing the construction of a dome over a square nave. Volumetric transitions and intersections are critical components of the geometry of architecture.

Built from 1420 to 1436 under the direction of Brunelleschi, the dome of Santa Maria del Fiore in Florence succeeded Hagia Sophia as the largest masonry dome in the world, a record it stills holds. Brunelleschi designed an eight-sided double dome shell without exterior buttresses that did not require any supporting framework during construction. Standing at about 165 feet above ground level, the interior of the dome is approximately 100 feet tall and spans 139 feet. The dome weighs more than 40,000 tons and required the use of more than 4 million bricks.

Domes became increasingly popular during the Renaissance, the Baroque era, and the nineteenth century. Influenced by the Pantheon and Santa Maria del Fiore Bramante, Michelangelo designed St. Peter's Basilica in Rome.

It has the world's tallest dome, and it inspired one of the most famous landmarks of Baroque architecture: the dome of the Invalides designed by Mansart. Completed in 1711, Wren's three-layer dome for St. Paul cathedral in London influenced the construction of the dome of



Hagia Sophia was built in the sixth century and is a masterpiece of Byzantine architecture.

the U.S. Capitol in Washington, D.C., which ultimately inspired the design of most U.S. state capitols.

Since the late nineteenth century, different materials, such as steel, wood, membrane, and reinforced concrete, have allowed the building of domes covering much larger spaces. The two monumental arches supporting the retractable roof of the Cowboys Stadium in Arlington, Texas, reach a height of 292 feet and span of 1225 feet—making it the largest domed stadium in the world.

Geodesic domes represent another modern type of dome, rejecting the classical arch principle. This type of dome is usually a partially spherical structure constituted of a network of triangular or polygonal facets that are in tension and compression. Because the thrust is equal in all directions, the dome can be anchored directly on the ground. Because of their cost-effectiveness and structural strength, hundreds of thousands of geodesic domes have been built all over the world—most often as a solution to provide shelter for poor families in developing countries, or to house people in extreme weather conditions. In 1960, Buckminster Fuller, who developed the mathematics of this type of dome, designed a geodesic dome two miles in diameter and one mile high at its top that would have covered Midtown Manhattan, and provided the whole district with permanent climate control.

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See Also: City Planning; Engineering Design; Symmetry.

Doppler Radar

Category: Weather, Nature, and Environment.
Fields of Study: Algebra; Geometry; Representations.
Summary: Doppler radar uses the mathematical characteristics of waves to track and predict weather patterns.

Radio detection and ranging, commonly known by the acronym "radar," was initially developed to detect and determine the distance of enemy aircraft when visual methods were insufficient, such as in poor weather or at night. It is commonly traced to the nineteenth century work of physicist Heinrich Hertz, who investigated the reflection of radio waves from metallic objects. Doppler radar is a type of radar that uses the Doppler effect to judge the speed and direction of distant objects. The Doppler effect (also known as "Doppler shift") is a physical property that applies to all types of waves, including sound and light. Mathematician and physicist Christian Doppler presented a paper on this effect in 1842, describing how frequencies of waves change in correspondence to the relative movement between source and observer. In 1948, Hippolyte Fizeau independently discussed the shift in the wavelength of light coming from a star in similar terms. Doppler radar has applications in many fields including aviation, meteorology, sports, and traffic control. For example, Doppler radar is widely used for detecting severe weather, and it is a critical component in wind-shear detection and warning systems for airports.

Mathematics of Waves

The Doppler effect relies on the mathematical properties of waves. Transverse waves, which disturb a medium perpendicular to the direction the wave is traveling, are described in terms of their wavelength and amplitude. Wavelength is the distance between two wave crests or troughs, while amplitude is the height of the wave. An example of this is light. Longitudinal waves produce a series of compressions and rarefactions in a medium and are described by their amplitude and frequency. An example is sound, where amplitude corresponds to intensity (or "loudness") and frequency corresponds to pitch.

A car with a siren emits a series of sound waves of constant frequency. If the car moves toward a stationary observer, the waves will seem to be "bunched up" (to have greater frequency), thus a higher pitch. The same siren moving away will have waves that appear "stretched out," with lower frequency and pitch. Similarly, an oncoming light source will appear more blue, while one moving away will appear more red, corresponding to higher and lower frequencies on the electromagnetic spectrum. The amount of change in frequency is relative to both speed and direction of the moving object. The speed of a moving object can be measured by shooting waves of a known frequency at the object, and then observing the frequency of the waves that bounce from the object to the source. The difference between the outgoing and incoming frequencies is used to calculate speed. Common examples are the handheld radar guns used to measure the speed of automobiles or a thrown baseball. Edwin Hubble, for whom the Hubble Space Telescope was named, used the Doppler effect to help measure the distances to other galaxies. Light from other galaxies looks more red, indicating they are moving away. This "redshift" is commonly used as evidence in favor of the Big Bang theory of the origin of the universe.

Weather Detection

Many consider Doppler radar to be the best tool available for detecting tornadoes, hurricanes, and other extreme weather in the twenty-first century. Weather stations commonly emit radio waves that strike objects like clouds or heavy rain, and reflect back. Meteorologists use this data to determine the speed and direction of a weather system, as well as for probabilistic models to predict the path and potential severity of a storm in a given geographic area. Mathematical algorithms produce color-coded weather maps, weather animations, and other visualizations for new programs or Web sites, indicating how a storm system is predicted to move through a geographic area. Some researchers have used input data from a single radar station and knowledge of the mathematical structure of hurricanes to construct three-dimensional maps.

In the twenty-first century, a system of 21 Atlantic and Gulf coast radar stations, starting in Maine and ending in Texas, gathers real-time data to mathematically estimate the characteristics of hurricanes within 120 miles of the coast. Previously, forecasters had to fly aircraft into oncoming hurricanes and throw instruments overboard to collect data, giving them a lead time of about half a day before hurricane landfall. Other mathematicians have explored numerical weather prediction using Doppler radar and a technique known as "four-dimensional variational data assimilation," which estimates model parameters by optimizing the fit between the solution of a given model and a set of observations the model is intended to predict.

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See Also: GPS; Traffic; Weather Forecasting.

Drug Dosing

Category: Medicine and Health.
Fields of Study: Algebra; Measurement; Number and Operations.
Summary: Mathematicians and scientists calculate optimal drug dosages to help ensure patient health.

Drug dosing is the administration of a particular amount of medication according to a specific schedule. There are two kinds of drugs: prescription drugs and nonprescription drugs (over-the-counter medicine). For prescription drugs, medical doctors normally prescribe the amount and time to take the medication. For over-the-counter drugs, information of drug dosing is usually recommended on the label of the medicine. Drug dosing is common in everyday life, but an error in drug dosing may claim lives or create serious medical burdens. According to a conservative estimate in 2006, drug errors injure more than 2 million Americans per year.

Dosage Measurements

Some drug dosing errors stem from inaccurate measurements and administering improper amounts of chemical compounds to the patient. The first mathematics-related issue is the measurement systems in treatment dosing. Drug dosing normally utilizes the metric system, the apothecary system, or the household system. These are the three main forms of measurement systems in the pharmaceutical industry.

The apothecary system is historically the oldest system in medicine measurement. It consists of grains, drams, ounces, and minims.

60 grains (gr) = 1 dram	8 drams = 1 ounce (oz)	1 fluid dram = 60 minims.

Although the apothecary system was widely used during earlier times, it is rarely used in the twenty-first century. The most widespread dosing measurement in liquid drugs in the twenty-first century is the household system, which is rooted on the apothecary system but uses relatively common items as measurement units. The household system primarily consists of teaspoons (tsp), tablespoons (tbsp), ounces (oz), pints (pt), juice glasses, coffee cups, glasses, measuring cups, drops, quarts (qt), and gallons (gal).

1 tablespoon (tbsp) = 3 teaspoons (tsp)	1 teaspoon (tsp) = 60 drops
1 ounce (oz) = 2 tablespoons	1 juice glass = 4 ounces
1 coffee glass = 6 ounces	1 glass = 8 ounces
1 measuring cup (c) = 8 ounces	1 pint (pt) = 2 measuring cups
1 quart (qt) = 2 pints	1 gallon (gal) = 4 quarts

The household system is convenient and commonly understandable, but it is just an equivalent measure without specific precision; for instance, the size of a coffee cup may vary. A more scientific and precise way is to measure with the metric system. The metric system is accurate, simple, and popular in most scientific experiments, including drug measurements, even though it is not as handy as the household system. It essentially consists of length, volume, and weight measures.

The basic metric length measure is meter (m). Along with the meter are the following:

1 kilometer (km) = 1000 meters	1 decimeter (dm) = 0.1 meter
1 centimeter (cm) = 0.01 meter	1 millimeter (mm) = 0.001 meter

The basic metric volume measure is liter (L). Along with the liter are the following:

1 kiloliter (kL) = 1000 liters	1 milliliter (mL) = 0.001 liter

The basic metric weight measure is gram. Along with the gram are the following:

1 kilogram (kg) = 1000 grams	1 milligram (mg) = 0.001 gram
1 microgram (mcg) = 0.001 milligram	

Each measuring system has its advantages and disadvantages. Administering a drug with a wrong measurement system could result in a fatal error. It is critical to distinguish the different systems and use them appropriately. The following are some basic conversions among the three drug measuring systems.

480 grains = 1 ounce (oz)	1 minim = 1 drop
1 milliliter (mL) = 15–16 drops	1 tablespoon = 15 milliliters

Dose Response, Drug Dosing, and Statistics

Besides dosage measurement, another important aspect in drug dosing is to understand that because of the immune system and drug resistance, efficacy does not necessarily increase as dosage increases. Factors such as body weight and age affect the shape of the dose response curve for each individual. To take account of population diversity, the expected effect within a population is principally considered as the guideline for the recommended drug dosage. For example, over-the-counter medication normally uses age or body weight of the patient as the guide to recommend efficient dosages.

Similar to the efficacy of a drug, for some medicines, side effects or toxicity of a drug need to be simultaneously considered in drug dosing. If the side effect or toxicity is too strong, administering the medicine may kill (rather than cure) the patient. In this regard, it is necessary to identify the maximum tolerated dose of a drug. The maximum tolerated dose is the largest dosage at which the toxicity/side effect has not reached the level to cause the specific harm to the patient, while the minimum effective dose is the smallest dosage to reach the expected treatment effect of the drug. If the minimum effective dose exceeds the maximum tolerated dose, the drug is normally not permitted. If the minimum effective dose is smaller than the maximum tolerated dose, the dosage range in which the drug is both safe and effective is called the therapeutic window of the drug. For example, if the minimum effective dose of a drug is 5 mg daily and the maximum tolerated dose is 12 mg daily, then the therapeutic window of the drug is 5–12 mg daily.

To make an inference on the efficacy and toxicity of a drug at the same time, statistical methods are used. After clinical trials (such as the double-blind experiment), simultaneous inference methods are used to estimate the minimum effective dose and the maximum tolerated dose. One of the well-known methods in identifying dose effects is Dunnett's method for multiple comparisons with a control, developed by statistician Charles Dunnett in the mid-twentieth century. Other effective techniques for identifying the therapeutic window of a drug have been explored by mathematicians and statisticians since that time.

Shelf Life

Mathematics and statistics also intertwine with drug dosing on the shelf life of a drug. For medications that emit chemical compounds over time, the drug effect may be affected by chemical half-lives well before the expiration date. In the United States, the Food and Drug Administration (FDA) requires companies to conduct stability analyses to establish the shelf life of new products. The same is true in many other countries around the world. The conclusions are generally based on statistical sampling and mathematical modeling of data, using estimation methods such as simultaneous confidence segments over time.

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See Also: Data Analysis and Probability in Society; Measurement in Society; Probability.

DVR Devices

Category: Communication and Computers.
Fields of Study: Measurement; Representations.
Summary: Mathematics is essential to the functioning of DVRs, including image processing, compression, and error correction.

Digital video recording devices (DVRs) have become an increasingly prominent factor in the television industry in the twenty-first century. The basic function of a DVR is to record television to a digital format on a disk drive, allowing it to be played back later. Combined with the timer and basic replay functions, this feature allows standard DVRs to perform many functions: store and play back television shows; automatically record specific television programs; and buffer live television to allow pausing and skipping.

Many DVRs can play and record the same program at the same time, a function earlier video recording devices lacked. A 1991 patent by father–daughter team

Eric and Romi Goldwasser is one of the earliest known for digital video recorders. One well-known brand of DVR is TiVo, introduced by engineer Michael Ramsay and computer scientist James Barton in 1999. Both previously worked at Silicon Graphics, Inc., which was a pioneer of computer workstations. Because of these roots in computers and the technology they utilize, some consider DVRs to be computers. Mathematics is essential to the functioning of DVRs, including image processing, compression, and error correction. It also plays a role in digital watermarking, which is widely used to enforce copyright laws.

Statistical analyses of television viewing habits by companies, such as ACNielsen, suggest that DVR use combined with online viewing are significantly changing the pattern of television delivery and assessments of popularity and marketability in the early twenty-first century. TiVo's Ramsay noted, ". . . it's forcing the industry to embrace the Internet . . . and once they embrace it, they will find that their business models change and new opportunities will arise."

Process and Functions

In DVRs, images are captured and stored in binary form. This process differs from older electromechanical systems, like videocassette recorders (VCRs). Raw video files tend to be very large and require sizable storage space, so DVRs use mathematical compression algorithms. Files must then be decompressed before viewing. Decompression is accomplished by hardware or software codec technology, which implements specific formats or standards. Motion Pictures Experts Group (MPEG) created the MPEG-1 format for digital storage in 1993 and MPEG-2 in 1994, which made high-definition television (HDTV) and digital versatile discs (DVDs) possible. The MPEG-4, released in 1999, facilitated digital video for Internet streaming and replaced some proprietary codecs in DVRs to facilitate file transfer.

MPEG compression is typically asymmetric; algorithmic encoders are more complex than their paired systematic decoders. Optimized compression to preserve image quality is achieved by mathematically controlling bit rates subject to constraints on variables like file size or transmission bandwidth. Quality applies not only to individual frames but also to the smoothness of transitions between frames, which affects the user's visual experience of motion. This approach can be formulated as a Lagrange minimization problem, named for mathematician Joseph Lagrange. Two- or three-pass encoding schemes are often used. A first pass collects complexity data for the entire video. Subsequent passes perform the actual encoding based on the information. Algebraic structures known as Galois fields, after mathematician Evariste Galois, are helpful in coding and error correction, and are sometimes paired with Fourier transforms, named for mathematician Joseph Fourier. This pairing is especially true in recorders that incorporate nonbinary, cyclic error correction, such as Reed–Solomon codes, named for mathematicians Irving Reed and Gustave Solomon, as well as for pseudo-random digital dither and randomized channel codes. Recording and compression are also affected by digital watermarking, where extra visible or invisible information is embedding in a digital signal. It can be used to identify ownership, track the file, and prevent recording. Watermarks may be classified by the embedding method, like quantization-type watermarks, which rely on quantization matrices.

Perhaps the best-known brand of DVR is TiVo, introduced in 1999. One of TiVo's features is its ability to employ statistical techniques, such as data mining, to generate recommendations. Viewers can rate shows they watch, and TiVo tracks the ratings, which are then examined for patterns. As of 2004, TiVo had accumulated more than 100 million user ratings on 30,000 different programs. The TiVo algorithm uses a collaborative filtering architecture, which relies on comparing viewer profiles and a viewer's past patterns using several thousand key details, like favorite actors and genres.

However, some users have complained about unusual or extreme matches resulting from this methodology and have intentionally subverted the algorithms by giving false or contradictory ratings. The server architecture is scalable and throttleable, which means as more server resources and user data become available, the system is faster for everyone and perhaps more efficient in finding recommendations for harder-to-match viewers.

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See Also: Digital Storage; MP3 Players; Nielsen Ratings; Predicting Attacks; Televisions.

E

Earthquakes

Category: Weather, Nature, and Environment.
Fields of study: Data Analysis and Probability; Measurement.
Summary: Earthquakes are measured in several ways, the most famous of which is the logarithmic Richter scale.

Earthquakes are the movements of Earth's crust resulting from tectonic plates colliding against each other. This sudden release in energy causes seismic waves that cause destruction. Depending on their severity, earthquakes range from being barely noticeable to causing permanent damage to infrastructure along with a significant loss of life. Most earthquakes are caused by the action of geological faults but they can also be caused by mine blasts, volcanic activity, and subterrestrial activity, such as injecting high-pressure water for geothermal heat capture. The focal point of the earthquake is called the "hypocenter." The point on the ground directly above the hypocenter is known as the "epicenter" of the earthquake. Philosophers, mathematicians, and scientists have long attempted to understand earthquakes. Thales of Miletus thought that earthquakes occurred because Earth rested on water. Mathematician, astronomer, and geographer Zhang Heng invented the first seismograph for measuring earthquakes in the second century. Mathematician Harold Jeffrey theorized that Earth's core is liquid after analyzing earthquake waves. Geologists use statistical methods to try to predict earthquakes.

Seismic Waves

A tremendous amount of energy is released from the epicenter radially outward. As the energy spreads, it is manifested in three forms: compression waves (P waves), shear waves (S waves), and surface waves.

P waves are felt first and do minimal damage. S waves follow the P waves and do minimal damage. It is the slower surface waves (also known as "Love waves") that cause the majority of the damage.

Measurement

The goal of earthquake measurement has been to quantify the energy released. Seismographs are highly sensitive instruments employed to record earthquakes. Conventionally, earthquake magnitudes are reported in the Richter scale. The Modified Mercalli Intensity Scale is commonly used to ordinally quantify (or rank) the effects of an earthquake on humans and infrastructure. Body wave or surface wave magnitudes are also used to measure earthquakes.

Richter Scale

The Richter scale quantifies the amount of seismic energy released during a quake. It is a base-10 logarithmic scale, which means that the difference between an earthquake of rating 2.0 on the Richter scale and 3.0 correlates to a tenfold increase in measured amplitude. Specifically, the Richter scale is defined as

$$M_L = \log_{10} A + B$$

where A is the peak value of the displacement of the Wood–Anderson seismograph (mm) and B is the correction factor. The wave intensity measurements are also logarithmic functions, using variables such as the ground displacement in microns, the wave's period in seconds, and distance from the earthquake's epicenter.

Modified Mercalli Intensity Scale

The Modified Mercalli Intensity Scale has 12 gradations: instrumental, feeble, slight, moderate, rather strong, strong, very strong, destructive, ruinous, disastrous, very disastrous, and catastrophic.

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See Also: Exponentials and Logarithms; Geometry and Geometry Education; Weather Forecasting; Weather Scales.

Educational Manipulatives

Category: School and Society.
Fields of Study: Connections; Problem Solving; Representations.

Summary: Some educators use objects to engage students' attention and encourage them to learn sensorially and experientially.

Educational manipulatives are physical, technological, or virtual objects that are intended to help students learn concepts by taking advantage of tactile and visual explorations.

Mathematical tools and technologies are common in mathematics education. Entire companies and sales catalogues are devoted to such mathematical products, and national and state curriculum standards emphasize their importance in schools. There is a rich history of tools and manipulatives in mathematics classrooms, and these have changed over time along with curricular, industrial, and technological needs and innovations. For instance, in the seventeenth century, the slide rule replaced logarithmic tables in scientific calculation and mathematics classrooms but these in turn became obsolete in the twentieth century because of calculators and computers. Educators, including classroom teachers and university researchers, along with professional designers, continue to create and refine manipulatives and research their effectiveness. Some also work for companies to develop or market these products and materials.

History

Two early developers of collections of learning manipulatives included Friedrich Fröbel (1782–1852) and Maria Montessori (1870–1952). Fröbel was a German educational researcher who is also referred to as the "inventor of kindergarten." He developed a set of manipulative tools called the Fröbel Gifts, which were intended for kindergarten play and learning in the nineteenth century. The fuller development of the manipulatives occurred after Fröbel's death. The Fröbel Gifts set contained objects such as balls, cubes, tiles, sticks, and framed figures that were built out of toothpicks and peas. They were designed to help young students explore mathematical concepts in two and three dimensions. Some of the surfaces were hung from string in order to highlight their cross-sections and symmetries. One focus of Frobel's kindergarten philosophy was free play, which was also carried out in different settings with other objects. For example, the Milton Bradley Company, an American game company established in 1860, sold a curvilinear set of pieces that could form a cylinder. In addition to free play, students ultimately learned to draw what they observed. Ideally, children would revisit concepts they learned using the manipulatives in increasingly sophisticated ways as they progressed through school. For example, in 1869, Edward Wiebe, who was an early proponent of kindergarten education in the United States, suggested that children could explore concepts like the Pythagorean theorem, named for Pythagoras of Samos, long before they understood the square of a number. Frank Lloyd Wright acknowledged the influence of Fröbel's Gifts on his career as an architect. Aspects of Fröbel's legacy continue to be found in manipulative design and in schools, although they have been greatly modified and adapted.

In the twentieth century, Italian physician and educator Maria Montessori, who is well known for the Montessori method of education, also focused on the importance of manipulatives in classrooms. She developed an integrated set of sensorial learning materials that included cylinders, cubes, rods, circles, triangles, polygons, boxes, and binomial and trinomial cubes. Montessori designed activities with educational outcomes in mind. Her ideas became popular in the United States and are still used in the twenty-first century. Montessori schoolteachers challenge students to arrange objects in specific ways so that the students will uncover concepts.

Examples

There have been a wide number and variety of other educational manipulatives created in the twentieth and twenty-first centuries, including polyhedral dice with varying numbers of sides for studying probability; multiplication blocks; algebra tiles that represented polynomials and polynomial operations; multicolored and interlocking Unifix cubes intended to teach number and operations concepts; pattern blocks for studying tessellations and fractions; tangrams for exploring geometry; and geoboards, which are pegged boards on which rubber bands could be placed and stretched to investigate concepts like perimeter and area. The abacus or counting frame that had been in use since antiquity found its way into U.S. schools in the nineteenth century. While it has mostly disappeared from twenty-first century classrooms, it remains important in a few educational contexts, like in classrooms for visually impaired children. Virtual manipulatives have replaced physical objects in some cases. There are even applets that mimic some of the physical manipulatives like pattern blocks, which teach similar concepts while providing different sorts of tactile and visual stimulation.

Effectiveness

There are diverse opinions regarding the effectiveness of manipulatives in mathematics education. In 2005, mathematician David Klein warned, "Too much use of them runs the risk that students will focus on the manipulatives more than the math and even come to depend on them.…Yet many state standards recommend and even require the use of a dizzying array of manipulatives in counterproductive ways." In the final report of the National Mathematics Advisory Panel in 2008, the panel cautioned that

> Despite the widespread use of mathematical manipulatives such as geoboards and dynamic software, evidence regarding their usefulness in helping children learn geometry is tenuous at best. Students must eventually transition from concrete (hands-on) or visual representations to internalized abstract representations. The crucial steps in making such transitions are not clearly understood at present and need to be a focus of learning and curriculum research.

Developmental psychologists and educators David Uttal, Kathyrn Scudder, and Judy DeLoache noted that

> . . . the sharp distinction between concrete and abstract forms of mathematical expression may not be justified. We believe instead that manipulatives are also symbols; teachers intend for them to stand for or represent a concept or written symbol.

Other researchers and teachers counter the claims that there is insufficient evidence; they cite a vast amount of educational literature and anecdotes regarding the benefits of hands-on activities, software, and manipulatives. Many students report that they enjoy the tactile manipulation. Students may also feel satisfied when they discover or confirm mathematical relationships, and this may help them connect to mathematics. Mathematics educators continue to study the effects of various manipulatives and the potential differences between physical and virtual manipulatives on student learning.

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See Also: Calculators in Society; Measuring Tools; Puzzles; SMART Board; Software, Mathematics; Visualization.

Educational Testing

Category: School and Society.
Fields of Study: All.
Summary: Mathematicians and researchers are constantly exploring the validity and reliability of educational testing.

Purpose of Testing

Educational testing is pervasive in modern education at the local, state, and federal levels, and mathematics is one of the most frequently tested areas. The purpose of educational testing is broad and multifaceted: to assess student progress and school accountability; to identify students' strengths and weaknesses, as well as their eligibility or need for special services; to make educational decisions about individuals and groups of students; to choose curriculum and instructional techniques; to reward teachers or schools for performance; and to formulate educational legislation and policies. Students are often placed in courses and special programs as a result of educational testing and may be required to pass tests to graduate from high school or be admitted to schools at all levels, especially colleges and universities.

While some educators, parents, and politicians cite standardized tests for their presumed objectivity in measuring achievement and other skills or attributes, these tests are frequently a source of anxiety and competitive pressure for students. There is an entire industry dedicated to helping students prepare for and pass or score well on these tests. At the same time, researchers are constantly exploring the validity and reliability of tests with regard to fairness for subgroups of students, as well as their actual predictive ability. For example, there is a broad body of research on whether measures like high school grade point average, SAT math scores, or mathematics placement tests are predictors of success in college mathematics courses.

Types of Testing

The decisions that can be made based on testing information depend on the type of test that is administered. There are two different types of tests that provide different types of information: norm-referenced tests (NRT) and criterion-referenced tests (CRT).

NRTs are created for the purpose of comparing students to a norming group, which is composed of students who are similar to the student being tested. The scores of the norming group create the very familiar normal (bell-shaped) curve. NRT scores are typically reported as percentiles, which indicate that a student scored above a certain percentage of the norming group. For example, a student at the 84th percentile scored the same or higher than 84% of the students in the norming group. It is a common misconception with NRTs that students are compared to all other students who have taken the test; however, most NRTs are normed every several years using a new norming group

Students may be placed in courses and special programs as a result of educational testing.

with which test-takers are compared. NRTs are typically very general in nature, covering a broad range of objectives. Items that have a variety of difficulty levels are chosen for NRTs, as these types of items encourage a wide variability in the scores, thus allowing evaluators to more accurately determine how a student compares to others. The SAT and many intelligence tests such as the Wechsler Intelligence Scale for Children are norm-references tests.

Unlike NRTs, which are used to compare students to each other, CRTs are used to determine if a student has mastered a given set of standards. CRTs are typically narrow in focus, testing only a few objectives, and are generally focused on those objectives that are deemed most important. Scores for CRTs are typically reported as percentage correct or as scaled scores. Proficiency is determined by comparing a student's score to an established cut point. Many schools regularly administer end-of-grade or end-of-course tests through which student achievement in mathematics subjects is measured.

Issues in Educational Testing

Two primary concerns with educational testing are the validity and reliability of the assessment. "Validity" in this context refers to whether a test is appropriate for the population being tested, as well as whether it appropriately addresses the content it is intended to measure. Educators from around the United States have expressed concern as to whether the tests that are currently being used to measure student achievement are valid and reliable. In an effort to address this concern, many states have undergone revisions of their tests in the past several years.

An additional concern with educational testing is in how student progress is measured over time. Statisticians have developed a variety of growth models to determine if individual students are improving as they move through school. These models may focus on improvement from grade level to grade level, or they may focus on student progress within a single school year (referred to as "value-added" or "teacher impact"). An ongoing issue with measuring student progress over time lies with the relationship between the assessments and the statistical measures that are used to analyze assessment data. Growth models are all based on certain assumptions about the assessments, which may or may not be met. In order to determine the impact of schools on student learning, one must ensure that the assessments and the statistical models used to analyze the data are compatible.

Test Analysis

Standardized educational tests undergo a variety of analytical procedures to evaluate their effectiveness at measuring a construct. Item analysis is frequently conducted to determine if items are functioning the way test developers intended. This analysis of student responses to items provides the difficulty index and the discrimination index. The difficulty index is simply the ratio of the number of students who answered the item correctly to the number of students who attempted the item; a higher difficulty index indicates an easier item. The discrimination index provides information on how well an item differentiates between students who performed well on the test and students who did not. A positive discrimination index indicates that those students who performed well overall on the test were more likely to answer the item correctly, while a negative discrimination index indicates that those students who performed poorly overall were more likely to answer the item correctly. For NRTs, item discrimination is particularly important, and test developers attempt to develop items that will have a high discrimination index.

Modern test analysis also uses a process called "item response theory" (IRT) to determine the effectiveness of a test or test item. IRT evaluates items based on the parameters of item difficulty, discrimination, and guessing and provides test developers with the probability that a student with a certain ability level will answer an item correctly. In addition, IRT allows for a more sophisticated measure of a test's reliability.

Trends in Educational Testing

Recent trends in educational testing have been focused around making international comparisons of student achievement. The most well known of these comparisons are the Third International Mathematics and Science Study (TIMSS), conducted in 2007, and the Program for International Student Assessment (PISA), conducted in 2006. The TIMSS included fourth-grade students from 36 countries and eighth-grade students from 48 countries. Participating countries submitted items for the test and the test was developed by a committee of educational experts from various nations. The TIMSS also collected information on students' background, including attitudes toward mathematics and science, academic self-concept, home life, and out-of-classroom activities. The PISA focused on problem solving in mathematics and science and on reading skills. The 2006 PISA included 15-year-olds from 57 countries. The goal of PISA is to determine students' abilities to analyze and reason and to effectively communicate what they know. Additional international studies involving educational testing include the International Adult Literacy Survey, the Progress in International Reading Literacy Study, and the Civics Education Study.

In the United States, the National Assessment of Educational Progress (NAEP) is used to compare student achievement across states. NAEP includes students from grades 4, 8, and 12 and is designed to provide an overall picture of educational progress. Schools are randomly chosen to participate and students within those schools are also randomly chosen. The NAEP tests students in mathematics, reading, science, writing, civics, economics, and history.

The public focus on educational testing in the United States sharpened with the implementation of the No Child Left Behind (NCLB) Act in 2002. For the first time in American history, schools were publicly designated as "meeting" or "failing to meet" state standards, and issues of educational testing were brought to the forefront. Organizations like Achieve began closely examining how schools were preparing students for college and the work force and began working with state officials and business executives to improve student achievement. Educational testing is a valuable tool for these types of organizations, providing information on the effectiveness of American schools.

Controversies in Educational Testing

Not everyone believes educational testing is useful or meaningful, and there are many arguments against the use of such tests. For example, studies have suggested that the SAT is both culturally and statistically biased against African Americans, Hispanic Americans and Asian Americans. Others have found that socioeconomic status is correlated with performance on the SAT, which is believed to be related to the fact that students from wealthier families can afford expensive test preparation courses or multiple retakes of the test, both of which have been demonstrated to improve test scores in some cases. Others have documented a gender gap in SAT mathematics scores that is not easily explained by issues like the difference in the number of male and female test takers.

On many tests, stereotype threat or vulnerability has also been shown to affect test scores when race, gender, or culture are cued before a test. In response, some have advocated that self-identification should occur after a test. Researchers have also shown that the structure or methodology of the test can have an effect on performance. For example, female test scores on tests of spatial ability can improve when "I don't know" is removed as an answer, or when ratio scoring or un-timed tests are used. Finally, there are many who believe that there are concepts that cannot be adequately measured by standardized assessments, even when the answers are not exclusively multiple choice and that using standardized tests as a primary method of assessment leads to "teaching to the test" rather than a broader educational experience for students.

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See Also: Curriculum, K–12; Diagnostic Testing; Learning Exceptionalities; Learning Models and Trajectories.

EEG/EKG

Category: Medicine and Health.
Fields of Study: Algebra; Data Analysis and Probability.
Summary: EEGs and EKGs visually convey important information about a patient's heart and brain.

Electrocardiography (ECG or EKG) and electroencephalography (EEG) are graphic representations of bioelectric activities of the heart and brain, respectively. EKG quantifies the rhythm of heart contraction—measurements that can be used to identify damage to various myocardial muscles. EEG is used in the diagnosis of epilepsy, seizure, and encephalopathy. The production of EKG and EEG signals is grounded in mathematical analysis. Diverse mathematical and statistical techniques, including applications of calculus and chaos theory, are also used to analyze and interpret signals related to conditions such as sleep disruptions, seizures, and mental illness.

EKG

EKG is a graphic representation of the myocardial contraction (systole) and relaxation (diastole) caused by depolarization of the heart. In the myocardial muscles, depolarization is an increase of membrane potential, and repolarization is a decrease of membrane potential. A typical EKG consists of P, Q, R, S, and T waves. Atrial depolarization normally begins at the SA node and is represented as the P wave. The depolarization proceeds to ventricles, which causes the ventricular depolarization (QRS complex) and then ventricular repolarization (T wave).

EKG was first systemically studied in humans by Augustus Walker in 1887. In 1903, Willem Einthoven created a reliable EKG device based on the galvanometer. Einthoven was awarded a Nobel Prize in 1924 for his invention. EKG provides information on heart contraction and the abnormality of EKG has been used to diagnose the area of myocardial damage. Heart rate variability is a quantification of fluctuations of EKG complex; a healthier heart has higher variability.

The production of EKG signals can be explained by an idealized model in which both intracellular and extracellular currents are confined to the direction parallel to the propagation of the plane wavefront. When there are no external currents, the relationship between the potential inside the membrane V_i and the potential outside the membrane V_o can be represented as

$$V_i = \frac{r_i}{r_i + r_o} U_m \quad \text{and} \quad V_o = -\frac{r_o}{r_i + r_o} U_m$$

where r_i and r_o are the intracellular axial resistance and extracellular axial resistance, respectively; and U_m is the membrane potential. During the depolarization, the transmembrane current i_m is

$$i_m = \frac{1}{r_i + r_o} \frac{\partial^2 V_m}{\partial x^2}$$

where the direction of positive current is defined as the direction of the positive x-axis. For the depolarization of cardiac tissue, a double layer appears at the wavefront with the dipole orientation in the direction of propagation.

A pair of electrodes can be used to produce one EKG signal; the output from the pair is called a "lead." Usually more than two electrodes are used and combined into pairs. Clinically, a 3-lead or 12-lead EKG is used to diagnose heart diseases. For a traditional 3-lead EKG, leads I, II, and III are defined as

$$\mathrm{I} = V_{LA} - V_{RA}$$

$$\mathrm{II} = V_{LL} - V_{RA}$$

$$\mathrm{III} = V_{LL} - V_{LA}$$

where *LA, RA,* and *LL* denote left arm, right arm, and left leg, respectively.

EEG

EEG is a recording of the electric potential of thousands or millions of neurons within the brain. The electrodes are placed on the scalp at certain anatomical locations. EEG was first systematically analyzed by Hans Berger in 1920, who introduced the term "electroencephalogram" to indicate fluctuations recorded from the brain. EEG waves are usually irregular and cannot be classified in the normal brain.

However, four characteristic frequencies have been identified: Alpha (8-13Hz), Beta (14-30Hz), Theta (4-7Hz), and Delta (below 3.5Hz) waves. Under pathological conditions, like epilepsy, distinct patterns can be observed and used to help predict the onset of the condition.

Using a simplified model of the brain and surrounding tissues as a sphere with several shells, it is possible to compute the EEG based on the measured intracerebral currents at the scalp. The field potential can be represented as a function of intracerebral currents or of the membrane potential. In an infinite, isotropic, and homogeneous medium, because of injected current densities $\vec{j}_i$ at a point $\vec{r}$, the electrical potential at a point $\vec{r}_0$ lying at a distance, R, from $\vec{r}$ ($R = |\vec{r} - \vec{r}_0|$)is the following:

$$V(\vec{r}_0) = -\frac{1}{4\pi\sigma} \int_{vol} \frac{\mathrm{div}\,\vec{j}_i}{R} d^3r$$

where σ is the conductivity of the medium; the operator div indicates differentiation of a vector. When the injected current densities originate at the cell membrane, by assuming that the neuronal membrane is equivalent to a double layer with an intracellular membrane potential V_m, the potential at a point $\vec{r}_0$ is given approximately by

$$V(\vec{r}_0) \approx -\frac{\sigma_i}{4\pi\sigma_e} \int_{surf} V_m(\vec{r})\, d\Omega(\vec{r} - \vec{r}_0)$$

where σ_i is the intracellular conductivity, σ_e the extracellular conductivity, and

$$d\Omega(\vec{r} - \vec{r}_0)$$

is the solid angle subtended by an infinitesimal surface on the membrane surface and seen from the extracellular point $\vec{r}_0$.

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See Also: Medical Imaging; Nervous System; Pacemakers.

Egyptian Mathematics

Category: Government, Politics, and History.
Fields of Study: Connections; Geometry; Measurement; Number and Operations; Representations.

Summary: Ancient Egyptians were adept at engineering and geometry and deeply dependent on accurate measurements of the annual Nile flood.

Our knowledge of Egyptian mathematics (3000–1000 B.C.E.) is based on hieroglyphic writings found on stone or as script (hieratic and demotic) in multiple papyri. Preserved in tombs and temples in the Nile valley, a papyrus is a narrow scroll of paper, about 15 feet in length, made by interweaving tiny strips of a water reed called *papu*. The key documents are the Moscow, Rhind, Rollin, and Harris papyri. These works are generally thought to be textbooks used by scribes to learn mathematics and solve problems.

In ancient Egypt, mathematics was used for many purposes necessary to everyday life: measuring time, drawing straight lines, measuring and recording the level of the Nile floodings, calculating land areas, and managing money and taxes. The Egyptians were also one ancient culture that came closest to determining the true length of Earth's year with mathematics. Perhaps most well known to the modern world are the fantastic tombs, pyramids, and other architectural marvels constructed using mathematics. Though their knowledge ranged from arithmetic calculations to algebraic rules to geometrical formulas to numerical ideas, historians consider the Egyptians' mathematical achievements to be somewhat less advanced compared to the Babylonians.

Egyptian Number System

Egyptian numbers are written using a simple grouping system whose symbols denote powers of 10. Their symbols included a vertical staff (10^0), heel bone (10^1), scroll (10^2), lotus flower (10^3), pointing finger (10^4), tadpole (10^5), and astonished man (10^6):

Using these symbols, a number was expressed additively. For example, the base-10 number 4501 was represented by a visual collection of 4 lotus flowers, 5 scrolls, and 1 vertical staff. As no place-value system is involved, these symbols can be written in any order or arrangement visually—they equal a numerical value as a group. Though able to represent large values of numbers with these symbols, the Egyptians' lack of place values deterred their ability to calculate proficiently using algorithms.

Again represented by hieroglyphic symbols, Egyptian fractions were restricted to unit fractions (numerator of 1) except for the special fraction 2/3. For example, the unit fraction 1/3 was represented by an ellipse (or dot) placed visually over 3 vertical staffs. The Egyptians had no symbol for zero as a place holder but such was not really needed because of their simple grouping system and use of distinct symbols for each power of 10.

Egyptian Arithmetic

Addition and subtraction are quite easy using the Egyptian numbers, involving only the union or removal of the grouped symbols. In addition, a symbol that appeared 10 times was replaced by the next higher level symbol; for example, 10 vertical staffs could be replaced by 1 heel bone. Similarly, in subtraction, a symbol could be traded in for 10 of the next lesser symbol if such was necessary. For example, to perform $23-8$, a heel bone could be traded for 10 vertical staffs so that 8 vertical staffs could be taken from the 13 vertical staffs.

Egyptian multiplication involved repeated addition, using a doubling process along with a counter. For example, to multiply 23×13, their process (in modern notation) would look like the following, with the counter on the right:

23	1*
46	2
92	4*
184	8*

Using the starred counters $(1+4+8=13)$, the product is obtained by adding the associated numbers $(23 + 92 + 184 = 299)$. The key to this multiplication is the distributive process, since

$$23\times 13 = 23\times(8+4+1) = (23\times 8)+(23\times 4)+(23\times 1)$$
$$= 184 + 92 + 23 = 299.$$

Thus, base two notation also is the underlying principle, since

$$13 = (1)(2^3) + (1)(2^2) + (0)(2^1) + (1)(2^0).$$

These processes of duplation and mediation (doubling and halving) remained as standard algorithms in Western mathematics until the 1500s.

Division required an inversion of the multiplication process. For example, to divide 299 by 23, the Egyptian scribe determined what number times 23 would produce 299, using a process like the following (in modern notation):

23*	1
46	2
92*	4
184*	8

Using the starred sums, $23 + 92 + 184 = 299$, the desired factor (or quotient) is obtained by adding the associated numbers, or $1 + 4 + 8 = 13$. The division process becomes complicated when no combination of the starred numbers equals the desired sum (for example, 300 divided by 23), requiring the use of unit fractions:

23*	1
46	2
92*	4
184*	8
1*	1/23

For more difficult divisions (for example, 301 divided by 23), considerable creativity was needed.

To aid in their computations, the Egyptians created tables for doubling and halving numbers, complemented by special $2/n$ tables that would help avoid odd-number situations. For example, the Rhind papyrus had a $2/n$ table for the odd numbers 5–101.

Egyptian Algebra

Though without an algebraic notation, the Egyptians solved numerous types of algebraic equations, known as "aha" calculations. The majority of their problems were linear equations with one unknown (called the "heap"). Their solution process involved the method of false position, where an initial guess is made, examined, and then adjusted to obtain the correct solution. This same process is now fundamental to the area of numerical analysis and is used extensively for scientific computing using computers.

Consider this Egyptian problem, "Heap and a seventh of the heap together give 19." In modern notation, the associated linear equation is $x + x/7 = 19$, while their step-by-step solution was the following:

Make a guess for heap, for example, 7

$$\text{Then } 7+\frac{7}{7}=8$$

$$\text{But } \left(2+\frac{1}{4}+\frac{1}{8}\right)(8)=19$$

$$\text{Thus, heap } =(7)\left(2+\frac{1}{4}+\frac{1}{8}\right)=16+\frac{1}{2}+\frac{1}{8}.$$

The processes of multiplication and division, as well as the law of associativity, play very important roles:

$$\begin{aligned}&\left(2+\frac{1}{4}+\frac{1}{8}\right)(8)\\ &=\left(2+\frac{1}{4}+\frac{1}{8}\right)\left(7+\frac{7}{7}\right)\\ &=\left(2+\frac{1}{4}+\frac{1}{8}\right)\left[(7)\left(1+\frac{1}{7}\right)\right]\\ &=\left[\left(2+\frac{1}{4}+\frac{1}{8}\right)(7)\right]\left(1+\frac{1}{7}\right)\\ &=\left(16+\frac{1}{2}+\frac{1}{8}\right)\left(1+\frac{1}{7}\right)\\ &=19.\end{aligned}$$

The majority of the Egyptians' "aha" problems created practical situations requiring the use of ratios and proportions, such as determining feed mixtures or combinations of grains to make bread. In some instances, the Egyptians did use special hieroglyphic symbols as part of their algebraic work, including "plus" (legs walking left to right), "minus" (legs walking right to left) and other ideograms for "equals" and the "heap."

Egyptian Geometry

The Egyptians' geometry was rooted in an algebraic perspective, devoid of any evidence of generalization or proof. Approximately one-fourth of the problems found in the papyri are geometrical—focusing on practical measurements, such as the calculation of land areas, or volumes of storage containers. Similar to the Babylonians, the Egyptians used prescriptive formulas. For example, they viewed a circle's area as equal to that of a square erected on 8/9 of the diameter. That is,

$$A=\left(\frac{8}{9}(2r)\right)^2=\frac{256}{81}r^2$$

implying their value of π approximated 3.160493827.

Historians agree that the Egyptians knew key formulas for computing the area of a triangle, the volume of a cylinder, some curvilinear areas, and even the volume of the frustum of a square-based pyramid. These formulas were apparently put to great use by the Egyptians in their accurate construction of the pyramids, feats that required a solid understanding of ratios, proportions, dihedral angles, and even astronomy. No evidence suggests the Egyptians knew of the relationships described by the Pythagorean theorem. Some of their geometrical prescriptions were also incorrect. For example, the area of a general quadrilateral (with ordered side lengths a, b, c, d) was calculated by the formula

$$A = \frac{1}{4}(a+c)(b+d)$$

which is correct only if the quadrilateral is a rectangle or square.

Signs of Advanced Mathematical Thinking

Egyptian mathematics was utilitarian in its direct ties to the solution of practical problems. Also, because their numeration system involved simple grouping with no place values, it is not reasonable to expect that the Egyptians had explored ideas such as factors, powers, and reciprocals. This limitation perhaps explains why no record has been found of tables involving Pythagorean triples. Nonetheless, they did apparently use some number tricks; when multiplying a number by 10, they merely replaced each hieroglyphic symbol by the symbol representing the next higher power of 10 (that is, replacing each vertical staff with a heel bone, each heel bone with a scroll, and so forth).

Problem 79 in the Rhind Papyrus suggests that the Egyptians did some recreational mathematics that had no real-world applications. The problem states, "7 houses, 49 cats, 343 mice, 2401 ears of spelt, 16,807 hekats." Historians assume that the scribe was creating a problem involving seven houses, each with seven cats, each of which eats seven mice, each of which had eaten seven ears of grain, each of which had sprouted seven grains of barley…wanting to know the total number of houses, cats, mice, ears of spelt, and grains. Mathematically, the solution of this problem would require some knowledge of powers of 7 and geometric progressions.

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See Also: Arabic/Islamic Mathematics; Babylonian Mathematics; Chinese Mathematics; Greek Mathematics.

Einstein, Albert

Category: Space, Time, and Distance.
Fields of Study: Algebra; Geometry; Measurement; Representations.
Summary: One of the most well-known physicists, Albert Einstein's work continues to influence many fields.

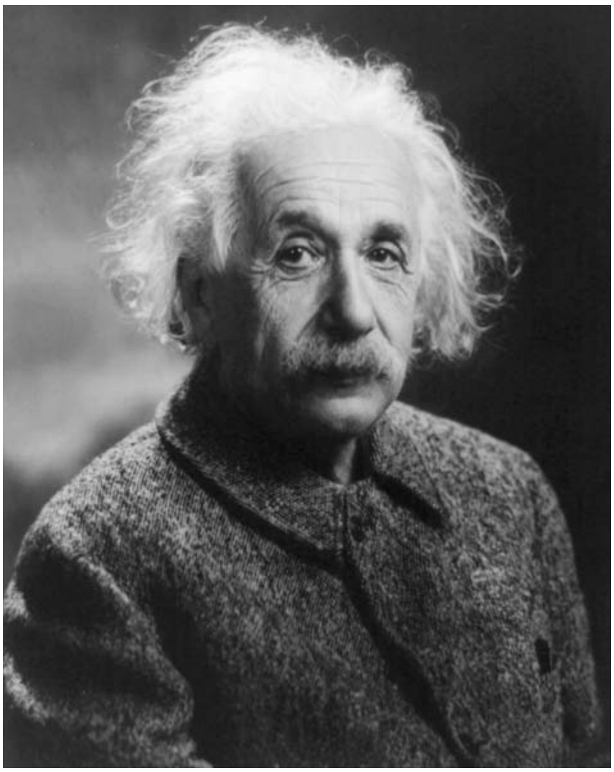
During the twentieth century, research in the fields of mathematics, physics, chemistry, information technologies, and engineering exploded. People's perception concerning the world and the universe around them changed dramatically within a faster and faster changing world. If one were to choose a single influential scientist to represent this era, some might choose Albert Einstein. During "the Age of Einstein," he introduced many original concept widely used in various fields, such as mathematics, science and technology, world politics, economics, and philosophy.

Early Life and Education

One common myth about Einstein was that he failed mathematics as a child. Albert Einstein was born at Ulm (Württemberg, Germany) on March 14, 1879. He studied in various places, including Munich, Italy, and Switzerland. His uncle, an engineer, presented him with questions about mathematics, such as a challenge to find a proof of the Pythagorean theorem. Einstein noted, "After much effort I succeeded." After viewing a *Ripley's Believe It or Not* headline about his proposed failure in mathematics, biographers note that Einstein replied, "I never failed in mathematics. Before I was 15 I had mastered differential and integral calculus." In 1896, he entered the Swiss Federal Polytechnic School in Zurich to study physics and mathematics. In 1901, he began working at the Swiss Patent Office. In 1905, he obtained his doctorate degree. He was a professor at various universities in Europe until 1933, when he immigrated to America because of anti-Jewish laws in Germany.

Accomplishments

One notable quotation attributed to Einstein is: "Do not worry about your difficulties in mathematics, I assure you that mine are greater." During his life, Einstein published a great amount of papers in several fields of the sciences. Many equations and laws are named for him, including: Einstein's absorption coefficient, Einstein photoelectric law, Einstein frequency condition, Einstein diffusion equation, Einstein–Bohr equation and Einstein coefficients, Einstein frequency and Einstein elevator, Einstein–Planck law, Einstein mass–energy relation, and so on. In chemistry, a synthetic radioactive chemical element having the symbol Es, the atomic number 99, and atomic weight 252.08 is called "einsteinium." Einstein's principle of relativity, the basic postulate of Einstein's special relativity theory, states that the laws of nature have the same form in all inertial frames of reference. Einstein based his general theory of relativity on mathematical ideas like mathematician Bernhard Riemann's geometric formulations. Gravity was now described to be curved spacetime, "Matter tells spacetime how to curve and curved spacetime tells matter how to move." He was also philosophical about the applicability of mathematics, saying, "How can it be that mathematics, being after all a product of human thought which is independent of experience, is so admirably appropriate to the objects of reality? Is human reason, then, without experience, merely by taking thought, able to fathom the properties of real things?" Einstein's field equations from general relativity and their solutions have been a fruitful research area in mathematics and physics, leading to concepts like metrics for black holes and the notion of Einstein manifolds, named for him. However, he complained that he had difficulty understanding the theory of relativity after mathematicians "invaded" it.



Albert Einstein once said "A man should look for what is, and not for what he thinks should be."

Conservation of Mass and Energy

While twenty-first-century researchers continue to investigate Einstein's field equations, Einstein's most famous equation is probably $E = mc^2$. In his paper on the equivalence of matter and energy, he deduced the equation. It meant that conservation laws can be unified into a single law of the conservation of mass-energy. This equation also predicted the development of nuclear power. However, Einstein was an opponent of nuclear weapons. In 1939, Einstein wrote and signed a letter to U.S. President Franklin D. Roosevelt to warn him about research on uranium and the possibility of the development of an atomic bomb. The president took his words seriously, which was the beginning of the Manhattan Project—the effort to construct a nuclear bomb. In a 1954 letter to his friend, Linus Pauling, Einstein confessed that his letter to Roosevelt was the one great mistake of his life. During his life, he made many contributions for peace. Einstein stated, "We have to divide up our time like that, between our politics and our equations. But to me our equations are far more important, for politics are only a matter of present concern. A mathematical equation stands forever." Just before his death, in a letter to Bertrand Russell, he still urged all nations to give up nuclear weapons.

A Public Person

Einstein was in his whole life a public person. Being a good-humored speaker, he took part in a large number of conferences and traveled in many countries. His name became the brand for "genius," and a large number of sayings and anecdotes are told on his account, such as "Pure mathematics is, in its way, the poetry of logical ideas." He always seemed to have a clear view of the problems to solve and the will to solve them. He remained a very curious person and taught his pupils not to be afraid of asking, trying, and failing. Failures, together with achievements, are merely stepping stones for the next adventure of discovery, he said.

In the 1949 publication of *Autobiographical Notes*, he stated, "In the beginning (if there was such a thing), God created Newton's laws of motion together with the necessary masses and forces. This is all; everything beyond this follows from the development of appropriate mathematics methods by means of deduction." After his retirement from Princeton, he continued to work on a theory of unification of the basic concepts of physics, natural sciences, mathematics, and religion. Albert Einstein received a Nobel Prize in physics along with honorary doctorate degrees in science, medicine, and philosophy from many universities. A crater on the moon is named after him.

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Simone Gyorfi

See Also: Clocks; Elementary Particles; Geometry of the Universe; Gravity; Hawking, Stephen; Measuring Time.

Elections

Category: Government, Politics, and History.
Fields of Study: Communication; Data Analysis and Probability; Number and Operations.
Summary: Mathematics can help explain and predict elections.

Long the domain of economists, political scientists, and philosophers, systems of government has emerged as a field ripe for the application and study of mathematics. Elections are typically classified under an emerging branch of mathematics called "social choice theory," though there are historical connections and applications in a number of areas, such as combinatorics and probability theory. Economist Duncan Black's 1958 book *The Theory of Committees and Elections* is credited with helping to revive modern interest in using mathematics to study election questions.

In a democratic society, such as the United States, elections are the primary vehicle for providing citizens a fair and equal voice in the machinations of federal, state, and local governments. As such, it is fundamentally important that elections be conducted in a manner that is perceived to be fair by the citizenry; that is, a governing body derives its legitimacy from the equitable interpretation and application of the voting power of the public.

Beyond the widely known popular elections (electing the candidate with the most first-place votes) there are a number of alternative voting methods; many of these allow voters to express more information about their preferences of various candidates. Since it is possible for different methods to produce different winners given the same voter preferences, a number of

Townspeople lined up to vote in rural areas of Guatemala for the 2007 national elections. In a democratic society, elections provide citizens with a voice in the workings of federal, state, and local governments.

voting properties have been postulated. Each property states a desired outcome or effect that a voting system should express. For example, a voting system should be "anonymous" in that individual voters should be able to exchange ballots without affecting the outcome; in other words, one person's ballot should not have special significance. A more challenging property is "independence from irrelevant alternatives," which requires the relative outcome of an election to remain unaffected if candidates are added or removed from consideration (provided this addition or removal does not change the relative way voters feel about the other candidates). Economist Kenneth Arrow demonstrated mathematically in his doctoral dissertation that no voting system can satisfy all the desired properties. Arrow's Impossibility Theorem was later published in his 1951 book *Social Choice and Individual Values.*

A particular type of voting system, weighted voting, arises when voters are assigned different numbers of votes. This system is usually employed to reflect a situation where some voters should have greater say or representation than others. The Banzhaf Power Index, named after John Banzhaf, is a tool that elucidates the voting power enjoyed by the voters in a weighted voting scheme and reveals that voting power is not always commensurate with a voter's number of votes. It is also sometimes called the Penrose–Banzhaf Power Index to include its original inventor, Lionel Penrose.

The U.S. Electoral College, an example of a weighted voting system, is used to elect a winner in U.S. presidential elections. The U.S. Electoral College illustrates a drawback of weighted voting in that a winning presidential candidate may not have received a majority of popular votes. This has sparked much interest in replacing the U.S. Electoral College in favor of the popular vote method but smaller states that enjoy more voting power with the U.S. Electoral College are likely to block attempts at Constitutional reform.

Exit polling, invented by statistician Warren Mitofsky, allows social demographers to understand the dynamics of an election and to predict the winner. Exit polling has become an increasingly important tool for media and news outlets as they scramble to retain and inform viewers on the eve of an important election. A number of studies have investigated the influence of exit polling while an election is taking place; for instance, polls broadcast in real time may influence voters who have yet to vote and hence possibly change the outcome of an election. Exit polling has also garnered interest in recent presidential elections when erroneous predictions caused media sources to prematurely, or incorrectly, identify a winning candidate.

The Ballot Box Problem is an interesting mathematical puzzle, proposed by Joseph Bertrand, which seeks answers about how an election may unfold as ballots are removed from the ballot box and counted. The solution to Bertrand's theorem is a Catalan number, named for Eugène Catalan. An elegant proof was derived by Désiré André.

Types of Elections

Though most people are familiar with the plurality election (also known as "popular vote") in which the candidate with the most votes (most first-place votes) wins, there are a number of alternative election methods. One of the most prominent is the Borda method, named for Jean-Charles de Borda, where voters are required to rank all candidates from their first choice to their last; points are then assigned to each candidate based on the candidate's rank on the each ballot. The sum of a candidate's total points is used to determine the winner. This method allows voters to specify more information about how they view the candidates, other than merely selecting their favorite.

In the Sequential Pairwise method, two of the candidates vie in a head-to-head competition (an imaginary election with only the two candidates) where the losing candidate is eliminated and the winner proceeds forward to battle another candidate. Again, voters rank candidates in preference listings, which are used to determine the winner between a particular pair of candidates. The winner can be inferred from the preference lists by assuming each voter would select the candidate that is higher on his or her list. A drawback of this method is that the order in which the candidates are selected for the individual competitions can change the ultimate outcome of the election.

A Condorcet Winner is a candidate who beats every other candidate in a head-to-head election. When one exists, a Condorcet Winner will obviously win the Sequential Pairwise election but not all sets of voter preference rankings produce a Condorcet Winner. The method is named for Marie Jean Antoine Nicolas de Caritat, Marquis de Condorcet.

In an Instant Run-off election, a plurality vote is taken and the candidate with the least number of first-place votes is eliminated. Then the election is repeated with the remaining candidates until only one winner remains. Again, voter preference rankings can be used to simulate the repeated elections in order to determine the winner without holding a series of actual elections.

Weighted Voting

Much of rationale behind the U.S. system of government is based on the principle of "one person, one vote" (each citizen should have equal say in the system of government). There are times, however, when it is appropriate to give certain individuals (or groups) more voting power than others. This type of voting system, often called "yes–no voting" or "weighted voting," occurs when voters are assigned a different number of votes or "weights" to their votes. Elections are between two alternatives; the winner is selected if the vote total exceeds a predetermined threshold. Each voter must use all available votes toward the same candidate or choice—votes cannot be split between the candidates or choices.

An example of a weighted voting system was the European Economic Community (EEC) established in 1958 as a precedent to the current European Union. The original six members were assigned votes in proportion to their population size:

Country	# Votes
France	4
Germany	4
Italy	4
Belgium	2
Netherlands	2
Luxembourg	1

A threshold is established to determine the number of total votes necessary to win an election. Though this threshold is often simple majority, in the EEC example, a threshold of 12 (of the total 17 votes) was established to pass certain types of legislation.

An interesting question arises as to the dynamics of weighted voting systems and, more specifically, an entity's ability to influence the outcome of an election. Several theorists have shown that voting power is not necessarily proportional to an entity's vote count. For example, it would be misleading to assume that France enjoys 23.5% (4/17) of the voting power in the EEC example.

Banzhaf applied a power index to argue a landmark case in Nassau County, New York, in 1965. His voting power calculations demonstrated the disenfranchisement of certain entities within weighted voting schemes and thus questioned the system's constitutionality.

Banzhaf's computation is based upon the notion of a winning coalition (a collection of voters whose vote total exceeds the threshold). Such a coalition (or "voting block") can win an election by all voting the same way. A voter is critical to a winning coalition if by removing that voter, the coalition no longer exceeds the threshold. A voter's Banzhaf Power Index (BPI) is related to the number of times that voter is a critical member of a winning coalition.

In the EEC example, France, Germany, Italy, and Belgium form a winning coalition since their vote total of 14 exceeds the threshold of 12. France, Germany, and Italy are all critical members because the coalition ceases to win without their votes. However, Belgium is not a critical member since France, Germany, and Italy together still form a winning coalition. The number of times each voter appears as a critical member of some winning coalition is computed as follows:

Country	# Critical	BPI
France	10	10/42 = 23.8%
Germany	10	110/42 = 23.8%
Italy	10	10/42 = 23.8%
Belgium	6	6/42 = 14.3%
Netherlands	6	6/42 = 14.3%
Luxembourg	0	0/42 = 0%

Each country's BPI is the number of times it is critical compared to the total number of critical instances. Here, there are 42 total instances where an entity is critical; Belgium has 6 of them and thus 14.3% (6/42) of the voting power. Thus, Belgium commands 14.3% of the voting power even though it has 11.8% of the votes. In this scheme, Luxembourg has no voting power—it is not able to influence the outcome of any possible election. It is common in weighted voting schemes of smaller size (20 or fewer members) for entities with a greater number of votes to possess greater voting power, while small entities (with a fewer number of votes) possess less voting power. As the number of voters increase, voting power tends to better approximate the proportion of votes. But such weighted voting systems are subject to arbitrary swings of voting power as new voters are added or removed, or as seemingly subtle changes to the weights are made.

An equally popular voting power computation was proposed by Lloyd Shapely and Martin Shubik in 1954. Instead of critical members in winning coalitions, their system identifies pivotal voters as the ones who enter a coalition and cast the deciding vote by doing so. A similar calculation ensues in which voting power is correlated with the percentage instances in which each entity plays the pivotal role.

U.S. Electoral College

The voting system responsible for electing the president of the United States, the U.S. Electoral College, is essentially a weighted voting scheme. A state's electors (or "votes") arise from the sum of their congressional representation: one vote for each of a state's two senators and one vote for each representative to the House of Representatives. The District of Columbia receives three electors to form a total of 538 (100 senators, 435 representatives, and three from Washington, D.C.). A presidential candidate needs a majority of the electoral votes—at least 270—to claim victory.

Under such a system, it is possible that the winning candidate need not garner a majority of first-place votes. In fact, U.S. presidential elections in 1824, 1876, 1888, and 2000 all produced a winner who lost the popular vote total.

Those elections and other issues have created an endless interest in reforming or removing the U.S. Electoral College and replacing it with a popular vote system. As recently as 2004, the Every Vote Counts Amendment proposed to replace the U.S. Electoral College with a popular vote initiative. Such a reform requires a Constitutional change and thus approval of 75% of the states.

It is unlikely such a measure would ever be adopted because small states enjoy significantly more voting power in the U.S. Electoral College than they would in a popular vote system. A state with few votes, such as South Dakota, would likely be ignored by campaigners since the voting population is too small to make a difference under a popular vote election.

The National Popular Vote Compact is an alternative attempt at election reform. In this compact, individual states would cast their electoral votes according to the national popular vote, not simply the tallies within the state. This has the effect of choosing a president elected by popular vote within the Electoral College system and thus bypassing the hurdle of constitutional reform. To date, this compact has been adopted by five states (61 electoral votes) with a number of others considering the compact in state legislature—enough states to compile 270 electoral votes would have to sign on to the compact in order to have the intended effect of electing a president by popular vote.

Exit Polling

An important factor associated with elections is the attempt to predict election outcomes through the surveying of voters as they leave the voting areas, a procedure known as "exit polling." This procedure contrasts with pre-election polls in that actual voters who have (presumably) just cast a vote are being sampled and thus results are typically more accurate than surveying people prior to an election who are "likely" to vote, or who may change their mind between being polled and actually casting a vote.

Although the science of predicting election outcomes has been around as long as elections themselves, it is at the beginning of the twenty-first century—with widespread electronic media coverage and more sophisticated polling techniques—that exit polling has garnered more national attention. A number of papers have been written about the effects of exit polling being broadcast in real time; the researchers hypothesize that exit polling influences voter behavior primarily by making an election seem closer or not closer than was previously perceived. This effect is especially true in the United States where, as a function of different time zones, voters in western states have access to more complete results of a national election unfolding across the country.

Exit polling has garnered an additional spotlight with the controversial presidential elections of 2000 and 2004. In both cases, especially the 2004 election, exit polling differed significantly from the actual vote tally, causing many media outlets to incorrectly, or prematurely, announce a victor.

Ballot Problem

There are several interesting mathematical puzzles based on elections and voting; perhaps the most well known of them is the Ballot Problem, originally presented by Joseph Bertrand in the late nineteenth century. Consider an election between two people, Alice and Bob, where Alice has received A votes and Bob B votes. Let $A > B$ so that Alice wins the election. The puzzle arises from the counting of the votes: what is the probability that as the votes are pulled randomly from the ballot box and tallied one by one, that Alice and Bob are tied in their vote total at some point after the first vote is read?

Survivor!

The popular television series *Survivor* nicely illustrates a ballot-box type of problem. In individual tribal councils, as well as the final vote for an overall winner, ballots are drawn from a ballot box and read aloud. It is easy to hypothesize that the ballots are not drawn in a random order but instead are selected so as to maximize the suspense of the election outcome. Another interesting question, related to information theory, is that ballots are read only until the election outcome is certain; unread ballots are not presented to the remaining tribe members, thereby depriving them of strategic information about the voting behavior of their fellow competitors.

The puzzle's solution is a creative argument based on combinatorics and probability. Sequences, a listing of votes as they are pulled from the ballot box, can be identified as those with ties and those without. The following is a sequence from an election with nine voters (A = 5, B = 4):

$$b\ b\ a\ b\ a\ a\ b\ a\ a.$$

In this sequence, the first tie occurs with the reading of the sixth vote, though there is also a subsequent tie. There is also a "matching" partial sequence in which the *a*'s and *b*'s exchange places up through the point of the first tie:

$$a\ a\ b\ a\ b\ b\ b\ a\ a.$$

Every such sequence of strings that produces a tie somewhere in the intermediate vote tally comes in matching pairs as shown. Out of each pairing, one sequence must start with an *a* while its match starts with a *b*. Since Alice wins the election, some of the sequences starting with an *a* will result in a tie but not all of them. However, every sequence that starts with a *b* must at some point achieve a tie since ultimately there will be more *a*s than *b*s. There are three categories of sequences:

- sequences that start with an *a* but never have a tie
- sequences that start with an *a* and achieve a tie at some point
- sequences that start with a *b* and achieve a tie at some point

The probability that any sequence is found starting with a *b* is

$$\frac{B}{A+B}$$

since there are B ballots out of $A + B$ total ballots where a *b* can be the first vote drawn. There are exactly as many sequences that start with an *a* and also achieve a tie because each one is matched with exactly one *b*-starting sequence. Therefore, the probability of reading the votes and achieving a tie along the way is exactly

$$\frac{2B}{A+B}.$$

This problem has spawned a number of related problems with interesting ties to Catalan numbers.

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See Also: Census; Congressional Representation; Game Theory; Gerrymandering; Government and State Legislation; Mathematics, Elegant; Voting Methods.

Electricity

Category: Architecture and Engineering.
Fields of Study: Algebra; Representations.
Summary: Electricity, arising from the flow of electrons, can be described mathematically.

Daily operations of modern industrial societies, including transportation, communication, heating, cooling, lighting, computing, and medical technology, rely on the use of electrical power. Power from batteries and electrical outlets is derived from the flow of electrons, known as "electric current." The term "electricity" refers to a variety of physical effects, both static and dynamic, that arise from electric charge. The mathematical description of electric and magnetic phenomena developed in the eighteenth and nineteenth centuries contributed to a rapid expansion of electrical technology, which is powered today by a vast grid of electric power stations and distribution systems.

Electric Charge and Coulomb's Law

Electric charge is a property of matter that can be negative (as in electrons), positive (as in protons), or zero.

Most matter has a net charge of zero, containing essentially the same number of electrons as protons. Two objects whose charges are both positive or both negative repel each other, while objects with opposite charges attract each other. Static electricity is created when electrons build up on or are depleted from the surface of a material, often by rubbing materials together. Effects of static electricity are seen, for example, in a rubbed balloon clinging to a wall, or in hair standing on end. In metals, electrons are not strongly bound to individual atoms but move freely through the lattice of protons. Materials with freely moving charges are known as "conductors." The force between two charged particles at rest is described by Coulomb's Law, named after French engineer Charles-Augustin de Coulomb (1736–1806). Coulomb's Law states that the magnitude F of the force exerted by one charged particle on the other is

$$F = \frac{kqq'}{r^2}$$

where q and q' are the magnitudes of the charges of the particles, r is the distance between the two particles, and k is a constant. This equation shows, for example, that if one charge is tripled, then the force is tripled, and if both charges are tripled, then the force becomes nine times as large. On the other hand, tripling the distance r between the particles multiplies the right-hand side of the equation by $1/3^2$, or 1/9, reducing the force to a ninth of its previous value.

Electric Field and Electric Current

The presence of charged particles creates an electric field that exerts a force on other charged particles in the region. An electric power generator, usually driven by a steam turbine fueled by coal or a nuclear reactor, creates an electric field between two terminals by building an over-supply of electrons (negative charge) in one terminal and a deficit of electrons (positive charge) in the other. The flow of electrons from a negative toward a positive terminal along a conducting path, such as a wire, is an electric current. In lightning, electrons from negatively charged clouds in the atmosphere are attracted to positively charged objects on the ground beneath the cloud. Here the electric field is so strong that electric current passes through air, which usually acts as an insulator that prevents the flow of electrons. Batteries operate by producing an electric current between oppositely charged terminals of chemical cells. A battery produces direct current (DC), where electrons flow in one direction, while a power generator creates alternating current (AC), where the direction of electron flow alternates rapidly, typically at a frequency of 60 hertz (cycles per second). The hertz is named for German physicist Heinrich Hertz (1857–1894), who made important advances in understanding the connection between electric and magnetic fields.

Ohm's Law

The energy that an electric field imparts to a unit charge moving from one terminal to another is the number of volts (V) between the terminals, named after Italian physicist Alessandro Volta (1745–1827). On electric bills, energy usage is typically given in kilowatt hours (kWh). The watt, named for British engineer James Watt (1736–1819), is a unit of power, or energy per time, and 1 kilowatt is 1000 watts. Multiplying power (in kilowatts) by time (in hours) yields energy, in kilowatt-hours. In an electric current, the current intensity (I) is abbreviated as "current" and is the quantity of charge that moves past a cross-section of the conducting path per unit time. As electric current flows through a material, the motion of the electrons is hindered by positive ions, creating electrical resistance (R). Resistance in the path of a current creates heat and light, as in appliances, such as stoves and light bulbs. Electrical energy can be transformed into mechanical energy to power motors as in cars, airplanes, power tools, kitchen blenders, and hair dryers when electric current passes through a coil of wire, inducing a magnetic field that sets the coil in motion.

Ohms's Law, formulated by German physicist Georg Ohm (1789–1854), states that for a metal conductor at constant temperature, the voltage (V) is $V = IR$, where I is the current, and R is the resistance. This equation shows, for example, that if the resistance is cut in half, then to maintain the same voltage, the current must be doubled. If too little resistance is present, the current may become so strong as to damage electrical equipment. Circuit breakers then sever the path of the current to avoid damage.

Electric Power from Generator to Consumer

High voltage generated at power stations is propagated along power lines almost instantaneously, over many miles, to substations near cities and towns. At the substations, the voltage is reduced and transmitted to electric distribution centers that channel the voltage to homes, offices, and other facilities. In standard electrical outlets in the United States, there are 120 volts between the wires leading to the two vertical slots. When an appliance is plugged into the outlet, the vertical prongs of the plug make contact with these wires, creating a pathway of current through the appliance. The third slot in the outlet carries a protective ground wire. In appliances with a three-pronged plug, the ground wiring is designed to provide a preferred pathway for escaped current so that it will not travel through the body of the person holding the appliance.

Large appliances, including most drying machines and ovens, operate at 240 volts, using a different type of outlet. Touching one or more openings in an electrical outlet or touching the prongs of a plug as it is inserted into the outlet may pass an electric current through the body that can be harmful or even deadly. At electrical facilities, "High Voltage" signs warn of the danger of electric shock because of the presence of high voltage.

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See Also: Elementary Particles; Light; Light Bulbs; Lightning; Microwave Ovens; Nanotechnology; Radiation.

Elementary Particles

Category: Space, Time, and Distance.
Fields of Study: Data Analysis and Probability; Number and Operations; Representations.
Summary: Various branches of mathematics are employed to study elementary particles, the smallest particles in the universe.

Particle physics is a branch of physics that seeks to describe and explain the universe on the smallest scales. The particles thought to be the fundamental building blocks of matter and force are called "elementary particles." Like all branches of physics, the study of elementary particles relies heavily upon many branches of mathematics, including calculus, geometry, group theory, algebra, and statistics. Particle physics also contributes to mathematical research by posing questions that give rise to new mathematical theories.

History

For thousands of years, scientists and philosophers have been asking the questions, "What is the universe made of?" and "Are there fundamental units that make up space, matter, energy, and time, or are these infinitely divisible?" As early as the fifth century B.C.E., Greek philosopher Democritus (c. 460–370 B.C.E.) hypothesized that all matter is made of indivisible, fundamental units called "atoms." Despite these early hypotheses, there was very little progress in this field until the dawn of the twentieth century.

The twentieth century saw the emergence of several new branches of physics. Among these was particle physics, a field that seeks to explore the universe on the smallest scales. Particle physicists try to identify the particles that form matter and force, describe their properties, and understand how these particles relate to each other. Some of these particles are not composed of any other particles and are therefore called "elementary particles." These elementary particles form the basic building blocks of the universe.

The understanding of particle physics at the beginning of the twenty-first century is embodied in the Standard Model of Particle Physics, an elaborate yet still incomplete model that attempts to list and describe all existing particles. Jokingly referred to as "The Particle Zoo," the Standard Model lists dozens of particles and includes elementary particles with exotic names such as "gluon," "muon," and "quark." Many of the particles in the Standard Model have yet to be detected experimentally, and their existence is conjectured based on theoretical work.

Mathematics Used in the Study of Particle Physics

Like all physical theories, particle physics relies heavily upon mathematics, which provides the theoreti-

NASA scientists detected a ring of dark matter that formed during a collision between galaxy clusters. Astronomers don't know what dark matter is made of; however, they believe it is a type of elementary particle.

cal framework physicists use to explain and describe physical phenomena. Mathematics also enables physicists to make predictions that can later be tested using modern tools, such as particle accelerators.

One of the most useful branches of mathematics is calculus, a field that has applications in practically all branches of the natural sciences, as well as in engineering and even in the social sciences. It is therefore not surprising that calculus occupies a central role in the theory of elementary particles. Differential calculus may be used to describe properties of particles at an instant, while integral calculus is used to describe cumulative effects of a particle or a system of particles over time and space.

Calculus is but one branch of the mathematical field of analysis that is useful in particle physics. Other branches of analysis—partial differential equations, complex analysis, and functional analysis—play important roles as well.

Geometry has traditionally been used to describe the universe on the grandest scales, those of galaxies, galaxy clusters, and the universe as a whole. Recently, geometry has found a place in elementary particle research as well. French mathematician Alain Connes (1947–) has described a theoretical model for particle physics that is based on noncommutative geometry, which is a geometrical representation of noncommutative algebras—systems in which the order of factors in an operation determines the value of the operation. For example, if a and b are real numbers, then it is always true that $a \times b = b \times a$, as multiplication is commutative for real numbers. However, if A and B are matrices, then generally $A \times B \neq B \times A$. Matrix multiplication is therefore noncommutative.

Symmetry, Group Theory, and Quantum Mechanics

One of the most fundamental mathematical concepts in elementary particles is symmetry. In mathematics, symmetry is defined as an operation on an object that leaves some of the object's properties unchanged. As an example, consider a square drawn in the plane and an axis of rotation that passes through the square's center, perpendicular to the plane. If the square is rotated by 90 degrees around that axis, the square will appear unchanged. Rotation by 90 degrees is thus called a "symmetry" of the square. The set of all symmetries of an object forms a mathematical construct called a group (a set with an operation that obeys several axioms). Group theory, a branch of algebra, plays an important role in particle physics, as properties of many elementary particles can be explained and described by the use of symmetry.

The chief group-theoretic structure in particle physics is the Lie (pronounced "Lee") group, named after Norwegian mathematician Sophus Lie (1842–1899). Lie groups are groups that posses the properties of geometric constructs known as "differentiable manifolds." Lie groups thus provide yet another connection between geometry and elementary particles.

One of the most important physical theories of the twentieth century is quantum mechanics, a theory that holds that, at the atomic and subatomic levels, behavior of particles is a statistical rather than a deterministic phenomenon. Since elementary particles obey quantum-mechanical laws, statistics and probability are invariably major components of the mathematical framework of elementary particles.

While physicists use mathematics as a tool for exploring the universe, the relationship between particle physics and mathematics is not one-directional. Research in particle physics drives the emergence of new mathematical theories, just as mechanics drove the emergence of calculus in the seventeenth century. In 1990, American theoretical physicist Edward Witten (1951–) won the Fields Medal, the highest honor in mathematics, for his many contributions to mathematics. He is the only non-mathematician ever to win the prestigious award. As both mathematicians and physicists continue to explore new horizons, the cross-fertilization of ideas will benefit both fields in decades to come.

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See Also: Gravity; Relativity; Symmetry.

Elevation

Category Space, Time, and Distance.
Fields of Study: Geometry; Number and Operations.
Summary: Various aspects of elevation can be calculated using mathematical techniques.

Trigonometry has long been used to measure height. Elevation is often the height of a point relative to sea level, and its measurement is called "hypsometry." Elevation affects air pressure, temperature, and gravity, all of which have noteworthy effects on people. Astronomers and mathematicians such as Blaise Pascal and Edmund Halley investigated relationships between barometric pressure and elevation.

Historical surveys of elevation include those who used barometers, like John Charles Frémont, who was at one time professor of mathematics of the Navy, and physician Christopher Packe. However, this method is sensitive to a number of variables. In the twenty-first century, detailed elevation data are available. Mount Everest is known as Earth's highest elevation. Topographical maps represent elevation by using contour lines, each line following a path of constant elevation. Transits were developed in the nineteenth century, and they can be used to calculate changes in elevation. Contour integrals and generalized contours for functions of two variables are investigated in multivariable calculus classrooms. Mathematicians and computer scientists have helped create realistic computer models of land elevation, called "digital elevation models." They have explored ideas like irregular-mesh grids or shifting nested grids in surface reconstruction. Other types of elevation studies also benefit from mathematical techniques, like using the ocean wave spectrum to investigate sea surface elevation peaks, or statistical techniques to investigate the impacts of elevation changes. Mathematician and astronomer Nilakantha Somayagi investigated the elevation of lunar cusps in the sixteenth century. The term "angle of elevation" in high school classrooms represents the angle between where an observer is standing and the line of sight to an object. The angle of elevation is found in many contexts, including in the Pyramids of Egypt, in the astrolabe, and in global positioning systems.

Highest Elevations on Earth

Elevations are nearly always computed relative to sea level, the average height of the ocean's surface. Sea level is an inexact measure since tides, temperature, wind, salinity, and air pressure affect the oceans. Mount Everest (above) in the Himalaya Mountains near the border of Nepal and Tibet is the highest mountain on Earth at an elevation of 29,035 feet as of 2010. Everest gains more than two inches of elevation per year because of the collision of tectonic plates and there are discrepancies in its listed height.

Earth is not spherically symmetric; its radius near the equator is more than 13 miles greater than its radius near the poles. Consequently, Mount Chimborazo in Ecuador holds the distinction of having the summit farthest from Earth's center. Lying about one degree south of the equator, where Earth is widest, Mount Chimborazo is approximately 20,561 feet above sea level, enough to make its summit more than a mile farther from Earth's center than Mount Everest's summit.

Topographic Maps

A topographic map is a two-dimensional map that conveys elevation information as well as other features of an area. Contour lines are the key to capturing elevation changes from a three-dimensional world on a two-dimensional map. A contour line is a path that follows a constant elevation. Early uses of contours date to the eighteenth and nineteenth centuries and include the work of engineer Jean-Louis Dupain-Triel and astronomer and mathematician John Couch Adams.

A contour line is drawn each time a predetermined elevation change is achieved. For example, a map may use 100-foot elevation increments, with one contour line following points having an elevation of 100 feet and the next marking an elevation of 200 feet. Consecutive contour lines always differ by 100 feet in elevation. As the mapped terrain climbs more steeply, the contour lines on the map will be closer together. The lines can mark elevations that increase and decrease, representing terrain that rises and falls intermittently. Contour lines can represent elevations that are zero, or negative numbers as when mapping an ocean floor.

A topographic map of an area with constant elevation at its boundary, such as an island bounded by the sea, will not have contour lines extending off the map's edge. In such cases, all contour lines will appear as closed curves. A curve is closed if it loops back to where it started. Typically, contour lines appear as simple closed curves that do not cross themselves. The pattern of contour lines as nonintersecting rings lying one within another is common on topographic maps.

Also common is to have two separate sets of nonintersecting rings contained within a single contour line, as when two hills are surrounded by a larger path of constant elevation.

The U.S. Geological Survey (USGS) has created a complete large-scale topographic map of the United States in more than 56,000 pieces. The National Elevation Dataset is noted as the "the primary elevation product of the USGS." The data set is updated regularly, and historic data sets are also available for investigations.

There is an ever-growing growing need for digitized maps, which allow a computer user to read elevation at any spot on the map. Some digitized maps enable the user to view a landscape from different perspectives, creating a three-dimensional view of the area's elevation changes, similar to what would be seen at the actual location. Data from existing topographic maps and aerial photography are used to create digitized maps. Improvements in technology will continue to affect the science of map making.

Effects of High Elevation

As elevation increases, air temperature drops because of a decrease in air pressure. At about 18,000 feet above sea level, for example, the air pressure is half that at sea level. In the troposphere, the lowest layer of Earth's atmosphere, a general rule of thumb is that air temperature drops 6.5 degrees Celsius for every 1000 meters of elevation gain, or roughly one degree Fahrenheit for every 280 feet of elevation gain in standard conditions. This phenomenon, which can be modeled with an equation, can be seen directly when an observer standing at a low elevation on a warm day views a tall mountain covered with snow.

Another consequence of this cooling is that water vapor in the air condenses, sometimes causing increased rainfall on the windward side of a mountain range and a "rain shadow" downwind from the mountains. Many deserts lie just downwind from a mountain range. For example, sand dunes in Death Valley, California, lie in the rain shadow of Mount Whitney, the highest peak in the continental United States.

Because of these differences in temperature and precipitation, tall mountains can have multiple climatic zones, with different plant species thriving near the summit than at lower elevations. Some animal species, such as Roosevelt elk, migrate seasonally to take advantage of elevation effects, climbing to cooler locations in the summer and descending to warmer valleys in winter.

The lower atmospheric pressure at high elevations makes breathing more difficult. Mountain climbers at high elevations use special apparatus to breathe. Some competitive distance runners train at high elevations in order to challenge their cardiovascular systems. When they race at a lower elevation, the air feels relatively dense and oxygen-rich, giving them a competitive advantage.

With the less-dense atmosphere at high elevations, the sun's rays can penetrate more easily, making sunburn possible even on cold days. Engines of naturally aspirated cars get less horsepower at higher elevations. Projectiles travel farther, a phenomenon known to golfers and baseball players. Standard equations for projectile motion sometimes assume a sea-level location; adjustments must be made to account for elevation.

The effect of gravity is reduced with travel to high elevations; mass remains the same but weight decreases slightly, primarily because of the increase in distance from Earth's center of mass. A person's weight would be less atop Mount Chimborazo than anywhere else on Earth.

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See Also: Curves; Gravity; Maps; Plate Tectonics; Temperature; Trigonometry; Weather Scales.

Elevators

Category: Architecture and Engineering.
Fields of Study: Algebra; Number and Operations.
Summary: Mathematics is used to quantify aspects such as the maximum speed and distance range of

elevators as well as model vibration and optimize traffic flow.

An elevator is a mechanism for vertical transport of persons or cargo. Mathematics is used to quantify aspects such as the maximum speed and distance range of elevators, determined by their purpose, such as lifting passengers, cars, or aircraft. Applied mathematical models focus on the dynamics and vibrations within different types of elevator mechanisms, such as hydraulic or rope systems. Mathematicians also investigate questions related to aspects such as waiting time, using probability models. Systems of multiple elevators are modeled as high-dimensional spaces using dynamical systems. The number of passengers in an elevator system constantly changes, making an optimal policy for what is referred to as an "elevator group control" mathematically interesting. At the end of the nineteenth century, scientist Konstantin Tsiolkovsky conceived of a space elevator. He was self-taught and worked as a mathematics teacher.

The National Aeronautics and Space Administration (NASA) holds an annual engineering competition to design a space elevator.

Hydraulic Elevators

The main concept related to why hydraulic elevators work is Pascal's Law, stating that when the pressure increases anywhere in a confined fluid, it equally increases everywhere. This, together with the fact that pressure (P) is equal to force (F) per unit area (A), can be exploited for an advantage of force. The elevator car stands on top of a piston ending in a wide shaft filled with oil, connected to a narrow shaft with oil. When a pump increases pressure in the narrow shaft, by applying a relatively small force, the equal pressure applies to the floor of the cabin, producing higher force because of the larger area: $P_1 = P_2$, and

$$\frac{F_1}{A_1} = \frac{F_2}{A_2}.$$

Hydraulic elevators are only used in relatively low buildings since the piston has to be as tall as the building to extend to the top floor but fully fit under the building when the elevator is on the ground floor. Digging as deep as a skyscraper is high to install an elevator is impractical. These elevators are mostly used for heavy loads in places such as car mechanic shops.

Roped Elevators

A mathematically interesting concept related to roped elevators is the conservation of energy. A roped elevator consists of two ends of a steel cable going around a pulley attached at the top, called a "sheave." The elevator car is attached to one end of the cable, and the counterweight, which weighs about the same, is attached to the other end.

When the elevator car is at the bottom of the shaft, the counterweight is at the top, and its potential energy converts to force, helping move the elevator car up. When the elevator car is higher than the counterweight, their roles are reversed. This way, it takes very little additional force to make the sheave rotate and the elevator car move up and down.

Logistics

In modern buildings with multiple elevators, computer programs determine how to dispatch elevators to minimize wait time and to save energy. For example, a sensor may detect that an elevator is near capacity and will not stop it for any additional passengers. An elevator going down may not open its doors for people who want to go up, avoiding carrying them back and forth. More sophisticated elevator software can take into account typical traffic patterns, directing elevators to the busiest floors.

Space Elevator

A space elevator is a structure for escaping the gravity well of a planet, transporting objects between the surface and a geostationary orbit. This proposed structure would consist of a large satellite counterweight in orbit and a cable connecting it to the ground. The inertia of the counterweight rotating around the planet will balance the gravitational pull on the cable, keeping the cable taut. The National Aeronautics and Space Administration (NASA) is working on several efforts related to construction of a space elevator, including an annual engineering competition. The technological problems include avoiding meteorites and dangerous atmospheric weather systems, developing materials strong enough for the cable, designing the counterweight, protecting passengers from radiation, and powering the elevator cars. In 2008, Japan announced plans to build a space elevator in the immediate future. Space elevators have frequently appeared in science fiction since the early twentieth century.

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See Also: Engineering Design; Interplanetary Travel; Pulleys; Spaceships.

Encryption

See *Coding and Encryption*

Energy

Category: Space, Time, and Distance.
Fields of Study: Algebra; Measurement.
Summary: Mathematics is used to study energy and energy conservation as well as to develop new sources of energy.

The concept of energy and transportation of energy are central to the survival of any civilization. As mathematical physicist Ludwig Boltzmann noted, “Available energy is the main object at stake in the struggle for existence and the evolution of the world.” At the start of the twenty-first century, human beings have accessed or created many forms of energy and power production, including coal-fired and oil-fired power plants, solar heating plants, wind farms, nuclear power plants, geothermal sources of heat, hydroelectric power produced by dams, biofuels that store solar energy, and tidal energy produced by gravitational interactions between Earth and the moon.

There are also potentially disruptive energy sources, including natural events, such as lightning, volcanoes, and earthquakes. Some global sources of energy and power that remain to be tapped by humans include the atmosphere’s expansion and contraction, ocean currents, and sea level differences. Various calculations of energy, including chemical reactions and nuclear reactions, invoke the principle of conservation of energy. In relativistic or quantum terms, the conservation of mass-energy is also important. Energy, work, and quantity of heat are all expressed in “joules,” a measure of work named for physicist James Joule. There is a vast array of energy problems that mathematicians research, and mathematics makes many contributions to energy issues.

Energy, Defined

Energy is found in nearly every system or process in the universe: mechanics, chemicals, heat, electricity, nuclear processes, and quantum effects. Mathematician and scientist René Descartes studied mechanics; centuries later, mathematician and philosopher Gottfried Leibniz criticized his ideas and developed what are referred to today as "kinetic energy," "potential energy," and "momentum." In mechanics, the kinetic energy (E) of an object is expressed as

$$E = \frac{1}{2}mv^2$$

where m is the object's mass and v is its velocity. Another form of energy found in mechanics is the energy of position called "potential energy." It has the units of joules. An example is the potential energy defined as work done in the compression of a coiled spring. The sum of all the kinetic and potential energies within a system comprises the mechanical energy of the system. Energy may be a conserved quantity within a closed system, or it may change forms, such as mechanical energy being converted to heat by friction. How energy in a system is measured is important. As noted, mechanical energy is measured as the sum of kinetic energy and potential energy, or energies of motion and position. Chemical energy is measured by the heat energy released in chemical reactions. Electrical energy is measured by work done in a system.

Energy Conservation

In general, the amount of energy of various types can be equated to an equivalent amount of heat energy. On an experimental scale, heat energy is the ability of work done to raise the temperature of water. The joule is a measure of thermodynamic energy and is the common unit of energy. James Joule is credited with experiments in the mid-1800s that demonstrated that work done on a system can be converted into heat. His experiments and those of others eventually led to the realization and statement of the "principle of conservation of energy" as a hypothesis, which was proved in certain restricted settings and generalized by induction. In 1865, mathematical physicist Rudolf Clausius worked on thermodynamics and stated his first law as, "The energy of the universe is constant." The principle of conservation of energy applies not only to certain mechanical systems but is also seen widely in systems where other forms of energy are considered. Thus, heat energy is produced by combustion and friction, radiant energy is from light and other forms of radiation, and chemical energy is stored in fuels and electrical energy. The principle is continually tested in new situations. This testing led to discoveries in the twentieth century in atomic physics. In the International System of Units, *Le Système International d'Unités* (SI), a joule is defined as a newton-meter, named for Isaac Newton. The systematic study of the relation of various physical quantities through an analysis of their dimensions is the subject of dimensional analysis. Richard Feynman noted, "For those who want some proof that physicists are human, the proof is in the idiocy of all the different units which they use for measuring energy."

One energy issue that has been important to mathematicians, philosophers, and physicists is the relationship between matter and energy. Some physicists wanted to assign matter-like properties to energy, such as Wilhelm Wien, who considered that energy might have a traceable motion. Mathematician William Clifford thought of matter and energy as types of curvatures. In the theory of special relativity of 1905, Albert Einstein proved an equivalence of mass and energy as expressed in his famous equation $E = mc^2$, where E is the energy equivalent of mass m, and c denotes the speed of light, 299,792,458 meters per second. There is no process available to human beings at the start of the twenty-first century in which matter can be converted completely into radiant energy.

For example, in a nuclear explosion, only a tiny fraction of nuclear material is converted into energy. The only known process of annihilating matter is to pair a particle of matter with a particle of anti-matter, with the result that two photons are formed with energies that are equivalent to the energies of the particles. This process is on a quantum scale. Fusion is one process for partially converting mass into energy and occurs naturally in stars. Many controlled fusion experiments have been performed but in the process of producing fusion, a greater amount of input energy is needed for the reaction than is ultimately released by the reaction. Only in uncontrolled thermonuclear explosions are large amounts of energy released by fusion.

Fusion

Scientists continue to explore novel sources of energy and power from sources that entail motion, heat, quantum uncertainty and other natural physical phenomena. One possible source of power is controlled fusion reactions, hot or cold. Controlled hot fusion reactions have not yet reached a break-even point where the energy of the reaction exceeds the energy input needed to trigger the reaction.

There are ongoing fusion experiments that use various solids and liquids with energy pumped into them by lasers in which fusion occurs but the fusion is not self-sustaining. The main problem is the energy input and inherent danger in heating suitable substances to temperatures at which fusion between atoms of hydrogen isotopes can occur. The hydrogen is in the form of deuterium or tritium, and the temperatures reached through compression must be on the order of millions of degrees, and there are often energetic byproducts that are dangerous to objects and people. In contrast to hot fusion, cold fusion (also known as "low-energy nuclear reactions" among the twenty-first-century research community) is the fusion of atoms at close to room temperature, generally through the use of supersaturated metal hydrides. These reactions produce heat, helium, and a very low level of neutrons. The energy output is greater than the input, leading many scientists and others to investigate this process as a viable solution to the energy needs of the future. Chemists Martin Fleischmann and Stanley Pons were the first, in 1989, to publicly announce that they had achieved cold fusion. Many competing scientific and mathematical models have been developed to explain how cold fusion works but many researchers and others remain skeptical regarding its existence or viability.

Other Mathematical Applications

Mathematicians and other scientists have long studied the various aspects of energy. The concept of energy is fundamental to many scientific and business theories, applications, and disciplines. For instance, mathematicians have modeled energy trading in financial markets, which is quantitatively interesting because, in such applications, energy possesses unique attributes as a non-storable and non-fungible commodity. They have also worked to design efficient shutdown schedules for electronic systems to address concerns related to energy conservation. Mathematics is important for explaining the cosmic phenomenon of dark energy. This type of energy, often modeled as a scalar field and inferred in large part from observation and mathematical analysis of gravitational fields, has implications for theories and measurement of universe expansion and dark matter. On the other hand, mathematicians such as Blake Temple have used mathematics to attempt to disprove the existence of dark energy and posit alternative explanations. Others have investigated the geometry of symplectic energy. Mathematicians are also influential in energy research and policy making via work at federal agencies like the U.S. Department of Energy. Mathematician J. Ernest Wilkins was a fellow at the Department of Energy's Argonne National Laboratory and physicist and mathematician Hermann Bondi was the chief scientific adviser to the Department of Energy. Mathematical analysis and computational methods have also been used to study energy problems related to equilibrium, stability, and energy transport.

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Julian Palmore

See Also: Einstein, Albert; Electricity; Geothermal Energy; Green Mathematics; Light Bulbs; Radiation; Solar Panels; Tides And Waves; Universal Constants; Wind and Wind Power.

Energy, Geothermal

See *Geothermal Energy*

Engineering Design

Category: Architecture and Engineering.
Fields of Study: Algebra; Geometry; Measurement.
Summary: Engineering design is a carefully regulated process to create optimal solutions for given problems.

Engineers design everything from automobiles and bridges to prosthetic limbs and sporting equipment. Designing is different than simply building in that it requires the adherence to a very systematic, yet iterative, process known as the "engineering design process." This process is to engineers what the scientific method is to scientists—guiding steps that help ensure that the end result is the best it can be. When a new product is created without following the steps of the engineering design process, there is a higher likelihood that the product designed will lack some important aspect: the end product may not appropriately account for the needs of its users, it may cost too much to manufacture, or it may not have been tested to ensure safety. Accordingly, the term "designing" refers to the entire process, such that an engineer "does design." The use of the term "design" as a noun may be used at different points in the process but may have very different meanings depending on what phase of the process the engineer is in. Design may really mean "design idea" during the brainstorming phase of the process or "model or prototype of the design" during the building phase of the process.

The engineering design process requires the application of mathematics in many of the steps. Throughout the process, engineers use basic mathematics concepts, including addition and multiplication to calculate costs; geometry to calculate surface areas for material needs; and measurements to ensure appropriate dimensioning. However, more sophisticated projects may require the application of higher-level mathematics, such as calculus and differential equations, to solve the technical engineering problems certain designs pose.

The Engineering Design Process

The engineering design process refers to the steps that are required to create the best possible solution to a problem. It is a process often undertaken by a team of engineers who work together, though it can be performed by an individual—trained or untrained as an engineer. Though there is no consensus as the exact breakdown and name of each step, the general design process is universally accepted.

In the first step of the engineering design process, the engineering team is presented with some type of problem or unmet societal need to be solved. Often, this problem is presented to the engineering team by a company that is trying to offer a product that better meets its customers' needs. The engineer must ask many questions to both the client and the user, as well as conduct background research, in an effort to establish the objectives and constraints of the design. The objectives are what the solution to the problem (the final designed product) should aim to accomplish. The constraints are the factors that limit the possible designs, such as time, money, or material restrictions. Time and money constraints are particularly important as they often drive the project and must be monitored throughout to ensure that the project is completed on time and within budget. At the end of this step of the design process, the engineering team fully understands the problem and has developed objectives and constraints to guide their possible solutions.

In the next step of the engineering design process, the engineers generate design ideas to solve the newly refined problem. Idea generation normally occurs through group brainstorming methods, with the goal of producing as many ideas as possible. There are a number of methods used to enhance the innovation and creativity of the ideas that come from the brainstorming session, including ensuring group diversity, drawing from existing stimulus and building off of each other's ideas. In this step of the process, some of the generated ideas will evolve into rough hand-drawn sketches. These sketches need to show perspective and relative size clearly.

The next step of the engineering design process is design selection. A method known as "decision analysis" is most commonly used for design selection. Decision analysis is a systematic process to objectively and logically choose the best idea to move forward with from the many generated through brainstorming. It is important because it reduces the likelihood of a designer's bias in selecting a design. As a first step, the brainstormed ideas must initially be narrowed down through discussion or other means to only the handful of ideas that appear to be most promising. These ideas are then compared through decision analysis. For the decision analysis, it is first necessary to create a list of design


Once the engineering team is satisfied with the final product, the design is executed through computer-aided design (CAD) drawings.

criteria and weight them based on their relative importance. As an example, as safety is paramount in design, the criteria of "safety" would be the most important criteria and would be weighted as 1.0 on a scale of 0–1. The criteria of "portability," on the other hand, might be desirable but not necessary, so it would be weighted as 0.5. There is no standard as to what weighting scale should be used but it is important to be consistent in its application. For each criterion, in addition to the determined weighted importance, a numerical range must also be established for rating each design with respect to the criterion. When possible, this range should be as objective and quantifiable as possible.

Each design being considered is then "scored" using the range for each criterion. The score is then multiplied by the relative criteria weight for a total score for each criterion and for each design. The total scores for each criterion are then summed for each design. The summed scores can be used to compare multiple designs, with the one scoring the highest being the one most likely to be successful.

After identifying a design to move forward with, refinement of the design is necessary. This step includes determining dimensions and materials that will be used to construct the chosen design. Detailed sketches, often drawn from multiple perspectives, are created and include the dimensions of each part to be made. Determining these dimensions often requires in-depth estimation and calculation. At the most simplistic level, dimensioning requires taking into account any necessary clearances or gaps in the design, especially when multiple parts need to be fitted together. It may also be necessary to determine the combinations of dimensions that ensure a specified surface area requirement is met, in which case algebra can be helpful. More in-depth designs may require that dimensions come from established tables of normative dimensions, such as anthropometric tables, providing typical measurements of different-sized people, or from engineering analysis, such as stress or buckling calculations. Deriving dimensions from engineering analysis methods often requires high-level mathematics and a technical background in engineering but ensures a stronger, safer product.

Once the design has been refined and the dimensions are known, building begins. For most designs, a scale model or a simplified prototype is created first to test for feasibility of the design before further time and money is invested. To create a scale model, all dimensions of the detailed sketches must be reduced by multiplying by some chosen scaling factor, often 1:2. Regardless of whether a full-size design or scale model is used, it is necessary to calculate the amount of each material that needs to be purchased to build the design. This requires thought and calculation, in particular when multiple parts could be cut from one piece of wood, metal, or fabric. Often, surface area is calculated according to the part's geometry to determine the total amount of material needed. Once material has been secured, building of the design can occur. Throughout building, it is essential to make careful measurements for all parts because almost all designs are made from multiple components that must fit together to function as one product. For example, if a piece of wood to be used for one leg of a chair is measured even ¼ inch shorter than the other legs, it will likely mean the finished chair will rock and wobble, and the design will be undesirable.

As a next step in the engineering design process, the constructed design is experimentally tested to determine its performance. This step helps to identify design strengths and weaknesses, which can be used to make recommendations for future refinement of the product. The specific experimental test performed is determined by the type of product designed and the design objectives. Regardless of the type of test conducted, measurements are taken throughout the experiment to record some aspect of the design's performance. Often, multiple trials will be taken, generating many data points. The data obtained from these measurements are then used to draw conclusions about the success of the design. Statistical analysis may also be employed to further assist in the interpretation of the data.

Almost always, the data collected during testing will suggest that the design could perform better if refined in some way. As such, it is common for the engineering team to return to the building stage and then iteratively cycle between it and testing steps until satisfied. At times, it may also be necessary to return to earlier steps in the engineering design process. Once the team is satisfied with the final product, final documentation is prepared to explain the design and share it with others. This is often done through computer-aided design (CAD) drawings and written technical reports.

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Kimberly Edginton Bigelow

See Also: Bridges; Green Design; Problem Solving in Society; Robots.

Equations, Polar

Category: History and Development of Curricular Concepts.

Fields of Study: Algebra; Communication; Connections; Geometry; Representations.

Summary: Polar coordinate systems were developed in the seventeenth century and have numerous modern applications.

The polar coordinate system is a coordinate system for the plane in which each point is determined by a distance from a fixed point, called the "pole," and an angle from a fixed direction, called the "polar axis." In normal usage, the pole is analogous to the origin in the Cartesian coordinate system, named for René Descartes. Both polar and rectangular (Cartesian) coordinates require two bits of data to place a point in the plane. While the Cartesian coordinate system requires knowing and placing two chosen lines to serve as axes, polar coordinates requires knowing one fixed point and one fixed ray. This characteristic makes polar coordinates useful in navigation. Students in twenty-first-century high schools are introduced to polar coordinate systems and the topic is further developed in college mathematics and physics classrooms.

History

The concept of using an angle and a radius may be dated to the first millennium B.C.E. There are references to Hipparchus of Rhodes (c. second century B.C.E.) using a type of polar coordinates to establish the positions of the stars that he studied. Archimedes of Syracuse describes his namesake spiral in the book *On Spirals*, as where the distance from a given point depends on the angle from a given radius.

In a number of articles about the development of polar coordinates, most notably the 1952 article "Origin of Polar Coordinates" by Julian Lowell Coolidge, further development of polar coordinates was generated by studying the Archimedean spiral. According to Coolidge's history, the first mention should go to Bonaventura Cavalieri in his 1635 treatise *Geometria indivisilibus continuorum* in which he studies the spiral of Archimedes. Cavalieri studies the area inside the spiral and relates it to other known areas.

Like all good stories in the history of mathematics, this assertion is not without disagreement. In 1647, Grégoire de Saint-Vincent in his work *Opus Geometricum* claimed that he was familiar with the method and had sent his work to Christopher Grienberger in 1625. Grienberger had died in 1636, and the priority of the work was the subject of an article by Moritz Cantor in 1900.

Spiral curves were of interest to many mathematicians, including Gilles Personne de Roberval, James Gregory, Descartes, and Pierre Varignon. Gregory, Descartes, and Varignon all used a type of transformation of coordinates that heralded the complete development of polar coordinates. It appears to be Jacob Bernoulli and Isaac Newton who most completely developed these transformations. Bernoulli worked on the lemniscate and introduced the terms "pole" and "polar axis." Newton investigated transformations between coordinate systems, including polar coordinates, in his work *Method of Fluxions*, which was written in 1671 but not published until 1736.

Applications

Polar coordinates are the basis for navigation and radar, since the direction of travel can be given as an angle and distance from the origin. The radar screen that is used in air traffic control uses the location of the radar transmitter/receiver as the pole and magnetic north as the polar ray, zero degrees. This aspect and the fact that the angles continue in a clockwise direction instead of a counterclockwise direction are the major differences between a navigational use and the mathematical system. This same radar is the basis for all weather radar that is available for viewing either on television or from the Internet. Each radar location (there are 178 National Weather Service Doppler weather radar locations that cover the United States) sets a pole and covers a specific area. Storms are located and their paths are computed using the overlaps. This information must be transformed from the polar system (how far from the radar site and at what angle) into GIS coordinate system and then placed on a map to go to television or to the Internet. One well-known measuring device is the polar planimeter, created by mathematician and physicist Jacob Amsler in the nineteenth century. It measured the area enclosed by a curve. Amsler switched careers to focus on mathematical instruments, and he produced thousands of Amsler planimeters.

Other examples of the use of polar coordinates are very simplified uses in planning sprinkler systems in a building, as well as in irrigation systems in landscape and farming. Each of the sprinkler heads serves as a pole, and different walls, boundary lines and such serve as polar axes.

Different microphones have different recording patterns depending on the specific purpose. The omnidirectional microphone is used when sound from all directions is to be recorded, such as a choir or a large group. A cardioid microphone is a unidirectional microphone, which would be used to record a performer but not the crowd. Bidirectional microphones are used in an interview situation where the voices of both the interviewer and interviewee need to be recorded. The pattern of sounds that are picked up by the microphone are a lemniscate—the figure studied by Bernoulli.

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David C. Royster

See Also: Climbing; Coordinate Geometry; Graphs; Maps; Transformations.

Escher, M.C.

Category: Arts, Music, and Entertainment.
Fields of Study: Geometry; Measurement; Representations.
Summary: The works of M.C. Escher are frequently used by mathematicians and mathematics educators to explore mathematical concepts.

Maurits Cornelis Escher (1898–1972) was a Dutch graphic artist perhaps best known for creating artwork with illusional and conceptual effects including woodcuts, lithographs, and mezzotints with meticulous detail. Despite the fact that he did poorly in mathematics in school, he accurately illustrated mathematical concepts in many of his works, which are frequently used by mathematicians and mathematics educators to illuminate and explore those concepts. He also wrote a paper called "Regular Division of the Plane with Asymmetric Congruent Polygons." About his own work and processes, Escher said:

> By keenly confronting the enigmas that surround us and by considering and analyzing the observations that I had made, I ended up in the domain of mathematics. Although I am absolutely without training or knowledge in the exact sciences, I often seem to have more in common with mathematicians than with my fellow-artists.

Early Work

As a student, Escher did not excel at any subject except drawing. After failing the final secondary school examinations, he enrolled in the Haarlem School of Architecture and Decorative Arts in 1919. Encouraged by one of his teachers, he shifted his interest from architecture to graphic arts. His first trip to the Mediterranean in 1922 made a strong impression upon him. He decided to leave the school and settle in Rome, where he married Jetta Umiker in 1924. In the following years, his fascination with Italian landscapes, combined with his passion for printmaking, resulted in a series of realistic woodcuts, *Castrovalva* being one of the most notable.

Hand with a Reflecting Globe (1935) marks the beginning of a period when the exploration of what he called an "inner vision" replaced his interest in the outward appearance of things. After fleeing from political turmoil to Switzerland, he testified that he was no longer inspired by his surroundings. During a visit to Spain in 1936, he worked extensively on copying the motifs from the Moorish mosaics in the Alhambra castle in Granada. The idea of creating patterns that would not only involve abstract shapes but also animal and human figures strongly obsessed him.

Art and Mathematics

In 1937 Escher created *Still Life and Street*, his first "impossible reality" image. In the same year, he moved with his family to Belgium, where he began to consider divisions of the plane using the work of mathematicians such as George Pólya regarding the 17 distinct plane symmetry groups. Using his own techniques, Escher explored questions such as the possible shapes for tiles that can produce a regular division of the plane, along with the various isometries that relate the edges of such tiles. Escher mapped adjacent tiles using translations, rotations and glide-reflections, all of which require the tiles' edges to be straight segments. This aspect became one of the central ideas of his art.

German occupation forced him to flee to Baarn, The Netherlands, in 1941, where he settled permanently. Two articles in *Life* and *Time* magazines in 1951 brought the world's attention to his work. Besides increasing demand for prints and numerous commissions, this recognition enabled him to start exchanges with many world-renowned scientists.

Escher's first ideas about infinity revolved around depicting decreasing figures as one moves toward the center of an image, as seen in his woodcut *Development II*. An article by a Canadian geometer Harold Scott MacDonald Coxeter made him reverse his point of view by creating a tessellation of a disc with tiles decreasing while moving toward the boundary of a disc. This approach produced some of his later prints, including *Circle Limit III*, and his last work, *The Snakes* (1969).

British mathematical physicist Roger Penrose, fascinated by Escher's lithograph *Relativity*, developed impossible objects known as the "Penrose tribar" and, together with his father, Lionel Penrose, the "Penrose staircase." After exchanging his ideas with the artist, these objects inspired lithographs *Waterfall* and *Ascending and Descending*. Mathematicians continue to investigate the mathematical details of Escher's work. Number theorist Hendrik Lenstra used the theory of elliptic curves and complex exponential functions to analyze aspects of Escher's *Print Gallery*.

Escher's most ambitious work, a 22-foot-wide woodcut, Metamorphosis III, was based on tessellations. Many other mathematical topics were also implemented in Escher's work: topology in depictions of the Möbius strip, the principle of self-reference in *Drawing Hands*, numerous polyhedra, concave and convex objects, irregular perspective, spherical geometry, optical illusions, and visual paradoxes, among others. Escher's creative interpretation of these subjects erased the boundaries between mathematics and art. He said,

> At first I had no idea at all of the possibility of building up my figures. I did not know any "ground

rules" and tried, almost without knowing what I was doing, to fit together congruent shapes that I attempted to give the form of animals. Gradually, designing new motifs became easier as a result of my study of the literature on the subject . . .especially as a result of my putting forward my own layman's theory, which forced me to think through the possibilities.

Legacy

M.C. Escher felt closer to mathematicians than to his peers. Although he frequently stated that he was a mere craftsman, not an artist, some of the images he created found their place in popular culture and mathematics, becoming icons of the twentieth century. Escher's son George noted that his father often did not seem to comprehend that his process of creation and exploration of the mathematical concepts he used in his work was in fact very much like a mathematician. His work tackles human understanding of the order of the universe and unveils it with unexpected beauty and refinement.

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Zoran Petrovic
Karim Salim

See Also: Crystallography; Geometry in Society; Optical Illusions; Symmetry; Transformations.

Ethics

Category: Mathematics Culture and Identity.
Fields of Study: Communication; Connections; Problem Solving.
Summary: Since the time of Plato, mathematicians have been analyzing and confronting ethical problems.

Mathematics and ethics have a long and tangled history. Philosophy has nurtured mathematical forms of thought that, in turn, have had a profound influence on ethical theorizing. For example, mathematics served as a model for Jeremy Bentham (1748–1832) whose goal in utilitarianism was to develop a calculus of pleasure and pain.

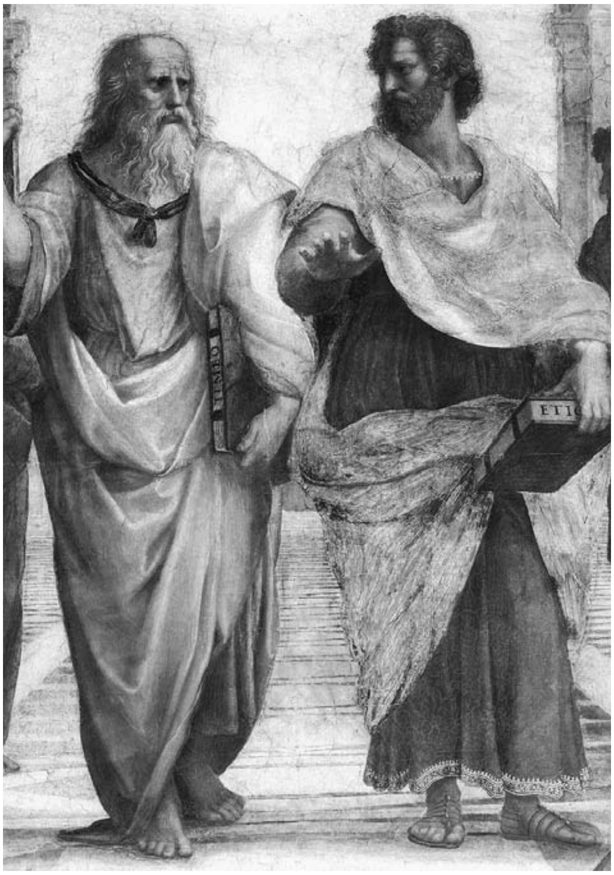
Several contemporary ethical theories are tied to the mathematics of game theory, especially the work of John Rawls (1921–2002). Ethical issues arise in mathematics teaching, research, industry, and government work. Mathematicians such as Lee Lorch challenge discriminatory practices and fight for human rights, justice, and equality. Other mathematicians have refused to work on projects they find ethically problematic. Ethical norms often change over time and for various contexts, leading to controversial applications of mathematics research, like the atomic bomb. In the face of increasing marketability of mathematical results, some have questioned the disparity between the academic tradition of making knowledge freely available and personal ownership of intellectual property. Many professional associations have developed, maintained, and revised ethical guidelines for their members, and mathematicians who wish to perform experiments must submit a proposal to an institutional review board for ethical review. In 2010, the National Science Foundation issued a program solicitation for an Ethics in Science, Mathematics, and Engineering Online Resource Center.

Mathematics and Ethics in Plato (429–347 B.C.E.)

Plato's *Republic* is the first systematic treatment of ethics. The best preparation for acquiring ethical knowledge is a firm foundation in mathematics. However, the connection between mathematics and ethics is much deeper. Methodologically, Plato develops his argument by building a simplified model of the state in the same manner in which a study of any geometrical figure is done in mathematics. Justice in the state is merely justice in the individual writ large. Thus, Plato appeals to similarity transformations. The argument is that, as a result of a uniform scaling operation, justice in the individual is similar to justice in the state. Further, within the Platonic tradition, mathematical and ethical knowledge have the same formal characteristics. They are both examples of purely intelligible objects grasped entirely by reason in an intellectual intuition and known as a result of a process of recollection. Thus, they are examples of immutable and unchangeable truths, which could not be other than what they are. Plato's very definition of justice contains a mathematical element, because justice is a type of equality. Justice is a matter of treating equal individuals equally and unequal individuals unequally. According to Plato, different political orders arise from the different conceptions of equality.

Mathematics and Ethics in Aristotle (384–322 B.C.E.)

For Aristotle, mathematics does not provide a model for ethics. However, mathematical concepts function in an analogical sense. Aristotle used a distinction between arithmetic and geometric proportion in his discussion of justice. Distributive justice is based on geometrical proportion, while rectificatory justice is based on arithmetical proportion. Issues of rectificatory justice arise when a judge must rectify a situation by attempting to restore equality to someone who has been injured. Issues of distributive justice arise when something has to be divided among two individuals.



A detail of Plato (left) and Aristotle from the painting The School of Athens *by Italian artist Raphael.*

Modern Moral Euclidian Philosophers

Both Thomas Hobbes (1588–1679) and Baruch Spinoza (1632–1677) incorporated the mathematical method of Euclid of Alexandria into their treatment of ethics. Hobbes thought that mathematical modes of thought could produce clarity in ethics and politics. However, it was Spinoza who most rigorously and consistently imitated Euclid's method. He begins each section of his *Ethics* with a set of definitions and axioms, which he then uses to demonstrate a series of propositions about the universe, human nature, and basic ethical precepts.

Mathematical Ethics

The guidelines of professional mathematical associations cover a wide range of topics. Creation, attribution, publication, and presentation of research, especially with regard to falsification and plagiarism, as well as skewed interpretations and one-sided "advertising" style arguments, are commonly addressed. These guidelines extend into the classroom, along with data sharing or loaning and responsible group work. Attention is also given to the nature of teacher-student and colleague relationships in which one individual has some level of authority over the other, especially when they involve professional decisions like hiring, granting tenure, issuing promotions, and conferring degrees.

Mathematician Philip Davis noted that ethics are typically derived from past experiences and so may do little good in addressing many future or even current dilemmas. Further, judging the past based on current criteria leads to additional difficulties. Arguments abound, for example, about whether statistical data gathered from Nazi medical experiments should be used or destroyed, or whether mathematicians can be held responsible for any future unanticipated uses of their work, such as computer viruses or code-breaking algorithms usurped by data thieves. The Manhattan Project exemplifies many of the moral dilemmas faced by mathematical scientists. Many participants have expressed profound regrets; others have not, citing the undeniable advances made in numerous fields and the need at the time to bring an end to the greater destruction of World War II. For example, the cyclotron was invented by Ernest O. Lawrence in 1931, who received the Nobel Prize in 1939 for this invention.

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Michael K. Green

See Also: Atomic Bomb (Manhattan Project); Game Theory; Genetics; Nanotechnology; Vietnam War.

Europe, Eastern

Category: Mathematics Around the World.
Fields of Study: All.
Summary: Eastern Europe has a long tradition of both mathematics research and education.

Throughout history, the countries of Europe have had shifting political and social boundaries. Eastern European mathematics evolved within the context of many mathematics traditions, including Soviet Union mathematics, over the past centuries. Historically, gifted young scholars from regions around the world completed their mathematical studies at Europe's well-known and respected universities. Studies of mathematicians' letters and scientific papers show that they often maintained connections with people in other countries who shared their fields of interest. The Soviet Union exercised broad social and political influence over most of eastern Europe and also impacted U.S. mathematics in the twentieth century. Within the Soviet Union, students from the far reaches of the nations within its boundaries were often brought to Russia for work or education, as well as sent to other parts of the Soviet Union to teach or to establish research centers. In the twenty-first century, students in the United States and around the worked attend study abroad programs, such as the Budapest Semesters in Mathematics. In the twenty-first century, the United Nations Statistics Division classified the following countries belonging to eastern Europe: Belarus, Bulgaria, Czech Republic, Hungary, Moldova, Poland, Romania, Russia, Slovakia, and Ukraine. The *CIA World Factbook* adds Estonia, Latvia, and Lithuania, which were among the member nations of the Soviet Union, though the United Nations classifies them as belonging to northern Europe. Geographical boundaries continued to change in the twentieth century because of post–World War II structures and, later, the breakup of the "Eastern Bloc" nations, which were once under the Soviet Union's political influence. Therefore, mathematics contributions of some people from eastern Europe may be included within the histories of other regions or countries.

History of Russian and Soviet Mathematics Education

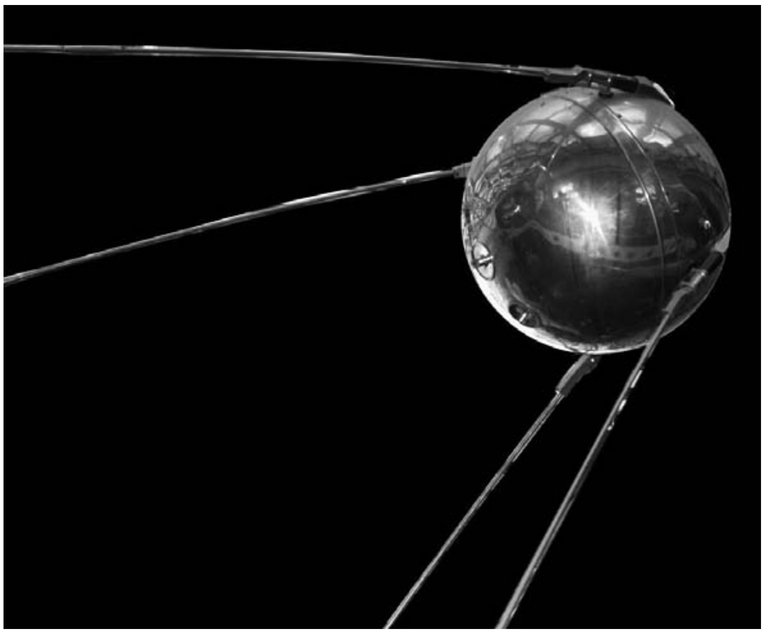
When examining past and present states of mathematics in Belarus, Moldova, Russia, Ukraine, Estonia, Latvia, and Lithuania, it is pertinent to acknowledge that they share a common sociopolitical root: they are all former member states of the Soviet Union. Further, the broader Eastern Bloc of Soviet Union allies included Bulgaria, Romania, Hungary, East Germany, Poland, Albania (until the early 1960s), and Czechoslovakia (which later split into the Czech Republic and Slovakia). The Eastern Bloc is sometimes known historically as "eastern Europe," versus the "western Europe" countries allied with the United States, a rival of the Soviet Union. During its several decades of existence in the twentieth century, the Soviet Union included many mathematicians who made significant contributions to the body of modern mathematical knowledge. Further, Russian and Soviet mathematicians were influential on many other countries.

One important landmark in mathematics education in Russia is the creation in 1701 of the School of Mathematical and Navigational Sciences in Moscow. Peter the Great, who had traveled widely in other parts of Europe to study the state of mathematics and science as part of his effort to modernize Russia and expand the empire, founded this school. It educated students in basic mathematics as well as more specialized subjects, such as astronomy and navigation. Notably, students from all social classes except serfs were admitted, and financial assistance was available. Graduates worked in the navy, as engineers, and as teachers in a variety of settings, so the school had a multiplier effect in terms of spreading mathematics education throughout Russia. Peter the Great also founded the Saint Petersburg Academy of Sciences in 1724, influenced in part by correspondence with mathematician Gottfried Leibniz, who also purportedly recommended a three-tiered educational system of schools, universities, and academies. Many eminent foreign mathematicians, such as Leonard Euler, Christian Goldbach, and Daniel Bernoulli, worked at the Saint Petersburg Academy.

As part of her goal of modernizing Russia in the European style, Empress Catherine the Great, who was born in Germany, established the first gymnasiums in Russia. These gymnasiums were schools meant to prepare students for higher education and were created in most major Russian cities in the nineteenth century. Nicolai Ivanovich Lobachevsky, one of the first Russian mathematicians to achieve international recognition, was a beneficiary of this expanded educational opportunity. He graduated from Kazan Gymnasium and Kazan University (in Tatarstan) and is most noted for his work in hyperbolic geometry, a form of non-Euclidean geometry. However, despite this considerable expansion, access to education was far from universal until the Soviet era. The Soviet Union was founded by revolution in 1917, when the monarchy of the Russian Empire was overthrown, but was not made official until 1922. The Saint Petersburg Academy of the Sciences evolved into the Russian and then Union of Soviet Socialist Republics (USSR) Academy of the Sciences. It reverted to the Russian Academy of Sciences following the dissolution of the Soviet Union, and remains an influential organization in the twenty-first century. Academies of sciences were also founded in most of the states of the Soviet Union. Universal compulsory education was established in 1919. Soviet schools had both political and educational goals but the expectation that all children would attend school rapidly increased literacy and played a key role in modernizing and industrializing the country.

In the Soviet Union, the study of mathematics and the sciences was emphasized, a choice that not only fostered rapid economic growth but also became a point of national pride, as by mid-century the Soviet Union was frequently seen to rival or even surpass the United States in scientific and applied research. When the Soviet Union successfully launched the satellite Sputnik in 1957, it raised concern in the United States not only because of the possibility that the Soviet Union was developing weapons for which the United States had no counter but also because it put into question the common assumption that the United States was the world leader in mathematics and science. One result of Sputnik in the United States was a substantial increase in federal funding for scientific education and research in the hope of catching up and surpassing the Soviet Union in the "space race."

As part of this concern that the Soviet Union was surpassing the United States, many studies were commissioned of the Soviet educational system and how it differed from the American system. Among the differences noted by researchers were the facts that in Soviet schools, specialists taught mathematics from the fourth grade onward, a uniform curriculum was used across the entire country, and much greater emphasis was



The launch of the first artificial satellite, Sputnik 1, by the Soviet Union on October 4, 1957, started the race to the moon.

placed on developing the talents of students who were identified as gifted in mathematics. The Soviet Union had “special schools,” which were free boarding schools at high school level for gifted students and specialized in particular subjects. Four such schools were devoted to mathematics. Correspondence courses in advanced mathematics were also available to increase the number of students studying those subjects. American observers noted that the level of mathematics required for university admittance during the Soviet period was much higher than what would be expected for entering freshmen in the United States. At the same time, other authors have noted that English-language sources often do not reflect the full scope and influence of Russian and Soviet mathematics. These omissions may be because of Cold War influences and a period of Soviet isolationism from the United States and much of Europe, a policy that contrasts strongly with earlier Russian connections and the growing collaborations following the Soviet era.

Notable Soviet and Russian Mathematicians

Andrey Kolmogorov (1903–1987) is known for his work in the fields of probability theory and topology, including the Kolmogorov axioms, Kolmogorov’s zero-one law, and Kolmogorov space.

Stefan E. Warschawski (1904–1989) studied at the University of Königsberg and Göttingen. His doctoral thesis was on the boundary behavior of conformal mappings.

Sergei Lvovich Sobolev (1908–1989) worked in mathematical analysis and partial differential equations. Sobolev spaces (named after him) can be defined by growth conditions on Fourier transforms.

Israel Moiseevich Gelfand (1913–2009) worked in the field of functional analysis. He is known for the Gelfand representation in Banach algebra theory; the representation theory of the complex classical Lie groups; contributions to distribution theory and measures on infinite-dimensional spaces; integral geometry; and generalized hypergeometric series. His name is linked to the development of mathematical education.

Igor Shafarevich (1923–) is the founder of the major school of algebraic number theory and algebraic geometry in the Soviet Union. He has also written well-known textbooks.

Grigori Perelman (1966–) declined the Fields medal, a prestigious award in mathematics often equated to the Nobel Prize, for his work on the Poincaré conjecture, named for Henri Poincaré. He cited inequities and reportedly noted, “If the proof is correct then no other recognition is needed.”

Other well-known Soviet or Russian twentieth-century mathematicians include Boris Pavlovich Demidovich, who worked on problems in mathematical analysis, and Yakov Isidorovich Perelman, who was a science writer and author of many popular science books.

Czech Republic and Slovakian Mathematicians

Kurt Gödel (1906–1978) proved fundamental results about axiomatic systems. Gödel’s Incompleteness Theorems are named for him.

Stefan Schwarz (1914–1996) studied semigroups, number theory, and finite fields and founded the Mathematico-Physical Journal of the Slovak Academy of Sciences in 1950.

Hungarian Mathematicians

Hungarian mathematicians of the twentieth century are well known in the mathematical world. Many of them immigrated to the United States after World War II.

Frigyes Riesz (1880–1956) was a founder of functional analysis. He produced representation theorems for functional on quadratic Lebesgue integrable functions, named for Henri Lebesgue, then introduced the space of q-fold Lebesgue integrable functions. He also studied orthonormal series and topology.

George Pólya (1887–1985) worked in probability, analysis, number theory, geometry, combinatorics, and mathematical physics. He wrote books about problem-solving methods, complex analysis, mathematical physics, probability theory, geometry, and combinatorics. He was regarded by many as a great teacher and influenced many mathematicians.

Cornelius Lanczos (1893–1974) worked on relativity and mathematical physics. He invented what is now called the Fast Fourier Transform, named for Joseph Fourier. He published more than 120 papers and books.

John von Neumann (1903–1957) worked in quantum mechanics, game theory, and applied mathematics, as well as helping pioneer computer science. His doctoral thesis was on set theory. His definition of ordinal numbers is the one commonly used in the early twenty-first century.

Rózsa Péter (1905–1977) is known for teaching, for her books on the history of mathematics, and for her series of theorems about primitive recursive functions.

Paul Erdos (1913–1996) is well known among mathematicians for his insatiable ability to pose and solve problems. It is often said that he lived on mathematics and coffee, touring the circle of his friends and pupils and giving lectures on combinatorics, graph theory, and number theory. He advocated for elegant and elementary proof. One of the most prolific mathematicians in history, he wrote more than 1500 papers.

Paul Richard Halmos (1916–2006) is known for his contributions to operator theory, ergodic theory, functional analysis (in particular Hilbert spaces, named for David Hillbert), and for his textbooks.

Alfréd Rényi (1921–1970) worked on probability theory, statistics, information theory, combinatorics, graph theory, number theory, and analysis.

László Lovász (1948–) published his first paper called *On graphs not containing independent circuits* when he was only 17 years old. He is a prominent figure of post–World War II mathematicians.

Notable Polish Mathematicians

Stefan Banach (1892–1945) worked on the theory of topological vector spaces, measure theory, integration, and orthogonal series. His doctoral thesis "On Operations on Abstract Sets and their Application to Integral Equations" (1920) marks the birth of modern functional analysis. He defined the "Banach space."

Benoit Mandelbrot (1924–2010) is known as the father of fractal geometry. The Mandelbrot set, a connected set of points in the complex plane, is named after him.

Mathematicians From Romania

János Bolyai (1802–1860) is perhaps the most famous Romanian mathematician because of his treatise on a complete system of non-Euclidean geometry in his book *Appendix*. In his own words, he created a new world out of nothing.

Caius Iacob (1912–1992) worked in the fields of analytic geometry, descriptive geometry, analysis, and complex functions.

Grigore C. Moisil (1906–1973) worked on differential equations, the theory of functions, and mechanics. He set up the first Romanian computer science course. Moisil was appreciated for his philosophy and humor.

Other important Romanian mathematicians include Dimitrie Pompeiu, Ferenc Radó, Isaac Jacob Schoenberg, Simion Stoilow, Gheorghe Titeica, Gheorghe Vranceanu, Octav Onicescu, Ion Colojoara, and Dan Barbilian.

Competitions and Contests

Building on eastern Europe's strong mathematics traditions, many mathematical contests are hosted frequently or entirely within the region, such as International Mathematical Olympiad, Romanian Master of Sciences (formerly called the Romanian Masters in Mathematics—it was expanded to include physics), Czech-Polish-Slovak Match, Bulgarian Competition in Mathematics and Informatics, Romanian National Olympiad, and the International Kangaroo Mathematics Contest (often called "Math Kangaroo") among others. Individuals from all over the world participate regularly in these competitions. There are also several winners of the Fields Medal who were born or worked in eastern Europe.

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Simone Gyorfi
Sarah Boslaugh

See Also: Europe, Northern; Europe, Southern; Europe, Western.

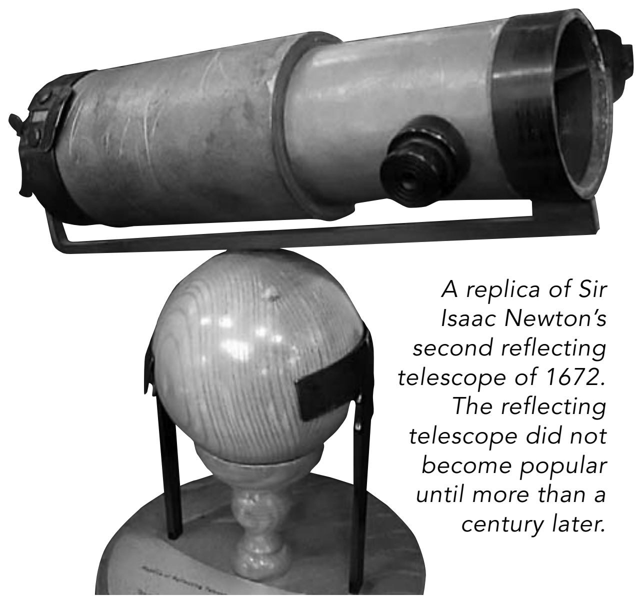
Europe, Northern

Category: Mathematics Around the World.
Fields of Study: All.
Summary: Since the Enlightenment, Northern Europe has made considerable contributions to mathematics research and continues to do so.

Northern Europe has produced many outstanding mathematicians and scholars in related fields, from the development of calculus by Isaac Newton in the seventeenth century to the cosmological models developed by Stephen William Hawking in the twentieth and twenty-first centuries.

Northern Europe also led the way in developing many practical applications of mathematics and later statistics, including taking a national census like the Domesday Book undertaken in England in 1183 and developing mathematical ways to measure the influence of personal habits on health as in the studies of Richard Doll and Austin Bradford Hill on the relationship between smoking and disease. In the twenty-first century, the United Nations category of northern Europe includes the Åland Islands, the Channel Islands, Denmark, Estonia, Faeroe Islands, Finland, Guernsey, Iceland, Ireland, the Isle of Man, Jersey, Latvia, Lithuania, Norway, Svalbard and Jan Mayen Islands, Sweden, and the United Kingdom of Great Britain and Northern Ireland. However, the changing political boundaries in many of these countries throughout history, as well as the rise and fall of the Soviet Union, which included countries like Estonia, Latvia, and Lithuania, mean that mathematical contributions of some individuals may be included within the histories of other regions.

Sir Isaac Newton was one of the most influential mathematicians of the modern era. He shares credit with Gottfried Leibniz for developing integral and differential calculus, and he also made major contributions in the fields of physics and astronomy. Newton's 1687 book *Philosophiae Naturalis Principia Mathematica* laid the groundwork for classical mechanics including a description of the three laws of motion and remains one of the most influential books in the history of science. He also built the first reflecting telescope and developed a theory of color based on the visible spectrum displayed when visible light is refracted through a prism. Through his work with the laws of gravity and Kepler's laws of planetary motion, named for Johannes Kepler, Newton was able to demonstrate mathematically the validity of heliocentrism, which is the scientific principle that Earth and other planets revolve around the sun.



A replica of Sir Isaac Newton's second reflecting telescope of 1672. The reflecting telescope did not become popular until more than a century later.

The nineteenth century saw several major breakthroughs in mathematics by scholars from northern Europe. In England, philosopher and mathematician George Boole developed the system now known as "Boolean logic," which has many practical applications and was instrumental in the development of modern digital computers. His most famous works are *The Mathematical Analysis of Logic* (1847) and *The Laws of Thought* (1854). His slightly younger contemporary, Norwegian Niels Henrik Abel, invented the field of group theory (contemporaneously with Frenchman Evariste Galois), which has many applications in mathematics and physics. Abel is well known for a proof he wrote at age 19 that there can be no general algebraic solution of an equation greater than degree four. In Ireland, Sir William Rowan Hamilton provided an important reformulation of Newtonian mechanics and invented an extension of the number system called "quaternions."

In the period 1910–1913, the British scholars Bertrand Russell and Alfred North Whitehead wrote the influential *Principia Mathematica* in which they attempted to derive the foundations of mathematics from a set of axioms and inference rules. Russell was also a prominent writer and political activist who won the Nobel Prize for Literature in 1950, while Whitehead was also noted as a philosopher. More recently, Andrew Wiles, who was born and educated in the United Kingdom but immigrated to the United States, achieved fame for proving Fermat's Last Theorem (named for Pierre de Fermat), one of the most famous previously unsolved problems in mathematics.

Honors

There is no Nobel Prize for mathematics but several different international awards are offered that have been termed the "Mathematics Nobel Prize" because of their prestige. The Fields Medal is awarded every four years to one or more mathematicians of age 40 or younger by the International Mathematical Union. Winners from the United Kingdom have included Klaus Roth (1958), Michael Atiya (1966), Alan Baker (1970), Simon Donaldson (1986), Richard Borcherds (1988), and Timothy Gowers (1998). Lars Ahlfors of Norway won in 1936, the first year the medal was given; Atle Selberg of Norway won in 1950; and Lars Hormander of Sweden won in 1962. Another major mathematical prize, the Abel Prize, is named after Norwegian mathematician Niels Henrik Abel and is awarded annually by the Norwegian Academy of Science and Letters. The Abel Prize has been awarded since 2003. Northern European winners include Michael F. Atiyah of the United Kingdom and Lebanon in 2004 and Lennart Carleson of Sweden in 2006.

The Wolf Prize in Mathematics has been awarded almost annually by the Wolf Foundation since 1978 and more than one prize may be given per year. Northern European winners include Lars Ahlfors of Finland (1981), Atle Selberg of Norway (1986), Lars Hormander of Sweden (1988), Lennart Carleson of Sweden (1992), Andrew Wiles of the United Kingdom (1995/1996), and David B. Mumford of the United Kingdom (2008).

Northern European countries have been regular competitors in the International Mathematical Olympiad, an annual competition held since 1959 for high school students. Each competing country sends a team of six students who are assigned six questions to solve. Individual students are awarded medals based on their scores, and countries are also compared based on the total score for their team.

There have been many medal winners from northern European countries. The United Kingdom began participating in 1967 and even hosted the 1976 and 2002 competitions. Ireland first participated in 1988. The northern Europe countries from the former Soviet Union—Estonia, Latvia, and Lithuania—first participated in 1993, which coincided with the removal of Russian troops from the area and other political reorganization throughout the former Soviet Union. Among the Scandinavian countries, Sweden first participated in 1967, Norway in 1984, Finland in 1965, Denmark in 1991, and Iceland in 1985. Sweden hosted the 1991 competition, and Finland hosted it in 1985.

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Sarah Boslaugh

See Also: Calculus and Calculus Education; Europe, Eastern; Hawking, Stephen; Lovelace, Ada; Wiles, Andrew.

Europe, Southern

Category: Mathematics Around the World.
Fields of Study: All.
Summary: Modern Western mathematics was developed in southern Europe and continues to thrive there.

The system of modern mathematics originated in southern Europe, with the ancient Greeks undoubtedly building on traditions already used in Egypt and by the Phoenicians. Like many areas of the world, the nations of southern Europe have had many different boundaries, names, and political alliances throughout history, and so the mathematical contributions of some individuals may be included within the histories of other regions. For example, many nations were member states of the former Soviet Union. The United Nations now includes Albania, Andorra, Bosnia and Herzegovina, Croatia, Gibraltar, Greece, Holy See, Italy, Malta, Montenegro, Portugal, San Marino, Serbia, Slovenia, Spain, and the former Yugoslav Republic of Macedonia in Southern Europe.

Ancient Greeks and Romans

The earliest Greek school of mathematics is ascribed to Thales (c. 640–550 B.C.E.), who came from Miletus, in present-day Turkey, and Pythagoras (c. 569–500 B.C.E.) who hailed from the Mediterranean island of Samos and later moved to Sicily. Archytas, who subscribed to the Pythagorean philosophy and worked on the harmonic mean, was from Tarentum in modern-day Italy. One of the most well-known Greek mathematicians of the ancient world, Euclid of Alexandria (c. 330–260 B.C.E.), was also not from the Greek mainland. He lived in Alexandria, in modern-day Egypt, and his work proved hugely influential to subsequent mathematicians with his detailed hypotheses and proofs. The great mathematician Archimedes of Syracuse (c. 285–212 B.C.E.) also studied in Alexandria but was from Sicily, where he spent most of his life.

These early Greek mathematicians were undoubtedly an influence on the Romans but the Romans themselves were seemingly more interested in applied mathematics—especially how it related to engineering and building—than in the pure mathematics that was favored by the Greeks. Mathematics was certainly taught in Roman schools and historians have long pondered why Roman mathematicians did not have more influence. This dearth of mathematical advancement has generally been ascribed to the Romans' lack of a designation for "zero" and their awkward system of numbers, which may have prevented any great advances in theory. The Roman Empire did, however, see a continual flourishing of mathematics in Greece and the Greek diaspora, in particular the city of Alexandria. Anicius Manlius Severinus Boethius (c. 475–525) was a well-known Roman mathematician who worked during the declining years of the Roman Empire.

The Renaissance

The Bishop of Seville, Isidorus Hispalensis (570–636), helped develop mathematics in Spain and there were great advances made in arithmetic with the Moorish invasions of Spain and the incorporation of many of the advances made in the Muslim world. The great trading cities of Genoa and Venice soon established themselves as important centers of finance, as did Florence during the Renaissance. Venice, in particular, because of its geographical position and its connections with the Arab world, saw the importation of many books and manuscripts on Arab mathematics—at that stage well advanced in pure mathematics theories compared to Europe. This Arab influence saw Leonardo Pisano Bigollo (c. 1170–1250), the son of an Italian merchant in North Africa, develop theories—the most well-known being the Fibonacci numbers, which were termed after his assumed name.

Several centuries later, the advent of the printing press also led to a republication of the works of Greek mathematicians such as Euclid, albeit in Latin translation. Cardinal Bessarion, the former Archbishop of Nicaea, helped bridge the link between Byzantium and Rome, helping to preserve some of the Greek learning that was lost when the city of Constantinople was captured and sacked in 1453. Leonardo da Vinci (1452–1519) developed mathematics theories, testing out some of them in siege machines designed for Cesare Borgia and others. Girolamo Maggi (c. 1523–1572), another Italian mathematician, was involved in designing military defenses in Cyprus. He was captured by the Ottoman Turks and executed in Constantinople but not before writing two major treatises from memory while in prison there.

The Renaissance saw a new interest in mathematics in Italy, with Galileo Galilei (1564–1642) being a well-known mathematician and scientist. He was a great influence on many subsequent mathematicians, including Alessandro Marchetti (1633–1714). Evangelista Torricelli (1608–1647) invented a barometer; Giovanni Ceva (1647–1734) proved Ceva's theorem in elementary geometry; and the Jesuit Franceso Cetti (1726–1778) helped connect mathematics to other scientific discoveries. Later Italian mathematicians include Giulio Ascoli (1843–1896) who taught in Milan, and Carlo Emilio Bonferroni (1892–1960) who developed the theory of Bonferroni inequalities. The Italian Mathematical Union was established in 1922 by Salvatore Pincherle and others, and its journal, the *Bollettino dell'Unione Matematica Italiana,* is widely respected around the world.

Professional Associations

Professional associations in the region other than the Italian Mathematical Union include the Bosnian Mathematical Society; the Croatian Mathematical Society; the Cyprus Mathematical Society; the Montenegro Mathematical Society; the Portuguese Society of Mathematics; the Mathematical Society of Serbia; the Mathematics, Physics, and Astronomy Society of Slovenia; and the Royal Spanish Mathematical Society. Mathematicians also gather from all over Europe in the European Mathematical Society. The International Mathematical Olympiad is a competition for high school students that originated in 1959. Albania first participated in 1993, Bosnia and Herzegovina in 1993, Croatia in 1993, Greece in 1975, Italy in 1967, Montenegro in 2007, Portugal in 1989, Serbia in 2006, Slovenia in 1993, Spain in 1983, Yugoslavia in 1963, and the former Yugoslav Republic of Macedonia in 1993. Greece was a host of the competition in 2004, Slovenia in 2006, Spain in 2008, and Yugoslavia in 1967 and 1977.

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Justin Corfield

See Also: Archimedes; Europe, Eastern; Greek Mathematics; Roman Mathematics.

Europe, Western

Category: Mathematics Around the World.
Fields of Study: All.
Summary: Western Europe has been home to many of the important astronomical and mathematical discoveries of the early modern age.

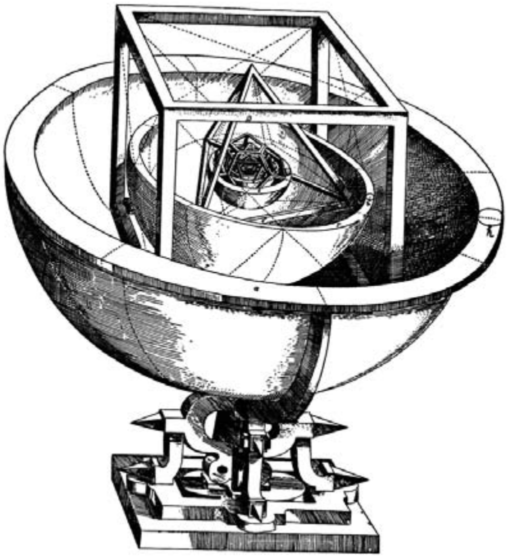
Historically, the term "western Europe" has had cultural and political definitions. For example, during the Cold War it was often used to designate a collection of noncommunist countries allied in some way with the United States. In the early twenty-first century, the United Nations Statistics Division for western Europe contains Austria, Belgium, France, Germany, Liechtenstein, Luxembourg, Monaco, the Netherlands, and Switzerland. There is a rich history of mathematics scholarship, education, and achievement in western Europe. Important work in a diverse array of mathematical areas like calculus, number theory, analytical geometry, probability, statistics, functional analysis, graph theory, logic, and number theory was produced by people from this geographic region, as well as many mathematical contributions to related disciplines like physics, astronomy, optics, engineering, and surveying.

Historical Contributions

Western European mathematicians have made major contributions to the development of mathematics and the application of mathematical theory to practical problems, from German mathematician and astronomer Johannes Kepler, who worked with Danish astronomer Tycho Brahe and helped established the laws of planetary motion, to French mathematician René Thom, who founded the study of catastrophe theory.

Much of modern science and mathematics has its roots in work done in Europe in the seventeenth century. Johannes Kepler studied at the University of Tubingen, where he learned both the geocentric model of astronomy (the view that Earth is the center of the universe, with the other planets revolving around it) and the heliocentric model of German astronomer Nicolaus Copernicus (the view that the sun is the center of the universe and the planets, including Earth, revolve around it). He later worked with Brahe and established the laws of planetary motion in several influential publications: *Astronomia Nova*, *Harmonices Mundi*, and *The Epitome of Copernican Astronomy*. Also in Germany, mathematician Gottfried Leibniz developed the field of calculus independent of Sir Isaac Newton in England.

In France, mathematician and philosopher René Descartes developed analytical geometry, including the development of Cartesian coordinates, did important work in optics, and was also one of the fathers of modern Western philosophy with influential books such as *Meditations on First Philosophy*, *Discourse on the Method* (which contains the oft-quoted statement



Johannes Kepler's platonic solid model of the solar system was published in 1596.

cogito ergo sum, or "I think, therefore I am"), and *Principles of Philosophy*. Also in France, the basics of probability theory were developed by mathematicians Pierre de Fermat and Blaise Pascal, while Fermat also did important work in number theory, analytic geometry, and optics. Fermat's Last Theorem, mentioned but not proved by Fermat in 1637 in the margin of a book, was among the unsolved problems in mathematics until British mathematician Andrew Wiles proved it in 1994. Pascal invented the mechanical calculator and the hydraulic press and is well known among middle school students for Pascal's Triangle, a presentation of binomial coefficients.

In the eighteenth century, Swiss mathematician and physicist Leonhard Euler spent much of his adult life working at the Russian Academy of the Sciences in St. Petersburg. He developed the concept of the function and the notation $f(x)$, one of several notation conventions he developed that are still used in the early twenty-first century (others include using the letter *e* for the natural logarithm, *i* for an imaginary unit, and the Greek letter *sigma* (Σ) for summation). He also made important contributions to calculus, number theory, graph theory (he solved the famous Seven Bridges of Konigsberg problem), and applied mathematics. French and Italian astronomer and mathematician Joseph-Louis Lagrange, who was born in Italy but worked primarily in France and Prussia, created the calculus of variations, developed a method of solving differential equations and transformed Newtonian mechanics into a branch of analysis, which facilitated the development of mathematical physics. He was also the first professor of analysis at the École Polytechnique, an elite engineering school founded in France in 1794. Also in France, mathematician and astronomer Pierre-Simon LaPlace played a key role in the development of Bayesian statistics, named for English minister and mathematician Thomas Bayes, and mathematical astronomy. He also posited the existence of black holes and gravitational collapse in the solar system.

In the nineteenth century, mathematician German Carl Friedrich Gauss made important contributions to several mathematical and physics fields including statistics, number theory, astronomy, surveying (he invented the heliotrope), and optics. The well-known normal distribution is sometimes referred to as the "Gaussian distribution" because he is often credited with discovering it. In France, Augustin-Louis Cauchy not only worked as an engineer but also pursued mathematical studies in his spare time and was appointed to the Académie des Sciences in 1816. He made numerous contributions to mathematics and physics, including his development of complex function theory, clarification of the principle of calculus, and development of the argument principle. In France, mathematician Evariste Galois proved, in parallel with the work of Norwegian mathematician Niels Henrik Abel, that there was no general method for solving polynomial equations of degree of greater than degree four.

In 1900, German mathematician David Hilbert gave an influential talk at the International Congress of Mathematicians in which he identified 23 unsolved problems in mathematics, which served as a spur for other mathematicians to focus on those problems (10 have been solved as of 2010). Hilbert is also well known for formulating the theory of Hilbert spaces, which are key to functional analysis, and did important work in mathematical logic and proof theory. Austrian mathematician Kurt Gödel, best known for his two incompleteness theorems, immigrated to the United States to escape World War II and spent his later years at Princeton University. A group of primarily French mathematicians, including Jean Dieudonne and André Weil, began publishing anonymously under the pseudonym "Nicolas Bourbaki." They are now known as the "Bourbaki Group" or "Association des collaborateurs de Nicolas Bourbaki" and have published several books in which they attempt to ground different areas of mathematics in set theory.

Awards and Honors

There is no Nobel Prize for mathematics but several different international awards are offered that have been termed the "Mathematics Nobel Prize" because of their prestige. The Fields Medal is awarded every four years to one or more mathematicians of age 40 or younger by the International Mathematical Union. Winners of the Fields Medal from western Europe include Laurent Schwartz of France (1950), Jean-Pierre Serre of France (1954), Rene Thom of France (1958), Pierre Deligne of Belgium (1978), Alain Connes of France (1982), Gerd Faltings of Germany (1986), Jean Bourgainof Belgium (1994), Pierre-Louis Lions of France (1994), Jean-Christophe Yoccoz of France (1994), Laurent Lafforgue of France (2002), Wendelin Werner of France (2006), Ngo Bao Chau of Vietnam and France (2010), and Cedric Villani of France (2010).

The Abel Prize, named after Norwegian mathematician Niels Henrik Abel, is awarded annually by the Norwegian Academy of Science and Letters. Western European winners include Jean-Pierre Serre of France (2003), Jacques Tits of Belgium and France (2008), and Mikhail Gromov of Russia and France (2009).

The Wolf Prize is awarded in several fields, including mathematics, by the Wolf Foundation. The first prizes were given in 1978 and it is awarded almost annually, with the possibility of more than one winner in a field in a given year. Western European winners include Carl L. Siegel of Germany (1978), Jean Leray of France (197), André Weil of France and the United States (1979), Henri Cartan of France (1980), Friedrich Hirzebruch of Germany (1988), Mikhail Gromov of Russia and France (1993), Jacques Tits of Belgium and France (1993), Jurgen Moser of Germany and the United States (1994/1995), Jean-Pierre Serre of France (2000), and Pierre Deligne of Belgium (2008).

Western European countries have been regular competitors in the International Mathematical Olympiad, held annually for students younger than 20 who have not yet begun tertiary education. There is both an individual and a team competition. Each country sends six students who are assigned six questions to solve. Countries are compared based on the total score for their team, while individual students may be awarded gold, silver, and bronze medals depending on how many problems they solve correctly. Germany has twice hosted the International Mathematical Olympiad and has participated since 1977.

East Germany also twice hosted the Olympiad and first participated in 1959, the year the Olympiad began. France began competing in 1967 and hosted the competition once. Belgium began participating in 1969. Austria began competing in 1970 and has served once as host. The Netherlands hosted the Olympiad in 2011 and has been competing since 1969. Luxembourg began competing in 1970, Switzerland began competing in 1991, and Liechtenstein began competing in 2005.

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See Also: Astronomy; Daubechies, Ingrid; Europe, Eastern; Europe, Northern; Europe, Southern; Mathematicians, Religious.

Expected Values

Category: History and Development of Curricular Concepts.

Fields of Study: Communication; Connections; Data Analysis and Probability.

Summary: The mathematical concept of "expected value" arose in the study of fairness in gambling but it has many scientific applications.

When people play lotteries or purchase insurance, they are investing money for a chance of some future financial return that may or may not occur. From the lottery or insurance company's perspective, money comes in from multiple purchasers and is paid out to the winners or claimants. Both sides may have questions regarding whether the investments are worthwhile or the payments are fair. These questions appear to date back to antiquity. Evidence of gambling games has been found in archaeological excavations of caves and in many ancient civilizations, including Egypt, Greece, and the Roman Empire. Babylonians used a form of maritime insurance and the Romans paid some investments in annuities.

A question concerning the fairness of certain gambling games spurred the development of probability theory in the seventeenth century. Mathematicians Blaise Pascal and Pierre de Fermat addressed fairness and related concepts while corresponding about a scenario in which two people wanted to quit playing a game and divide the winnings fairly, given that one player had a better chance of winning the game than the other. Mathematician Pierre-Simon Laplace seems to have first defined expected value in his 1814 work *Essai Philosophique sur les Probabilitiés*, writing, "This advantage in the theory of chance is the product of the sum hoped for by the probability of obtaining it.…We call this advantage mathematical hope." Expected value is the long-term average of the possible outcomes of a random variable or process, like tossing a six-sided

Expected Value

Consider a game in which a player rolls a standard six-sided die one time. If the result is a six, the player wins \$4. If the result is any number from one to five, the player loses \$1. If the player continues to play the game many, many times, will the overall outcome be a profit, a loss, or will the player break even? Mathematical calculations of expected value can be used to find an answer to this question and determine whether the game is fair to both sides.

Let x = the outcome of a single roll of a six-sided die, expressed as financial gain or loss

Die Roll	Outcome x	Probability $p(x)$
1	Lose \$1	1/6
2	Lose \$1	1/6
3	Lose \$1	1/6
4	Lose \$1	1/6
5	Lose \$1	1/6
6	Win \$4	1/6

The expected value would be

$$\text{Expected Value} = E(x) = \sum xp(x)$$
$$= -1\$\frac{1}{6} - 1\$\frac{1}{6} - \$1\frac{1}{6} - \$1\frac{1}{6} - \$1\frac{1}{6} + \$4\frac{1}{6} \approx -\$0.17.$$

This value means that over a large number of times playing the game, the player should expect to lose 17 cents per play, on average. However, in the short run, a player might win or lose more, since a winning a single roll could yield \$4 or a series of losses could cost several dollars. If the amount received for winning were \$5 instead of \$4, the expected value would be \$0 and the game would be "fair" in the sense that neither side would have a monetary advantage. This notion of fairness is different than the fairness or equal chances of the die rolls, which determine the probabilities and could also affect the expected value.

die. Mathematically, expected value is computed as the weighted sum of the outcomes, where the weights are the corresponding probabilities. For discrete random variables, expected value is a summation; for continuous variables, it is an integration. While computing means for data is very common beginning in middle school classrooms in the twenty-first century, finding expected values for random variables is more commonly part of high school and college curricula. Though initially motivated by notions of fairness, expected values have many important applications in probability and statistical theory and practice.

Applications

Scientific problems involving measurement were an inspiration for many mathematical advances in probability and applied data analysis. Astronomers in the eighteenth century often computed arithmetic means (or averages) for data to estimate parameters and describe distributions of "errors," like those they found when taking multiple measurements of the same astronomical distance. These averages were likely to be close to the true distance or value, or so they generally believed. This technique was used without proof for a long time, though mathematician Thomas Simpson had shown that an average was a better measure than a single observation in a very limited set of cases. Some issues in finding a suitable proof stemmed from the fact that probability distributions commonly used for describing errors at that time presented mathematical difficulties when trying to find expected values for averages versus expected values for individual observations. Work by mathematicians Abraham de Moivre and Laplace led to the Central Limit Theorem, derived by Laplace in the nineteenth century and later extended by other mathematicians such as Francis Edgeworth. This result is sometimes called the "DeMoivre–Laplace theorem" and was given its more common name in work by George Pólya in the early twentieth century. The primary impact of the Central Limit Theorem with regard to expected values is that it defined the expected value for the sampling distribution of the mean, given sufficiently large sample sizes. It established a theoretical basis for estimation and a later hypothesis testing for various parameters.

There are many different probability distributions that mathematicians, statisticians, and others have found, derived, named, and studied. For many years the normal distribution, credited to mathematician Carl Friedrich Gauss, played a central role in error modeling and other applications. However, approaching the twentieth century, increasing application of probability and statistics in a wide variety of fields, including biology, business, genetics, and psychophysics, led investigators like statistician Karl Pearson to research non-normal or skewed distributions to better represent phenomena they encountered. The problem then became to estimate parameters for these distributions and discover their mathematical properties. The method of moments estimates parameters like variance and skew using expected values. It primarily considers deviations of points from the distribution mean, called "central moments," which are conceptually related to the idea of moment or torque about a point in physics. Deviations are raised to various powers so that the k-th moment corresponds to the k-th power. The first central moment is zero, since it essentially sums all deviations from the mean or expected value. Variance is the second central moment, which is the expected value (the weighted sum) of all squared deviations from the mean. The third moment quantifies skew or asymmetry and is the expected value of all cubed deviations from the mean. A symmetric distribution has skew of zero. The fourth moment is called "kurtosis" and measures whether the distribution is taller or shorter and has thicker or thinner tails than a normal distribution with the same variance. Mixed moments can be found for two variables together to quantify the covariance and, by extension, correlation. Measures of skewness and kurtosis based on moments are credited to Pearson.

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See Also: Data Analysis and Probability in Society; Dice Games; Game Theory; Lotteries; Measures of Center; Probability.

Exponentials and Logarithms

Category: History and Development of Curricular Concepts.
Fields of Study: Algebra; Communication; Connections.
Summary: Exponential and logarithmic functions are used to study and analyze a variety of mathematical relationships.

Much of the language and notation of mathematics involves a very advanced shorthand. As ideas grow and become more complex, mathematicians seek ways to express highly condensed thought in relatively simple terms. Exponents are an elementary example: if one wants to multiply the number 2 times itself 10 times, rather than write "$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$" one can write "$2^{10}$" instead. From these beginnings, which date to ancient Egypt and Babylon, the remarkable worlds of exponential and logarithmic functions emerge. When one develops the understanding of what it means to take 2 to any real number power, one naturally considers the function $f(x)=2^x$, an example of what is called an "exponential function." For larger and larger positive x, the function grows amazingly fast: $2^{10}=1024$, $2^{20}=1{,}048{,}576$, and $2^{30}=1{,}073{,}741{,}824$.

The exponential function $f(x)=e^x$, where e is the so-called "natural base," an irrational number whose decimal approximation is $e \approx 2.71828$, is an important exponential function. With e in homage to the great Swiss mathematician Leonhard Euler (1707–1783), this special exponential function $f(x)=e^x$ might rightly lay claim to the title of "the most important function in all of mathematics." Exponential growth and decay functions, along with the number e itself, have a wide variety of uses and applications.

In classrooms in the twenty-first century, the logarithm of a number is defined as the exponent or power to which a stated number, called the "base," is raised to obtain the given number. The development of logarithms in the seventeenth century led to a revolution in scientific calculation, especially when the slide rule replaced tables of logarithms. While the advent of calculators and computers eliminated the need for calculation by logarithms in the latter part of the twentieth century, logarithms remain important in order to understand financial and natural processes. For instance, the Richter scale to measure earthquakes, named for Charles Richter, is a logarithmic scale. In chemistry, the pH scale is based on the negative logarithm of the concentration of free hydrogen ions. Students in the middle grades investigate exponential notation while high school students explore exponential and logarithmic functions.

Archimedes of Syracuse investigated that the addition of what he called "orders" corresponded with their product, known today as the "first law of exponents." The number e may have first appeared in the early seventeenth century in an appendix to John Napier's work on logarithms. This number also arose in the work of Christiaan Huygens in the mid-seventeenth century when he was exploring the area under the hyperbola $xy=1$. Finally, in the late seventeenth century through work involving continuous compound interest, Jacob Bernoulli was led to consider the expression

$$\left(1+\frac{1}{n}\right)^n$$

for large values of n, and this expression approaches e as n grows without bound. Mathematicians explored many issues related to e and exponentials, including such people as Euler, Gotthold Eisenstein, and others, who investigated the convergence of sequences of iterated exponentials. Bernoulli may also have been the first mathematician to realize that the number e was intricately linked to emerging ideas with logarithms.

The Natural Exponential Function

Because any exponential function can be written in terms of e, one finds that functions of the form $P(x)=Me^{kx}$, where M and k are constants that depend on the context, arise in many natural settings. Exponential cell and population growth, as well as exponential decay in radioactive materials, are modeled by functions of this form. Once the values of M and k are identified, the function easily indicates the corresponding output for any input value x. For example, if a car is initially valued (at time $t=100$) at \$10,000 that depreciates at a certain continuous rate, one might use the function $P(t)=10000e^{-0.2t}$ to model the worth of the car in year t.

Functions like this generate very natural questions, including ones like "At what time t will the car's value be \$3,000?" Before trying to answer this more complicated question, consider some simpler ones. For instance, what value of t makes $10^t = 17$? Since $10^1 = 10$, while $10^2 = 100$, it seems like there ought to be a number between 1 and 2 such that 10 raised to that power is 17. But what is the number? Here, some very considerable mathematical ideas are involved: the function $y(t) = 10^t$ is continuous; the range of y is all positive real numbers; and $-y$ is always increasing, making it a one-to-one function. All these facts together combine to indicate that one can pick any positive real number y and know that there must be one and only one real number t that satisfies the equation $10^t = y$. In other words, there is a function h that takes any positive real number y, and to this value y associates the real number t so that 10 raised to the power t is y. This explanation is how teachers usually describe to students where logarithms come from—the logarithm is the very function that accomplishes this association. It is all a matter of perspective; if t is known and y is sought, the exponential function is used, while if y is known and t is sought, the logarithm function is used. Expressed in words, it is "y equals 10 to the power t" and "t is the power to which we raise 10 to get y." Babylonian clay tablets presented similar questions.

Historical Development

Historically, the further development of logarithms arose very differently. In the late fifteenth century and early sixteenth century, both John Napier and Jost Burgi, who were each interested in key problems in astronomy, developed logarithms for a much different use: as a new tool to help do arithmetic with large numbers. Their approach to logarithms was fundamentally geometric, as algebra was not yet sufficiently well developed to aid their work, although Napier's approach was more algebraic than Burgi's methods. Napier noted, "Seeing there is nothing that is so troublesome to mathematical practice, nor doth more molest and hinder calculators, than the multiplications, divisions, square and cubical extractions of great numbers, which besides the tedious expense of time are for the most part subject to many slippery errors, I began therefore to consider in my mind by what certain and ready art I might remove those hindrances." In 1624, Henry Briggs published logarithm tables in *Arithmetica Logarithmica* and he is noted by some as perhaps the man most responsible for popularizing logarithms among scientists. The development of the slide rule made logarithms easy to use, since they reduced the reliance on tables. In 1620, Edmund Gunter noted logarithms on a ruler by marking the position of numbers relative to their logarithms. William Oughtred placed two sliding logarithmic rulers next to each other and by 1630, the portable circular slide rule reduced multiplication computations to the act of lining up two numbers and reading a scale. Logarithms remain a useful way to deal with large numbers in the early twenty-first century, because the logarithm of a large number is a much, much smaller one. R. C. Pierce Jr. noted, "It has been postulated that logarithms literally lengthened the life spans of astronomers who had formerly been sorely bent and often broken early by the masses of calculations their art required." Modern mathematicians have also come to fully understand the connection between logarithms and the area under the curve $xy = 1$, which was explored by Huygens in the 1600s.

Using Logarithms to Solve Exponential Functions

Perhaps the most powerful property of logarithms is that they "undo" exponential functions. For example, for the natural logarithm of base e, denoted "ln," one obtains $\ln(e^5) = 5$. Remember, $\ln(e^5)$ means "the power to which one raises e to get e^5." This power, of course, is 5. The general property that holds here is that for any real number t, $\ln(e^t) = t$. This rule proves to be immensely useful in solving exponential equations. To see how, consider an earlier example: the function $P(t) = 10000e^{-0.2t}$ (the value of a car in year t). At what time t will the car's value be \$3,000? This question is equivalent to solving the equation:

$$0.3 = e^{-0.2t}.$$

Taking the natural logarithm of both sides of the equation "undoes" the effects of the exponential function and hence gains more direct access to the variable t: $\ln(0.3) = \ln(e^{-0.2t}) = -0.2t$.

Dividing both sides of the last equation above by -0.2, one finds that

$$t = \frac{\ln(0.3)}{-0.2} \approx 6.0199$$

so that the car's value will be $3,000 in just over six years. The natural logarithm of 0.3 is central to answering the question.

While the motivation for the need for logarithms can be seen in relatively elementary terms—solving exponential equations—the actual mathematics that explains what logarithms really are and how they work is deep and is best supported using some sophisticated ideas from calculus. Even with exponential functions, there are some big questions without answers: how is *e* to the 5th power calculated? How is the natural logarithm of 0.3 computed? Until the invention of personal computers in the 1970s, such computations were all done by hand, usually with the assistance of elaborate tables, or with slide rules. At one point in history, entire books were written that held nothing but tables of values for logarithms. People now use inexpensive handheld calculators, computer algebra systems like *Maple* or *Mathematica*, or even Google, and each returns a value almost immediately. These modern technological tools rely on a rich and beautiful mathematical theory of exponential and logarithmic functions. Beyond their interesting mathematical properties, exponential and logarithmic functions remain important for their many applications, such as the key role that exponential functions play in the study of differential equations, including those that model vibrations in bridges and buildings, thus forming a central component of modern civil engineering.

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See Also: Calculators in Society; Carbon Dating; Earthquakes; Functions; Mathematics, Elegant.

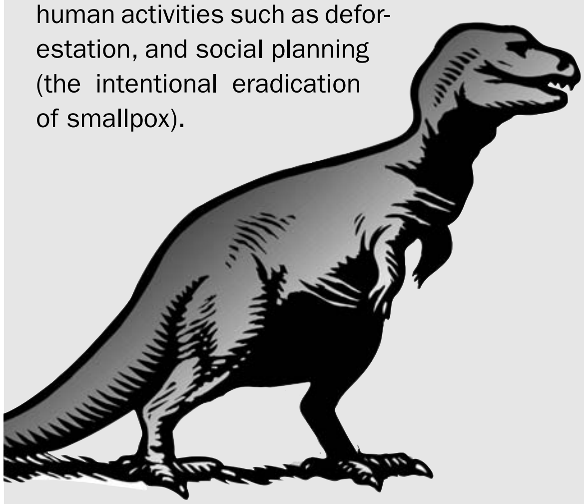
Extinction

Category: Weather, Nature, and Environment.
Fields of Study: Algebra; Problem Solving.
Summary: Causes and factors of extinction can be quantified and modeled using mathematical and statistical techniques.

Extinction occurs when the last member of a species dies. A species survives for much longer than any of its members. For example, a human can live up to about 120 years, whereas the human species (*Homo sapiens*) is thought to have existed for hundreds of thousands of years. It is not known how long our species will endure and indeed most species on Earth have already become extinct. There are many causes of extinction, some natural and others as a result of human activities. Many factors influence whether an endangered species can avoid extinction. These factors can be quantified and modeled using mathemati-

Causes of Extinction

A species can become extinct for various reasons, including intense competition with other species, disease, or failure to adapt to changing climatic conditions, as well as the disappearance of a species' prey. Anthropomorphic reasons for extinction include over-hunting by humans, habitat loss from human activities such as deforestation, and social planning (the intentional eradication of smallpox).



cal and statistical techniques. A species can disappear in some parts of its habitat but not in others. Not all species have existed on Earth for the same length of time—some appear only briefly while others manage to persist for incredibly long periods of time. Human activities may be increasing the rate at which other species become extinct.

Rise of Extinction

A species is endangered when it consists of a small number of members. In such cases, individuals may have trouble finding each other because of geographical separation. For a species that is endangered, it is of interest to know whether the species is likely to become extinct. It is customary to let $N(t)$ represent the size of a population at time *t*. The fact that the species is endangered implies that $N(t)$ takes positive values close to zero. If $N(t)$ is eventually measured to be zero, then the species has become extinct. However, if $N(t)$ rebounds to larger positive values, then the species persists. In general, stochastic effects largely determine whether an endangered species will become extinct. Given population data $N(t)$ at different times *t*, one may compute the mean (μ) of the population growth rate.

$$R(t) = \ln\left(\frac{N(t)}{N(t-1)}\right).$$

For example, if $t = 10$ then

$$\mu = \frac{(R(1) + R(2) + \cdots + R(10))}{10}.$$

A positive (or negative) value of μ indicates that the population is growing (or declining) on average. Combining this information with the standard deviation (σ) of $R(t)$ allows one to assess the risk for extinction, which is typically highest when μ is negative and σ is small. Complex models of population dynamics exist to predict whether a species will persist or become extinct. These include geometric growth models in which a population multiplies at a fixed rate, logistic growth models in which populations slowly attain steady-state sizes, and Lotka–Volterra predator-prey models for interactions between multiple species, named for Alfred Lotka and Vito Volterra.

Local Extinction

A species can become extinct in one area (such as an island) and still persist elsewhere (such as a continent). If the species is able to recolonize the former area, then this is known as a "rescue effect." If local extinction events become synchronized—as a result of global climate change, for example—then the risk of a species becoming globally extinct is much higher.

Rate of Extinction

Scientists estimate that there may be 10 million species alive today and yet they account for fewer than 1 in 1000 species that have ever lived. The average time to extinction for a species, as measured from the time of its first appearance, is close to 10 million years. When the time to extinction for a species is much longer, such as more than 100 million years, then later members are said to be living fossils.

Mass Extinction

A mass extinction occurs when a large number of species become extinct in a short period of time. Although rare, the fossil record indicates that these events have occurred at least five times, the most famous being the mass extinction of non-flying dinosaurs 65 million years ago in what was probably a meteor impact. Many scientists believe that we are currently in the midst of a sixth mass extinction, with up to 40,000 species becoming extinct each year—a rate that is roughly 100–1000 times higher than in prehistoric times.

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See Also: Animals; Climate Change; Deforestation; Forest Fires; Mathematical Modeling; Predator–Prey Models.

Extreme Sports

Category: Games, Sport, and Recreation.
Fields of Study: Algebra; Geometry.
Summary: The emphasis on fast motion, tricks, and personal expression in extreme sports makes geometry especially relevant to athletes.

There is no single definition of extreme sports, though they generally include dangerous sporting activities that involve a substantial risk of injury, like Buildings, Antennae, Spans, and Earth (BASE) jumping, cliff diving, street luge, or even the traditional running of the bulls in Pamplona, Spain. Extreme sports are believed to be attractive to participants because of the challenge and adrenaline rush and to spectators because the results are typically unpredictable.

The popularity of extreme sports grew rapidly in the latter part of the twentieth century. The television network ESPN created the Extreme Games, now called the "X Games," in 1995, making extreme sports more visible to the general public. Other networks have also begun to televise these types of competitions and some extreme sports events have been included in the Olympic Games. Mathematics is important in extreme sports. Knowing and applying concepts from geometry and probability helps participants be safe and successful. Innovative equipment manufacturers use concepts and techniques from many areas, including geometry, statistics, modeling, and simulation, to prototype and refine their designs, resulting in greater safety and effectiveness.

Skateboarding

Skateboarders perform tricks using a wheeled board, either on a flat surface or using equipment like ramps or rails. Many stunts rely on differential pressure applied by the rider's feet to various parts of the skateboard to tilt or flip it, often rotating both board and rider in one or more axes. Lip tricks require a vertical orientation and transitional edge like the lip of a swimming pool or ramp. In aerial tricks, the rider leaves the ground completely, using counterpressure of hands and feet to maintain control of the board while spinning or flipping.

Tony Hawk is one of the most well-known extreme athletes and a vertical skateboarding pioneer. He was the first person to competitively perform an aerial turn of two and a half rotations, or 900 degrees, at the 1999 X Games. In the past, he has done 720 degree turns. For the 900, he exerted greater takeoff force in the direction of the turn, producing more rotational velocity. Tony Hawk's Project 8 video game used motion capture technology to smoothly animate professional skaters, while Tony Hawk Ride allowed players to simulate the sport using a skateboard-like controller.

Snowboarding

Snowboarding is similar to skateboarding and involves standing on a board and sliding down a snow-covered hill. Snowboarding became an Olympic sport in 1998, with giant slalom and half pipe competitions taking place. The giant slalom is a speed race in which athletes speed down a steep hill with gates that require them to zigzag between. Determining an optimal path from one gate to another without crashing or wasting time requires mathematics, especially geometry. A half-pipe consists of two quarter-cylinders connected by a flat space and topped by a small lip. The competition is a more artistic event, with athletes generating enough speed using the curves of the pipe to become airborne and do tricks. These may include multiple rotations, both twisting and somersaulting. At the 2010 Olympics, Shaun White executed a record-setting 1260-degree trick consisting of two flips and three and a half spins.

BMX Biking

In bicycle motocross (BMX), athletes ride specially designed smaller bicycles that enable them to shift their center of mass to make precision movements. BMX courses often use steep hills to launch the rider into the air to perform tricks. Other tricks and spins may be done on flat ground. The sport was added to the list of events for the 2012 Summer Olympic Games. Billy Gawrych is a professional BMX competitor who performs intricate routines, often set to music, with tricks linked together in a series of connected, flowing patterns.

When a skateboarder performs an ollie, the forces acting on the board are the weight of the rider, the force of gravity on the board, and the force of the ground pushing up on the board, which balance out to zero net force.

Sports Engineering and Equipment

Sports engineering is a growing interdisciplinary field that draws from mathematics, engineering, biology, physics, materials science, and many other disciplines to study characteristics of athletes and equipment, as well as their interaction. The focus is on performance and safety. For example, engineer Mont Hubbard described the motion of skateboards with riders using two mathematical models, and mathematicians develop new models using techniques and theories from areas like trigonometry, physics, differential equations, and probability. Quality function deployment is a method of quality control that attempts to translate often subjective customer requirements into mathematical engineering specifications. One research group studied the subjective perception of the "feel" of snowboards. They used field evaluations and laboratory data to create matrices of parameters. Snowboards for freeride and freestyle, the two primary types of snowboarding, have somewhat different designs; however, issues of flexibility, torsional stiffness, and curvature were the important factors affecting feel and performance for both styles. Equipment for sports of all kinds is subjected to statistically designed tests to evaluate safety, and data from accidents and failures helps fuel further research.

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See Also: Mathematical Modeling; Probability; Trigonometry.

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Fantasy Sports Leagues

Category: Games, Sport, and Recreation.
Fields of Study: Algebra; Data Analysis and Probability.
Summary: Fantasy sports leagues employ a variety of algorithms to predict player performance and to rank players and teams within each league.

In fantasy sports leagues, players act as the owners and managers of virtual sports teams that are typically composed of real players who are active in a given sport during a competitive season. Performance statistics for individual athletes on a fantasy owner's roster, who usually belong to many different teams in the sport in real life, are mathematically combined to produce "fantasy points" for the owner. Often, owners may trade athletes or must make other types of decisions about who on their roster will be counted as "active" for a given period of competition, just like real managers. Fantasy baseball and fantasy football have historically been the most popular but fantasy leagues have evolved for many other sports, including basketball, golf, hockey, soccer, auto racing, and even cricket. Different leagues, even within the same sport, use a variety of formats, statistics, and weighting schemes to compute fantasy points. Season winners are usually the owners who have accrued the most fantasy points. While such games have existed in one form or another since at least the end of World War II, the development of the Internet drastically changed the nature and popularity of fantasy sports leagues by providing real-time access to data and tools for automated computation, making the activity more accessible for a broader range of participants. There are estimated to be millions of fantasy sports players in the United States alone. In the twenty-first century, mathematicians and others study fantasy sports leagues, and they have become a tool in mathematics classrooms as well.

History

Fantasy sports leagues grew from other types of sports simulator games that used data from past seasons and random number generation to determine the outcomes of simulated games. One of these was Strat-O-Matic, a board game using player statistics cards and dice that was developed by Hal Richman. It premiered in 1961 and still exists in both card and computerized forms. Richman began developing the game as a child because he "loved baseball and numbers" and disliked what he saw as unrealistic randomness in other baseball board games. He released the game while earning his undergraduate mathematics degree. John Burgeson, an IBM computer programmer, created a computer fantasy baseball simulator in 1960 that used random numbers and player statistics to generate a play-by-play description of a game between two teams. Many real baseball managers reportedly played fantasy-style games when they were young. According to writer Alan Schwartz, "That's how they learned how to apply the mathematics of risk-taking."

In the 1970s and 1980s, early fantasy leagues began to emerge for baseball and football. Writer Daniel Okrent developed Rotisserie League Baseball, which was named after the restaurant where players conducted the first draft. "Rotisserie baseball" is now a standard term for this widely-used format. It differed from most older games by using current-season statistics and data as they occurred rather than past seasons' statistics. This style of play became popular after an *Inside Sports* magazine article described the rules of the game and discussed the league's first season. Statistician George William "Bill" James also developed the analytical methodology of sabremetrics around this time and his *Bill James Baseball Abstract* was widely used by fantasy players. Similar mathematical analyses were produced for fantasy football by *Fantasy Football* magazine, which evolved into the print and online *Fantasy Football Index* (and also *Fantasy Baseball Index*). These publications and many others provided mathematically modeled variables, such as dollar values, statistical projections, and optimization strategies, for fantasy players. Sometimes the modeling proved useful enough that the writers went on to advise real teams.

Before the Internet, coordinating fantasy sports and calculating points could be time consuming. Data came largely from print sources, which were time delayed. A standard 162-game baseball schedule required near-daily computations for each owner in the league. Fantasy football was somewhat less challenging because of the smaller number of games in a season, but most fantasy methods had to restrict the number of variables used. Some commercial statistical services started to fill this need by compiling databases of sports statistics and providing services to calculate points—for a fee. Results were mailed or faxed; later, they could be sent electronically. The development of the World Wide Web in the mid-1990s facilitated and often automated the process of tracking player statistics and calculating points and league standings. Fantasy players could also quickly communicate with each other using e-mail, message boards, and chat rooms, resulting in online communities and worldwide leagues. Researchers have modeled this growth using sociologist Everett Rogers's diffusion of innovation theory. The curve of fantasy players over time exhibits the classic *S*-shape of slow initial growth among early innovators and adopters, a middle period of accelerated growth, and a saturation of the market leading to a leveling off or slower growth period. The rapid growth of fantasy sports in the late twentieth century led to issues related to its potential classification as gambling, fairness in prizes, and the legal rights of players or teams to control the dissemination and use of statistical information about professional athletes, especially when outside companies were making a profit from such use.

Mathematical and Social Connections

The line between fantasy sports and real sports is often blurred and mathematical methods used in one are often applied to the other. For example, mathematicians have explored a concept often called the "magic number" or elimination number, which quantifies the number of games a team must win to avoid being eliminated from the championship. The problem is popular in computer science classes. A common solution is to compare the number of games a team has left to play to the win-loss difference of the nearest rival. Researchers found that the numbers for all teams may be found simultaneously as they are a function of the number of games won plus the number of games left to play. Other mathematicians investigate optimal strategies for drafting players to teams using methods such as stochastic dynamic programming and deterministic dynamic programming coupled with various types of mathematical modeling and decision making. Some have researched the extent to which players rely on mathematical modeling and statistical methods instead of on heuristics and personal preferences. Mathematics teachers have found some success in using fantasy sports to motivate students and to help them succeed. Additional evidence suggests that fantasy sports may help reduce gender gaps in mathematics achievement. Some girls have stated that fantasy sports are "cool" and help them relate to boys as equals, and women are involved in the creation and management of fantasy leagues. For example, Jordan Zucker, who has an undergraduate degree in mathematics, created the

Girls' Guide to Fantasy Football Web site and manages an all-female fantasy football league.

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See Also: Baseball; Betting and Fairness; Ethics; Football; Lotteries; Rankings.

Farming

Category: Weather, Nature, and Environment.
Fields of Study: Algebra; Geometry; Measurement.
Summary: As a fundamentally important human activity, agriculture has long been a motivator for mathematical and statistics research.

Farming, also called "agriculture," is the production and distribution of plant and animal products. Farming methods range from organic farming to industrial agriculture. Farming operations are also categorized by their products, including foods, pets, decorative plants, pharmaceuticals, building materials, fibers, resins, and bioplastics. Agriculture has long been a motivator for mathematical and statistic research. Mathematical concepts and models have helped advance many agricultural methods beyond simple arithmetic calculations of quantities of seed and fertilizer. Many consider Ronald Fisher to be the father of modern statistics. Much of his research in statistical methods originated from his work with more than 60 years of agricultural data at Rothamsted Experimental Station, one of the oldest agricultural research institutions in the world. Methods pioneered by Fisher are still widely used in the twenty-first century, including hypothesis testing, analysis of variance, maximum likelihood estimation, and factorial experimental design. Mathematician Michael Weiss has worked in several mathematical areas with applications in agriculture, including nonlinear and chaotic dynamics, fuzzy set theory, and topological and algebraic entropy. Some applications of his work include a model of crop yields as a two-dimensional stochastic process, called "random surfaces," and assessing revenue risk as a probabilistic function of foodborne disease outbreaks. Precision farming models spatial variability in farmland and the resulting changes in yields as geometric surfaces.

Numerical characteristics of the farmland, such as fertilizer needs, are assigned to surfaces by functions and mapped to other surfaces by operators using modeling software. The so-called cobweb theorem relates price and production for situations in which there is a time lag between the marketing of a product and initially obtaining price information to determine production. This is common in agricultural markets, since prices in one year tend to influence planting in subsequent years.

The Role of Agriculture in the History of Mathematics and Science

Agricultural development shaped the history of humankind, including the growth of science in mathematics. This impact is acknowledged in the historical tradition of naming major farming breakthroughs "revolutions," since the changes they produced in society were large and relatively fast. The neolithic revolution started circa 8000 B.C.E. and included the development of permanent settlements. The resulting architecture and centralized management systems required abstract thought and systems of knowledge, including writing, mathematics, and science. The Arab agricultural revolution took place in the eighth through the thirteenth centuries C.E. and included the development and distribution of international knowledge exchange, sophisticated algebra and geometry, and astronomy for farming and navigation, as well as the scientific method and the modern number and computational system in mathematics. The British agricultural revolution started in the seventeenth century. It codeveloped with the Industrial Revolution and included the heavy use of mechanical tools and developments in the natural sciences, including chemistry and biology. This industrialization of agriculture continued into the twentieth century, driving the research in organic chemistry and genetics known as the green revolution.

Domestication of local crops, such as rice in China around 8000 B.C.E., allowed for both population growth and population concentration in villages and, later, towns. Planting, harvesting, and other timed activities required relatively exact time and weather observation, which in turn led to the development of astronomy and the development of sophisticated time measurement tools and calendars. Circa 5000 B.C.E., the people of Mesopotamia employed intensive farming methods, including monocrop fields, aggregation of crops for trade, and complex irrigation. Such methods called for and enabled major technological developments, such as better plows. It is hypothesized that the complex division of labor, distribution, and observation of water levels and calendars required for this type of agriculture led to the development and relatively widespread use of writing.

Mesopotamian clay tables show that quadratic and cubic equations, the Pythagorean theorem, and other topics currently found in algebra, geometry, and calculus were already widely used circa 2000 B.C.E. in problems related to agriculture, such as astronomy-based calendars to time flooding and harvesting or the distribution of products. Some of this knowledge later was lost and then rediscovered by other cultures, and some continued to be used in the original form. For example, the practice of measuring time based on 60 minutes in an hour and 60 seconds in a minute comes from the Babylonian sexadecimal (base 60) number system. The number "60" was a convenient one for the Babylonians being highly composite (with more divisors than any number less than 60).

Agriculture promoted the development and spread of increasingly complex mechanisms, such as waterwheels in China. Excess crops supported the development of trade and transportation, from the domestication of draft and pack animals in ancient times to sophisticated spice trade fleets circa the sixteenth century. Starting in the eighth century C.E., Muslim traders established an extensive network of trade routes among Asia, Europe, and Africa, enabling the diffusion of agricultural techniques and crops beyond their places of origin. This Arab agricultural revolution led to the development and distribution of science and mathematics, including the Arabic numerals used around the world in the twenty-first century. For example, one of the first documented uses of the scientific method comes from thirteenth-century work on medicinal plants and agronomy (the farming of plants).

The Industrial Revolution, starting in the eighteenth century, included the increasing mechaniza-

Combines harvesting crops at precise intervals with each row overlapping slightly. Combines were invented in 1834.

tion of agriculture. Agricultural machines, such as the tractor, both decreased the number of people required for farming and increased productivity. The scientific advances associated with these developments primarily took place in engineering and chemistry. The green agricultural revolution of the second half of the twentieth century promoted advances in chemistry, genetics, and bioengineering, which led to high-yield, disease- and pest-resistant cultivation of major crops. The sustainability of these practices is not yet clear at the start of the twenty-first century.

Measurements in Agriculture

Metrics used in farming focus on average production of different cultivars of plants, breeds of animals, or farming methods; resource intensity of practices; efficiency of distribution; nutritional value of food products and industry-specific values of fibers, fuels, and lumber; environmental impact and sustainability; and the role of agriculture in local and global economy.

The global production levels, by crop type, are measured in tons per year. For example, cereals was the number one category of agricultural product, with worldwide production at around 2 billion tons per year in the early twenty-first century, while meat production at this same time was around 250 million tons per year. The total and per capita rates of production are frequently compared between years. For example, the total agricultural production grew by a factor of 16 between the early 1800s and 1970, while the world population grew by a factor of seven. This means that per capita consumption of agricultural products more than doubled during that period but not necessarily because of food items. Fiber or farmed trees for paper and construction are also included.

Farm yields are measured in crop weight per area for plants; in the ratio of seed input to seed output for grains; or in meat, fiber, or egg production per animal for animals. The yields are estimated using statistical methods of random sampling, or total outputs of a farm. In the United States, for example, corn yields averaged about 30 bushels per acre in the early 1900s and around 130 bushels per acre in the early 2000s. Food anthropologists estimate the minimal ratio of grain input to output necessary for sustaining farming as the main source of food as 1:3. For each grain planted, farmers get three grains, one of which is planted and two of which are either eaten by people or fed to farm animals. Yield metrics can be used to compare different methods of farming. For example, irrigation can raise corn yields by a factor of four or five. Industrial farming in developed countries produces yields that are about 10% greater than organic farming in nondrought years and about 70% less in drought years, netting about the same average yields over decades.

Resource intensity is measured by the outside input required per area of crops, per individual animal in meat or egg farms, or per unit of farm product output. For example, it takes about 1000 liters of water to produce 1 liter of corn-based ethanol. Resource intensity is one of many sustainability metrics used in farming. Other mathematical metrics of sustainability include nutrient leaching into water systems, which may cause proliferation of algae; biodiversity of farms; and pollution of soil, water, and air with herbicide and pesticide residues; as well as the carbon footprint of farming practices. For example, livestock production is currently responsible for about one-fifth of the total carbon footprint of humanity.

Farming and the Economy

Agricultural systems include production, processing, packaging, distribution, marketing, and consumption. The proportion of resources and energy required for these activities varies with farming practices. For example, eating local foods reduces the resources expended in transportation; operating monocrop farms reduces labor per unit of production; eating processed foods increases packaging costs.

Agricultural economics is the study of resource allocation and distribution related to agriculture. It uses mathematical statistics for data analysis and trend prediction and mathematical modeling for research and development. Many general economic mathematical models were first developed in agricultural economics, for example, the cobweb model, which explains the cycles of price fluctuations through analyzing lags within the production chains, such as planting and harvesting.

Factory farming uses economies of scale by raising livestock in confinement and with high population densities. The calculations involved in factory farming include cost-output analysis and bioengineering of animals to optimize product output as well as the logistics of supplying food in to each animal in place and disease prevention through administering antibiotics. There are several measurements of factory farming impacts. For example, there are metrics involved with animal welfare, such as the degree of confinement, measured in area of pen per animal. Human health impact measures and research include studies of pesticide, antibiotic, and growth hormone levels in farm products and statistical studies of the impact of food on human health. Environmental impact measures are standard for all operations and include levels of specific air, water, and soil pollutants produced by the farm and its carbon footprint. Capital redistribution is the measure of movement of money among communities, which is relatively high for factory farming because of its centralized nature.

Industrial marketing and distribution models do not work well for organic farming because most organic products are not scalable. In the early 2000s, organic farmers developed a variety of peer-to-peer credence and distribution models, network marketing models, and sharing economy (mesh) models. Such modern models support decentralized production and disintermediated distribution. Some organic farmers join together in cooperatives and use economies of scale. Community-supported agriculture (CSA) is an economic model that provides a way to share the benefits and risks of farming. In a typical CSA, consumers buy farm shares and receive a weekly delivery of farm outputs.

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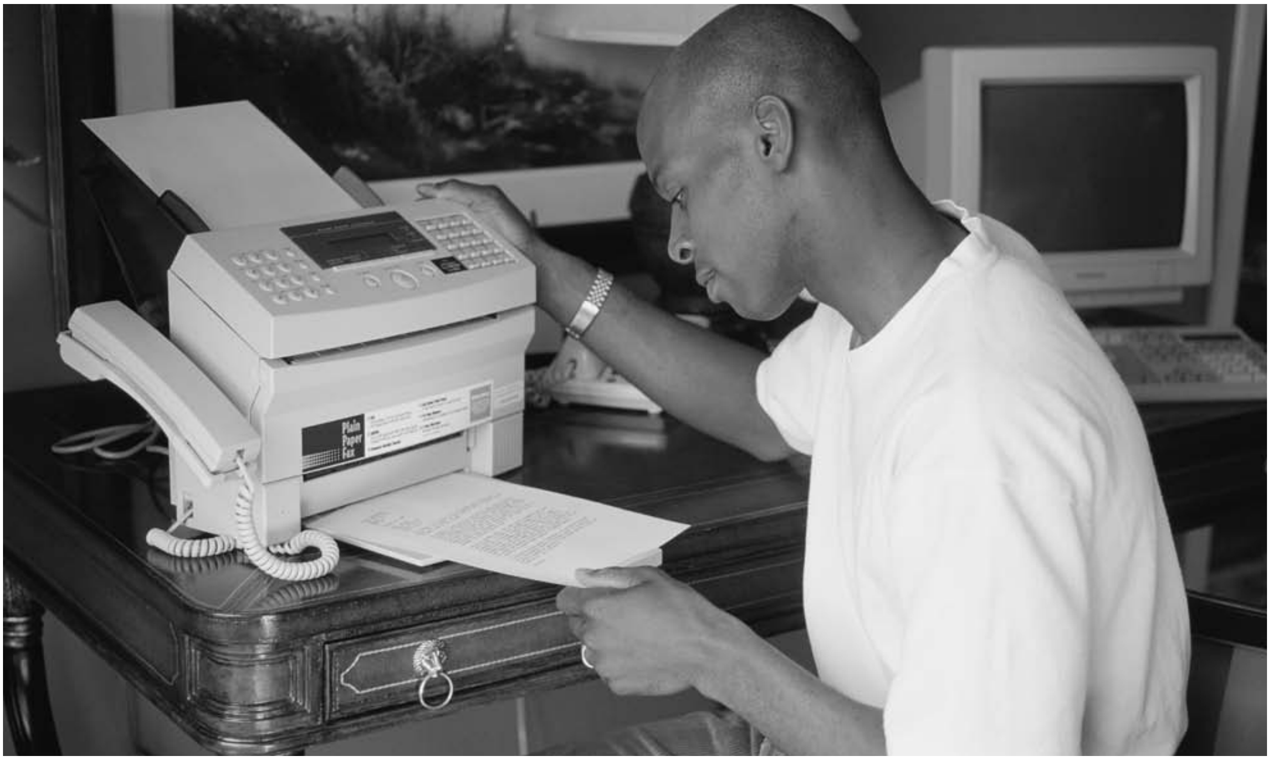
See Also: Animals; Arabic/Islamic Mathematics; Calendars; Carbon Footprint; Chinese Mathematics; Deforestation; Green Design; Industrial Revolution; Measuring Tools; Nutrition; Quality Control.

Fax Machines

Category: Communication and Computers.
Fields of Study: Number and Operations; Representations.
Summary: Fax machines revolutionized the process of sending and receiving documents.

A fax machine enables documents, including illustrations and other graphical elements, to be transmitted over a distance and reproduced by the receiver. The roots of the word "facsimile" are from the Latin words *facere*, meaning "to make," and *similis*, meaning "like." In the nineteenth century, Alexander Bain developed what some refer to as the first fax machine. His system transmitted information using analog telegraph lines. The sending and receiving equipment was timed using matched pendulums. At the receiving end, an electrically powered stylus recorded messages on a roll of paper. Current from the stylus turned the chemical coating on the paper blue, transcribing the signals' dots and dashes. Frederick Bakewell demonstrated a chemical fax machine at the 1851 London Exhibition, and the first commercial telefax service began operation in 1865, predating the telephone.

A more modern ancestor is the radio facsimile, developed in 1924, which used radio waves to wirelessly transmit images and is still used in the early twenty-first century to transmit weather information. Modern fax machines scan an input sheet line by line to produce rows of pixels. Algorithms used in fax machines take advantage of the fact that there are white and black pixels in order to compress the data. For example, David Huffman's variable-length lossless codes and their variations, originally invented in the 1950s, assign binary codes to patterns of pixels using probabilistic methods. The codes are shorter than the strings they replace, reducing overall file size. To optimize compression, symbols with higher probabilities or frequencies of occurrence are assigned shorter codes. The International Telecommunications Union, based in Geneva, Switzerland, makes recommendations for data compression standards. To derive one code called the "Group 3 code," the organization applied the Huffman algorithm to eight representative samples to assign a code to each run length. Fax machines transmit documents in minutes instead of hours thanks to compression algorithms.



The sending fax machine uses a sensor to scan the document and to convert the pattern of black and white elements into a code. The receiving machine decodes and uncompresses the information and prints it.

Fax Machine Technology

Modern fax machines utilize the technology of the telephone and the copy machine. Fax machines developed in the 1970s could scan a document and encode and transmit it over telephone lines to another fax machine, which could record and reproduce the document. Fax machines became common in offices as they replaced the need to send paper documents by messenger service or mail, were much quicker than retyping a document for telex, and could send any type of graphical information. Japan played an important role in developing modern fax machines, which used electronic circuits to replace mechanical parts and greatly increased the speed of transmission and reduced the size and price of the machine. Because the Japanese language incorporates many Chinese characters (*kanji*), the ability to transmit graphical images was particularly useful in that country.

Sending a document by fax requires two fax machines—one to send the document and one to receive it. The sending machine uses a sensor to scan the document, usually line by line, and to convert the pattern of black and white elements on the page into a code (several coding standards exist). The fax machine is not "reading" text—in the sense of converting the letters into meaning—but only recording their shape. For this reason, fax machines are as adept at sending images and diagrams as they are at sending text. The scanned data are compressed in order to reduce the number of bits to be transmitted and thus to speed up the process. The speed of transmission depends in part on how much information, such as text or diagrams, as opposed to blank space is contained on the page being scanned. The receiving fax machine decodes and uncompresses the information and uses it to re-create and to print the sent document. In the 1980s, most fax machines used thermal printing, which required the use of special paper that turns black when exposed to heat. However, in the twenty-first century, most fax machines print on standard white copy paper using either laser or inkjet printing technology.

Internet fax (efax, or online fax) technology has supplemented and, in some cases replaced, the use of traditional fax machines. There are a number of different services offering Internet fax capability, and although they differ in some details (for instance, can the machine receive, send, or both) the principle is the same: they provide a means to transmit facsimile documents to and from computers either as e-mail attachments or through a dedicated phone number or Internet site.

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See Also: Coding and Encryption; Digital Images; Internet; Telephones.

Fertility

Category: Medicine and Health.
Fields of Study: Algebra; Data Analysis and Probability.
Summary: Individual fertility cycles can be mathematically predicted and national fertility rates are a useful statistical measure for analyzing population demographics.

The term "fertility" has been used historically in a variety of contexts, including the richness of croplands with respect to producing food, the creativity of the human mind and imagination, and the ability of people to have children. The term "fecundity" is often interchanged with fertility when discussing human reproduction. However, nineteenth-century physician Matthews Duncan, who researched birth statistics and fertility, differentiated the two terms by defining fecundity in an essentially binary fashion as the capability of bearing children or not, versus fertility, which he used to quantify the number of children a woman had borne. Demographers often use fertility rate as a standardized metric to describe the number of children borne per person, couple, or population and to make comparisons across populations. Many collections of global statistics, like the *CIA World Factbook,* include fertility rates, which have been connected by mathematical and statistical models to economic measures such as individual income or a country's gross domestic product. Others study relationships to medical and social variables, such as the availability of birth control and assisted reproduction or attitudes about single parenting. Some rates adjust for women in specific age groups or other variables. At the start of the twenty-first century, organizations such as the United Nations also began to turn serious attention to the issue of population decline in many nations and its potential effects on national economies, workforces, and social security systems. Mathematicians, statisticians, demographers, and others continue to research the reciprocal relationships between fertility and other measures to attempt to determine causes and effects and to forecast future trends as well as to contribute to the development of technologies related to fertility and reproduction. Statistician Leslie Kish was awarded the American Statistical Association's Samuel S. Wilks Award for his work on the World Fertility Survey, which "illustrates his impact as an international ambassador of statistics and a tireless advocate for scientific statistical methods."

Fertility Rates

In the years immediately following World War II, many countries, especially the United States, Canada, Australia, and New Zealand, saw a marked increase in the number of babies borne. This "baby boom" generation has been widely researched and continues to have an impact on society and social policy. There are many ways to quantify fertility. For example, birth rate is typically the number of live births per thousand people per year for a given population. The total fertility rate of a population is an estimated measure based on observed age-related fertility rates during a given time period and assuming a woman lives throughout her entire likely reproductive span, or roughly to age 50. It is intended to represent the average number of live births per woman in a given population. However, since human reproduction requires genetic contributions from both males and females and social conventions typically restrict who may reproduce with whom, the male-female ratios in populations can affect actual fertility. Net reproduction rate quantifies the number of daughters borne to a woman, using statistical estimation methods similar to the total fertility rate. This statistic is often used in researching countries that exhibit strong preferences for one sex of child over another or that practice sex selection. Some other possible estimates include gross fertility rate, generational or cohort fertility rate, or completed family size.

In 2010, Russian president Vladimir Putin publicly addressed the growing concern of Russia's declining population, which he attributed to both declining fertility and high death rates, calling it "the most acute problem of contemporary Russia." Sub-replacement fertility rate is a threshold value of the total fertility rate where the number of births is not large enough to replace or maintain a given population at its current level. In theory, each couple must produce two children to replace themselves or, referring to net reproduction rate, each woman must have one daughter to replace herself. In reality, not all people pair and reproduce and early mortality and other factors affect population sizes. Mathematical and statistical models have been used to model average behavior and account for such variables. In the early twenty-first century, the global replacement fertility rate was about 2.3 children per woman: the theoretical value of two, plus a fractional value that adjusts for mortality and other factors. Anything below this value is sub-replacement, leading to a declining overall population. In developed countries, the value was about 2.1 children per woman, while in some developing nations, the replacement rate has been calculated to be as high as 3.3 children per woman. Leslie models, named after population biologist Patrick Leslie, often include fertility matrices based on age groups to model population growth. They are also related to Euler–Lotka equations of population dynamics, named for mathematical demography pioneer Alfred Lotka and mathematician Leonhard Euler.

Fertility Cycles

Individuals seeking to improve their own fertility often rely on various methods to either predict when a woman will be fertile, such as measuring and charting basal body temperature, or to study the viability and motility of male sperm. In the late nineteenth century, physician Mary Putnam Jacobi was among the first to observe biphasic patterns in basal body temperature during menstrual cycles, though the connection with ovulation was not made until the early twentieth century. Studies by many researchers throughout the twentieth century statistically determined patterns in ovulation and fertility, such as the frequency of ovulation, the most probable window of ovulation during the menstrual cycle, and associations between fertility and observable physical characteristics, such as temperature, pain, and mucosal secretions. Many of these studies were the basis for calendar-based methods of fertility planning, such as basal body temperature (BBT) graphs. Beginning in the mid-twentieth century, physicians and others mathematically analyzed and interpreted BBT charts, though some techniques required complete data over long periods, which was considered not to be practical for use by individual couples. In the 1960s, neurologist John Marshall proposed the "three over six" prediction method: a pattern of any three plotted daily temperatures higher than the previous six was a sign of likely ovulation. This method was still in common use at the start of the twenty-first century, though with advances in computing technology, mathematical algorithms for detecting patterns may be used. Alternatively, the Billings method, named for physicians John and Evelyn Billings, is a scoring or quantification system for rating and graphing characteristics of cervical mucus to predict ovulation.

Greater understanding of the biomechanics of conception resulted in new studies of the male role in fertility. Male fertility is often quantified by sperm count or sperm concentration, which is the number of sperm cells per unit fluid volume. The term "oligozoospermia" refers to a sperm count that falls below "normal" as compared to statistically derived reference standards set by the World Health Organization and other agencies. Sperm cells may also be analyzed for abnormal morphology or geometry, which is one of the factors that affects their motility (rate of motion). Mathematical analyses have been used to explore motility. For example, mathematicians David Smith and John Blake created a mathematical model of a swimming sperm cell that they used to explore the fluid dynamic forces between sperm cells and surfaces. Understanding normal sperm motility via such models may help correct motility problems in infertile men and suggest future clinical practices.

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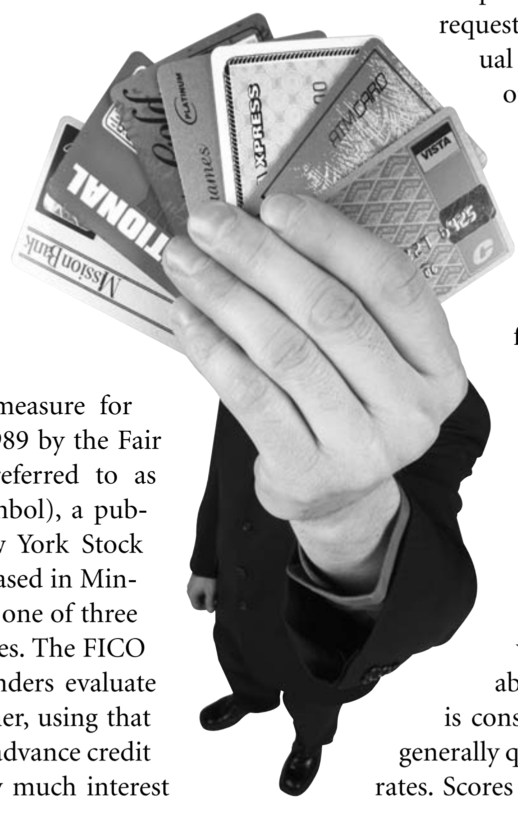
See Also: Census; Forecasting; Pregnancy.

Fibonacci Tuning

See *Pythagorean and Fibonacci Tuning*

FICO Score

Category: Business, Economics, and Marketing.
Fields of Study: Algebra; Measurement; Representations.
Summary: A person's FICO score helps lenders mathematically evaluate risk.



The FICO score is a standard measure for credit risk. It was developed in 1989 by the Fair Isaac Corporation (commonly referred to as FICO, which is also its ticker symbol), a public company (traded on the New York Stock Exchange) founded in 1956 and based in Minneapolis, Minnesota, and Equifax, one of three major U.S. credit reporting agencies. The FICO score is one of the chief ways lenders evaluate the credit risk posed by a consumer, using that information to decide whether to advance credit or a loan to them and, if so, how much interest to charge. Consumers judged to pose greater risk of default are generally charged higher interest rates to compensate for the high probability that they will not repay their debts. Models based on FICO scores are blamed in part for the housing crisis that occurred in the early twenty-first century. In the 1990s, subprime mortgage lenders began relying more on automated underwriting and quantitative models. These suggested that subprime borrowers were improving in terms of average FICO scores. This suggestion, coupled with the historical performance of subprime mortgage securities, was interpreted as a sign of strength in the subprime market, which proved not to be true.

Calculating the FICO Score

The formula for calculating the FICO score is proprietary and is regularly revised, but it can be described in general terms. The FICO score is calculated from data in a person's credit report. The importance placed on the different categories of information varies but for an average customer the weights are approximately as follows: payment history is 35%, amounts owed is 30%, length of credit history is 15%, types of credit used is 10%, and new credit is 10%. FICO does not calculate the report itself; instead, when a lender requests a credit rating for an individual FICO, software is used by one of the three major national credit reporting agencies (Equifax, Experian, and TransUnion) to calculate the FICO score. These calculations may differ, since the three credit agencies often include different information. The score is therefore time dependent, and changes in a person's financial and credit situation can be expected to result in changes to their FICO score as well.

The range of a FICO score is from 300 to 850, with higher scores denoting greater creditworthiness. The median score is about 725 and a score above 700 is considered good; a score above 770 generally qualifies people for the best credit rates. Scores lower than about 660 generally qualify people for only limited credit at much higher interest rates.

History of the FICO Score

The FICO score is a modern solution to a long-standing issue in business: managing the risk of lending money to an individual or business not personally known to the lender. Systems of credit reporting have been in existence for over 100 years and throughout their history, credit reporting systems have had to deal with the tension between lenders and merchants who wanted to protect their assets and consumers and businesses who wanted fair access to credit, which would help them expand their businesses or purchase major assets, such as a house. Credit reporting was largely unregulated until the 1960s, so there were no legal restrictions over what could be included (information about sexual preference and alcohol consumption were sometimes included, for instance) and individuals had no right to see what was in their records or to challenge incorrect information. Some criticize the FICO calculations, saying that the proprietary nature makes them unfair, they are inexact and poorly quantify risk for some subgroups of borrowers, information is not updated frequently, and the burden of correcting misinformation falls on the individual rather than the companies.

In 1971 Congress passed the Fair Credit Reporting Act (FCRA), which gave individuals the right to view their records and to dispute or correct any mistakes in their records. At the same time, credit reports began to include positive information (for instance, loans repaid on time) as well as negative information and, in 2001, consumers gained the right to see their credit scores rather than simply the information in their reports. The importance of the ability to view and to challenge information in one's credit report was underlined in a 2004 study by the U.S. Public Interest Research Group, which found that 79% of credit reports have errors (usually outdated information or information that pertained to a different person), including about one-quarter with errors serious enough to justify the denial of credit. Although there are many criticisms of the process of computing credit scores, few would be willing to discontinue their use because they are an important tool for risk assessment, help ensure equitable treatment, and make the credit market more efficient.

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See Also: Bankruptcy, Personal; Credit Cards; Home Buying.

File Downloading and Sharing

Category: Communication and Computers.
Fields of Study: Algebra; Measurement; Number and Operations.
Summary: Mathematicians work on developing compression algorithms and resolving security issues to make file downloading and sharing faster and more secure.

The words "downloading" and "uploading" began to enter mainstream usage in the 1970s. Bulletin board systems, a precursor to the Internet, were among the first systems that allowed computer users to access an external system. At the start of the twenty-first century, e-mail was commonly uploaded and downloaded from remote servers. The term "file sharing" came into popular usage later, especially in reference to peer-to-peer file sharing systems, like Napster. File sharing refers to providing multiple users access to digitally stored information, usually from a remote system. Streaming differs from downloading, since data that is streamed is not stored but used as soon as it is accessed. The amount of time that is required to upload or download a file is, in part, a function of its size. Compression algorithms make data faster and easier to transfer. Mathematician Claude Shannon formulated a theory of data compression in the late 1940s using concepts from entropy and probability, including theoretical limits on lossless and lossy compression that depended, in part, on a function expressing the allowable distortion error. This theory is also known as "source coding theory." Mathematicians work on reliability and security issues, such as detecting and preventing file sharing worms. Mathematical models of file sharing systems created using techniques from areas such as graph theory and statistics help study connections, patterns, and probabilities. The Gfarm Grid File System was developed in the early twenty-first century as a federated and scalable virtual file system designed to facilitate the high performance petascale-level computing and data mining problems, such as those that result from theoretical particle physics. Mathematicians Duncan Watts and Steve Strogatz made mathematical connections between the behavior of network nodes using look up protocols and human participants in Stanley Milgram's experiments on the small world phenomenon.

In the early twenty-first century, the term "file sharing" is sometimes used specifically with reference to the illegal proliferation of copyrighted material, which may be attributed, in part, to widespread publicity about this issue. There are several important variables related to the prominence or frequency of illegal file sharing: the availability of Internet access; the growth of typical Internet connection speeds; the development of new file formats that resulted in smaller sizes for high-quality music files; and peer-to-peer file sharing systems. Napster, released in 1999, was the first widely used peer-to-peer file sharing system. It was developed by Shawn Fanning and enabled mostly anonymous sharing of music files with other users through a centralized server, including a search function to locate songs. Though it was shut down by court order only two years later, half a dozen similar programs had been released in that time and the Bit Torrent client was released shortly thereafter. Napster was purchased by Best Buy in 2008 and is now a pay service. Mathematicians research topology and traffic in distributed networks, like Napster and Gnutella, with methods from graph theory and scheduling algorithms, among other tools. They are often seen as advantageous because they reduce or eliminate reliance on centralized servers. These highly connected network nodes are often critical failure points. They also use statistical methods and other types of mathematical modeling to study the economic impacts of peer-to-peer file sharing on retailers and artists as well as user behaviors with regard to their willingness to pay for digital music or movies.

The Bit Torrent client was nearly as large a step forward in file sharing as Napster had been, because it was not a service but a protocol, or a method of sharing files, and is not exclusive to sharing music files. The essential innovation of Bit Torrent, developed by Bram Cohen, was that file seekers were connected to many peers at once, instead of just a single peer. Pieces of the file are simultaneously downloaded and then reassembled on the user's computer. Furthermore, all peers downloading the file were capable of sharing the pieces they have, even before they have the complete file. A complete copy of a file is called a "seed." There must be at least one seed involved for downloads to successfully complete. Once a more-popular file has propagated many locations, the network of peers broadens, increasing the piecewise download speed. Unlike Napster, the Bit Torrent protocol does not utilize a central server, making it difficult to detect downloading, though servers called "Bit Torrent trackers" are the targets of law enforcement. Random ports also help users avoid detection. Mathematical methods, such as stochastic differential equations, have been used to model network environments and peer behavior and mathematically based peer-to-peer simulators can be used to evaluate and test new algorithms and solutions before they are implemented.

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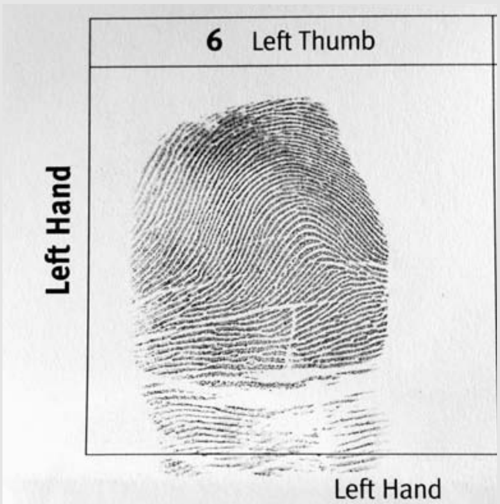
See Also: Cerf, Vinton; MP3 Player; Servers.

Fingerprints

Category: Medicine and Health.
Fields of Study: Geometry; Representations.
Summary: Mathematical algorithms help professionals use fingerprints as a means of identification.

The study of fingerprints could be considered both science and art. Fingerprint interpretation and analysis have grown over the twentieth and twenty-first centuries along with the development of new technologies and mathematical tools for imaging processing. Fingerprinting is a recognized method for personal identification and is used worldwide.



Brief History of Fingerprinting

In 1788, a German scientist, J. C. A. Mayer, presented the theory that each person produces a unique fingerprint. Almost 50 years later, Johannes Purkinje explained that the fingerprints could be classified in patterns that could be recognized. It was the beginning of fingerprints being used to identify individuals. In 1892, anthropologist Sir Francis Galton, cousin of Charles Darwin, published that fingerprints remain unchanged for a person's life and they are permanent. This led to the official use of fingerprints for criminal identification at Scotland Yard. In the early twenty-first century, fingerprint verification has been used as one of the most reliable personal identification methods for criminal investigation or to access control applications.

A fingerprint is an impression left by the raised portion of the epidermis on the fingers. The epidermal ridges are small corrugations of the skin (with an average of 0.5 mm in breadth) without hair or sebaceous glands but with numerous sweat glands, also found in the toes. The epidermal ridges in a particular area of the inner hands and bottom of the feet have two functions: to provide traction to help people grab objects (the sweat glands moisten the skin, augmenting the security of contact) and to enhance the sense of touch by the stimulation of the underlying nerve. Humans are not the only species with epidermal ridges; some primates, including gorillas and chimpanzees, and koala bears have their own unique prints.

Fingerprint Patterns

There are three general groups of fingerprint patterns: arch, loop, and whorl. They may be divided into subgroups by means of the smaller differences existing between the patterns in the same general group. Fingerprint groups may be also divided into male and female and by age. Historically, the identification of these patterns was done manually in a tedious and time-consuming approach requiring ink, paper, and sufficient knowledge and training of the fingerprint examiner. In the early twenty-first century, automatic fingerprint identification systems can quickly verify a person's identity by searching millions of records in a matter of seconds. Advanced mathematical algorithms are used in forensic science and other areas such as biometric identification, the science of identifying a person using some unique physical characteristic. Correlation-based methods rely on identifying characteristics of print patterns and positioning those characteristics within the pattern, using what are called "registration points." Another mathematically interesting problem is to reconstruct a fingerprint from a partial print or a blurred print.

Other Advances

Other methods to identify humans are used in addition to fingerprints: biometric technology voiceprint, retina/iris scan, hand geometry, and facial recognition. However, fingerprinting is the easiest to use and it provides an average accuracy of 98%.

Wavelets have become an important mathematical tool for fingerprint recognition. This method could be an efficient solution for fingerprint recognition systems because it eliminates the necessity of preprocessing the images, reducing the time required for analysis.

Fingerprint identifications play a vital role in many criminal investigations but there are still challenges, such as identifying the body of a victim of a fire with parts of the fingers burned. Mathematical equations and operators have been used for the calculation of fingerprint probabilities based on individual characteristics, such as only a partial print. The use of digital fingerprints requires more work in description and analysis to avoid ambiguities in identification, such as wrongly convicting an innocent person to prison.

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See Also: Accident Reconstruction; Crime Scene Investigation; Daubechies, Ingrid.

Firearms

Category: Government, Politics, and History.
Fields of Study: Algebra; Measurement; Number and Operations.
Summary: Mathematicians have long studied and analyzed firearms and projectile motion to create more accurate weapons.

The successful construction and use of many types of offensive weaponry rely on mathematical principles. Ancient people typically used body-powered projectiles, like spears and stones thrown from slings, which required judgments of force and angles to achieve the correct parabolic motion to hit the target. Archimedes of Syracuse designed weapons like mechanical catapults to defend Syracuse from attack by the Romans. The development of gunpowder-propelled field artillery, the successor to mechanical missile weapons like catapults and ballistae, created a demand for sophisticated mathematics. Mathematicians studied and solved problems of ballistic velocities and trajectories to increase accuracy and impact. Handheld firearms of all types rely on similar principles. There are a number of interesting mathematical properties related to firearms, including weapon caliber, rate of fire, rifling, muzzle velocity, and propulsion, as well as telescopic sights and other characteristics. Mathematics training or degrees are suggested for firearms identification and bullet matching, which are increasingly used to match weapons to crimes, and mathematics skills are one of the requirements cited for careers in firearms repair.

Brief History of Firearms

As artillery and projectiles began to play a much larger role in warfare, kings, generals, and powerful concerns in society began looking for more powerful and more accurate weapons. They called upon scientists and mathematicians to address the problem. Niccolo Tartaglia, Galileo Galilei, Evangelista Torricelli, Rene Descartes, Isaac Newton, and Johann Bernoulli are some of the people who worked on the problem of projectile trajectories. Two of the foremost mathematicians to work in this area were Benjamin Robins and Leonhard Euler.

Tartaglia published an important work on cannon trajectory in the sixteenth century. Using the science and mathematics of the time (Aristotelian dynamics, named for Aristotle, and Euclidean geometry, named for Euclid of Alexandria), he thought of the flight of a cannonball as moving from a line with slope determined by the angle of the cannon, the final trajectory by a vertical line and a circular segment on which the apex of the trajectory occurs joining these two lines. In his 1537 text *La nova scientia* and his 1546 text *Questi et inventioni diverse*, he indicated that this was only an approximation to the actual trajectory. However, it was such a good approximation—and so comparatively easy—that it was used by artillery groups well into the eighteenth century. His model took into account the practical knowledge gained through working with gunners and their experience in the field.

A collection of World War II firearms. Groups of mathematicians were employed by the war departments both in World War I and World War II for tasks such as creating tables of trajectories for army artillery units.

Galileo stated that if there is no air resistance, the trajectory of a projectile is a parabola. This conjecture appears first in the work of his student Bonaventura Cavalieri in 1632 and, later, in Galileo's 1638 work *Discorsi e dimostrazioni matematiche: intorno a due nuoue scienze*. Torricelli also worked with Galileo. His book *De motu* contained a geometric method for computing the range of a projectile. Galileo asserted that the path of the trajectory and the shape of a hanging curve (the catenary) are the same, leading him to work with the idea that the trajectory curve is symmetric. This idea results in erroneous computations for range.

In the next era, the important work of Christiaan Huygens, Bernoulli, and Newton on air resistance set the stage for great strides forward in understanding projectile trajectories. The first to explicitly consider air resistance was Benjamin Robins, an English mathematician who was a student of Henry Pemberton and a protégé of Newton. Robins became interested in military engineering in the 1730s from his work on Newton's fluxions and their utility in describing objects in motion. In 1736, he wrote a detailed critique of Euler's *Treatise on Motion* and his extensive use of algebra versus geometry. He was subsequently barred from an appointment as mathematics professor at the new Royal Military Academy in Woolrich in 1741 because of a political dispute. In order to bolster his application for this position, he returned to his work on ballistics and in 1742 published *New Principles of Gunnery*. In 1747, the Royal Society awarded him its prestigious Copley Medal for his work in ballistics. A major contribution of Robins was in determining that the important consideration for ballistics was the initial velocity of the projectile and the effect of air resistance, not the range, which was a function of initial velocity. Experiments showed that the assumption of Huygens and Newton that air resistance was proportional to the square of the velocity was true only at low velocities. Also, Robins hypothesized that lateral deviations were caused by random spinning of the projectile. He advocated the use of rifled barrels with ovoid (rather that spherical) bullets to control this effect.

Robins's work ultimately had a broad impact. In a time of very poor English-Continental mathematical relations, Euler himself found Robins' work from 1742 so important that he translated it into German in 1745 and made extensive additions. He contributed and acclaimed the work of Robins. Napoleon Bonaparte, an avid student of mathematics who is widely considered to have revolutionized the use of field artillery, had Euler's translation translated into French for his study. Euler is credited with bringing the study of trajectory motion into the modern mathematics realm. In his 1753 work, he described motion in terms of second-order differential equations, allowing him to make appropriate changes in assumptions about air resistance and to give better approximate solutions that matched experimental results. Work was undertaken to create tables of trajectories for army artillery units. As technology advanced, mathematics had to evolve to keep pace. There were groups of mathematicians who worked for the war departments in both World War I and World War II. For example, British mathematician John Littlewood improved and simplified calculation formulas for range, flight time, and angle of descent of projectiles and updated ballistics tables. The Applied Mathematics Panel in the United States in World War II looked at various trajectory issues, including aerial dogfights and projectile trajectory. The U.S. Navy maintained the Aberdeen Proving Grounds after the war and had panels of mathematicians there to help model projectile motion and explosions.

There are a number of other interesting mathematical connections related to artillery and firearms, such as caliber and barrel rifling. The caliber of a firearm is the approximate diameter of the barrel and the projectile used in it, usually measured in inches or millimeters. Rifling is traditionally the process of making helical grooves down the entire length of a firearm's barrel to impart a spin to the projectile. Polygonal rifling is another method that shapes the interior of the barrel like a polygon with rounded edges to achieve a similar effect, most commonly with hexagons but sometimes with octagons or decagons. Overall, rifling gives the projectile gyroscopic stability and improves its trajectory. Since a rifled barrel is noncircular, as opposed to a smoothbore (nonrifled) weapon, there are different ways of measuring caliber. In the case of helical rifling, measurements may be taken of the bore diameter, which is the diameter across the lands or high points in the rifling, or the groove diameter, which is the diameter across the grooves or low points. Rifling grooves create striations on the bullet, which, together with caliber, are used in forensics to identify the firearm that shot a bullet. Twist rate for rifling is the distance the projectile must travel down the barrel to complete one full revolution about its own axis, which is often given in units of turns per inches or centimeters. A shorter distance indicates a higher turning rate and a faster spin.

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See Also: Archimedes; Artillery; Crime Scene Investigation; Infantry (Aerial and Ground Movements); Missiles.

Fireworks

Category: Architecture and Engineering.
Fields of Study: Algebra; Geometry; Number and Operations.
Summary: Firework mathematics involves the timing and rhythm of burning, rocket flight, and explosions.

Fireworks are explosions for entertainment with design elements of light, sound, and smoke. Chemical additives are used to color fireworks, which originated in ancient China. The province of Liuyang is known as the home of fireworks. Fireworks as an art are temporal, like dance or animation; therefore, much of firework mathematics has to do with the timing and rhythm of burning, rocket flight, and explosions. Mathematicians around the world have modeled and quantified various aspects related to fireworks, like the path and maximum height. In the seventeenth century, Claude Dechales published what became a popular textbook on mathematics that included pyrotechnics. Engineer Amédée-François Frézier, whom some also refer to as a mathematician, worked on the theory of fireworks in the eighteenth century. The process of mathematical induction has been likened to a sequence of connected fireworks. In the United States, the Bureau of Alcohol, Tobacco, Firearms and Explosives classifies and regulates fireworks.

Patterns of Explosions

Most fireworks shot into the air explode in spherical patterns. By modifying the composition of fireworks, it is possible to add or remove tail effects, change the speed of individual parts, and produce delayed explosions to parts, filling spheres with radial lines or creating expanding spheres. Less frequent are fireworks that burn sustained, extending, two-dimensional shapes, such as rings or hearts.

Ratios and Proportions of Shells and Mortars

Many fireworks are packed into shells and fired out of special mortars, or small cannons. Larger shells are fired out of larger mortars with higher speeds and also fly higher. As with any projectile, the path and height of the firework shell, until the explosion, obey the quadratic equation of gravitational deceleration and the shell flies following the path of a parabola. On the other hand, because of the physics of the black powder or pyrex used to propel the shells out of the mortars, the relationship between the size of shells, mortars, and their initial speed is linear. The relationship between the size of the shells and the maximum height they fly is also linear. Pyrotechnician formulas approximate 100 feet of the shell's maximum flight height per every inch of its diameter. The explosion of the firework has to be timed so it happens when the shell is high up in the air, which is achieved through solving the height equation and matching the time of chemical reactions in the shell to that height.

Fireworks Color and Temperature Gradients

There are two distinct ways to color fireworks. The first method is based on the same physical process used in incandescent light bulbs and the second on that used in neon lights. The first method uses blackbody radiation—the property of objects to emit more light with higher temperature. Blackbody radiation emits light over a broad spectrum. As metals heat, they start to become red to the human eye because the majority of

Many of today's fireworks are made in much the same way they were hundreds of years ago.

the spectrum is light at infrared wavelengths human beings cannot see. When the temperature rises, the emission of the light in the visible spectrum increases and the object becomes first yellowish and then white, the mixture of all visible-light wavelengths. Thus, fireworks that depend on blackbody radiation for their color can only be dull red, pale yellow, or white.

The second method of firework coloring is based on the so-called atomic emission. Atoms in the firework material, before the firework is fired, are in a stable state, corresponding to particular orbits of electrons. If atoms are electronically excited, they emit photons to return to that stable state. When photons are in the visible spectrum, the human eye sees a color as the atomic emission takes place. Some elements have a narrow spectral band in their atomic emissions, allowing particular pure colors to be pinpointed. For example, sodium emits bright yellow and barium emits green when electronically excited. Copper salts emit pure blue but they are so unstable at high temperature that people only recently learned to use them safely in fireworks.

If the firework material burns too hot, the blackbody radiation process takes over. Therefore, to produce pure colors of the atomic emission process, pyrotechnicians create mixtures that burn relatively cool. The chemistry breakthrough allowing this to happen was the substitution of potassium chlorate, which burns at around 120 degrees Celsius, for potassium nitrate, which burns at 560 degrees Celsius. Fireworks contain coolants that prevent burning from reaching higher temperatures, for example, by releasing some water and carbon dioxide, as sodium bicarbonate does.

Pyrotechnic Competition and Measurements

At competitive events, fireworks are measured based on several criteria, mostly qualitative and artistic. The quantitative criteria include purity and brightness of color and the appropriate explosion height. The timing of the intended fireworks effects, such as the change of shape and color, is also taken into consideration—it has to follow a recognizable temporal pattern and to form a pleasing rhythm.

Competition judges add points for technical difficulty, celebrating innovations in fireworks. For example, when strobe effects were first discovered, fireworks using them were awarded technical difficulty points at competitions. After a few years, as strobe effects became well researched, judges stopped awarding points for them.

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See Also: Chinese Mathematics; Energy; Light; Temperature.

Fishing

Category: Games, Sport, and Recreation.
Fields of Study: Algebra; Geometry.
Summary: Fishing tactics, management, and measuring all require the sophisticated use of mathematical principles.

Mathematics has proven to be a useful tool in understanding the impact of a variety of factors that influence fish populations. Other mathematical techniques have been used to analyze photographs of fish and to generate useful estimates of the fish's weight. Mathematics has also demonstrated its utility in the creation of tools for locating and catching fish.

Fishery Management

The estimation and regulation of the striped bass and bluefin tuna populations along the East Coast of the United States are examples of important fishery management issues with serious economic implications.

Mathematics as an ecosystem-based management tool has been used to formulate population models that attempt to account for very complex environmental factors, including variations in water quality and temperature; fluctuations in the availability of important forage species upon which the targeted species depend for food; the presence (or lack thereof) of appropriate spawning areas; the impact of fish farming on wild fish populations; the interplay of commercial fishing and sport fishing; the introduction of invasive species; and the impact of diseases. Regulations regarding the timing, size, and number of fish that are to be harvested are based, in part, on mathematical models. Presumably, understanding the likely consequences of changes in these and other factors will lead to improved management decisions. An alternative management approach has been suggested by analysis of the history of the sardine fishery in California's coastal waters. Such evidence has led some mathematicians to believe that fluctuations in fish populations are best explained by utilizing branches of mathematics known as "complexity theory" and "chaos theory."

Weight Estimation

Mathematicians were called upon when the National Freshwater Fishing Hall of Fame faced controversy over its listing of the record muskellunge as a fish caught in 1949, reported to be 63.5 inches in length and weighing 69 pounds. Three photographs of the angler holding the fish in front of him documented the catch. The question arose, since the height of the angler in the photograph was known: could the length of the fish be accurately estimated? In fact, projective geometry together with some precise measurements gleaned from the photographs could provide very good estimates of the length of the fish. However, a difficulty remained: was there a way of accurately estimating the weight of a muskellunge based upon its length, without knowing its girth? In fact, an algebraic formula has been developed for estimating the weight of a muskellunge that requires only a precise measurement of the length of a portion of the fish's body. The formula is

$$W = \frac{L^3}{2800}$$

where W is the weight in pounds and L is the length in inches.

Tools for Locating and Catching Fish

The electronic devices often utilized in locating fish include flashers, LCD graphs, and global positioning systems. Each of these items depends upon mathematical underpinnings. However, mathematics also plays an important role in the creation of the nonelectronic tools used in sport fishing.

The design of reels, fly lines, and fishing rods depends upon mathematics. The role of geometry is especially apparent in the building of traditional split-bamboo fly rods. For example, in a two-piece split-bamboo rod, each of the two sections of the rod requires that six strips of bamboo be cut and planed to a precise taper such that each strip has cross sections along its length that are equilateral triangles of diminishing size. When these strips are properly glued together, hexagonal cross sections result. The rod blank so created is the foundation of a bamboo fly rod. The builder must still decide where to place the line guides along the length of the blank in order to produce a fishing rod that will both cast well and enable the fisherman to quickly capture hooked fish. Not only does the distance between consecutive guides increase from the rod tip toward the butt of the rod but also those distances change in a precise way. The initial placement of the guides on the rod is accomplished by using an idea from algebra known as "arithmetic progression." The fine-tuning of the guide placement on the rod then depends upon measuring the arc through which the rod bends when placed under a predetermined load.

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Philip McCartney

See Also: GPS; Knots; Marine Navigation; Mathematical Modeling; Predator–Prey Models; Problem Solving in Society; Tides and Waves.

Floods

Category: Weather, Nature, and Environment.
Fields of Study: Algebra; Data Analysis and Probability; Measurement.
Summary: Engineering has always been engaged with flood protection and the containment of floodwaters; mathematics is also used to predict flooding.

Although some floods occur with little to no warning, overall patterns of flooding along rivers or streams can be determined based on measurement and the statistical analysis and extrapolation of gathered data, such as a river's historical and current discharge, stage, and flood-stage levels. The resulting data can be used to find the probability of future flooding. Mathematicians and engineers are actively engaged in developing systems to model, predict, and control floods, especially for low-lying areas of the world like the Netherlands and the Mississippi Valley in the United States. Flood prediction and flood control are vital because of floods' potentially devastating impacts—floods are among the leading natural disasters in terms of loss of life and property damage.

Flood Prediction

One of the first steps in flood prediction is the measurement of a river's discharge, stage, and flood stage. The size and flow of rivers are measured using a variety of different methods. Key determinations include the discharge or flow, which measures the volume of water passing through a section of the river in a particular time frame, such as cubic feet per second; the stage, or water surface level over a set criteria, such as sea level; and the flood stage, when a river's overflow will result in widespread inundation or heavy impacts on life and property. Determination of the area of inundation during a flood stage must also take into consideration the topography of the nearby area, such as its slope. During a particular flood, analysts also determine the peak or crest, when the river reaches its highest stage.

Scientists then create flood forecasts based on calculations determined from the statistical analysis of the gathered data. The mathematical calculation of

An aerial photograph of flooding in Port-au-Prince, Haiti, caused by Hurricane Ike in 2008. The U.S. Navy photographed the flood while providing disaster relief support to Haiti.

the relationship between an area's precipitation levels and the discharge of nearby rivers and streams relies on a number of complex factors. Geographical factors can include the topography; types of bedrock, soil, and vegetation; and area of the drainage basin. Meteorological factors can include the intensity and duration of precipitation on average, as well as before and during a particular storm. Because of the complexity of the data, forecasters rely on calculating probability based on historical data of peak discharge frequency.

Statistical analysis of the probability of exceeding the average annual peak discharge in a specified time frame can be made for drainage basins for which a series of records of maximum annual discharges (peak flow) are available and ranked from largest to smallest. The calculated probabilities include the probability that a peak flow will be equaled or exceeded within one year, known as the "exceedence probability" and expressed as a decimal fraction; and the recurrence interval, which is the average number of years between past events. The recurrence interval can also be defined as the number of years in which analysts expect a one-time flow that will equal or exceed a peak flow.

The recurrence interval for a particular location can be used to determine the probability of a flood at that location, expressed by the formula

$$P = \frac{1}{T}$$

where P is the probability of a flood and T is the recurrence interval. For example, a 100-year recurrence interval would produce a 1% probability of a flood of equal or greater magnitude in a given year. Engineers, scientists, forecasters, and the public must be aware, however, that the resulting probability is an average. For example, a 100-year flood is not statistically expected to occur exactly once every 100 years and two such floods may occur in close proximity.

Graphing and Modeling Floods

Analysts use these statistics in the construction of graphs and tables known as "frequency distributions," which show the probability of various discharges for particular locations and thus the probability of a flood in a particular area. Analysts can utilize a variety of mathematical equations to carry out the statistical analysis needed to create frequency distributions. The most common equations include Normal Distribution, Log-Normal Distribution, Gumbel Distribution, and Log-Pearson Type III Distribution.

Different mathematical methods are used to determine frequency distributions in those locations where recorded data of discharge is unavailable or incomplete. In some cases, analysts use flood frequency estimates from nearby or similar areas with complete data to create estimates for areas that lack data. One commonly used method is the rational method, which utilizes the relationship between peak discharge and the product of drainage basin area, precipitation intensity level, and a standard coefficient based on the drainage basin's land use or ground cover. Other methods allow for the incorporation of changes in a river's discharge over time as well as its peak discharge. The increasing availability of flood-modeling software allows analysts to input data into computers, which then produce flood probabilities and frequency distributions as well as the effects of environmental impacts, such as deforestation and global climate pattern changes, on future flood patterns.

Applications of Flood Models

Meteorologists use flood probabilities and frequency distributions to aid in the issuance of flood watches and warnings. Engineers use flood probability estimates of both magnitude and frequency when constructing and managing flood control structures, such as dams and levees, as well as nearby structures, such as roads and bridges.

The information is also useful when planning to divert or change the course of rivers or streams that frequently flood, increase the slope of the surrounding topography to lessen inundation, create floodway channels, or determine when to lower dam reservoir levels. Governments and other groups use flood probabilities and frequency distributions when planning the location of residences, towns, and industries along rivers and streams.

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Marcella Bush Trevino

See Also: Earthquakes; Forecasting; Hurricanes and Tornadoes; Landscape Design.

Football

Category: Games, Sport, and Recreation.
Fields of Study: Data Analysis and Probability; Geometry; Measurement.
Summary: Football coaches use statistics to inform their decisions while the National Football League analyzes the effects of its rules.

Though a physical battle between two teams of talented athletes, football can be analyzed using mathematical ideas and techniques. Pertinent to coaches, players, fans, and betting agents, these analyses focus on all aspects of football—the physical aspects and performance of players, game elements (passing, running, defense, and kicking, as well as strategies), and the geometry-based physics surrounding the game. Mathematical analysis can impact the game positively or negatively. Nonetheless, football remains a physical competition between two teams, despite the use of mathematics to identify patterns of strengths and weaknesses, suggest optimal strategies, provide rankings, stimulate discussions, and possibly resolve arguments.

Quarterback Rating

The National Football League uses a mathematical formula to rate quarterbacks. Data are collected for each game and for the season relative to a quarterback's pass completion percentage (P), touchdown pass percentage (T), pass interception percentage (I), and average gain per attempt (G). Using a few boundary conditions, a quarterback's rating (Q) is determined by the formula

$$Q = \frac{5}{6}\left(P + 4T - 5I = 5G + 2.5\right).$$

The formula's derivation in terms of four independent variables involves multiple regression techniques.

Overtime Rules

The National Football League also uses Markov chain techniques to analyze its overtime rules in response to the "statistical fact" that too many football teams were winning important games with a field goal on their first overtime possession. Thus, the "winning" team, after a hard-fought game, is influenced too greatly by a single coin flip that determines team possession, with minimal differences accounted for by a team's ability to score on the first possession. Effective in 2011, the rules for play-off games were changed to prevent the game ending with a field goal on the first possession of overtime.

Though difficult to implement practically, geometry, trigonometry, and calculus all play strong roles within a football game and its situations. Examples include the following:

- Use of a quarterback's physical characteristics to determine the best angle and release points for throwing a pass, assuming it must reach receivers in different field locations and at multiple distances
- Use of the law of cosines to both understand and improve passing angles, timing, and patterns run by receivers
- Determination of an optimal efficiency for punters on each kick, or the ratio of the actual kick's distance to the maximum possible distance using the same force
- Prior to kicking a field goal, determination of success in terms of the angle subtended by the two goal posts
- Determinination of a defensive lineman's stance to maximize centers of gravity and potential force on impact with an opposing linemen

By gathering and analyzing the available data provided by a game, probabilities can help examine the particular events happening within a game, such as the following:

- Likelihood of a team making 0, 1, or 3 points after a touchdown score
- Reality of a quarterback having a "hot hand" in his or her completion of successive passes
- Success of making a field goal, given it will or will not result in a change in who has the leading score
- Probability of scoring during a fourth-and-goal
- Monitoring a coach's decisions in calling plays, especially if "conservative"
- Probability of a record being broken, either by a team or by a player

Similarly, mathematical statistics provide perspectives that explain game occurrences, provide comparative rankings of teams and players, and assist in decision making by coaches and team management. The usual sources of statistics are data regarding passing, running, defense, kicking, turnovers, and time management. Examples include the following:

- Use of ratios, means, and medians as descriptive statistics for a player, a position, a game, or a season
- Use of logistic regression models to calculate end-of-game point differentials, based on independent variables such as turnovers, passing yardage, running yardage, penalty yardage, number of first downs, and number of completed passes
- Impact of icing a place-kicker at crucial times within a game
- Correlations between a player's characteristics and training regimens relative to game performance
- Trend analysis, based on either a player's or a team's performance in specific ways over the past 5, 10, and 15 games
- Winning tendencies based on connections to lead changes during a game or knowledge of the team leading at the end of the third quarter
- Impact of rules changes on team scoring and defenses within the sport itself, such as observed effects of initial field positions subject to penalties or punts out of bounds
- Determining the "best" all-time player in a particular position (for example, quarterback, tight end, halfback, linebacker, or field-goal kicker), at a particular time in a game (such as the last quarter) or in an era
- The use of digraphs and "mysterious" statistical formulas to determine weekly rankings and placement of teams in a bracketed tournament (such as the Bowl Championship Series), directly affecting betting pools with stated odds
- Selection of players by professional teams during the annual draft, using historical data for each player's performance in conjunction with physical data
- The use of statistical data as part of contract negotiations between players and management, or even the release or trading of players based on team needs
- The questionable yet significant correlation between stock market performance and the Super Bowl's winning team

Mathematical game theory is evident in a coach's decision-making process, such as on each play within a football game, hoping to choose optimal tactics. The specific decisions range considerably and include the following:

- A coach's choice of designed offensive plays and defensive set-ups, relative to the down, position on the field, time of game, score, and opponent
- A coach's calling of time-outs and play reviews at opportune times
- A coach's use of techniques to motivate specific players
- A team's selection of players during a draft, dependent on the players' apparent abilities, the inferred needs of other teams, and the specific draft round
- Contract negotiations involving players, agents, and team management

Finally, using these statistical data and mathematical modeling techniques, one can create realistic simulations of football games, possibly using computer animations.

At the collegiate and professional levels, coaches increasingly use mathematics to remain competitive, even hiring mathematical statisticians as important parts of their staff. However, some authors and fans suggest that the football team with the best players and coaching will usually win, despite any use of sophisticated mathematics.

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Jerry Johnson

See Also: Baseball; Basketball; Hockey; Kicking a Field Goal; Soccer.

Forecasting

Category: Business, Economics and Marketing.
Fields of Study: Data Analysis and Probability; Number and Operations; Problem Solving.
Summary: The science of prediction is grounded in statistics, data analysis, and modeling, applied to such areas as traffic, sales, and the stock market.

Forecasting essentially means predicting. Prediction of various phenomena has been of interest to mankind ever since humans have inhabited the planet. One prediction of early human beings may have been that the sun would rise the next day, along with where animals or other food might appear. These predictions would be based on observation and experience. Once human beings began to investigate natural laws, certain predictions like the rising of the sun came to be regarded as certainties by many scientists. Generally, all predictions are based on experience but may be formulated with varying degrees of mathematical rigor that involve different levels of probability or uncertainty. "Prediction" may refer to guessing about the past but "forecasting" is always used to mean guessing events that may or may not happen in the future. Forecasting may be qualitative or quantitative, and events may not occur at all or may only occur after a very long period of time.

Mathematicians and statisticians have explored forecasting in a variety of fields, such as traffic flow, ocean waves, and asset price forecasts. Many mathematicians have contributed to weather forecasting, such

Business Forecasting

Forecasting is used extensively in business. Data required for mathematical forecasting models in business come from a variety of subjective, judgmental, or objective sources, some of which must be coded or quantified. Some common problems in business include predicting the following: the number of people who are interested in or are likely to buy a particular new product, sometimes called a "market survey" or "demand forecast"; the amount of sales a company will make during a given fiscal period, called "sales projection"; or "consumer satisfaction." Mathematician John Paulos's book, *A Mathematician Plays the Stock Market*, addressed the issue of predicting stock market behavior, widely regarded as one of the most mathematically challenging forecasting problems.

His discussion included criticisms of some mathematical methods of stock market forecasting, like Elliott waves. Accountant Ralph Elliott studied stock market data and investor psychology. He theorized that the market moved in probabilistic cycles that could be analyzed and predicted using Fibonacci numbers, named for mathematician Leonardo Fibonacci. Many investors engage in pattern trading based on Elliott wave methods.

as Johann Werner, James Glaisher, Lewis Richardson, Vilhelm Bjerknes, and Edward Lorenz. With regard to business, economics, and marketing, in the eleventh century, Shen Kua explored price forecasting and the theory of supply and demand. Statisticians George Box and Gwilym Jenkins in 1970 published a book on time series analysis for forecasting. In 2001, George Tiao won the Samuel Wilks Award of the American Statistical Association, in part for his work in forecasting, and in 2003, David Wallace won the same award, in part for his research on forecasting elections.

Forecasting Models

Forecasting models are created using a wide variety of analytical and computational methods from mathematics and statistics. In general, the quantification and reduction of uncertainty are required to make forecasting models accurate enough to help businesses make sound decisions.

Several issues arise while forecasting, including the time range of the forecast (the time until which the forecast may be applicable) and the availability and reliability of the data. Some traditional data analytic methods must be modified to account for the serial correlation common in data resulting from processes observed repeatedly over time. A large class of mathematical forecasting models involves applying weighted smoothing methods to fit functions or trends to historical data. Smoothing constants and other parameters may depend on choices made by the forecaster, so different models based on exactly the same data might produce varying forecasts.

Autoregressive moving average (ARMA) models, sometimes called "Box–Jenkins models" because they are estimated using a methodology developed by Box and Jenkins, along with integrated moving average (ARIMA) models, are widely applied to what are known as observable, nonstationary processes with serially correlated data. Financial data commonly falls into this process category. They may also use adaptive filtering, widely found in other applications such as signal processing, to remove noise.

Decomposition forecasting models mathematically separate overall trend, seasonal, and random components in data. Scatterplots, simple linear regression, and curve fitting may be useful for explorations and some modeling. Simulation methods facilitate dynamic models and exploration of "what-if" scenarios. The cross-impact matrix method explicitly takes into account the fact that the occurrence of one event can impact the likelihood of other events, so probabilities can be assigned to produce an intercorrelational structure to examine relationships between system components. Multiple regression is also used to examine multifactor influences. In general, the greater the interdependence of components, the more difficult it becomes to make a prediction about any single component. Decision trees, game theory, and chaos theory are other mathematical areas that have been used to explore systems to make forecasts.

Forecasting Validity

Ultimately, forecasts are usually judged by their accuracy, often in a subjective manner, and there are many theories regarding how to measure the utility of forecasts. One criterion is to assess whether the forecast differs from pure randomness. Another is to quantify the magnitude of error. Decision scientist Spyros Makridakis has stated that in many situations, judgmental forecasting by human experts has been shown to be superior to mathematical models. However, in terms of optimization, he also noted that forecasting many complex problems is unfeasible without computer modeling. For example, simultaneously forecasting inventory levels for thousands of items for sale at a major retailer or needed by a manufacturing company is likely beyond the scope of subjective judgmental forecasting. Computer technology also allows for the creation of complex decision algorithms with subsystems and feedback loops.

Stability in the system being modeled is also an important factor in determining whether model extrapolation will be valid and reliable for forecasting. Developmental inertia is the idea that some systems are less variable and therefore more easily predictable than others. For example, the rapidly changing fashion industry is a low-inertia or unstable system and new trends are difficult to predict mathematically. Decisions also need not be dichotomies but rather probabilities along multiple paths. Mathematical concepts from decision theory and utility theory, such as expected value, have also been incorporated into forecasting modeling and decisions.

Forecasting Ethics

An ethical consideration raised by forecasting is whether probabilistic inferences actually create the future, since decisions made today by individuals, businesses, and policy makers undoubtedly affect actions taken later. In his 1970 novel *Future Shock*, sociologist Alvin Toffler discussed the impact of evolving technology on humans and asserted the need for value impact forecasting, which is the idea that social forecasting must incorporate cultural and societal values. Mathematicians and others continue to study and debate these theories and problems and to seek ways to quantify psychological and qualitative variables considered essential by many forecasters.

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Sreenivasan Ravi

See Also: Data Mining; Ethics; Inventory Models; Predicting Preferences; Probability; Scheduling.

Forecasting, Weather

See *Weather Forecasting*

Forest Fires

Category: Weather, Nature, and Environment.
Fields of Study: Algebra; Data Analysis and Probability; Geometry; Measurement.
Summary: The spread of forest fires has been modeled for decades to guide firefighting decisions.

Forest service officials have often used controlled burns to reduce the risk of fires spreading by burning the dry vegetation that builds up on the forest floor. Predicting the spread of a fire, whether a controlled burn or a wildfire burning out of control, is of great interest in forest management. Mathematical models take into account various parameters, indices, and activity levels.

Brief History of Forest Fire Modeling

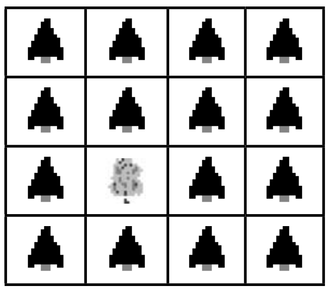
According to Forest Service (a branch of the U.S. Department of Agriculture) documents, the first mathematical model of the spread of fires was developed in 1946 by W. R. Fons. Fons's model was based on approximating the spread as a series of ignitions, with the key elements being ignition time and the distance between particles. Over the years, fire models became more sophisticated, using increasingly complicated mathematical equations, as in Richard C. Rothermel's 1972 differential and integral equation model of fire spread.

With the development of high-speed computers in the last quarter of the twentieth century, simulation models that use large numbers of relatively simple probabilistic and geometric relationships have become more common. In these models, forest fires are represented by a grid of trees where a variety of parameters are set for each tree.

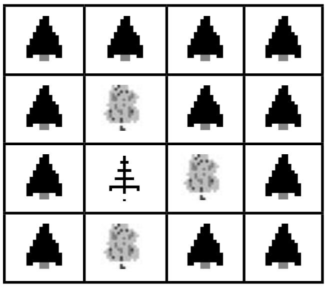
Examples of Forest Fire Models

A very simple simulation of a forest fire can be modeled with a grid of evenly spaced trees and a number cube. Set a forest dryness factor—a set of numbers that, when rolled on a six-sided die, indicate that a tree will catch fire if one of its four neighbors is on fire. For example, a dry forest might be represented by the numbers 1, 2, 3, and 4. In this example, 4/6 or 2/3 of the time, the fire would spread to neighbor trees.

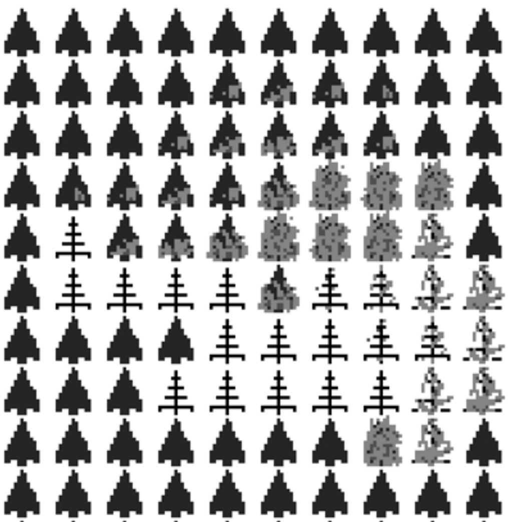
To see how such a simple model works, set the tree in position (3,2) on fire in the grid below and then roll the number cube for the trees in positions (3,1), (2,2), (3,3) and (4,2) to see if they will catch fire as the original tree "burns out."



Suppose the number cube rolls are 5, 2, 3, and 2, respectively; then the fire would spread as pictured, with the tree in position (3, 1) remaining unlit.



To complete the simulation, continue rolling the number cube to see if the trees adjacent to the three now on fire will burn. As with all models involving probabilities, it is important to run the simulation a large number of times and to look at the average or most common results rather than to rely on one run of the simulation, such as the forest fire simulation screen shot below:



The simulations in use today to model forest fires are very sophisticated. They include thousands of trees and hundreds of parameters, such as tree size, distribution, and dryness; wind speed and direction; humidity and ambient air temperature; leaf litter buildup; heating, ignition, and burn time; and the geometry of the terrain. These parameters contribute to the calculation of the probability that a tree will catch fire when its neighbors are on fire. Computer visualization software of the early twenty-first century allows programmers to build sophisticated user interfaces for these models in which the spread of the fire can be watched on screen and users can interact with the model, clearing a firebreak or starting a backfire.

Applications of Forest Fire Models

These models can be used to predict how a hypothetical fire might behave or to determine the best intervention in an existing fire, provided the parameter values used in the model accurately reflect the real conditions in the forest. Estimating these parameters poses a challenge to forestry officials—terrain and tree size and distribution are constant in a given forest at a specific time but other parameters, such as tree dryness, wind speed and direction, humidity, and ambient air temperature, vary over time, sometimes significantly.

Failure to accurately gauge parameters in a model can lead to disastrous results. In 2000, the National Park Service developed a fire plan for a controlled burn at the Bandolier National Monument in New Mexico. Now known as the Cerro Grande fire, the wind shifted and strengthened unpredictably, causing the fire to rage out of control, damaging more than 200 homes and 48,000 acres of land in and around the town of Los Alamos.

Agencies and firefighters use a wide variety of National Fire-Danger Rating System (NFDRS) indices and activity levels to monitor and make decisions about fires. For example, the Occurrence Index predicts the potential fire incidence within a rated area. Fire behavior researchers, like George Byram, defined many quantitative measures of fire behavior, such as the definition for fire intensity as the rate of heat energy release per unit time per unit length of fire front, which is defined independently of the depth or width of the fire. The Burning Index (BI) is commonly used to indicate the amount of effort that is needed to contain a given fire. The BI is calculated based on the material that is burning and other factors, including a modification of an equation defined by Byram for flame length. Some people have criticized agencies for failure to use historic data in making future predictions of wildfire hazards, such as recent burn areas in which wildfire is rarely likely to spread.

Newer mathematical models may improve fire forecasts and replace indices like the BI. Statistician Frederic Schoenberg collected and analyzed historic wildfire data in order to build statistical models that clarify relationships such as the apparent linear association between wildfire hazard and average temperature for those that fall below 21 degrees Celsius. While drought is a demonstrated predictor of fires, climatology statistician Sam Shen, atmospheric physicist Robert Field, and earth scientist Guido van der Werf also linked fires in Indonesia with changes in land use and population density. These types of studies have led to quips that only mathematics can prevent forest fires.

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Holly Hirst

See Also: Earthquakes; Floods; Forecasting.

Fuel Consumption

Category: Travel and Transportation.
Fields of Study: Algebra; Measurement.
Summary: Vehicle fuel consumption and efficiency are often mathematically investigated.

Fuel consumption can be defined as the amount of fuel used for each unit of measurement (usually time or distance). An often mistaken meaning is fuel economy, which is the reciprocal of fuel consumption: the amount of distance or time for each unit of fuel used. In addition to people in all walks of life using mathematics to measure fuel consumption, mathematicians research fuel optimization.

In one case, Pontryagin's Maximum Principle, named after mathematician Lev Pontryagin, characterizes optimum values that determine a trajectory, such as fuel consumption or flight time. On the other hand, the counterintuitive assertion that greater fuel efficiency often results in increased fuel consumption is sometimes known as the Jevons Paradox, after economist and logician William Jevons. Mathematicians are also involved in research for alternative fuel sources for vehicles, such as biodiesel and electrical power.

Fuel consumption is calculated for various reasons, including budgeting and maintenance. If a business that uses fuel knows the average amount of time or average distance traveled by its machines and the fuel consumption for each unit, it can calculate the approximate amount of money needed for purchasing fuel over the next fiscal period. Tracking fuel consumption on a regular basis can indicate a potential breakdown of internal engine parts before the issue becomes a major repair or hazardous situation.

Calculating Fuel Consumption

Calculating fuel consumption is a fairly simple process if you have a way to measure both the time or distance the machine was used and the amount of fuel used to refuel the machine. For example, machines that are designed for travel, like cars, trucks, vans, or tractor-trailers contain an odometer to record the number of miles or kilometers traveled. Many even have a trip odometer that can be reset after refueling. To calculate fuel consumption, start by having the vehicle completely filled with fuel and the odometer reading recorded or reset.

After using the vehicle, fill its tank with fuel and measure the amount of fuel that has been added. The assumption here is that the amount of fuel added to bring the tank back to its full position would approximate the amount of fuel used since the last time the vehicle was fueled. At the same time that the vehicle is refueled, also record the odometer. A trip odometer indicates the exact distance traveled since the last fill-up (the distance traveled since it was last reset). If not using a trip odometer, take the current total distance traveled and subtract the previous reading taken at the last fill-up.

Now that the distance and amount of fuel has been measured, calculating fuel consumption is the simple division problem $C = F \div D$, where C is the fuel consumption, F is the amount of fuel used, and D is the distance traveled.

For example, a vehicle that traveled 400 miles on 20 gallons of fuel has a fuel consumption of $20 \div 400 = 0.05$ gallons per mile, meaning that five-hundredths of a gallon (6.4 fluid ounces) of fuel was used to travel each mile. In Europe, Australia, and other countries (like Canada and China) fuel consumption is calculated in liters per 100 kilometers traveled.

A vehicle that traveled 600 kilometers on 75 liters of fuel would have a fuel consumption of $75 \div 600 = 0.125$ liters per kilometer. To get liters per 100 kilometers, multiply the result by 100 to get 12.5 liters per 100 kilometers. When looking at fuel consumption, a lower number is better than a higher number, meaning you use less fuel to achieve the same distance.

Some countries use fuel economy; for example, the United States uses miles per gallon and Japan uses kilometers per liter. The formula for fuel economy (E) is $E = D \div F$.

In the above examples, $400 \div 20 = 20$ miles per gallon (mpg), and $600 \div 75 = 8$ kilometers per liter. For fuel economy, a larger number is desired, meaning a greater distance can be traveled using the same amount of fuel.

Not all machines were designed to travel, such as forklifts and construction equipment. Generally, these machines do not measure the distance they have traveled but rather the number of hours the machine has been in use. Many of these machines have an "hour meter" that measures the time the machine operates. For example, if a forklift uses 5 gallons of fuel over an 8-hour shift, fuel consumption is found by the formula $C = F \div T$ where T is the time the machine is in use. In the above example, $5 \div 8 = 0.625$ gallons per hour.

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Chad T. Lower

See Also: Auto Racing; Energy; Measurements, Length.

Function Rate of Change

Category: History and Development of Curricular Concepts.
Fields of Study: Algebra; Calculus; Communication; Connections.
Summary: The rate of change of a function is a key focus of differential calculus.

Though calculus has a reputation for impenetrability compared to algebra and geometry, one of its two main branches, differential calculus, is concerned with derivatives, or rates of change. Rate of change is intuitively understood: the stock market is falling, but how fast? The rolling ball is slowing down, but when will it stop? A derivative is the rate of change of a mathematical function, discovered through a process called differentiation.

History and Language of the Study of the Rate of Change

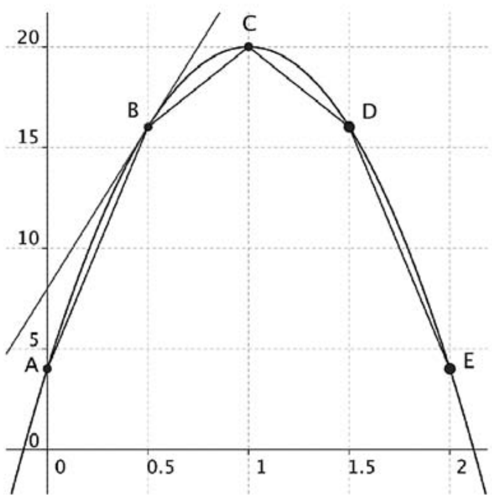
The ancient Greeks wrestled with the concept of change. Parmenides of Elea asserted that change is impossible, while Heraclitus of Ephesus believed that everything changes and nothing remains still. Aristotle accepted some forms of change but he denied a "change of change" related to motion. Historians have commented that a lack of recognition of a rate of change of a velocity was the major stumbling block to the development of calculus by Archimedes of Syracuse almost 2000 years before Sir Isaac Newton.

The language that describes changing quantities can still be confusing. A politician announces that a proposed spending bill will "cut the deficit" because it will "lower the rate at which the deficit is growing." Geologists announce that global oil production continues to increase and that "the rate at which production is growing is decreasing." Does this mean that the rate of global oil production itself may soon also decrease? Reflecting on a particular twenty-first century recession, economists shared that "the rate at which jobs are being lost is decreasing." Are these announcements good news or bad news?

Average and Instantaneous Rates of Change

In order to understand these statements, several key ideas about functions must be investigated; in particular, it is important to know what it means to say that a function is "increasing" or "decreasing," as well as whether a function's rate of change is increasing or decreasing. A familiar physical situation is helpful to consider. Let a ball be tossed into the air and follow its height above the ground at time *t*.

In the late 1600s, Sir Isaac Newton correctly conjectured that the ball's height can be modeled by a certain quadratic function. Consider the function $y = h(t) = -16t^2 + 32t + 4$, where h is measured in feet and t is measured in seconds, and observe its graph below. The goal is to study how the function changes as time moves forward and, hence, to understand what is meant formally by "the rate of change of the function."



From our understanding of quadratic functions, we can observe that at time $t = 0$, when the ball is tossed, its height is $h(0) = 4$ feet. The ball lands when its height is $h = 0$, which occurs for the positive value of t that satisfies $-16t^2 + 32t + 4 = 0$; the quadratic formula indicates that the positive t that satisfies this equation is $t = 1 + \sqrt{5}/2 \approx 2.118$ seconds.

Finally, since the vertex of the parabola occurs at $t = -32/(2\cdot(-16)) = 1$, it follows that the maximum height the ball reaches is $h(1) = -16\cdot 1^2 + 32\cdot 1 + 4 = 20$ feet. Clearly the ball is going up on the interval from $t = 0$ until $t = 1$, and the ball is going down thereafter. Perhaps a more interesting question is "*how* is the ball going up and going down?" Or, "how fast is the ball rising or falling at a particular moment?" For instance, consider the interval $[\frac{1}{2}, 1]$.

On that interval, the ball rose 4 feet, since $h(1) - h(1/2) = 20 - 16 = 4$. In addition, half a second of time elapsed. This knowledge shows that the function's "average rate of change" on the time interval $[\frac{1}{2}, 1]$is

$$\frac{h(1) - h(\frac{1}{2})}{1 - \frac{1}{2}} = \frac{20 - 16}{\frac{1}{2}} = 4 \times \frac{2}{1} = 8 \text{ feet per second.}$$

The units on this quantity are important: the numerator is measured in feet, while the denominator is in seconds, so the overall units are "feet per second," reflecting the rate of change of height with respect to time. The algebraic form of the average velocity on the time interval $[a, b]$,

$$\frac{h(b) - h(a)}{b - a}$$

is reminiscent of another familiar quantity: the slope of a line that passes through the points (x_1, y_1) and (x_2, y_2) is given by

$$\frac{y_2 - y_1}{x_2 - x_1}$$

Straight line segments are used to model and approximate the parabolic function, for example, from point B to point C in the graph. Hence, it is seen that the average rate of change of the function h on a given interval is understood visually to be the slope of a line that passes through two points on the graph of *h*.

The average rate of change of the function quantifies how fast the ball is rising or falling and will vary on different intervals. This accurately reflects that the ball is "falling faster," since its average rate of change is more negative than on the preceding interval. For instance, on the interval $[0, \frac{1}{2}]$, the average rate of change of *h* is

$$\frac{h(\frac{1}{2}) - h(0)}{\frac{1}{2} - 0} = \frac{16-4}{\frac{1}{2}} = 12 \cdot \frac{2}{1} = 24 \text{ feet per second,}$$

while on the interval $[\frac{1}{2}, 1]$, the average rate of change is

$$\frac{h(1) - h(\frac{1}{2})}{1 - \frac{1}{2}} = \frac{20-16}{\frac{1}{2}} = 4 \cdot \frac{2}{1} = 8 \text{ feet per second.}$$

These quantities confirm numerically what we can see visually from the graph: the ball is rising faster during the first half-second than it is during the second half-second. What happens in the subsequent half-second?

Similar computations, shows that on the time interval $[1, \frac{3}{2}]$, the average rate of change is

$$\frac{h(\frac{3}{2}) - h(1)}{\frac{3}{2} - 1} = \frac{16-20}{\frac{1}{2}} = -4 \cdot \frac{2}{1} = -8 \text{ feet per second.}$$

Here, for the first time, a negative average rate of change is encountered; the minus sign is extremely important, as it is the numerical indicator that the ball is falling. From the symmetry of the parabola, one can expect (and can calculate to check) that on $[\frac{3}{2}, 2]$, the average rate of change of *h* is −24 feet per second. This result accurately reflects that the ball is "falling faster," since its average rate of change is more negative than on the preceding interval.

It is next natural to seek to understand the difference between the ball's average rate of change on a time interval and its "instantaneous" rate of change at a single value of *t*. By taking average rates of change on smaller and smaller time intervals, one encounters a remarkable phenomenon. For instance, consider the average rates of change on [0.5, 1], [0.5, 0.6], [0.5, 0.51], and [0.5, 0.501]. The average rate is 8 feet per second on the first interval; on the next interval, the function's average rate of change is

$$\frac{h(0.6) - h(0.5)}{0.6 - 0.5} = \frac{17.44 - 16}{0.1} = 14.4 \text{ feet per second.}$$

On [0.5,0.51], similar computations reveal that the average rate of change is 15.84 feet per second, while on the final interval, [0.5, 0.501], the rate is 15.984. Here, despite the fact that one is dividing by numbers that are getting closer and closer to zero (0.5, 0.1, 0.01, 0.001), it can be seen that the resulting quantities themselves seem to be settling down, nearer and nearer a single number. Calculus is the mathematics that allows these ideas to be made precise. The notion of limits and other key related ideas allow mathematicians to move from the notion of average rate of change to instantaneous rate of change and indeed the instantaneous rate of change of the ball's height with respect to time at the time $t = 0.5$ is 16 feet per second. By considering the corresponding line segments that pass through two points on the curve, the so-called secant lines actually approach a single line that is "tangent" to the curve at the point (0.5, 16), as pictured in the graph as point B. The red line touches the curve only at (0.5, 16), has slope 16, and represents the instantaneous rate of change of the ball's height with respect to time at the moment $t = 0.5$.

The Beginnings of Calculus

This idea of moving from average rates of change to instantaneous ones is the starting point for the entire subject of differential calculus. Abu Arrayhan Muhammad ibn Ahmad al-Biruni investigated instantaneous velocity and acceleration approximately 1000 years ago, and Isaac Barrow may have been the first to draw tangents to curves in 1670. The development of calculus led to a rich collection of concepts centered on the idea of a rate of change, many of which were introduced by Isaac Newton in his attempts to develop a universal theory of gravitation. For instance, Newton attempted to avoid the use of the infinitesimal by forming calculations based on ratios of changes and he determined the area under a curve by extrapolating the rate of change. In fact, Newton's second law states that the rate of change of momentum of a body is equal to the force acting on the body in the same direction. Gottfried Leibniz also investigated concepts related to rate of change. He explored maxima, minima, and tangents in 1684, but mathematicians had difficulty understanding his six-page work.

Other notions of rate of change are also important in mathematics and in real-life applications. For example, in 1847, Jean Frédéric Frenet assigned a frame of vectors to each point on a curve and described the twists and turns of the curve by the rate of change of the frame. Joseph Alfred Serret independently considered similar ideas in 1851. The Frenet–Serret frame continues to be useful in the early twenty-first century when it is impossible to assign a natural coordinate system.

Many real-life problems, such as population growth, can be expressed and modeled as an equation involving a quantity and its rate of change. All of these ideas rest in some way on the fundamental concept of slope, which is investigated beginning in the middle grades, while the notion of rate of change is first developed in high school. Other methods to solve these types of problems are studied in the field of differential equations, which is usually introduced in college.

Applications of Rates of Change

Returning to two of the original questions about the meaning of certain statements and whether they are good news or bad news: a proposed spending bill will "cut the deficit" because it will "lower the rate at which the deficit is growing." This is not great news, since the deficit is still growing, but a deficit growing at a decreasing rate is better than one growing at an increasing rate. It would be much better to hear that the budget deficit itself was decreasing. Next, the information that "the rate at which oil production is growing is decreasing" may likely mean that the rate of oil production could be leveling off and soon start to decrease—the concept of "peak oil" (when the rate of daily global oil production reaches its maximum)—and many analysts believe humans have just passed this peak and that the rate of oil production will only continue to fall from here.

With the Earth's human population growing at a present rate of 83 million people per year, as well as so many other changing quantities, collective efforts to understand resource allocation and management require sound understanding of rates of change and trends in data. Calculus and its language of change are a key tool in building a sustainable future for humanity and the planet.

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Matt Boelkins

See Also: Calculus and Calculus Education; Climate Change; Functions; Linear Concepts; National Debt; Water Distribution.

Functions

Category: History and Development of Curricular Concepts.
Fields of Study: Algebra; Communication; Connections.
Summary: There are many different types of functions that arose in a variety of historical contexts.

A mathematical function expresses the idea that one quantity can be completely determined by another quantity. As such, a function is a well-defined rule between two sets, A and B, where each element x of A is assigned to one element y of B. The careful study of the implications and applications of this definition comprises much of mathematics. Functions are ubiquitous throughout nearly all fields of mathematics and their importance cannot be overstated. Since a function expresses a relationship between an independent variable and a dependent variable, many real-world phenomena are modeled using functions. Functional correspondence between variables can be expressed verbally, algebraically, or graphically. Although the formal definition of a function is a relatively recent development, this concept has been implicit since the beginning of mathematics.

Examples of Functions

Functions can be classified in various ways and many of these specific classes of functions are the bases of entire fields of study within mathematics. Below are a few examples to illustrate some areas encompassed by the definition of a function. The algebraic equation $y = x^2$ expresses y as a function of x. No matter what value has been chosen for x, only one y value will be produced.

The expression $y = \pm x$ does not represent y as a function of x. If $x = 1$ is entered into the equation, then two y values are obtained, $y = -1$ and $y = 1$.

The expression $y = \cos(x)$ is a trigonometric function. For any given angle expressed in radian measure, the cosine function uses a rule from trigonometry to produce a real number from -1 to 1.

An example of an exponential function is given by the formula $y = 2^x$. For instance, when $x = 2$, $y = 4$. Exponential functions are useful for situations in which the rate of growth of a population is directly proportional to the size of the population, such as modeling the growth of bacteria or the decay of a radioactive sample.

A piecewise function is defined by different formulas for different values of x that are entered into the function. The Heaviside function is an example of a piecewise function: $y = 1$ for $x \geq 0$, and $y = 0$ for $x < 0$.

If one assigns "on" to 1 and "off" to 0, this function models the behavior of turning on a switch at $x = 0$ and leaving it on. It is also used in the study of electric circuits to indicate the surge of an electric current.

A barcode scanner at a supermarket can be thought of as a function. After a particular barcode is scanned, only one price will be displayed.

A computer program that obtains the five-digit zip code for an address acts as a function because every address in the United States has only one zip code assigned to it.

Equivalent Formulations

A helpful way to think about a function is as a machine that produces exactly one output y for each input x. This "machine" could simply be a description or a list of the pairings between x and y. Although this may be easier conceptually, in practice it is unfeasible to list all of the pairings when there is a large number (or even an infinite number) of x values. When one considers an infinite number or a large number of x values, it is more advantageous to have a mathematical formula that precisely relates x and y.

Graphs of Functions

Another way to represent a function is by using a graph. One begins with a set of x values from any subset of the real number line and a function. The function then specifies a y value for each x. This results in a collection of pairings (x, y). Each of these pairs denotes a point, which is plotted on the xy-plane in the two-dimensional Cartesian coordinate system. The collection of all points produced by this process is the graph of the function.

By virtue of the definition of a function, every x value on the graph is paired with, at most, one y value. Thus, any vertical line that is drawn will cross the graph of the function at most one time. This is known as the "vertical line test": a curve in the xy-plane is the graph of a function if—and only if—no vertical line crosses the curve more than once. As a consequence of the vertical line test, given a curve, it is relatively easy to determine if it is the graph of a function. All non-vertical straight lines are graphs of functions. Circles are not graphs of functions.

History

The notion of a function has been implicit throughout the history of mathematics. Addition is the most fundamental arithmetical operation and, although it was not initially formulated as such, it is a function of two variables. The pair of numbers to be added is the input and the resulting sum is the output of the addition function. Ancient cultures, such as the Babylonians, developed extensive tables of mathematical calculations of the reciprocals and square roots of positive whole numbers. These calculations involve specific functions but were not formulated using the function concept.

In the fourteenth century, Nicole Oresme had a rudimentary grasp of the idea that one changing quantity can be dependent upon another. He depicted this relationship graphically using a method he called the "latitude of forms." This depiction was the first known attempt of the graphical representation of a function. Throughout the Middle Ages, the latitude of forms continued to be studied; however, further development of the function concept was hampered by the absence of a suitable algebraic framework.

The formal study of functions began in the late seventeenth century with the discovery of calculus. Although in 1692, Gottfried Leibniz first introduced the word "function" in association with the tangent problem in calculus, its first definition did not emerge until 1718 with Johann Bernoulli. The primary representation of a function at that time was from curves that were connected to physical problems.

As the eighteenth century unfolded, algebraic equations were increasingly being used to represent two-dimensional curves. As a result, the emphasis and focus of the function concept evolved from a graphical setting to that of algebra. This shift was evident in Leonhard Euler's 1748 treatise on functions, *Introductio in analysin infinitorium*. Euler's definition of a function was that of an "analytic expression" or an algebraic formula that could contain any combination, the five arithmetic operations: exponentials, logarithms, trigonometric ratios, derivatives, and integrals. Euler's emphasis on the algebraic formulation of functions was evident, as the first volume of the *Introductio* contains no graphs.

The middle of the eighteenth century saw a development of the function concept when a controversy arose over the solution to the Vibrating String Problem. Given an elastic string with fixed endpoints and deformed into an initial shape, the string was released and began to vibrate. The problem was to determine a function that would describe the shape of the string at any future time. In 1747, Jean Le Rond d'Alembert produced a solution in the form of an algebraic equation. A year later, Euler verified that this solution was correct but he disagreed that it was the most general. He claimed that d'Alembert had neglected several initial shapes of the string that could be drawn freehand and for which there were no algebraic expressions. Euler also pointed out that other initial shapes could be obtained by piecing together simpler curves. This critique led to the acceptance of functions produced from freehand drawing for which there may not be any algebraic formula and piecewise defined functions.

Another solution to the Vibrating String Problem further complicated matters. In 1753, Daniel Bernoulli solved the problem differently than Euler and d'Alembert and arrived at a seeming contradiction: different mathematical expressions defined the same function. The controversy was not resolved at the time; it remained for Joseph Fourier to expand upon Bernoulli's idea. Fourier's solution to the Heat Conduction Problem in 1807 resulted in a revolution of the understanding of a function. Fourier demonstrated that a function could be expressed as an infinite series of sine and cosine functions, now known as a Fourier series. These series demonstrated that two different expressions could define the same function. Following the development of Fourier series, the connection between the geometric and algebraic forms of a function was further solidified. Furthermore, ideas from calculus were reexamined in a new light.

In 1837, Lejeune Dirichlet suggested a definition of "function" that was closely related to the modern definition. Dirichlet emphasized that a function provides a relationship between two variables but allowed for freedom in describing the rule that describes how x and y are related. To show how pathological a function can become, Dirichlet introduced the function $y = c$ for x an irrational number, and $y = d \neq c$ for x a rational number.

This badly behaved function cannot be sketched and there is no algebraic equation defining it.

Since the nineteenth century, it was a natural evolution to recast Dirichlet's definition by using set theory. The modern definition for a function now provides a correspondence between two sets, which may or may not be numerical; for example, functions between algebraic structures like groups or geometric objects like surfaces.

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See Also: Calculus and Calculus Education; Function Rate of Change; Functions, Recursive; Limits and Continuity; Number and Operations.

Functions, Recursive

Category: History and Development of Curricular Concepts.

Fields of Study: Algebra; Communication; Connections.

Summary: Recursive functions describe the output value of a function in terms of other output values of the function.

Recursive functions, by iterating certain defined procedures, can provide a succinct way of describing a multistep algorithm or calculating an intricately defined number. Long used in mathematical problem solving, recursive functions became indispensable with the advent of computer programming, as they provide a method of concisely encoding repetitive processes.

Many mathematicians explored ideas related to recursive functions, including nineteenth-century mathematicians Richard Dedekind and Giuseppe Peano. In the early and middle part of the twentieth century, the development of recursion theory is tied to questions about computability and the foundations of mathematics.

Alan Turing and Kurt Gödel preferred the term "computable" over "recursive" but the latter terminology has been common since the 1930s. Mathematician Rosza Peter is noted as a founding mother of recursion theory who, along with other researchers like Alonzo Church, Gödel, Jacques Herbrand, Stephen Keene, Andrey Markov, Emil Post, and Turing, developed the area. In the twenty-first century, high school students explore recursive functions in mathematics classes, and college students apply ideas from a related topic, the principle of mathematical induction, in proofs.

What Is a Recursive Function?

Recursive functions describe the output value of the function in terms of other output values of the function, usually for smaller input values. In this case, to avoid an infinite recursion loop, one must explicitly specify at least one specific output value. For instance, one could say that the value of a certain function at some counting number is equal to that input value multiplied by the value of the function at the next smaller input value—algebraically,

$$f(n) = n \times f(n-1).$$

Then one would need to define a specific value of the function. Say, at 0, the function is equal to 1; that is, $f(0) = 1$. This specification effectively defines the value of f at each whole number, although it may require a few steps to get there. For example, if one wanted to determine $f(3)$, one would first see that

$$f(3) = 3 \times f(2).$$

However, $f(2) = 2 \times f(1)$, and $f(1) = 1 \times f(0) = 1$.

So, working backwards, $f(1) = 1$, $f(2) = 2$, and $f(3) = 6$. This example, known as the "factorial function," plays a key role in combinatorics and probability, where $f(n)$ is written $n!$ and equals the number of ways to arrange n objects in an ordered list.

Early Uses of Recursion

Dating from ancient Egypt (c. 1650 B.C.E.), Problem 79 of the Rhind (or Ahmes) Papyrus describes an estate containing 7 houses, 49 cats, 343 mice, 2401 heads of wheat, and 16,807 hekat measures (of grain) and gives the total of all these numbers as 19,607. The list contains powers of the number 7. In 1907, Moritz Cantor interpreted this list as a possible precursor of a modern nursery rhyme. He proposed: an estate has 7 houses; each house has 7 cats; each cat can catch 7 mice; each mouse eats 7 heads of wheat; and each head of wheat produces 7 hekats of grain. What is the total of these numbers? Or, for a different question, because of all the mouse-eating cats in all the houses, how many hekats of grain were saved on the estate? This calls to mind the familiar "As I was going to St. Ives" nursery rhyme, with its final question, ". . . kits, cats, sacks, wives, / How many were going to St. Ives?" [*The answer is one.*]

The Rhind Papyrus problem can be posed as a simple recursive function via iteration in which the output of a function is used as the same function's next input value and this process is repeated a preordained number of times. In this case, one could use the function that multiplies the input value by 7: $f(x) = 7x$. To obtain the number of houses, input the number of estates into the function, obtaining $f(1) = 7$. Then, to determine the number of cats, input the number of houses into the function: $f(7) = 49$.

Thinking recursively, this is calculating $f(f(1))$. The number of mice is then

$$f(f(f(1))) = f(f(7)) = f(49) = 343, \text{ and so on.}$$

Simply put, to obtain the next term in the sequence, perform the function on the previous term.

A similar problem appeared in Leonardo Fibonacci's 1202 work, *Liber Abaci*. In the same work, he also posed another famous problem to determine how many pairs of rabbits there would be at the end of 12 months, starting with one mature breeding pair and assuming that each mature pair breeds one pair of offspring each month and that the new offspring must wait one month until they become a mature breeding pair. The sequence of the total numbers of pairs of rabbits in each month proceeds 1, 1, 2, 3, 5, 8, 13, ... Fibonacci noted that the number of rabbit pairs, from the third month on, is equal to the sum of the number of pairs in each of the two previous months. Today, this sequence is called the "Fibonacci sequence" and its entries are called "Fibonacci numbers."

Often, the Fibonacci sequence is defined as a recursive function, this time with two starting values: $f(1) = 1$ and $f(2) = 1$. Then, the Fibonacci sequence is

$$f(n) = f(n-1) + f(n-2)$$

for all positive integer values of n greater than 2. The Fibonacci sequence has appeared in botany, specifically in phyllotaxis, the method of leaf formation. The number of spirals in a sunflower head, a pineapple, or a pinecone are often Fibonacci numbers.

Recursion in Computer Programming

Recursive functions play a key role in computer programming, as they allow the programmer to encode a possibly lengthy algorithm in a relatively short number of steps. For instance, to calculate the factorial function mentioned above using a computer, the "non-recursive" approach could be to store several values of the function in memory and return those values when needed.

This approach could take an unlimited amount of memory, because each value of the function would require its own memory space. The recursive approach is simpler—the factorial of a number can be encoded in two statements, thus allowing the computer to calculate the factorial function for any positive integer efficiently. Some sample "pseudocode" follows:

```
Function Factorial(input):
If input = 0, then Factorial(0) = 1;
Else, Factorial(n) = n × Factorial(n – 1);
End Factorial.
```

Some computer games require players to demonstrate recursive programming skills. For example, Robozzle asks the player to program a spaceship to collect all the stars on the screen but to do so with a limited number of commands. The Tower of Hanoi puzzle, marketed by French mathematician Edouard Lucas in 1883, is often used as an example of recursion in classrooms. Often, recursion is necessary to complete the task. For instance, to move the rocket ship forward indefinitely using recursion, one could simply enter a command to move the ship forward and a command to go back to the beginning of the program, which would then move the ship forward and then go back to the beginning of the program again and again. It is often surprising to see the intricate patterns that can be programmed relatively succinctly using recursive functions.

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Christopher Goff

See Also: Egyptian Mathematics; Functions; Middle Ages; Sacred Geometry; Social Networks.

G

Game Theory

Category: Games, Sport, and Recreation.
Fields of Study: Algebra; Number and Operations.
Summary: Game theory models various real-world and hypothetical situations as "games," the play and strategy of which can be analyzed mathematically.

Game theory is the branch of mathematics dedicated to analyzing strategic behavior in different situations. It attempts to describe situations in which several people or entities must make choices even when the outcomes of their decisions rely on the choices made by others. While game theory can be used to address situations typically thought of as games, such as checkers and poker, it can also be used to study situations that are extremely practical and important, such as strategies to use in military operations or auctions and the evolution of species. As with many areas of mathematical modeling, approaching a problem in game theory first typically involves quantifying the objectives and options in terms of algebraic equations and then finding the choice that gives the highest possibility of maximal success.

History of Game Theory

People have studied games and strategies for centuries, but game theory came into its own as a branch of applied mathematics when John von Neumann proved what is known as the "minimax theorem" in 1928. This theorem considers games played between two players in which each player chooses one of a finite number of options, and—depending on the choice made by each player—one of the players gives a certain amount of money to the other player. This is commonly referred to as a "zero-sum game," as the losses incurred by one player exactly equal the gains won by the other player. Von Neumann was able to prove that there is a unique strategy that will maximize a player's winnings (or minimize losings), and one can find this strategy by considering the worst-case outcomes of each of the player's choices and choosing the best-possible, worst-possible outcome. In particular, one would typically like to choose an option that leaves the player indifferent to the choice made by his or her opponent. This work was later expanded by von Neumann and Oskar Morgenstern in their book *Theory of Games and Economic Behavior*, which introduced game theory as a valuable tool for economists.

Rock-Paper-Scissors

An example of the type of strategy that von Neumann wrote about comes up when playing the children's game of rock-paper-scissors. In this game, each of two players chooses one of three possible options (rock, paper, or scissors), and—depending on the choice made by each player—one of the two is declared the winner. In particular, rock beats scissors, scissors beats paper, and paper beats rock. If the two players make the same choice, the game is declared a tie. No matter which choice an opponent makes, one of a player's three options will result in a win, one will result in a loss, and one will result in a tie. Therefore, if the player does not have any inside knowledge of what the opponent will choose, the player will do best by choosing one of the three options at random, each with a probability of one-third.

The Prisoner's Dilemma

The most famous problem in game theory is the Prisoner's Dilemma. The Prisoner's Dilemma is a non-zero-sum game in which there are two participants, each choosing one of two possible outcomes. It is most often described by the following type of story: two criminals, Alice and Bob, are arrested after committing a crime. The police isolate the two prisoners and interrogate them separately. Each criminal must choose whether to confess or to deny the crime, without communicating with the other prisoner. If both confess, they will each get three years in jail. If both deny the crime, there will not be enough evidence to convict them of the felony, but both will get one year in jail. If Alice confesses and Bob denies the crime, then Alice will go free and Bob will go to jail for five years, but if Bob confesses and Alice denies the crime, then Bob will go free and Alice will go to jail for five years. One can see that no matter what Alice chooses to do, Bob will be better off confessing and no matter what Bob chooses to do, Alice will be better off confessing. Because they cannot communicate, one is led to suspect that they will both end up confessing, even though they would both be better off if they both chose to deny the crime. This situation's key principle is how much the criminals trust their partner to deny the crime, rather than do what is in their own self-interest. While this story may seem contrived, it turns out to have many applications in areas such as economics, biology, and political science.

Applications of Game Theory

Much of the research on the Prisoner's Dilemma, as well as other areas of game theory, has taken place at the RAND Institute, a nonprofit think tank originally set up by the United States Army and the Douglas Aircraft Company with a mission "to help improve policy making through research and analysis." Along with then defense secretary Robert McNamara, they developed the game theoretic concept of mutually assured destruction (MAD), which leads to a military doctrine of nuclear deterrence. The idea is that if one country launches a nuclear attack on another, then the conflict quickly escalates until the whole planet is destroyed, and, therefore, such an attack will never take place. This concept has been critiqued by many scholars, but is still an influence on foreign relations today.

John Forbes Nash, Jr.

John Forbes Nash, Jr., is a mathematician who worked extensively in game theory, in addition to work in algebraic geometry and topology. His best-known work involved finding solutions to games that are not zero-sum games so that players can collectively get better outcomes if they work together than they will get if they work against one another. Nash was born in Bluefield, West Virginia, in 1928 and received his undergraduate degree from Carnegie Mellon University. His dissertation, completed in 1950 at Princeton University, defined the concept that has become known as "Nash Equilibria," which are pairs of choices that two players can make in which neither player is tempted to change their choice. His theory was that most games will eventually work their way to such a situation if they are played repeatedly.

This work, along with subsequent work in this area, led to Nash's being awarded the Nobel Prize for Economics in 1994. In addition to being a mathematician, Nash was a schizophrenic and has spent much of his life dealing with treatments for paranoid schizophrenia, including several prolonged stays in mental hospitals. His life story is the subject of the book and film *A Beautiful Mind.*

A Game of Pistols

Three people have decided to settle a conflict by firing at each other with pistols. Mr. Pink has a one-third chance of succeeding and killing his opponent, while Mr. Blue has a two-thirds chance, and Mr. Orange is guaranteed to succeed. To even this score, the men will take turns with Mr. Pink taking first shot, followed by Mr. Blue, and then Mr. Orange. A natural question in this situation is whether Mr. Pink should shoot at Mr. Blue or Mr. Orange first. It turns out that the answer is neither. Using game theory, one can show that Mr. Pink has a better chance of surviving if he shoots into the air and intentionally misses both of the other players.

While most games in the real world deal with situations in which the players do not have full information or in which there is an element of chance, there is also a strong mathematical study of perfect information games such as checkers and Go. One famous example of such a game is Nim, a game played between two players starting with a number of objects in different piles. On each player's turn, they can remove any number of objects from a single pile. The players alternate turns, and the player to remove the final object loses. This game has been extensively studied and written about by game theorists, such as Elwyn Berlekamp and John H. Conway. It turns out that one of the two players is guaranteed to have a winning strategy, but which player it is depends on the number of piles and the number of objects.

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Darren Glass

See Also: Baseball; Basketball; Board Games; Cold War; Tic-Tac-Toe; Voting Methods.

Games

See *Board Games; Video Games.*

Garfield, Richard

Category: Games, Sport, and Recreation.
Fields of Study: Connections.
Summary: Mathematician Richard Garfield applied his understanding of combinatorics to create Magic: The Gathering, the game responsible for the collectible card game craze.

Richard Garfield was born on June 26, 1963, and is a mathematician and an inventor of card and board games. He holds a B.S. degree in computer mathematics and a Ph.D. in combinatorial mathematics and has worked as a professor at Whitman College. Garfield is probably best known for creating Magic: The Gathering, the first widely popular collectible card game, but he has also designed board games and many other card games—collectible or not.

It should not be surprising that a mathematical background would be useful in designing games. The connections between mathematics and music, architecture, dance, and other forms of art is somewhat well understood. The role of mathematics in the art of game design is at least as direct. The aesthetics of a game come from the dynamics and combinatorial interaction of its rules and various components. This principle is especially true of so-called collectible card games, such as Magic: The Gathering, in which each player's creativity in designing his or her own deck is

Players at the 2009 Magic: The Gathering World Championships. Most countries sends their top four players to the tournament, though nations with smaller Magic-playing communities may send just one player.

part of the game. The rules and the library of cards must have sufficient flexibility to accommodate a large variety of strategies and styles (keeping play interesting and dynamic), but sufficient control to prevent a single overpowered card or combination of cards from "breaking" the game.

Collectible Card Games

In a collectible card game, such as Magic: The Gathering, players buy packages containing random cards from a large universe of possible cards. Some cards are much more rare and others much more common. Players organize their cards into decks according to certain guidelines and play casually against friends or competitively at official tournaments. In order to keep the game dynamic, the universe of cards periodically grows as expansions are released and older cards are retired. Expansions are still developed today, though Garfield's direct involvement is limited and intermittent.

Much has been written about the mathematics that underlies the gameplay of Magic: The Gathering. Many relevant mathematical ideas come from combinatorics and probability, and a recurring theme is trade-offs. Including many copies of a card increases the chances of drawing it at a crucial time, but at the expense of having a smaller variety of different cards to deal with unexpected situations. An important game mechanic in Magic is that cards come in five different colors. Different-colored cards tend to have different types of effects and require different "energy" in order to function. Multicolored decks can be much more flexible, but at the expense of being much more likely not to have the energy you need at a key moment.

Richard Garfield created several other collectible card games, including Vampire: The Eternal Struggle, Netrunner, BattleTech CCG, and the Star Wars Trading Card Game. Though these have enjoyed some success and favorable opinions from critics, none can boast the mainstream attention that Magic: The Gathering received. In Richard Garfield's most famous board game, Robo Rally, players navigate robots around an obstacle course. Players construct programs for their robot out of instruction cards like "turn left" and "forward two spaces," then all robots simultaneously attempt to execute their instructions. If a player has miscalculated, or if multiple robots attempt to use the same paths at the same time, the results can be unexpected. This is a challenging game, requiring players to develop skills in game theory, logic, sequential and spatial reasoning, and the basic concepts of computer programming.

Not all of Richard Garfield's creations require the player to use mathematical skills in a conscious way. In The Great Dalmuti, players match cards from a special deck, racing to empty their hands and get the most prestigious status among the group. After each hand, players' relative rank may change, causing changes to the seating order and their privileges within the game. The elegance of the game is that it is easier to advance to a higher rank than it is to consistently hold on to the highest rank. In this case, the mathematical structure operates "quietly in the background," ensuring that the game remains dynamic and engaging.

In some cases, the mathematics in Richard Garfield's games is more explicit. One such game is Complex Hearts, a complex-number-themed variation on the classical game of Hearts. As in the original game, players score points based on which cards they take. However, in this version, the scores can be positive, negative, or imaginary, depending on the cards and card combinations, so that each player's total score is a complex number. The goal of the game is to keep the magnitude of one's score as low as possible.

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See Also: Board Games; Competitions and Contests; Game Theory; Probability.

Genealogy

Category: Friendship, Romance, and Religion.
Fields of Study: Algebra; Connections; Data Analysis and Probability; Geometry; Representations.
Summary: Mathematical methods are used to investigate genealogies in a variety of ways, including creating probability models and simulations to determine the likelihood of common ancestors.

Genealogy is the study of families, often motivated by the desire to tell the story of lineage and to place family history in a larger historical context. For instance, among Americans there is often a great interest in determining one's pre-American roots. The study of genealogy requires not only an understanding of history and the ability to work with historical primary sources of data but also mathematical structures. Ancestral charts double with every generation, and this geometric progression grows to large numbers quickly, so mathematical techniques have been fundamental in organizing and presenting family connections, both visually and in numerical formats. Mathematicians may use probability models and simulations to investigate the likelihood of common ancestors. Mathematicians also construct their mathematical genealogy, where parentage is redefined using the adviser and student relationship.

Genealogy Graph and Visualization Formats

Though people tend to think of a "family tree," genealogical graphs may overlap or be shaped differently than tree-like structures. Other representations of the data include hourglass charts, which are centered on an individual, and spread both upward and downward to show direct ancestors and descendants, eliminating relations like cousins. Exponential crowding and edge crossing are common challenges in visualizing family data, and some researchers propose a multitree arrangement.

Genealogical software typically presents a variety of visualization options. Numbering systems have long been used to identify individuals. Methods from graph theory are important in analyzing the data for connections and patterns. Another genealogical challenge is the integration of information from disparate sources, such as census information and individual recordkeeping.

While standardized software and Genealogical Data Communication (GEDCOM) files may on one level make it easier to share information, the dynamic nature of huge online genealogical graphs presents new mathematical challenges. One method used to simplify such graphs has been to deemphasize individuals who enlarge a tree but do not increase the complexity.

Brief History of Genealogy

Historically, most genealogy was the study of the kinship and descent of royal and noble families—in this form, many of the earliest histories in Egypt and ancient Rome are genealogies mixed with mythology. As a study of royalty, genealogical research generally had the ultimate goal of demonstrating or undercutting claims of legitimacy or determining a line of succession. Early American genealogical research was associated with efforts to prove kinship to noble families and was thus part of the British class system, which the egalitarian republic had outgrown. The New England genealogist and historian John Farmer (1789–1838) may have been the first to change this, as his work on local histories was seen as a way to honor and glorify the work of early Americans and the story of America's growth from loosely affiliated royal colonies to an independent nation. Farmer referred to his work—the combination of genealogy and local history—as "antiquarianism."

The trend he helped to popularize led to the creation of the New England Historic Genealogical Society (NEHGS), the oldest genealogical society in the United States, in 1845, six years after his death. Many such societies opened throughout the country, notably the Genealogical Society of Utah (1894), now associated with the Church of Jesus Christ of Latter-day Saints (LDS), which has since developed the most extensive genealogical records in the world. Because LDS beliefs focus strongly on the sealing of family units together so that they may copersist in eternity, genealogy is an especially critical concern for the faith and necessary for religious ceremonies. Later in the twentieth century, the revival of interest in ethnic identity and in ties to ethnic roots long abandoned or forgotten in the 1960s and 1970s led to a revival of interest in genealogy. This interest was furthered in the following decades as software and genetic research provided new genealogical tools, while the Internet provided a new source of information sharing.

Genealogical Numbering Systems (GNS)

A variety of numbering systems are used to quantify family relationships. One descending numbering system that traces the line of an earlier ancestor is the Register System, which was developed by NEHGS in 1870, for the purpose of simplified recordkeeping in the New England Historic and Genealogical Register.

The system groups generations separately and uses both Arabic and Roman numerals, assigning each parent a unique Arabic numeral and using lowercase Roman numerals to enumerate progeny of each parent:

1 Parent
    2 *i* Child
       ii Child (no progeny)
    3 *iii* Child

(2nd Generation)
2 Child
    4 *i* Grandchild
3 Child
       i Grandchild (no progeny)

(3rd generation)
4 Grandchild
    5 *i* Great-grandchild

Along with the Register System, the most popular GNS in the United States is the NHSQ System, named for the National Genealogical Society Quarterly, and often called the Record System. It is derived from the Register System but assigns Arabic numbers to children without progeny as well. If a new child is discovered, the family numbers must be recalculated.

An older GNS is the Ahnentafel ("ancestor table"), published by historian Michael Eytzinger in 1590. Unlike the Register and Record systems, the Ahnentafel is an ascending numbering system, beginning with "1" in the present generation and increasing as generations are traced backward through time:

1	Subject
2	Father
3	Mother
4	Father's father
5	Father's mother
6	Mother's father

7 Mother's mother
8 Father's father's father

The Ahnentafel results in the following mathematical relationship: the number of an individual's father is double that individual's number, while the number of the individual's mother is double plus one. Apart from #1, all even-numbered persons are male, and all odd-numbered persons are female. It is plain to see why this version did not become the dominant system in the United States: it is principally concerned with demonstrating to which families one has a blood relation, without accounting for siblings in any generation: nor can it be continued through the subject's children and their descendants. It was a popular system for European nobility to display the noble families to whom they were related, along with their coats of arms.

Common Ancestors

Common ancestors are individuals who are the genealogical ancestors of every person in some given set of people. Genealogists and mathematicians often try to determine how many generations into the past a tree must be traced to find such common ancestors. These models rely on statistical estimates, such as the average human lifes pan at different points in time, and average length of time between successive generations, as well as rates of reproduction. In the early twenty-first century, computer scientist Douglas Rohde, writer and editor Steve Olson, and statistician Joseph Chang collaborated to create mathematical models to estimate the most recent common ancestor (MRCA) of every human currently alive. Their initial probabilistic model, designed primarily for theoretical insight, assumed an unrealistic random mating scheme to facilitate an explicit analytical solution. A second, more realistic model required the researchers to mathematically express historical population dynamics and conduct Monte Carlo simulations to produce a distribution of feasible results with associated probabilities. According to these models, the MRCA likely lived just a few thousand years ago, perhaps during the reign of Tutankhamen, or even as recently at the start of the first century C.E.

In 2008, the genealogical Web site Geni allowed people with common ancestors to merge trees. Privacy was maintained by defining a set distance from the ancestor in which the information would be viewable. This defined distance has changed over time. The rate at which the trees enlarged often increased as the size of the tree increased. One particularly large, connected component of Geni's graph is known as the "big tree" and represented over 35 million people in 2010.

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See Also: Graphs; Mathematics Genealogy Project; Six Degrees of Kevin Bacon; Social Networks.

Genetics

Category: Medicine and Health.
Fields of Study: Data Analysis and Probability; Number and Operations.
Summary: Bioinformatics and probability theory come into play in the study of genetics.

Issues related to genetics are no longer exclusively discussed in academic circles. The lay community every day accesses a large amount of information through the mass communication vehicles that enable the socialization of knowledge related to heredity and biotechnology. Paternity tests, transgenic plants, early diagnoses in medicine, gene therapy, and cloning are no longer exclusive subjects of specialized research centers and can be easily researched in the media and found in movies, cartoons, and on the Internet. These are examples of how closely aligned this area of science is to modern society and how broad the possibilities are for development. Mathematical tools are essential for the analysis and interpretation of data related to these genetic processes that otherwise would become empty of real meaning. From the simple knowledge of probability to the most powerful algorithms associated with genetic engineering techniques, probability is necessary to elucidate the most difficult questions surrounding the genetics field. Sharon Grossman is one mathematician who has notably contributed to genetics research through her investigations of gene groupings based on geographic location.

Genetics is the field of biology that studies the chemical nature of hereditary material and the mechanism responsible to transfer information contained in genes. In general, reproduction is constituted by a series of events that result in a randomized combination of gametes. This process involves the mixing of thousands of information packets and results in the production of a new living being.

Early Findings

The first steps of genetics were performed by Austrian Gregor Mendel (1822–1884), who, for many years, crossed varieties of peas. After obtaining numerous generations of these plants, he observed differences in the types of progeny formed and identified the proportion of each of these features in future generations. His main findings showed that specific factors were transmitted by parents to offspring. He also found that these factors occur in pairs and that their descendants receive one from each parent. Crosses made with peas (called "hybrids") had particular characteristics, like seed color. By calculating their frequencies, Mendel realized that the prevalence of these factors was different in several generations. Some manifest themselves only when appearing in double dose (recessive), while others in a single dose determined the characteristic (dominant). These findings served as the basis for developing laws on inheritance, which came to be called the "first and second laws of Mendel."

Genetic Probability

Probabilities are used to express the chance of occurrence of an event. They represents a possibility, not a conviction. The probabilities can be expressed in several ways, including fractions, percentages, and decimals. For example, the chance of occurrence of a biological event can be expressed as "50%," "0.50," or "1/2." Many genetics calculations are solved using probability. Mendel used mathematical rules previously used for common events, such as a coin toss (individual events), or combined events, such as the simultaneous release of multiple dice.

Genotype is the set of genes from one living being, the frequency of these genes, and can be calculated mathematically. The calculations performed in the theory of probability do not determine the appearance of a particular genotype—they merely represent the chance this event will occur. In practical terms, genetic calculations allow one to determine the probability, for example, of two individuals with dark eyes to conceive a child with blue eyes (a recessive gene). This event is possible if both parents

Powerful computers can analyize a patient's DNA data, and have revealed that most of the DNA molecule is not involved in protein-coding.

are hybrids, in which case the probability in each pregnancy is 25%.

The application of rigorous scientific method and careful statistical research of some characteristics led Mendel to conclusions that still underlie modern genetics in the early twenty-first century. Not until in 1900 could the work of three independent researchers—Hugo de Vries, Karl Correns, and Erich Tschermak—show that Mendel's conclusions were correct.

Modern Genetics Research

For a long time it was believed that protein was the molecule that contained genetic information. Biochemical studies allowed the identification of a molecule able to replicate and thereby allow a flow of identification information: deoxyribonucleic acid (DNA)—the molecule associated with heredity. In 1953, James Watson and Francis Crick published in *Nature* the model of the DNA molecule. Understanding the complex spatial geometry of DNA allowed researchers in the early 1960s to prove that the code was formed by groups of three nucleotides that were repeated in complementary sequences. It was noticed that sequences of the DNA molecule were able to be expressed as proteins with the participation of another nucleic acid: ribonucleic acid (RNA). Studies of the most primitive life forms, like bacteria, also led to knowledge related to peculiarities of DNA activity, as well as its transmission and their biochemical behavior.

The challenge became the elucidation of the genome (the entire set of genetic information that is found in the chromosomes) from a living organism. After advancing with some simple life forms, such as bacteria and protozoa, the Human Genome Project (1988–2003) arose. International cooperation efforts were necessary to decipher the sequence of 3 billion base pairs of DNA subunits found in human chromosomes. Powerful computer programs and the use of combinatorial analysis revealed that most of the DNA molecule is not involved in protein-coding. It is now known, however, that the role of this DNA is very significant, especially for matters pertaining to evolution, and it is responsible for many adaptive differences between species.

Genetic Variability

The prevalence of certain genes in a population depends on how the expression of a particular feature is selected by the environment and is related to the presence of other genetic variation factors, such as genetic mutations, numbers of crosses, or natural events that abruptly decrease the frequency of certain genes in a population (for example, earthquakes, fires, or floods).

How is it possible to evaluate this natural dynamic that sometimes takes decades or even centuries to occur? Since a group within the set of genes undergoes a random process of transmission, it cannot be adequately studied without resorting to mathematical tools to assess the frequency of certain genes in a population and the possible consequences of this variability for that group. Wild populations (animals, plants, or, specifically, humans) are subject to phenomena—such as gene recombination, mutation, and gene conversion, which is the change of position of genes within a chromosome—that lead to the emergence of genetic variability. Genetic mathematics aims to understand how genetic changes occur for individuals both within species and over time.

Several phenomena are responsible for genetic variability. Crossing-over, for example, is a phenomenon in which parts of chromosomes are broken and glued in different positions, generating a larger mix of information and expression of phenotypes (physical or physiological), which results in an increased possibility of adapting to the environment in which the individual belongs. Random events observed in gene transfer result in the formation of functional characteristics and patterns that may often cause trouble and injury, but that is partly responsible for the possibility of evolution.

Bioinformatics

Genetics is an area of study that uses the theories of probability and the handling of large volumes of data. The difficulties in performing complex calculations—far more advanced than the calculations made by Mendel—necessitated the use of information technology in studies of biological and genetic research. Bioinformatics is the application of computer systems in the processing of biological and biomedical data. It is an interdisciplinary science that aims to develop and apply computational techniques to study genetics, molecular biology, and biochemistry. Without this tool, it would be impossible to perform thousands of mathematical operations in real time.

In bioinformatics, gene sequences are analyzed and stored in databases, manipulated, and analyzed using specific software. Databases allow scientists to get information from other laboratories and also to share the genetic sequences. Despite the efforts of international collaboration in this area, the patenting of genes clashes science with ethical issues regarding the detention of the natural heritage of knowledge as private property. Laws regarding other issues related to cloning and gene manipulation vary according to country.

Genetic engineering uses principles formulated many years ago. The development of refined methods using molecular biology techniques allowed the manipulation of genetic material, known as "recombinant DNA technology" or "genetic engineering." Once DNA fingerprinting had become associated with the identification of individuals, great hopes arose regarding the possibility of isolating and cloning genes to replace defective genes as therapy.

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See Also: Math Gene; Pregnancy; Probability.

Geometry and Geometry Education

Category: History and Development of Curricular Concepts.
Fields of Study: Communication; Connections; Geometry.
Summary: Geometry has been studied since ancient times and continues to develop today.

The word "geometry" is derived from the ancient Greek words *geo* (Earth) and *metron* (a measure). In addition to its practical origins, it is also associated with the language and theory of geometric figures, spaces, and forms. Ancient and medieval civilizations from all around the world contributed to the development of geometric concepts, including mathematicians in Babylonia, Egypt, China, India, Mesoamerica, Greece, and the Islamic and Arabic world. At times, the prominence of geometry has declined, such as in Western Europe during the Middle Ages and in certain research areas and undergraduate courses in North America in the twentieth century. Some curricular concepts that were once the focus of investigations have declined in relevance, such as spherical trigonometry, having been replaced in curricula by new fields or notions. Geometry research and education continue to evolve in response to changing emphases. At the beginning of the twenty-first century, students explore the properties of geometric objects and transformations. They learn about deductive geometry, coordinate geometry, and algebraic connections. Visualization and geometric history and applications are also a focus. Some of the curricular topics have been fundamental for millennia, like the Pythagorean Theorem, named for Pythagoras of Samos (c. 569–475 b.c.e.), while others, like vertex-edge graphs, are relatively recent inclusions.

Early Geometry

Some of the first indications of geometry in terms of geometric patterns appeared about 25,000 years ago. These indications have been found in a number of prehistoric sites, such as Stonehenge, spirals in Europe (Ireland and Italy), and various places in Mesoamerica and North America. Geometry also appears in the designs of the pottery, baskets, and mat weaving of many older civilizations and aboriginal peoples in the world. For instance, African tapestry and pottery are filled with symmetric figures. Civilizations around the world, including Egypt, Mesopotamia, China, India, Mesoamerica, and later civilizations, also used geometry to help produce calendars, which, at the zenith of their power, were quite accurate. The Greek historian Herodotus of Halicarnassus (c. fifth century b.c.e.) credited the Egyptians with having originated the subject of geometry, but there is evidence that the Babylonians, the Hindu civilization, and the Chinese knew much of what was passed along to the Egyptians. The earliest extant written records of geometry come from the predynastic Egyptians and Sumerians as early as the fifth millennium B.C.E. Many have connected these papyri and cuneiform stone tablets to art, decoration, and construction rather than to the systematic investigation of figures, patterns, forms, and quantities that has come to be associated with deductive geometry. Some historians and mathematicians caution against interpretations that examine earlier knowledge without considering the contextual language and culture or through the lens of later work in geometry. In many cases, the evidence that survives likely represents an incomplete geometric record.

Egypt

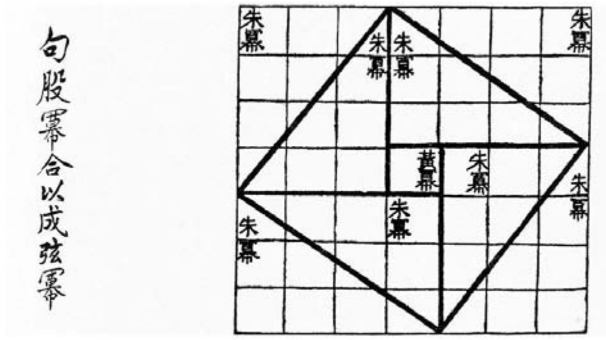
The Egyptians were extremely accurate in construction, making the right angles in the Great Pyramid of Giza precise to what is noted as one part in 27,000. Most of what is known about Egyptian mathematics comes from two Egyptian documents from about 1650 B.C.E., the Rhind Papyrus and the Moscow Papyrus. There are only a limited number of problems from these ancient Egyptian works that concern geometry. The examples therein demonstrate that the ancient Egyptians computed areas of triangles, rectangles, and circles; surface areas of hemispheres; and volumes of cylindrical granaries, rectangular granaries, and pyramids.

Babylon

In the late twentieth and early twenty-first centuries, scholarly work on some of the thousands of extant Babylonian mathematical clay tablets has led to revisions and insight in the understanding of Mesopotamian mathematics and geometry. Some tablets illustrate problems related to lengths and areas of fields, trapezoids, rectangles, right and isosceles triangles, circles, and irregular quadrilaterals. A number of clay tablets provide evidence of knowledge of the Pythagorean theorem long before the Greeks. The Babylonians also computed volumes and used geometric techniques to solve algebraic problems, like completing the square.

China

In the *Story of Civilization* series, Will and Ariel Durant state that "Chinese mathematicians apparently derived algebra from India, but developed geometry for themselves out of their need for measuring the land." Geometry was also an integral part of cosmology and astronomy in China. For instance, astronomers from the Confucian time had correctly calculated eclipses



The Chinese Pythagorean theorem identified the 3-4-5 triangle in the Chou Pei Suan Ching, 500–200 B.C.E.

and created a basis for the Chinese calendar. One early work is from the Mohists, where one finds a definition of point as the smallest indivisible component, one that cannot be divided into smaller parts. In contemporary terminology, they also explored the congruency of two lines of equal length and provided definitions for the comparison of lengths, parallels, circumference, diameter, radius, and volume. There remain some disagreements about document dating in the history of mathematical development in China. For example, some have dated *Zhoubi suanjing* (The Arithmetical Classic of the Gnomon and the Circular Paths of Heaven) from approximately 1200–1000 B.C.E. during the Han dynasty, but many scholars believed that early versions were written during 300–250 B.C.E. The *Zhoubi suanjing* has a diagram of the Gougu Theorem (Pythagorean Theorem) that is well-known in twenty-first-century classrooms. The best known of the Chinese mathematical classics may be the *Jiuzhang suanshu* (Nine Chapters on the Mathematical Art). The book had many applied geometry problems, such as finding areas for squares and circles, the volumes of various solids, and the use of the Pythagorean theorem. Included are mathematical surveying techniques in order to calculate distance measurements of depth, height, width, and surface area. There are also formulas for the areas of planar figures and the volumes of solids that were known by the time of the Han dynasty (202 B.C.E.–9 C.E.). Jesuit missionaries introduced the Chinese to Western mathematics during the Ming dynasty. As part of the Jesuit's program, part of Euclid's *Elements* was translated into Chinese in 1607. The translation of this ancient Greek textbook on deductive geometry led to research and comparison of early geometric knowledge in both cultures, work that continues in the twenty-first century.

India

The Sulbasutras (c. 700–400 B.C.E.), which can be translated as "cord-rules," are often referenced in the twenty-first century as a source of ritual geometry from India. This and earlier texts gave precise rules for the construction of sacrificial fire altars. The Sulbasutras, specifically the Baudhayana Sulbasutra, may contain the earliest extant expression of the Pythagorean Theorem: "The rope which is stretched across the diagonal of a square produces an area double the size of the original square." Many historians agree that the statement predates the Pythagoreans in Greece. Some of the material in the Sulbasutras may have originated from the Babylonians or the Chinese or been passed to the Chinese or Greeks. The Sulbasutras also contain other geometric constructions that preserve areas, as well as lists of Pythagorean triples and statements about squaring the circle or "circling the square" needed to construct certain altars for the rituals. There are various theories about the association between the geometric constructions and the religious rituals, such as whether the rituals inspired the geometry or vice versa. Aryabhata's *Aryabhatiya* (499 C.E.) includes the computation of numerous areas and volumes. Brahmagupta wrote his astronomical work *Brahmasphutasiddhanta* in 628, which included his famous theorem on the diagonals of a cyclic quadrilateral as well as his formula for the area of a cyclic quadrilateral. The Bakhshali manuscript written on birch bark and found in 1881 near the village of Bakhshali in what is now Pakistan is another mathematical manuscript from the Indian subcontinent. The date is uncertain, but many scholars agree that it contains information that is older than the document itself. It includes some geometric items such as the volumes of irregular solids.

Greece

The ancient geometry was passed on to the Greeks, who furthered and transformed the field into an essential component of a liberal arts education. The beginnings of deductive and axiomatic geometry have traditionally been attributed to Thales of Miletus (624–547 B.C.E.). Having studied in Egypt, he was likely familiar with the computations handed down from Egyptian and Babylonian mathematics. The deductive approach was continued over the next two centuries by Pythagoras of Samos (569–475 B.C.E.) and his disciples. Their foundation of plane geometry was brought to a conclusion around 440 B.C.E. in a treatise by the mathematician Hippocrates of Chios (470–410 B.C.E.). Plato (427–347 B.C.E.) founded "The Academy" in 387 B.C.E., which flourished until 529 C.E. and is noted to have included an inscription at the entrance stating the importance of geometry as prerequisite knowledge: "Let no one who is unversed in geometry enter here." Theætetus of Athens (417–369 B.C.E.) was a student of Plato, who developed solid geometry and the Platonic solids. Menaechmus (380–320 B.C.E.) discovered and developed the conic sections. He was the first to show that ellipses, parabolas, and hyperbolas are obtained by cutting a cone in a plane not parallel to the base.

Euclid of Alexandria (325–265 B.C.E.) collected the theorems of Pythagoras, Hippocrates, Theaetetus, and other predecessors and included discoveries of his own into a logically connected formal axiomatic system, the *Elements*. So completely did Euclid's work supersede earlier attempts at presenting geometry that few traces

Mesoamerica

Mesoamerican civilizations apparently had no written language, which is a measure of a civilization. The mathematics that can be identified in the Olmec, the Mayan, the Incan, and the Aztec civilizations is quite remarkable. In addition to sophisticated calendars, these civilizations were great builders, and there are indications that the Aztec builders had a compass as one of their standard tools, though very little is known about how they used it. Additional research has shown that they were knowledgeable of right triangles with the angles that would result in a 3-4-5 right triangle. Fractal geometry patterns have also been observed. Too little is known or left existing in the twenty-first century for historians to be able to be more certain about the full extent of geometric knowledge in Mesoamerica.

remain of these efforts. His approach to geometry has dominated the teaching of the subject for over 2000 years. Moreover, the axiomatic method used by Euclid is the prototype for what is now called "pure mathematics." Euclidean geometry was certainly conceived by its creators as an idealization of physical geometry. The entities of the mathematical system are concepts suggested by, or abstracted from, physical experience but differing from physical entities as an idea of an object differs from the object itself. Centuries later, the philosopher Immanuel Kant even took the position that the human mind is essentially Euclidean and can only conceive of space in Euclidean terms.

Some attributed the eventual decline of Greek mathematics to events like the destruction of the library at Alexandria and political and economic factors, and others to a lack of algebraic notation, which was developed by Arabic and Islamic mathematicians and was later to revolutionize geometry theory and applications. However, some Greek mathematicians after Euclid continued to explore concepts that became a fundamental part of school curricula. For instance, Archimedes of Syracuse (287–212 B.C.E.) is regarded as one of the greatest Greek mathematicians. He found the areas and volumes of many objects and explored semiregular polyhedra. Apollonius of Perga (262–190 B.C.E.) was known as "The Great Geometer" for his work on conics and other geometric concepts. Menelaus of Alexandria (70–130) developed spherical geometry in his only surviving work, the *Sphaerica*. Pappus of Alexandria (290–350) is considered one of the last of the great Greek geometers. His major work in geometry was the *Synagoge*, or *The Collection*, a handbook on a wide variety of topics: arithmetic, mean proportionals, geometrical paradoxes, regular polyhedra, the spiral and quadratrix, trisection, honeycombs, semiregular solids, minimal surfaces, astronomy, and mechanics.

Islamic World

Mathematicians in the medieval Islamic and Arabic world preserved and extended classical geometry and astronomy from India, Persia, Syria, and Greece, and developed and applied geometric concepts. Geometric design was found in mosaic tessellations in mosques. Mathematicians extended the astrolabe by adding circles for azimuths on the horizon in order to solve problems in spherical astronomy and trigonometry. Scholars, such as Abu Ja'far Muhammad ibn Musa al-Khwarizmi (c. 780–850) and Omar Khayyam (1048–1131), developed algebraic and trigonometric concepts and applied them to geometric notions. Arabic and Islamic mathematicians explored many geometric topics, including conic sections, constructions, spherical projections, and the parallel postulate. Scholars like Ibrahim Ibn Sinan (908–946) wrote works on geometric analysis and problem solving. Abu Arrayhan Muhammad ibn Ahmad al-Biruni (973–1048) calculated an extremely accurate radius of Earth using the law of sines. There are examples of Greek works that would have been lost if not for copies that were preserved in Islamic libraries or translated as a part of Arabic treatises and commentaries. As Europe emerged from the Dark Ages, these works were translated into Latin and this paved the way for geometry's return to Europe.

The Changing Nature of Geometry Education and Research Since the Seventeenth Century

While Euclid's *Elements* has been standard in mathematics education for thousands of years, geometry curricula were impacted by the development of many new research areas since the seventeenth century. For example, René Descartes (1596–1650) and Pierre de Fermat (1601–1665) explored analytic geometry in the seventeenth century. This exploration allowed for the representation of geometric objects in terms of coordinates and two-variable equations, a topic that begins in primary schools in the twenty-first century and is fundamental in many real-life applications. In addition, analytic geometry is typically paired with calculus courses. However, axiomatic or synthetic perspectives continued, such as through the work of Girard Desargues (1591–1661) and Jean-Victor Poncelet (1788–1867) on projective geometry. While projective geometry declined in some contexts, such as in undergraduate education in the twentieth century, students continue to learn about both coordinate geometry and deductive perspectives. Gaspard Monge (1746–1818) emphasized descriptive geometry at the École Polytechnique, a French technical university, by exploring three-dimensional geometry through two-dimensional images. Descriptive geometry remained important in architecture, engineering, and mathematics classes.

The discovery of non-Euclidean geometry, which can be found in some twenty-first-century high school and college classrooms, was a revolution in geometry.

It fell to three different mathematicians independently to show that Euclid's fifth postulate is not provable from the other axioms and what is derivable from them. These mathematicians were Karl Friedrich Gauss (1777–1855), Nicolai Ivanovich Lobachevsky (1792–1856), and János Bolyai (1802–1860). Gauss's work appears only in a letter to Franz Taurinus (1794–1874) in 1824, but he seems to have foreseen the results of the other two. Lobachevsky published his work in a Russian journal in 1826, and it was not until 1848 that it came to be published more widely in German. Bolyai's work received the widest initial distribution, being published in 1831 as an appendix to his father's algebra textbook. Each of these men, independently, assumed the negation of Euclid's fifth postulate and developed a consistent geometry: "worlds out of nothing," as Bolyai described it. Following the pioneering work of these mathematicians, the pieces of geometry began to fall into place. More was learned about non-Euclidean geometries—hyperbolic and elliptic, or doubly elliptic (spherical). For instance, Eugenio Beltrami (1835–1900) helped rigorously establish the subject, and the elliptic geometry studied by Bernhard Riemann (1826–1866) gave rise to Riemannian geometry and manifolds, which gave rise to differential geometry and then to relativity theory. Students may explore Henri Poincaré's (1854–1912) disk model of hyperbolic geometry. Some undergraduate and graduate students take courses in differential or Riemannian geometry. Mathematicians also started looking at finite geometries (from the standpoint of an algebraic geometry and from an axiomatic process), which led to areas of combinatorics and graph theory that are a curricula part of schools and colleges.

Geometry research continued to revolutionize school geometry. For instance, Felix Klein (1849–1925) greatly influenced geometry through his Erlangen Program, which attempted to unify geometry through symmetries; middle grades students in the twenty-first century explore transformations and symmetries. Kurt Godel's (1906–1978) work on consistency shook the foundations of axiomatic geometry, and David Hilbert's (1862–1943) axioms are now explored in some high school classrooms. Donald Coxeter (1907–2003) was noted as preserving the tradition of classical geometry, which then remained a core area in primary school through high school. Likewise, following World War I, the French mathematicians Pierre Fatou (1878–1929) and Gaston Julia (1893–1978) began looking at objects that later came to form the foundation of fractal geometry, introduced in the late twentieth century by Benoît Mandelbrot (1924–2010). Some middle grades students are exposed to fractals, and undergraduate students may take courses focusing on fractal geometries.

Educational theories about geometric learning also had an effect on school geometry. For example, English educator John Perry (1850–1920) advocated an intuitive, inductive approach to teaching geometry, such as graph paper measurements to test Euclid's propositions. George Bruce Halsted (1853–1922) noted that geometry "always relied upon for training in the logic of science, for teaching what demonstration really is, must be made worthy [of] the world's faith. There must be a text-book of rational geometry really rigorous." Halsted's textbook was based on Hilbert's axioms rather than on Euclid's. One well-known geometric learning model was the van Hiele model of geometric thought, which originated in 1957 through the work of Dutch educators Dina van Hiele-Geldof (d. 1959) and Pierre van Hiele (1909–2010). The model encompassed five levels: visualization, analysis, informal deduction, deduction, and rigor. Educational research on the van Hiele model in the Soviet Union during the 1960s and 1970s led to curriculum based on the theory. Some have criticized the structure of the levels and created other geometric learning models.

Recent Developments

In the twentieth century, geometry education was fundamentally transformed because of computers, calculators, and other devices. Geometry for navigation, like spherical trigonometry computations, was built into computer programs or global positioning systems, and so the related topics were eliminated from the curriculum. Other geometric topics were introduced, such as fractal geometry and computational geometry. Mathematicians at places like the Geometry Center for the Computation and Visualization of Geometric Structures produced videos and applets. Teachers and mathematicians discussed topics and shared resources on the Internet. For instance, what began as the Geometry Forum in 1992 was extended in 1996 to the Math Forum. The development of dynamic geometry software programs encouraged mathematical discovery. Students could manipulate geometric constructions while preserving the mathematical relationships that defined the figure. This method enabled students to uncover invariants like the angle sum in a triangle and allowed for inductive educational approaches. Two school programs that originated in the 1980s and remain in use at the beginning of the twenty-first century are Cabri Geometry and the Geometer's Sketchpad. Jean-Marie Laborde (1945–) headed a team to develop Cabri in order to explore geometric relationships. Nicholas Jackiw (1966–) created Sketchpad as part of a visual geometry project headed by Eugene Klotz and Doris Schattschneider (1939–). Geometry educational software continues to be developed, including open source versions. Graphing calculators and computer algebra programs allowed for easy visualization of sophisticated curves and surfaces.

There has long been debate in geometry education regarding which topics should be taught, including a tension between practical applications and theoretical considerations. For instance, in some locations and time periods around the world, educators concentrated on geometric techniques for construction, surveying, and navigation, while in others Euclidean geometry was the focus in order to train the mind. Teachers point to Euclid's philosophy, as noted by commentator Proclus Diadochus (411–485): "They say that Ptolemy once asked him if there were a shorter way to study geometry than the *Elements*, to which he replied that there was no royal road to geometry." Educators continue to debate how to teach geometry, such as whether two-dimensional perspectives should be taught before three-dimensional perspectives.

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See Also: Conic Sections; Coordinate Geometry; Curves; Geometry in Society; Geometry of Music; Geometry of the Universe; Graphs; Measurements, Area; Measurements, Length; Measurements, Volume; Measuring Tools; Painting; Parallel Postulate; Perimeter and Circumference; Pi; Polygons; Polyhedra; Proof; Pythagorean Theorem; Ruler and Compass Constructions; Squares and Square Roots; Surfaces; Transformations; Trigonometry; Vectors; Visualization.

Geometry in Society

Category: School and Society.
Fields of Study: Connections; Geometry.
Summary: Geometry permeates society from its many applications in daily life to its usefulness as a framework for deductive inquiry.

Geometry has long been useful in society for both practical purposes and as deductive inquiry. The word itself is a combination of two ancient Greek words: *geo* (Earth) and *metron* (a measure).

Thus, a direct translation might be "Earth-measuring." Geometry developed from practical needs in ancient cultures, such as the taxation of lands and the construction of monuments. In many settings, geometry played an important role in both aesthetic quality and stability. For instance, in art and architecture, beautiful geometric figures tiled surfaces. Stability notions, like the center of mass, could be calculated using geometry, and a camera's tripod has three legs because three points determined a plane, so three legs made it more convenient to find a stable position on an arbitrary surface. The Greeks explored geometry as an axiomatic system, and for thousands of years geometry was an essential part of a liberal arts education. Along with fields such as algebra and analysis, it also formed a core area in research. However, the role of geometry in school has changed over time, reflecting the priorities of society, researchers, and industry. In addition, educators have long debated which geometric topics should be taught. In some college settings in the twentieth century, the prominence of geometry declined. Some topics from courses like discrete geometry were taught in other departments, like computer science. Emerging fields, like algebraic geometry, were associated with algebra programs. While geometry was no longer a core area in some undergraduate mathematics curricula, it remained important in all levels of school in one way or another because it could be used in so many occupations. Construction, design, and architecture are just a few of the jobs that make use of geometry.

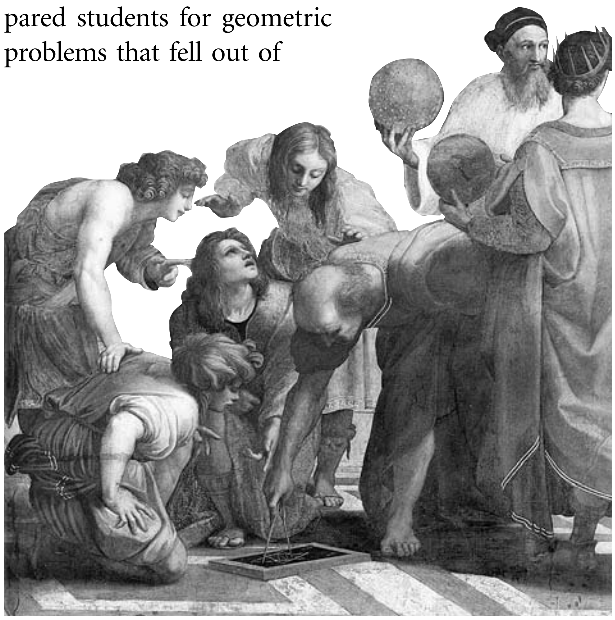
Early History

One development from the history of geometry and measurements of length, area, and volume can be found about 3000 years ago, when peoples in ancient Egypt farmed along the Nile River. King Sesostris is noted as having divided the land into rectangles. He taxed farmers based on the area of the land they occupied. But there was a problem: every year, the Nile River flooded the surrounding area. After flooding, a large portion of the lands allocated to farmers was destroyed. Hence, Sesostris had to exempt the tax on the destroyed lands. To do this, he had to measure the exact area of destroyed land. Another problem that naturally arose was how to divide the land among a number of farmers. Covering a given region by pieces is called a "tessellation" or a "tiling" of the region. Precisely, a tessellation of the plane is a set of plane figures or tiles that cover the plane without any overlaps and gaps. Tessellations were also found in mosaics as well as in floor and wall coverings.

Historians theorize that axiomatic investigations arose in ancient Greece because there was a prevalence of debate and justification in Greek society. However, even though the Greeks are noted as transforming geometry into a deductive branch of mathematics, they were still interested in practical applications. Plato is noted as believing that "for the better apprehension of any branch of knowledge, it makes all the difference whether a man has a grasp of geometry or not."

Geometry Education Since the Seventeenth Century

Ideas about the utility of geometry have spurred some changes in the way that geometry has been taught over the years. Most students who went to college prior to 1800 came from some type of preparatory school or had private tutors. As more universities opened in the United States and Europe, the preparation of the students needed to be considered. In some locations, Euclidean geometry was taught directly from a translation of Euclid of Alexandria's *Elements* as a second-year course in college. Students were expected to learn how to prove everything in the *Elements* in the same way that Euclid had outlined the proof. This process developed a strong sense of proof and logical structure in the student, but may not have prepared students for geometric problems that fell out of



In The School of Athens *painting by Raphael, Euclid performs a geometric construction with a compass.*

the direct line of proofs in the *Elements*. The argument about the utility versus the deductive nature of geometry was reflected in the diverse foci of geometry education around the world: was geometry to prepare students in the formal axiomatic method offered by geometry, or was geometry to teach students about how geometry could be used? In some locations, solid geometry and spherical geometry and trigonometry for surveying and navigation were the focus, while in others, it was Euclid's planar geometry and axiomatic perspectives.

In 1794, Adrien-Marie Legendre wrote a textbook in which he rearranged the material from Euclid and added other concepts, such as measurement. This textbook was adopted by Claude Crozet and brought to the United States Military Academy (USMA) in 1817. In 1819 Charles Davies, a mathematics professor at the USMA, translated this textbook into English and started making changes to include the type of geometry useful in mensuration and navigation that the United States Army and Navy wanted of its leaders. This type of geometry was adopted by most of the other military schools in the United States. This course came to be known as "descriptive geometry," which then led to engineering drawing. By the 1840s, universities decided that students who desired entrance needed to have had a course in Euclidean geometry in high school. This requirement moved the course in geometry into the K–12 curriculum.

Bernhard Riemann, for whom "Riemannian geometry" is named, considered that: "It is well known that geometry presupposes not only the concept of space but also the first fundamental notions for constructions in space as given in advance. It only gives nominal definitions for them, while the essential means of determining them appear in the form of axioms. The relationship of these presumptions is left in the dark; one sees neither whether and in how far their connection is necessary, nor a priori whether it is possible. From Euclid to Legendre, to name the most renowned of modern writers on geometry, this darkness has been lifted neither by the mathematicians nor the philosophers who have laboured upon it." At the turn of the twentieth century, Felix Klein, who revolutionized the understanding of geometric spaces by investigating them through their transformations or symmetries, noted: "Everyone who understands the subject will agree that even the basis on which the scientific explanation of nature rests is intelligible only to those who have learned at least the elements of the differential and integral calculus, as well as analytical geometry." The debate about how geometry should be taught has continued into the twenty-first century.

Applications

Geometry is a broad subject, hence it casts a broad shadow. Henri Poincaré, whose name is attached to the Poincaré disk in hyperbolic geometry, stated "by natural selection our mind has adapted itself to the conditions of the external world. It has adopted the geometry most advantageous to the species or, in other words, the most convenient. Geometry is not true, it is advantageous." Recent work has shown that geometry may be innate and form some core knowledge in the brain. For example, some researchers have reported that indigenous tribes in the Amazon River basin have a much deeper geometric intuition—without any formal education—than Western schoolchildren. Studies of animals, including fish and chimpanzees, have indicated that they may have a Euclidean map of their home territory in their brains. There are several types of geometry that illustrate the wide variety of applications:

- *Euclidean plane geometry* is the plane geometry of Euclid. It has close connections with computational geometry, computer graphics, discrete geometry, and some areas of combinatorics. It is the geometry of engineering drawing and architecture.
- *Euclidean solid geometry* describes three-dimensional space. It is used in solid modeling, constructive solid geometry, computer graphics, engineering design, and architectural design, among other fields.
- *Differential geometry* has become increasingly important to mathematical physics and cosmology because of the work of Albert Einstein on general relativity. The objects that are considered in differential geometry are smooth objects—objects without sharp corners or edges. Differential geometry is used in econometrics in economy; to solve problems in digital signal processing in engineering; to analyze and describe geologic structures in geology; to analyze shapes in computer vision; and to analyze and process data in image processing.

- *Discrete geometry* focuses on the properties of finite or discrete objects, like lattice points. It is used in robotics, computer graphics, crystalline theory, packing theory, and configurations of objects, among others.
- *Computational geometry* is a field that includes researchers from computer science and mathematics and investigates algorithms, data structures, and computational issues related to geometric structures and operations. It is used in robotics, computer graphics, geographic information systems (GIS), computer-aided design, medicine, and machine learning, among others.

Geometry in Design and Manufacturing

Geometry is used in the planning, layout, and production of most items that are manufactured. The design process may involve finding the optimal way to lay out a pattern on a piece of cloth or on a piece of wood, plastic, or metal so as to minimize waste. Computers are used to find the best place to divide large sheets of wood in the manufacture of cabinets, flooring, and paneling so as to generate the maximal use from that wood. Areas that use geometry in this manner are quite diverse and include the following:

Architecture includes home planning, interior design, and landscape architecture.

Assembly planning involves objects manufactured using an automated assembly line or robotic manipulations. In robotic manufacturing, the constraints of the robots determine the motions that can be made, and the motions can determine the programming and design of the robotics to be used.

Computer-aided design (CAD) includes many commercial and open source programs used by architectural and manufacturing firms to complete the design of items from motherboards to cars.

Grasping and fixturing answers the question: where does one place obstacles, such as robot fingers or fixtures, to prevent some object from moving?

Machinists are professionals who work on computer numerical control (CNC) machines to make parts in the manufacturing process. They can understand the process better if they have a deeper understanding of solid geometry. The cutter on one of these machines is controlled by the computer that is reading from a design that has been programmed—probably digitized and programmed. Because of the manner in which the machine operates, most instructions do not come from reading in the standard Cartesian coordinate system, but in cylindrical or spherical coordinates, or at times in a newly developed coordinate system designed just for that machine. The tool and die makers for manufacturers across the nation must take designs—and sometimes the designs are only outlines—from the engineer and create a prototype for the part. These prototypes can now be designed in the computer using CAD and then printed on a three-dimensional printer. The geometry for "printing" these parts is complicated, but allows for faster prototyping and manufacture.

Geometry in Graphics and Visualization

Computer graphics is an area that continues to expand from its beginnings attempting to represent geometric shapes (consider the 1982 movie *Tron*) to the extensive work of Pixar and other computer-generated imagery (CGI) groups in the movie industry to bring to life entire worlds that look realistic (consider the 2009 movie *Avatar*). Shapes and figures are first designed, digitized, and then rendered as nets. Once the basic figure is digitized, it is manipulated by computers according to the movie script. Once the entire script is done, the figures are finalized to give them a more realistic appeal. Advances in this area seem relatively simple, yet the example of making Sulley's hair move realistically in the 2001 movie *Monsters, Inc.* or the realistic appearance of the water in the 2005 movie *Madagascar* took a great deal of effort to develop.

Printing and the graphic arts involve issues of layout and form. The optimal use of geometric shapes on a page or palette, relative size of objects, and perspective are some of the relevant geometric considerations.

Geometry in Information Systems

Cartography and geographic information systems (GIS) are used by most local and state governments in the United States for maintaining property and road records and for making maps.

Voronoi diagrams answers the question: given a collection of objects (for example, fire stations) to be located throughout a city, how does one allocate these objects so that each person in the city is closer to one than any of the others? The Voronoi diagram, named for Georgy Voronoi, is a geometric partition of a space. Voronoi diagrams are used in situations such as models

(Left) Graduate students in Cornell University's mathematics department try to make a tiling, which is fit to one student's body.

(Right) A sample tiling covering made by Professor Thurston.

of crystal and cell growth, locations of limited facilities, and reservoir simulations.

Geometry in Medicine and Biology

Protein and virus modeling investigates the shape of a protein or virus and its motions, which are important in understanding its behavior and in developing treatments.

Medical imaging uses lower dimensional information, such as two-dimensional images, to reconstruct the shapes of organs, bones, or tumors. The reconstruction of three-dimensional shapes from slices is a geometric problem.

Geometry in Physical Sciences

Astronomy is one of the oldest uses for solid geometry. Computational geometry problems come about in observation planning and shape reconstruction of irregular shapes, such as asteroids.

Scientific computation involves the application of computer visualization and simulation.

Physics has long been intertwined with geometry. For example, symmetry is an important concept in both fields. Physicists have used geometric ideas to model the world and the universe, and geometers have investigated physical problems.

Robotics

Computer vision is the ability of a robot's computer to recognize the shape and geometric features of an object before it can interact with the object, such as picking up a part from a manufacturing line to be used in the assembly of a larger component.

Robot motion planning is an issue in robot design. While the engineer and the planner know what they want the robot to do in a manufacturing or other type of process, the composition of the robot and the components used in its manufacture put restrictions on what movements it can actually perform. An understanding of this "movement space" and what can be reached, held, moved, and so forth is a consideration of the geometry of the robot.

Geometry in Fashion Design

In March 2010, there was a fashion show of a Japanese fashion designer, Dai Fujiwara for Issey Miyake. It was not an ordinary fashion show but a place where fashion and advanced mathematics met. Dai Fujiwara was inspired by a legendary mathematician, William P. Thurston. Human bodies are beautiful geometric figures, which are curved in quite complicated ways. Covering these geometric objects with pieces of clothing in various types is certainly a place where mathematics can have a great influence.

A body is a surface of variable curvature. The top of the head, or shoulders, are positively curved parts, like spherical surfaces. The armpit is an example of a negatively curved part of the body, like a hyperboloid or saddle shape. Divide a circle into three arcs with equal-length and three points, A, B, and C, form endpoints of the arcs. Suppose there are hinges at A, B, and C so that the angle between two adjacent arcs can be changed. By changing the angle, one can make the deformed circle fit to a part of some surface. If the curvature is locally constant on some neighborhood of a point on the surface, and the size of the circle is small enough to be contained in that neighborhood, then this is possible. The curvature of the surface there should be same with the sum of all angle changes at the hinges. This idea was originally proposed by the great German mathematician, Carl Friedrich Gauss. A similar idea was proposed by Thurston. His idea was the following. Instead of a circle, consider Y-shape pieces. The three legs have the same length, and the angle between each pair of adjacent legs is 120 degrees. Suppose the size of the Y-shape piece is small enough. Connect endpoints of many Y-shape pieces by adding hinges, and let the hinges have some appropriate angles, and the result could fit on various types of surfaces. The angles that the hinges make determine the local curvature of the surface. If the surface is curved dramatically or the curvature of the surface is very large, then much smaller Y-shape pieces would be needed. This is one of the simplest ways to obtain a tessellation of a surface. Fashion designers have made use of these ideas in order to make beautiful coverings for the human body.

Geometry in Other Applications

In character recognition, a document is scanned and read on a computer; a computer is able to distinguish characters since they have certain configurations. If the image is clear, the recognition is simple. When the image is not clear, recognition becomes a much harder problem and geometry is brought to bear to try to differentiate characters. The algorithms used must be fast, however.

Social network theory involves the connections that people make in their social networks, which form a part of what can be studied using finite geometries. A 2009 survey of "friends" on Facebook showed that there was an average of 6.5 connections between any two randomly chosen participants.

Occupational Connections

Geometry is connected to a number of occupations and is used often in industry.

Carpenters, cabinetmakers, and construction managers are professionals who need to know, understand, and use the concepts of angle measurement, parallel lines, quadrilaterals, the Pythagorean Theorem (named for Pythagoras of Samos), area, and volume and need to know how to make and read three-dimensional drawings.

Surveyors, cartographers, photogrammetrists, and surveying technicians are professionals who need to know, understand, and use the concepts of angle measurement, congruent triangles, the triangle inequality, parallel lines, quadrilaterals, similarity, the Pythagorean Theorem, right-triangle trigonometry, circles, constructions, area, volume, and transformations and need to know how to make and read three-dimensional drawings.

Firefighters are professionals who need to know, understand, and use the concepts of area and volume.

Forest, conservation, and logging workers are professionals who need to know, understand, and use the concepts of angle measurement, congruent triangles, right-triangle trigonometry, area, and volume and need to know how and read three-dimensional drawings.

Automotive service technicians and mechanics are professionals who need to know, understand, and use the concepts of angle measurement, area, and volume.

Geometry has been useful in a wide variety of other professions also, including printing and the graphic arts, heavy equipment operation, fashion and apparel design, navigation, painting and paperhanging, engineering, home planning, plumbing and pipe fitting, outdoor advertising, landscape technology, and architecture and drafting, as well as optical technicians, machinists, cement workers, electricians, general contractors, and surveyors.

In the twenty-first century, geometry is connected to many occupations and fields within and outside mathematics. Students investigate geometric topics throughout their school experiences. Sometimes these experiences are in separate geometry courses, but often they are integrated with numerous mathematical perspectives and applications. In the nineteenth century, algebraist James Joseph Sylvester explained that

> Time was when all the parts of the subject were dissevered, when algebra, geometry, and arithmetic either lived apart or kept up cold relations of acquaintance confined to occasional calls upon one another; but that is now at an end; they are drawn together and are constantly becoming more

and more intimately related and connected by a thousand fresh ties, and we may confidently look forward to a time when they shall form but one body with one soul.

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See Also: Animation and CGI; Engineering Design; Geometry and Geometry Education; Geometry of Music; Geometry of the Universe; Medical Imaging; Origami; Painting; Pythagorean School; Sacred Geometry; Symmetry; Visualization.

Geometry of Music

Category: Arts, Music, and Entertainment.
Fields of Study: Communication; Connections; Geometry; Representations.
Summary: The mathematical principles of symmetry and scaling play important roles in musical composition.

Musical information can often be represented naturally with shapes, allowing insights to be gained from geometric techniques.

One indication of the close connection between music and geometry comes from the fact that Euclid of Alexandria, who wrote *Elements of Geometry* (300 B.C.E.), a founding document of geometry, also wrote a comprehensive treatise on the mathematics of musical pitches, *Theory of Intervals.* The eighteenth-century mathematician Leonhard Euler also developed geometric tools for music analysis.

Symmetry is one of the most powerful ideas in geometry. No less so in the geometry of music, where symmetries abound. Geometric techniques can be applied to musical scales, chords, and melodic lines. Because of the concept of octave equivalence, the 12 pitches of the equally tempered chromatic scale are inherently cyclic in nature. Thus, the geometric theory of cyclic groups plays a major role in the mathematical description of scales and chords. Similarly, geometry can play a role in the analysis of musical rhythm, particularly in musical forms based upon a repeating rhythmic motif. In twentieth-century atonal music, geometric ideas have been proposed as unifying theoretic structures to fill the role once played by tonal harmonic concepts.

Symmetries in the Twelve Pitches of the Equally Tempered Scale

Two fundamental principles of modern musical analysis are "octave equivalence" and "equal temperament." Octave equivalence refers to the perception, believed to be universal in developed music cultures, that two pitches separated by an octave are members of the same "pitch class." Equal temperament refers to the system of musical intonation by which the 12 chromatic half steps within the octave represent uniform frequency scaling—given a pitch with frequency f, the pitch one half step above has frequency $2^{1/12}f$. In the equally tempered scale, enharmonically spelled notes, such as C♯ and D♭, represent the same pitch.

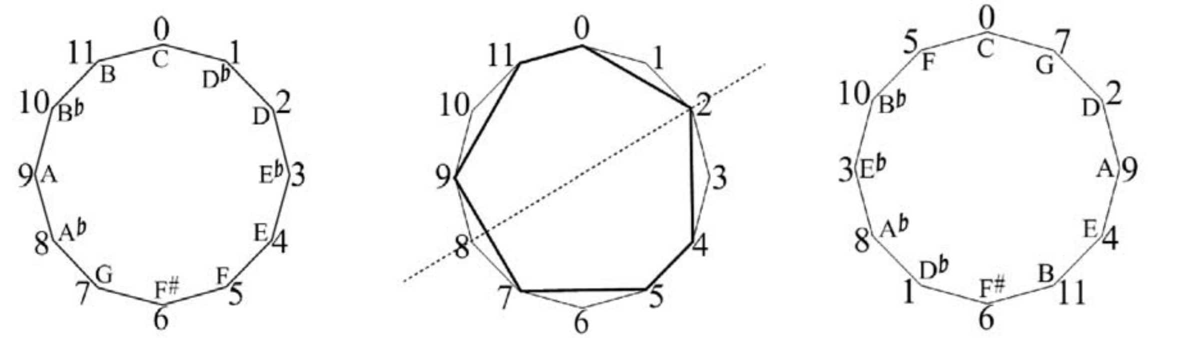
The twelve pitch classes are inherently cyclic. This principle is represented in the left view of Figure 1, which is identical to an analog clock face, with the traditional "12" replaced by "0." The diatonic scale is represented by the vertices of the inscribed polygon in the center view of Figure 1. This arrangement of the seven diatonic pitches is the most even spacing possible for seven pitches in the 12-tone octave. The evident symmetry about the 2–8 axis puts the complicated diatonic sequence of half steps and whole steps into a simpler conceptual framework. The figure illustrates that the Dorian Mode (which begins and ends on the second diatonic scale degree, given here as "D" or "2") is unique among the diatonic modes in that it follows the same sequence of intervals both ascending and descending.

The six pairs of diametrically opposite pitch classes in the clock representation are separated by the interval of a "tritone," so named because it contains three whole steps. In tonal music, the tritone is considered the most dissonant-sounding interval. If the three odd-numbered pitch class pairs on the clock face are reflected diametrically, the result is the "circle of fifths" shown in the right view of Figure 1. The circle of fifths is familiar to music students as a mnemonic device for learning the musical key signatures: the number of sharps increases by one (or alternatively, the number of flats decreases by one) at each step in the clockwise direction, while the number of flats increases (or sharps increase) at each step in the counterclockwise direction. The circle of fifths is used extensively as an analytical tool for twentieth-century music in the work of American composer and music theorist Howard Hanson.

Representing Musical Structure in Geometric Spaces

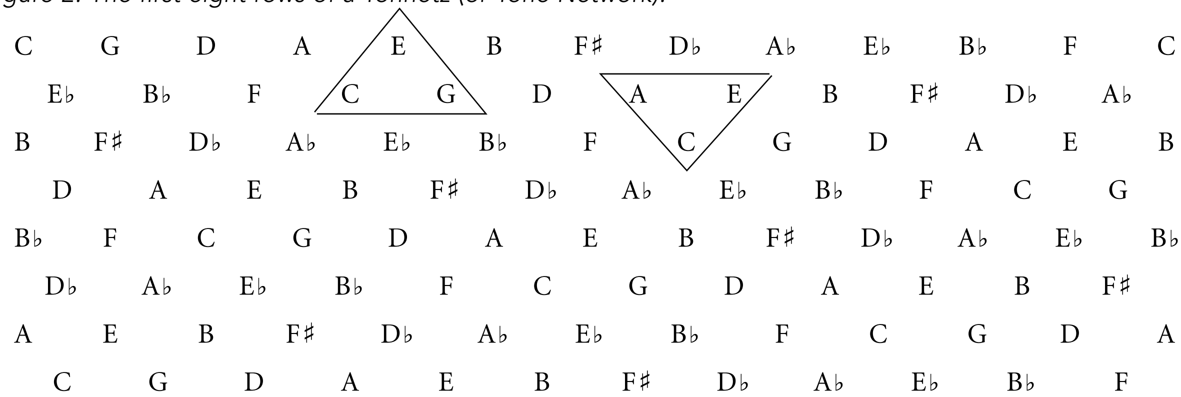
Beginning with the musical writings of Euler and continuing at least through the work of the influential music theorist Hugo Riemann (not to be confused with the mathematician Bernhard Riemann) in the nineteenth century, the representation of harmonic concepts in a two-dimensional array called a "Tonnetz" (Tonal Network) has guided the understanding of tonal harmony. In the tonnetz shown in Figure 2, the rows are simply the entries of the circle of fifths, while the columns are the 12 diatonic pitch classes arranged chromatically (by half steps). The result is that the diagonals are made up of pitch classes separated by minor thirds (in the southeast direction) and major thirds (in the northeast direction). In this arrangement, the sonorities of tonal harmony can be represented by polygonal groupings of the adjacent symbols: triangles for major and minor triads, parallelograms for major and minor seventh chords, and similar structures for diminished, augmented, and dominant seventh chords. The musical theory of "modulation" (changing from one tonal center to another in the course of a musical composi-

Figure 1. The 12 pitch classes.



Left: The 12 pitches of the equally tempered chromatic scale arranged on a circle. Center: The vertices of the inscribed polygon represent the pitches of the diatonic scale. The diatonic arrangement is the most evenly spaced distribution of seven vertices in a 12-sided figure. Note the symmetry inherent in the Dorian Mode, which begins and ends on pitch 2 (D). Right: Diametric reflection of the odd-numbered pitches results in the circle of fifths.

Figure 2. The first eight rows of a Tonnetz (or Tone Network).



The pitch classes of the circle of fifths are arranged horizontally. The vertical alignment of the pitch classes is chromatic. Diagonals in the southeast direction progress by intervals of the minor third. Northeast diagonals progress by major thirds. All tonal sonorities are given in this representation by polygons containing adjacent pitches. For example, major triads are given by triangles with vertex at top and minor triads are given by triangles with a vertex at the bottom, as shown above for the C major and A minor triads.

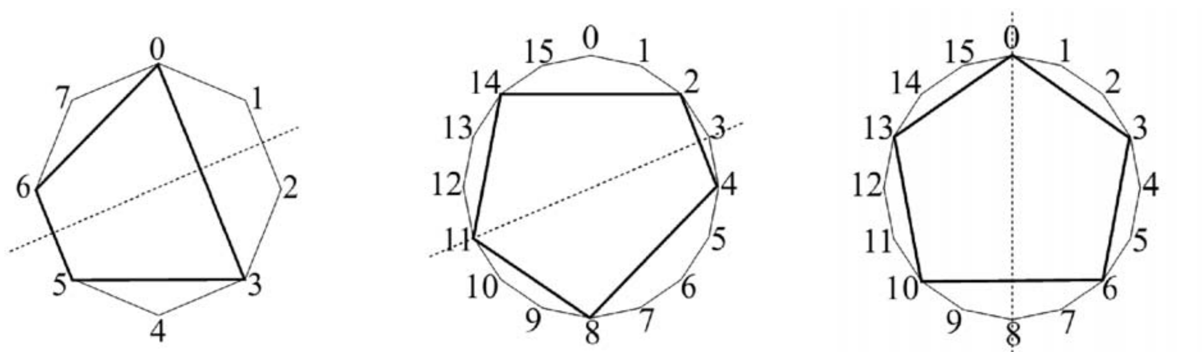
tion) is aided by the geometric perspective of a Tonnetz. Tonal networks such as the one shown here are precursors of the contemporary musical theory of "pitch class spaces."

Recently, chords have been modeled as points in geometric spaces called "orbifolds." Music theorists analyze the symmetry of chords inside of the space with respect to translation, reflection, or permutation and look at short line segments between structurally similar chords.

Rhythmic Symmetry

Like the 12 pitch classes, the metrical organization of music in time is also highly cyclic, allowing similar geometric techniques to be applied to rhythm. The left view of Figure 3 shows the eighth-note subdivisions of a 4/4 measure. The vertices of the inscribed polygon represent the rhythmic placement within the measure of the handclap rhythm from the iconic 1956 Elvis Presley recording of "Hound Dog." This complicated rhythm has a simple symmetric structure when viewed

Figure 3. Eighth-note subdivisions of rhythmic units arranged around a circle.



Left: The vertices of the inscribed polygon represent the handclap rhythm heard in the Elvis Presley recording of "Hound Dog." Center: The vertices of the inscribed polygon represent the well-known clave rhythm heard in Afro-Cuban music. Right: The bossa nova cowbell rhythm heard in Quincy Jones's "Soul Bossa Nova."

geometrically. Similarly, the center view in Figure 3 shows the clave rhythm familiar to listeners of Afro-Cuban music, with its line of symmetry. The left view of Figure 3 shows a characteristic bossa nova rhythm (which can be heard on the cowbell in Quincy Jones's "Soul Bossa Nova") and its line of symmetry.

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Eric Barth

See Also: Composing; Harmonics; Scales.

Geometry of the Universe

Category: Space, Time, and Distance.
Fields of Study: Geometry; Measurements; Representations.

Summary: Characteristics of the universe such as size, shape, and composition have long concerned mathematicians and astronomers and over the course of history various models have been offered.

The shape of the universe and its geometry have been the topic of human interest for millennia. Researchers in scientific disciplines such as physics, astronomy, and cosmology, along with mathematicians, especially those working in geometry, are seeking to discover what shape the universe is, whether it is finite or infinite, and how many dimensions it has. Not only do researchers investigate this topic; it is also popular for philosophical debates in the media and educational settings, for example, as the theme of Mathematics Awareness Month in 2005. Generally speaking, the density of the universe determines its geometry. The shape of the universe could therefore be estimated by measuring the average density of the matter within it, assuming that all matter is evenly distributed—though there might be considered distortions caused by very dense objects with mass accumulated locally, such as galaxies. This assumption is well justified by cosmological observations showing that, while the universe appears to be weakly inhomogeneous and anisotropic locally, on average it is homogeneous and isotropic. Therefore, all considerations about the geometry of the universe have to be seen from two perspectives: the local geometry that is related to the observable universe and the global geometry related to the universe as a whole, where also that is included which humans have yet to be able to measure in the early twenty-first century.

Measurements are closely related to the origins of geometry, a discipline flourishing more than 5000 years ago from the early stages of the human civilization in ancient Egypt and later in ancient Greece, and are practical and necessary in connection to the geodetic measurements of Earth. Later, developed as a theoretical abstract branch of mathematics, geometry offered mathematical background for the description of geometric abstract spaces with more dimensions, which cannot be visualized in the three-dimensional spaces, but can be used as models in modern physical and cosmological theories describing the possible form, structure, and principal laws of the universe.

From the History

For thousands of years, people believed that the universe revolved around Earth, and astronomers created mathematical models to explain observations in the sky. Eudoxus of Cnidus created a model containing rotating spheres centered about Earth. With this model, Aristotle was able to partially explain some of the planetary motions by rotating the spheres at different velocities, but other observations, such as differences in brightness levels, could not be resolved.

In the next century after Aristotle, Euclid of Alexandria expressed the parallel postulate. While it was not linked with models of the universe at the time, it was to eventually take on an important role in the geometry of the universe. Euclid is the author of the famous *Elements*, one of the earliest and most influential works in the history of mathematics, consisting of 13 books. Here, all principles of the geometric space, today called "Euclidean," were deduced in the form of mathematically proved propositions and constructions from a small set of postulates and definitions. Postulates were not proved or demonstrated, but considered to be self-evident and true. They described all basic relations and measures between ideal geometric figures as points, lines, triangles, circles, or solids, and also numbers that were treated geometrically as line segments with various lengths. The introduced list of postulates referred to the following five groups of relations: incidence, congruence, order, continuity, and parallelism.

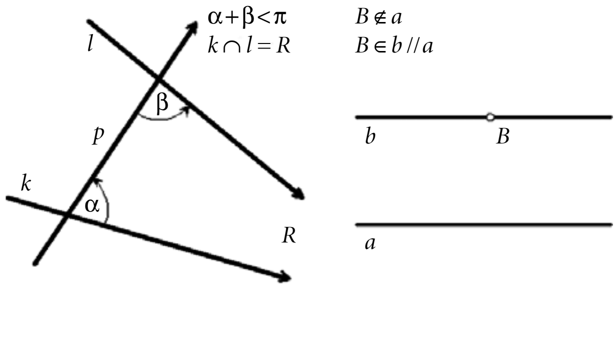
The fifth postulate about parallelism says: "If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles lesser than the two right angles." From the time of its publication until the late nineteenth century, this postulate, apparently different from all others and of more complicated form, attracted mathematicians, who strived to prove it as a consequence of the first four groups. New equivalent formulations of this famous parallel postulate appeared. The most familiar form is this: "Through a point not on a given straight line, at most one straight line can be drawn that never meets the given line" (see Figure 1).

All efforts to prove the fifth parallel axiom appeared to be pointless. On the contrary, different possible formulations of this special property were introduced, as the negations of Euclid's postulate, revealing thus the existence of new kinds later called "non-Euclidean" geometries with unusual properties emerging from these formulations.

Geometric Spaces

Even in the face of overwhelming evidence, it took a long time for humanity to accept that Earth is not at the center of the universe because this revolution required an imaginative leap that surpassed problematic religious and philosophical implications. In his famous work, the *Almagest*, Claudius Ptolemy, a second-century philosopher, refined and improved an Earth-centered model based on the earlier work of Apollonius of Perga and Hipparchus of Rhodes. In the Ptolemaic universe, planets now moved along epicycles, which had circles attached to the spheres around Earth, and yet this model still did not completely resolve contradictions with astronomical observations. Aristarchus had suggested a heliocentric system, and in the sixteenth century, Nicolaus Copernicus gave substance to Aristarchus's ideas by carrying out the detailed mathematical calculations. His model still utilized epicycles in order to explain the circular motion of the planets, but it placed a motionless sun close to the center of the universe. Johannes Kepler revolutionized astronomy by finally overthrowing the stranglehold of purely circular motions. His introduction of elliptical orbits

Figure 1. Axiom on parallelism.



Triangles in Different Geometric Spaces

Among many consequences of the validity of Euclid's postulate on parallelism, the most striking one is about the sum of the triangle interior angles. The measures of the interior angles of a triangle in the Euclidean space always add up to exactly 180 degrees. This property is related to the planar triangles located in a flat plane. Examples of triangles in non-Euclidean geometries are spherical triangles and hyperbolic triangles. Here, the sum of measures of the interior angles of a triangle is always more than 180 degrees in the elliptical non-Euclidean space, while in the hyperbolic non-Euclidean space it is always less than 180 degrees.

together with his other two laws of planetary motion form the basis of celestial mechanics to this day. They were also critical in the formulation and verification of Sir Isaac Newton's laws of gravity and of motion, which in turn became the basis for cosmology for the following two centuries.

Around 1830, Hungarian mathematician János Bolyai and Russian mathematician Nikolai Ivanovich Lobachevsky published their papers on non-Euclidean geometry, independently and unaware of each other—hyperbolic geometry is therefore also called Bolyai–Lobachevskian geometry. The famous mathematician Johann Karl Friedrich Gauss explored such geometry about 20 years earlier, but he never published his work. Lobachevsky developed a theory of a new geometric space, in which the fifth postulate was not true, by negating the Euclid's postulate about the existence of a unique parallel to a given line. He stated a new, nowadays called the Lobachevsky, axiom of parallelism: "Through a point not on a given straight line, at least two different lines can be drawn that never meet the given line." Lobachevsky based this reasoning on his own findings received from measuring distances of stars calculated from their trajectories traced on the celestial sphere because of the movements of Earth in the solar system. In his gigantic triangles, the sum of the interior angles measured less than 180 degrees. Bolyai worked out a geometric theory whereby both the Euclidean and the hyperbolic geometry were possible, depending on a special introduced parameter. Bolyai wrote in his work that it is not possible to decide whether the geometry of the physical universe is Euclidean or non-Euclidean through mathematical reasoning alone, and he regarded this to be a task for the physical sciences.

Bernhard Riemann was a German mathematician who founded a new field of geometry, later called the "Riemannian geometry," in his famous lecture in 1854. He constructed an infinite family of non-Euclidean geometries by giving a formula for a family of Riemannian metrics on the unit ball in the Euclidean space. His theory of Riemannian surfaces—which can be divided into three types: hyperbolic, parabolic, and elliptic or spherical corresponding to negative, zero, or positive curvature—can be generalized by his uniformization theorem in terms of conformal geometry. Every connected Riemann surface X admits a unique complete two-dimensional real Riemannian metric with constant Gaussian curvature equal to −1, 0, or 1 inducing the same conformal structure. The surface X is then called "hyperbolic," "parabolic," and "elliptic," respectively, according to its universal cover.

Later on, Riemann's remarkable work was elaborated by German mathematician Felix Christian Klein, who established a new classification of geometric spaces based on algebraic theory of the underlying group of transformations and their invariants, which is known as the "Erlangen program" presented at the University of Erlangen in 1872. Basic properties of a specific geometry can be represented as sets of invariant properties of the space figures under a given group of transformations. This definition of geometric spaces encompassed both Euclidean and non-Euclidean geometry in a unifying theory of geometric spaces, taking into consideration not only geometric figures and the space dimension, but also specified geometric transformations and their invariants.

The development of non-Euclidean geometries was inevitably important to physics in the twentieth century. Modern geometry shows multiple strong bonds with physics, exemplified by the links between Riemannian geometry and relativity. In 1917, Albert Einstein used Bernhard Riemann's mathematics in order to present a model for the universe that was consistent with his theory of relativity. His model was based on a finite spherical universe. Geometry, where the curvature changes locally from point to point, is the Riemannian geometry of continuous manifolds. One of the youngest physical theories, string theory, is also very geometric in flavor.

Dimensions: Shape of the Universe

There is a direct link between the geometry of the universe and its shape. The homogeneous and isotropic universe allows for a spatial geometry with a constant curvature, and three different possible types of geometric spaces can be distinguished, depending on the sign of the curvature.

If the density of the universe equals exactly the critical density, then the geometry of the universe is flat, like a plane. One has to consider a geometric space with zero curvature and Euclidean geometry as described by Euclid. As Euclid's fifth postulate on parallelism is true, the sum of the triangle's inner angles equals exactly 180 degrees, and light photons traveling on parallel lines never meet each other (see Figure 2).

Figure 2. Planar parabolic Euclidean geometric space.

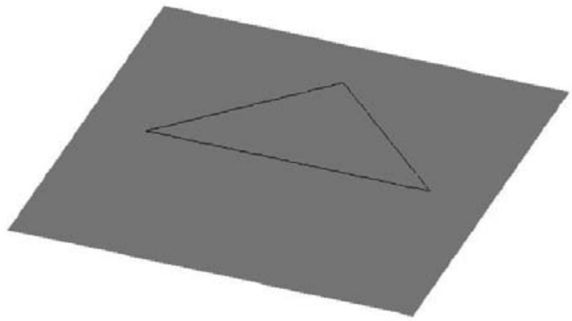


Figure 3. Spherical non-Euclidean geometric space.

If the density of the universe exceeds the critical density, then the geometry of space is closed and positively curved like the surface of a sphere. No parallel lines exist, and the sum of the triangle's inner angles is more than 180 degrees. This implies that, initially, parallel photon paths converge slowly. Eventually, they cross and return back to their starting point if the universe lasts long enough (see Figure 3).

If the density of the universe is less than the critical density, then the geometry of space is open, negatively curved like the quadratic surface called "hyperbolic paraboloid." Infinitely many parallels exist through a point to a given line and the sum of the triangle's inner angles is less than 180 degrees. Parallel photon paths can be considered as traveling to infinity in different directions from one starting point (see Figure 4).

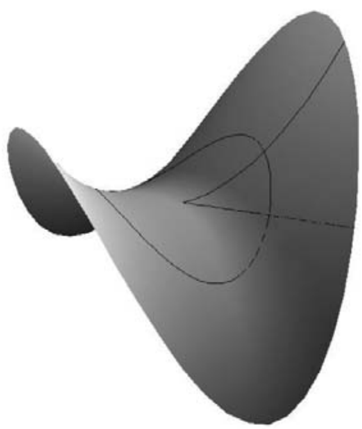
Global geometry describes the topology of the whole universe—the observable part and beyond. For a flat spatial geometry, any topological property may or may not be directly detectable, as the scale of all such properties is arbitrary. Probability to detect the topology of spherical and hyperbolic geometries by direct observation depends on the spatial curvature. Using the radius of curvature as a scale, a small curvature of the local geometry, with a corresponding scale greater than the observable horizon, makes the topology difficult to detect. In a hyperbolic geometry, the radius scale is unlikely to be within the observable horizon, while a spherical geometry may have a radius of curvature that can be detected.

There are three primary methods to measure curvature: luminosity, scale length, and density. Luminosity requires an observer to fix some standard source of light, such as the brightest quasars, and follow them out to high red shifts. Scale length requires determination and usage of some standard size, which can be the size of the largest galaxies. Density is a number of galaxies in a chosen box as a function of distance. Recently, all these methods have been inconclusive because the size and number of observable galaxies and their brightness are changing with time in unpredictable ways. As of 2011, the cosmological measurements were consistent with the model of a flat universe, based on data from sources such as NASA's Wilkinson Microwave Anisotropy Probe (WMAP). NASA has declared the universe to be flat within a 2% margin of error.

Two following investigations are decisive in the study of the global geometry of the universe:

- Whether the universe is a compact space or it is infinite in extent
- Whether the topology of the universe is simply or nonsimply connected

Figure 4. Hyperbolic non-Euclidean geometric space.



Both of these topological properties depend on the mass distribution and, therefore, on the total strength of gravitation within the universe. However, each of them implies a different history and future development of the universe:

1. If the universe is a space with negative curvature, there is insufficient mass to cause the universe to cease expansion. Therefore, the universe has no boundaries, and it will continue expanding forever, ending in a Heat Death. This model of the universe is presented as an "open universe."
2. If the universe is a space with zero curvature, there is exactly enough mass to stop its expansion, but this will take an infinite amount of time. In this case, the universe has also no bounds and will expand forever; but after an infinite amount of time, the rate of its expansion will be gradually approaching zero. This is a "Euclidean flat universe" model.
3. If the universe is a space with positive curvature, there is more than enough mass to stop its expansion. The universe is not infinite, but it is endless. The present expansion of the universe might eventually stop and turn into a contraction, and the universe will start collapsing on itself. This model is called a "closed universe."

Scientists still do not know which of these three scenarios of the future of the universe could be correct, as they have not yet been able to determine exactly how much mass is in the universe.

If the three-dimensional manifold of a spatial section of the universe is compact, then the universe has a definable volume, as on a sphere. If the geometry of the universe is not compact, then the universe is infinite in extent with no definable volume, such as the Euclidean plane. Therefore, if the spatial geometry is spherical, then its topology is compact, while for a flat or a hyperbolic spatial geometry the topology can be either compact or infinite.

Particle physics, quantum field theory, and cosmological theories led to a revolution in thought and new paradigms of subatomic matter that require the existence of a so-called hyperspace, which is an ultimate universe of many dimensions. In an ongoing quest for a synthesis of quantum mechanics and relativity physics into a superstring theory of universe unifying four fundamental forces (gravity, electromagnetism, and the strong and weak nuclear forces), the idea of a Theory of Everything has been born. This unified field theory, as it is understood in the early twenty-first century, does not preclude any of such hypotheses as, for instance, the existence of superstrings, black holes, wormholes, other parallel universes, and time travel ideas. Modern physics still needs a more powerful

Dimensions

Dimensionality of the geometric space is an intrinsic characteristic that is understood and perceived differently for the space inhabitants (locally) than from the global point of view of the outside observers. Inhabitants of the three-dimensional space cannot easily realize the fourth dimension, similarly to the behavior and abilities of inhabitants of a two-dimensional space, Flatland. English mathematician and writer Edwin A. Abbott explored the nature of dimensions in his novel *Flatland: A Romance of Many Dimensions* that appeared in 1884, where he predicted the possible existence and reality of the fourth dimension of the universe. Flatlanders are not able to imagine the third dimension existing outside their living environment of the two-dimensional space, which is, however, quite natural for the three-dimensional space inhabitants. His work inspired mathematicians to develop considerations of how higher dimensions could appear to human beings as inhabitants of the universe, provided this can be considered as a three-dimensional surface of a four-dimensional space-time. Many books and films appeared, describing the idea of dimensionality and its perception; for example the short film *Flatland* produced by Seth Caplan in 2007, or the computer animated film *Flatland* directed and animated by Ladd Ehlinger, Jr., in Lightwave three-dimensional software.

mathematical theory and topology of the 10-dimensional space in order to understand completely our expanding and evolving cosmos. The theory of hyperspace introduced by American mathematician Michio Kaku may be the leading candidate for the Theory of Everything, for which Albert Einstein spent the last years of his life searching.

When, in 1990, scientists sent the Hubble Space Telescope into space, they did not expect to find that the expansion of the universe was speeding up, nor did they realize the existence of the black matter and the dark energy that became the dominant force in the universe, recently accelerating its expansion. The James Webb Space Telescope, NASA's next orbiting observatory and the successor to the Hubble Space Telescope, is scheduled to be launched in 2014 to distant orbits. This infrared telescope detecting infrared radiation will be capable of seeing wavelengths of light difficult to observe from Earth, thus opening new horizons of the visible universe. It is hard to imagine and predict what discoveries and answers to the mysteries of the universe scientists will gain using its observations in the future.

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Daniela Velichová

See Also: Black Holes; Gravity; Parallel Postulate; Universal Constants.

Geothermal Energy

Category: Weather, Nature, and Environment.
Fields of Study: Algebra; Data Analysis and Probability; Measurement.

Summary: Geothermal energy can be harnessed for domestic heating or to produce electricity via steam turbine.

"Geothermal" refers to heat from the interior of Earth generated from the forces that led to the planet's creation and the ongoing slow radioactive decay that continues to generate thermal activity. While Earth's surface is relatively cool, temperatures increase dramatically with depth, which is known as a region's "geothermal gradient." The interiors of continents tend to have lower gradients than "spreading center" regions, where continental tectonic plates are slowly separating. A prime geothermal area is along the Ring of Fire rimming the Pacific Ocean's eastern, northern, and western coasts.

High geothermal gradients make prime candidates for geothermal energy projects. However, the average gradient is approximately 2.5–3 degrees Celsius per 100 meters. Approximately 6000 kilometers beneath the surface, molten rock reaches temperatures of approximately 5000 degrees Celsius. A small portion of this extreme heat makes its way to the surface as steam through cracks and fissures. Geothermal leakage to the surface leads to dramatic volcanic eruptions as well as to the formation of hot springs and geysers. Geothermal-warmed, mineral-rich waters have long been considered to be sacred or to have healing properties by many people. Geysers such as Old Faithful in Yellowstone continue to attract visitors from around the world.

Mathematicians, geothermal engineers, geologists, and other scientists use mathematical methods to research various aspects of geothermal processes, such as the deformable, porous properties of soil and rock that allow geothermal heat to make its way to the surface. These studies have broad applications in many scientific areas, including the way brains deform during neurosurgery and in industrial injection molding. In other cases, Lagrangian–Eulerian flow models, named for Joseph Lagrange and Leonhard Euler, are used to model characteristics such as precipitation and transport, which have applications for engineering geothermal reservoirs and isolating radioactive waste.

Stochastic models for system optimization and control as well as geometric models also help mathematicians understand geothermal heat. Many are working on computer models to update, integrate, and expand the U.S. Geological Survey's MODFLOW, a three-dimensional finite-difference groundwater flow model first published in 1984 and widely used for research and industrial applications.

Geothermal Heating

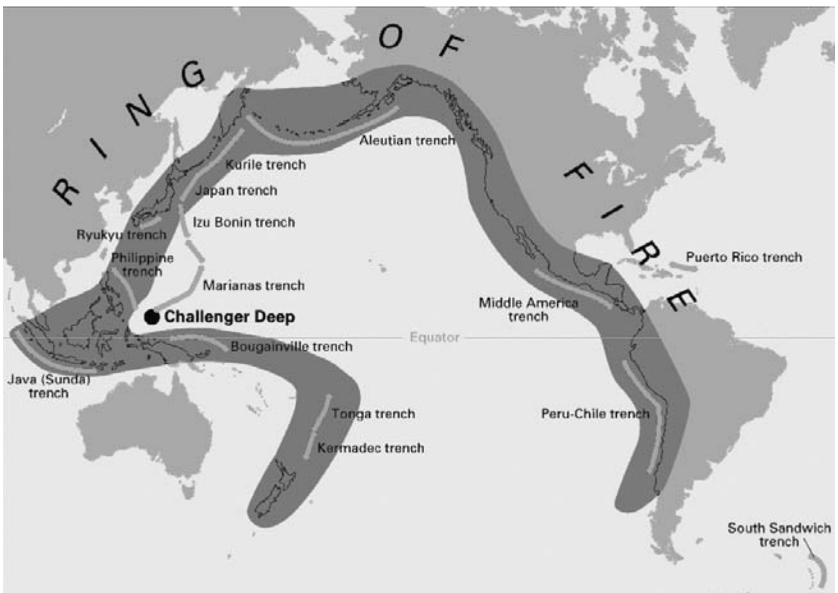
As long ago as the nineteenth century, scientists and engineers began to develop geothermal-based applications for chemistry and heating, though there is evidence that even prehistoric people built dwellings around naturally occurring geothermal heat sources. With abundant geothermal resources, Iceland began to emerge by the late 1920s as a world leader in the use of geothermal energy for domestic heating and cooling. Advances since that time have led to the development of geothermal heat pump systems. During cold periods, heat pumps transfer to buildings heat from either the ground (beneath the frost line) or from the bottom of ponds. During warm periods, the process is reversed and heat is taken from buildings and put into the ground or ponds. However, purposeful movement of water on a large scale can have geological consequences. For example, in Venice, the removal of subsurface water resulted in subsidence (settling of loose, porous soil), which lowered some buildings. Adding or subtracting water from one part of a geothermal field can affect all aspects of the field, including system pressure and surface vents. Seismologists use mathematical models describing the behavior of deformable porous rock and soil to predict where events like earthquakes might occur as a result of water-pumping activities.

Geothermal Electricity

Geothermal resources can also be used to produce electricity. The first geothermal electric power plant was built in Larderello, Italy, in 1904. Japan and the United States followed suit in 1910 and 1921, respectively. The spread of geothermal energy has been slow in the decades since. However, because of concerns regarding global warming and a quest to develop nongreenhouse gas (GHG)–emitting energy technologies, geothermal power generation has received more attention.

There are two types of geothermal power plants, both of which rely upon the production of steam to drive the conventional turbines that create electricity. Electricity can be produced directly from steam if the temperatures are at a minimum of 95 degrees Celsius (200 degrees Fahrenheit), and higher outputs are possible after temperatures crest at 175 degrees Celsius (350 degrees Fahrenheit). At the Geysers geothermal power plant in California, steam at a temperature of approximately 235 degrees Celsius (455 degrees Fahrenheit) is used to directly drive turbines. At lower temperatures, geothermal heat can still be used, but it relies upon specialized fluids that have a low boiling point capable of producing high pressures, rather than natural steam.

While the capital costs are high for both types of geothermal electricity, once in production it has several advantages over other forms of electricity generation. Like wind, its fuel costs are negligible. Similar to wind and nuclear power, once constructed, geother-



Volcanic arcs and oceanic trenches partly encircling the Pacific Basin form the Ring of Fire, a zone of frequent earthquakes and volcanic eruptions.

mal plants produce far fewer GHG emissions than traditional fossil fuel plants. Geothermal also has advantages over other alternative energy producers. Unlike wind, which is intermittent because of its dependency on weather conditions, geothermal electricity can be relied upon to produce consistent baseload power. Geothermal plants are also less intrusive visually than large wind farms and tend to draw less public attention.

Geothermal also has two key advantages over nuclear generation. Nuclear power plants are dependent upon a finite resource (uranium), and nuclear waste disposal is both controversial and costly. In contrast, geothermal generation depends on a virtually infinite source (heat generated in Earth's interior), and there are no long-term waste issues.

Popular and government interest in geothermal energy and its advantages over both traditional and alternative electricity generating options led to a 20% increase in global geothermal electricity production between 2005 and 2010. In addition, there has been a 52% increase from 2007 to 2010 in the number of countries developing geothermal resources.

Despite the increasing numbers, geothermal energy production continues to significantly lag behind other electricity sources at the start of the twenty-first century. In part, this lag is the result of a perception that there are a limited number of high-quality geothermal sites that would enable geothermal energy to become a major producer. In addition, there are technical, permitting, and electric transmission issues that drive up capital costs and inhibit substantial expansion.

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Jason L. Churchill

See Also: Climate Change; Electricity; Energy; Volcanoes; Wind and Wind Power.

Gerrymandering

Category: Government, Politics, and History.
Fields of Study: Data Analysis and Probability; Geometry; Number and Operations.
Summary: Mathematical algorithms are used in the process of redistricting and to help evaluate whether or not gerrymandering has occurred.

Gerrymandering is a form of political boundary delimitation, or redistricting, in which the boundaries are selected to produce an outcome that is improperly favorable to some group. The name "gerrymander" was first used by the *Boston Gazette* in 1812 to describe the shape of Massachusetts Governor Elbridge Gerry's redistricting plan, in which one district was said to have resembled a salamander. In the United States, congressional and legislative redistricting occurs every 10 years, following the decennial census. The aim of redistricting is to assign voters to equipopulous geographical districts from which they will elect representatives, in order to reflect communities of interest and to improve representation.

Both redistricting and gerrymandering can be characterized as mathematical optimization functions. For good-government redistricting, the optimization function is based on measures of representation and fair political outcomes. These measures may include the number of expected majority-minority districts and the number of competitive districts as well as bias and responsiveness of the expected seats-votes response curve. In contrast, a gerrymander may aim to minimize the number of districts in which a racial or ethnic minority can elect a representative, maximize the number of partisan seats, protect incumbents by creating districts that are not competitive, or obtain some other improper advantage.

Forms of Redistricting

Redistricting is the process of dividing a larger geographical unit into a fixed number of regions (known as districts). The formal aim of redistricting is to create the set of districts that yields the optimal results—as measured by some cost/benefit criteria—while at the same time meeting a set of constraints. The generalized redistricting problem applies to a variety of fields, including the assignment of sales territory; the site selection for warehouses, fire stations, and schools; and the division of political territories into election districts.

In political redistricting, a larger political unit, such as a state, is divided into a number of districts containing roughly equal numbers of people (or, in some jurisdictions, voters). The voters in each district will have the right to elect a fixed number of candidates to represent that district. In addition, the district plan must satisfy various legal requirements such that districts are geographically contiguous; composed from undivided subunits, such as counties, or census blocks; be nonempty; and do not overlap.

Mathematical Representation

In mathematical terms, both redistricting and gerrymandering are readily represented as a type of combinatoric partitioning problem. (The optimization problem is combinatoric because the rules for redistricting typically require that districts be constructed only from whole census blocks.) There are many specific formulations of this problem—all equivalent—including set-partitioning, integer-programming, polygon-divisioning, and graph-partitioning.

The law typically requires that each district is contiguous and has roughly equal population. For legislative districts, equal population may be within 10% of the "ideal" population; for congressional districts, only minimal differences are permitted. Thus, a common characterization of the redistricting problem is the weighted graph partition problem: find a partition of the entire graph (for example, state to be redistricted) that induces connected subgraphs (guaranteeing contiguity) of equal node-weight (guaranteeing equal population) and that maximizes some goal function.

The choice of goal function depends on the objectives of the redistricter. For example, a redistricter intent on creating a partisan gerrymander might use a goal function that estimates the expected probability of one party controlling the legislature under a given plan, or alternatively, the expected number of party-controlled seats—a crude estimate of this is the number of districts with party registration over 55%. In contrast, the goal function for a more fair-minded redistricter might be the number of expected competitive seats, the expected bias of the expected seats-vote curve, or another measure of political representation.

Computational Issues

The behavior and characteristics of a district can be readily predicted based on the properties of the units it contains. For example, it is relatively straightforward—using modern statistical modeling and computational methods—to predict the number of seats each party is likely to capture in the next election, given a particular districting plan.

However, although each plan may be easy to evaluate, and the problem of choosing the "best" plan is easy to formulate, actually finding the best plan is extremely difficult. In fact, it is provably "NP-complete." NP-complete problems are generally considered by computer scientists and mathematicians to be computationally intractable. Surprisingly common forms of redistricting to optimize neutral, good-government, or partisan objectives (including compact, contiguous, equipopulous plans, proportionally representative plans, and partisan-seat maximizing plans) are all computationally intractable.

While algorithms exist to solve these problems precisely, reliably, and with certainty, the time required to obtain such a solution grows exponentially as the number of problems grow. Thus, it is impossible to use reliable solution methods for practical problems. Redistricting problems are instead solved computationally using heuristics (problem-solving procedures that provide no guarantees of yielding "good" solutions, although they may produce acceptable solutions in certain circumstances). In other words, when districts are created manually or with a computer, one usually cannot know whether these are the best districts possible.

Distinguishing Gerrymandering and Redistricting

In theory, and in U.S. law, a gerrymander is distinguished from a legitimate redistricting through its effect and the intent of the redistricter. If the intent of the redistricter is to produce an improper outcome and is effective in achieving that outcome, a gerrymander has occurred. In practice—except in more extreme cases—distinguishing gerrymanders from ordinary redistricting is challenging for three reasons. First, although it may seem easy to identify gerrymanders by district shape alone (and many measures of shape "compactness" have been proposed), in fact, none of these measures is related strongly either theoretically or empirically to improper political intent or effect. Politically relevant groups are not uniformly distributed in space. Further, partisanship and demographics are often strongly correlated. For example, members of some parties tend to live in cities, and the poor are often clustered in neighborhoods. As a result, geographical compactness measures that may seem neutral on their face have predictable political biases when applied. Thus both scholars and the courts have declined to accept measures of geographical compactness for gerrymander detection.

Second, it is not feasible to determine the optimal plan for a given objective, or the statistical distribution of possible redistricting plans, because the problem is too difficult to compute. This makes it challenging to determine whether a redistricter intended or achieved maximization of a particular goal, or not. Third, there is generally a lack of consensus on how to measure the various dimensions of political representation. Thus even good-government redistricters may disagree as to the best "goal function" to use when creating a plan. These three issues makes it challenging to use statistical and quantitative methods to determine whether the properties of a proposed plan are extreme, to determine the intent of the redistricter, and to determine whether a particular plan has unambiguously violated representational values.

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Micah Altman

See Also: Graphs; Packing Problems; Permutations and Combinations.

Global Warming

See *Climate Change*

Golden Ratio

Category: Arts, Music, and Entertainment.
Fields of Study: Measurement; Number and Operations; Representations.

Summary: The golden ratio of roughly 1.618 is found throughout nature and art.

It was Euclid of Alexandria, a well-known Greek mathematician, who in his book *The Elements* (300 B.C.E.) first wrote about the golden ratio. The golden ratio is denoted by the Greek letter ϕ (phi) and known also as the "golden section," the "golden mean," and the "divine proportion."

This last name was given to ϕ because of the frequency with which the ratio exists in the natural world—leading many to hold it up as a mystical number. The golden ratio is, as all ratios are, a comparison. In his description, Euclid describes the golden ratio through the division of a line segment. A line segment whose length is A is divided into two smaller pieces, one of length B and the other of length C, such that the ratio of the original segment to the larger piece is equal to the ratio of the larger piece to the smaller piece. Mathematically, this ratio would be represented as the following:

$$\frac{A}{B} = \frac{B}{C}.$$

A perfect rectangle is a rectangle in which the ratio of the length of the longer sides to the length of the shorter sides yields ϕ. Alternatively, the ratio may be expressed as follows:

$$\frac{1+\sqrt{5}}{2}$$

and it is approximately equal to 1.16180339877. . . . As this is an irrational number, there is no end to its digits and no pattern among them.

The golden ratio may be used to create a golden spiral. Golden spirals are common in nature and can be found on shells, the caverns of the inner ear, the horns of various animals, and even some flowering plants. A golden spiral is a spiral that gets wider by a factor of ϕ for every quarter turn it takes as it opens outward from the point of origin (see Figures 1–2). If one considers the origin to be the eye of a hurricane, the spiraling out can be seen in the shape of the hurricane (the circling of winds that opens outward from the eye), and this provides yet another example of the golden ratio's appearance in nature.

Figure 1. A golden spiral.

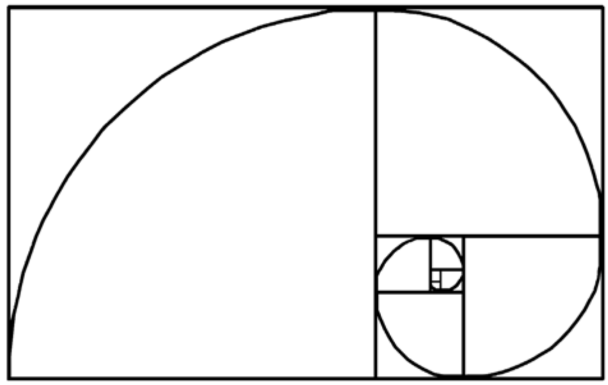
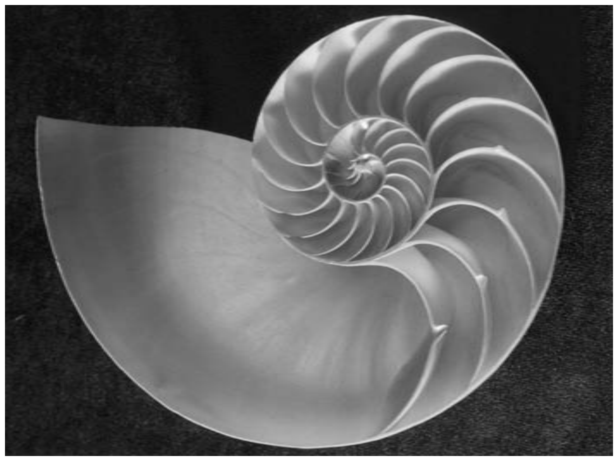


Figure 2. A golden spiral in a seashell.



The golden ratio appears in many other areas as well, including science, art, and nature. For example, the work of Herodotus (fifth century B.C.E.), considered the first historian, indicates the use of the golden ratio in the construction of the pyramids (see Figure 3). Phiddias (490–430 B.C.E.), a sculptor, is said to have used the golden ratio in the creation of sculptures that were later found in the Parthenon. The Parthenon itself consists of many uses of the golden ratio, a simple example being the length and width of the building. Similarly, the golden ratio appears in modern architecture, such as the United Nations Building in New York City. Here the ratio of the height of every 10 floors as compared to the width of every 10 floors also yields the golden ratio.

The work of Leonardo da Vinci is also said to incorporate the golden ratio, including in the definition of the proportions in the Mona Lisa (see Figure 4).

Figure 3. The golden ratio in pyramids.

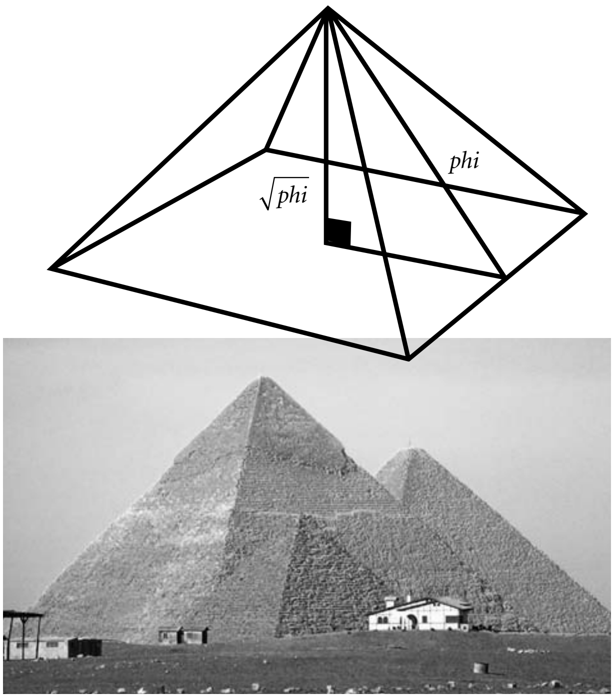
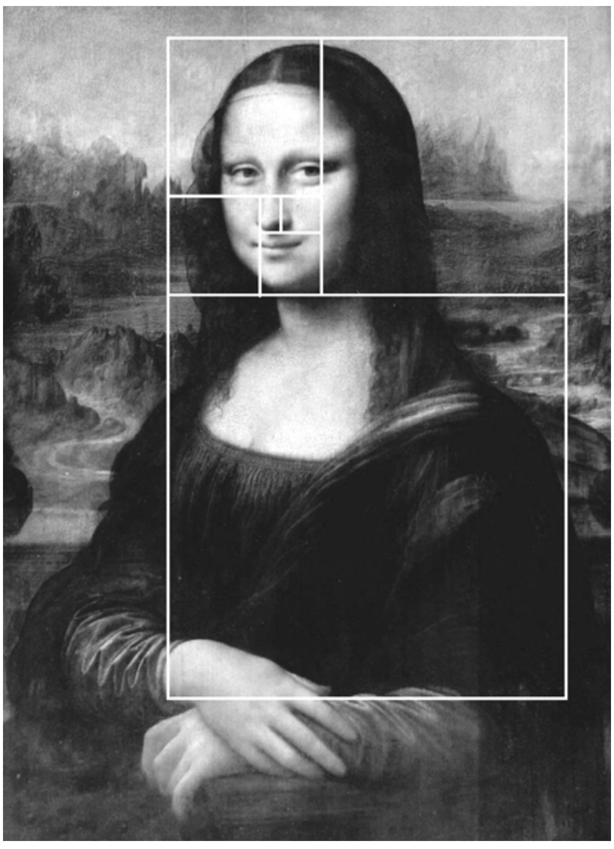


Figure 4. The golden ratio in the Mona Lisa.



The use of the golden ratio in art and architecture is common, especially when one considers that the ratio is pleasing to the eye. Gustav Fechner (1801–1887) performed many experiments with respect to this ratio. He found that rectangles, books, buildings, and other objects were more pleasing to individuals when they contained the golden ratio.

Music is another place where the golden ratio plays a vital role. Mozart's piano sonatas use the golden ratio in the arrangement of sections of measures that make up individual pieces. Mozart's piano sonatas are made up of two sections called the "exposition" and the "recapitulation." In one 100-measure composition, Mozart divided the pieces into two sections between the 38th and the 62nd measures. The measures in the pieces, when compared, yield the closest approximation to the golden ratio that can be made when dividing a 100-measure composition into two sections. However, the pieces do not always make use of the golden ratio throughout. That is, subsections do not always include the golden ratio, leading some to question whether Mozart was conscious of his use of it. In addition, in many of the most successful musical pieces, the climax of the piece occurs in accordance with ϕ. That is, the ratio between the length of the piece prior to the climax compared to that after the climax yields, once more, the golden ratio.

Further, the golden ratio is apparent in proportions in the human body. If the distance from the navel to a person's foot is considered to be "1," then the height of the person is approximately ϕ. The ratio of the distance from the navel to the top of the head to the length of the head also approximates ϕ. In the idealized human face (that which is said to be most beautiful in terms of proportions, ϕ comes up when comparing the length of the face to the width; the length of the mouth and the width of the nose, and many other comparisons.

The golden ratio is also related to the Fibonacci sequence—a numeric sequence in which each successive term (except for the first two) is obtained by adding the two prior terms. This yields 1, 1, 2, 3, 5, 8, 13, 21, ... When the ratios between successive terms in the sequence are found, they approach the golden ratio.

Some question whether the golden mean is a number that is preferred or significant in nature or whether the number is so prevalent because the mathematical meaning of the number influences or biases perceptions of the applicability. The diversity of systems in which it appears, including multiple developmental markers of human growth, suggests that it may be broadly advantageous. Analysis shows that the ratio's logarithmic spiral is a system that could theoretically self-replicate indefinitely. It also minimizes wasted space and gives new growth maximum exposure to necessary resources, such as sunlight. This makes a golden spiral an optimal and efficient design for growth in biological systems.

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Lidia Gonzalez

See Also: Geometry of Music; Hurricanes and Tornadoes; Painting; Sequences and Series.

Government and State Legislation

Category: School and Society.
Fields of Study: Connections; Representations.
Summary: Legislation shapes the conditions in which mathematics education and research take place and mathematics quantifies the impact of proposed laws.

Government and state legislation impacts mathematics and mathematics education in many ways. For instance, legislators may guide research or teaching or mandate state or federal testing. They set funding levels that affect raises, the hiring or firing of teachers, and the daily operations at many state-assisted schools, colleges, and universities. Federal funding for mathematics programs at organizations like the National Science Foundation or the Department of Education, as well as state funding through Boards of Education, is often given as grants with the hope that they will lead to innovations in research and teaching. Some scholarship programs or economic incentives are designed to increase the number of graduates in science, technology, engineering, and mathematics (STEM). U.S. House or Senate resolutions bring attention to mathematical events like Mathematics Awareness Month or π-Day. Professional mathematical societies organize or co-organize policy and advocacy committees that lobby the U.S. Congress and provide testimonies on issues related to mathematics. Another way that mathematics impacts legislation is through the quantitative knowledge of legislators. Scientists and mathematicians also serve on state or national committees like those at the National Academy of Sciences, which advise the federal government on STEM issues.

Structure and Representation

When citizens are not voting directly on legislation, they rely on elected representatives to give voice to their preferences. The constitutional democracy implemented in the United States was formulated expressly to prevent any one individual or group from exerting too much influence over the citizenry. Power sharing is manifested in the United States by partitioning governing responsibilities across the three branches of the federal government: judicial, executive, and legislative. The familiar system of "checks and balances" allows each branch to exert some measure of control over the other two.

Most state governments are structured in a similar way. The legislative branches at the federal and state levels implement further power-sharing measures in that they are often "bicameral," meaning two separate bodies deliberate on laws and policies. Reflecting one of the great political compromises of American government, these two legislative bodies are formulated on two distinct representative principles. The U.S. Senate, for example, has equal representation from each state to ensure that each, especially smaller states, has equal voice in new policy formation. The U.S. House of Representatives features representation that is in proportion to the population size of each state, thereby ensuring that larger states have a voice that fairly represents their larger constituency. In a system of representation, a single representative usually stands in for a population of citizens. The primary technical and mathematical challenge in this system of representation is that not all representatives will represent the same number of citizens. That question forms the basis of the apportionment problem, which is a topic of great historical and theoretical mathematics study.

Government-Sponsored Mathematics Education

Many federal and state agencies impact STEM fields through legislated acts such as those related to funding or establishment of responsibilities. For instance, in 1867, the U.S. Department of Education was created in order to collect data on schools. The 1890 Second Morrill Act, which required states to prove that race was not a factor in granting college admissions or to land-grant institutions, led to new responsibilities for the Department of Education. As a result of the launch of Sputnik, Congress passed the National Defense Education Act (NDEA) in 1958: "To help ensure that highly trained individuals would be available to help America compete with the Soviet Union in scientific and technical fields, the NDEA included support for loans to college students, the improvement of science, mathematics, and foreign language instruction in elementary and secondary schools, graduate fellowships, foreign language and area studies, and vocational-technical training." In 1980, Congress established the U.S. Department of Education as a cabinet-level agency.

The Department of Education continues to impact mathematics education in the twenty-first century by focusing on educational excellence and equal access. Legislative funding and policies are an important aspect of curriculum changes through state or local agencies such as Boards of Education or Departments of Public Instruction. For example, in the late 1990s, concerns about student achievement led the state of California to adopt mathematics standards and the state's legislature appropriated $1 billion for new instructional materials. Local agencies impact mathematics education through funding for teachers, charter schools, or voucher programs.

Another important federal agency for mathematics is the National Science Foundation. Under President Harry Truman, Congress established the National Science Foundation in 1950 via Public Law 81-507. The National Science Foundation provides grants and supports research and education in STEM. The agency attributes its founding to a response to the contributions of research scientists who helped win World War II, for example, with the creation of penicillin and the atomic bomb.

Professional Organizations

Mathematicians in professional mathematics organizations, such as the National Alliance of State Science and Mathematics Coalitions, track federal and state legislation, help lobby legislators, and review legislation for potential positive and negative impacts. One well-known example of mathematics legislation with mathematical errors at the state level relates to the concepts π and squaring a circle. House Bill 246 read: "A bill for an act introducing a new mathematical truth and offered as contribution to education to be used only by the state of Indiana free of cost by paying any royalties. . . ." This erroneous bill did not become law because of the intervention of mathematics professor C. A. Waldo.

Mathematics in Government

The extent of mathematics and scientific knowledge among legislators has long been a concern. Plato advocated the idea that learning to calculate "is a kind of knowledge which legislation must make a subject of study; and we must endeavor to persuade those who are in positions of authority in our State to go and learn arithmetic, not as amateurs, but they must carry on the study until they properly understand the nature of numbers; nor again, like merchants or retail-traders, with a view to buying or selling, but for the sake of their military use, and of the mind itself; and because this will be the easiest way for it to pass from the world of becoming to that of truth and reality." Under President Abraham Lincoln, an act of Congress established the National Academy of Sciences in 1863 in order to conduct experiments on scientific issues and advise any department of the government that needed them to do so. The National Academy of Sciences created the National Research Council in 1916.

An example of how the National Academy of Sciences has impacted legislation related to mathematics is the twenty-first-century report "Rising Above the Gathering Storm: Energizing and Employing America for a Brighter Economic Future," also known as the "Augustine Report." Congress had requested an economic competitiveness study and Normal Augustine, also a member of the President's Council of Advisors on Science and Technology, chaired the resulting National Academy of Sciences committee. He was educated as an engineer and served as chairman and chief executive officer of Lockheed Martin Corporation. Through the report, the committee highlighted the ties between STEM innovations and the global economy and made international comparisons. It advocated improved education in mathematics and science as well as an increase in the number of students in the STEM pipeline. This report led to the American Competitive Initiative of 2006, which was enacted into law in 2007 as the America Creating Opportunities to Meaningfully Promote Excellence in Technology, Education, and Science Act, or the America COMPETES Act. It sets targeted federal funding levels for STEM, such as doubling funding for the National Science Foundation.

Congress has investigated many issues related to mathematics, which often first arose out of related congressional and National Academy of Sciences committee work. The twenty-first century Committee on Science and Technology or the historic Committee on Coinage, Weights, and Measures is one example. Congress passed the Metric Act of 1866: "It shall be lawful throughout the United States of America to employ the weights and measures of the metric system; and no contract or dealing, or pleading in court, shall be deemed invalid or liable to objection because the weights or measures expressed or referred to therein are weights or measures of the metric system." Additional relevant legislative actions include the House Resolution: Expressing Support for Mathematics Awareness Month, or House Resolution 224, that supported the designation of March 14 as "π-Day" to help publicize mathematical events. Congress has also investigated or held hearings on issues such as how to close the gender gap in STEM or whether to relax H1B1 visa caps so that technology firms can hire more foreign workers. Mathematicians, scientists, and business leaders testify before Congress on STEM issues. Presidents can also issue executive orders related to mathematics, such as when President George W. Bush created the National Mathematics Advisory Panel in 2006 to advise both him and the Secretary of Education regarding best practices in mathematics education.

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Matt Kretchmar

See Also: Curriculum, College; Curriculum, K-12; Educational Testing; Mathematics Literacy and Civil Rights; Professional Associations.

GPS

Category: Travel and Transportation.
Fields of Study: Geometry; Measurement; Number and Operations.
Summary: Global positioning systems have been made available to the private sector but depend on satellites originally placed into orbit for military purposes and require precise calculations.

The global positioning system (GPS) is a satellite-based navigation system comprised of a network of satellites placed into orbit by the U.S. Department of Defense in 1973. GPS was originally intended for military applications to accurately determine locations worldwide in all kinds of weather. In the 1980s, the U.S. government made the system available for civilian use. GPS is used as a navigation and positioning tool in transportation, such as fleet cars and commercial trucking, in surveying, and for almost all outdoor recreational activities. In the scientific community, GPS plays an important role in geology, meteorology, wildlife studies, archeology, and many other areas. Mathematics was critical in

Once a user's position has been determined, a GPS unit can calculate not just location but speed, bearing, track, trip distance, distance to destination, and sunrise and sunset times, among other things.

the development of this system and mathematicians work on many ongoing issues, such as precision and error correction.

There are three parts that form the GPS: the space segment (satellites), the user segment (the receiver), and the control segment (control stations). The control segments are on the geoid (a three-dimensional model of Earth). The first segment of the system consists of a constellation of satellites, orbiting 20,000 kilometers above Earth in 12-hour circular orbits. While the exact number of satellites in operation varies at any given moment, at least six groups of four satellites are necessary to ensure that they can be detected from anywhere on Earth's surface. Each group is assigned a different path, creating six orbital planes that completely surround Earth.

Trilateration

The satellites transmit signal information to Earth. GPS receivers take this information and use trilateration to calculate the user's exact location. Each satellite continuously transmits a data stream containing orbit information, equipment status, and the exact time. GPS receivers contain computer chips that then calculate the difference between the time a satellite sends a signal and the time it is received. The unit multiplies this time of signal travel by the speed of travel to get the distance between the GPS receiver and the satellite. Since these are radio waves, the speed used is the speed of light. One satellite gives a sphere on which the receiver sits. Two satellites give two spheres on which the receiver sits. The intersection of two spheres (and they must intersect) is a circle. Adding a third satellite gives the receiver one of two points at which the sphere will intersect the circle. Using the geoid as the fourth solid, the receiver fixes the point of location. Despite this, there is still some possibility for error if the clock on the receiver has a slight error. A clock error of only one-thousandth of a second causes a position error of almost 200 miles. The solution is to use geometry. If one more satellite is added, then even if the clock in the receiver is off, it is off for all of the satellites by the same amount. The receiver lies on a line from each of the satellites. If all clocks are exact, then the receiver will sit at the intersection of the lines. However, the error in the receiver clock will cause the lines to intersect in different points, resulting in a polygon surrounding the receiver. The receiver can be calculated to be at the center of this polygon.

GPS Capabilities and Accuracy

A GPS receiver must be locked on to the signal of at least three satellites to calculate the latitude and longitude and to track movement. With four or more satellites, the receiver can determine the user's latitude, longitude, and altitude. Once the user's position has been determined, the GPS unit can calculate other information, such as speed, bearing, track, trip distance, distance to destination, sunrise and sunset times, and more. Most GPS receivers are accurate to within 15 meters on average. Newer GPS receivers often come with wide-area augmentation system (WAAS) capability that can improve accuracy to less than three meters on average. No additional equipment or fees are required to take advantage of WAAS. Users can also get better accuracy with differential GPS (DGPS), which corrects GPS signals to within an average of three to five meters. The U.S. Coast Guard operates the most common DGPS correction service. This system consists of a network of towers that receive GPS signals and transmit a corrected signal by beacon transmitters. In order to get the corrected signal, users must have a differential beacon receiver and beacon antenna in addition to their GPS.

Possible sources of error include the following:

- *Ionosphere and Troposphere Delays*. Different layers of the atmosphere have different impacts on the speed of the satellite signal through those layers. Mathematicians have been working on creating better models of these atmospheric layers in order to give smaller errors.
- *Geoid Error*. The receiver uses a mathematical model of the surface of Earth, the geoid. Better mathematical models can improve the accuracy as long as they are relatively easy to use in computation.
- *Signal Multipath*. The GPS signal may be reflected off objects, increasing the travel time of the signal, thereby causing errors. Mathematicians are working on developing models to account for multipath based on the relative location of receiver.
- *Orbital Errors*. Inaccuracies in the satellite's reported location are handled by the control segment, which tries to keep each satellite on track.
- *Number of Satellites Visible*. If only three satellites are visible, the receiver gives a position with a warning that it is likely to be very inaccurate.
- *Satellite Geometry/Shading*. Differences in the relative position of the satellites at any given time may cause errors. Ideal satellite geometry exists when the satellites are located at wide angles relative to each other. Poor geometry results when the satellites are located in a line or in a tight grouping.
- *Intentional Degradation of the Satellite Signal*. Selective Availability (SA) is an intentional degradation of the signal previously imposed by the U.S. Department of Defense. SA was intended to prevent military adversaries from using the highly accurate GPS signals. The government turned off SA in May 2000, which significantly improved the accuracy of civilian GPS receivers.

GPS Signal Transmission

GPS satellites transmit two low-power radio signals, designated "L1" and "L2." Civilian GPS uses the L1 frequency of 1575.42 MHz in the UHF band. A GPS signal contains three different bits of information: a pseudorandom code, ephemeris data, and almanac data. The pseudorandom code is simply an identification code that identifies which satellite is transmitting information. Ephemeris data, which are constantly transmitted by each satellite, contain important information about the status of the satellite (healthy or unhealthy), current date, and time. The almanac data tell the GPS receiver where each GPS satellite should be at any time throughout the day. Each satellite transmits almanac data showing the orbital information for that satellite and for every other satellite in the system.

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David Royster

See Also: Geometry of the Universe; Marine Navigation; Satellites; Trigonometry.

Graham, Fan Chung

Category: Mathematics Culture and Identity.
Fields of Study: Geometry; Representations.
Summary: Fan Chung, a role model to mathematics students, has done key work in Ramsey theory.

Known professionally as Fan Chung, Fan R. K. Chung Graham (1949–) is a Taiwanese-American mathematician specializing in combinatorics. She earned her doctorate from the University of Pennsylvania in 1974, where she was a student of Herbert Wilf (1931–), then spent two decades at Bell Labs and Bell Corp. In 1983, she married Ron Graham, a famous mathematician in his own right; she has two children from a previous marriage. In 1994, she left industry and returned to academia as an endowed professor at the University of Pennsylvania. Three years later she accepted her current position as a professor at the University of California, San Diego.

Fan Chung loved mathematics from a young age in Taiwan and decided in high school to be a mathematician. She gravitated to combinatorics because the problems were fun and "many problems . . . were easily explained, you could get into them quickly, but getting out was often very hard." She is a role model to mathematics students, especially young women entering mathematics. Fan Chung advises students, "Don't be intimidated!" and emphasizes the importance of seeking and exploring the connections between different topics in mathematics and applications. "It is like playing a game of Go. . . . If your territory is all connected together, then each piece is strong and useful."

Another theme in Fan Chung's discussions about mathematics is the importance of communication. "As an undergraduate in Taiwan, I was surrounded by good friends and many women mathematicians. We enjoyed talking about mathematics and helping each other." At Bell Labs, she was intimidated at first by some of the research mathematicians and scientists with whom she worked. However, her interest in diverse mathematical problems led her to inquire about others' work. Then, as she said, "You make mathematical friends and share the fun!"

Fan Chung's primary research interests are in random graphs, spectral graph theory, and extremal graph theory. She has also made many contributions in discrete geometry, communication networks, and algorithms. Her generalization of the Erdos–Rényi model for random graphs has applications to the study of large information networks. At the same time, she has blended and balanced her work and family lives. When she became pregnant with her second child, she reassured others, "Since I already had one at home, I thought what's the problem with one more? . . . I just took four weeks vacation and wrote one paper in between." She has also said that it is "quite wonderful" to have a supportive spouse with whom she can share her ideas and challenges. Recreationally she paints, including portraits of mathematicians she has known. In 1999, the Graham home, which has a unique circular design, was named "Home of the Year" by *Décor & Style* magazine and was also featured on the television program "Extreme Homes."

Ramsey Theory

Fan Chung's doctoral dissertation and much of her work since come under the general heading of Ramsey theory, named for British mathematician and economist Frank Ramsey (1903–1930). This branch of combinatorial mathematics deals with the inevitability of certain types of order and patterns. The simplest nontrivial result says that, in any group of six people, there are either three where all know each other or three where none know each other. If there are six vertices with a line connecting each pair, and each line is colored either red or blue, there will necessarily be either a red triangle or a blue triangle. A fundamental result of the theory says if there are N vertices, with each pair connected by a line, and if each line is colored in any of k colors, then there will be some n vertices that are all connected in the same color, provided N is large enough in terms of n and k. It is very difficult to estimate well how large N must be, given values of n and k. Ramsey theory is not limited to people. The objects of study may be, for instance, stars or sequences of random numbers. There are also connections to number theory and implications in scheduling problems.

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Michael "Cap" Khoury

See Also: Cocktail Party Problem; Graphs; Mathematics, Theoretical; Scheduling.

Graphs

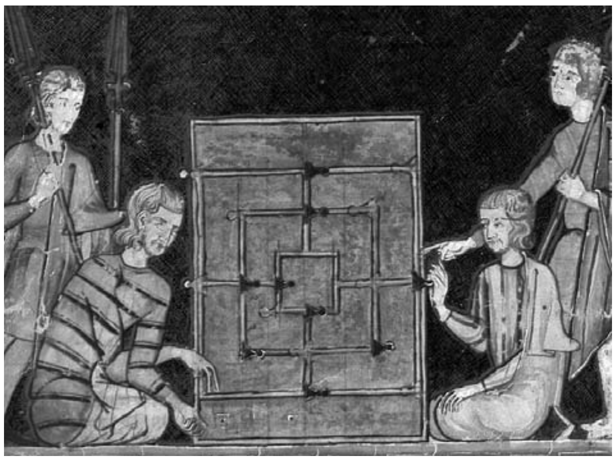
Category: History and Development of Curricular Concepts.
Fields of Study: Communication; Connections; Geometry.
Summary: Graphs and diagrams are one way to represent mathematical information and may convey it more clearly than other methods or reveal interesting patterns and relationships.

Graphical representations have been found since antiquity in such places as cave drawings and maps. Modern graphs are fundamental to the organization and presentation of information. The concept of a graph developed along with advances in printing, mathematical theory, and empirical observations, especially in such fields as astronomy, cartography, chemistry, crystallography, calculus, geometry, probability, and statistics. Quantitative information, such as data points or functions, is often exhibited and analyzed in graphs. In twenty-first-century classrooms, students of all ages explore various types of graphs. Graph theory is a branch of mathematics that studies mathematical graphs in which vertices or nodes, representing objects, are connected by edges that represent relationships between the objects.

Early Graphs

Attempts to depict familial relationships led to a variety of family trees and graphs, some of which survived from the Middle Ages. Family trees have long been of historical and personal interest in tracing ancestry and nobility relationships. The rise of genealogical social networks at the beginning of the twenty-first century led to huge family trees. Researchers and software developers have created new ways to visually represent ever-changing family relationships, including divorce and remarriage.

Some graphical representations arose in the context of puzzles or games. For instance, variants of a game known as Men's Morris have long appeared in carvings on Roman buildings and in cathedrals in medieval England. In the thirteenth century publication of Alfonso X of Castile, the *Libro de los Juegos* (Book of Games), an illustration, below, shows a Morris game board with nodes that represent the positions of game counters and connections between them that represent the moves. The beginnings of graph theory are often attributed to eighteenth-century mathematician Leonhard Euler. In 1736, he presented a solution showing that it was impossible to continuously traverse the seven bridges of Konigsberg, Russia, without retracing the same path or lifting the writing utensil. However, his paper does not contain any graphs,



An illustration from the 1283 LIbro de los Juegos *depicting a group of men playing Nine Men's Morris.*

although it does contain maps of Konigsberg. Continuous figure tracing also appeared in Danish folk puzzles, as well as in the Angola, Zaire, and Zambia region in Africa, and in the New Ireland and Vanuatu regions in Oceana. Euler also did not use graphs in his 1759 work on a Knight's Tour, where a knight must traverse each square on a chessboard without repetition. In 1771, Alexandre-Theophile Vandermonde used a graph drawing in this context.

One common notion of a graph is a pictorial representation of a function. The graph of a function passes the vertical line test, so that each input has one assigned output. Egyptologists Somers Clarke and Reginald Engelbach noted that an ancient Egyptian architect's diagram showed a curve with vertical lines and coordinate measurements expressed in units of cubits, palms, and digits. The graphical depiction of changing quantities where one quantity depends on another can be found in the fourteenth-century publications of Nicole d'Oresme and in *De latitudinibus formarum* (the Latitudes of Forms), which may also have been written by d'Oresme. The development of coordinate geometry, coordinate axis systems, and the notion of a function in the seventeenth and later centuries, through the work of René Descartes, Pierre de Fermat, Gottfried Leibniz, Peter Dirichlet, and others, allowed for the graphical representations of algebraic formulas, curves, and other mathematical objects. Thomas Hankins noted that graphs started appearing in 1770 in the context of

> . . . the statistical atlases of William Playfair, the indicator diagrams of James Watt and the writings of Johann Heinrich Lambert. . . . That leaves us with the question of what is to count as a graph. If we include maps and geometrical and astronomical diagrams, graphs are very old indeed. What was new in the late eighteenth century was a diagram with rectangular coordinates that showed the relationship between two measured quantities. Lambert called them *Figuren*, Watt called them "diagrams," and William Playfair called them "lineal arithmetic." William Whewell, who seemed to rename everything that he came into contact with, called them the "method of curves."

Gaspard Monge's eighteenth-century work also influenced the development of graphs as well as fields like architecture and engineering. He is known as the "father of descriptive geometry," which studies three-dimensional geometry through two-dimensional images.

The earliest known uses of the terms "graph," "graph paper," and "graph theory" originated in the nineteenth and twentieth centuries. Mathematician James Sylvester is noted as the first to use the term "graph" in the publication *Nature* in 1878 when he described a chemical graph. Graphs in chemistry originated earlier, such as in the eighteenth century when chemist William Cullen referred to an "affinity diagram" to model molecular forces. Alexander Brown depicted molecules as graphs in 1864. Mathematician Arthur Cayley developed graph theory in the 1870s in the context of chemistry. Some have cited Julius Peterson's late-nineteenth-century work as the start of the field of graph theory. The Peterson graph that is named for him is explored in graph theory classes. George Chrystal referred to the "graph of a function" in his 1886 algebra text: "This curve we may call the graph of the function." Graph paper was originally known as "squared paper" or "coordinate paper" and was patented by Dr. Buxton in the late eighteenth century. The use of "graph" as a verb may date to an 1898 work on applied mechanics, in which John Perry advised: "Students will do well to graph on squared paper some curves like the following . . . in each case calculate y. Plot the values of x and y as co-ordinates of points on squared paper, and draw the curve passing through the points. . . ."

Types of Graphs

The study of logical statements, their implications, and their relationships resulted in a variety of different types of diagrams. Young children use Venn diagrams, which represent set containments and intersections using overlapping circles. These were named for philosopher and mathematician John Venn because of his nineteenth century work to formalize and generalize them. The concept of a Eulerian Circle, named for Euler, is related. Aristotle's square of opposition is named for the ancient Greek philosopher. Aristotle analyzed deductive logic among various statements. Fourth-century mathematician and philosopher Anicius Manlius Severinus Boethius also explored the logical relations.

In some versions, the square of opposition was presented as a square diagram that contained propositions that were represented inside circles. Lines that connected the circles represented the relationships between the propositions. College students and researchers in fields like logic, topology, algebra, and geometry use commutative diagrams with arrows or other symbols to represent mappings or logical relationships.

Educational Graphs

Students in the twenty-first century investigate a wide variety of graphical and diagram representations. In primary schools in the United States, students represent and analyze problems using graphs, charts, data, and functions; the graphs also serve as a subject of study themselves. William Playfair's 1786 publication *The Commercial and Political Atlas* is noted as the beginning of charts, such as bar charts and line charts, and perhaps the first appearance of statistical time series graphs. He also invented the pie chart in 1801. In the middle grades, students also generalize patterns with graphs and identify and contrast linear and nonlinear graphs.

In addition, they convert between symbolic algebraic formulas and graphical representations and learn about graphical features, such as the slope or intercept of a line and the changing quantities in a graph. In high school, students continue to create graphical representations and they approximate the rate of change of a function from its graph. In calculus, students use graphs to further understand the properties of functions, such as their derivatives, integrals, and the notion of concavity. The integral is defined as the area under a curve, and students use Riemann sums, named for nineteenth-century mathematician Bernhard Riemann, to approximate the area using rectangles.

The widespread use of graphing calculators and computer software in the late twentieth century changed the way that students explored graphs. They were able to quickly graph complex equations and large amounts of data to look for patterns. Students and teachers explore candidates for categories like the most beautiful graph, the funniest graph, or the worst graph, which some define as the most misleading and others as the most confusing.

Debate continues regarding what is the desired balance between by-hand graphing skills versus a reliance on graphical methods on the computer or calculator. Some teachers argue that if students do not understand how to create graphs, they will not be able to fully understand misrepresentations or analyses. Another area that has taken on new prominence in twenty-first century schools and colleges is discrete mathematics and graph theory.

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See Also: Coordinate Geometry; Curves; Function Rate of Change; Functions; Maps; Visualization.

Gravity

Category: Space, Time, and Distance.
Fields of Study: Algebra; Measurement.
Summary: Our understanding of gravity has changed considerably over time, such that a history of gravity is virtually a history of physics. Researchers study many different effects and conceptualizations of gravity, some of which are very far from Isaac Newton's falling apple.

On the surface of the Earth, every object has some weight, which is simply the gravitational force that Earth exerts on it. In reality, minuscule gravitational forces are exerted on every atom of every object, the net effect of which is the same as the effect of a single force (the weight) acting at a single point, the center of gravity (CoG). If the object is sitting on a table, the downward force of gravity is balanced by the upward force provided by contact with the table, and there is no movement. Likewise, when a person holds an object like a barbell, the person must provide an upward force equal to the barbell's weight to keep it from falling. Mathematics shows how Sir Isaac Newton's second law of motion can explain a very complex set of observations. Scientists and mathematicians also study other conceptualizations of gravity, such as energy extraction from gravitational fields, quantum gravity, topological gravity, and supersymmetric gravity.

Properties of Gravity

Gravitational force is peculiar in that it does not depend on motion (unlike, for example, muscle forces or aerodynamic forces). The force of gravity is the same whether the object sits on a table or is allowed to fall. For an object in free fall, Newton's second law dictates: downward acceleration = net downward force ÷ mass, and if aerodynamic forces are small enough to be neglected, net downward force is equal to weight, so that downward acceleration = weight ÷ mass.

Another peculiarity of gravitational force is that it is directly proportional to mass. Therefore (weight ÷ mass) is the same for all objects; it is approximately 9.81 m/s^2 near the surface of Earth, called "acceleration due to gravity," generally denoted by g.

Any object accelerates as it falls downward. Starting from rest (speed = 0), its speed after t seconds will be $g \times t$. So,

$$\text{average speed } = \frac{0+g\times t}{2} = \frac{g\times t}{2}.$$

Therefore, the distance traveled (d) can be calculated as d = average speed × t, which can be expressed algebraically as

$$d = \frac{g\times t}{2}(t) = \frac{g\times t^2}{2}.$$

This gravitational force provides a simple method for measuring a person's visual reaction time: have the subject hold a ruler at the top and let it hang vertically. Let the subject bring his thumb and forefinger near to but not touching a known reading on the ruler, ready to grab it when it falls. At a random time, let the ruler fall. Measure the distance d it fell before it was grabbed and compute t, the reaction time, from the above equation. For d in centimeters:

$$t \text{ in milliseconds } = 45.15\sqrt{d}.$$

When gravity is the only force, whether the object is moving up, down, or at an angle, its velocity vector changes continually but its acceleration vector remains constant (magnitude g, pointing downward). The distinction between the velocity and acceleration vectors is fundamental to dynamics. The space shuttle circling Earth has constant downward acceleration when it is not firing its rockets, though its velocity—never downward—changes direction continually. Mathematics allows one to calculate what its speed must be so that the change in direction would correspond to the known constant acceleration. This speed (about 17,500 miles per hour) then determines that the period of making a complete circle around Earth is approximately 90 minutes. Farther away from Earth, gravity is weaker, so that g is smaller. It is proportional to

$$\frac{1}{r^2}$$

where r is the distance from Earth's center ("inverse square law"). Taking this factor into consideration, one can determine that a circular orbit at an altitude of 22,236 miles will take 24 hours to make a complete circle. This is, indeed, where communications satellites are located, so that they would seem not to be moving as seen from the rotating Earth. Similarly, the distance to the moon's orbit can be related to its period of revolution.

The same ideas can be applied to the gravitational forces between the sun and the planets, leading to remarkably accurate descriptions of the shapes the orbits of planets can take, the change in speed as the orbit is traversed, and the relation between period of revolution and distance from the sun. All this follows from Newton's second law and a rule of how much the gravitational force weakens with distance.

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Ziaul Hasan

See Also: Interplanetary Travel; Planetary Orbits; Satellites.

Greek Mathematics

Category: Government, Politics, and History.
Fields of Study: Algebra; Connections; Geometry; Reasoning and Proof; Representations.
Summary: Greece provided the deductive foundation for many mathematical concepts.

Historians of mathematics and ethnomathematicians have noted that we do not know what all early civilizations did in mathematics. From the evidence that is available, however, it seems that ancient Greece in the late half of the first millennium b.c.e. was the first known civilization to specifically study pure mathematics—mathematics for its own sake, mathematics as aesthetically beautiful. There are occasional examples of pure mathematics in earlier civilizations, notably mathematical proportions in art and design in Egypt and elsewhere, but the earlier peoples used mathematics mostly for practical applications, even if those applications related to religion and art.

Most of the earlier civilizations had subsistence economies, where successful life depended on success in producing food and shelter, so mathematical thinking was used to contribute to these ends. Life was difficult for most people and required full-time concentration, so there was little time for the relaxation that would allow contemplation of mathematical relationships as beauty. However, by 600 and 500 b.c.e., Greece had become prosperous, with strong markets and trade ties around the eastern Mediterranean. There was subsistence work to be done, but the upper-class elite did not have these responsibilities and could devote time to philosophy and learning for its own sake. The trade also brought ideas from other areas, and the open marketplaces encouraged the exchange of ideas and the defense of one's own. These encounters set the stage for studying mathematics beyond the everyday uses and also for the idea of deduction to prove statements.

Early Greek Mathematicians

One of the earliest mathematicians known by name was Thales (624–547 b.c.e.) of Miletus (in modern Turkey). He was an early user of formal deduction in geometry and was known for demonstrating several basic geometric properties: that a diameter bisects a circle, that base angles of an isosceles triangle are equal, and that vertical angles formed by the intersection of lines are equal. He also used angle-side-angle and angle-angle-side triangle congruences and showed that an angle inscribed in a semicircle is always a right angle. In practical geometry, he recognized that the North Star (Polaris) could be used for navigation, and, most impressively, he is said to have predicted a solar eclipse in 585 b.c.e. (though some doubt this). He was also a businessman and bought oil-press mills when his predictions showed a good year for olives.

Pythagoras (572–497 b.c.e.) is more famous, and, for many, more interesting. After traveling as a young man, he settled in Crotona (in what is now southeastern Italy) and gathered followers in a secretive cultlike organization of number worshippers. They believed that whole numbers and ratios of whole numbers are central to everything—numbers rule the universe! They studied geometry, astronomy, and music, but linked all to numbers (including noticing how a

Artists' representations of Greek mathematicians, Thales (left) and the more famous Pythagoras (right).

plucked string sounds an octave higher when it is half as long, and that other common fractions of the length also make harmonic tones). Their worship led them to the beginnings of number theory as they studied odds and evens, prime numbers, and figurate numbers (numbers of objects arranged into squares, triangles, or other shapes). Some of the questions of number theory that they investigated remain as unsolved problems even in the early twenty-first century.

The most famous mathematics connected with Pythagoras and his group is the theorem of the relationship of the lengths of the sides and hypotenuse of right triangles. Others, notably the Egyptians and the Babylonians, also recognized this relationship, at least in simple cases such as the 3-4-5 triangle for the Egyptians, more such triples for the Babylonians, and, independently, the Chinese. However, the Pythagoreans were probably the first to prove the relationship in general, and hence, in Western mathematics, it is called the Pythagorean Theorem, $a^2 + b^2 = h^2$, where a and b are the lengths of the right triangle legs with the right angle between them, and h is the length of the hypotenuse across from the right angle. This theorem has been described as the first nonobvious theorem of mathematics.

The simplest example of the Pythagorean Theorem is a right triangle with each leg one unit long. This triangle has a hypotenuse of the square root of 2. Unfortunately for the whole-number-worshipping Pythagoreans, the square root of 2 can never be expressed as the ratio of any two whole numbers. Today, it is called an "irrational number," with an infinite, nonrepeating decimal expansion. An irrational number is contrary to the beliefs of the Pythagoreans—such a serious discrepancy that they kept this result secret. More broadly, the issue of irrational numbers caused a crisis in Greek mathematics. Some have even credited this problem to the general shift of Greek mathematics from numbers to a basic geometry that does not use measurement. The geometry of the Greeks became one that allowed figures to be constructed using only a compass and an unmarked straightedge.

Three Construction Problems

Three construction problems challenged the Greeks and many others in later centuries. One was the task of constructing a square with exactly the same area as a given circle—the hope was that this would aid in finding areas of round shapes. This would require finding a way to construct a line $\sqrt{\pi}$ units long. Another was to construct a cube of volume double that of a given cube, which would need a line of length the cube root of 2. The third problem asked for a trisection of a given angle—bisecting an angle was easy, but this asked for the angle to be cut into thirds. The problems were never solved by the Greeks, but their efforts led to interesting insights in geometry. The Greek mathematicians were redeemed in the nineteenth century when all three constructions were proved to be impossible, but there are still some skeptics who erroneously claim to have produced proofs for these constructions.

Deductive Reasoning and Euclid

This geometry and the use of deductive arguments became the standard not only of mathematics but also of clear thinking and logic. Plato's Academy posted a sign that said only those with a knowledge of geometry could enter—deductive geometry was the prerequisite knowledge for philosophy, government, and critical thinking in all areas. Greek civilization greatly expanded under Alexander the Great late in the fourth century B.C.E., reaching as far east as modern Afghanistan and south into Egypt. The city of Alexandria was established at the mouth of the Nile and became a center of trade—and a scholarly center with the construction of the library (also called museum) of Alexandria.

One of the early leaders of the library was Euclid (c. 300 B.C.E.), a mathematician whose life is little known, but his work is one of the most published works in all of mathematics. Probably drawing on the work of earlier scholars, he set up an axiomatic, deductive structure of geometry that became the basis for much future mathematical research. He began with five postulates that mostly drew upon the rules of geometric construction, plus some fundamental obvious truths and some basic definitions. From these, he developed deductive proofs of more geometric properties.

From these early theorems, further deductions eventually led to a "tree" of proven statements, each traceable back to the original theorems. His book, *The Elements*, is said to have been published more than any book except the Bible, and remains the framework for the introductory study of formal geometry even today. His fifth postulate did not come from constructions and defined parallel lines, leading to the difficult use of infinity—noting that parallel lines would not even meet no matter how far they were extended. It seems Euclid himself was worried about the issue of infinity and hesitated using this postulate as long as possible. Two thousand years later, challenges and changes to the fifth postulate would lead to the development of non-Euclidean geometries in the nineteenth century.

Archimedes

Archimedes (287–212 B.C.E.) is often considered the greatest of the ancient Greek mathematicians and one of the greatest in all of history. Unlike many mathematicians, he was recognized even in his lifetime. His achievements are especially notable in that he worked in both pure and applied areas of mathematics. In pure mathematics, Archimedes came close to developing integral calculus more than 1800 years before Newton and Leibniz. He wanted to find ways to calculate areas and volumes of round shapes and used the idea of dividing the shapes into very small slices, much like the similar slices used to integrate areas and volumes in calculus. He found volumes of spheres, cones, and cylinders and discovered an interesting relationship when these shapes have the same diameter and height: the volumes of these special cones, spheres, and cylinders form a 1:2:3 ratio.

Also using calculus-like techniques, he found the value of π by inscribing and circumscribing regular polygons inside and outside a circle and then increasing the number of sides on the polygons so they would close in and estimate the circumference of the circle. He calculated the value of π to be between 3 1/7 and 3 10/71. To help handle large numbers, he greatly expanded the numeration system.

Archimedes lived in Syracuse on the island of Sicily, and his applied work often was related to his life there. He studied the mechanics of simple machines such as levers, pulleys, and screws. He was reputed to have used some of this knowledge to help the king repulse an invasion from the Romans. Once the king asked him to check the authenticity of gold in a crown. He knew he could compare densities of pure gold and an alloy, but to do so, he needed to know the volume of the very irregularly shaped crown. As he entered his bath, he noticed the water level rise to compensate for his own volume; from that he recognized that he could measure the volume of the crown from the amount of water it would displace. The story says he jumped out of the bath and ran through town naked shouting "Eureka!" (I have found it!) in his excitement at the discovery.

Although Archimedes had helped fight off the Romans, they returned when he was an old man. Legend says he refused to leave the geometry he was writing in the sand when a Roman soldier told him to go. At the refusal, the soldier killed him. In some sense, this is symbolic, in that not only did Archimedes die at the hand of a Roman soldier but much of the Greek civilization fell to the expanding Roman Empire. The Romans were good engineers and built a network of roads and aqueducts, but they mostly used existing mathematics and contributed little beyond the work of the Greeks.

Other Greek Mathematicians

However, across the Mediterranean Sea, Alexandria and its library did not fall. Following from Euclid, the Alexandria library continued to be a center for Greek mathematics that would continue even several centuries after the decline of the overall Greek civilization. Some of the work was in astronomy. As early as 200 B.C.E., Eratosthenes calculated the circumference of Earth fairly accurately (incidentally, also indicating that he knew the Earth was round) by comparing the angle of the sun at noon in Alexandria and at Cyrene and using geometrical comparisons to do the calculation.

Later, other Greek astronomers, notably Ptolemy (100–178 C.E.), found more measurements of the movements of the planets. Some of their work led to the erroneous belief that Earth was the center of the solar system, but other studies provided a sound mathematical basis for early astronomical research.

Three other names of mathematicians bring the story of ancient Greek mathematics to a close in the early centuries of the Common Era. Hero (also called Heron) in the first century designed a device that, if constructed, could have been the first steam engine, but it did not get built. He also found a remarkable formula for the area of a random triangle when only the lengths of the three sides (a, b, and c) are given:

$$\text{Area} = s\sqrt{s(s-a)(s-b)(s-c)}$$

$$\text{where } s = \frac{a+b+c}{2}$$

the semiperimeter. Like the Pythagorean Theorem, this formula is considered one of the important early non-obvious theorems and is also useful in practical applications.

Diophantes, who lived in the mid-third century, has sometimes been called the "Father of Algebra." He broke from the Greek interest in geometry and studied numerical problems with techniques that resemble later algebraic methods. He was especially interested in problems whose statements and results were all whole numbers, thus restricting the range of solutions but offering challenges that led to creative work.

Hypatia (370–415) was famous as a mathematics researcher and teacher in Alexandria. Notably, Hypatia is one of the earliest important women mathematicians known in history. Originally taught by her father, who was also a mathematician, Hypatia wrote commentaries and expansions on earlier Greek work, a common type of mathematical research of the time. She was also especially noted as a teacher. However, she inadvertently was caught up in the religious politics of her time and was captured and killed by a mob. Thus, two phases of Greek mathematics ended in tragic deaths: Archimedes at the hands of Roman soldiers approximately marked the end of Greece's Golden Age in mathematics, while the mob killing of Hypatia came near the very end of Greek mathematical work.

Overall, Greek mathematics had continued for nearly 1000 years, providing an unequaled example for future mathematical work. The Greeks did important work in the applied areas but are especially recognized for laying the foundations for pure mathematics.

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Lawrence H. Shirley

See Also: Archimedes; Golden Ratio; Pythagorean School; Pythagorean Theorem; Roman Mathematics.

Green Design

Category: Architecture and Engineering.
Fields of Study: Data Analysis and Probability; Geometry; Measurement.
Summary: Green design requires evaluating the life cycle of a product or material and the cost of that life cycle in energy and other resources.

Green design, also called "environmental" or "sustainable design," is a set of design principles for optimizing environmental impact. This includes reducing pollution, promoting ecological and economical sustainability, using reusable resources, and promoting harmony between people and natural environments. Mathematics plays a significant role in both designing green solutions to a variety of problems and measuring the impact of green solutions. Many colleges offer degree or internship programs in green design, which requires strong science and mathematical skills.

Impact Measures

Ecological design employs a series of metrics for evaluating the degrees of sustainability. A mnemonic used for types of sustainability is "Three Rs": reduce, reuse, and recycle. Reducing waste, pollution, and resource use involves calculations of the impact of production, packaging, transportation, and disposal, as well as renewability of resources. Some design movements, such as Tiny Houses, are predominantly based on the principle of reducing space and resources. Reuse design principles allow objects to be used multiple times, possibly for different purposes. Recycling is the ability to turn objects into materials for making other objects.

The notion of life cycle is central to measuring environmental impact. For example, product life cycles include research and development, main use, and disposal after use. Different stages in the cycle require different types of impact measures. Green design has to address all the stages, from sustainable research practices to possibilities of reuse and recycling at the last stage of the product's life.

There are numerous rubrics and point systems for measuring environmental impacts of industrial, product, or architectural designs. For example, products, activities, or organizations can be measured by their resource intensity, with amount of resources used per unit cost. A toy designer can calculate liters of water spent during manufacture per dollar of the toy's cost. The inverse of resource intensity is resource productivity, measured in quantity or price per unit of resource spent. In this example, resource productivity is the price, in dollars, of toys produced using one liter of water.

Leadership in Energy and Environmental Design (LEED) is an international green building certificate. To give a building or a community its score, LEED combines metrics, such as the carbon footprint, as well as energy and water efficiency. LEED has separate ratings for construction of commercial buildings and homes, interior design, maintenance of existing buildings, and neighborhood development. In each category, the maximum score is 100 points, with certification levels of Platinum (more than 80 points), Gold (60–79 points), Silver (50–59 points), and Certified (40–49 points).

There is a global mathematical problem involved in measuring and reducing environmental impact of design. Namely, there are money and environmental price differences between different design types, and noticeable costs of certification and measurement. The overall sustainability measures have to include all these costs and optimize the total. Because many current economical practices are standardized in nonsustainable manners, the economy of scale makes their use cheaper than the corresponding green designs. This phenomenon is being addressed at the government level by changing price and tax structures to promote sustainable practices.

Cherokee Mixed-Use Lofts is an urban market-rate housing project and was the first LEED-certified building in Hollywood, California.

Green Urban Design

New urbanism is an example of urban design that includes several green principles, including jobs within walkable distances, bike-friendly roads, shared public and housing spaces, diverse communities, and matching local terrain and conditions in landscaping. Geometries of new urbanist designs are concentric and include discernible centers for neighborhoods, such as a historical artifact or a town square, with a transit node tied to this center for optimized logistics. Houses of different types, matching a variety of family and economic situations, are situated within the five-minute walk radius (about one-half kilometer) from this center, and commercial properties surround the houses. The design of roads uses network science to slow down car traffic, minimize travel, and place important administrative, educational, and religious public buildings in traffic network nodes. This relatively compact design, the opposite of urban sprawl, also helps make electricity, water, and gas distribution more efficient, because less energy is spent on delivering these resources and less is lost in transit.

Models from Nature

One of the principles of green design is the use of models found in nature to build products or systems. For example, thermoeconomics models the design of social structures on the laws of thermodynamics. Economical entities are considered on the basis of energy, matter, and information involved in them. Production and use of goods and services are seen as energy and mass exchange, and scarcity has to do with entropy.

The concept of exergy is especially important in industrial design. Exergy is the maximum work theoretically possible as a system reaches energy equilibrium with its surroundings. The second law of thermodynamics says that systems tend to dissipate energy or increase entropy. This loss of exergy is called "anergy." Green designers use both energy and exergy efficiency. Energy efficiency measures how much energy is lost during industrial processes. Exergy efficiency has to do with minimizing anergy, that is, the loss of exergy.

Some social designers consider the total exergy of Earth or even the solar system, working toward designs at these large scales. For example, burning oil or coal produces heat, but these fuels also required inputs of exergy in their making. A mathematical model can approximate the history of the fuels and incorporate their current use, computing energy and exergy efficiency of our actions with regard to Earth, and the sustainability of Earth, over time.

Biomimicry, biomimetics, and bionics are direct uses of design ideas and principles found in nature. For example, engineers studied birds and insects to develop flying devices. More recent examples have to do with efficiency and sustainability. The shape of nautilus shells, mathematically related to the Fibonacci sequence named for mathematician Leonardo Fibonacci, is used to minimize friction in fans, conserving energy. The mechanism of water condensation used by desert beetles can be applied on the human scale. The ways termites keep their mounds warm at night and cool during the day are studied to produce sustainable air conditioning in houses.

Designers and engineers rarely repeat natural designs completely but rather analyze them to find appropriate elements and include elements into the design. There are three directions for such analysis. Designers can incorporate methods of manufacture found in nature, such as the strong material of the mussel's shell. They can mimic mechanical or thermodynamical principles found in nature, for example, the way butterfly wings are colored as the basis of energy-efficient displays. Finally, designers can look at the global organizational principles found in nature, such as modeling a robotic cleaner on insect scavenging behaviors or building artificial intelligence based on the ways brains work.

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See Also: Carbon Footprint; City Planning; Energy; Engineering Design; Fuel Consumption; Landscape Design; Solar Panels; Wind and Wind Power.

Green Mathematics

Category: Weather, Nature, and Environment.
Fields of Study: Algebra; Data Analysis and Probability; Measurement; Number and Operations.
Summary: Modeling, analysis, and computation are used to promote environmentally conscious practices.

Green mathematics is the use of mathematical modeling, analysis, and computation to promote ecologically sound practices, such as sustainable production or reduction of pollution. Green mathematics is an increasingly popular and controversial topic with ties to other contentious social, scientific, and political issues, such as recycling laws and global warming. It is a rich area of research and development for mathematicians and scientists. For example, computer scientists Young Choon Lee and Albert Zomaya have developed and patented an Energy Conscious Scheduling (ECS) algorithm. The ECS software maps the assignment of computational tasks in high-performance computer systems as a function of the dynamic voltage scaling capability of the processors. It optimizes scheduling to decrease task completion time and energy use. Green mathematics also appeals to many mathematics educators at all levels for its apparent applicability, real-world connections, and the ability to connect to academic curriculum in other areas like history and science. In 2010, Roger Williams University's student mathematics fair was organized around the theme "Designer Math Goes Green! Mathematics and the Environment!" College programs for ecology and sustainable development rely heavily on mathematics and statistics for research and applications. On the other hand, "green math" can have negative connotations for some people, especially when it affects taxpayer dollars and restrictive changes in public policy. In some cases, this reflects an incomplete understanding regarding the basis for such calculations and the methods by which final figures are derived, often because such information is not presented to the public. In others, this may result from inappropriate extrapolation or the political "spin" attached to such calculations.

Green Measurements and Metrics

Measurements of sustainability and environmental impact apply to persons, groups, products, and events. Carbon footprint, for example, is the measure of total emission of greenhouse gases, mostly carbon dioxide, involved in an event or in the lives of people over a given time, usually a year. Energy consumption measures how much energy a process or product takes over its lifetime and accounts for sources of energy, such as atomic or fossil fuels. Ecological metrics may also include emissions of chemical pollutants, such as heavy metals, strength of potentially harmful electromagnetic fields, and intensity of light pollution. The units of measure vary by type of pollution; for example, weight per volume is used for water quality measurements, but light pollution is measured in changes in sky brightness.

Quality standards in ecological measurements include some of the same principles that apply to measurement in general, such as precision and accuracy, together constituting validity. In addition, the measurement of ecological impact requires holistic, systemic approaches, taking into account interactions among multiple variables and their relative weight for particular ecosystems. For example, different ecosystems have different resource balances. Polluting a scarce resource, such as the only water source in a desert oasis, has higher environmental impacts than polluting an abundant resource. This can be reflected in mathematics equations by applying different coefficients to different types of impacts, according to the particular situation within each ecosystem.

It is difficult to weigh different types of environmental impacts against one another. For example, producing paper from trees grown for that specific purpose takes less energy than recycling paper, but involves more water and air pollutants.

Computational Modeling in Ecology

A mathematical model is an idealized system of variables, parameters, and equations governing relationships, assumed to be close enough to a real system for the purposes of prediction or explanation. Mathematical models in ecology are typically based on observations of sets of data from real environments and involve hypothesizing about sets of data that would result if variables changed.

Models can predict developments of ecological systems if the outputs of models, taken over time, fit the corresponding changes in variables of the real ecosystem closely enough. Evaluation of a model includes its accuracy, based on a statistical metric of closeness between observed and predicted data. Nonparametric statistics is the field that deals with evaluating the accuracy of models when the data is limited and not all mathematical assumptions can be tested.

The explanatory power of a model is based on the claim that the model preserves cause-effect relationships within the ecosystem. In mathematical models, such relationships are expressed as algebraic or differential equations among variables of the model.

The possibility of general patterns (models) in ecology has to do with two global problems, or hypotheses: contingency and complexity. The contingency hypothesis says that causal relationships in any given ecosystem are so numerous that projection from one system to another is not possible. The complexity problem is that the number of variables and their weak interactions in any given ecosystem are beyond the computational power theoretically available, making systems immeasurable, their equations insoluble, and the models unable to be interpreted. That is, contingency and complexity are theoretical and philosophical challenges to the possibility and validity of ecological modeling.

Environmental Considerations by Type of Mathematics

Different areas of mathematics allow different approaches to environmental problems. Algebraic reasoning, for example, assumes functional dependencies among variables and known operations. It is most appropriate in cases where algebraic relationships among variables are stable over time and can be established with empirical measurements. For example, producing one megajoule of energy by burning coal emits 92 grams of carbon dioxide. One can compute the carbon footprint of heating a house by coal algebraically by measuring the energy consumption and multiplying it by 92 grams of coal.

Calculus is the study of rates of change in variables and limits of change. In green mathematics, calculus methods are most appropriate when algebraic relationships between variables and their changes over time are measurable. For example, rocket propulsion consumes fuel stored within the vehicle, making the vehicle lighter with time. The efficiency of rocket engines can be computed by applying integrals over time to equations connecting changes in mass and momentum resulting from the engine.

Differential equations involve the study of unknown functions by known values and their rates of change, that is, derivatives—a situation frequently encountered in ecology. Differential equations are extensively used in green mathematics to model interactions within systems, such as predator-prey dynamics, fluid dynamics in natural and human-made water and gas systems, radioactive decay, or economic growth.

Statistical methods deal with the organization and interpretation of data that include random elements. Descriptive statistics summarizes patterns in data collected from some group of objects or events, called "population." It may include data calculations such as mean or frequency. Descriptive statistics is useful for comparing systems that include randomness, such as per capita consumption of energy in different countries or recycling behaviors in neighborhoods of a city.

Inferential statistics predicts patterns in the whole population based on data observed in a sample of the population. It is extensively used in biology, ecology, and economics because collecting data about every element in the population is rarely possible. One of the most powerful methods of inferential statistics is the analysis of correlations within data. For example, the levels of air pollution in cities correlate with the incidence of asthma among the population. Notably, even strong correlations between two variables do not necessarily mean particular cause-effect relationships. The first variable may depend on the second, or the second on the third, or both may depend on another factor. For example, in children younger than 6, problem-solving abilities strongly correlate with foot size. The reason is that both foot size and problem-solving abilities increase with age.

Data visualization is an interdisciplinary area spanning descriptive statistics; grid and graph use from algebra and calculus; specific representation methods from more narrow areas of mathematics, such as tree diagrams from combinatorics; psychology of perception and learning; and design. Visual literacy combines the ability to understand and critically analyze visualizations produced by others and to create quality visualizations for the purposes of analyzing and sharing messages. Because green mathematics frequently deals with controversial issues, individuals and groups promoting different agendas use and often abuse data visualization to make their point. Visual literacy is one of the "twenty-first-century skills" whose importance is growing with heavier use of mathematics in ecology and growing emphasis on ecological approaches in all areas of life.

Green Economics and Sustainability

Mathematics is used to describe, plan, model, and predict green economy, which is economy based on ecological and social sustainability. Sustainability is a system's capacity to endure over time, measured by a variety of indices and metrics. For example, the biodiversity index measures the number of plant and animal species in an ecosystem. Using an old-growth forest for lumber and replanting trees may produce the same amount of biomass, but such "farmed" forest typically has a much-lower biodiversity index. Air quality indices assign point values to combinations of air pollutants, such as dust, ground-level ozone, and sulfur dioxide. Higher values of an air quality index correlate with higher incidents of asthma and other adverse health effects. Factories and other entities and events can be evaluated by their effects on an air quality index.

Carrying capacity of an environment, with respect to a species, is the number of individuals the environment can sustain. In differential equations, carrying capacity is the stable state of the system: populations over carrying capacities decrease over time, and populations under carrying capacities grow. Carrying capacity for humans changes depending on their practices. For example, hunter-gatherer tribes need larger areas for sustenance than groups that practice agriculture. The classic mathematical models of carrying capacity were developed for animal populations in relatively small and closed ecosystems. Because people actively change their environments, travel, and exchange resources globally, such models need significant modifications for applications to humans. Current mathematical models are based on evaluating population growth and resource use over time. For example, mining for groundwater can dramatically increase agricultural outputs and thus support population growth until the water runs out, at which time famine can lead to a population collapse.

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See Also: Carbon Footprint; Climate Change; Deforestation; Farming; Fuel Consumption; Green Design; Nutrition; Probability; Randomness; Recycling; Temperature.

Gross Domestic Product (GDP)

Category: Government, Politics, and History.
Fields of Study: Data Analysis and Probability.
Summary: Gross domestic product is a figure used frequently in economics to discuss a country's complete economic output.

Until the Industrial Revolution, population size was a dominant factor in economic output. With the coming of technology, notions of productivity changed. The gross domestic product (GDP) is currently the most widely accepted and broadest indicator of aggregate economic activity.

The GDP represents a country's overall economic output, the dollar value of all final goods and services produced over a period of time within a nation's domestic boundaries. Many assert that the concept of quantifying a nation's economic output can be traced back to newspaper articles written in 1939 by British economist John Maynard Keynes, who was concerned about how Britain would manage its very limited resources at the start of World War II. The Keynesian formula for GDP was the sum of a country's consumption, investment, government spending, and exports, minus its imports.

In the United States, the GDP is calculated and released quarterly by the Department of Commerce. In general, the GDP is used to define emerging economic trends, devise appropriate policies, and gauge the effectiveness of current economic policies. More specifically, corporations use the data to forecast sales and adjust production and investment accordingly. Social scientists monitor the GDP as an indicator of well-being and as a proxy for individuals' voting and investment decisions. In 2003, economists Sir Clive Granger and Robert Engle won a Nobel Prize for their innovative, sophisticated methods of statistical time series analyses that enhance the understanding of market movements and economic trends. In 2010, mathematicians developed an objective quality of life index that uses linear functions and dimensionality reduction to combine four well-studied and widely used indices, including per capita GDP, to produce a relative ranking of countries.

Economists have devised three distinct methods of calculating a nation's GDP. While these approaches derive the same value, each views the GDP differently. The "product method" represents the market value of final goods and services newly produced within a nation during a particular time frame. The "expenditure method" is the national expenditure on goods and services within a specific time frame. The "income method" is the total of wages, rents, dividends, interest, and profits received by producers during a specified time frame. Regardless which method is used, the outcome is referred to as the "nominal" gross domestic product. When the nominal GDP is adjusted for inflation, it is called the "real" gross domestic product. The real GDP is used to measure the growth of a country's economy and real GDP per capita is often used as an indicator of aggregate standard of living.

The Product Approach to Measuring the GDP

The simplest and most direct way to calculate the GDP is the product approach. The product approach calculates the GDP as the market value of final goods and services newly produced within a specific nation. Goods and services produced throughout the year may be classified as either intermediate or final goods. Intermediate goods and services are those that are consumed during the production of other goods and services and are not counted when calculating the GDP; only the final value of a good or service is included in total output. This avoids an issue often called "double counting," in which the total value of a good is included multiple times in national output. The following equation is used to solve for the GDP using the product approach:

$$GDP = P - C$$

where P is the market price of final goods and services and C is intermediate consumption.

The Expenditure Approach to Measuring the GDP

The expenditure approach works on the principle that all of the products must be consumed, therefore the value of the total product must equal the people's total expenditures. The four main components in calculating the GDP via the expenditure method are consumption expenditures by households (C), gross private investment spending (I), government purchases of goods and services (G), and net exports (exports minus imports, $EX - IM$).

The expenditure approach can be represented in the following equation:

$$GDP = C + I + G + (EX - IM).$$

The Income Approach to Measuring the GDP

The income approach to measuring the GDP assumes that expenditures on final goods and services are eventually received by households and corporations as income. A key to calculating the GDP using this method is a concept known as "national income." The national income consists of five types of income: compensation of employees (W), proprietor's income (P), rental income (R), corporate profits (C), and net interest (I). Thus, national income = $W + R + I + P + C$.

Once the national income is calculated, several adjustments must be made before arriving at the GDP. The first is an adjustment for the taxes paid by businesses to the government (indirect business taxes). Next, depreciation, or the consumption of fixed capital, is taken into account. Finally, the net foreign factor income (*NFI*) is included as an adjustment. The *NFI* is the difference between payments received from the foreign sector and payments made to the foreign sector for domestic production. The *NFI* represents the key difference between gross domestic product and gross national product. The following equation is used to solve for the GDP using the income approach:

GDP = Compensation of Employees + Rent + Interest + Proprietor's Income + Corporate Profits + Indirect business taxes + Depreciation + *NFI*.

Nominal Versus Real GDP

If using GDP to examine production over time, the effects of price increases and inflation must be taken into account. The real GDP is the total value of all goods and services adjusted to eliminate the effects of changing prices. The nominal GDP is calculated by using current market prices. Hence, the real GDP is the value of all goods and services produced by an economy in a given year in dollars of constant purchasing power.

*Figure 1. Top 10 Countries by GDP in 2009 (**International Monetary Fund, World Economic Outlook Database).*

Country	Year	Units	Scale	GDP
United States	2009	U.S. $	Billions	14,256.28
Japan	2009	U.S. $	Billions	5,068.06
People's Rep. of China	2009	U.S.$	Billions	4,909.28
Germany	2009	U.S. $	Billions	3,352.74
France	2009	U.S. $	Billions	2,675.92
United Kingdom	2009	U.S. $	Billions	2,183.61
Italy	2009	U.S. $	Billions	2,118.26
Brazil	2009	U.S. $	Billions	1,574.04
Spain	2009	U.S. $	Billions	1,464.04
Canada	2009	U.S. $	Billions	1,336.43

Mathematics concepts have also been used in recent years to debate related economic concepts that are rooted in mathematics, such as the principle of comparative advantage. Taken in its simplest form, it states that if two or more countries have already expanded their respective GDPs as far as possible under some set of international trade restraints, they can expand them further by relaxing those restraints. This has been formulated and proven mathematically, using techniques like convex analysis. However, one June 2000 letter to the editor of *SIAM News* (the monthly magazine of the Society for Industrial and Applied Mathematics) argues against such practices for "soft" social science concepts. Motivated by a then-recent protest of global free trade policy, the author stated, "... you can criticize the application of a theorem not only by questioning the validity of the hypotheses, but also by questioning the interpretation of the conclusion. … international trade in addictive drugs and guns being only the most glaring and brutal counterexamples to the 'goodness' of increasing GDP."

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See Also: Accounting; Mathematics, Applied; Measurement in Society; Statistics Education; Unemployment, Estimating.

Growth Charts

Category: Medicine and Health.
Fields of Study: Algebra; Measurement.
Summary: Children's development both mentally and physically is modeled using data-based norms, some of which are indicated by growth charts. .

When a parent brings his or her child to a physician for a checkup, a number of measurements are taken to help the physician assess the health and development of the child. For children up to 36 months of age, three typical measurements include height, weight, and head circumference. The healthcare professional will use these measurements to decide whether the child is on track developmentally. These measurements are expected to vary depending on the gender and age of the child. Considering weight, for instance, younger children tend to weigh less than older children and girls tend to weigh less than boys. However, there is even considerable variability in these measurements for children within the same gender and age group. There are individual differences from child to child resulting from genetic and environmental factors, including diet and physical activity habits.

Percentiles

To make a judgment about whether the child's development is on track, the relevant question to pose is where the child's measurements fit in relation to other children of the same age and gender in the population. Percentiles are typically used to facilitate this comparison and growth charts summarize these quantities in graphs. If a young boy's weight is at the 75th percentile, this means that of the boys the same age in the population, about 75% of them weigh less and about 25% of them weigh more than this boy. If parents are told that one of their child's measurements is at the 99th percentile, should they be concerned? Very high or very low percentiles may be a sign of something abnormal. For example, a child's weight at the 4th percentile may be a sign of malnutrition. Extreme measurements indicate to the healthcare professional that further follow-up may be necessary. Generally speaking, measurements under the 5th percentile or over 95th percentile or growth patterns that shift considerably in terms of their percentiles over time require further assessment.

Growth Charts

Growth charts are graphical summaries of mathematical functions that are developed based on extensive body measurement data collected on large groups of children from the population of interest. They provide benchmarks for comparison and are widely used by the health community to monitor and track the growth and development of children. According to the Centers for Disease Control and Prevention (CDC), growth charts have been used in the United States since 1977. Prior to 1977, there were child development references in use, but they did not adequately represent the population. As of 2011, the charts used in the United States are the 2000 CDC Growth Charts. The infant (0–36 months of age) charts include smoothed percentile curves of weight by age, length by age, head circumference by age, and weight by length for boys and girls. The children and adolescent (2–20 years of age) charts include weight by stature, weight by age, stature by age, and body mass index by age for each sex.

In order to find a percentile based on these charts, one needs to be able to plot a point on the graph. For instance, consider the weight-by-age infant chart for boys. Suppose the boy is 18 months and his weight is 25 pounds. Find 18 months along the horizontal axis and 25 pounds along the vertical axis of the graph. Mark this point. Based on the chart, this point falls between the 25th and 50th percentile curves. As demonstrated by this example, not all percentile curves are summarized in the charts. If a measurement falls somewhere between the 3rd, 5th, 10th, 25th, 50th, 75th, 90th, 95th, or 97th percentile curve, the professional reading the chart will need to interpolate between curves to approximate the percentile value.

The percentile curves summarized in the 2000 CDC Growth Charts were developed by the United States National Center for Health Statistics (NCHS) based on the results from a number of large health surveys conducted based on representative groups from the U.S. population. Data analysis was used to estimate percentiles for the various growth measurements and statistical modeling was used to smooth the estimates into the percentile curves to facilitate comparisons.

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See Also: Disease Survival Rates; Graphs; Life Expectancy.

Guns

See *Firearms*

Gymnastics

Category: Games, Sport, and Recreation.
Fields of Study: Algebra; Data Analysis and Probability; Geometry.
Summary: Performing gymnastics depends upon an understanding of geometry and forces.

Gymnastics is an athletic performance activity that depends on balance, flexibility, and strength for producing graceful movements. Gymnastics can be recreational or competitive. There are also numerous forms of gymnastics, including artistic, acrobatic, and aerobic. The main mathematical topics involved in gymnastics include the mechanics of motion, patterns in choreography, and competition scoring systems. Mathematics has sometimes been described as "mental gymnastics."

Rotations

Many gymnastics routines include rotations. The key mathematical characteristic of a rotating body is its angular momentum, which is equal to the product of the mass, the velocity, and the distance between the center of mass and the axis of rotation. When there are no external forces, the angular momentum is conserved—it does not change. Gymnasts cannot change their mass, but they can reposition their center of mass relative to the axis of rotation, making the speed change to preserve the momentum. When a rotating gymnast tucks in closer to the center of rotation, the speed increases. For example, a gymnast can hold onto a bar by the hands and keep the body straight, making a relatively slow rotation around the top uneven bar called "giant swing." As the gymnast tucks his or her limbs in closer to the bar, the center of mass becomes closer to the axis of rotation, and the gymnast spins faster. Mathematics also helps determine the optimal angle at which the gymnast should release from the bar in order to perform subsequent transitions and maneuvers.

While simpler routines can be performed intuitively, through trial and error, competitive gymnasts develop complex sequences of moves that involve detailed calculations of mass, momentum, velocity, position of the apparatuses, and so on. Conversions between rotation and moving along straight lines are a part of many routines, with speeds and directions determined by conservation of momentum laws. For example, a gymnast runs to a springboard, accumulating momentum. As the gymnast jumps off the springboard, the vectors of the momentum generated by the springs and the momentum of the run are added together, propelling the gymnast forward at about a 45-degree angle to the floor. The gymnast can then push off a horse ahead, converting momentum into the angular momentum of rotating the body around the horse. At the highest point of this rotation, the gymnast can tuck the limbs in, moving the mass close to the axis of rotation and accelerating for a flip in the air. For example, a triple back somersault involves two and three-quarter body rotations before landing. Before landing, the gymnast straightens out, moving the limbs farther from the axis of rotation and slowing down the rotation, allowing for a soft, safe landing on the feet.

Scoring of Artistic Gymnastics Competitions

The current system of scoring in artistic gymnastics is relatively complex. It assigns a difficulty score to the attempted routine and then subtracts from that score for mistakes in execution. The score is analytic—it is based on decomposing gymnastic routines into individual elements. Existing elements are summarized in the illustrated Table of Elements, and given difficulty ratings from A (0.1 points) to G (0.7 points). Additions to the Table of Elements are frequently named after gymnasts who first performed them successfully. Such new elements are submitted by the competing gymnasts ahead of the competition event, to be evaluated by an international committee.

Eight highest difficulty values of the routine, added together, form the difficulty value (*DV*). Skills from five required Element Groups are awarded 0.5 points each, for the maximum 2.5 points in composition requirement (*CR*). Finally, an additional 0.1 or 0.2 points are given for each element if elements are connected, which adds to connection value (*CV*). The difficulty score (*D*) is the sum of these points: $D = DV + CR + CV$.

In addition to the difficulty score, there is an evaluation of the artistry and execution called "E-score." The judges take away points from the perfect 10.0 E-score for technical or artistry mistakes. Each fall costs 1 point.

Trampolining and Conservation of Energy

Many gymnastic apparatuses are somewhat springy. Trampolining is a type of gymnastics that occurs entirely on trampolines and uses flight-like moments between contacts with the surface for striking routines. Trampolining involves the accumulation of energy. First, the kinetic energy of the gymnast's limb flexes and motions is converted into the potential energy of the stretched trampoline fabric. Then, the gymnast is thrown in the air, converting this potential energy into the kinetic energy of the motion. As the gymnast gains height, the kinetic energy is converted into the potential energy again. Gravity pulls the gymnast down with acceleration, converting to kinetic energy, which converts to the potential energy of the stretched trampoline upon contact, and so on.

From the point of view of mechanics, the trampoline is a device for storing the gymnast's potential energy between jumps. This view can explain, for example, why gymnasts cannot jump infinitely high, adding more and more energy to the trampoline. The maximum stretch of the trampoline limits the amount of energy stored in it. This can also be used to compute the theoretical maximum height of a trampoline jump.

Different types of gymnastics are easier to perform with different body types. A lower body-mass-to-height ratio makes it easier to twist during movements and to hide momentum transitions in the twisting, so tall, skinny people are better suited for artistic gymnastics. In trampolining, twists and transitions are not as crucial as higher rotation speeds and are easier with a higher body-mass-to-height ratio. Also, both take-offs and landings on trampolines require significant bursts of energy and muscle strength. Therefore, shorter, stockier athletes are better suited for trampolining.

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See Also: Body Mass Index; Cheerleading; Joints; Transformations.

H

Harmonics

Category: Arts, Music, and Entertainment.
Fields of Study: Algebra; Measurement; Number and Operations; Representations.
Summary: Harmonics are sonic components that are periodic at and integer multiples of the fundamental frequency.

Harmonics are components of a musical sound with well-defined frequency relationships to one another. For a pitch of frequency f, typically measured in units of cycles per second, or hertz (Hz), the nth harmonic has frequency $n \times f$. In this context the frequency f is referred to as the "fundamental frequency." Harmonics are closely related to overtones (or partials), which are defined to be secondary pitches that audibly resonate when a fundamental pitch sounds. The number and strength of these secondary pitches are responsible for the distinct timbres perceived in different instruments or voices. The overtone series in music (also called the harmonic series at the risk of confusion with the infinite sum of the same name) refers to the sequence of ascending harmonics with frequencies $2f$, $3f$, $4f$. . . . With only a few exceptions, the pitches of the lower harmonics match well with the frequencies of 12 pitches of the equally tempered scale. Further along the overtone series, the pitch spacing becomes very small—smaller than the traditional half step—and these upper harmonics, if heard, would sound distinctly out of tune. With the discovery of the overtone series by Jean-Philippe Rameau in the eighteenth century, the notion of musical consonance as the exclusive natural and rational sonic phenomenon—pursued by mathematicians from Pythagorus of Samos to Leonhard Euler—began to fade. There is a close physical relationship between the harmonic frequencies and the length of the vibrating medium. This relationship is exploited in the performance practices of musical instruments.

Vibrating Media and the Overtone Series

For vibrating strings (such as violins and guitars) and open vibrating air columns (such as the Western concert flute and some organ pipes), the words "harmonic," "partial," and "overtone" are essentially synonymous, with a slight difference in the enumeration: the fundamental pitch (frequency f) is referred to as the "first harmonic." The first overtone (frequency $2f$) refers to the second harmonic, and so on. In stopped air columns (such as the clarinet and some organ pipes), the overtone series omits certain harmonic frequencies. For vibrating membranes (such as percussion instruments), overtones may exist at nonharmonic frequencies.

It is therefore a slight abuse of terminology to refer, as is commonly done, to the sequence of harmonics as the "overtone series." Physically, the overtone series is seen by observing the motion of a vibrating string of length L and natural frequency f. If forced to vibrate at frequencies $n \times f$ (for $n = 2, 3, \ldots$), $n-1$ stationary points (nodes) appear along the string, at intervals of L/n. In effect, the string moves as n strings of length L/n joined end to end. String performers utilize this fact by lightly stopping the string at lengths $L/2, L/3, \ldots L/n$ to produce flute-like harmonic tones (sometimes called "flageolet tones").

From the overtone perspective, only lower harmonics are perceptible to the hearer of a fundamental pitch. The first six harmonics are perceived by the modern hearer as in tune within the 12 pitches of the equally tempered scale, in which the octave (the distance between the first and second harmonic) is divided into 12 equal half-step intervals. The frequency difference between successive pitches in this 12-tone system is given by

$$f_{n+1} = 2^{1/12} f_n .$$

The second, fourth, and eighth harmonics, at octaves above the fundamental, sound perfectly in tune. Upper harmonics can sound significantly out of tune, however. The seventh harmonic sounds uncomfortably flat compared to its nearest corresponding equal temperament pitch. The 11th harmonic has a frequency almost equidistant between adjacent notes of the equally tempered scale, causing it to sound very out of tune—likewise for the 13th and 14th harmonics.

These considerations are significant for period-instrument brass performers, whose instruments, like the so-called "natural trumpet," are nothing more than long tubes without the length-changing system of valves of modern trumpets. Performers play tunes on these instruments by producing overtones, typically between the 3rd and 16th in the series.

While skillful performers can compensate for the most problematic overtones, composers in the baroque era typically avoided these notes or used their sonic character for special effect. Modern composers have experimented with specially tuned pianos and electronic instruments to directly explore the sonorities of harmonics. The first 24 harmonics are listed in Table 1 with fundamental pitch taken as the *A* below middle *C*. Harmonics with frequencies that differ significantly from the equally tempered scale are indicated in bold type.

Table 1. The first 24 harmonics of a selected fundamental frequency. Also listed is the nearest pitch in the equally tempered scale. Note that some upper harmonics deviate substantially from pitches of the 12-tone scale.

Nearest Pitch (Hz)	Frequency (Hz)	Harmonic	Nearest Pitch (Hz)	Frequency (Hz)	Harmonic
A (220)	220	1st harmonic	**F♯ (2960)**	**2860**	**13th harmonic**
A (440)	440	2nd harmonic	**G (3136)**	**3080**	**14th harmonic**
E (659)	660	3rd harmonic	G♯ (3322)	3300	15th harmonic
A (880)	880	4th harmonic	A (3520)	3520	16th harmonic
C♯ (1109)	1100	5th harmonic	A♯ (3729)	3740	17th harmonic
E (1318)	1320	6th harmonic	B (3951)	3960	18th harmonic
G (1568)	1540	7th harmonic	C (4186)	4180	19th harmonic
A (1760)	1760	8th harmonic	C♯ (4435)	4400	20th harmonic
B (1976)	1980	9th harmonic	**D(4698)**	**4620**	**21st harmonic**
C♯ (2218)	2200	10th harmonic	**D♯ (4978)**	**4840**	**22nd harmonic**
D (2349) **D♯ (2489)**	**2420**	**11th harmonic**	**E (5274)**	**5060**	**23rd harmonic**
E (2636)	2640	12th harmonic	**E (5274)**	**5280**	**24th harmonic**

Other Uses of the Word "Harmonic" in Mathematics

In mathematics, the word "harmonic" appears in a number of contexts, all of which trace their origins to the overtone series and associated physical vibrations. A harmonic progression is defined as the term-by-term reciprocal of an arithmetic progression. For example, the arithmetic sequence $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, . . . , $a_n = n$ gives rise to the harmonic sequence $h_1 = 1$, $h_2 = 1/2$, $h_3 = 1/3$, . . . , where $h_n = 1/n$. In this example, the arithmetic sequence gives the frequency multiples for the overtone series, and the harmonic sequence corresponds to the wavelengths of the respective overtones. The harmonic mean is the reciprocal of the arithmetic mean of reciprocals.

For example, the harmonic mean of two numbers x and y is defined as $2\left(1/x + 1/y\right)^{-1}$. The harmonic series in mathematics is the infinite sum $1 + 1/2 + 1/3 + \ldots$, providing the canonical example of a series whose terms approach zero, but nevertheless, the sum diverges. The harmonic oscillator is a differential equation whose solutions are sinusoidal functions that can be used to model musical sounds. Harmonic analysis is the study of functions (or signals) by decomposition into fundamental component functions by means of the Fourier transform or other techniques. In the study of complex variables, harmonic functions are generalizations of the sinusoidal functions that model fundamental vibrations.

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Eric Barth

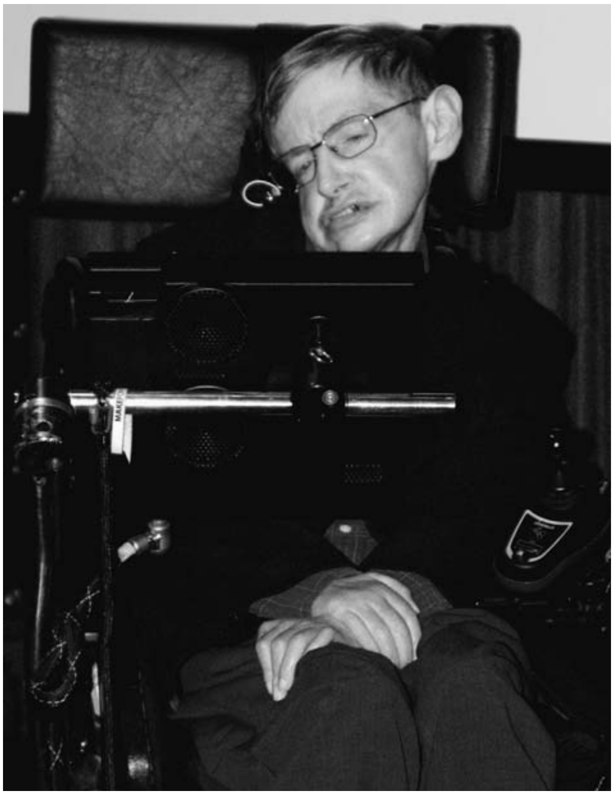
See Also: Geometry of Music; Pythagorean and Fibonacci Tuning; Scales; Wind Instruments.

Hawking, Stephen

Category: Space, Time, and Distance.
Fields of Study: Algebra; Geometry; Measurement.
Summary: Stephen Hawking's work has popularized the mathematics of the universe.

Stephen Hawking is a theoretical physicist recognized for his groundbreaking scientific work concerning the relationship between black holes and the beginning of the universe. He studies the basic physical laws governing the universe in an effort to understand how the universe began. Hawking uses complex mathematical models to explore his ideas and develop his scientific theories. He believes anyone can understand the basic ideas of his research and endeavors to share his excitement about science with anyone who is interested, regardless of academic background. He is the author of several books written for nonscientists explaining the concepts of his research on black holes and the universe. His best-selling book *A Brief History of Time*, first published in 1988, sold over 9 million copies as of 2010 and is published in more than 30 languages. His personal story is also a human interest story. At the age of 21, while attending Cambridge University, Hawking was diagnosed with amyotrophic lateral sclerosis (ALS), also known as "motor neuron disease." This degenerative disease affects voluntary muscle coordination. ALS does not affect brain function and Hawking is able to continue working in spite of the disabling effects of the disease. His confinement to a motorized wheelchair, his use of a computer-generated voice synthesizer, and his many appearances in the media have made Hawking one of the most recognized scientists around the world.

Born January 8, 1942, in Oxford, England, Hawking realized at a young age that he wanted to study science. He attended Oxford University and planned to study mathematics, but chose to study physics because the university did not have a program in mathematics. After completing his studies at Oxford, Hawking earned a Ph.D. in cosmology from Cambridge University. While working as a research associate at Cambridge, Hawking became interested in the study of black holes and the history of the universe. In his dissertation, Hawking theoretically proved that the universe began as a single point of infinite density, known as a "singularity." As a theoretical physicist, Hawking relies on mathematical



Stephen Hawking during a press conference at the National Library of France in 2007.

models to describe and build scientific theory, which can then be supported or refuted through observation. In particular, Hawking applies known mathematics to study particular objects in the universe, called "black holes," and to study the universe itself. His work concerning the origins of the universe is built from the mathematical model of the general theory of relativity. This theory, developed by Albert Einstein and published in 1915, describes gravity in terms of its geometric relationship to space and time.

Using the mathematics of general relativity, Hawking demonstrated that the equations imply the universe had a beginning as a singularity. This moment at the beginning of the universe is known as the "Big Bang." However, theoretical physicists such as Hawking have been unable to determine the precise conditions that enable the Big Bang to occur because the mathematics of time and space become undefined at the point of a singularity. Singularities also occur when stars collapse under their own gravitational force and become black holes. Applying the mathematics of general relativity, Hawking also demonstrated that time should come to an end inside a black hole, although the equations are again undefined at the point of a singularity. His research into the structure of black holes further led to his development of a theoretical model concerning radiation emitted from black holes, which is known as "Hawking radiation."

Hawking states he has always been intrigued by life's big questions and wants to find scientific answers to those questions. His extensive work in the mathematical exploration of black holes and the structure of the universe has led to profound insights in the fields of theoretical physics and cosmology. Among his many academic honors and awards, he held the prestigious Lucasian Chair of Mathematics at Cambridge University from 1979 to 2009, a post once held by Isaac Newton. He continues to work toward his goal of achieving a complete understanding of the universe and why it exists as it does. He and other theoretical physicists are searching for mathematical models to combine quantum mechanics (the study of subatomic particles) and general relativity. He claims that he does not particularly enjoy working with complex mathematical equations because he does not find them intuitive. Rather, he thinks about his ideas geometrically by envisioning mental pictures and visual images. It is these mental pictures and images he uses to try to convey his theoretical ideas. Since writing *A Brief History of Time*, he has continued his efforts to share his ideas and has written several books for the nonscientist, including *Black Holes and Baby Universes and Other Essays* and *The Universe in a Nutshell.*

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See Also: Black Holes; Einstein, Albert; Geometry of the Universe; Relativity.

Helicopters

Category: Travel and Transportation.
Fields of Study: Algebra; Geometry.
Summary: Helicopters apply vertical thrust to overcome their weight.

A helicopter is a type of aircraft that overcomes gravitational force by employing spinning blades to generate vertical thrust. The ideas of vertical flight can be traced back to the Chinese and to Leonardo da Vinci. Thomas Edison studied several different propeller designs and concluded that a feasible helicopter needed a lightweight engine that could produce a large amount of power. Mathematicians such as Theodore Karmen and George de Bothezat also worked on helicopter design in the early twentieth century. In modern helicopters, downward force is supplied by an engine driver rotor. A helicopter has many advantages over a fixed-wing aircraft, such as the ability to take off and land vertically, to hover, and to fly backwards and laterally in the air. As the main rotor spins, it generates a torque that could set the helicopter into a fatal spin. To compensate for this, helicopters have a smaller rotor and blades on their tails.

Flight Controls

A helicopter has four main flight control inputs that enable it to perform various aerial maneuvers: the cyclic control, the collective pitch control, the anti-torque pedals, and the throttle. The cyclic control changes the pitch of the rotor blades cyclically, enabling the helicopter to move in the desired direction. The collective pitch control controls the altitude of the rotorcraft. The anti-torque pedals change the pitch of the tail, altering the amount of thrust.

Mathematically Modeling Helicopter Flight

Helicopters fly by sucking air from above their rotors and forcing it downwards with a thrust equal to (if hovering), greater than (if climbing), or less than (if descending) their weight. The pressures at various points around a helicopter are given by

$$P_0 + \frac{1}{2}\rho v_{out}^2 = P + \frac{1}{2}\rho v_{in}^2 + \frac{1}{2}\rho v_{out}^2$$
$$= P + \Delta P + \frac{1}{2}\rho v_{in}^2$$

here P_0 is the rest pressure of the air far above the rotors, $P + \Delta P$ is the pressure below the rotors, v_{in} is the velocity of the air as it is sucked in, and v_{out} is the velocity of the air as it is forced down.

There are also equations governing the stability and flight of a helicopter. These take into account the inertial velocities in the moving axes system, the Euler rotations defining the orientation of the fuselage axes with respect to Earth, and the aircraft mass. In the early twenty-first century, mathematicians model areas of helicopter flight and performance, such as aerobatic maneuvers that push the limits of the system and that help inform improvements and future designs of new helicopters.

Transverse Flow and Ground Resonance Effects

In forward flight, because the air is being accelerated for a longer period of time as it travels to the rear of the rotor system, air passing through the rear portion of the rotors has a greater downwash angle than the air passing through the forward portion. This pressure difference causes a decrease in the angle of attack, resulting in less lift in the rear of the rotorcraft, increased angle of attack, and more lift in the front. This is called the "transverse flow effect" and it causes easily recognizable vibrations.

When a helicopter is resting on the ground with its rotor spinning, a destructive harmonic vibration called "ground resonance effect" can develop and is caused by a reaction of the rotor blades to the lateral motion of the helicopter. Ground resonance effect develops when the rotor blades move out of phase with each other and cause the rotor disc to become unbalanced.

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See Also: Aircraft Design; Airplanes/Flight; Geometry in Society; Wind and Wind Power.

Highways

Category: Architecture and Engineering.
Fields of Study: Algebra; Geometry; Measurement; Number and Operations.
Summary: Highway design requires an adequate model of anticipated traffic and a determination of the grade.

In the early twentieth century, a series of Federal Aid Highway Acts aimed to create a national highway system. Considerations for a highway design include government design specifications and speed limits, the planned route's geographical and geological features, water drainage requirements, land use issues such as environmentally sensitive areas, driver comfort and safety, and maximization of the highway's life span. Planners and engineers also gather data and determine the minimum and maximum expected traffic volumes based on number of standard axles, vehicle types, expected uses, driver visibility requirements, and the minimum radius of bends and curves. The mathematics used in designing the combination of horizontal and vertical, and straight and curved, sections of a proposed highway results in a design plan that construction crews follow as they build and maintain the highway. Mathematicians also investigate questions related to highways such as mileage, distance, and traffic issues. Mathematician and physicist Louis Roberts served as director of energy and environment at the Transportation System Center in Massachusetts, a division of the U.S. Department of Transportation that researches and develops transportation-related energy conservation practices.

Modeling Highways

Highway designers utilize mathematics to create a three-dimensional layout when planning the horizontal and vertical sections that comprise a highway. The plan view (x and z coordinates) shows the proposed highway's horizontal alignment, which is comprised of straight sections known as "tangents" and the horizontal curves that connect them. The profile view (y axis) shows the proposed highway's vertical alignment, which is comprised of the various slopes known as "grades" at points along the highway. Computer software programs enable modern engineers to create visual models of the plan route and aid in the mathematic calculations involved.

One of the key calculations of highway design and construction is the determination of the necessary grade along the various sections that comprise the highway, defined as the measure of the highway's slope. The grade of a section of highway is calculated using the equation grade = (rise ÷ run) × 100. This equation divides the highway's height increase along that section, known as the "rise," by the horizontal distance a vehicle on a level highway section travels, known as the "run." Designers express distance as stations, whereby one station is 100 feet of highway alignment distance.

The resulting decimal calculation gives the ratio of rise-to-run, which is the grade of that particular section of highway. The decimal grade is then converted to and expressed as a percentage through multiplication by 100. Grade calculations are used to ensure smooth traffic flow along the highway and along the intersections between highways and other roadways as well as to ensure proper water drainage. Designing the proper grade can also help reduce fuel consumption and prevent accidents. During construction, crews move and level the dirt along the right-of-way to create the desired grades.

Vertical Curves

There are two types of vertical curves used in highway design: sag vertical curves and crest vertical curves.

Highway Safety

Vehicles traveling along a highway must be able to safely transition between the different gradations and straight sections of a highway. Designers incorporate horizontal and vertical curves to ensure a gradual transition. Designers use mathematical calculations to ensure that the centrifugal forces created by driving along curved surfaces will not adversely affect the vehicle. The calculations involved in designing curves take a variety of data into account, including designed vehicle speeds, geological features, highway and vehicle types, grade, driver sight line obstructions, stopping distance, and connections with other roadways.

The difference between the two is the measurement between the tangent grades at the starting and ending points of the curve, expressed as a percentage. An ending tangent grade that is higher than the beginning tangent grade defines a sag vertical curve, while an ending tangent grade that is lower than the beginning tangent grade defines a crest vertical curve. Thus, a sag vertical curve has a negative value and a crest vertical curve has a positive value. These measurements and calculations combine to create the completed highway design plan.

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See Also: HOV Lane Management; Smart Cars, Street Maintenance; Traffic.

Hitting a Home Run

Category: Games, Sport, and Recreation.
Fields of Study: Algebra; Data Analysis and Probability; Geometry.
Summary: Home runs in baseball can be mathematically analyzed according to numerous factors, including ballpark design, altitude, and initial velocity.

A home run in baseball happens when the batter circles all the bases in a single play. This typically results from the ball being hit over the outfield fence. In modern baseball, a home run rarely occurs as a result of hitting the ball so that it is still in a state of play inside the field—an "inside the park" home run. Home runs

Albert Pujols of the St. Louis Cardinals hitting a home run, one of the most exciting plays in baseball.

are considered to be some of the most exciting plays in baseball, and a great deal of time and effort is spent trying to help batters achieve this skill.

A number of factors are at work in hitting a home run, including the player's stance and swing, the flight path of the ball, and the characteristics of the outfield wall, which are not standardized in U.S. baseball stadiums. For example, the Boston Red Sox stadium is renowned for its left field wall, named "The Green Monster," which is much taller than average outfield walls, but it is only a little more than 300 feet from home plate—a fairly short distance in professional baseball. Probability and statistics are also used to analyze home runs, though differences in the game over time make some mathematical comparisons challenging. The 1961 race between Roger Maris and Mickey Mantle to break George Herman "Babe" Ruth's home run record was widely followed and highly controversial, in part because the increased number of games in the season made direct comparisons of the number and rate of home runs problematic.

There are various techniques, training schools, and methods to improve a batter's chances of hitting a home run. Contributing factors considered in some of these methods include the mass of the baseball bat and the speed at which the bat is swung. A projectile equation is used to model the motion of the ball as a parabola, using these variables as input. The distance traveled and the greatest height achieved both depend on initial conditions starting from when the ball hits the bat: height, angle, velocity in units of distance per second, and other factors such as altitude above sea level. There is also a great deal of research in sports medicine and kinematics. Some of this research focuses on batter and performance variables, such as age, bat grip, bat speed and velocity, reaction time, and visual cues, though the fundamental mathematical analyses of trajectory do not differ.

Mathematicians and scientists have also developed computer simulations designed to model batting. These often allow multiple parameters to be modified dynamically and quickly to explore and to visualize results. One such simulator shows that when a batter hits a baseball, the air resistance, speed, and angle all have an effect on where the ball goes. It further allows the user to choose a stadium location and then alter the speed, angle, and altitude to observe the success or the failure.

Baseball statistics may be more familiar to the wider sports audience, from the numbers that appear on the backs of baseball cards to the advanced mathematical analyses of sabermetrics. For example, in September 2007, four Los Angeles Dodgers players hit four home runs in a row. This was only the fourth time this had ever happened in over a century of major-league baseball. A sports-business professor calculated the odds as 1:3,300,000, a number that gained wide attention in the media.

Then, roughly a month later, four Boston Red Sox players repeated the exceptionally rare feat, spurring alternative calculations and discussions among statisticians and sports analysts. Mathematician Howard Penn used statistical hypothesis testing to determine whether the Colorado Rockies' practice of humidifying their baseballs (to counter the beneficial effects of high altitude on distance), actually reduced their overall number of home runs. He concluded that there was a statistically significant decrease, though the park was still "home run friendly."

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See Also: Arenas, Sports; Baseball; Basketball; Fantasy Sports Leagues; Football; Kicking a Field Goal.

HIV/AIDS

Category: Medicine and Health.
Fields of Study: Algebra; Data Analysis and Probability; Measurement.
Summary: Epidemiological models track and estimate human immunodeficiency virus (HIV) and acquired immunodeficiency syndrome (AIDS) while immunological studies generate probable values for use in immune dynamics models to evaluate possible treatments.

The human immunodeficiency virus (HIV) is a type of retrovirus that targets the immune system. Retroviruses replicate by encouraging host cells to make copies of their own ribonucleic acid (RNA) after invading them. The immune system is designed to fight viruses and infections, but HIV targets the immune system and progressively destroys the body's ability to fight infections and certain kinds of cancer. People with HIV may get life-threatening diseases called "opportunistic infections," and they can later develop what is known as acquired immunodeficiency syndrome (AIDS). Mathematical and statistical techniques are used to track the spread of disease, estimate the number of cases, define various parameters for describing incidence and prevalence, and evaluate the clinical tests that are used to identify HIV and AIDS.

Historically, AIDS has been defined as a syndrome of several different illnesses that occur when the immune system fails. AIDS was first clinically identified in the 1980s and called "gay-related immune deficiency" (GRID), because the illness initially appeared in men who had sex with men. The cause was not known, but there were many theories, including cytomegalovirus and certain drugs. Later, AIDS cases emerged in both males and females who had received blood transfusions, suggesting an infectious agent with bodily fluids as a transmission vector. As early as 1983, scientists isolated a virus later named HIV, which was ultimately correlated with AIDS. As with most diseases, cause is inferred though animal testing and by making comparisons between people with and without the proposed causal agent. In 1993, a more precise definition was adopted by the Centers for Disease Control and Prevention (CDC), which added the criterion of a person having less than 200 CD4+ T-lymphocytes/μL or less than 14% CD4+ T-lymphocytes. In 2009, evidence suggested the human form of HIV developed sometime between 1884 and 1924, far earlier than originally believed.

Prevalence

HIV prevalence refers to the overall percentage of a population that has HIV, while HIV incidence refers to the rate of new infections that occur in a given year. Historically, tracking HIV has been difficult because AIDS surveillance registries relied upon AIDS cases that were reported, and data were then extrapolated with statistical models to estimate the prevalence in the wider population. Other data collection alternatives are available. Some methods involve surveys of different groups of people, including high-risk groups. Others, like prospective cohort studies, track samples of people over time. Globally, at the start of the twenty-first century, heterosexual women are most at risk of acquiring HIV. In Western countries, men who have sex with men are considered a high-risk group. Prisoners are also at risk because inmates may engage is high-risk behaviors. HIV is considered a pandemic, and a typology was developed to classify geographic regions according to the type of epidemic. Generalized type means that HIV prevalence is greater than 1% in pregnant women. Concentrated type means the prevalence is greater than 5% in some subpopulation but less than 1% in pregnant women. The low-level type has a prevalence of less than 5% in any subpopulation and less than 1% in pregnant women.

Epidemiology

There are many types of epidemiological models that are used in HIV/AIDS tracking and estimation. They take into account different variables, including the size of the overall population; the proportion of people who are already infected; the size of sexually active or other at-risk subgroups; the number of people who leave the population for various reasons, including deaths from AIDS; and the number of new sexual partnerships that people may form. They may calculate quantities such as risk of transmission between individual members of the population or overall population rate of transmission. A statistic called the "basic reproductive number" is often used to quantify transmissibility. A value less than 1 for this measure implies that a disease will eventually die out, assuming that the values entered into the model do not change over time. A value greater than one implies that the infection will spread. Very large values imply an epidemic, which may be difficult to control. Treatments and other interventions can reduce the infectiousness of HIV, which affects values like the basic reproductive number.

Many models are simplified, assuming that all individuals in a hypothetical population have the same patterns of sexual behavior. Factoring in individual differences in sexual behavior may increase realism in statistical models of HIV risk and infection. Many parameters could change, depending on factors like age, particularly sexual behavior and infectiousness of the virus. Some other variables that could be considered are types of sexual activity, the number and type of sexual partners, condom use, and HIV testing. The role of treatment in mitigating transmission could also be considered. Researchers from the Amsterdam Cohort Study created a more complex mathematical model that described the spread of HIV in one high-risk group. The researchers took into account many individual behavioral variables and were able to show that the majority of new HIV infections were because of main partners, not casual partners as was previously assumed. This had important implications for targeting risk-reduction messages to men in long-term relationships.

Testing

In the twenty-first century, politicians and reporters have questioned whether mandatory HIV testing should be required. In 2003, the CDC initiated the Advancing HIV Prevention: New Strategies for a Changing Epidemic program and tested the possibility of rapid HIV tests in some emergency rooms. HIV tests have a high sensitivity rating, with some tests listed as 99.7% accurate, meaning only 0.3% of the people with HIV will falsely test negative. The remaining people with the disease will correctly test positive. However, even with this degree of accuracy, one concern about universal testing is the possible number of false positives (people who test positive but do not actually have the disease). The percentage of false positives with these tests is also small. Yet, in the United States, because most of the population is HIV negative, the small percentage would be multiplied by the very large number of people in the HIV-free population and would result in many false positives, perhaps more than true positives. Repeated testing or the development of more accurate tests can mitigate the impacts of false positive results.

Immunology

Immunology is the study of the immune system's response to a pathogen, and mathematical models of HIV immune dynamics can be constructed. Data from experiments can be used to find plausible values for the mathematical model, such as the expected life of CD4+ T-cells. These models are particularly useful when evaluating or predicting the success of treatments that interfere with the replication of retroviruses. Mathematical models of immune dynamics are very complex and require revisions as knowledge about HIV changes. Collaboration between mathematicians and clinicians is important so that models can be maximally effective in preventing HIV spread and improving health outcomes for people infected with the disease.

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See Also: Disease Survival Rates; Diseases, Tracking Infectious; Life Expectancy; Sample Surveys; Viruses.

Hockey

Category: Games, Sport, and Recreation.
Fields of Study: Data Analysis and Probability; Geometry; Measurement.
Summary: Playing hockey is an application of geometry, as players in constant motion determine angles of approach, plot routes through opponents, and visualize the vector of the puck.

Ice hockey is a team sport played on an ice rink by skating players using sticks to move a rubber disk called a "puck" into the opposing team's goal. Field hockey and street hockey are usually played on foot, either on grass fields or street surfaces, using a ball. There is evidence that hockey-style games have existed for millennia, and ice hockey has long been popular in parts of the world that are cold enough for long-lasting seasonal ice. The basic rules of modern ice hockey were developed in Canada in the late 1800s, and the National Hockey League of North America (NHL) dates back to the early 1900s. The growing prevalence of indoor ice rinks has allowed hockey to expand into warmer places, like Florida and California, with mixed success. Ice hockey is highly geometric, in terms of both player action and the surface on which it is played. Mathematics and statistics are also used to model various aspects of game play and to develop improved equipment.

Geometry

A hockey rink is in some ways more geometric than other sports surfaces. Overall, the ice is essentially rectangular. North American professional rinks have corners that are rounded on a circle with a radius of 28 feet. Rinks have mirror symmetry end-to-end and side-to-side, including five circles used for face-offs. The goalie primarily occupies the space in front of the goal known as the "crease," which is a half-circle with a six-foot radius in international play. In North American professional rinks, the crease is truncated to eight feet wide by transecting lines drawn one foot on either side of the six-foot-wide goal. Aside from the crease, goalies in some professional leagues may play the puck only in the goaltender's trapezoid. This symmetrical region has one 18-foot base formed by the goal line and another 28-foot base determined by the boards (the wall behind the goal).

Hockey also requires an awareness of geometry for competitive play. Players are in constant motion and thus always calculating the best angle at which to approach an opponent, based on the opponent's speed and trajectory, as well as the best route through the moving players. Turning and stopping on ice require different applications of forces than sports played on foot, with arcing turns or various radiuses being more common than point pivots and sudden reversals. Being a hockey goalie is an ongoing exercise in mathematics and physics. Geometric ideas like circumferences, radiuses, and angles are very important, as is the ability to visualize vectors. Goalies shift within the crease in response to the continuously changing locations of other players in the plane of the rink to simultaneously minimize opponents' possible angles of attack and maximize their ability to intercept the puck. Time series analyses of several decades of data have shown that NHL games steadily average about 30 shots on goal per 60-minute game. There have been vocal critics of the artificial intelligence used for hockey goalies in some video games, with assertions that the programming fails to accurately mimic the sort of continuous precision adjustments used by real goalies. Hockey terminology has been used with some students to motivate and teach geometric concepts.

Statistics

Sports fans have become increasingly interested in studying sports statistics for prediction and deeper analyses. Operations researchers Jack Brimberg and William Hurley investigated the common belief that the first goal in the game "sets the tone" for the rest of the game. They calculated that the team that scored first was more likely to win, especially if the first goal was scored later in the game. Others have analyzed the way in which the NHL determines which teams will compete in the play-offs. There are 82 games in the regular NHL season. Points are awarded to the teams as follows: two points for winning the game, zero points for losing in a regulation 60-minute game, but one point for losing if the game went to overtime. No other league rewards a team differentially for losing in overtime. The intent is purportedly to keep tied teams playing competitively in the third period. However, data suggest that teams tend to rein in play and allow the game to go into overtime, which mathematical game theory suggests is the better move, because the reward for winning is the same, but the penalty for losing is reduced. A European system changes optimal strategy because the winner gets only two points in overtime versus three.

Hockey Equipment

Hockey equipment also benefits from mathematics. Helmets have become mandatory in most hockey leagues, and researchers are continually seeking ways to better disperse the powerful kinetic energy of blows and collisions. Iconic player Robert "Bobby" Hull is credited with introducing curved blades on hockey sticks, which improves control and accuracy. Many players still use traditional wooden hockey sticks, but researchers have also developed flexible, lightweight composites and aluminum sticks, often involving statistical analyses and modeling. Physicists have also used mathematical models to analyze the characteristics of different hockey shots.

Other Connections to Mathematics

In climate science, Michael Mann, Raymond Bradley, and Malcolm Hughes quantitatively reconstructed temperature trends for the last 1000 years, producing a controversial graph called the "hockey stick graph," since its changes in slope resemble the bend of a hockey stick. One theorem regarding diagonals in Pascal's Triangle, named for Blaise Pascal, is also sometimes known as the "hockey stick theorem" for the shape it produces.

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Bill Kte'pi

See Also: Arenas, Sports; Fantasy Sports Leagues; Skating, Figure.

Home Buying

Category: Business, Economics, and Marketing.
Fields of Study: Algebra; Data Analysis and Probability; Number and Operations; Problem Solving.
Summary: Interest rates on mortgages are set using sophisticated mathematical techniques.

Homes are the largest single purchase most people make in their lifetimes. Buyers usually take out a home loan, called a "mortgage," rather than pay cash. When the desired property is identified and funds for a down payment (typically 20%) are acquired, home buyers work with a bank or other lender to finance the purchase. When determining what a person can afford to borrow, lenders consider several variables. In the past, these judgments were often highly subjective decisions made by individual lender agents, but the increased popularity and availability of credit cards in the latter half of the twentieth century, as well as federal legislation designed to combat discrimination in lending, required more objective methods of assessment.

For example, FICO scores mathematically measure risk of nonpayment. Created by Fair Isaac Company (FICO), founded by engineer Bill Fair and mathematician Earl Isaac, an individual's FICO score is a weighted combination of variables such as previous credit performance, current debt, and length of credit history. Also, lenders may use debt-to-income ratios to indicate the size of payment a borrower can afford. Comparing home loans can be challenging because different lenders may use this information differently. Also, interest rates, closing costs, and additions to the base payments need to be considered in the comparison. The process of buying a house may involve additional expenditures beyond the mortgage. Buyers routinely hire a home inspector to independently assess the condition of the home, and many such inspectors charge a fee based on the square footage. In some areas, radon tests or soil analyses might be required. If problems are found, either the buyer or the seller may have to hire a structural engineer or other professional to rectify the problem before an agreement is reached or the loan approved. Property taxes, based on the assessed value of the home and land, and homeowners insurance, which is also a function of the assessed value and replacement cost of the home and its contents, are also part of almost all home-buying transactions. Home buying tax credits or reduced interest rates may offer the buyer additional options and are designed to stimulate the economy.

FICO Scores and Credit Ratings

Each of the three major credit bureaus (Experian, TransUnion, and Equifax) calculate a credit score based upon advice from the Fair Isaac Company, an independent company that specializes in business analysis, including risk assessment. However, not all three companies use identical inputs, and each may yield a different result. A FICO score is between 300 and 850, with higher scores indicating better risks. The exact formula used for FICO score calculation is proprietary and changes periodically, but the personal data incorporated in the FICO formula include, in order of importance, payment history; amounts owed; length of credit history; new credit applied for; and types of credit used. The FICO score influences requests for credit, and many banks charge higher interest rates to people with lower scores.

Debt-to-Income Ratio

Most standard loan applications request information on both income (annual income, bank account balances, investments such as stocks and bonds) and debt (amount owed on loans, credit cards, and standard monthly bills like car payments, utilities, insurance). This information can be used to determine the percentage of income already committed to be spent each year: debt-to-income ratio = total debt ÷ total income.

If a prospective borrower's expected debt-to-income ratio is higher than the bank's cutoff (most banks have a limit between 32% and 40%), a loan may be denied, particularly if the prospective borrower's FICO score is low. A high FICO score might result in the prospective borrower being granted the loan even with a higher debt-to-income ratio.

Calculating a Mortgage Payment

The principal (amount to be borrowed), the annual interest rate, the payment schedule (for example, monthly or bimonthly), and the length of the loan (10, 15, 30 years, for instance) all factor into the payment. The formula for the payment (R) is given below, where P is the principal, r is the adjusted interest rate (the annual rate divided by the number of payments in one year), and n is the number of payments to be made over the life of the loan

$$R = \frac{rP}{\left(1-(1+r)^{-n}\right)}.$$

Additions to this base payment include the following:

- PMI: Borrowers paying less than 20% down may be required to purchase private mortgage insurance (PMI) to protect the lender's investment. The cost of this insurance is added to the loan payment.
- Escrow: Many banks require that payments be made into an escrow account to accrue funds to pay property tax and insurance.

Finalizing a Mortgage Loan

Transaction fees for processing a mortgage are more commonly called "closing costs"; hence finalizing a home loan is called "closing." Charges to be paid when closing on a loan typically include an origination fee for the lender, appraisal fee for the appraiser, title search and recording fees for the attorney, and points. Points are up-front interest fees charged by the lender, with one point costing 1% of the principal. Banks often give borrowers the option of purchasing additional points at closing to decrease the interest rate on a mortgage.

The reduction of interest for purchasing a point can vary from bank to bank; one point may reduce the rate by as little as 0.1% or as much as 0.25%. The cost of the points purchased at closing is tax deductible, as is any interest paid over the life of the mortgage.

Amortization of Loans and Fixed Rates

Loan payments include a portion that reduces the principal balance and a portion that the lender keeps—the interest. The amount of interest included in a payment varies over the life of the loan, but can be determined by remembering that each payment includes "simple interest payable on the balance." Calculating the schedule of payments, including the split between principal and interest for each payment, is referred to as "amortizing." The Latin roots of the term mean "death pledge," indicating linguistically the willingness to forfeit something of great value if the debt is not paid. In this case, failure to pay the debt results in foreclosure by the bank and the loss of the property.

To illustrate, suppose a home buyer borrows $100,000 at 6% fixed annual interest (the interest rate does not change over the life of the loan) payable monthly for 30 years (360 payments). Using the loan formula, the monthly payment is $599.55, assuming no PMI or escrow. Since the homebuyer is paying 6% annual interest and 12 payments a year, the adjusted (monthly) interest rate is 0.5% for all payments.

Thus, the homebuyer owes the lender 0.5% of $100,000 in the very first payment; $500 will be kept by the lender as interest and $99.55 will be used to reduce the principal. For the next payment the balance is $99,900.45; the interest will be 0.5% of that balance or $499.50. These calculations can be summarized in an amortization table, which is usually provided to the buyer as part of the mortgage agreement. The first and last several rows for this example are presented in Table 1. As the loan progresses, the interest portion decreases and the remaining amount applied to the principal increases.

Table 1.

Payment #	Interest Owed	Payment	Principal Paid	Balance
Closing				$100,000.00
1	$500.00	$599.55	$99.55	$99,900.45
2	$499.50	$599.55	$100.05	$99,800.40
3	$497.99	$599.55	$101.56	$99,497.24
...	...	...	...	...
358	$8.90	$599.55	$590.65	$1,190.17
359	$5.95	$599.55	$593.60	$596.57
360	$2.98	$599.55	$596.57	$(0.00)

Variable Rates

The example uses a "fixed" interest rate, but many lenders also offer variable rate loans, meaning that the interest rate may be changed according to some economic indicator (called the "index"), such as the prime rate. There are legal restrictions on this practice: the lender must inform the borrower of the size and frequency of such changes (called the "interval"), and the maximum (called the "cap") for the rate. For example, a two-year adjustable rate mortgage (ARM) payable monthly over 30 years might have the following particulars: 6% to start indexed on the 6-month U.S. Treasury Bill, and adjustments of at most 2% are allowed every two years with a cap of 10%.

The initial calculations, such as down payments or the payment amount, do not change in this case. For this two-year ARM, the first two years of the amortization table above does not change. One important question to ask the lender when considering an adjustable rate loan is what happens to the payment when the interest changes? Most redo the amortization calculations starting at the next payment, so a new rate would mean a new payment amount.

The housing crisis of 2007–2009 resulted in many homeowners finding themselves with homes whose values had decreased to the point that they were worth less than the amount owners still owed on their mortgages. There were many factors that influenced this outcome. Prior to the Great Depression, home ownership was much more rare than in the early twenty-first century when homes were often financed with balloon-payment mortgages in which a loan is amortized over only part of its lifetime, leaving a large principal payment due at the end. The federal push to open the housing market using fully amortized, fixed-interest mortgages required lenders to assume much greater financial risk, which can be mathematically modeled but not perfectly predicted. To manage that risk, mortgages became financial commodities in the larger financial marketplace. Housing prices, interest rates, and other aspects of financial markets are highly variable, and some people blamed the housing crisis on too much reliance on sophisticated mathematics.

In general, it was probably not the models themselves but the sometimes-incorrect ways in which the models were often used. In addition, many lenders ignored reliable risk predictors, such as FICO scores and debt-to-income ratios, resulting in more people taking on higher loan payments than they could afford. Home prices rose from demand to the point where properties were extremely overvalued. They later decreased in value, so the property was worth much less than the balance on the loan, leading to a large increase in foreclosures. Homeowners defaulted on loans, ruining their credit ratings, and banks paid large foreclosure fees. There were also more short sales, where banks agreed to accept less than the mortgage balances when homes sold to avoid foreclosure charges and poor credit ratings for the homeowners.

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Holly Hirst

See Also: Bankruptcy, Personal; Budgeting; Comparison Shopping; Loans.

Houses of Worship

Category: Friendship, Romance, and Religion.
Fields of Study: Connections; Geometry.
Summary: Sacred spaces have often been designed with special attention paid to their geometry.

Houses of worship are places dedicated to spiritual and religious practices. Because spatial metaphors are universally important to humans, many faiths tie their religious ceremonies or practices with special architecture, decoration, and visual symbols that promote spiritual contemplation or changed states of consciousness or, in some cases, serve to teach aspects of the religion. Mathematically rich areas of study and practice related to houses of worship include sacred architecture and sacred geometry. The mathematical patterns from these studies are frequently investigated within both mainstream science and religions, for example, under the umbrella of ethnomathematics.

Houses of Worship as Motivators for Mathematics and Science

Because houses of worship typically had special meanings, they were often constructed differently than other buildings. At times, this elevated the architectural and aesthetic demands on both designers and builders, who found themselves challenged with erecting unusual and often very large structures, which promoted applied mathematics. Their methods remain a source of debate in modern scientific circles, since it is not clear exactly which engineering methods were used to lift the enormous monoliths that make up Stonehenge in England, to fit together large blocks forming the ancient Egyptian Luxor temples or pyramids, or to orient Sumerian temples with compass directions. However, it is clear that the desire for special sacred architecture could provide justification to spend time, materials, and other resources in mathematically rich ways.

A Sampler of Mathematical Features in Houses of Worship Through History

Sophisticated mathematical ideas and principles can be found in Sumerian ziggurats, which were built in ancient Mesopotamia. These structures have a characteristic "gigantic step" shape made of two to seven receding tiers. The top tier, where historians assume rituals were performed, could only be reached through narrow ramps. This feature would have isolated the priests and made the defense of the top tier easy. This was important, as ziggurats also served as city administrative centers, and the shrine complexes contained within them housed kings who performed rituals or who may have been considered deities themselves by the people they ruled. Such ties between ruler and divinity, or the concept of the divine right of kings, were found in many of the ancient cultures, including Egypt, as well as in many European monarchies. This principle may also have played some role in design. The ziggurat tier shape is still frequently used in modern architecture.

In Japan, Shinto shrines and Buddhist temples encouraged congregants to bring small wooden tablets called *sangaku* as offerings to gods. They have been traced back to the beginning of the Edo period of the seventeenth century. Temple visitors painted colorful sangaku to share Euclidian geometry puzzles, named for Euclid of Alexandria, or solutions and variations on puzzles others shared earlier.

Hindu temples were based on vedic mathematics. Their shape was usually square, with sides divided into eight or nine parts. This defined 64 or 81 smaller squares within the temple, dedicated to different gods. The whole temple, in a fractal manner that appears in many houses of worship, represented both the universe and the inner space of a person and the idea that the two are similar. Town plans often followed these temple plans, to add another level of recursion. Mathematical formulas in India were frequently used as descriptions or metaphors for sacred ideas. For example, one of the words for "temple" is *vimana*, which literally means "well-proportioned." Ratios within the temple symbolized the harmony within the universe and were strictly followed during construction. Even images within temples displayed the sacred ratios of "iconometry."

Ancient Egyptian temples, built strictly along the east-west line, were believed to be symbolic models of the universe. For example, the floor was slant upward from outer to inner courts, symbolizing the rise of the world out of primordial waters. Precise measurements and astronomical observations were required for construction of the temples, which also served as libraries and education centers.

Islamic architecture typically includes repeating geometric patterns, symbolizing Allah's infinite power. The mathematics underlying some of the patterns is so intricate that research papers connecting it to recent discoveries are still published in the twenty-first century. For example, girih design consists of tessellating polygons overlaid with networks of lines, described by the areas of mathematics now called "quasicrystalline structures" and "Penrose tilings," named for Roger Penrose.

These designs can be infinitely extended without repetition. Muquarnas (or stalactite vaults) are nested, self-similar structures consisting of niches hanging from the ceiling. Their two-dimensional projections consist of tessellating shapes of decreasing size, with ratios set in such a way that shapes still fit together. As with compass orientations and direction in some other sacred spaces, the alignment of mosques toward Mecca, the direction of which varies by geographical location, is another topic that is of interest to mathematicians and others.

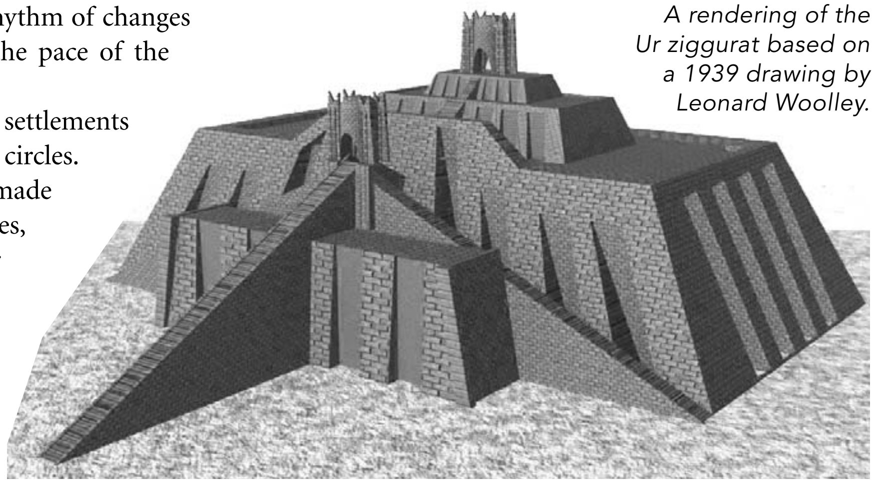
The sweat lodges used by Native Americans for purification ceremonies employ the effects of raised temperature, humidity, and plant smoke rather than specific architectural elements to achieve spiritual effects by affecting the body. Sophistication of the lodge experience comes from matching the rhythm of changes in temperature and humidity to the pace of the ceremony.

Traditional southern African settlements display a fractal structure based on circles. The village is built in a large circle made of smaller extended-family circles, and the large circle in the middle for the chief's family that includes circular huts for honoring the spirits of ancestors. Within each house, the shape is repeated on a miniature scale, with a circular sacred altar in the middle.

Light and Sound

Stained glass windows have long been used in temples and churches. Geometric elements of stained glass windows include reflections, rotations, symmetry, and tessellations. The chemical knowledge necessary for coloring glass and for connecting it with metal strips helped promote scientific development. Dividing large stained glass windows into panels for structural stability was an engineering problem. The colored light, filtering through stained glass, which is variable depending on time of day and weather, is also an important element of the internal atmosphere of the religious space. Another application of light is in the Abu Simbel temples in Egypt, the design of which was reportedly astronomically aligned so that the light rays would reach the innermost sanctum on the birthday of Ramses II.

The dynamic play of light and shadows may also lead to mathematical investigations. Mathematician Thomas Banchoff has commented on the abundance of geometry in houses of worship, which provided him with both spiritual and mathematical inspiration. For example, as a student, he noted, "there was plenty of time to contemplate the shadows advancing across the tiles at the base of the altar rail. When we first arrived, the narrow of the altar rail covered only a small portion of the triangular tiles, and by the end of Mass, almost the entire triangle was in shadow. When, I asked myself, did the shadow cover half the area? I hadn't studied any formal geometry yet, but I figured that if you cut an isosceles right triangle in half



A rendering of the Ur ziggurat based on a 1939 drawing by Leonard Woolley.

by a line perpendicular to the hypotenuse, then one of those halves could be rotated to give the triangle that remains when the shadow was covering half of the original triangle. It surprised me that the line did not pass through the centroid of the triangle! To this day, I still use that example when I teach calculus students about centroids."

Sound as well as light is an important feature of houses of worship. The acoustics of houses of worship are determined by the architecture, including the shape and size of spaces as well as construction materials. Resonance is the property of a system in which it oscillates with larger amplitude at some frequencies than at others. Very high ceilings used in many houses of worship amplify sound and make group singing more resonant and subjectively powerful.

Domed ceilings and other shapes may reflect sound back toward the speakers or singers at many different angles at the same time. This reflection causes interference and reverberation in the sound waves that may make individual words less intelligible, which may promote a feeling of the merging of individuals into the congregation.

Size

Places of worship tended to be some of the largest buildings within each settlement, since they were often the only gathering place for the community for both spiritual and secular uses. Historically, group trading and entertainment also attracted large numbers of people. To symbolize the higher importance of the house of worship compared to secular buildings, some places also required it to be the highest building. This led to sacred architectures that had tall narrow towers, such as steeples on churches and minarets on mosques, soaring high over the town. Large constructions could also provide security, such as the Sumerian ziggurats. Another reason for height was the need for sounds related to religious practices to carry through the settlement, such as calls to prayer from minarets or tolling bells marking the start of services.

Conversely, some spiritual practices call for small, temporary houses of worship. These ephemeral structures evoke meanings opposite to large, permanent houses of worship, such as unity in hard times or intimacy within families; for example, some Native American sweat lodges and the Jewish sukkah—a temporary structure decorated with branches and gourds that is used for a week-long harvest festival. It symbolizes joyful but temporary shelter from the wilderness.

Unusual Modern Designs

Several houses of worship built in the last decades of the twentieth century and the first decade of the twentieth-first century feature interesting mathematical concepts. The Baha'i House of Worship in Delhi, India, also known as the Lotus Temple, is based on a nine-sided polygon. It looks like a half-open lotus flower, with all walls consisting of curved "petals." Modern software allows for the design of such complex surfaces. Wotruba Church, in Austria, consists of 152 asymmetrical concrete blocks and resembles an abstract sculpture. Balancing such blocks and calculating safe loads within the structure presented interesting spatial geometry problems. The Cathedral of Christ the Light in California employs many traditional features in unique ways. From the outside it appears as a truncated cone composed of many semiopaque windows in a steel grid. In the tradition of stained glass windows, the cathedral uses light to create atmosphere and convey images. The church's Omega Window is a representation of a traditional Christian symbol known as Christ in Majesty, which often includes a mandorla frame. The image was taken from an eleventh-century stone sculpture and digitally converted into a pixel-like pattern of 94,000 holes that were drilled into aluminum panels. The holes' varying diameters transmit different degrees of sunlight to create the image.

This is one of many light effects created by the curved internal and external geometry and features like curved beams, folded gothic-style arches, and slats that tilt to manipulate light.

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Maria Droujkova

See Also: City Planning; Mathematics and Religion; Sacred Geometry; Vedic Mathematics.

HOV Lane Management

Category: Travel and Transportation.
Fields of Study: Algebra; Data Analysis and Probability; Measurement; Problem Solving.
Summary: The decision to designate a traffic lane as a High Occupancy Vehicle lane is based on traffic analysis, computer simulations, and mathematical models showing the effects of implementation.

High occupancy vehicle (HOV) lanes are intended to improve automobile transportation efficiency by reserving certain traffic lanes for vehicles carrying at least two or three people. The idea is to encourage carpooling by allowing cars with multiple occupants to use a dedicated lane and thus reduce the number of cars on the road relative to the number of people traveling. Sometimes traffic lanes are designated HOV only at certain times of the day, or they may be used under special circumstances by buses, hybrid power vehicles, or other single-passenger vehicles. HOV lanes have been tested or used in many countries, including the United States, Canada, Spain, the United Kingdom, Norway, Austria, Indonesia, Australia, and New Zealand. Mathematical modeling, data analysis, and computer simulation are widely used for making decisions regarding when and where to use HOV lanes, for designing their construction and geometric properties, and for evaluating their safety and effectiveness. Many mathematical modelers are using cross-disciplinary concepts and approaches to analyze traffic. For example, engineer Morris Flynn and mathematicians Aslan Kasimov, Jean-Christophe Nave, Rodolfo Rosales, and Benjamin Seibold modeled traffic jams using continuous density and flow functions similar to those used for modeling fluid flow and the propagation of detonation waves. Analogous to the traveling nonlinear wave solutions called "solitons," they christened traffic waves "jamitons."

HOV lanes are typically most useful in regions that have severe traffic congestion and many vehicles carrying only the driver. The opportunity to use less-congested, quickly moving HOV lanes is intended as an incentive to encourage drivers to decide to use carpooling or to carry passengers, with the overall intent of reducing traffic jams and accidents caused by traffic volume and lane changing. Studies of HOV lane usage and effectiveness showed that, as of 2008, 21 U.S. states had HOV lanes for a total of 1,745.14 miles with an average density of 833 vehicles per lane per hour and a total of over 276 million miles of vehicle travel. Exclusive HOV lanes were most common (993.27 miles) and carried the highest density of traffic (an average of 906 vehicles per hour), followed by normal lanes designated HOV in certain periods (545.82 miles, 790 vehicles per hour) and shoulder or parking lanes designated HOV in certain periods (206.6 miles, 596 vehicles per hour).

Experience with HOV lanes is mixed, although it should be noted that this is a relatively new method of organizing transportation and that local variation in conditions and implementation could explain why some projects were more successful than others. An example of a successful HOV implementation was that introduced in 1998 near Leeds, United Kingdom (the first HOV lanes in the United Kingdom). Prior to HOV lane implementation, 30% of the cars had two or more occupants, and a journey that should take three minutes if traffic were moving freely regularly took more than 10 minutes. After implementation of the HOV lanes, traffic was reduced 10% to 20%, journeys were quicker for both HOV and non-HOV traffic, lane violations were low, casualties were reduced 30%, and noise reduction was noticeable—although little change was noted in air quality. In the United States, an HOV lane scheme near Washington, D.C., for vehicles carrying four or more occupants, proved successful, with the HOV lanes operating at twice the speed of travel of the regular lanes. However, a study of HOV lanes in San Francisco, California, found that they actually increased congestion. HOV lanes have also been criticized on grounds of safety, because of the differing speeds of traffic in adjacent lanes, and as a violation of the right of motorists to freely use highways paid for with their tax dollars.

Mathematicians continue to investigate issues for HOV lane design, implementation, and management. Analyses using concepts from fields such as geometry, graph theory, and statistics help designers optimize features like lane setbacks, entrance and egress paths, gates and signals, and shoulder widths. Speed contour plots can be used to visualize recurrent blockages, while probability models and scatterplots can be used to quantify and display spatial distribution of accidents as functions of one or more variables. Other mathematicians seek to simplify existing multiparameter models, which may rely on unobservable quantities, using smaller sets of physical and measurable variables in order to study the impact of design features and traffic behavior. Yet others have used logit-type models to investigate economic concerns, like converting HOV lanes to high occupancy toll (HOT) lanes.

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Sarah Boslaugh

See Also: Climate Change; Highways; Smart Cars; Traffic; Travel Planning.

Hunt, Fern

Category: Mathematics Culture and Identity.
Fields of Study: Algebra; Data Analysis and Probability; Geometry.
Summary: Fern Hunt is a prominent mathematician at NIST with diverse research interests.

Fern Hunt is an applied mathematician employed as a prominent researcher in the Mathematical Modeling Group at the National Institute for Standards and Technology (NIST). The daughter of Jamaican immigrants, she earned a Ph.D. in mathematics in 1978 from the renowned Courant Institute of Mathematical Sciences at New York University. She recognizes that, "I am here because of the sacrifice of other black people. I am aware of that and immensely grateful." Before assuming her current position at the NIST, she taught as a professor. Her interest in education and in inspiring and mentoring students has not diminished; in addition to her extensive and varied research, she continues to give mathematics lectures at universities across the country and to work directly with students during the summer.

Fern Hunt's research interests and applications are highly diverse. Her early work was in mathematical biology, including models of behavior of certain bacteria and models of the genetic evolution of populations in a deteriorating environment. At NIST, she studies the physical and chemical properties of many materials used in industry. She says, "I think of myself as your average Jane and the fact that I can discover these connections—every now and again!—gives me a great deal of satisfaction. It means I'm participating in something that's at the root of the universe. Mathematics gives you the opportunity to create." Her work has drawn from many areas, including chaos, dynamical systems, and probability.

A notable example is her work with physicist Robert McMichaels on modeling the Barkhausen effect. The Barkhausen effect, or "Barkhausen noise," is a phenomenon in which the magnetic output of a metallic object has a jumpy, erratic response to a change in magnetic force (the term "noise" is appropriate, since these erratic jumps can be amplified and heard on a loudspeaker as a static-like click pattern). Using sophisticated mathematical tools, Fern Hunt developed a new, much more accurate statistical model of the phenomenon; the new model was able to explain subtle, experimental observations that the previous model could not. A better understanding of this effect has wide practical applications to all the ferromagnetic data storage devices in society, including disk drives and the magnetic stripe on credit cards.

Another important set of projects for Hunt deals with paints and other surface coverings. She studies paints and other such materials at a microstructural level, both measuring and modeling properties such as light-scattering behavior. One innovation of her research program is the use of computer-rendering software to understand and control much more closely how materials will actually appear to the human eye "in real life." Research of this kind is expected to lead to improvements in the materials used by industry.

In addition to the applied research problems arising from the NIST projects, Fern Hunt actively studies ergodic theory and dynamical systems. She has expressed the belief that some mathematical research for its own sake, not directly connected to a current project, is very important—it serves to stimulate creativity and to strengthen one's command of mathematical ideas. This belief is especially important for an applied mathematician such as Hunt, whose NIST projects require the use of mathematical ideas from very diverse and unpredictable parts of mathematics.

Ergodic theory, Fern Hunt's primary area of theoretical mathematical research, is the study of how certain types of systems evolve over time. A simple ergodic system is the circumference of a circle that is being rotated in increments of one radian; if one follows the trajectory of any single point over time, it will eventually come arbitrarily close to every point on the circle. Ergodic theory turns out to have deep connections to geodesic flow, number theory, representation theory, harmonic analysis, and probability theory. The connection to probability theory, in fact, is through Markov chains, a mathematical tool that Fern Hunt has used frequently in her research, such as her improvements to existing models of the Barkhausen effect. This research area is closely related to the mathematics of chaos and fractals.

Fern Hunt has been dedicated to service, and is a member of a number of important committees and boards; she advises, "be in service to others and the world itself. Also try to look beyond day-to-day difficulty and look at maximizing opportunities here and now. This is what keeps me going." She has served on the board of trustees for the Department of Energy and for the Biological and Environmental Research Advisory Committee and has also been part of the American Mathematical Society Committee for Education.

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Michael "Cap" Khoury

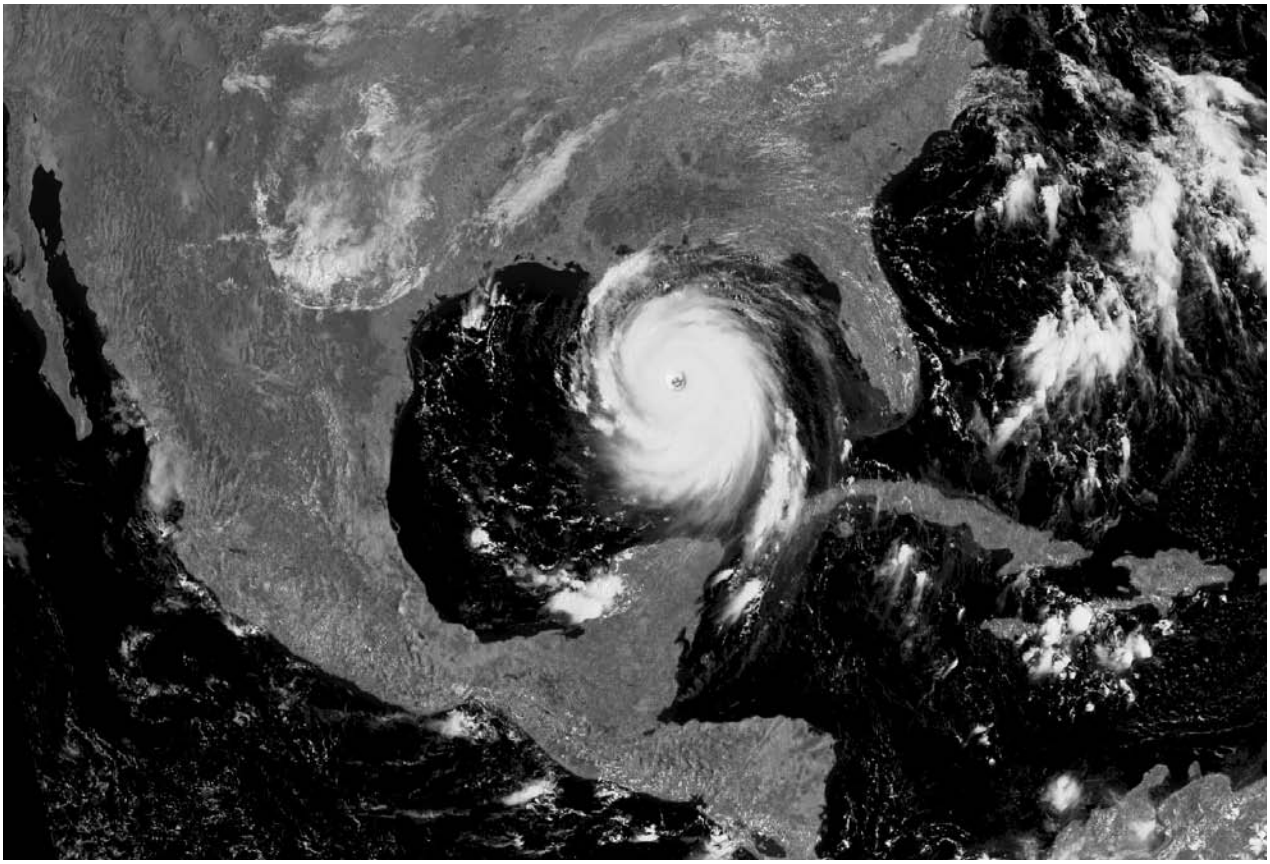
See Also: Careers; Mathematical Modeling; Mathematics, Applied.

Hurricanes and Tornadoes

Category: Weather, Nature, and Environment.
Fields of Study: Data Analysis and Probability; Geometry; Measurement; Problem Solving.
Summary: Mathematical analysis and modeling have been used to attempt to predict and simulate hurricanes and tornadoes.

Hurricanes and tornadoes are both potentially catastrophic types of storms that cause billions of dollars in damage and claim many lives each year. Predicting major weather events of these types is difficult, though mathematical modeling and computer power have allowed mathematicians and scientists to make advances in storm science. The term "cyclone" is often erroneously applied to tornadoes; it properly refers to the class of storms originating over water that includes hurricanes, typhoons, and tropical cyclones. Tornadoes and cyclones are characterized by revolving forms and high winds, but tornadoes are typically smaller, faster spawning, shorter lived, and their damage is usually more focused. Mathematical analysis and modeling of storms draws from many fields.

For example, vector calculus plays a substantial role in analyzing and modeling these storms, since both pressure and humidity can be represented as scalar fields and wind as a vector field. Theories and equations from physics for conservation of mass and energy, along with angular momentum and shear, are also quite important. Historically, challenges in storm description, prediction, modeling, and simulation have often been related to data collection and computing power. One of the earliest systematic data collection and prediction efforts was conducted in the 1880s by John Finley of the U.S. Army Signal Corps, but for a variety of sociopolitical reasons, federal research lagged until about World War II. The emergence of Doppler radar advanced storm science, as did computers in the 1970s that were capable of generating three-dimensional models. However, even in the twenty-first century, no one can perfectly predict the emergence, path, strength, or damage of a hurricane or tornado. Even with multiple stations and satellites, data are still sometimes sparse or difficult to integrate across sources, and this type of research raises theo-



A weather satellite image of Hurricane Katrina in the Gulf of Mexico. Starting as a slight pressure difference, hurricanes grow into large spiraling storm systems of low pressure, complete with high winds and driving rain.

retical questions about the limits of predictability. At the same time, early warning systems that give even a few hours of notice regarding approaching storms are widely considered to be beneficial, and mathematicians continue to contribute to this area. Actuaries are also involved in calculating the costs of these storms, in terms of both money and lives.

A tornado is a rotating column of air that is in contact with both the ground and a cloud. Tornadoes are generally spawned by thunderstorms. The United States has the highest incidence of tornadoes of any country in the world, in part because of the confluence of cold air from Canada, warm, moist air from the Gulf of Mexico, and dry air from the Southwest. A related phenomenon is water spouts, which are essentially tornadoes that form over water, especially in tropical areas. A hurricane is a powerful, spiraling storm that begins over a warm sea, near the equator. "Hurricane" is, in fact, just one name for the kind of storm scientists refer to as a "strong tropical cyclone." Depending on where they occur, hurricanes are given a different label. If they begin over the Atlantic Basin (Atlantic Ocean north of the equator, the Caribbean Sea, the Gulf of Mexico) or the Northeast Pacific Ocean, they are called "hurricanes."

When the same kind of storm occurs in the western North Pacific Ocean, it is called a "typhoon." In the southwest Pacific Ocean and the Indian Ocean, the storms are referred to as "cyclones." No matter what it is called when a hurricane, typhoon, or cyclone hits land, it can do great damage through fierce winds, torrential rains, inland flooding, and huge waves crashing ashore. A powerful hurricane can kill more people and destroy more property than any other natural disaster. Hurricanes and other cyclones form in the tropics during summer and fall.

Predicting Major Storms

A few very important characteristics of hurricane are as follows:

- Hurricanes form under weak, high-altitude winds
- Hurricanes have no fronts
- Hurricanes main energy source is the latent heat of condensation
- The center of a storm is warmer than the surrounding air
- Hurricane winds weaken with height
- Strongest winds are near the Earth's surface
- Hurricanes weaken rapidly over land

As global weather patterns become more erratic as evidenced in the early twenty-first century, it has become difficult to accurately forecast hurricanes. However, mathematics allows forecasters a thorough insight into the mechanisms of weather features, including large-amplitude water waves and sustained winds cloud structure. Moreover, statistical models built from historical data perform with greater precision. Also, scientists use high-quality time series data along with less precise time series data using a Bayesian approach, which does not require data to have uniform precision. This way, scientists have been able to forecast U.S. hurricanes six months in advance.

Wind engineer Herbert Saffir and meteorologist Robert Simpson introduced the very popular Saffir–Simpson wind scale, which is a 1–5 categorization based on the hurricane's intensity at the indicated time. This scale is an excellent tool for alerting the public about the possible impacts of various-intensity hurricanes. However, the scale does not address the potential for other hurricane-related impacts, such as storm surges, rainfall-induced floods, and tornadoes.

The estimation of hurricane-generated waves and surges in coastal waters is of critical importance to the timely evacuation of coastal residents and the assessment of damage to coastal property in the event that a storm makes landfall. Tornado wind speed or intensity is rated using the Fujita scale, named for Tetsuya Theodore Fujita. It is based on the subjective assessment of the damage caused to human and vegetation structures by the tornado. Its original development was linked to the Beaufort wind force scale, named for Francis Beaufort. Ratings range from a minimum of "F0" to a maximum of "F6." It is also sometimes called the Fujita–Pearson scale to recognize contributions of Allen Pearson, who was director of the National Severe Storms Forecast Center at the time. The scale has since been revised by data gathered from structural engineers and others that suggested that the original wind speeds were too high for categories F3 and above.

To provide accurate estimates for wave height, scientists use Wave Model (WAM). WAM is built around the solution to the action balance equation in terms of an action density function. With the aid of FORTRAN and other programming languages today, WAM is an extremely efficient model.

Hurricane size (extent of hurricane-force winds), local bathymetry (depth of near-shore waters), topography, the hurricane's forward speed, and its angle to the coast are all factors that affect the surge that is produced. Mathematicians and scientists are working hard to develop a reliable technique for prediction of storm surges. The capability for prediction of hurricane surges is based primarily on the use of analytic and mathematical models, which estimate the interactions between winds and ocean, also taking into account numerous other factors. One of the models used for storm-surge modeling is known as the Advanced Circulation Model (ADCIRC). This is a finite-element circulation model based on the two-dimensional, depth-integrated shallow-water equations representing the conservation laws for mass and momentum. The momentum equations are combined with the continuity equation and result in the generalized wave continuity equation. ADCIRC is implemented in spherical coordinates for this application. As expected, many parameters can be set to optimize running the model for specific applications and locations.

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Kumer Pial Das

See Also: Climate Change; Clouds; Coral Reefs; Geothermal Energy; Green Mathematics.

I

Incan and Mayan Mathematics

Category: Government, Politics, and History.
Fields of Study: Connections; Measurement; Number and Operations; Representations.

Summary: The Incan and Mayan civilizations had a variety of mathematical achievements, including number systems and calendars.

The Inca Empire existed from 1438 until 1533 C.E., when it was conquered by the Spanish and the last Inca emperor, Atahualpa, was murdered. At its height, the Inca Empire comprised most of present-day Peru, Bolivia, and Ecuador, as well as parts of Colombia, Chile, and Argentina. It was a culturally diverse but politically centralized empire, based in the capital of Cuzco. Having no written words, the Incas invented a clever method of recording numbers, usually for administrative purposes, using knotted cords called a *quipu*.

The Maya civilization flourished between 250 and 900 C.E. The homeland of the Mayans was the Greater Yucatan Peninsula, including present-day Guatemala and Belize, as well as parts of Mexico, Honduras, and El Salvador. In contrast to the Inca Empire, the Maya civilization was never a political entity but consisted of a multitude of independent city-states. Among the many remarkable accomplishments of Mayan culture were hieroglyphic writing, a vigesimal and duodevigesimal number system, the invention of a symbol for zero, an elaborate system of calendars, and highly accurate astronomical observations.

Incan Quipus

A quipu is a bundle of colored, knotted cords. Every quipu has a main cord that is thicker than the others. Pendant cords are tied to the main cord, and subsidiary cords are tied to pendant cords or other subsidiaries. Quipus have been found with as many as 2000 pendants and six levels of subsidiaries. The pendant and subsidiary cords carry knots. Three types of knots are used: simple knots, figure-eight knots, and long knots with two to nine turns. To record numbers, the Incas used a decimal number system. Each digit other than the units is represented by a cluster of the appropriate number of simple knots. The Incas did not have a special knot for zero but simply left an empty space on the cord.

Units are represented by a long knot with the appropriate number of turns. If the unit is one, however, a figure-eight knot is used, since a long knot with only one turn is identical to a simple knot. For example, the number 701 is represented by a cluster of seven simple knots, an empty space, and a figure-eight knot. The digits are ordered with the units away from the main cord.

Since the units are distinguished from the other digits, the same cord can carry several numbers. The colors of the cords and the topology of pendants and subsidiaries do not contribute to the numerical information but signify the item that is being counted. There are about 800 quipus in museums today. The largest number found on a quipu is 97,357.

Quipus are not suitable for performing arithmetic. In 1590, Spanish Jesuit missionary José de Acosta described how the Incas carried out difficult computations by moving around maize kernels. A Peruvian drawing from about 1615 shows a tablet, called a *yupana*, that might have been used for this purpose. This yupana is divided into smaller squares, each containing 1, 2, 3, or 5 dots, which could be maize kernels. Acosta explicitly mentioned the numbers 1, 3, and 8. This has led to speculations that the Incas used so-called Fibonacci numbers in their calculations since 1, 2, 3, 5, and 8 are the first such numbers.

Mayan Numbers and the Invention of Zero

The Mayan number system is neither a pure grouping system, like Roman or Aztec numbers, nor a pure positional system, like Hindu–Arabic numbers, but a mixture of the two, like Babylonian or Incan numbers. Numbers from 0 to 19 are written with dots representing 1, lines representing 5, and a symbol for 0 resembling an eye. Thus, 17 is written as two dots and three lines. For numbers larger than 19, a base-20 and, at one place, a base-18 positional system is used. The first place represents units, and the second place represents multiples of 20.

The third place, however, does not represent multiples of $20 \times 20 = 400$ but multiples of $18 \times 20 = 360$. From then on, the fourth place represents multiples of $20 \times 360 = 7200$, the fifth place multiples of $20 \times 7200 = 144{,}000$, and so on. Mayan numbers were originally written vertically with the units at the bottom. For convenience, Mayanists write them horizontally with the units to the right. Thus, the Mayan number 9.12.11.5.18 means the following:

$$9 \times 144{,}000 + 12 \times 7200 + 11 \times 360 + 5 \times 20 + 18$$
$$= 1{,}386{,}478.$$

After the Babylonians, the Mayas or possibly their Olmec predecessors were the first culture in the world to invent a symbol for zero. The earliest known occurrence of this zero symbol is found on a stela in Uaxactun, Guatemala (357 C.E.). The earliest indisputable inscription using the Hindu–Arabic decimal system including a symbol for zero is from Cambodia (683 C.E.).

Mayan Calendars

The Mayas used three different calendars: the Tzolkin, the Haab, and the Long Count. A typical Mayan date looks like the following:

9.12.11.5.18 6 Etznab 11 Yax.

Here, "9.12.11.5.18" is the Long Count date, "6 Etznab" is the Tzolkin date, and "11 Yax" is the Haab date. This was the day of death of the great ruler, Pacal, of the city-state, Palenque, corresponding to August 29, 683 C.E.

The Tzolkin calendar is based on two independent cycles of 13 and 20 days, respectively. A Tzolkin date consists of a number from 1 to 13 followed by one of the following 20 names of days:

Ahau	*Kan*	*Lamat*	*Eb*	*Cib*
Imix	*Chicchan*	*Muluc*	*Ben*	*Caban*
Ik	*Cimi*	*Oc*	*Ix*	*Etznab*
Akbal	*Manik*	*Chuen*	*Men*	*Cauac*

Both the number and the day name change daily such that the calendar runs as follows: 1 Ahau, 2 Imix, 3 Ik, and so forth. Every possible Tzolkin date occurs once during the Tzolkin year of $13 \times 20 = 230$ days. This follows from the so-called Chinese Remainder Theorem, which the Mayas must have known at least in some special cases, and the fact that 13 and 20 have no common divisors.

The Haab calendar consists of 18 months of 20 days, followed by five extra days. The length of the Haab year is thus $18 \times 20 + 5 = 365$ days. The names of the months are the following:

Pop	*Tzec*	*Chen*	*Mac*	*Kayab*
Uo	*Xul*	*Yax*	*Kankin*	*Cumku*
Zip	*Yaxkin*	*Zac*	*Muan*	
Zotz	*Mol*	*Ceh*	*Pax*	

The days of each Haab month are numbered from 0 to 19. The Haab calendar thus runs as follows: 0 Pop, 1 Pop, 2 Pop, and so forth. The final five days, called

Uayeb, are numbered from 0 to 4; these days were considered unlucky.

The least common multiple of 260 and 365 is $73 \times 260 = 52 \times 365 = 18{,}980$, which means that the combined Tzolkin–Haab calendar repeats itself after 73 Tzolkin years, or 52 Haab years, or 18,980 days.

The Mayas believed in a cycle of eras of $13 \times 144{,}000$ days or approximately 5125 years, each era ending with a time of great change. A Long Count date is a five-digit Mayan number recording how many days have elapsed since the last transition of cycles. There is a unique correspondence between the last digit of the Long Count date and the Tzolkin day name. If the last digit is 0, the day name is Ahau; if the last digit is 1, the day name is Imix, and so forth. According to various Mayan sources, the previous era ended on the following date:

13.0.0.0.0 4 Ahau 8 Cumku.

The problem of translating Long Count dates into dates in the Gregorian calendar is known as the Correlation Problem and has been a topic of considerable controversy. Today, most Mayanists believe that

The Mayas were known for building complex and highly decorated ceremonial structures, including temple-pyramids, palaces, and observatories, all contructed without the use of metal tools.

13.0.0.0.0 4 Ahau 8 Cumku corresponds to August 11, 3114 B.C.E. The Mayans thus expected the next cycle change upheaval to occur on 13.0.0.0.0 4 Ahau 3 Kankin, corresponding to December 21, 2012 C.E., when the present Long Count cycle ends.

Mayan Astronomy and the Dresden Codex

The Dresden Codex is one of only four original Mayan books that have survived to the present day. It contains astronomical tables in which the number 584 figures prominently; this is the best integer approximation to the average period of Venus, as seen from the Earth, of 583.92 days. In the Codex, 584 is divided into parts of 236, 90, 250, and 8, reflecting the phases of Venus.

First Venus appears as the Morning Star for 236 days, then it disappears on the far side of the sun for 90 days, then it reappears as the Evening Star for 250 days, and finally it disappears again for eight days while it is between the Earth and the sun. The difference between 90 and 8 is explained by the fact that, as seen from the Earth, Venus moves more slowly relative to the sun when it is on the far side of the sun. The difference between 236 and 250 is thought be because of a local difference between the eastern and western horizons.

It is a strange coincidence that $584 = 8 \times 73$ and $365 = 5 \times 73$ have the large common prime factor of 73. This implies that five Venus periods correspond very closely to eight Haab years, and indeed the Codex contains a Venus table of this length of time. The Mayas knew, however, that this correspondence was not exact. To compensate, they subtracted either four days after days, giving a period of 583.93 days, or eight days after days, giving a period of 583.86 days.

It has been suggested that the Mayas used the first correction four times and the second correction once, thus subtracting a total of 24 days after 301×584 days, which gives a Venus period of exactly 583.92 days. This explanation, however, was questioned by the famous physicist, Nobel laureate, and amateur Mayanist Richard Feynman.

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David Brink

See Also: Astronomy; Calendars; Central America; South America; Zero.

Income Tax

Category: Government, Politics, and History.
Fields of Study: Measurement; Number and Operations.
Summary: Mathematics is used to compute income tax returns and analyze income-tax fraud.

Albert Einstein once quipped that preparing a tax return was an activity too difficult for a mathematician and was better suited for a philosopher. Many would point to the complex and ever-changing laws regarding taxation, rather than the underlying mathematical concepts, as being the problematic part of understanding income taxes.

Numbers and their operations, along with algebra, are very useful in the calculation of the taxes owed by individuals and corporations. In addition, probability, statistics, and geometry are among the fields used by those interested in the analysis of the process and outcomes of taxation, such as tax irregularities and evasions, tax burden, and the effects of taxation on overall economic welfare.

History

In 1861, the U.S. Congress imposed a tax on personal incomes to help finance the Civil War. Prior to that time, it had depended mainly on excise taxes and customs duties. The first income tax was a proportional (or flat) tax: anyone who made an income of more than $800 per year had to pay a fixed 3% of that income in taxes. The next year, a two-tiered progressive rate structure was put into place. Taxable incomes up to $10,000 were still taxed at 3%, while higher incomes paid 5%, though people were allowed to take various deductions from their incomes before calculating the tax. Taxes were also withheld by employers for the first time. This tiered taxation method became the standard for income tax, although some countries in the early twenty-first century use a dual income tax system in which individuals and corporations are taxed at a low rate while labor income is taxed at a high rate.

Income taxes were abolished in 1872; but after a great deal of legal debate, they returned permanently with the passage of the Sixteenth Amendment in 1913. Everyone who earns income in the United States is subject to federal individual income tax and, in most cases, state income tax as well. Some municipalities also charge local income tax. Employers are required to withhold money from taxpayers' paychecks and to remit the funds to the appropriate government agencies. Self-employed taxpayers are required to submit quarterly payments.

Calculating the Income Tax

As can be seen from the Internal Revenue Service 1040 individual income tax form, a series of calculations are required to determine the amount of income tax owed.

Total Income: All sources of taxable income must be added to calculate total income, including not just wages but also funds accrued from sources such as tips, interest earned, alimony, capital gains, retirement withdrawals, royalties, and business income.

Adjusted Gross Income: Certain types of expenses can be subtracted from the total income, including some expenses related to moving, business, education, alimony paid, self-employment, and student loans. After subtracting the allowable expenses, the result is the adjusted gross income.

Taxable Income: Additional deductions and exemptions are subtracted from the adjusted gross income to arrive at the taxable income.

- *Deductions*: Taxpayers can elect to take the standard deduction, which is a set amount depending on filing status, or they can itemize their deductions to see if a tally of the allowable deductions results in more than the standard amount. People who paid mortgage interest, significant medical costs, large charitable donations, and/or business expenses will often find that itemizing produces a larger deduction than the standard.
- *Exemptions*: The federal government allows taxpayers to deduct a fixed amount for each dependent in the household; in 2009 that amount was $3,650 per dependent.

Tax owed: The tax is then looked up in the tax table, reading the appropriate column depending on filing status (single, married filing jointly, married filing separately, head of household), unless the taxable income is over $100,000, in which case a tax computation worksheet is used. In general, single people pay more taxes than married couples filing jointly with the same income.

Understanding the Federal Tax Tables and the Tax Computation Worksheet

The government defines a series of tax brackets, which are percentages linked to income ranges. The income ranges for a specific tax bracket vary depending on the filing status of the taxpayer.

The federal government sets different ranges for the following categories: single, married filing jointly, married filing separately, and head of household. In 2009, for example, the tax brackets were 10%, 15%, 25%, 28%, 33%, and 35%. The range for a single tax payer in the 10% bracket was $0 to $8,350 in taxable income. For a married couple filing a joint return, the income range for the 10% bracket was $0 to $16,700.

The tax table and tax computation worksheet values do not correspond directly to the tax brackets. For example, a single person earning $62,025 in 2009 would appear to fall into the 25% tax bracket ($33,950 to $82,250). However the tax shown in the tax table is $11,694, which is less than $17,250 (25% of $69,000). The tax table value was determined by applying the tax brackets to the taxable income in stages. In 2009, the tax brackets for a single taxpayer were

- 10% bracket: $0 to $8,350
- 15% bracket: $8,350 to $33,950
- 25% bracket: $33,950 to $82,250

The first $8,350 of the taxable income earned falls into the 10% bracket, yielding $835 in taxes. The next $25,600 ($33,980 – $8,350) of the taxable income falls into the 15% bracket, yielding $3,840. The last $28,075 ($62,025 −$33,950) falls into the 25% tax bracket, yielding $7,018.75.

The total tax is $835 + $3,840 + $7,018.75 = $11,694 (rounded to the nearest dollar).

The tax tables are provided in $50 increments, so anyone earning between $62,000 and $62,050 would pay the same amount of tax.

The tax computation worksheet calculations work the same way. For a single person with a taxable income of $130,000 (28% tax bracket), the worksheet calculation is to multiply by 0.28 and then subtract $6,280. The $6,280 figure is subtracted to compensate for the lower taxes paid on the portions of the $130,000 income that fall into the lower tax brackets.

Other Methods of Calculating Taxes

Some groups are concerned that the federal tax code is too complicated, confusing, and unfair. There are those who advocate simplifying the tax code and leaving the graduated tax bracket structure, and others who advocate a flat tax—one percentage rate for all with no exemptions or deductions.

Most states follow the federal government's lead and have a series of tax brackets. In 2009, Colorado, Illinois, Indiana, Massachusetts, Michigan, Pennsylvania, and Utah all had flat taxes ranging from 3 percent to 6 percent. Alaska, Florida, Nevada, South Dakota, Texas, Washington, and Wyoming did not collect any individual income tax.

Mathematical Modeling

The impact of taxation is of great personal and political concern. Income taxes, in particular, can generate a great deal of debate, and many people feel personally and directly affected by changes in these taxes. Mathematical methods are used to model a variety of phenomena related to taxes. For example, equilibrium modeling seeks to explain and predict the broad economic repercussions of different market factors, including taxes.

These complex models take into account the flow of cash, commodities, and other goods between various people and businesses, which have different motivations and constraints. Other potential variables can include prices, interest rates, and taxes. A system (like the U.S. economy) is in equilibrium when the inflows and outflows, or supply and demand, are balanced. These models are computationally intense and generally solved using numerical methods, graph theory, geometry, and stochastic simulation.

Several countries and U.S. states, as well as companies and accounting firms, use software based on Benford's Law to check income tax returns for fraud. Benford's Law is named for engineer and physicist Frank Benford. According to stories about Benford, he was inspired by the fact that pages of logarithm books associated with numbers starting with the digit 1 were dirtier and more worn than other pages. Thinking that it was unlikely that scientists had some special preference for these numbers, he analyzed over 20,000 sets of data from a wide variety of sources, such as baseball statistics, numbers he found in magazine articles, and atomic weights.

All of these data sets followed a similar pattern in terms of the first digits of the numbers. About 30% of the time, the first digit of the numbers was a 1. Each subsequent numeral 2 through 9 occurred less and less often as the initial digit, such that the probability of any number n from 1 through 9 being the first digit is the following:

$$\log\left(1+\frac{1}{n}\right).$$

One simple way that data can be tested is by comparing the observed first-digit counts to Benford's Law. For example, accountant Mark Nigrini examined 169,662 IRS files and found that they follow Benford's Law, with an allowable statistical margin of error. Former president Bill Clinton and (as of 2010) Secretary of State Hillary Rodham Clinton's tax returns for several years were also analyzed.

Nigrini concluded that the Clintons may have used some rounded-off dollar estimates rather than exact numbers, but his test did not uncover any fraud. Generally, studies show that fraudulent data contain too few numbers starting with 1 and too many starting with 6.

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Holly Hirst

See Also: Accounting; Data Mining; Sales Taxes and Shipping Fees.

Individual Retirement Accounts (IRAs)

See *Pensions, IRAs, and Social Security*

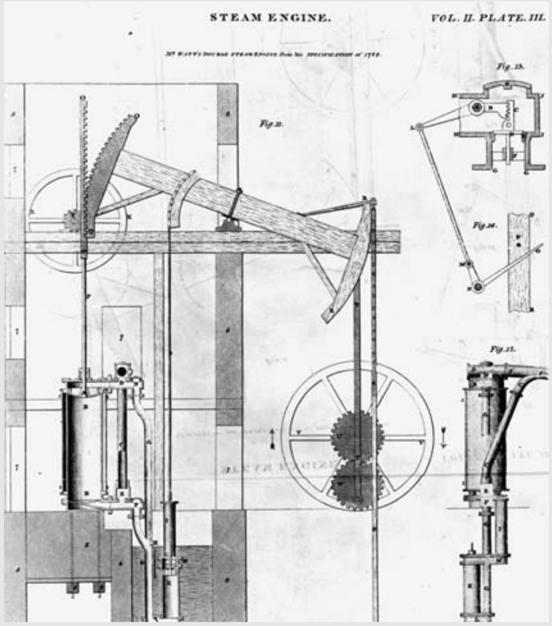
Industrial Revolution

Category: Business, Economics, and Marketing.
Fields of Study: Measurement; Number and Operations; Problem Solving.
Summary: New energy sources, management styles, and more intensely divided labor revolutionized manufacturing and technology.

The term "Industrial Revolution" refers to the great social transformation, beginning in the mid-eighteenth century, during which manufacturing replaced agriculture as the center of productive activity. This transition had profound implications for economic and political institutions and international relations, as well as for the landscape and environment, family, education, and culture. Its two main dimensions were technological innovation and the social organization of production. The Industrial Revolution was facilitated by the increased use of realistic perspectives in painting and drawing that flourished in the Renaissance, as well as by the invention of the printing press in the fifteenth century, which spurred intellectual growth in many fields, including mathematics. These developments allowed for better visual representation and distribution of mathematical ideas and inventions to a much broader audience than the older master-apprentice models.

Characteristics

Some historians question the use of the term "revolution," since these developments indisputably occurred incrementally over a period of a century or more. Nonetheless, their cumulative impact dramatically



Engraving of gears and pumps from James Watt's double steam engine specifications of 1782.

Watt and Horsepower

As Watt marketed his steam engines, he developed the standard unit of "horsepower" to demonstrate the superiority of his product over the horses traditionally used to power a mill wheel. Based on his observations, he calculated that one horsepower was equal to approximately 33,000 ft-lb/min. The "watt," which came into use as a unit of power in the late nineteenth century, was named for James Watt.

changed virtually every aspect of life, first in Great Britain and eventually worldwide. New technologies both drew on existing mathematics and prompted its further development. New institutions of intellectual life also fostered the emergence of increasingly abstract mathematics.

The key technological feature of the Industrial Revolution was the application of new sources of power: first the steam engine (late eighteenth century), and later electricity and the internal combustion engine (late nineteenth century). As the Industrial Revolution spread in the late twentieth century, nuclear energy and emerging "green energy" sources have been developed. A crucial problem of the early Industrial Revolution was the means of transmitting power from the steam engine to the machines used in production itself. This problem gave rise to the mathematical theory of linkages.

Equally important to the Industrial Revolution was the large-scale organization of labor. In England, the Enclosure Acts (1760–1845) forced small farmers into urban areas, while vagrancy laws, poor laws, and workhouses (places where those who were not able to support themselves could seek shelter and employment) instilled labor discipline. A large labor pool was thus created for the new factories. Market competition impelled factory owners to use the cheapest possible labor—children as young as 5 as well as adult women and men—and to maximize profits by extending the working day to 14 hours or more per day, seven days per week.

The vastly larger scale of production made possible by mechanization and the steam engine created a qualitatively distinct industrial organization of labor. It intensified the division of labor, de-skilling some jobs and creating new forms of specialization.

The Industrial Revolution therefore meant profound changes in work, residence patterns, family relations, and urban life. This in turn sparked interest in social statistics. Edwin Chadwick (1800–1890) and Friedrich Engels (1820–1895) pioneered the use of quantitative measures to describe social problems. Belgian mathematician Adolphe Quetelet applied the statistical techniques previously used in astronomy to social problems, further developing them and helping to institutionalize the discipline of statistics.

James Watt and the Steam Engine

James Watt (1736–1819), the grandson of a mathematics teacher, possessed the combination of manual dexterity and an aptitude for mathematics. He trained as a maker of mathematical instruments, securing a position at the University of Glasgow, a major center of the British Industrial Revolution, where he first encountered the inventive yet inefficient Newcomen steam engine. While the Newcomen engine served to pump water from coal mines, Watt's improvements turned the steam engine into a practical means of supplying power to factories and of transporting manufactured goods to market.

James Watt's parallel motion mechanism (1804), in particular, allowed the force of an engine to act in both push and pull directions, converting rotary motion to linear motion. This provided an empirical, though imprecise, solution to the geometrical problem of constructing a straight line without tracing a straight line. In Euclidean geometry, it is axiomatic that a straight line can be produced, but—in contrast to the circle—no method existed to do so.

Following Watt, a spatial linkage that traced exact straight lines was created by mathematician Pierre-Frederic Sarrus in 1853 and proved geometrically by Charles-Nicolas Peaucellier in 1864. The mathematical theory of linkages was further developed by Pafnuty Chebyshev, James Joseph Sylvester, Alfred Kempe, and Arthur Cayley.

Mathematics and the Industrial Revolution

The late eighteenth and early nineteenth centuries were extremely fruitful in the development of modern mathematics. However, the connections between this work and the Industrial Revolution are mainly indirect.

A notable exception was Charles Babbage (1791–1891) and his work on some of the earliest computing machines. Numerical tables used in applied mathematics were calculated by hand and often contained many errors. Babbage sought to replace these human "computers" with machines, as so many manufacturing jobs were being mechanized. He began work on his first "difference engine" in 1822, moved on to a programmable "analytical engine," and continued experimenting with steam-powered computing machines for much of the rest of his life. Ada Lovelace, generally credited as the first computer programmer, created a program that could have run on Babbage's machine, had it been built.

Some technical problems that arose in connection with the Industrial Revolution proved amenable to solution via abstract mathematics developed in other contexts. For example, analysis of electrical circuits, waves, and oscillations is simplified by using complex numbers, originally explored in relation to the solution of algebraic equations.

In France, the École Polytechnique, founded by mathematicians Lazare Carnot and Gaspard Monge in 1794 to train military engineers, supplied technical training and expertise for emerging French industries. Its faculty, students, and examiners included many of the most influential French mathematicians of the nineteenth century, and its textbooks, such as the calculus texts of Adrien-Marie Legendre and Sylvestre-Francois Lecroix, influenced mathematics instruction internationally.

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Bonnie Ellen Blustein

See Also: Electricity; Lovelace, Ada; Measurement in Society; Painting; Renaissance.

Infantry (Aerial and Ground Movements)

Category: Government, Politics, and History.
Fields of Study: Data Analysis and Probability; Geometry.
Summary: Mathematics has long played a significant role in infantry operations, including influencing cryptography, logistics, and military strategy.

The oldest military unit and still the backbone of most modern armies, infantry units consist of soldiers who engage the enemy face-to-face. Historically, infantry units marched from one location to another. In modern times, infantry units may be deployed in a variety of ways, including overland in trucks; by sea, such as the troops landing on Omaha Beach on D-Day; or by air, either from planes or helicopters. Paratroopers are often considered elite among infantry units. In general, infantry are distinct from other land-based mobile units, such as cavalry, employing different tactics and strategies.

Mathematics has always played a major role in warfare, including infantry movements. Early Babylonian clay tablets show evidence of sophisticated mathematical calculations of the volume of dirt that would be needed for siege ramps and what sort of minimum manpower would be required to accomplish the task. The sophistication of mathematics in ancient Greece was no doubt in part because of its usefulness to war—the Greeks may have left a legacy of philosophy and art but spent much of their time and resources at war among themselves and with their neighbors.

Napoleon Bonaparte is widely considered to be a military genius who revolutionized the use of light infantry and artillery. He was also an avid mathematics student and was often accompanied in the battlefield by mathematicians, including Joseph Fourier. He discussed his own solutions to mathematics problems with notable mathematicians, such as Lorenzo Mascheroni, Pierre Laplace, and Joseph Lagrange, including what is known as Napoleon's Theorem. He was quoted as saying, "The advancement and perfection of mathematics are intimately connected to the prosperity of the state." Many modern officers have been educated at the U.S. Military Academy at West Point and other military academies, which emphasize mathematics and engineering in their curriculums, and both military and civilian mathematicians continue to play critical roles in infantry tactics and deployment, especially in the modeling and simulation of twenty-first-century combat strategies.

History

Archimedes, one of the most famous ancient mathematicians, applied his knowledge of geometry, the estimation of weights and volumes, and three-dimensional rotations to defending the city of Syracuse from siege by Roman forces (214–212 B.C.E.). In addition to the standard trick of cutting holes into the walls for archers to fire arrows through, Archimedes helped to design the catapults used by the Syracuse artillery units, and

Scientists such as Luis Alvarez helped create the Identification Friend or Foe (IFF) radar camera, shown above, and improved antenna systems to identify friendly aircraft without using visual confirmation.

he called for traps to be built in the walls to drop heavy stones on approaching ships. Cranes were even used to drop grappling hooks onto ships and capsize them. The siege took much longer than it otherwise would have, and the Roman commander reportedly ordered that Archimedes's life be spared out of respect for his intellect—an order that was ignored, and Archimedes was killed when the siege finally succeeded.

The Renaissance was a time of flourishing mathematics, with applications in a wide variety of sciences, including cartography. While the Age of Discovery certainly was one cause for the demand for increasingly more precise maps, so too was the desire to accurately direct the movement of troops and ships while at war. Accurate chronometers were developed at the order of the military, which also called for more precise ways of determining latitude in order to increase the usefulness and accuracy of maps.

Modern Warfare

Eventually, mathematics would be used to more accurately determine the velocities and paths of projectiles, which in turn influenced not only the behavior of artillery units but also the design of infantry firearms, which became increasingly critical in conflicts like the U.S. Civil War and World War I.

World War II, because of its extraordinary size and resource consumption, put mathematicians to use in all areas of the military, a close relationship that has continued and been further assisted by the development of modern-day computers. The advent of paratroopers in World War II added a new level of complexity to the deployment of infantry troops, taking into account not only point-to-point movement on the ground but also precision insertion via parachute. Humans leaping from a moving plane do not fall straight down, so calculations had to be made to take altitude, speed, and other factors into account in order to determine when, where, at what altitude, and at what intervals paratroops should deploy to successfully land on a predetermined spot. A hybrid transportation algorithm that first mathematically computes an ideal solution, which is then used for stochastic simulations, has been successfully used to model deployment of troops and equipment.

Other investigations into this problem often use numerical methods, fluid dynamic equations, 3-dimensional flows, mesh resolution techniques, and simulation methods. The use of aircraft for combat reconnaissance was also largely pioneered during World War II, though it was hampered by their limited speed and at times by unreliable radio communications, which did not facilitate the rapid decisions infantry commanders in the field were required to make.

Modern communication methods allow for rapid computer modeling and real-time decision making, virtually as soon as the data are collected. Military radar was also in its infancy in World War II, though work by mathematicians and scientists such as physicist Luis Alvarez would improve its utility. For example, Alvarez helped create transponders, then known as Identification Friend or Foe (IFF) radar beacons, and improved antenna systems, which identified friendly aircraft without visual confirmation and facilitated precision delivery of troops and bombs even in poor weather.

Mathematics at War

The quantification of troops, inventory, and distances as well as the order of battle and the estimations of travel speeds and damage to fortifications have likely always played a role in warfare. The term "order of battle" originally referred to the order in which troops were positioned relative to the position of the commander but has come to refer to the composition of the forces involved in a field operation, including their command structure, personnel, disposition (the geographical locations of the headquarters of units and subunits), and equipment.

In U.S. Army practice, an order of battle prepared for an intelligence report also includes information on personalities (known enemy personnel and relevant information pertaining to them), unit history relevant to the current situation, a logistics report on how units obtain supplies, and a combat effectiveness section that is prepared using combat modeling applications based on sophisticated algorithms. Orders of battle are fundamental to a military commander's situational awareness. Commanders depend more on combat effectiveness projections as modeling techniques have become more sophisticated and data from field operations have been applied in order to continually evaluate them.

In essence, the same mathematics responsible for governing the artificial intelligence of enemy forces in video games like Call of Duty is used—albeit with a great deal more data and more powerful processing—to evaluate enemy forces in real life. These models draw on a diverse array of mathematical methods. Game theory in general is concerned with modeling strategy. Statistical analysis, Andrey Markov chains, business logistics, and fluid dynamics have all played significant roles. During World War I, mathematician Frederick Lanchester devised Lanchester's Laws, which use systems of ordinary differential equations to determine which of two sides will remain at the end of a battle, as functions of the defenders' strengths and time, assuming neither side breaks off combat. They continue to be the basis for many modern simulations. Some models simplify problems or address only small portions of a vastly complex problem, including trying to quantify "soft" or qualitative aspects of combat, though hybrid modeling with both discrete and continuous components is a growing way to reliably model critical subsystems and also their interactions with one another. Mathematical analysis of satellite data and images is also used for detecting landmines and improvised explosive devices, which are some of the greatest threats to troops on the ground.

Perhaps the biggest impact of mathematics on the infantry is that the use of combat modeling means the ability to predict—if not always accurately, at least with a greater degree of accuracy than in the past—the outcome of various combat scenarios and, thus, to manage risk and reward when allocating troops. Military effectiveness can be maximized at multiple levels, from the allocation of funds at the budget stage to recruitment techniques to the command structure of the armed forces to troop movements.

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Bill Kte'pi

See Also: Archimedes; Artillery; Predicting Attacks; World War I; World War II.

Infinity

Category: History and Development of Curricular Concepts.
Fields of Study: Calculus; Communication; Connections; Number and Operations.
Summary: Infinity is an important part of the curriculum and has a rich and interesting history.

Counting comes naturally to humans. Children as young as 2 years old begin to associate numbers with groups of objects: 1, 2, 3, 4, and 5 are quickly understood. The concept of "plus one" also develops early in life. Given any whole number, there is always a "next number," the one achieved by adding one. As such, early in life we face the reality that there is no largest number, for given a number of any size, adding one to it produces a number that is yet bigger. That is, the set of natural numbers $\{1, 2, 3, 4, 5, \ldots\}$ is infinite. While the concept of infinity is fundamental in mathematics, cosmology, and theology, many of the advances in understanding infinity were met with severe criticism or worse. For example, according to some stories, Hippasus, a member of the Pythagorean order, was drowned for divulging the existence of infinite nonrepeating decimals. Revolutions in philosophy and mathematics resolved many of the fascinating paradoxes related to infinity, but infinity continues to challenge and interest us today.

A Hotel Example

Infinity is a concept, but it is not itself a number. To illustrate how the notion of infinity is different, it is helpful to turn to one of the great mathematicians of all time, David Hilbert (1862–1943). In 1900, he spoke to the International Congress of Mathematicians about 23 unsolved problems that he considered to be the most important to the progress of mathematics—the search for solutions to these problems shaped a great deal of twentieth-century mathematics, and some even remain open to this day. Besides being a leading mathematician, Hilbert was also a thoughtful teacher, and he was reputed to have used the following paraphrased story to challenge his students to think about the curious nature of infinity.

> A mathematician owned an unusual hotel, one with infinitely many rooms. Each room was assigned a natural number—Room 1, Room 2, and so on—and on one occasion, it happened that every room in the hotel was filled. A customer seeking a room walked into the lobby and asked the manager if there were any openings. The manager reported that every room was full but that there was a way for the customer to get a room.
>
> The occupant of Room 1 was asked to move to Room 2; the occupant of Room 2 moved to Room 3; and in general, the person in Room N stepped next door to Room $N + 1$. The customer who had requested a room at the entirely full hotel was now able to occupy Room 1.
>
> The next day, when the hotel was still completely full, an unusual charter bus arrived, carrying infinitely many passengers, all seeking rooms. At first, the members of this group were disheartened to learn that the hotel was completely booked. But the mathematically savvy manager once again had a solution.
>
> The occupant of Room 1 was asked to move to Room 2; the occupant of Room 2 moved to Room 4; the person in Room 3 went to Room 6; and in general, the person in Room N stepped down the hall to Room $2N$. The customers getting off the bus were now able to move into all of the odd-numbered rooms, as rooms 1, 3, 5, 7... were all open.

While this story may seem far-fetched because there are only a finite number people alive on Earth, it illustrates some remarkable properties of natural numbers and raises concerns, such as whether more natural numbers exist than there are natural numbers.

Infinite Sets

Georg Cantor's revolutionary ideas on the sizes of such infinite sets form the basis of many ideas in modern mathematics, including the fields of analysis and calculus. For example, removing the odd natural numbers from the set of all natural numbers

$$\{1, 2, 3, 4, 5, 6 \ldots 2n, 2n + 1, \ldots\}$$

leaves the set $\{2, 4, 6 \ldots 2n, 2n + 2, \ldots\}$

which is yet another infinite set. Galileo Galilei believed that the sizes of infinite sets could not be compared or contrasted. However, Cantor and mathematicians today agree that since a first even natural number can be identified, a second even natural number, and so on, just as a first natural number can be identified, a second natural number, and a third, then there are the same number of even natural numbers as there are natural numbers since they can be put in one-to-one correspondence. Cantor also proved that there are uncountable sets that have a different measure of infinity, such as the real numbers. However, Cantor did not receive the recognition during his lifetime that he has today. Some theologians believed his work challenged the uniqueness and infinity of God, and both mathematicians and theologians strongly objected to his work at the time.

Limits

A question that has intrigued many people over the centuries is whether or not the numbers 1 and $0.\overline{9}$ are the same. In fact they are, as the following argument shows. If we consider the number $0.\overline{9}$, observe the following:

$$0.\overline{9} = 0.9999\ldots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000} + \cdots$$

Certainly, two numbers can be added, three numbers, four numbers, and indeed, as many finite numbers as likened be. From this, observe the following:

$$\frac{9}{10} + \frac{9}{100} = \frac{99}{100};$$

$$\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} = \frac{999}{1000};$$

$$\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000} = \frac{9999}{10000}; \text{ and,}$$

$$\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots + \frac{9}{10^n} = \frac{10^n - 1}{10^n}.$$

Since this last sequence of numbers converges to the number 1, one concludes that the infinite sum is 1. That is,

$$0.\overline{9} = 0.9999\ldots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000} + \cdots = 1.$$

At first glance, this may seem strange to a person unaccustomed to the role of limits in mathematics. But, as was perhaps first understood by Archimedes in antiquity, limits are the bridge from the finite to the infinite, and they are indispensable to mathematics and the mathematician. Understanding infinity allows for the understanding that the numbers 1 and $0.\overline{9}$ are the same.

Paradoxes

Certainly the concept of infinity presents some challenges and unusual situations. Greek philosopher Zeno of Elea was known for posing paradoxes that challenged mathematicians for centuries. For instance, in Zeno's Paradox, a person walks toward a wall by each time stepping half the remaining distance, thus taking
time stepping half the remaining distance, thus taking
an infinite number of steps but (theoretically) never actually reaching the wall.

Another example is Gabriel's Horn, an infinite surface that can be easily generated by revolving a simple curve about an axis. Interestingly, the surface is not named after its discoverer, Italian physicist and mathematician Evangelista Torricelli, but is rather named after the Archangel Gabriel in order to connect the infinite with theology. This infinite surface can be shown to contain finite volume yet have infinite surface area. In other words, Gabriel's Horn, if filled with paint, would require only a finite volume, yet that paint could not cover the surface of the horn. While situations like these initially seem impossible, mathematics provides interesting and satisfying explanations of these phenomena.

Modern Developments

In the twentieth and twenty-first centuries, mathematicians continue to grapple with the concept of infinity. French mathematicians Émile Borel, René Baire, and Henri Lebesgue explored rationalist ideas, while a group of Russian mathematicians led by Dmitry Egorov linked mathematics to philosophy and theology. Building upon the French work and using mystical insights gained during their religious practice of Name Worshipping, they founded descriptive set theory, which transformed mathematical analysis.

However, this did not resolve the contradictions of infinitesimals in calculus, which Sir Isaac Newton, Gottfried Leibniz, and Bishop Berkeley had wrestled with during the development of that subject in the seventeenth and eighteenth centuries. Abraham Robinson created the field of nonstandard analysis in 1960 when he gave a rigorous definition of an infinitesimal number, and mathematicians continue to explore the implications of both standard and non-standard analysis. Besides there being infinitely many natural numbers, there are even infinitely many prime numbers. Primes form the building blocks of numbers and in many ways the very foundation of mathematics. In a similar way, calculus rests upon the notion of limit, which at its core involves infinite processes. Because so much of the subject naturally involves the infinite, mathematicians have had to face, understand, and conquer infinity; more than this, the presence of infinity in the world guarantees that there will always be more mathematics to explore, discover, and comprehend.

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Matt Boelkins

See Also: Archimedes; Calculus and Calculus Education; Number Theory.

Insurance

Category: Business, Economics, and Marketing.
Fields of Study: Algebra; Data Analysis and Probability; Number and Operations.
Summary: Society has long used mathematical methods to quantify risk and protect against loss, and professionals like actuaries help make these decisions.

Insurance involves the exchange of a fixed amount of money or sequence of payments (called premiums) by the insured to an entity or group for indemnification of the insured from specified losses. Thus, insurance involves trading a small but certain cost (the premium) for payment of a potentially large but uncertain loss in the future.

It is used to manage risk of loss in uncertain situations by hedging the risk (for example, by pooling money with others and sharing losses) or transferring it to some entity, like an insurer, for a price. Because the price paid today must cover future costs and future uncertain indemnification payments, the insurance industry employs many mathematicians to calculate and predict expected future costs and payments.

Importance

Risk transfer and risk pooling via insurance are very important. Following the government, insurance is probably the second most important mechanism available to alleviate social upheaval and to reduce risks to citizens. Social upheaval is reduced by supplying a financial safety net in times of loss. Risk reduction

Partial copies of the Code of Hammurabi exist on clay tablets, like the one above in the Louvre Museum.

is achieved since insurance establishes risk reduction incentives, such as lowering the cost of insurance, for those who undertake risk reduction behaviors. Examples of risk reduction behavior include premium reduction in automobile insurance for defensive driving classes or having air bags; lowering premiums and providing loss control consulting to business firms concerning risk exposures; and lobbying governments for stronger safety standards.

Insurance allows entrepreneurs to create new products, explore new energy alternatives, and engage in selective risk-taking beneficial to society, such as creating new pharmaceuticals, which might be too uncertain or create potential liability exposure consequences too great to be undertake if not insured. Through insurance, cash flows of firms are stabilized, bankruptcy likelihood is reduced, and the cost of capital to firms is lowered.

History

Because of the individual and societal benefits of insurance, it is no wonder that the rudiments of insurance date back millennia—although the modern approach to insurance awaited the development of mathematical tools to create the logical underpinning of the industry. The Code of Hammurabi (c. 1750 B.C.E.) details how early Babylonian merchants who had a loan on cargos or vessels could pay a little extra so that if the ship were lost at sea, the loan would be forgiven—an early example of risk transfer. Early civilizations also had arrangements wherein members pooled resources, and if one suffered a loss, such as a building burning down, others would pitch in and furnish materials and labor to rebuild the member's lost structure—an example of risk pooling. Before formal life insurance companies were developed, people in England in the seventeenth century would band together in groups called friendly societies, each contributing a small sum such that if an emergency or death occurred, the group would pay medical expenses, funeral costs, and sometimes give a stipend to the widow. Some of these friendly societies later developed into insurance companies.

Mathematics of Premiums

A crucial element in insurance is determining the insurance premium. The premium is the amount of money to be paid by the insured whose risk of loss is being indemnified, but needs to be an amount sufficient for the insurer selling the insurance to both cover potential loss costs and make a profit. Indeed, many early insurance-type organizations failed from the lack of correct assessments of risk and potential exposures to financial loss by the group furnishing the insurance—an incorrect quantification of risk. Without quantification of risk, the expected lost costs cannot be formalized and monitored. It is in this area of risk quantification that mathematics of insurance arises, mostly in the area of probability and statistics, which deal with the quantification of uncertainty.

The mathematics of insurance, known as "actuarial science," had its birth amid the incredible growth in mathematics in the seventeenth century. Most major mathematicians of the seventeenth and eighteenth centuries contributed to insurance mathematics in a variety of ways, such as calculating annuity tables based on interest rates and tables listing the probability of death at each age (called "life tables"). Some, such as Abraham DeMoivre, made a living, in part, by consulting on the calculation of annuity values. The first life table was constructed in 1694 by mathematician and astronomer Edmund Halley, now most famous for identifying Halley's Comet.

The development of modern probability theory—an essential element of the quantification of risk needed to price insurance—is usually attributed to French mathematicians Blaise Pascal and Pierre Fermat from a series of letters from 1654 concerning games of chance left unfinished. Using this new mathematical theory, the fair price of insurance could be rationally developed for the first time. For example, if, in the case of the occurrence of an event having a probability p, a benefit B is to be paid at some future time T, then the fair price today is pBv^T where v is the "discount rate" accounting for interest available on money invested today and paid at time T, expressed algebraically as the following:

$$v = \frac{1}{1+i}.$$

In this formula, i denotes the annual interest rate on invested money. Subsequent developments in mathematics have allowed for uncertainty in B, v, and T, enabling one to obtain the fair value of the insurance in more-complex risk transfer situations.

A mathematical foundation for insurance lies in the Law of Large Numbers (LLN), developed by mathematician Jacob Bernouli, and the Central Limit Theorem (CLT), developed by Abraham de Moivre and extended by mathematician Pierre Simon Laplace. The LLN is fundamental to insurance since it proves that the empirical relative frequency with which an event occurs in a risk pool will, as the size of the sample increases, approach the "true" probability of the event.

This allows insurance companies to objectively obtain the likelihood of loss-producing events from their experience in large collections of policyholders. The CLT proves that the average of a sample of homogeneous independent observations, such as losses within a pool of risks, will be well approximated by the bell-shaped Gaussian distribution as the number in the pool increases. From this idea, the setting of premiums for insurers who are appropriately confident of remaining solvent can be calculated.

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Patrick L. Brockett

See Also: Betting and Fairness; Data Analysis and Probability in Society; Expected Values; Life Expectancy; Normal Distribution; Probability.

Intelligence and Counterintelligence

Category: Government, Politics, and History.
Fields of Study: Algebra, Number and Operations; Problem Solving.
Summary: Quantitative data, mathematical models, cryptography, data analysis, and social network analysis have proved powerful tools in intelligence.

The intelligence industry is tasked with gathering information and predicting or inferring past, present, or future behavior based on that information. While code-breaking is the most popularly known intersection of intelligence and mathematics, lattice theory is at least as relevant and various forms of data analysis are constantly relied upon. Math can "connect the dots" to maximize the usefulness of a small set of data.

"Mathematicians Won the War"

During World War II, the mathematics underlying cryptography played an important role in military planning. Winston Churchill admired Alan Turing, the Cambridge University mathematician who had mastered the Nazi codes, recognizing him as the man who had perhaps made the single greatest individual contribution to defeating Germany. After the first frosts of the Cold War descended in the Soviet East, approximately $2 billion was spent in the development of game theory.

After the Cold War came the "war on terror." The adversary uses rational strategies to attack, so rational strategies are needed for defense.

The "War on Terror"

The National Security Agency (NSA) is a riddle wrapped in a mystery inside a code—a black palace of glass located in Fort Meade, Maryland. It dwarfs the location of the Central Intelligence Agency (CIA). Its budget is unknown, and it is the world's largest employer of mathematicians, primarily number theorists, whose work depends integrally on the presumed complexity of factoring large numbers.

In May 2006, one of the NSA's secrets escaped. *USA Today* reported that the phone companies AT&T, Verizon, and Bell South had handed customer records over to the agency—not transcripts of calls, they said, just who was calling whom. Technically, only telephone numbers were being recorded, but one could easily obtain a name from a phone number. This information was being used to determine who might be a terrorist. With the NSA data, one can draw a picture or a graph with "nodes" (or dots) representing individuals and lines between nodes if one person has called another. The field of social network analysis (SNA) deals with trying to determine information about a group from such graphs, such as who the key players are or who the cell leaders might be.

Even if everyone in the graph is a known terrorist, graphs do not directly portray information about the order or hierarchy of the cell. SNA researchers look instead for graph features like centrality—they try to identify nodes that are connected to many other nodes, like spokes around the hub of a bicycle wheel. Indeed, Monterey Naval Postgraduate School researcher Ted Lewis, in his textbook *Critical Infrastructure Protection*, defines a critical node to be such a central hub.

There are two problems in creating such a graph. First, the "central player" might not be as important as the hub metaphor suggests. For example, Jafar Adibi of the University of Southern California looked at e-mail traffic between employees of the company Enron before its famous collapse and drew a graph. He found that if you naively analyzed the graph, you could mistakenly conclude that one of the "central players" was CEO Kenneth Lay's secretary. Second, as the journal *Studies in Conflict and Terrorism* reported in 2003, one can capture all the central players in a terrorist cell and leave the cell with a complete chain of command still capable of carrying out a devastating terrorist attack.

Critical Infrastructure Protection

The U.S. government tried to prevent the publication of a study showing how the U.S. milk supply could be poisoned by terrorists, an analysis that uses queuing theory. Similar mathematics has been used to study the threat of dirty bombs.

Which border do you guard? Which border do you want the terrorist to think is weak? You want to funnel him toward your snare, thinking the field is open. Reflexive theory—a branch of mathematical psychology developed by the Soviet military and funded by the U.S. State Department—gives a quantitative method to address these questions. The same mathematical analysis could potentially be used to alleviate the problem of improvised explosive devices in Iraq. Phoenix Mathematics, Inc. is developing software tools to help border patrols allocate personnel and spread disinformation to the adversary.

Lattice Theory Applied to the "War on Terror"

While it is true that NSA expert Kathleen Carley of Carnegie Mellon University was twice able to correctly predict who would take over Hamas when its leaders were assassinated (Hamas, the Palestinian Islamic Resistance Movement, is considered a terrorist organization by the U.S. government), her analysis uses detailed information about the individuals in the organization, not just which anonymous nodes were linked with which. Since terrorist cells are composed of leaders and followers, it is important to utilize lattice theory, which takes into account order and hierarchy.

Formal concept analysis (FCA), a branch of applied lattice theory, helps identify persons of interest. Individuals who share many of the same characteristics are grouped together as one node, and links between nodes in this picture, called a "concept lattice," indicate that all the members of a certain subgroup with certain attributes must also have other attributes. For instance, one might group together people based on what cafés, bookstores, and houses of worship they attend and then find out that all the people who go to a certain café also attend the same church, but maybe not vice versa. At Los Alamos National Laboratory, the laboratory that helped build the first atomic bomb, formal concept analysis has been used to mine data drawn from hundreds of reports of terrorist-related activity and to discover patterns and relationships that were previously in shadow—connections that human analysts could not have easily found without something like FCA.

Tools from lattice theory can be applied to help intelligence agencies determine whether they have disrupted a terrorist cell. In early June 2005, the Pentagon announced plans to revise its strategy in the "war on terror." While then U.S. president George W. Bush repeatedly cited that 75 percent of Al Qaeda's leadership had been killed or captured, Al Qaeda remained active. The Pentagon shifted its target to mid-level captains and foot soldiers. Lattice theory, along with some extramathematical analysis, will help law enforcement agencies determine which individuals in a terrorist cell should be captured first, in order to maximize the chances of disrupting a cell by expending as few resources as possible. Lattice theoretical methods tell us the probability that a terrorist cell has been disabled based on how many terrorists have been captured and what rank they held in the organization.

Social choice theory has been applied to the hierarchical relationships within terrorist cells, determined from the direction of communications traffic, to model network formation. Researchers at New York University have identified two types of coalitions. They have found that the detection of one type of cell is more effective in disrupting networks, whereas the detection of the other type of cell is more effective in identifying all the members of the cell. They have also used the lattice theory to try to determine the leaders from the graph of a terrorist network. Lattice theory and graph theory can even account for gaps in one's knowledge of the structure of a terrorist cell by making assumptions about how the "perfect" terrorist cell must be organized. The knowledge of the structure of the perfect terrorist cell could also be used by terrorists to counter intelligence efforts.

Winning the Battle for Hearts and Minds

Former U.S. defense secretary Donald Rumsfeld stated in a *USA Today* article on October 22, 2003, "Today, we lack metrics to know if we are winning or losing the global war on terror. Are we capturing, killing, or deterring and dissuading more terrorists every day than the madrassas and the radical clerics are recruiting, training, and deploying against us?" To model the growth of a terrorist network, one could use the same differential equations that govern the spread of an infection, like severe acute respiratory syndrome (SARS). Such models could be used to help the government understand, and eventually contain, the spread of a terrorist insurgency.

On March 16, 2003, then U.S. vice president Dick Cheney predicted on *Meet the Press* that Americans would be "greeted as liberators" in Iraq. Ideas from statistical physics have been used to model the battle for the hearts and minds of the people of Iraq. Just as a magnetic pole may be north or south, a person could be either for the occupation or against it. The model shows that there can be a tipping point in the evolution of public opinion. It may seem as if much of the population is with one side (for example, the United States) but then, dramatically, a wave of hostility sweeps down, and one witnesses the birth of an insurgency.

Terrorism of the Futures Market

When bombs explode, the stock market drops. Mathematician Stefan Schmidt of the Technical University in Dresden, Germany, has attempted to quantify the impact on the market of a terrorist incident. The only people who know when a bomb will explode are, of course, the terrorists. By playing the market, they may already have obtained as much money as they need, thus stifling U.S. Treasury Department efforts to cut off their funding. The terrorism of the futures market may be the terrorism of the future.

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Jonathan David Farley

See Also: Coding and Encryption; Risk Management; Social Networks; Strategy and Tactics.

Intelligence Quotients

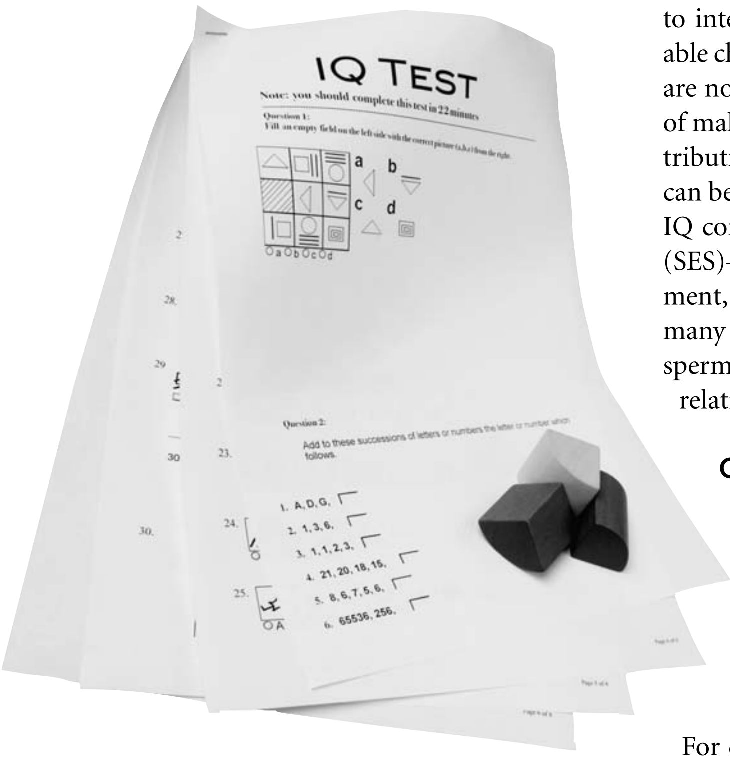
Category: Medicine and Health.
Fields of Study: Algebra; Geometry; Number and Operations; Problem Solving.
Summary: Intelligence tests are created and analyzed using mathematics.

The term "intelligence" is broadly synonymous with the term "cognitive ability." Intelligence tests are tests designed to measure cognitive abilities. According to Ian Deary and David Batty, cognitive abilities are mental abilities "that are not principally sensory, emotional or conative (related to the will)." Standardized intelligence tests produce a score called the "Intelligence Quotient" (IQ). IQ tests are usually copyrighted, and to prevent people from practicing for them, they must be administered in supervised conditions. Many tests that claim to measure IQ have appeared on the Internet but may not have been validated by professional psychologists. Intelligence, or cognitive ability, has been defined in different ways but broadly refers to people's ability to process complexity "on the spot."

Since psychologists such as Alfred Binet (originator of the test that later evolved into the Stanford–Binet) and David Wechsler (creator of the Wechsler Adult Intelligence Scale and Wechsler Intelligence Scale for Children) began measuring cognitive abilities over 100 years ago, nearly all measures of cognition have been shown to correlate. This fact is interpreted as evidence for a general factor, called *g*, representing general intelligence. At the beginning of the twenty-first century, no test of cognitive ability has been created that does not correlate with other cognitive ability tests. In practice, this means that people who are good at processing complexity in one area tend to be good at processing complexity in another. A person's IQ score is a numerical representation of their level of *g*.

Most IQ tests are designed to have a mean of 100 and scores are normally distributed. However, the standard deviation varies across different tests. The interpretation of the standard deviation is that it represented the average distance from the mean, in either direction. To understand and interpret a person's IQ score, it is necessary to know the standard deviation of the test they took. Common standard deviations are 15 or 16, and the range of IQ scores is generally between about 55 and 145 for a test with a standard deviation of 15. Further, about two-thirds of individuals will have scores within one standard deviation of the mean and about 95 percent will have scores within two standard deviations of the mean. For this reason, IQ scores are sometimes evaluated using percentile scores, which divide the normal distribution into 100 parts so that 1% of the scores are in each part. For example, admission to the high-IQ society Mensa requires a person to score in the 98th percentile or higher on several different validated IQ tests. This requirement means about one in 50 people would be eligible to join.

Percentile IQ scores can be useful, but they can be misinterpreted since the distance between each percentile is not equal. In contrast, standard deviations are the same distance apart, sometimes making it more sensible to compare individuals in terms of average distance from the mean. Also, IQ tests are imperfect measures of intelligence because they generally do not produce the exact same score for the same person, even if the test is taken more than once. This inaccuracy is quantified by the standard error of measurement and represents how much variability an individual person's scores would have if they took the test many times. For example, if a person scored 100 on an IQ test that had a standard error of 2, the person's true IQ score would often be



IQ tests are usually copyrighted and must be administered under supervised conditions.

interpreted as being somewhere between 96 and 104. Some researchers and others have suggested that the average of three IQ tests provides a better indication of a person's true IQ score than a single test.

There are three features of general intelligence that are important because they negate arguments that IQ scores have no meaning: their stability, their heritability, and their correlation with external phenomena. First, IQ scores are remarkably stable across the life course from childhood to old age. Data to demonstrate this are exceptionally rare, but one exception can be found in Scotland. During one day in 1932, every 11-year-old in the country took an IQ test. They were retested 66 years later, and the scores were found to correlate highly with childhood IQ score (0.76), providing evidence of stability of IQ scores over time. Second, IQ scores are highly heritable. The heritability of individual differences has been estimated as between 30% and 80%, illustrating that genetics contributes strongly to IQ scores. However, no single gene or set of genes has been identified. This suggests that the genetic contribution to intelligence is multifactorial, as with other observable characteristics (phenotypes), such as height. There are no sex differences in IQ, although the distribution of males' scores is slightly wider at both ends of the distribution. Third, IQ scores correlate with variables that can be considered external, or outside the IQ test itself. IQ correlates with indicators of socioeconomic status (SES)—a indication of factors like educational attainment, income, and occupational social class—and with many biological variables, including brain size, height, sperm quality, and mortality. The causes of these correlations are disputed.

Content of IQ Tests

The content of IQ tests differs, depending on the specific cognitive abilities they are intended to measure. Some tests have been criticized as being culturally biased because they ask questions that require culturally specific knowledge. Tests that do not evaluate "general knowledge" are considered more "culture fair." For example, Raven's Matrices is a test that contains no written information, requiring abstract reasoning skills. This test contains no culturally specific information, so that it is not possible to learn how to take the test. Similarly, tests of reaction time are considered indicators of *g*, because they reflect speed of information processing. These do not assess culturally specific information or knowledge. Clifford Pickover imagined how aliens might test human intelligence and designed related mathematics and logic puzzles. Other intelligence researchers argue that knowledge is a reliable indicator of *g* and should therefore be included in IQ tests. IQ tests also differ in the extent to which it is necessary to complete every question. Traditional IQ tests are designed using classical test theory. In these tests, the IQ score is more reliable for people with an average level of IQ. Since people with high IQs find many questions easy and people with low IQs find many questions difficult, fewer relevant questions are answered by people at either end of the IQ distribution. More recently, computerized adaptive tests have been developed and informed by item response theory, which addresses these problems. These tests can alter the difficulty of test items, so that people with high IQs receive a larger number of difficult items. Reliability is improved and testing length can be reduced because respondents do not have to answer every question.

Age and Intelligence

Although IQ scores are relatively stable, cognitive decline typically occurs with increasing age. This fact is important because cognitive decline may indicate mild cognitive impairment and risk of dementia. When considered over time, specific kinds of cognitive abilities appear to deteriorate at different rates. Fluid intelligence, referring to processing speed (particularly of new information), declines from age 26 onward. In contrast, crystallized intelligence, referring to speed of recall of existing knowledge (for example, vocabulary and general knowledge) is relatively stable.

For this reason, standardized tests of word recognition, such as the National Adult Reading Test (NART), are useful at estimating premorbid IQ in patients suspected of having dementia. A discrepancy between IQ as estimated by the NART and IQ estimated from another test could indicate that cognitive decline has occurred. Cognitive decline can result in mild cognitive impairment and dementia or Alzheimer's disease, which have high mortality, morbidity, and treatment and care costs.

Research into the prevention of cognitive decline is ongoing, but several risk factors have emerged consistently, such as cigarette smoking and physical inactivity. Consumption of fish oils, either from oily fish or fish oil supplements, may help prevent cognitive decline. Prior IQ is a strong protective factor, such that a higher initial IQ appears to protect against cognitive decline in later life. Claims that IQ can be changed are controversial. Although brain plasticity is known to be greater than once thought—and there is evidence that children exposed to cognitive stimulation enjoy increases in IQ—it is not clear how stable these gains are. Furthermore, attempts to increase IQ in adults have not been successful.

Lower IQ scores are associated with earlier mortality and higher morbidity. This association provides further evidence for the validity of IQ tests. It is noteworthy that the relationship between IQ and mortality often remains after adjusting for indicators of socioeconomic status (SES), such as income, educational attainment, and occupational social class. Given that IQ is largely stable after childhood, this relationship is unlikely to be explained by societal factors. Evidence suggests that IQ contributes strongly to health literacy, which, according to the World Health Organization, refers to "the cognitive and social skills which determine the motivation and ability of individuals to gain access to, understand, and use information in ways which promote and maintain good health." People with inadequate health literacy skills tend to have unhealthier lifestyles, adhere less well to medical regimens, and do not understand written health information or the need for regular screening for diseases.

Access to healthcare does not solely explain the IQ-health relationship because it can also be found in countries that have free healthcare, such as the United Kingdom, which has the National Health Service (NHS). Managing chronic diseases, such as diabetes, involves repeating many complex tasks, such as monitoring blood sugar and planning activities around meals. Without supervision and support, the risk of making dangerous mistakes could accumulate over time. Many areas of life involve repeating a set of unpredictable, complex tasks, which can damage health in the long term. The field of cognitive epidemiology studies why IQ is linked to worse health outcomes and the role that literacy plays in this relationship.

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See Also: Educational Testing; Measurement in Society; Normal Distribution; Psychological Testing.

Interdisciplinary Mathematics Research

See *Mathematics Research, Interdisciplinary*

Interior Design

Category: Architecture and Engineering.
Fields of Study: Algebra; Measurement; Problem Solving.
Summary: Mathematics is involved in the layouts and color schemes of interior design.

Interior design is a career that combines mathematics and art and is the art and craft of making living spaces that bring positive emotional and aesthetic effects to the inhabitants. Mathematics has long been connected to interior design. One of the 10 books of Marcus Vitruvius Pollio's mathematical work, *De Architectura*, is focused on interior decoration. These books heavily influenced the Western scientific, engineering, and mathematical developments in the 2,000 years since they were written. Mathematician Jamshid al-Kashi approximated the surface area of a *muqarnas*, a decoration made of flat and curved polygons that covers building joints, while the Art Deco design movement of the 1920s and 1930s relied on geometric forms.

Computations to estimate the amount of materials and their cost, such as the area of a surface that will be covered in fabric, tile, or paint, underlie interior design. The International Mathematics & Design Association was founded in 1998. It publishes a journal focusing on areas such as computer-aided design, computational geometry, mathematical modeling, visualization, and system media design.

Design Principles and Elements

Lists of overarching design principles tend to sound very mathematical. Figure 1 shows a widely accepted example, with mathematical elements listed next to each principle.

The principles are achieved through the combination of design elements. With the extensive use of software and digital media for home decorating, expressing design elements in formulas, graphs, tables, and other mathematical representations has become commonplace. A typical list of elements includes the following: line, shape, direction, size, texture, color, and value.

Shape and Logistics

A circle has the maximum possible area to perimeter ratio of all two-dimensional shapes. This characteristic is the reason circles were adopted as elements of living space structures whenever exposure needs to be minimized. Houseboats may have circular windows to minimize leak danger. In noisy cities or harsh natural environments, architects opt for circular shapes of houses, or their key parts, to minimize the contact with the outside. This design was used in many cultures, such as in traditional Mongol yurts, Celtic roundhouses, Lakota tipis, and Lesotho's mokhoros.

City and road building dictated cuboid houses for several reasons. Many tools and materials make it easy to mass-produce rectangular structures, such as boards, bricks, and panels. Also, circles do not tessellate (tessellation occurs when a repeated shape covers a plane without any gaps or overlaps), making it impossible to build circular houses adjacent to one another, as is done in cities. While hexagons tessellate, the edge of a block of hexagonal houses is not straight, making it problematic to build roads. Another tessellating floor shape candidate, a triangle, has sharp corners that are inconvenient to use and psychologically problematic.

Figure 1.

Design Principle	Corresponding Mathematics
Balance	Symmetry, center of mass, and equivalence
Rhythm	Pattern, algebraic group, gradient, tessellation, sequence, and growth
Proportion	Ratio and proportion, golden ratio, geometric series, scale, and dimensions
Dominance	Ratio and proportion, categories (similarities and differences), extreme value, and frequency
Unity	Categories (similarities and differences), shape, pattern, continuity, similarity, density and proximity, vectors, and alignment

Order in Complexity: Tessellations and Fractals

Repeating patterns satisfy the design principles of balance, rhythm, and unity. Since they are practical to make and use with a variety of simple tools, they are used in home decorating in all human cultures. Tessellations appear in mosaics, on all parts of buildings, and in designs of rugs, coverings, and wall hangings. Traditional designs often combine beauty, cultural and spiritual meanings, and utility through modular units.

The mathematics known to artisans is still being formally described. This process of rediscovery and formal mathematical description is called "ethnomathematics." For example, medieval Islamic mosaics masters described very complex symmetric, never-repeating patterns, made of standard polygons. These patterns were rediscovered by mathematicians in the 1970s and named "quasicrystalline" and "Penrose tilings." Many traditional African villages are laid out to form fractals, with the village shape repeated in the house clusters, houses, and interiors of each house.

These villages were first mathematically described in the 1990s. Mandalas—appearing in several cultural traditions, such as Hinduism, and in work of modern artists, such as M.C. Escher—exhibit elements of projective geometry, as their tessellating shapes shrink toward the edge of the circle. Once the underlying mathematical principles of complex patterns are understood, software can be programmed for further experimentation and discovery. Complex computer-generated patterns, often incorporating ancient artisan traditions, are now ubiquitous in home decorating materials such as wallpaper, flooring, textiles, and tiles.

Color Models, Circles, and Schemes

Colors are defined by spectral wavelengths; for example, the wavelengths of reds are approximately 630–730 nanometers. Color models define colors as additions or subtractions of primary colors and are used to pinpoint precise colors for decorating projects, often using software. The additive model known as "red, green, blue" (RGB) can be physically implemented in overlapping lighting with different colors. The subtractive model known as "cyan, magenta, yellow, key black" (CMYK) can be implemented by mixing pigments and is used in color printing, including wallpaper, yarn, and fabric dyes, as well as in mixing household paints. A color wheel is a traditional artist and designer infographic used to visualize color models. The wheel has the primary colors positioned at three equidistant points around it and color mixes between the primaries, with the position signifying the ratio of the mix.

A color scheme is a combination of two or more colors that work well together. Home decorators use special terms to describe colors, with each term having mathematical meaning in color models. The terms for describing colors include "warm," "cool," "hue," "intensity," "contrast," and "tone." On the other hand, colors can be described metaphorically, which is used more frequently in consumer-oriented product names such as "Light in the Leaves" or "Chilled Chardonnay."

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See Also: Engineering Design; Painting; Quilting; Textiles.

Internet

Category: Communication and Computers.
Fields of Study: Algebra; Number and Operations; Problem Solving.
Summary: Many properties and problems of the Internet are studied and modeled using mathematics.

The Internet is a worldwide computer network connecting other computer networks in government, business, academia, and other public and private sources. Communications are facilitated by the Internet Protocol Suite (TCP/IP), originally proposed by Vinton Cerf and Robert Kahn in 1974. The Internet is used for implementing various applications including electronic mail, pioneered in the late 1960s, and the World Wide Web (WWW) of linkable documents. The idea of networks connecting information nodes appeared in futuristic scientific writings and science fiction beginning in the early twentieth century.

The work of mathematicians, computer scientists, cyberneticists, and many other scientists contributed to the emergence of the Internet and the World Wide Web by the end of the twentieth century. Researchers and teachers in nearly every discipline use the Internet to further their work, and many study the properties of the Internet itself using mathematics. One problem explored by mathematicians and computer scientists is mapping the Internet, often undertaken to understand the nature of connections and to reduce stress on routers. The field of hyperbolic geometry has proven to be highly useful in creating such maps, especially with regard to assessing global stability and developing efficient routing methods. Mathematicians also consider the theoretical and computational challenges posed by the massive graphs that result from Internet mapping, which test the limits of even the largest and fastest computers. Others examine society's increasing dependence on the Internet for a range of critical everyday tasks (like banking and medical recordkeeping) along with the risks and vulnerabilities (like identity theft) that this reliance may create.

Codevelopment of Mathematical Sciences and the Internet

Mathematicians including John Von Neumann, Alan Turing, and Norbert Wiener contributed to the development of both the hardware and the software necessary to implement computer networks and the Internet. The precursors of the Internet were networks such as the telegraph, telephone, radio, and television. Even early electronic computers had systems for data input, computation, and output. In the late 1960s, individual computer "nodes" were connected to one another, building on the technology for connecting subsystems within the same computer. These early stages of building computer networks promoted the development of the mathematics-rich fields of cybernetics, informatics, and artificial intelligence.

Mainframe computers enabled countless historical achievements and facilitated research and problem-solving in mathematical fields such as cryptography, simulation, and genetics. In the late 1970s and early 1980s, the introduction of the first personal computers changed the face of computing by creating applications and giving access to new groups of users. In the 1980s, the National Science Foundation (NSF) funded five supercomputer centers connected by NSFNET, which built on Computer Science Net (CSNET) and the Department of Defense's Advanced Research Projects Agency Network (ARPANET). Demand during the first year was so great that the system had to be upgraded almost immediately, and uses for the new network continued to expand, as did the mathematics research needed to meet user demands for functionality. At the same time, national computer networks such as ARPANET and NSFNet, the Japanese JUNET, and the European CERN remained isolated from one another. The big challenge at the time was to make these separate networks compatible and interoperable.

Adoption of several dozen international protocols, such as the TCP/IP for the Internet, facilitated interlinking. In the early 1990s, the idea of common protocols enabled the system of file hosting, accessible by anyone at all times and called the World Wide Web. The explosive evolution of the Internet and the Web in the next decade is well documented. In the United States, efforts were aided by several pieces of legislation. For example, the High Performance Computing Act (HPCA) of 1991 reset priorities for computing research and education. President Bill Clinton stated that he believed such legislation enabled collaborations "critical for assuring American prosperity, national and economic security, and international competitiveness in the twenty-first century." Computer scientists Eric Bina and Marc Andreessen developed the first widely used graphical browser, Mosaic, released in 1993 and funded by a program associated with the HPCA. Tim Berners-Lee, the creator of several WWW protocols, was knighted in 2004 by Queen Elizabeth II "for the invention of the World Wide Web."

Mathematical Problems

One mathematical problem that had to be solved in order to build computer networks was packet switching, which is grouping data of all types into blocks known as "packets" of size that are appropriate for network transmission. Network nodes or routers have algorithms that decide how to queue, buffer, and deliver individual packets as a function of network traffic patterns. This is a different, mathematically more complex model from circuit switching, which was used in older telephone networks to transmit information bits at a constant rate. Computer scientists Paul Baran, Donald Davies, and Leonard Kleinrock pioneered packet switching networks. Baran's work was shaped in part by Cold War concerns about maintaining communications in the face of nuclear attack. Donald Davies worked with Alan Turing at the National Physical Laboratory and is reputed to have found mistakes in Turing's groundbreaking paper "On Computable Numbers." Kleinrock, a recipient of the U.S. National Medal of Science, said of his work, "Basically, what I did for my Ph.D. research . . . was to establish a mathematical theory of packet networks."

In the late 1960s, mainframe computers had message systems among their different users, who all had to be online at the same time to communicate. In the early 1970s, the message system software was modified to include new computer networks. The ability to deliver messages to offline users, make different systems compatible, and uniquely identify users were significant research problems. The compatibility issue, still important in the twenty-first century, was resolved in part by creating software and hardware gateways that connect different systems. BITNET was cofounded by Ira Fuchs and Greydon Freeman primarily for research and academic communities, while FidoNet was implemented for personal computers and bulletin board systems by Thomas Jennings. Unique identification of users is a complex mathematical problem, since for any string length there is a finite number of possible letter and symbol permutations.

Similar concepts apply to the study and selection of secure electronic passwords. A system developed in

A mathematical problem that had to be solved in order to build computer networks was packet switching—grouping data of all types into blocks known as "packets" that are sized appropriately for network transmission.

Erdös–Rényi Graphs

One mathematical discovery of network science is that large-scale networks like the Internet are structured in ways that do not appear to be random, though some researchers initially thought they would produce Erdös–Rényi graphs, which are random graph models having bell-shaped degree distributions. They are named for mathematicians Paul Erdös and Alfréd Rényi. Instead, large social networks have degree distributions with no peaks and heavy tails, proportional to a power function. This means that most nodes have very few connections, and only a few nodes, called hubs, have many connections.

For example, the majority of Wikipedia editors have edited only one or two articles, and the majority of Web pages have one or no links leading to them. In 2004, physicist Mark Newman and his colleagues studied scientific coauthorship networks using these models. Mathematician Paul Erdös, acknowledged by many as one of the founders of graph theory, was a highly prolific collaborator—a node of very high degree in the network of published mathematicians, who often compute their personal Erdös number to describe their closeness to Erdös.

the early 1970s assigned registration codes to domains and then to users within domains in the form "user@domain." This method and the use of "@" are credited to Raymond Tomlinson. At the start of the twenty-first century, the mathematical structure of domain names is a type of tree, with multiple hierarchical levels. Minimally, there are two levels. Each domain name ends with the top-level domain including generic ones, such as ".com" and ".edu," and country code ones, such as ".us" or ".uk," with a period on the left. To the left of that period comes the second level domain name; for example, "wikipedia.org" or "google.com."

If there are more domain levels, they appear on the left of the second-level domain and are separated by periods as well; for example, "simple.wikipedia.org" or "groups.google.com." There is no limit to the number of domain levels. This syntax and structure was first published in the 1980s in connection with the Advanced Research Projects Agency Network (ARPANET). IP addresses are the numerical representations of individual computers, mapped to domain names. They consist of four bytes of information displayed as numbers. Each byte has eight bits and can be any integer from 0 to 255. With the exponential growth in Internet users, assigning unique identities to users, domains, and computers continues to be a challenging problem, especially since many users have multiple e-mail and IP addresses. For computer users, off-line message delivery is achieved by storing messages on digital servers until the recipient accesses them.

E-mail programs typically employ the Internet Message Access Protocol (IMAP), developed by Mark Crispin, or the older Post Office Protocol (POP) to retrieve mail. Simple Mail Transfer Protocol (SMTP) is also used for sending and receiving functions. Mathematical algorithms enable the queuing, encryption, authentication, and filtering of e-mail, and mathematicians continue to contribute new developments and improvements. Many agencies are responsible for making assignments and tracking Internet protocols. The Internet Assigned Numbers Authority was headed for nearly 30 years by computer scientist Jonathan Postel, who codeveloped and documented many of the key Internet standards, including SMTP and Domain Name System (DNS) servers.

The Growth of Networks

Other mathematical problems of Internet development sprang from the incredibly fast growth of networks. To compare the rate of growth of different networks, researchers use metrics such as time per number of users. They have determined, for example, that it took only five years for the Internet to reach 50 million users, versus 13 years for television and 38 years for radio. As the number of users and domains grew, search algorithms became a prominent field in computer science and mathematics, with several major developments such as clustering and relevance rankings. There are many search engines, many of which initially used the content of Web pages to rank results. Google's PageRank method was among the first search protocols to use sophisticated mathematical modeling, including directed graphs and stochastic matrices, to explore links between pages hierarchically. The Page Rank algorithm is named for Google cofounder and computer scientist Lawrence Page.

In 2009, Google research scientist Kevin McCurty noted that successful search engines continually improve by employing mathematical methods that quickly find relevant material and eliminate irrelevant factors that can skew results. Along with better ranking schemes, Internet speed is critical in effective searching and content delivery. The original packet switching and data routing problems have become even more complex as the Internet has grown. Mathematicians and computer scientists model Internet traffic flow using many mathematical and statistical techniques, taking into consideration many variables, including the type of content being exchanged. Photos, videos, music, text, e-mail, and online gaming all require different resources. Based on these models, algorithms to optimally route traffic can be designed and implemented, reducing congestion and slowdowns. For example, the traffic load on a given Website's computers can be reduced by storing some content at other servers that provide more optimal access patterns, a process known as "network caching."

Some twenty-first century models are starting to use concepts from disciplines like economics, such as equilibrium theory. One example is called "congestion-dependent pricing," which would route packets depending on users' willingness to pay more for privileged Internet access during periods of congestion. Given the number of packets in even a small text file, this is a mathematically complex problem that still requires a great deal of research.

A separate set of science problems has to do with hardware and the various means of connecting to the Internet. As of 2010, it is possible to connect to the Internet through both land-line and cell phones, radio, satellites, dedicated fiber-optic lines, and television cables. While similar in many ways, each has a unique set of issues related to speed, security, data transmission, compatibility, and bandwidth, especially when considering that people are connecting to the Internet with many devices other than personal computers. Mathematicians, computer scientists, and others work on both the hardware and the software solutions.

Network Science

Network science predates the Internet, having its root in graph theory. It is interdisciplinary and includes mathematics, engineering, computer science, biological sciences, sociology, and other disciplines interested in studying various types of networks. It flourished with easy availability of empirical data from computer and social networks made possible by the Internet and the high demand for applications in all aspects related to the Internet. Concepts and methods from graph theory, such as centrality, betweenness, and closeness are used to quantify and describe networks. Centrality is a measure of the importance of a node within a network. Betweenness measures the quality of paths through the node, such as the number of shortest paths between pairs of other nodes. Closeness is the topological measure similar to distance, usually defined as the average number of nodes in the shortest path between a given node and all other nodes in a network that connect to it.

Maps of networks help mathematicians and others analyze vulnerabilities, such as critical nodes that lie between many other nodes and whose loss would sever connectivity, and deprecated connections, where use of outmoded software or features affects speed or leaves the users open to attack. In addition to graph theory, hyperbolic geometry adds to Internet mapping by considering geometric coordinates of nodes in space, not simply the map of connections. The added information can then be used to quantify the issue of closeness from a geometric point of view. In graph theory, each node of a network has a degree, which is the number of other nodes connected to it. Degree distribution is a statistical measure showing the probability distribution of various node degrees over the network. Statistical sampling strategies are often used in network research, since the problems and networks examined are typically far too vast for complete data collection.

Economics and the Internet

In the 1990s, many people believed that the Internet would bring about fundamental changes in the landscape of the business world. Starting in the mid-1990s, venture capitalists were investing heavily in new Internet businesses, sometimes called "dot-coms." During this time, many Internet companies operated at annual losses, expanding in anticipation of future revenues. This worked for relatively few companies, such as Amazon and Google. In 2001, this "dot-com bubble" burst, with many Internet-related businesses declaring bankruptcy.

The promises of the Internet that survived the dot-com bubble became clearer toward the end of the first decade of the twenty-first century. For example, researchers found that in many cases, product popularity obeys a frequency distribution law similar to the degree distribution of network nodes. The majority of customers use a few most popular products, with the majority of products liked by small minorities of customers.

In the early 2000s, several companies realized large profits by reaching these so-called long tails (named after the characteristic shape of the distribution curve) of niche customers and redefined their industries. Apple changed the music industry by selling individual tracks online; Netflix had a similar effect on movie rentals. Mathematical algorithms for determining customer preferences and making recommendations were driven in large part by Internet commerce. Recommender systems use complex relevance metrics, evaluating content such as texts or video based on statistics of past behavior of all users within the system.

These systems use explicit data, such as rank preferences given by users, as well as implicit data, such as actions other similar users have done before. Over time, these systems accumulate large amounts of data and increase the accuracy of their recommendations. Mathematics involved in creation of these algorithms includes statistical analysis and linear algebra for working with matrices defining closeness of users. Illustrating how lucrative good algorithms are from the business perspective, in 2009 the Netflix Prize awarded $1 million to the developers of an improved filtering algorithm for recommending movies.

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See Also: Cerf, Vinton; File Downloading and Sharing; Personal Computers; Raghavan, Prabhakar; Search Engines.

Interplanetary Travel

Category: Space, Time, and Distance.
Fields of Study: Geometry; Measurement; Problem Solving.
Summary: Space exploration requires mathematics to plan trajectories and to navigate in space, as well as to measure and to analyze massive amounts of data.

Interplanetary travel can be defined as any spaceflight—manned or remotely guided—to the various bodies of the solar system, including planets, their satellites, and asteroids. Such space exploration required new mathematics to plan trajectories and navigate in space, as well as to measure and to analyze massive amounts of data. These flights have had a great societal impact and have radically changed human attitudes toward the outer space surrounding the Earth.

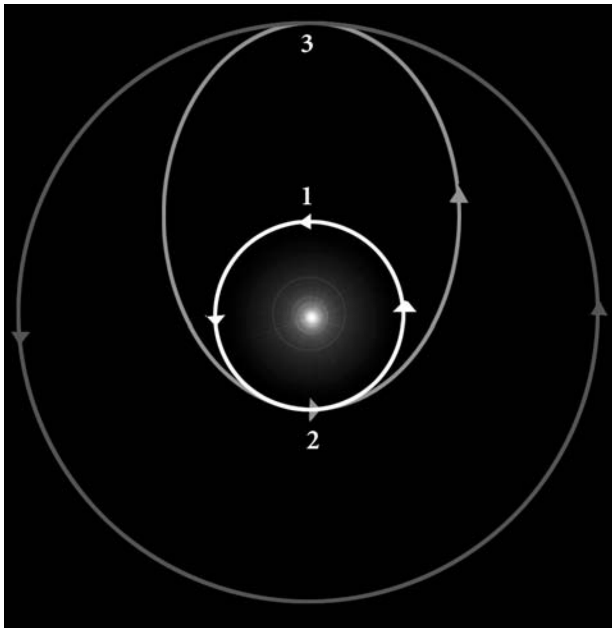
History

A scientific possibility of interplanetary travel was discussed for centuries after Isaac Newton wrote *Principia* in 1687, in which he unified terrestrial and celestial dynamics by discovering the force of gravity as an important source of motion, including the movement of celestial bodies. Step by step, an important new mathematical branch of astronomy emerged and received the title "celestial mechanics." In its formative days, celestial mechanics played an outstanding role in the progress of mathematics, demanding and inspiring novel and efficient mathematical tools. Among the pioneers of celestial mechanics were prominent mathematicians such as Leonhard Euler (1707–1783), Alexis-Claude Clairaut (1717–1765), and Joseph-Louis Lagrange (1736–1813). Today, the branch of celestial mechanics dedicated to spaceflight is usually termed astrodynamics.

For many years following Newton's discovery, the topic of interplanetary travels mainly remained the subject of science fiction writers. In the nineteenth century, among the most influential science fiction writers were Jules Verne (1828–1905) with his books *From the Earth to the Moon* and *All Around the Moon* and H. G. Wells (1866–1946) with his book *War of the Worlds*. Verne's work contained a great deal of mathematics discussion, much of which was reasonably accurate based on the knowledge of the time.

To put interplanetary travel into practice, it was necessary to realize some significant preconditions, including designing spacecraft with the capacity for maneuvering, designing technologies for boosters to reach escape velocity, developing a theoretical base for space navigation, and creating systems for long-distance radio communications. These technological developments were not made until the beginning of the space era in 1957.

Figure 1. Hohmann Transfer Orbit.



Mathematical Development

From a mathematical viewpoint, the most interesting part of interplanetary travel is space navigation. An appropriate example of a solution with respect to navigational problems is the Hohmann transfer orbit. In 1925, Walter Hohmann calculated that the lowest-energy route between any two celestial bodies is an ellipse that forms a tangent to the starting and destination orbits of these bodies. Such a transfer orbit between the Earth and Mars is graphed in the following illustration. A spacecraft traveling from Earth to Mars along the Hohmann trajectory will arrive near Mars's orbit in approximately 18 months. Just a small application of thrust is all that is needed to put a space probe into a circular orbit around Mars. The Hohmann transfer applies to any two orbits, not just those with planets involved (see Figure 1). In the figure, Hohmann Transfer Orbit (light gray oblong ring), Earth's orbit is represented by the white circle, and Mars' orbit is represented by the darker gray circle. A spaceship leaves from point 2 in Earth's orbit and arrives at point 3 in Mars's.

Another example of navigational technique is routinely called the "gravitational slingshot." It utilizes the gravitational influence of planets and their moons to change the speed and direction of a space probe without the application of an engine. In this case, a spacecraft is sent to a distant planet on a path that is much faster than the Hohmann transfer. This would typically mean that it would arrive at the planet's orbit and continue past it. However, if there is a planetary mass between the departure point and the target, it can be used to bend the path toward the target, and in many cases the overall travel time is greatly shortened. Prime examples of the gravitational slingshot are the flights of the two spacecraft of the American Voyager program, which used slingshot effects to redirect trajectories several times in the outer solar system. Astrodynamics considers many other interesting approaches. Several technologies have been proposed that both save fuel and provide significantly faster travel than Hohmann transfers; most are still theoretical.

Because of astrodynamics limitations, travel to other solar systems bodies is practical only within certain time windows. Outside of such windows, these bodies are essentially inaccessible from Earth using current technology. Mathematicians helped design the Interplanetary Superhighway, a network of low-energy trajectories, in order to find efficient routes through space; these mathematical foundations originated with French mathematician Henri Poincaré.

Achievements and Obstacles

The modern accomplishments in interplanetary travels are extraordinary. Remotely guided space robots have flown past all of the planets of the solar system from Mercury to Neptune, and the National Aeronautics and Space Administration's (NASA's) spacecraft New Horizons is scheduled to fly past the dwarf planet Pluto in 2015. The five most distant spacecraft (including the American ships *Pioneer-10*, *Pioneer-11*, *Voyager-1*, and *Voyager-2*) were scheduled to leave the solar system at the beginning of the twenty-first century. Artificial satellites have orbited Venus, Mars, Jupiter, and Saturn. Spacecraft have landed on the Moon, Venus, Mars, Saturn's moon Titan, and asteroid 433 Eros. The first probes to comets (European *Giotto*, Russian *Vegas*, American *Stardust*) were fly-by missions. In 2005, the *Deep Impact* probe hit the comet 9P/Tempel to study the composition of its interior.

Great achievements took place in manned interplanetary travels once mathematicians, scientists, and engineers understood the mathematical principles required to launch spacecraft outside Earth's atmosphere and to maneuver in the microgravity environment of space. NASA also recruited astronauts with strong academic credentials in science and mathematics. America's Mercury and Gemini programs put humans into space and Earth orbit and taught them how to change trajectory in space to move to a new orbital altitude or to dock with other spacecraft, while the Apollo program took them to the moon. After missions in which men orbited the moon and returned, Apollo 11 landed astronauts Neil Armstrong and Edwin "Buzz" Aldrin on the moon in 1969. There were six successful manned American expeditions to the moon from 1969 to 1972.

Further development of interplanetary travel has many obstacles that will require a great deal of mathematical analysis to model, simulate, and solve. For example, astronauts must be protected from extreme radiation exposure in the Van Allen belt, a torus-shaped region of space surrounding the Earth and other planets named after geophysicist James Van Allen of Iowa.

The larger outer radiation belt is about four Earth radii (RE) above the surface of the Earth and the inner is about 1.6 RE, with a gap at roughly 2.2 RE. Apollo astronauts were briefly exposed to this radiation on trips to the moon. Conspiracy theorists who disputed the notion that humans landed on the moon cited the Van Allen belt as evidence that the astronauts would have died from radiation, but simple calculations and the data collected by radiation sensors worn by astronauts (similar to those worn by scientists and hospital workers who may be exposed to radiation) demonstrated that the speed and design of the Apollo capsules protected astronauts during these relatively short trips.

If the Earth was the main focus of many sciences (geodesy, geology, geophysics, geochemistry, and oceanography) for millennia, interplanetary travel created a new important branch of research—comparative planetology—which is essential for understanding the history of Earth and its evolution.

Among many other difficult problems of interplanetary travel is developing adequate human life support. A breathable atmosphere must be maintained, with adequate amounts of oxygen, nitrogen, controlled levels of carbon dioxide, trace gases, and water vapor. It is also necessary to solve the problem of food supply.

At some point in time, all of these problems may be overcome. Incentives for future expansion of interplanetary flights include the possibility of colonizing other portions of the solar system and utilizing resources.

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See Also: Planetary Orbits; Ride, Sally; Spaceships; Weightless Flight.

Inventory Models

Category: Business, Economics and Marketing.
Fields of Study: Data Analysis and Probability; Measurement; Number and Operations.
Summary: Mathematical inventory control models help businesses make decisions, and they are widely studied in the discipline of operations research.

In an ideal world, retail stores would stock all of the products that customers are interested in buying and stock these in sufficient quantity to cater to all customers. In reality, store area is limited and a company would not benefit by stocking an excess of each product. The problem, then, is to calculate the optimal amount of supply.

These decisions take into consideration how many units should be kept so that most, if not all, customers can be served on a particular day, because if customers do not find what they want, they will shop elsewhere. At the same time, a store does not want too many units on hand, as there are costs attached to storing excess units, and they may remain unsold, which also reduces profit.

The problem can be considered in manufacturing, where a product consists of many small components, and a business has to decide how many components it must order and store so that fabrication runs smoothly. Similar examples exist in service industries and military ships. Mathematical inventory control models help businesses make decisions, and they are widely studied in the mathematical discipline of operations research.

Mathematics of Inventory

Computational logistics is a mathematical and businesses field concerned with planning the flow and storage of goods, services, or information from the point of origin to the point of use. One key planning consideration is the trade-off between transport and inventory costs, a factor recognized at least as early as the mid-1880s. Mathematicians, computer scientists, and others continue to develop new inventory management and optimization models as well as the algorithms and software necessary to implement them. Mathematician Samuel Karlin was awarded the John von Neumann Theory Prize in 1987, as well as the National Medal of Science in 1989, for diverse mathematical contributions, including inventory theory.

Inventory models used to calculate optimal order quantities and reorder points, often broadly called economic order quantity (EOQ) models, existed long before the arrival of the computer. Advances in mathematical methods and computer technology have facilitated more realistic models that account for more variables. Optimizing inventory depends on factors such as storage space, storage cost, demand rate, time between demands, cost of ordering, time for retrieving stored item or receiving an ordered item, discounts for bulk orders, and many other real-world costs.

Just-in-time models are based on the idealized principle that items are available exactly when they are needed, with zero storage time or delay. Just-in-time inventory management and lean manufacturing ideas existed as far back as Henry Ford's Model T factories but became widely feasible in the late twentieth century with advances in technology that affected variables, like the lead time required to place an order for more stock. Reduction of process variability, using better monitoring, waste reduction, or inventory buffers, are typically seen as key to achieving optimal models under this system. A just-in-time model can save money by reducing inventory, but tighter constraints make them consequently more vulnerable to disruptions that violate the constraints.

Many basic EOQ models are simplified by assuming that variables such as demand are fixed or uniform across some period of time. These deterministic models are easy to solve analytically but may produce unrealistic results. They are often useful for theoretical study or businesses with greater variability tolerances. Many variables that influence inventories, such as demand and delay times for orders of new goods, are more realistically modeled as random variables. As a result, inventory models are often probabilistic or stochastic. Constraints tend to be operationalized as costs. For example, the physical area available for storage, such as square footage of shelf space or warehouse volume, can be reformulated as a cost constraint by calculating a cost per unit area or volume. Cost may also be parameterized into components like procurement and maintenance costs. Markov chains and linear programming techniques are useful for formulating and solving various types of inventory models. Statistical methods are used to obtain valid data for modeling and simulations.

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See Also: Budgeting; Deming, W. Edwards; Probability; Quality Control; Shipping.

Investments

See *Mutual Funds*

Irrational Numbers

See *Numbers, Rational and Irrational*

Islamic Mathematics

See *Arabic/Islamic Mathematics*

J

Jackson, Shirley Ann

Category: Architecture and Engineering.
Fields of Study: Connections.
Summary: Shirley Jackson is a noted physicist and mathematician who has performed important research in semiconductor systems.

Shirley Jackson (1946–) is an American physicist. She was always interested in mathematics, in part because her father shared his talent in mathematics and science with her. She excelled in school, graduating as valedictorian in 1964 after completing accelerated programs in both science and mathematics. She advises, "to do science takes a cumulative background. You can't do advanced mathematics if you don't know calculus; if you don't know trigonometry, geometry, algebra and you certainly can't do those things if you can't add, subtract, multiply, divide, no fractions, et cetera." She went on to the Massachusetts Institute of Technology for both her bachelor's degree and, in 1973, a Ph.D. in physics under the direction of James Young, thereby becoming the first African-American woman to receive a Ph.D. from that institution. Her research as a student centered on solid-state physics and elementary particle theory, both of have continued as motivating research interests throughout her career. Shirley Jackson's research interests are diverse, but largely relate in some way to semiconductor systems, particularly new or unconventional types of semiconductor systems. Her research includes electronic, optical, and magnetic aspects, as well as Yang–Mills gauge theory, which is an extremely important topic in applied mathematics. Jackson has been highly acclaimed for her achievements in science, education, and public policy. She has been elected to both the American Philosophical Society and the American Physical Society, among others. She noted, "how does a young woman, eager for success, but also desirous of support and respect, respond to . . . the limitations associated with racial and gender stereotypes? I will tell you. I chose a 'trade.' I chose physics!" In 1998, she was formally inducted into the National Women's Hall of Fame. In 2002, *Discover* magazine named Shirley Jackson one of the 50 most important women in science.

Professional Life

After receiving her doctoral degree, Jackson obtained a position as a research associate at the Fermi National Accelerator Laboratory in Batavia, Illinois, where her work concentrated on hadrons, a class of subatomic particles including protons, neutrons, pions, and kaons, among others. Later she was a visiting scientist, first at the European Organization for Nuclear Research (also known as CERN) and subsequently at the Aspen Center for Physics. She noted, "I like everything about begin a physicist: thinking about a problem, solving it, writing about it, working with my colleagues, and giving talks all over the world." Jackson was a respected lecturer in physics at the Stanford Linear Accelerator Center. In 1976, Jackson became part of the Theoretical Physics Research Department at Bell Laboratories. From 1991 to 1995, she was a professor on the faculty of Rutgers University, but continued to consult with Bell Laboratories.

President Bill Clinton appointed Jackson to the position of chairman of the U.S. Nuclear Regulatory Commission (NRC) in 1995. At that time, she was both the first woman and the first African American to hold this position. In 1999, after her term at the NRC expired, she became president of Rensselaer Polytechnic Institute, a private university in Troy, New York, dedicated to scientific and technological research. She continues to hold this position to the present time and her current contract extends until 2020.

She has also served on the boards of directors of a large number of organizations, including the New York Stock Exchange, FedEx Corporation, Marathon Oil Corporation, IBM Corporation, the Massachusetts Institute of Technology, and the Smithsonian Institution. She is an active member of the National Academy of Sciences, the National Science Foundation, and the American Association for the Advancement of Science. She has described her purpose on these committees as supporting American innovation by (1) increasing support for education, (2) bringing in the best international talent, and (3) promoting the participation of women, minorities, and other underrepresented groups in science-based careers. She feels that, "We have to have more degreed teachers—teachers with actual degrees in science, mathematics and engineering." In 2009, she was appointed by President Barack Obama to the President's Council of Advisors, which focuses on important matters of public policy.

Shirley Ann Jackson speaks at the 2010 Annual Meeting of the New Champions in Tianjin, China.

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Michael "Cap" Khoury

See Also: Careers; Mathematics, Applied.

Joints

Category: Medicine and Health.
Fields of Study: Algebra, Geometry.
Summary: Joints allow bones to move—a movement that is modeled and analyzed using mathematics.

A joint (where bones join) generally allows motion of those bones relative to each other. The motion, typically, is a rotation about the joint. Such rotations underlie almost all the movements humans perform in everyday life. Mathematics plays a crucial role in understanding the causes and consequences of the joint rotations, singly or in combination, and also in estimating the forces to which the joints are subjected.

Simple Joint Movement

Suppose, for simplicity, that rotation is confined to the elbow joint. Then the forearm would move in a plane, and the position of the hand would be represented by extrinsic (x,y) coordinates that involve trigonometric—sine and cosine—functions of the elbow angle. When many joints participate, such as the shoulder, elbow, and wrist, the description of a hand movement, like reaching for a cup, involves combinations of trigonometric functions of the joint angles. The relationship between changes in the joint angles and the resulting changes in the extrinsic coordinates is expressed in the form of a matrix (called the "Jacobian matrix"), consisting of rows and columns of trigonometric functions. The methods of matrix algebra can be used for understanding the consequences of a sequence of changes in joint angles.

The inverse problem of finding the joint angles when the extrinsic coordinates are given can have an infinite number of solutions, called "kinematic redundancy." For example, there are many ways of configuring an arm so as to get a finger to touch one's nose. Why a person chooses a certain configuration is not known, though various hypotheses have been proposed. This is a crucial issue also in robotics, where "joint" angles have to be computed in order to reach a prescribed position in space. Various mathematical methods have been utilized for picking an "optimal" solution to this problem.

Three-Dimensional Joint Movement

The importance of mathematics in understanding and describing joint function is further emphasized when considering motions in three-dimensional space because certain phenomena arise that are far from intuitive. As an example, assume the shoulder to be a ball-and-socket joint and imagine the following two sequences of 90-degree rotations about the right shoulder, starting each time with the arm horizontal and stretched out to point to the right:

- Rotation about the vertical axis (bringing the arm to point to the front), followed by rotation about the left-right axis (raising the arm up, above the head)
- The opposite sequence of rotations, first about the left-right axis (twisting the arm about its long axis), followed by rotation about the vertical axis (bringing the arm pointing to the front)

The two sequences lead to different configurations. The dependence of the final outcome on the sequence of the rotations is expressed by mathematicians as the "noncommutativity" of rotations in three-dimensional space. It means that rotations can not be described simply by three numbers, unless the sequence is also specified. Certain ways of specifying the sequence have been standardized, such as a rotation being described by three "Euler angles" (yaw, pitch, and roll). There are also several other mathematical techniques, involving matrices, for dealing with rotations in 3-dimensional space, as matrices too have the property of noncommutativity ($A \times B \neq B \times A$). Another technique, which uses four rather than three numbers to represent a rotation, is the method of "quaternions." These abstract entities were proposed originally as extensions of complex numbers. Incidentally, the designers of computer visualizations, like video games, utilize quaternions for programming the rotational motions of the objects.

Forces

Motions about joints result from muscle and external forces. It is the moments of these forces that matter for rotation. In multijoint movements, a muscle moment about one joint can cause motions about several joints; specifically, even a fully relaxed joint would flop when there is motion about nearby joints. This phenomenon is described by rather complicated differential equations, which the neural control system takes into account in its planning. But the force with which the bones at a joint push against each other cannot be determined simply from the moments of forces. This force (called "joint loading") depends upon both external and muscle forces, and is typically many times greater than any external forces. The wear and tear of the joint—natural or artificial—depends upon the loading. Also, joints being nearly frictionless, slippage occurs if the load has a substantial component parallel to the surface of contact. Noninvasive techniques

A National Institute of Biomedical Imaging and Bioengineering (NIBIB) medical imaging study.

for estimating the joint loading force are highly computational. Given the external forces and observed motions, one determines the needed muscle torques at each joint, and then, knowing the anatomical layout of the muscles and their strengths, one estimates the distribution of forces among the muscles. With all other forces thus known or estimated, one can derive the joint loading.

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Ziaul Hasan

See Also: Matrices; Robots; Trigonometry; Video Games.

K

Kicking a Field Goal

Category: Games, Sport, and Recreation.
Fields of Study: Algebra; Data Analysis and Probability; Geometry.
Summary: Probability, statistics, and physics govern a football coach's decision to attempt a field goal.

Football is one of the most popular sports in the United States, and successfully kicking a field goal is one of a few ways that football teams can score points. Every kick that is made is unique, and a solid knowledge of mathematics is necessary to tailor the kick to be successful. The path of the ball is parabolic in nature and may differ based on environmental conditions. Geometry plays a major role in the strategy used when kicking a field goal. Coaching staffs gather large amounts of data not only from their own team but from others as well to understand how these conditions alter kicking success. The data are used to create information about the probability of success to then determine if and when a field goal will be attempted. Mathematicians have also studied and modeled these problems using a variety of methods such as partial derivatives and concepts from algebraic topology. Many models focus on such critical variables as goal post placement with respect to inbounds lines, goal post height and distance between uprights, the kicker's distance from the goal post base, wind factors, and kicking angles. These models can also be applied to other sports that have goals, like hockey and soccer.

Process

American football has been played for over 100 years and has evolved over the years into the game played today. Scoring can occur in a variety of ways including the field goal, which gives the kicking team three points. A field goal is scored when the ball is kicked from the ground and through a goal. The goal is made up of a horizontal crossbar that is 10 feet off of the ground and two upright side posts that are 18.5 feet apart in the National Football League and 23 feet 4 inches apart for college and high school games. The ball must pass between the two upright posts while going over the crossbar, which is located 10 yards behind the end zone line. The ball is kicked from a position on the field that is either where the last down was marked (if the ball is in the center of the field) or on hash marks that are lined up with the two side posts of the goal.

Thus, the path the ball must travel to score a goal is not a set path, but rather is based on the starting position of the ball; as long as the ball clears the crossbar between the two side posts, it is successful. If the kick is not successful, the other team gets possession of the ball at the location of the ball prior to the kick. Because of this potential exchange of possession of the football,

Sergi Güibas attempts a field goal during a Seattle Mariners game. Coaches study their own team and others, using geometry to understand how different conditions alter kicking success.

the likelihood that a kick will be successful must be well understood.

Flight Path Factors

The path that the ball travels is very close to a parabola, with differences potentially being created by environmental conditions such as wind, rain, or altitude. Wind can act to either increase or decrease the distance of flight, depending on the wind direction. Rain acts to decrease both the ball's distance of flight and height. Higher altitudes mean there is less air resistance, so the ball will travel both farther and higher when games are played in stadiums high above sea level. Other factors that influence the ball path include aerodynamic forces related to ball spin. These can lead the ball to move to one side or the other and make a ball that has the appropriate distance and height fail to score. These forces can also shorten a ball distance if the ball is spinning backward.

Coaches gather large sets of data about successful and unsuccessful kicks in different venues to better understand how the environmental conditions might alter the kick path. A team with an outdoor stadium often has an advantage over a team whose home field is in an enclosed dome. Along with venue information, coaches study information about their kickers' practice kicks and the opponent's defense to determine the probability of a kick leading to a score. Collectively, this information helps coaches decide whether they want to take the risk of kicking a field goal or whether punting the ball down the field is the wiser decision.

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Michele LeBlanc

See Also: Hockey; Mathematical Modeling; Soccer.

King, Ada (Countess of Lovelace)

See *Lovelace, Ada*

Knitting

See *Crochet and Knitting*

Knots

Category: Games, Sport, and Recreation.
Fields of Study: Geometry; Representations.
Summary: Mathematical knots are useful in physics and biochemistry.

Since ancient times, knots have been used in sailing, building, textiles ("knit" comes from "knot"), climbing, and in recreation, as well as serving as symbols for spiritual or religious concepts like eternity or wisdom. Topology generalizes the idea of a knot to an embedded circle in 3-dimensional Euclidean space. In knot theory, a knot is a tangled-up loop, like a piece of string with the ends fused together. The simplest is the unknot, simply an untangled loop like a rubber band. Two knots are the same if one can be manipulated (transformed) into the other without breaking the loop or passing the string through itself. In 1926, Kurt Reidemeister demonstrated that all such transformations were made up of a sequence of just three basic moves called Reidemeister moves. Deciding whether two knots are the same via a sequence of such moves is a member of a host of problems involving changing one object into another without breaking or tearing, which have long stumped topologists. Topologists find it difficult to assure themselves that failing to transform one knot into another truly reflects impossibility, or rather just their own failure. In modern times, mathematical knots are useful in physics and biochemistry.

Invariants and Links

To wrestle with this problem, topologists have created an assortment of invariants, mathematical entities that can be unambiguously computed for each knot. If a particular invariant has different values on two knots, then those knots are different. Unfortunately, different knots can have the same invariants. In 1928, James Waddell Alexander II created a method for associating a polynomial to a knot, now called its Alexander polynomial. In 1983, Vaughan Jones, studying a simplified model of phase transitions, such as freezing, discovered a second invariant, the "Jones polynomial." Another mathematician, Edward Witten, soon noticed that the same polynomial could be computed from an invariant on particular three-dimensional spheres, providing insight into another difficult classification problem. Witten and Jones shared part of the Field's Medal in 1990 for these discoveries. Victor Vassiliev has since created a host of new invariants. The Vassiliev invariants are infinite in number, and it is conjectured that any two different knots will differ in at least one such invariant.

Not all invariants are polynomials. Henri Poincaré created a topological invariant called the "fundamental group." Applied to knots, it is called the "knot group" and is actually computed on the complement of the knot, that is, the abstract concept of all space with the knot removed. Poincaré's invariant was the seed of an area that grew into a central focus of twentieth-century mathematics called "homological algebra."

Knots, and their close cousins, links, have proven useful in a branch of physics called "topological quantum field theory." For this application, physicists use particular guidelines to trace knots in two dimensions.

The knot diagrams then portray scenarios in which particles are created, interact, and are finally annihilated. By appropriately labeling pieces of knots, mathematicians can realize the Jones and other invariants via important modern mathematical constructs, including the Yang–Baxter equations and quantum groups. Mikhail Khovanov has created a new type of invariant on links, keeping this topic at the very forefront of contemporary mathematics.

Applications in Biochemistry

The application of knot theory to DNA molecules has helped to elucidate their biochemistry. The DNA molecule of a bacterium closes into a circle, which bends and twists itself into a knot. This knotted structure can block DNA replication. Using electron microscopy or gel electrophoresis, the biologist can determine an individual molecule's crossing and unknotting numbers, two numbers that classify knots. Enzymes called "topoisomerases" release the knots as a preliminary step to DNA replication. By carefully examining the knots that arise, molecular biologists have determined that there are two different topoisomerase molecules. Topoisomerase I releases the knot by cutting both strands of the molecule, and Topoisomerase II nicks just one strand and twists the cut strand around the other.

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Michael Klucznik

See Also: Elementary Particles; Game Theory; Genetics; Surfaces.

L

Landscape Design

Category: Architecture and Engineering.
Fields of Study: Algebra; Measurement; Problem Solving.
Summary: Landscape design is an application of geometry, shaping an outdoor environment to something pleasing.

Landscape design is the combination of gardening and architecture for making outdoors environments more aesthetically pleasing, ergonomic, and useful.

It is a synthetic occupation, requiring the knowledge and skills of horticulturists, engineers, architects, and visual artists.

Mathematical calculations underlie many aspects of landscape design, such as how many plants are needed to fill a bed or landscape or how to build a landscaped terrace that will resist erosion. Landscape architects often include design elements based on symmetry and other geometric features of areas, surfaces, and three-dimensional elements. More advanced mathematical forms, such as fractals or labyrinths, are incorporated in some landscapes, like crop circles. Peter Schaar was an applied mathematician for many years before turning to a career as a landscape designer. He noted that the notion of an elegant solution is common to both mathematics and garden design.

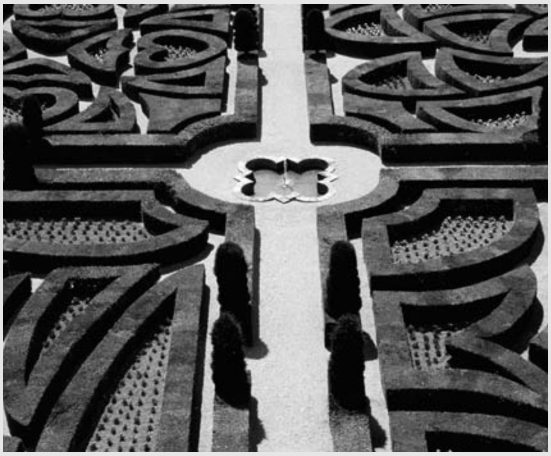
Design Elements and Principles

Landscape design, like other forms of design and decorating, uses design principles and elements that are mathematical in their nature. The Western traditions of landscape design typically use lists of elements, including the following:

- Line
- Shape
- Size
- Texture
- Color

Every element is expressed through natural or architectural media, including plants, stones, and ground shapes. Straight or curved lines and shapes are created using hedges, paths, flower borders, and shapes of bushes and trees. Sizes of landscape elements, including stones, plants, and built structures, can match or contrast. Textures and color can be natural, such as foliage, water, grass, and stone, or modified by people, such as cut bushes, polished stones, and painted structures.

Likewise, the artistic principles, such as repetition, balance, and focal points, are achieved with the combination of human-made and natural elements. For example, traditional landscaping focal points include sculptures, fountains, and flower beds.



Labyrinths and Mazes

A labyrinth is an elaborate landscape structure consisting of live or stone hedges or mosaic ground patterns, with a winding single path leading to its center. Unlike mazes, which have many possible paths and serve as spatial puzzles, labyrinths are easy to navigate. Labyrinths are used as pleasing places for conversation or meditation. Mathematically similar patterns for labyrinths appear in archaeological finds on all continents. Solutions of mazes and constructions of labyrinths have to do with topology, graph theory, and knot theory in mathematics.

Sacred Traditions and the Development of Mathematics

Building, gardening, and designing landscapes were connected to spiritual practices by many cultures around the world. The resulting complexity of habitats often elevated mathematical and scientific knowledge, as well as the arts within the cultures practicing these traditions.

For example, feng shui is the Chinese design tradition connected with the development of astronomy and precise measurement instruments, such as magnetic compasses and astrolabes. Mathematical ideas involved in feng shui symbols include binary numbers, powers, and combinatorics.

Some mid-African cultures use fractal structures in village design, where the shape of the village is repeated in shapes of house clusters, then houses, then rooms within houses. The shape is connected to the beliefs of the people and reflected in the lore while at the same time being practical for the needs of the village.

Ancient Egyptians used the concept of *gnomon*, which is a specially constructed geometric shape corresponding to a regular polygon, in their area and architecture calculations. When a gnomon is added, the ratio of polygon sides is maintained. Osiris was associated with this idea of the constant ratio, in the myth as the God of Sun, growth, and constant change, and was often drawn on a square throne expanded with the L-shaped gnomon. These geometric traditions were inherited by the Greeks, formalized as Euclid's geometry, and entered the Western knowledge base.

Budgets and Rates

Landscaping expenses include the price of material and labor for construction and maintenance. It is estimated that in the United States, a house with its landscape design rated "excellent" by experts can sell for 5% to 10% more than the same house with its design rated "good." Therefore, it may make financial sense to spend money landscaping the property. These calculations are performed by developers and real estate agents when deciding landscaping budgets.

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Maria Droujkova

See Also: African Mathematics; Chinese Mathematics; City Planning; Egyptian Mathematics; Geometry in Society; Green Design; Knots; Mathematical Puzzles; Mathematics, Elegant; Perimeter and Circumference; Sacred Geometry.

LD50/Median Lethal Dose

Category: Medicine and Health.
Fields of Study: Algebra; Data Analysis and Probability.
Summary: The median lethal dose of a compound is determined through experiment and statistical estimation.

Toxicity often needs to be compared across various chemical compounds and other noxes. The detailed and complex dose-response curve describes the relationship between the dose of a compound and its harmful effect. Frequently, a simple summary in the form of a single number is needed for practical purposes. Median lethal dose, or "LD50" is most popular in this context. It is defined as the dose at which 50% of exposed individuals die.

In order to be meaningful, such a definition implicitly assumes certain features of the dose-response relationship, namely its monotonicity, the fact that mortality increases with dosage. Although the concept is defined for a theoretical dose-response curve, its practical application is strongly related to statistical estimation of the dose-response curve model based on data obtained from an experiment with many animal or other nonhuman organisms randomly assigned to various doses.

Toxicological Testing

In toxicology and related disciplines, such as food safety and environmental risk assessment, one often needs to quantify how toxic or dangerous a substance is. A quantification of the harmful effect is needed for many practical comparisons; for instance, to compare the toxicity of different substances or to compare them with a standard. Although there are many possible aspects of how dangerous a compound is, survival of exposed individuals is frequently of interest. The survival is assessed experimentally in the "quantal response trial."

It is based on a set of animal or other nonhuman organisms, whose randomly selected groups are exposed to different doses of the tested compound. The outcomes are summarized as the percentages or proportions of those that survived in each dose group. Mortality would be just the complement of the proportion of survivors, expressed as: Mortality = (number of individuals dead after exposure) ÷ (total number of exposed individuals).

Based on common sense, one would expect that the mortality would increase with the dose of a toxic compound. Most typically this is indeed the case and the mortality obtained from an experiment with a large total number of exposed is monotonic, meaning it increases with the dose.

Dose-Response Curve

When mortality is taken as a function of dose, one can plot the so-called dose-response curve. Dose-response curve has lower asymptote at 0 since no exposure-related death can occur when no exposure is applied. Similarly, it has upper asymptote at 1, since exposure-related death will always occur with a large enough dose. The asymptotes are shown as horizontal dashed lines.

Note that often one needs to go over several orders of magnitude of doses in order to observe transition from zero effect to the full effect, so the dose-response is then plotted as the mortality versus logarithm of the dose. Since the logarithm is a one-to-one function, nothing is lost by the transformation, and the plot is more readable.

LD50

Because the dose-response curve is a rather complex quantity, many possible features might be compared across different compounds. It might be cumbersome in practice to compare curves, however. A simple summary is often all what is needed. Median lethal dose, or LD50, is the most popular characteristic. It is defined as the dose at which 50% of exposed individuals die. When a dose-response curve is available, an LD50 is constructed by drawing a horizontal line at 0.5, finding its intersection with the dose-response curve, drawing vertical line at the intersection, and reading off the value where it crosses the horizontal axis.

Statistical Estimation

In practice, one does not have the dose-response curve at hand. It needs to be estimated from experimental data by statistical means. In fact, the mortalities obtained from two experiments with the same doses would be very likely somewhat different, as a result of random errors. For example, different randomly selected experimental animals would react differently to a given dose.

Nevertheless, when the size of the experiment increases, increasing both the number of animals in every dose group and increasing the number of different dose groups, random errors would tend to decrease in line with the law of large numbers. In fact, for a very large experiment, the mortality estimates get close to the probabilities of survival. Since not all of the infinite possible doses can be explored in a real experiment, a model relating the survival probability to the dose is assumed in order to be able to interpolate between the doses actually used in the experiment. An interpolation is typically needed when calculating LD50. Parameters of the model are then estimated by various statistical means. Very often, logistic regression is used to this end.

Other Uses of LD50

While the definition of LD50 is directly related to lethality, the mathematical concepts used in LD50 testing and modeling can be applied to many other less-dramatic outcomes. In general, these models are useful when the relationship being explored involves a binary response variable, like yes/no or pass/fail, predicted by a quantitative explanatory variable, as long as the relationship is bounded and monotonically increasing in the same manner as before. For example, rather than finding the dose that induces mortality, researchers may wish to model what dose of a medicine will cause 50% of exposed individuals to show a certain, nonlethal symptom.

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Marek Brabec

See Also: Curves; Functions; Limits and Continuity; Probability.

Learning Exceptionalities

Category: School and Society.
Fields of Study: All.
Summary: There are a variety of ways in which students can perform especially well or poorly in particular areas of mathematics and ways for schools to address their needs.

Learning exceptionalities in mathematics include both difficulties in learning mathematical concepts and mathematical giftedness. In both cases, exceptionalities take many forms and may manifest themselves at nearly any stage of life. In some cases, students may even excel in one area while displaying a deficit in another. The neurobiology of learning mathematics is not yet fully understood. Research in these areas is ongoing, often using sophisticated medical imaging to identify and map mathematical associations and processes, such as calculations, visualization of polyhedrons, proving theorems, or pondering number theory problems. There are also some difficulties in devising tests to reliably identify specific types of exceptionalities, and many people, especially those with difficulties, may not be diagnosed until very late in their academic careers.

Educational institutions often struggle with appropriate ways to serve students with exceptionalities so that all students may reach their maximum potential. These range from specific classroom instruction techniques all the way up through broader policies or legislation that addresses the needs of these subgroups of students. There are currently many formal systems in place by which students are assessed and accommodation plans are developed, most of which require periodic reassessment and revision. Plans for students with disabilities typically fall under Section 504 of the Americans with Disabilities Act and are commonly referred to as "504 Plans." In recent years, the term "dyscalculia" has emerged as a broad term to encompass the set of mathematics learning disabilities.

Mathematical Disabilities

Sadly, unlike reading disabilities, many mathematical disabilities go undiagnosed, primarily because of social acceptance of the idea that certain people either have or have not mathematical abilities. For many students, the perception that they are "not good at math" provides a reason for them not to strive for success in the mathematics classroom. Researchers and practitioners alike indicate that if teachers truly want all students to succeed, a new perspective must be adopted: "all students can do math, should do math, and will do math."

With this new perspective in mind, the focus shifts to identifying mathematical disabilities. According to David Geary, researchers in the field of mathematical disabilities have attempted to identify disabilities by studying normal mathematical development theories and using those theories to study children who demonstrate difficulties in mathematics despite having average or better IQs. Since most of the research has focused on students in the elementary grades, it becomes even more important for teachers and parents to be alert for mathematical difficulties early in school.

Currently, children with mathematical disabilities are defined as children with at least average IQ scores who also score at or below the 10th percentile on mathematics achievement exams. Research indicates that 6% to 7% of elementary school children demonstrate persistent mathematical difficulties in the area of number and arithmetic. It is important to note that current research studies indicate these difficulties persist regardless of IQ, motivation, and other factors that influence learning. What makes this area of research perplexing is that these children may have very specific deficits that make only certain aspects of mathematics difficult. For example, a child may have difficulty with counting but show a strong ability in geometry. Because standardized tests, which are frequently used for making decisions about whether a child should be recommended for special services, assess a wide variety of mathematical skills, a child's particular mathematical disability may not be immediately identified. Adding to the difficulty is that children who score at approximately the same level on standardized tests may have vastly different mathematical deficits. Unfortunately, current methods of assessing mathematics knowledge are not sufficient for identifying mathematical disabilities, as assessments that focus on specific number and arithmetic skills are needed.

Learning Basic Numbers

Several mathematical disabilities have been identified. First, children may have a disability in learning basic number skills. Geary states that the "learning of basic number skills is much more complicated than many adults would assume." In order to learn basic number skills, children must learn the English number words (known as "word tags") and the Arabic numbers in the correct sequence, and learn to translate between the two. Children must then learn the quantities associated with the number words and number symbols, as well as develop an understanding that numbers can be decomposed into smaller numbers or combined into larger numbers.

The learning of place value in the base-10 system is a key component of developing number sense, and children with this particular type of mathematical disability may not be able to comprehend that 12 is actually 10 + 2, leading to later difficulty with basic arithmetic skills.

Counting Skills

A second mathematical disability is in the area of counting. While children do not typically have difficulty learning the basic counting sequence, they may have difficulty learning the basic concepts that enable them to count objects effectively. Geary identifies these concepts as the following:

- *One–one correspondence*: When counting, one does not count and tag the same item twice
- *Stable order*: The order of the word tags remains constant across counted sets
- *Cardinality*: The value of the final word tag represents the quantity of the items in the counted set
- *Abstraction*: The concept that objects of any kind can be collected together and counted
- *Order-irrelevance*: Items within a set can be counted in any sequence

Geary notes that having children count does not provide an indication of a child's understanding of the counting rules, as children may learn the sequence of counting without developing the understanding of applying the word tags to objects. An additional complexity to this mathematical disability is that children may have difficulty remembering information during the act of counting; therefore, they may understand the counting rules but may forget numerical information during the counting process.

Arithmetic Skills

A third area of mathematical disability is that of arithmetic skills. Children with arithmetic disabilities typically have difficulty remembering as many basic arithmetic facts as other children and may not recall basic facts as quickly. This memory difficulty may be the result of children having trouble storing basic facts in long-term memory, or it may be the result of other arithmetic facts inhibiting the child's ability to recall. For example, a child may see a problem like 4 + 5, and the child may correctly remember 9, but also may remember 20 (or 4 × 5), causing the child to take longer to recall the correct fact. Children with arithmetic difficulties also may not use highly developed problem-solving procedures to solve arithmetic problems but may rely on procedures typically used by younger children.

In general, children with mathematical disabilities use less mature strategies in their approach to mathematics, resulting in more errors and delayed acquisition of advanced mathematical thinking. Finally, children may verbally show an excellent grasp of mathematical concepts but have difficulty translating that understanding into paper and pencil assessments. These children struggle with paying attention to operations and sequencing steps in complex operations. Interestingly, many students who show difficulty with arithmetic skills in the elementary grades become "good math students" in the higher grades where conceptual understanding is emphasized more heavily.

The Language of Mathematics

Fourth, some children may have difficulty with the language of mathematics. These children easily confuse mathematics terminology and struggle with verbally communicating their mathematical thinking. This deficit can inhibit students from making progress in advanced mathematics, as they may not have the

When learning exceptionalities are mentioned, most people automatically think of learning disabilities. However, there is another group of students that has exceptional needs: gifted mathematics students.

verbal skills necessary to track the steps needed for complex calculations.

Visual-Spatial Skills

Finally, children may be disabled in their visual-spatial skills. These students frequently have difficulty with complex problems, as they may not be able to maintain a logical, coherent sequence of steps on a piece of paper. Additionally, these students have difficulty with pictorial representations, making mathematical topics such as graphing and trigonometry especially challenging.

The National Council of Supervisors of Mathematics (NCSM) offers several recommendations for teachers of students with mathematical disabilities in a 2008 position paper. First and foremost, teachers must reconsider their perceptions of what students with mathematical disabilities can and cannot do and maintain high expectations for all students. Teachers need to be better educated about mathematical disabilities, particularly about the diagnostic language that is used to describe the needs of the mathematically disabled student. If teachers develop a conceptual framework for what students with mathematical disabilities need, they can incorporate effective interventions and accommodations in the classroom. NCSM also suggests that mathematics teachers should establish collaborative relationships with special education teachers. Mathematics teachers should focus on using teaching strategies that enable students to move from the concrete to the abstract and that allow students to demonstrate understanding through a variety of methods. Mathematics education activities should be meaningful and connected to a number of mathematical topics, thereby enabling struggling students to make connections between mathematical concepts.

Mathematical Giftedness

When learning exceptionalities are mentioned, most people automatically think of learning disabilities. However, there is another group of students that has exceptional needs: gifted mathematics students. These students are typically described as having "natural mathematics ability" and frequently are left to their own devices as teachers spend the majority of their time and attention on struggling students. While the reality of the classroom is that teachers focus more on students with difficulties, the needs of the gifted students are just as important.

M. Katherine Gavin points out that three main issues exist regarding gifted mathematics students. First, just as with students with mathematical disabilities, gifted mathematics students demonstrate a wide variety of aptitude, and abilities. Some students learn concepts quickly, which makes mathematics easier to learn and apply. Other students show great persistence in problem-solving, while still others demonstrate an ability to apply mathematical concepts in new ways.

Second, elementary teachers typically do not have specialized training in mathematics and may not know how to address the gifted student's needs in the elementary grades. The response of many elementary teachers is to keep gifted mathematics students occupied with puzzles or advanced curricular materials, which typically do not advance the gifted student's mathematical ability.

Third, current grade-level curricula are lacking in materials that are challenging and substantial enough for the gifted mathematics student. Therefore, gifted mathematics students may be given materials that do not allow for the development of critical thinking skills and the conceptual understanding of complex mathematics concepts.

For classroom teachers, it can be difficult to meet the needs of gifted mathematics students. Dana Johnson offers the following suggestions:

- Pre-assess students to determine which students already have mastered the material. For students who demonstrate mastery, provide instructional materials with advanced content and a problem-solving focus.
- Utilize a variety of assessment techniques, providing students with opportunities to show differences in understanding, creativity, and accomplishment.
- Choose textbooks with a variety of enriched opportunities. Use multiple resources to meet the needs of gifted mathematics students.
- Be flexible in expectations about pacing. A student may be gifted in one area of mathematics but struggle in another.
- Use hands-on, discovery-based teaching strategies as well as higher level questions.
- Provide opportunities for students to participate in mathematics contests, such as Mathematical Olympiads and Math Counts.

According to Gavin, the implementation of such strategies in the classroom will allow gifted mathematics students to develop their cognitive skills while maintaining the joy of doing mathematics.

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Calli A. Holaway

See Also: Curriculum, K–12; Educational Testing; Succeeding in Mathematics.

Learning Models and Trajectories

Category: School and Society.
Fields of Study: All.
Summary: Various models of learning mathematics suggest in turn different approaches to teaching.

In October 2010, educator Jill Biden, the wife of Vice President Joseph Biden, chaired the first White House summit on community colleges. She restated an idea that is increasingly in the forefront of both political and educational discussions: "The nations that out-educate us today will out-compete us tomorrow." This idea is considered to be particularly true in the fields of science, technology, engineering, and mathematics (STEM). The philosophy and methods of mathematics education are driven not only by perceived societal needs and events, such as the Industrial Revolution, the Cold War, the civil rights movement, and the coming of the digital age, but by research and theories on the way people learn. Biologists, psychologists, mathematicians, and others have all contributed to the current body of knowledge on how both children and adults learn mathematics, and research in these areas is active and ongoing. In turn, organizations like the Mathematical Association of America and the National Council of Teachers of Mathematics synthesize this knowledge and make recommendations that shape curriculum at all levels.

The Piaget Model

Epistemologist Jean Piaget reportedly believed that what distinguishes human beings from other animals is the ability to reason with abstract symbols. The model of cognitive development that bears his name includes four hierarchical stages that mathematics educators have analyzed with regard to the development of mathematical concepts like spatial skills or abstract reasoning. In Piaget's model, infants in the first stage can link numbers with objects and may have some understanding of counting. In the second stage, toddlers and young children can recognize the concept of closeness and other topological ideas, as demonstrated in experiments and puzzle-solving activities. However, perceptions at this level are often restricted to one aspect or variable at a time. For example, Piaget poured liquid from one container into a similar container and then into a wider container while children watched. The children failed to recognize that the volume of liquid was the same because the height of the liquid in the new container was lower. In the third stage, elementary school children and early adolescents develop logical operations skills like classification or seriation, the ability to order objects based on a variable like height, and they can analyze many variables at the same time. However, experience and training combine with the cognitive stage to determine the level of advancement, such as how successfully an early adolescent can analyze mental images of rotated objects. Hands-on activities can help students at this stage to connect abstract concepts and symbols with concrete objects. In the fourth stage, adolescents and adults can fully develop abstract arguments using symbolic notation. They can learn to analyze and evaluate logical arguments and to apply deductive reasoning.

Piaget's theories have had broad impact on mathematics education, but not without criticism. His ideas are used to develop puzzles that are designed for toddlers. Elementary and middle grades students use mathematical manipulatives, like blocks, Pythagorean theorem puzzles, algebra tiles, or tangrams. Abstract courses, like algebra, are usually taught to adolescents because Piaget's model suggests that abstract reasoning is developed then. Critics believe Piaget's model may under- or overestimate the abilities of children and adolescents. For instance, children ages 3–5 sometimes notice incorrect counting sequences and can develop ingenious strategies to solve problems related to some higher stage concepts if they have age-appropriate task design and instructions. Middle grade children may not be ready in the way that Piaget asserted, though Piaget recognized that the level and time in a developmental stage varies with each child. As a result, different representations might be more meaningful to some children than to others.

The Van Hiele Model

Another influential model that has especially impacted geometry curricula is the van Hiele model, developed by Dutch educators Dina van Hiele-Geldof and Pierre van Hiele. The five levels of geometric thinking are visualization, analysis, informal deduction, deduction, and rigor. The levels are sometimes labeled as 0–4 and other times as 1–5. Children master one model level before progressing to the next. This occurs with repeated exposure and experience. For instance, students do not usually satisfy the deductive level after only one proof-oriented course. Unlike Piaget's model, the van Hiele model is not age dependent. During the 1960s and 1970s, Russian (Soviet) researchers may have been the first educators outside Denmark to extensively experiment with the theory. They found that grade 8 students in a curriculum based on the van Hiele model demonstrated geometric sophistication on par with students in grades 11 and 12 under the old curriculum. This research and other worldwide studies of the van Hiele model have led mathematics educators in many countries to develop activities, textbooks, and curricular innovations. However, some have critiqued the rigidity and linearity of the van Hiele structure and asserted that students weave between levels or reason at multiple levels at the same time.

Neurobiology

Right- versus left-brain learning is often discussed in mathematics education. This differentiation appears to have some basis in biology, and many people do exhibit preferences for one style over the other when tested. Critics point to people called "middle-brain" thinkers, who flexibly switch between styles depending on the situation. Some argue that brains, especially those in children, are much more malleable than previously believed, so that anyone can be a flexible learner with training or a variety of methods of engagement. The neurobiology underlying mathematics learning is not yet well understood. In the past, researchers relied largely on verbal descriptions of how people solved problems.

Visualization methods like magnetic resonance imaging allow researchers to make connections between brain components and specific processes, some of which have been found to activate parts of both the left and the right sides of the brain, often in surprising ways. For example, scans of infant brains showed that they seem to detect changes in the number of objects in an array, suggesting they have number sense. Overall, the right brain is commonly associated with holistic, subjective, intuitive learning as well as with artistic skills. Mathematics is often considered to be a left-brain activity, since the left brain is associated with logically and objectively analyzing parts or sequences to understand the whole. People have also researched teaching styles according to right- and left-brain theory. Teachers classified as left-brained more often used highly outlined lectures and discussions in their classes. They also assigned more independent problem solving or research than teachers classified as right-brained, who were more likely to use less-structured, hands-on activities or group projects that included manipulatives, art, visuals, role playing, and music. In many educational settings, teachers are encouraged to consciously consider both the ways they teach and the ways in which their students may learn in order to design a breadth of teaching and assessment methods.

Constructivism

Educational reformers in countries such as the United States, the United Kingdom, Canada, Germany, and Taiwan began strongly promoting constructivism in the late twentieth century. The constructivist framework rejects objective reality; learning is experiential, and the instructor is more of a facilitator than a teacher. The foundations can be traced to Socrates and the term "constructivism" was coined by Giambattista Vico in the eighteenth century, though many consider Piaget to be the first educational constructivist. The United Kingdom mandated constructivism in the 1980s and the National Council of Teachers of Mathematics 1989 Curriculum and Evaluation Standards for School Mathematics endorsed constructivism for U.S. schools. There are many vocal critics of constructivism, sometimes known as the "back to basics" movement. Opponents argue, among other things, that constructivism fails to systematically instill fundamental skills required for true mathematics mastery. Also, constructivist approaches can be very time consuming and difficult to assess fairly, especially in an environment of increasingly common standardized tests. Many constructivists assert that mathematics is a cognitive process shaped by sociocultural context, as well as a sociocultural phenomenon created by the community of active learners. Mathematics learning is therefore seen as a function of prior knowledge; perceptions of what others know; methods of knowledge sharing; norms of participation in the classroom or community of learners; what it means to "do mathematics"; and methods by which mathematical validity is determined.

Learning Trajectories

If all learners are unique, then schools must take into account the many ways individuals might learn mathematics. Educators have to consider what it means to know and to do mathematics, both in school and beyond, before they can develop curriculum and select teaching strategies. In some cases there seems to be a natural progression, similar to the way children learn to crawl, then walk, then run. A hypothetical learning trajectory is a hypothesized typical path that students might follow when learning a set of interrelated concepts and skills, including ways in which learning will be facilitated by the instructor. Research suggests that learning trajectories can be effective for early-grades mathematical concepts, such as counting and arithmetic. Additional research is needed on mathematics topics from later in the standard school curriculum, like patterns, as well as for more sophisticated ideas addressed in high school and beyond. Learning trajectories are also empirically linked to teacher development. For example, training in learning trajectories increased knowledge in teachers as well as motivation and achievement in students. Some researchers assert that students should be explicitly included in the formation of learning trajectories to better anticipate individual responses and divergences from the typical path.

A hypothetical learning trajectory begins with the students' current knowledge and is targeted toward a specific "big idea" or goal, such as the idea that geometric shapes can be analyzed, described, transformed, composed, and decomposed into other shapes. The learning trajectory also includes a sequence of tasks designed to guide students in learning concepts and building upon their previous learning, taking into account that some students may think about ideas in different ways or learn them in a different order. It may also include remediation for students who begin with insufficient knowledge or extensions for students who reach the goals quickly. Consider, for example, counting. Young children first learn the words and sounds associated with numbers. Then they put those words in order, though not always completely, before they begin to associate words with objects on a one-to-one basis. Eventually they can count objects, determine why counting is important and what "how many" means, and finally acquire a true sense of cardinality. A teacher would select tasks, teaching methods, and assessments to address each stage in turn, while working with students to determine whether they are learning and adjusting accordingly. At most grade levels, students are simultaneously involved in multiple learning trajectories.

Scaffolding

Scaffolding is one teaching technique closely associated with learning trajectories. The term "scaffolding" is a metaphor for the teacher's supporting role with respect to the student. Parents seem naturally to use scaffolding with babies and young children. Research suggests that scaffolding provides individualized instruction that engages and motivates students while also improving learning and retention. However, the method can take a great deal of time to implement in the classroom and relies on trained teachers who have access to appropriate educational materials. Further, teachers must be willing to relinquish some degree of control in order to promote students' independence and those students must be carefully and differentially assessed at the beginning and throughout the process. In an environment where standardized testing is the norm and teachers are assessed based on student performance, or for teachers used to a more traditional approach to classroom management, the issue of control can be a difficult one to manage. Scaffolding is widely used in business and sports applications. In many settings, peer working groups serve as teachers. From an employer's point of view, scaffolding seems well suited to promote the lifelong independent learning skills that are needed in the rapidly changing twenty-first-century job market.

In scaffolding, a student learns independently as much as possible. The teacher structures tasks and provides help with concepts or techniques that are just beyond a student's current capability. Scaffolding usually involves several steps. First the student and teacher agree on the goal. Then the student focuses on the concepts and tasks as a whole, not as a sequence of discrete steps. The teacher is available to provide quick help and feedback. Rapid response is intended to minimize frustration and wasted time while encouraging the student's self-efficacy. The teacher helps only with immediate needs in areas where it is truly needed—the teacher does not repeat knowledge the student has already mastered and over time intervenes less and less. The teacher may also give an explicit example as an "expert model." All of this takes into account different student approaches and the student's current state of knowledge.

Computer software can also include scaffolding to facilitate online or independent learning, though these scaffolds often provide static, versus dynamic, interaction with a teacher. Different sorts of scaffolds that have been explored for software include conceptual scaffolds, which help students organize ideas and connect them to related information; strategic scaffolds, which help students ask more specific questions about concepts and processes; and procedural scaffolds, which clarify tasks. These scaffolds might include suggested readings, templates for presentations or note-taking, journals, and interactive essays.

An Australian dance conference called Moving On 2000 included an interesting application of scaffolding. At an initial workshop, participants created the beginnings of a dance piece. The dancers met again later to further the work. Not every person remembered each step and sequence, so other participants assessed what they did know (prior knowledge) and then modeled the forgotten components as the learners followed along. This was done without explicit direction from anyone. Eventually the students no longer needed the teachers or "experts"; their goal of knowing the whole dance had been met. Then the dance was extended ever further by participants, who later modeled and taught the new moves to others at successive sessions. In this context, people were both teachers and learners in turn.

Conclusion

Many models and theories continue to shape mathematics education. Elements of behaviorism, cognitivism, and humanism appear in some educational approaches. One often-promoted theory is psychologist Howard Gardner's multiple intelligences, which posits that intelligence is divided into several parts, including logical-mathematical intelligence and spatial intelligence. Italian physician and educator Maria Montessori's early-twentieth-century philosophies about children's self-guided, sensory learning also persist. In Montessori schools, shaped and textured beads, sandpaper numbers, and segmented rods help students explore basic mathematical concepts like numbers, place value, operations, geometrical relationships, and algebra, such as the binomial and trinomial theorems.

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See Also: Curriculum, K–12; Educational Manipulatives; Mathematical Modeling.

Legislation

See *Government and State Legislation*

Levers

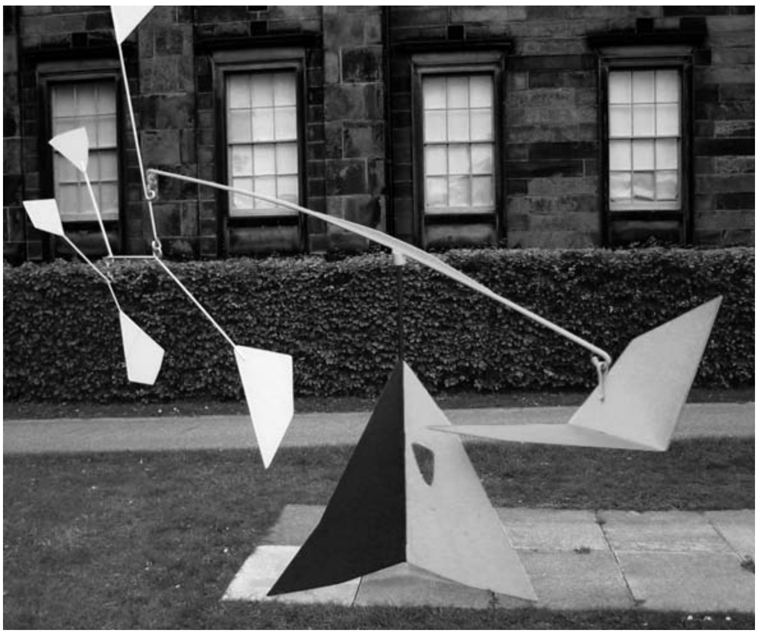
Category: Architecture and Engineering.
Fields of Study: Algebra; Geometry.
Summary: Levers negotiate forces in ways useful in engineering.

Levers are rigid beams that pivot around a point called the "fulcrum" to mediate three forces: an applied effort, a load to be moved, and the fulcrum's reaction. Depending on how the load, effort, and fulcrum are placed along the beam, either force or travel distance can be increased and the other decreased in proportion. There are three classes of lever, distinguished by the placement of the effort, load, or fulcrum. Levers of the first class have the fulcrum between the effort and load, like a see-saw, for changing direction of force and travel distance and increasing or decreasing either of them. The second class has the load in the middle, like a wheelbarrow for increasing force. The third class has the effort in the middle, like a pair of tongs for increasing travel distance. As these examples illustrate, levers are everywhere in the mechanical world and have been for the entirety of civilization.

Levers also occur in animals: the bones in limbs function as rigid rods and fulcrums, with muscles pulling hard close to a joint (the fulcrum) to move the extremity through greater distances than the contracting muscle can cover but exerting a force weaker than the muscle exerts on the bone. A train of three levers—the hammer, stirrup and anvil bones—magnify tiny acoustic displacements as they transmit sound from the eardrum to the cochlea.

Early Study

Our present formulation of levers derives from the *Equilibrium of Planes* of Archimedes, who determined that "Magnitudes are in equilibrium at distances reciprocally proportional to their weights." Using levers, Archimedes investigated the volumes of spheres and cones. Archimedes imagined the cone or sphere divided into thin slices: if a slice is hung on one side of a lever, what cylinder slice must be hung at what position to maintain equilibrium? By working through the entire volume of the cone or sphere, Archimedes constructed a cylinder of equal volume, thus giving the sphere's and the cone's volume. Levers also appear in Galileo's 1638 book of mechanics,



Alexander Calder's mobile sculpture L'empennage *displayed in the sculpture garden of the Scottish National Gallery of Modern Art.*

Two New Sciences. Whereas Archimedes had abstracted the lever as a perfectly rigid line, Galileo considered it as a three-dimensional, flexible object, leading to the first theory of beams. Combinations of levers, constrained in various ways, became a research topic during the Industrial Revolution. "Linkages," as these devices are called, were important for converting the rotation of steam engines into linear motion. Researchers in the nineteenth century took a mathematical approach to the problem. Among the best-known linkages is the Peaucellier cell, invented in 1864. The Peaucellier cell also plays theoretical roles in computer science.

Applications

Levers feature in mobiles and, notably, in the sculptures of Alexander Calder, who often places the fulcrum slightly above the beam that assists in balancing. The raised fulcrum has long featured in balances for weighing; the pivot point is above the lever's center of gravity so that, when the pans pull with equal torque, torque from the displaced beam's own weight will pull it level. Not all balances rely on this feature. Chinese pharmaceutical balances, for example, require the operator to look for nonrotation rather than perfect leveling.

More generally, nonmechanical levers exploit length to multiply distance. Optical levers rely on a mirror doubling an angle and a long travel distance for the light ray to register a large displacement. Social, financial, intellectual, and political resources can be metaphorically "leveraged" by using them to achieve outcomes larger than the resource itself, though the metaphor generally neglects to acknowledge the loss required for a mechanical lever to provide any gain.

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Alistair Kwan

See Also: Animals; Industrial Revolution; Pulleys; Vectors.

Life Expectancy

Category: Medicine and Health.
Fields of Study: Algebra; Data Analysis and Probability.
Summary: Estimating life expectancy in present populations relies on actuarial tables.

Life expectancy for an individual is the average number of years remaining until death. It is often used to quantify risk of certain characteristics or behaviors as well as to evaluate and compare populations in terms of economics and health. For example, in the United States the life expectancy for a single female currently age 35 is 50.1 years using the 2010 Social Security mortality table. Life expectancy can also be applied to machines or appliances, for product development, to manufacturing quality control, and for the determination of warranty periods. Most incandescent light bulb packages have the life expectancy printed on the packaging. A typical value is 900 hours of use. In this type of application, life expectancy is used as a measure of quality. The calculation of life expectancies can be as simple as taking averages, but normally it uses more advanced mathematics or sampling.

Human Life Expectancy

For human populations, factors affecting life expectancy include resource availability, sanitary practices, healthcare quality, war and sociopolitical factors, cultural and behavior factors, genetic and demographic factors, environmental factors, and epidemics. An increase or decrease in life expectancy may be quoted to describe the risk of a behavior or activity. As an example of using mathematics to make decisions, mathematician James Stein provides the statistic that each hour driven on an interstate highway decreases life expectancy by 19 minutes, while each hour flying decreases life expectancy by only 13 minutes, thus illustrating that flying may be a safer mode of transportation. To quantify the risk in smoking, the U.S. Centers for Disease Control and Prevention (CDC) states that the average life expectancy for a smoker is approximately 14 years less than for a nonsmoker.

Comparing Populations

The life expectancy of newborns is often quoted to compare the relative health of populations in different geographic areas as well as for differences between ethnic or socioeconomic groups, sexes, historical periods, or age groups. The populations being compared may differ in time, geographic region, or demographic characteristics. To compare populations from different time periods, the life expectancy of a newborn in the United States in the early 1900s was about 47 years, improving to about 60 years by the mid-1930s, and further improving to about 78 years by 2009. Life expectancy can vary by gender and race. Historically, females have typically exhibited a higher life expectancy than males. The life expectancy of a newborn female in the United States was estimated to be 80.2 years in 2006, compared to just 75.1 years for a newborn male. Also in 2006, a newborn white male had a life expectancy of 75.7 years, compared to 69.7 years for a newborn black male. According to the United Nations World Population Prospects 2006 Revision, the world life expectancy for a newborn in 2005–2010 is estimated to be 67.2 years, with Swaziland exhibiting the lowest life expectancy at birth for an individual country—approximately 40 years. The latter is often attributed to the high HIV/AIDS mortality and poor healthcare and socioeconomic conditions in sub-Saharan Africa.

For populations that lived in the past, the life expectancy can be calculated by taking the average of the age at death for all of the individuals who lived in the population of interest. For this type of calculation, one normally needs detailed records of dates of births and deaths for the entire population. The first life tables constructed in this way are attributed to John Graunt (1620–1674), who also provided estimated life expectancies in his tables. Following Graunt, a notable life table constructed from birth and funeral data for the purpose of determining life annuity values was published in 1693 by Edmund Halley (Halley's Comet is named after him) for the city of Breslaw, Poland. Halley used this city for his table because he thought Breslaw was representative of an average European population at the time. Interestingly, Halley provided his own definition of "life expectancy" in describing the third use of his table. In Halley's description, the expected future years a person of a certain age can reasonably expect to live is the proposed number of years upon which an even wager, which is a bet with a 50-50 chance of being won, can be made that the person arrives at that age before he dies. Halley's description is that of the median future lifetime, which differs mathematically from the more modern definition of life expectancy.

Sampling and Estimation

In the absence of complete data, modern statistical methods, including sampling, are used to estimate the average age at death. Similar statistical methods are used to estimate the life expectancy of appliances, components, and machines. In the case of inanimate objects, life expectancy may be interpreted as the average time to failure. To estimate the average time to failure, a sample may be taken and tested in a laboratory environment, or failure statistics may be kept after the product goes to market. The failure rates obtained from such data not only provide a basis for determining the life expectancy of the product, but also can be used in determining the cost of a warranty or guarantee issued by the manufacturer.

In modern populations, actuarial tables are developed that estimate the probability of death at any particular age. These probabilities are used to calculate the life expectancy for an individual at his or her current age. For example, suppose a male age 96 is within a population whose mortality table indicates the probability of a male age 96 dying before age 97 is 0.45; the probability of surviving to age 97 and dying before age 98 is 0.35; and the probability of surviving to age 98 and dying before age 99 is 0.2. Then the expected age at death is calculated as the expected value,

$$96\left(0.45\right)+97\left(0.35\right)+98\left(0.1\right)+\frac{1}{2}=97.25\,.$$

Hence, the life expectancy is 1.25 years. The term "1/2" in the expected age at death calculation reflects the assumption that the individual dying within the year lives on average one-half the year.

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Kevin L. Shirley

See Also: Disease Survival Rates; Expected Values; Forecasting; Insurance; Measures of Center; Quality Control.

Light

Category: Weather, Nature, and Environment.
Fields of Study: Algebra; Representations.
Summary: Now understood as both a particle and a wave, light is a recurring subject of interest in physics.

Light, a form of electromagnetic energy, mediates the electrostatic interactions between particles. Under some experimental conditions, it acts as a particle, and under others, as a wave. Attempts by physicists to reconcile this dual nature and to otherwise exploit this duality have been the impetus for the development of large areas of mathematics.

Particle or Wave?

Isaac Newton advocated the particle nature of light, initiating the study of geometric, or ray, optics. This form of optics treats light as rays that travel in straight lines, though capable of bending near objects. It is based on two laws. The law of reflection states that when light is reflected from a surface, the angle of incidence equals the angle of reflection. The law of refraction says that light will bend when it passes from one medium to another according to Snell's Law, named for mathematician Willebrord Snell, a relation between the angles of incidence and refraction and light's speed in the two media.

At about the same time, Christiaan Huygens discovered polarized light and explained it with a wave theory. From this beginning, Thomas Young and Augustin-Jean Fresnel developed physical optics. The resulting mathematics allowed engineers to construct extremely faithful lenses; its close cousin, wave acoustics, helped architects design performance halls. Scientists pursued these optics to ever-finer scales. Eventually, they developed the electron microscope, which permits biologists to see individual DNA molecules. Physical biochemists use a related technique called "crystallography." When X-rays are shot through crystals of protein molecules, they form a diffraction pattern, which when transformed by a technique called Fourier analysis (named for mathematician and physicist Joseph Fourier) allows the precise determination of the protein's atomic structure. Many owe their Nobel Prizes to this transformation.

The wave theory of light provides the most natural explanation for the spectrum of visible light. What the physicist calls "light" varies from about 10^{23} cycles per second, corresponding to gamma rays, down to roughly 1000 cycles per second for the electron waves in plasma. What humans can see is but a small part of this, varying from purple at a wavelength of 380 nanometers (nm) or 7.8×10^{14} cycles per second, to red at about 780 nm, or 3.8×10^{14} cycles per second.

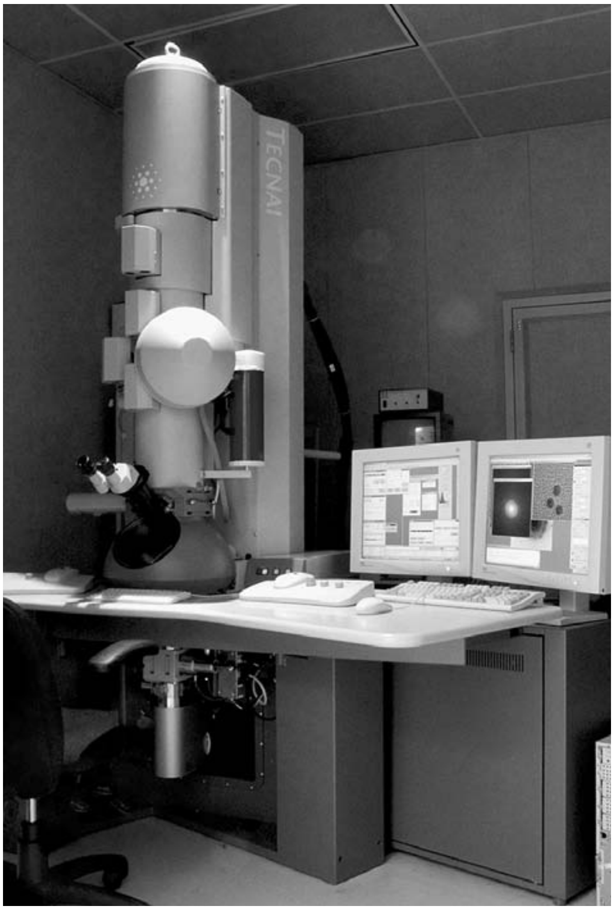
Light Speed

In 1861, James Clerk Maxwell wrote down his famous equations describing the interactions between electric and magnetic fields in terms of their sources. Four years later, he derived from them an electromagnetic wave equation, which physicists soon understood to be a description of light waves. In 1907, Edward Rosa and Noah Dorsey used these equations to calculate the speed of light at 299,784 km/sec. The accuracy of this calculation was not matched by experiment until 1926, when Albert Michelson obtained a value of 299,796 km/sec. In 1983, the 17th Conférence Général des Poids et Mesures established a new standard for the length of the meter by fixing the speed of light at 299,792,458 meters/second.

In the 1890s, Hendrik Lorentz, George Fitzgerald, and Joseph Larmor noticed that Maxwell's equations did not change under a certain type of transformation. Henri Poincaré called these "Lorentz transformations" and noticed that they formed a group of symmetries on four-dimensional space-time. Albert Einstein incorporated this symmetry into his theory of special relativity. One of the key postulates is that light travels at the universe's speed limit and so nothing can travel faster. Hermann Minkowski developed from these theories a four-dimensional geometry called "Minkowski space," in which Einstein's famous theory is understood as geometric properties of the space.

Quantum Phenomena

Light held yet further mysteries. Nineteenth-century physics predicted that heated bodies should radiate infinite amounts of energy and that an atomic electron should plunge into the nucleus. Max Planck eliminated the first problem by postulating the quantization of light. Einstein used this idea to explain properties of the photoelectric effect, the phenomenon behind solar panels. Niels Bohr expanded these ideas into an explanation of why electrons in atoms do not continuously radiate light until they collapse into the nucleus. All three won Nobel Prizes for their work and quantum physics was born.



The National Center for Electron Microscopy houses this microscope with a 2048x2048 CCD camera.

John von Neumann developed a mathematical description of these quantum phenomena involving Hilbert spaces and operator algebras. As a result, research into Operator Algebras became a major research focus of the last half of the twentieth century. To further explain quantum behavior, von Neumann and Garrett Birkoff developed quantum logic, a subject pursued not only by mathematicians but also by many philosophers. In a high point of this endeavor, John Bell developed the Bell inequalities in 1966. Sixteen years later, Alain Aspect confirmed that quantum systems do violate these inequalities, and provided strong evidence that the mysterious results of quantum mechanics are not solely because of our difficulties in measuring systems on such a fine scale but are because of the very nature of these small-scale systems. These experiments exploited a quantum property called "entanglement." Richard Feynman hypothesized this entanglement might be exploitable as a computational resource. In recent decades, Peter Schor, Lov Grover, and others have developed algorithms based on Feynman's idea and created the field of quantum computing.

Quantum mechanics has, in the last half century, developed into quantum field theory (QFT). QFT attempts to explain all particles and forces by equations that are modeled on Maxwell's. In developing their models, mathematical physicists rely on physical properties to perform manipulations mathematicians find objectionable because of their lack of rigor. Many great mathematicians have taken up the challenge of developing a rigorous axiomatic basis for QFT. Lying at the intersection of philosophy, mathematics, and physics, many mathematicians see this as one of the great challenges of the twenty-first century.

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Michael Klucznik

See Also: Crystallography; Einstein, Albert; Electricity; Elementary Particles; Radiation.

Light Bulbs

Category: Architecture and Engineering.
Fields of Study: Algebra; Connections.
Summary: Light bulbs are ideally designed for great luminous efficacy, emitting more light than heat.

Light bulbs are common sources of electric light. The light bulb's evolution is not entirely certain. Historians cite more than 20 contributors, dating back to roughly 1800, who made discoveries prior to inventor Thomas Edison's 1879 patent for an incandescent bulb. Some attribute Edison's success to the fact that he also invented an entire electricity distribution system.

Traditionally, light bulbs work on the principle of incandescence. The filament inside an incandescent bulb resists the flow of electrons supplied by an electrical source, causing the filament to heat up and emit radiation. Approximately 90% of the power consumed by an incandescent light bulb is, in fact, emitted as heat rather than as visible light. The wavelength of the emitted radiation determines the color of the light. In common household incandescent bulbs, the emitted radiation is primarily in the infrared region of the spectrum, which humans cannot see, along with the visible red, orange, and yellow wavelengths nearest the infrared. This characteristic gives the bulb its characteristic yellowish color.

Compact fluorescent light bulbs, which are intended to replace incandescent bulbs, operate on a different principle. Electricity excites mercury vapor to produce light, but little heat. The emitted spectrum of traditional fluorescents is much closer to the blue end of the visible spectrum, though there are now a variety of models that closely mimic natural light. In addition to quantifying the emitted radiation spectrum, mathematics is used to calculate other important features of light bulbs, such as electrical rating and efficiency.

Rating

Incandescent bulbs are normally rated according to their electrical power. Common household sizes in the United States range from 15 watts, often found in refrigerators and other appliances, to 150-watt bulbs used for reading or to light large areas. As the bulb is purely resistive (its inductance and capacitance are insignificant), the electrical power can be computed as $P = V \times I$, or $P = I^2 \times R$, where P is the electric power in units of watts, V is the potential difference in volts, R is the resistance of the filament in ohms, and I is the current in amperes or "amps," named after André-Marie Ampère, a French mathematician and physicist considered the "father of electrodynamics." Household voltage in the United States is usually 120 volts, so higher wattage bulbs require more current to operate, which makes them more costly to use. Because compact fluorescents operate on a different principle than resistance, they typically draw less current to produce the same perceived intensity of light.

Luminous Efficacy

Another metric used to distinguish light bulbs is luminous efficacy, defined as

$$\text{LES} = \frac{F}{P}$$

where F, the flux in lumens, is the total useful amount of visible radiant light, and P is the power. A weighted luminosity function adjusts for the human eye's response to different wavelengths of light when flux is calculated. If total electric power consumed by a bulb is used in this computation, it is referred to as "luminous efficacy of a source" (LES). LES is a good indicator of source's ability to provide visible light from a given amount of electricity. For example, a 40-watt incandescent bulb has an LES of roughly 12.6 lm/W, and a flux comparable to a 9- to 13-watt compact fluorescent. A 100-watt bulb has a flux comparable to 17.5 lm/W, versus a 23- to 30-watt compact fluorescent.

Humor

Light bulbs are also a source of humor, with hundreds of light bulb jokes of the general form, "How many (fill in the blank) does it take to screw in a light bulb?"

Many of these jokes are intended to satirically poke fun at the subjects; mathematicians are no exception. For example, “How many mathematicians does it take to screw in a light bulb?” The answer is “None. A mathematician can’t screw in a light bulb, but he can easily prove the work can be done.”

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Ashwin Mudigonda

See Also: Carbon Footprint; Electricity; Energy; Green Mathematics; Light.

Lightning

Category: Weather, Nature, and Environment.
Fields of Study: Data Analysis and Probability; Geometry; Measurement; Representations.
Summary: Lightning is studied, modeled, and predicted using mathematical techniques.

Lightning is an electrical phenomenon of nature that has been observed by people around the world for thousands of years. Thunder is the sound of lightning, created by the intense heat of a lightning bolt. Many people may have learned as children a simple calculation for estimating the distance of lightning based on the sound of thunder. Since thunder travels about one mile in five seconds, a 15-second delay between the time lightning is seen and the time the thunder is heard indicates that the lightning strike was about three miles away.

Lightning strikes occur frequently around the globe, with an estimated 25 million cloud-to-ground strikes per year in the United States alone. Lightning has a large number of religious associations, and it is often used as a metaphor for sudden insight or inspiration. Mathematician Carl Friedrich Gauss is reported to have said, regarding a problem he had been working on, “Like a sudden flash of lightning, the riddle was solved.” Lightning is studied by mathematicians, often in collaboration with scientists in other fields, to better understand the various facets of this complex phenomenon.

Among the several types of lightning that occur, the most commonly seen and the most dangerous is cloud-to-ground lightning, caused by the discharge of electrons into the Earth from thunderclouds in the atmosphere. The voltage released by a bolt of cloud-to-ground lightning is on the order of 1 million times the voltage in a standard electrical outlet.

The excess of electrons at the base of a thundercloud repels electrons on the ground deep into the Earth, inducing a strong positive charge on the ground below. While air usually acts as an insulator, preventing the flow of electric current, the strong electric field between a storm cloud and the Earth can reach tens of thousands of volts per inch, pulling air molecules apart into negatively charged electrons and positive ions. This creates pathways of ionized air known as “streamers.” The freely moving charges in the ionized air allow electric current to flow through it.

A lightning strike occurs when a streamer carrying electrons from the cloud toward the Earth meets a shorter, positively charged airstream reaching up from an object on the Earth. This creates a complete conductive pathway between the cloud and the ground and a sudden and massive discharge of electrons into the Earth.

Between an average thundercloud and the Earth, there are an estimated 10^8 volts, reaching 10^9 (1 billion) volts in more-intense strikes. For perspective, one may compare 1.2×10^8 volts between a thundercloud and the Earth to the 120 volts delivered by a standard electrical outlet in the United States:

$$\begin{aligned}&\text{Voltage between cloud and ground}\\&= 1.2 \times 10^8 \text{ volts}\\&= 1.2 \times 10^2 \times 10^6 \text{ volts}\\&= 120 \text{ volts} \times 10^6\\&= \text{Voltage in standard electrical outlet} \times 1 \text{ million.}\end{aligned}$$

The heat created by the electric current in a bolt of lightning reaches temperatures up to 30,000 kelvins (K), more than five times the temperature of the surface of the sun and hot enough to melt rock and fuse soil and sand into glass. The temperature on the Kelvin

Lightning has been observed around the world for thousands of years. It has been spotted not only during thunderstorms but in volcanic eruptions, intense forest fires, heavy snowstorms, and large hurricanes.

scale is the temperature in degrees Celsius plus 273.15. The intense heat in a channel of lightning causes the air within the channel to expand rapidly, sending out a shock wave that weakens into the acoustic wave of thunder. The electric current and heat of a lightning strike can start forest fires, damage a property, destroy electrical equipment, and cause serious or fatal injuries to people and animals. According to estimates by the National Weather Service, lightning causes on average about 60 deaths and 300 injuries in the United States each year.

Statistics collected by NASA satellites have found that most of the eastern half of the United States sustains about eight flashes of lightning per square mile per year (decreasing to less than one per square mile per year toward the West Coast). Since 1 $mi^2 = 640$ acres, this translates to eight flashes per 640 acres per year, or one flash per 80 acres per year. Accordingly, a one-acre lot in this region would be struck by lightning on average once every 80 years.

Mathematical research can help to predict the behavior of lightning strikes based on weather patterns and other variables; for example, by modeling probabilistic distributions of lightning strikes according factors such as time, geography, and strength. The mathematical theory of highly optimized tolerance (HOT) is useful in controlling forest fires caused by lightning. This theory suggests optimal placement of fire breaks: if data or other evidence suggests that lightning strikes some areas of a forest more frequently than others, then large fires can best be prevented by purposefully cutting fire breaks that create sections whose sizes are inversely proportional to the rate at which lightning strikes. Other mathematicians are interested in studying the patterns and geometry of lightning. Mathematician Benoit Mandelbrot, known for his study of fractal patterns, noted that lightning does not travel in a straight line but rather in patterns reminiscent of fractals. Techniques of fractal modeling are used to study fractal patterns in the ionized plasma structures of lightning streamers. Morphological filtering and gradient detection can be used to help visualize lightning in satellite imagery and separate it from other visible effects, such as city lights.

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See Also: Earthquakes; Electricity; Elementary Particles; Energy; Forest Fires; Light.

Limits and Continuity

Category: History and Development of Curricular Concepts.
Fields of Study: Algebra; Calculus; Communication; Connections.
Summary: One of the key concepts of calculus, the limit is the value a function approaches as its input approaches a given value.

The concepts of limit and continuity are fundamental in calculus, analysis, and topology. Their inception can be traced back more than 2000 years to Greece, China, Babylon, Egypt, and other places. During the inception of calculus, introduced independently by Isaac Newton and Gottfried Wilhelm Leibniz, these concepts were still vague and controversial. In the twenty-first century, these concepts are explored in high school. The modern limit of a function $f(x)$ is the value the function tends to when x changes in a structured way; for instance, x approaches a specific value. Many different definitions of limits in calculus can be combined into a single definition of limit in topology. Moreover, a function is continuous if it preserves closeness or, equivalently, if it preserves limits, and topology is the study of continuous functions and the properties they preserve. These ideas underpin many mathematical results and can be used to organize and simplify mathematical processes. Limits are also useful in real life to understand such concepts as demographics, finance, or terminal velocity. Modeling discrete data with a continuous function and the notion of continuous payments are also important. Leibniz defined a principle of continuity, or *lex continuitatis*, which inspired philosophers such as Charles Peirce. The notions of limit and continuity are still debated philosophically, as in whether growth spurts are continuous over time.

The Ideas of Limit and Continuity in the Ancient World

The idea of limit in the ancient world was related mainly to two activities: one was more practical, like measuring length, area, and volume, and the other was more abstract, such as making sense of numbers that are not rational. For example Archimedes from Greece and Liu Hui in China used regular polygons, inscribed in a circle, increasing the number of sides of the polygons, in order to compute the length of the circumference and the area of a circle. In the process, approximations of π were computed. Eudoxus from Greece created his theory of proportions to legitimize irrationals like $\sqrt{2}$.

This theory is expounded in Book Five of Euclid's *Elements.* It is also a precursor of the contemporary theory of the real numbers. Ancient mathematicians, such as Zeno of Elea and Aristotle, wrestled with the notion of continuity. They debated whether motion, time, and space are continuous. The paradox about Achilles and the tortoise illustrated the interplay between the ideas. The paradox states that Achilles can never overtake the tortoise if the tortoise is given a head start, because by the time Achilles reaches its initial position the tortoise has farther advanced and so on; infinitely many segments of time are necessary.

The Calculus of the Infinitesimals

In the middle of the seventeenth century after significant advances in science, particularly in physics, mechanics, and geometry, the methods of infinitesimal calculus were introduced independently by Isaac Newton in England and by Gottfried Wilhelm Leibniz in Germany. Newton assumed that geometric magnitudes are generated by continuous motion, and some historians suggest that he may have been the first to present a limit argument using an infinitesimal like *epsilon*. Leibniz explored a principle of continuity but is not thought to have explored the derivative as a limit. He viewed the ocean as continuous.

The quest to find an acceptable, rigorous foundation for the new calculus was ongoing. There were attempts made to follow the method of exhaustion from the Greeks or to use series instead of infinitesimals to introduce the derivative. Jean le Rond d'Alembert stressed the importance of a firm foundation for limits and explored a geometric limit of secant lines. But it was Augustin-Louis Cauchy who introduced contemporary definitions of limit and continuity and placed them as a cornerstone of calculus. It is interesting to mention that another mathematician, Bernard Bolzano, from Prague and a contemporary of Cauchy, came up with the same definitions first, but he was more isolated and his work did not get the same recognition that the *Cours d'analyse* enjoyed. The German mathematician Karl Weierstrass solidified the rigorous definitions and made significant contributions to the development of analysis.

Contemporary Definitions

The limit of a function is a dynamic concept. The input of the function varies and the output varies as well. Intuitively speaking, one says that $\lim_{x\to c} f(x) = L$ when the values of f become closer and closer to the number L when c gets closer and closer to c. This intuitive concept is easy to grasp and also not difficult to observe if one has a graph of the function f. But this should be expressed rigorously, so that there is a tool to verify whether the limit exists.

Definition: The $\lim_{x\to c} f(x) = L$ if and only if for each $\varepsilon > 0$ there is a $\delta > 0$, such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

This definition enables mathematicians to verify the existence of limit or to make an argument that there is no limit and is also a tool to prove many properties about limits. The number c does not have to be in the domain of the function, but one should be able get δ close to it from the domain for any positive δ. The concept of limit is used to define a continuous function.

Definition: A function f is continuous at a point c if c is in the domain of the function and for any positive ε there is a positive δ, such that if $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.

The concept of limit is used to define the definite integral and to measure area.

Contemporary Developments.

Limits of other objects, such as sequences and geometric spaces, can be defined and are important in many disciplines of mathematics and continuity is still explored in the field of topology. One twentieth-century development that goes back to the history of limits occurred around 1960 when Abraham Robinson entertained the idea that the advantages of infinitesimal calculus can be utilized as soon as the infinitesimals are defined in a rigorous way. This would eliminate the use of limit in the way it is known and make the analysis very much like algebra, as soon as the number system is extended to permit infinitely small and infinitely large numbers. This is exactly what Robinson did using tools from logic, a development called "non-standard analysis."

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See Also: Function Rate of Change; Functions; Numbers, Real; Proof.

Linear Concepts

Category: History and Development of Curricular Concepts.
Fields of Study: Algebra; Communication; Connections.
Summary: Linear relationships are a fundamental concept of mathematics.

From ancient civilizations to modern societies, people use linear concepts in a multitude of ways, including statistical analysis, for advanced mathematics, and in scientific applications, many of which are designed to solve real-world problems. The fundamental idea of a linear relationship involves comparing two or more quantities which form a straight line when graphed. A basic linear relationship comparing two quantities is represented by the linear equation $y = mx + b$, where x and y are the quantities which vary in direct proportion to each other, m represents the slope of the line, and b represents the value where the line crosses the y-axis on the coordinate plane. This idea implies that the quantities in a linear relationship depend upon each other. One of the most common ways to represent a linear relationship is the linear equation. Linear equations are equations involving one or more unknown values, called "variables," which are of the first degree. Linear equations are the simplest type of equation, since the unknown quantities are always raised to the power of one. Spaces of lines, such as a plane, are also the simplest type of geometric space. However, linear equations and the methods of finding the solutions become more complex as the number of unknowns increases or when solving more than one linear equation at a time. Students begin to explore linear concepts in the primary grades, and these are built upon and extended throughout high school and college.

Linear equations are used extensively in applied mathematics, particularly in modeling and representing real-world phenomena. Linear relationships are also used in advanced mathematical applications and modeling, typically by reducing nonlinear equations to linear equations or by constraining events within a set, or "system," of linear equations. The development of general methods for solving linear equations was a slow process because of the limitations of communicating and representing the unknown quantities in linear relationships. These equations were initially solved using the elementary operations of addition, subtraction, multiplication, and division. The problems and their solutions were written using words. Algebraic methods of solving equations were not developed until a system of symbolic notation replaced the use of words. The modern practice of using variables in place of unknown values did not gain widespread use until the sixteenth century. Before that time, problems were typically written using only words or by using a limited set of symbols.

Linear Equations

The earliest known linear equations and methods of solving them are found in several ancient civilizations.

A portion of the Rhind Mathematical Papyrus, one of the most famous surviving papyri.

These societies used linear equations and systems of linear equations to solve problems arising in everyday life, particularly based on civic and government needs. Although the historical information and records that exist from these ancient civilizations are fragmented, there exists enough evidence to show how the Babylonian, Egyptian, Chinese, and Islamic civilizations used and solved linear problems. Within the Babylonian civilization, the need for computational techniques beyond simple counting arose in areas of commerce, taxation, and construction. The Babylonians wrote their problems on clay tablets and included many examples of solving linear equations and systems of linear equations. These numerical problems were expressed rhetorically, without symbolic notation, and were provided to show the method of solution for a particular example. Reasons and explanations were not given, nor were any general methods of solution. The Egyptians were also concerned with commerce, taxation, and construction. They also described the methods used to solve linear equations arising from everyday life, such as dividing loaves of bread or a given amount of grain. They wrote their problems on papyrus, which was made from a reed plant, very few of which exist today. The most famous surviving papyri are the Rhind Papyrus and the Moscow Papyrus. One particular procedure the Egyptians devised for solving linear equations is known as the "method of false position." This procedure began with guessing a value for the unknown and then adjusting the value until the correct result was found. This method was also used by other ancient civilizations and continued to appear in elementary algebra textbooks until the nineteenth century. The Chinese used a similar method of false position but made two guesses for the unknown rather than one guess. This method is known as the "method of double false position" and was later used by Islamic mathematicians. This approach continued to be used in Europe until the 1600s when advances in symbolic notation made solving linear equations a much more simple process. It took many years for a symbolic system to develop and allow for the development of general solutions to linear equations.

Linear Modeling

Linear relationships appear extensively in modeling applications. In statistics, simple linear regression is commonly used to model the relationship between two variables when that relationship appears to be generally linear. That is, when plotted on the coordinate plane, the data tend to cluster about a straight line. It can also be used to make predictions for situations when the value of one variable is known and the other is not. The concept of linear regression was developed by Sir Francis Galton in the late nineteenth century while investigating genetic inheritance. A description of this method was published in the article "Regression Towards Mediocrity in Hereditary Stature." The term "regression" was actually a reference by Galton to observed effects in the data, not to the method itself, yet the statistical process of fitting a line to data still bears this name. An extension of the method, called "multiple linear regression," is used to model relationships between several variables in *n* dimensions.

Many types of advanced mathematical models also rely on the use of linear relationships. For example, linear programming is used in business applications as a way to model important decisions that lead to maximum profit. This modeling is accomplished by constraining the variables, such as production costs, within a system of linear equations. Linear programming is also used in a modeling process known as "linear optimization." This modeling process has a wide variety of applications in areas such as business, finance, engineering, and industry. Linear programming and linear optimization are based on the mathematical procedure of defining all of the related variables as linear relationships. This process was first developed in the 1940s and is one of the few mathematical applications that has a wide range of practical uses as well as a theoretical development of the mathematics.

Many definitions in mathematics rely on linear approximations. The derivative of a function of one variable at a point is the slope of the tangent line (the slope of the line that best approximates the curve at a given point). Mathematicians such as Isaac Barrow and Sir Isaac Newton made linear concepts a fundamental part of their work in the development of calculus. In higher dimensions the derivative is a linear transformation that is represented as a matrix. In geometry, a surface is defined as a space that locally looks like a plane. Georg Friedrich Bernhard Riemann defined higher dimensional spaces, now called "manifolds," as locally looking flat and possessing shortest paths that are straight. In 1917, Albert Einstein used Riemann's mathematics in order to present a model for the universe that was consistent with his theory of relativity.

Linear Algebra

Linear algebra is a subject that is fundamental to modern mathematics and applications. It arose from the study of coefficients of systems of linear equations, and linear concepts are fundamental in this area. For example, Arthur Cayley explored linear maps or transformations, and Giuseppe Peano was the first to give an abstract definition of the algebraic structure of linear vector spaces.

The late development of a symbolic notational system used in solving linear equations slowed the development of finding general methods for solving these equations. The first breakthrough in using algebraic techniques to solve linear equations occurred in the sixteenth century when Jacques Peletier proposed a general rule of algebra. This general rule involved setting up linear relationships as equations and finding the roots, a method still used in the teaching of algebra. With the adoption of a system of symbolic notation, the applications of linear equations continue to evolve and to be used in numerous ways, from basic equation solving to advanced mathematical techniques in both pure and applied mathematics. Linear algebra has a long history in mathematics, and linear concepts are considered one of the most important concepts in mathematics because of their appearances in so many levels of both pure and applied mathematics and in a multitude of real-world applications.

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See Also: Algebra and Algebra Education; Algebra in Society; Babylonian Mathematics; Egyptian Mathematics; Function Rate of Change; Graphs; Mathematical Modeling; Scatterplots.

Literature

Category: Arts, Music, and Entertainment.
Fields of Study: Communication; Connections.
Summary: Since ancient Greece, literature has drawn on mathematical imagery.

Literature and mathematics share many characteristics despite their apparently different natures. The work of the mathematician is similar to that of the writer, and mathematics has inspired many works of fiction, biography, satire, and mystery. Because of their intelligence and interesting personalities, mathematicians appear as characters in many works of fiction. Biographies of mathematicians and tales of famous mathematical problems also provide fascinating narratives because of their interesting characters and the characters' struggles with mathematical and personal problems.

At times, mathematics has been a target for attack by satirists for mathematicians' tendency to overuse mathematics by reducing social and economic issues to mere mathematical equations. Literature is also an effective tool in mathematics education, especially for small children.

The Mathematics–Literature Connection

Few fields of human activity seem, at first glance, as distant as mathematics and literature. Mathematics is a field of rigor, exactness, and absolute truth. It involves formulas, equations, laws, and theorems that do not leave much room for opinion, subjectivity, or individuality. On the other hand, literature is the realm of emotions, characters, imagination, and subjectivity. The author, unconstrained by the strict laws of nature, creates worlds, people, and events as the imagination desires. The resulting stories are usually told by a human or anthropomorphic narrator, and the narrator's tone and style affect the story and the reader.

Despite these differences, there are many connections and commonalities between mathematics and literature. Mathematics describes relationships between numbers, functions, sets, and other mathematical objects; literature is concerned with relationships between characters. Mathematics tries to describe how nature works; literature describes how people behave. Mathematics often describes paradoxes, unintuitive concepts, and unsolved problems; literature often depicts irrational behaviors, impossible situations, and other situations that defy explanation.

The work of a mathematician is similar to that of an author in many ways. Both mathematics and literature require imagination and creativity, albeit of a somewhat different type; both are mostly individual endeavors; both require intuition and insight; both require a significant amount of time, patience, and persistence; and both provide an immense sense of accomplishment and exhilaration when the product—be it a novel or a proof—is complete.

Early Influences

The relationship between mathematics and literature can be traced back to ancient Greece, the cradle of both modern mathematics and the liberal arts. Greek thinkers were philosophers (lovers of wisdom) and pursued knowledge and beauty in all forms. These philosophers were thus interested in the arts as well as in scientific questions.

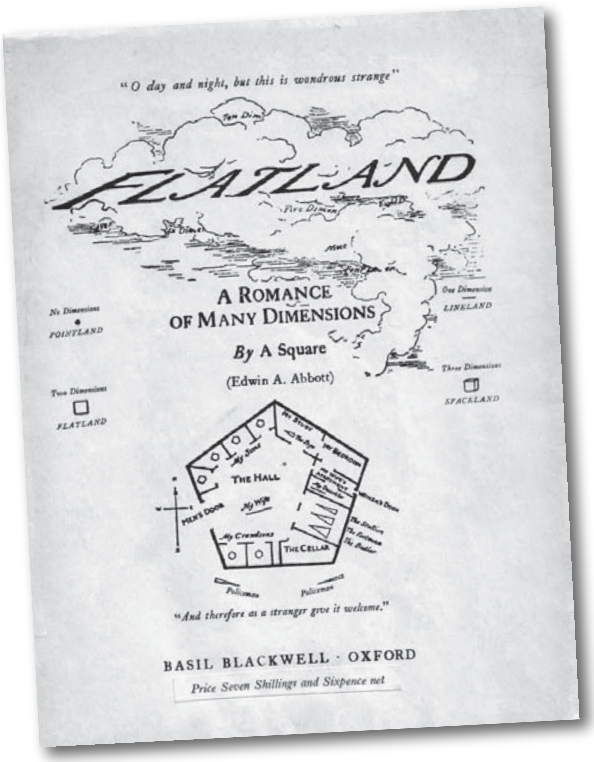
This intellectual environment was conducive to cross-fertilization of the arts and sciences, and the great mathematician Pythagoras was among the first to seek the literary and metaphysical meanings of numbers. For the Pythagoreans (followers of Pythagoras), numbers were not merely abstract tools for counting and measuring but also symbols with mystical meanings. For Pythagoras, all things were essentially numbers.

Pythagoras's notions on the mystical meaning of numbers have little relevance in modern science and mathematics, but they advance the idea that numbers may be used as literary objects. These ideas paved the way for other thinkers seeking greater meanings for mathematics than those constrained within the realm of science.

Mathematical Imagery in Fiction

Mathematics is a field rich in shapes, structures, and relations, and writers may find in mathematics a vast resource of imagery, analogy, and metaphor. Examples of mathematical imagery in fiction abound, and while some are explicit and obvious, others require varying degrees of mathematical knowledge to be fully appreciated.

Edwin Abbott's 1884 *Flatland* is perhaps the most famous novel whose characters are mathematical objects. This witty and influential novel takes place in a two-dimensional universe whose denizens are anthropomorphic lines and polygons. The narrator, a square, describes social classes, political unrest, and practical issues of life in two dimensions. He then describes visits to lower dimensional worlds and to Spaceland, the world of three dimensions. The narrator then conjectures the existence of higher dimensional worlds. The novel won renewed recognition near the end of the twentieth century in part because of the development of physical theories, such as string theory, which suggests that the universe may have more than the three spatial dimensions that are visible to us.



Edwin Abbott's 1884 witty and influential novel Flatland, *whose characters are mathematical objects.*

The popular 1865 fantasy novel *Alice's Adventures in Wonderland* was written by a mathematician, Charles Lutwidge Dodgson, who wrote it under the pen name Lewis Carroll. The novel contains several mathematical themes, such as apparently faulty multiplication (4 times 5 is 12) that can be rationalized by using a different base (4 times 5 is 12 in base 18). Logic (or lack thereof) also plays a role in the novel. During a tea party, the Mad Hatter reproaches Alice for committing the logical fallacy of assuming that a statement implies its converse. There are many other possible mathematical themes in the book; however, because of the light-hearted and fantastic nature of the works, it is impossible to determine which of those were intentional.

Argentinean author Jorge Luis Borges uses many mathematical themes in his stories. In his 1941 short story *Library of Babel*, he tells the story of a library filled with an infinite number of books, each containing exactly 410 pages. The story incorporates diverse mathematical ideas and concepts ranging from combinatorics to geometry and topology. The concept of infinity is also a recurring theme in the story. The story is so rich in mathematical imagery that it inspired a 2008 book, William Goldbloom Bloch's *The Unimaginable Mathematics of Borges' Library of Babel*, dedicated to the exploration of these themes.

In his 1869 novel *War and Peace*, Leo Tolstoy argued that history is not driven by major historical characters but rather by the infinitesimal contributions of many people. He uses an analogy with mathematical integration, where the sum of an infinite number of infinitesimal terms is taken, thereby giving the integral its value.

Fictional Mathematicians in Literature

In popular culture, mathematicians are considered to be highly intelligent individuals who possess an investigative mind and a good sense for problem solving.

Mathematicians also have a reputation for eccentricity and lack of social skills. These attributes appeal to many authors and readers and make mathematicians interesting literary characters.

In 414 B.C.E., Athenian comic playwright Aristophanes incorporated a fictional mathematician into his play *Birds*. In the play, the characters decide to build a utopia in the sky in order to escape the routine of Athenian life. Meton, a geometer, joins them and proposes to survey the skies and parcel them into lots. While describing his planned layout, he mentions his plan to circle the square, a mathematical problem that occupied several Greek mathematicians.

The American poet and author Edgar Allan Poe was a science and mathematics enthusiast and used many scientific and mathematical themes in his stories. In the 1843 short story *The Gold Bug*, the protagonist uses mathematical intuition, common sense, and rudimentary principles of cryptanalysis (code breaking) to decipher an encoded message that describes the location of a buried treasure.

In Poe's 1841 story *A Descent into the Maelström*, the narrator uses his knowledge of solid geometry and fluid physics to escape death in a giant whirlpool that is sinking his ship. Thinking of a fabricated result in fluid mechanics that Poe attributes to Archimedes, the narrator recalls that solids subject to whirlpool display differential flotation based on their shape and that cylinders sink slower than other solids. He then saves his life by attaching himself to a water cask and throwing himself into the water with the cask.

Fictional characters may use mathematics as a pastime or as a source of pleasant diversion. In John Cheever's 1966 story *The Geometry of Love*, Charlie Mallory distracts himself from his unhappy marriage and unsatisfying professional and personal lives by trying to create Euclidean models of his relationships. To Mallory, these models are simple, elegant, and stable structures, while his life is often unpredictable and turbulent. Through these models, Mallory receives the stability and equanimity that are lacking in his life.

Aleksandr Solzhenitsyn's 1968 book *The First Circle* is a quasi-autobiographical novel set in a gulag in the Soviet Union. The novel's protagonist, Gleb Nerzhin, is a mathematician incarcerated in the gulag and forced to work with other scientists and engineers on secret state projects. The novel takes a close look at the difficult choices that scientists and mathematicians have to make when they have little or no control over their lives or the products of their research.

Arthur Conan Doyle's iconic detective Sherlock Holmes was a master of deductive reasoning. An expert at deriving surprising conclusions from evidence and clues, Holmes treated crime mysteries as mathematical puzzles and derived great pleasure from solving them. It is fitting that Holmes's arch-nemesis, Professor James Moriarty, was a mathematician. Moriarty was a genius villain and head of a large crime organization that pervaded England throughout Holmes's career. In the 1893 short adventure *The Final Problem*, Holmes and Moriarty are engaged in hand-to-hand combat as they fall to their deaths from a gorge.

Literature in Mathematics Education

In recent decades, literature has been increasingly used in mathematics education, especially with small children. Children's literature provides an approachable path to learning mathematics through the use of friendly characters, ample graphics, and age-appropriate narrative. Literature may assist children in learning basic mathematical concepts, such as numbers, counting, shapes, sizes, and arithmetic.

Literature can also be used to teach children that mathematics is not merely a field of dry equations and laws, but rather that it has many practical uses in everyday life. By reading about characters learning and using mathematics, children can learn to appreciate the many roles mathematics plays in their lives.

An example of children's books that may be used as mathematics education tools is the series *Sir Cumference* by Cindy Neuschwander. The books present children with mathematical and geometrical notions such as circumference, radius, and diameter through the use of similarly named characters, such as the series' namesake.

Not all mathematical geniuses realize their potential, in fiction or in real life. Aldous Huxley's 1924 short story *Young Archimedes* is a tragic tale of a mathematically gifted boy who falls victim to the unscrupulous and selfish behavior of adults. Huxley suggests that the tragedy is not only the victim's but also that of a society that fails to give its geniuses the environment they need in order to thrive.

Science Fiction

With its emphasis on science and technology, science fiction is a natural genre for mathematical themes. This theme is particularly true with the advent of "hard" science fiction, a branch of science fiction that stresses scientific rigor and theoretically possible technologies. A notable example of hard science fiction is Greg Egan's 2002 novel *Schild's Ladder,* which uses themes from advanced mathematics and physics. The novel describes a futuristic civilization that is forced into perpetual migration and the discord that develops within that civilization.

Isaac Asimov's classic *Foundation* series of novels portrays a fictional mathematician, Hari Seldon, as an influential character. Seldon is a brilliant mathematician who developed a branch of mathematics known as "psychohistory," which he uses to predict the collapse of the Galactic Empire.

Biographies and Memoirs

Many biographies of mathematicians are available in the literary market. While some of these biographies appeal mostly to mathematicians and historians, many appeal to the general public because of their historical narrative and the extraordinary characters they describe.

Perhaps the most popular biography of a mathematician is *A Beautiful Mind*, written in 1998 by Sylvia Nasar. The book tells the touching and tragic story of John Forbes Nash, a mathematical genius who was diagnosed with paranoid schizophrenia. Nash's stellar rise in the ranks of mathematics and his tragic fall provide a fascinating juxtaposition of mathematical genius and mental illness. Nash won the Nobel Prize in Economics in 1994 for his work on game theory.

Masha Gessen's 2009 book *Perfect Rigor: A Genius and the Mathematical Breakthrough of the Century* tells the story of Grigori "Grisha" Perelman, a Russian mathematician who proved the century-old Poincaré Conjecture in 2003. Perelman's proof was ensued by a distasteful affair that was fueled in part by Perelman's reclusive personality and eccentric behavior, as well as by the controversial conduct of his fellow mathematicians. Disappointed and disillusioned, Perelman withdrew from mathematics at a fairly young age. Perleman's story sheds light on both the often-overlooked world of scientific politics and intrigues and the colorful individuals who supply them. In 2006, Perelman was awarded the Fields Medal, the most prestigious prize in mathematics, for his ground-breaking work. He declined the award.

Memoirs and autobiographies of mathematicians also abound. Notable among these is G. H. Hardy's *A Mathematician's Apology*, a 1940 philosophical memoir that discusses the beauty of mathematics and the life of a mathematician, with its inevitable joys and sorrows. The memoir is highly influential among mathematicians and among laypersons who want a glimpse into the mind of a mathematician.

Satire

Because of the wide-ranging utility of mathematics, there exists a tendency to overuse it and to attempt to reduce social, political, and economic problems to mathematical equations. This attempt at oversimplifying serious societal problems raises the ire of some authors, who use their pens to strike back. By using *reductio ad absurdum* (reduction to absurdity, also known as "proof by contradiction"), a popular technique for proving mathematical theorems, authors may attempt to defeat mathematicians on the mathematicians' turf by showing the absurd results of the overuse and abuse of mathematics. The resulting satires describe these absurd results in an entertaining yet serious fashion.

In the 1726 novel *Gulliver's Travels*, Jonathan Swift describes the people of Laputa as obsessed with mathematics. They describe everything, even the beauty of women, in mathematical terms, and their constant political bickering reminds the narrator of the mathematicians of Europe. Swift made a similar—albeit less obvious—attack upon mathematical reductionism in his 1729 essay *A Modest Proposal* in which he proposes that the poor sell their children for food. By offering a preposterous yet simple solution to the problem of poverty, Swift was arguing that not all social problems can be solved by the use of deductive reasoning and mathematical thinking.

In his 1854 novel *Hard Times*, Charles Dickens criticizes an education system that is based solely on learning of facts, with no room for fancy, imagination, feelings, or arts. The ideal of fancy is embodied in Sissy Jupe, a poor schoolgirl who struggles with a curriculum obsessed with facts. Her frustration rises when she is asked to calculate the percentage of dead if 500 of 100,000 voyagers perished at sea. Jupe, confused and embarrassed, answers that the percentage is nothing, so far as the loved ones of those killed are concerned. Sissy Jupe is thus portrayed as humane and emotional, a person capable of seeing the people behind the numbers. Her schoolteachers, on the other hand, are emotionally paralyzed and see numbers as satisfactory descriptions of everything.

Famous Mathematical Problems

Stories of famous mathematical problems, whether open or solved, make for fascinating reading material for their mathematical content as well as for their narrative. By telling the tale of a particularly difficult mathematical problem, the author can braid an exposition of a difficult mathematical subject with stories about the history of the problem and the lives and personalities of famous mathematicians who tried to solve it.

The Riemann Hypothesis, the most famous open problem in mathematics, has inspired several books. In the 2003 book *Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics*, author John Derbyshire uses odd-numbered chapters for mathematical exposition and even-numbered chapters for discussion of the history of the problem and the people behind that history.

In the 1997 book *Fermat's Enigma*, author Simon Singh tells the story of Fermat's Last Theorem, a mathematical riddle that tantalized mathematicians for four centuries. Singh tells the tale of the famous conjecture, from its formulation by Fermat in 1637 to its proof by British mathematician Andrew Wiles in 1995. The book combines mathematical exposition with stories about the many mathematicians who struggled with the problem throughout the centuries.

Creativity in Mathematics and in Literature

Fiction writers often face the question "How do you come up with your ideas?" Similarly, mathematicians are often asked how they concoct the brilliant ideas that allow them to solve mathematical problems. Psychologists and neuroscientists have been trying to identify the sources of creativity for a long time. New discoveries are published regularly, and advances in technology, such as brain-mapping magnetic resonance imaging (MRI) machines, may shed further light on the subject. Until scientists elucidate the sources of creativity, the experiences and opinions of creative individuals writers, mathematicians, and others—may provide glimpses into the workings of creative minds.

In his 1846 essay *Philosophy of Composition,* author and poet Edgar Allan Poe analyzes the creative process he used to compose his famous poem *The Raven*:

> It is my design to render it manifest that no one point in its composition is referable either to accident or intuition—that the work proceeded step by step, to its completion, with the precision and rigid consequence of a mathematical problem.

Another glimpse into the creative process is provided by mathematician Jacques Hadamard in his 1945 work *The Psychology of Invention in the Mathematical Field*, in which he discusses the psychological processes of discovery and invention in mathematics. While Hadamard acknowledges the crucial role of conscious, logical thought, he contends that mathematical invention is a multi-step process in which intuition, inspiration, and unconscious thought are integral.

Poe's and Hadamard's views provide an interesting juxtaposition: while the poet describes his creative work in terms of a mathematical problem, the mathematician emphasizes creative processes that are usually associated with artistic work.

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Or Syd Amit

See Also: Curriculum, K–12; Movies, Mathematics in; Poetry; Science Fiction.

Loans

Category: Business, Economics, and Marketing.
Fields of Study: Algebra; Number and Operations.
Summary: Determining the terms of a loan so that they are fair but compensate for risk is a challenge of algebra.

Most people have personal experience with one or more types of loans, such as home mortgages, car loans, or home equity loans. In each case, the general format of the loan is the same: the lender provides temporary funds to a borrower, and the borrower repays these funds over a prespecified period of time, according to a prespecified pattern. As it is for any financial asset or liability, mathematics is a critical tool for determining the appropriate parameters of loans, including the periodic payment necessary for the borrower to completely pay off the loan by the end of the loan's life.

Mathematicians work on many problems related to loans. For example, individuals who take out large loans, like mortgages, are often required to purchase insurance for those loans. Actuaries use mathematical and statistical methods to assess lending risk to decide whether insurance is needed and how much. They also work on more complex problems related to interest rates and credit, such as deciding what constitutes usury (unreasonably high interest rates) for loans whose yield rate is not fixed or determining the reliable predictors of credit risk.

History

Loans appear to have been a part of economic activity ever since economies began to become sophisticated. In response to certain historical unfair lending practices, a number of proscriptions against usury were recorded in ancient sources, such as the Old Testament, and works by Aristotle and Tacitus. More generally, an active lending market is important to an economy, as it facilitates the availability of funds for investment.

Loans, like other financial instruments, are two-sided transactions. There is a lender and there is a borrower, and cash flows are made between them—what one party pays, the other receives. Algebraically, this process is usually reflected by identifying the cash flows as either positive or negative; a positive cash flow for the lender would be a negative cash flow of the same magnitude for the borrower, and vice versa. For the lender, the loan transaction is essentially an investment, and thus an asset. For the borrower, the loan represents a liability and ultimately needs to be paid back.

The most common method in the twenty-first century of paying off a loan is via amortization, in which interest and a portion of the original borrowed principal are paid back in each of the periodic payments. There are a number of parameters associated with the typical amortization loan, including the following:

- The original amount borrowed (B)
- The length or term (n) of the loan (for personal loans, such as mortgages and auto loans, the length of the loan is typically

For the lender, a loan is basically an investment and an asset. For the borrower, the loan is a liability and needs to be paid back.

measured as the number of monthly payments to be made by the borrower to the lender; theoretically, however, payments can be made according to any schedule, such as weekly, annually, or uneven periods of time)

- The periodic (for example, monthly) interest rate (i) on the loan, which determines the amount of interest paid by the borrower to the lender
- The periodic (for example, monthly) payment (R) made by the borrower to the lender

In the most common type of amortized loan, the payment made by the borrower each period is constant over time. Each payment consists of two components: an interest payment and a partial principal repayment. Across the life of the loan, the sum of all of the n partial principal repayments is equal to the total original amount borrowed, B. As each payment is made, the outstanding balance of the loan is lessened by the amount of the partial principal repayment in that payment.

The effect of this approach is that, while each payment R is of the same size, the split between the interest component and the principal component of each payment changes over time. More specifically, as time moves on, the principal component increases and the interest component decreases. This is because the indebtedness (the outstanding balance) of the loan decreases over time, and thus the periodic interest charged on the loan (which is equal to the interest rate multiplied by the loan's outstanding balance) also decreases over time.

To illustrate, suppose that $1,000 is borrowed, and this four-year loan is to be paid off with four equal annual payments of R, one at the end of each of the four years during the life of the loan. Suppose that the effective annual interest rate $i = 0.10$, or 10%. In this situation, the annual payment R can be determined by the formula

$$R = \frac{iB}{1 - \left(\frac{1}{1} + i\right)^{n}}$$

where $i = 0.10$, $B = \$1{,}000$, and $n = 4$. Thus, $R = \$315.47$.

This value for R can be verified by considering the impact of each annual payment separately. For example, consider the first payment of R. During the first year, the borrower incurs interest charges of 10% of the outstanding balance at the beginning of the year, or $100. Thus, $100 of the $315.47 first payment covers the interest for borrowing the original $1,000 during the first year; the remaining $215.47 of the first payment then serves to partially pay off the loan, leaving an outstanding loan balance, or indebtedness, of $\$1{,}000 - \$215.47 = \$784.53$. During the second year, the borrower incurs interest of 10% of that new outstanding balance, or $78.45. That portion of the second payment of R covers this interest, and the remainder ($\$315.47 - \$78.45 = \$237.02$) serves to further pay down the loan. Thus, after the second payment, the borrower has loan indebtedness of $\$784.53 - \$237.02 = \$547.51$. Continuing this process through the fourth and final payment will reveal that, after that final payment, the original $1,000 loan has been completely and precisely paid off.

Occasionally, people will pay off installment loans before their final due date by making early payments or paying slightly more than is due at each installment. In this case, they may be entitled to a rebate on some of the originally computed interest. Rebates can be figured using several methods, including variables such as how the interest was originally computed and the way in which the regular and extra payments were divided between principal and interest. The actuarial method of calculation is generally more favorable to the borrower than rebates calculated under other methods, such as the Rule of 78s.

There are other ways of paying off loans; for example, paying the interest regularly and then paying off the entire principal at the end of the loan term. In fact, this process is essentially how a specific type of financial instrument, a bond, works. When corporate or governmental entities issue bonds, they are borrowing money. More precisely, they are borrowing an amount equal to the price of the bond from the investor or investors who purchase the bond. The issuing organization pays periodic interest to the investors (in the form of coupons) and at the expiration date of the bond pays back to the investors a lump sum, known as the "redemption value."

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Rick Gorvett

See Also: Accounting; Home Buying; Money; Pensions, IRAs, and Social Security.

Logarithms

See *Exponentials and Logarithms*

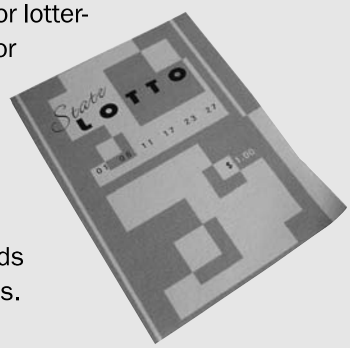
Lotteries

Category: Games, Sport, and Recreation.
Fields of Study: Data Analysis and Probability; Number and Operations.
Summary: A successful lottery depends on assuring the randomness of its selections and maintaining the perception of fairness.

History of Lotteries

In Athens during the fourth and fifth centuries B.C.E., lotteries were used to select political office holders. In Rome, the emperor Gaius Julius Caesar Augustus rebuilt his empire's infrastructure with money raised through lotteries. Lotteries also helped to fund the building of the Great Wall of China. Governments throughout much of Europe, notably in England and France, have raised essential funds with lotteries over the past few centuries.

George Washington supported lotteries as a means of funding transportation and educational systems in a fledgling nation. In the United States in the twenty-first century, most states sponsor lotteries. The jackpots for Powerball and for Mega Millions, two popular multistate lotteries, sometimes run into the hundreds of millions of dollars.



Lotteries, which can be thought of as games that involve a winner selected by chance, have played an important role in the development of societies for more than 2000 years. Lotteries can include those run by political bodies, like states, where the winnings are money, or those run by a sports entity, like the National Basketball Association (NBA) Draft Lottery, where teams get to select new members. The U.S. government runs a Green Card Lottery program and selects winners using a computer-generated drawing. In most lotteries, very few people win anything substantial, and the purchase of a lottery ticket usually amounts to an unfair bet, in that the price of a single ticket is less than the average payoff across all tickets.

Nevertheless, lotteries are quite popular and consequently can raise substantial funds or allocate a small number of goods, services, or sought-after players among a large number of people or teams. The mathematical concepts of "randomness" and "expected value" are fundamental to the operation of lotteries and perceptions of fairness. Probability methods, especially combinations and permutations, are used to compute the odds or chances of winning, given certain conditions.

Distribution of Winnings

If lottery commissions somehow redistributed all of the ticket sale money into winnings for each game, then, at least in a cumulative sense, the purchase of lottery tickets would constitute fair bets—the average payoff would equal the average ticket price. An example of this would be if each player paid a dollar for a ticket that went into a hat, and then a winning ticket was chosen from the hat, with the purchaser of that ticket winning all of the money that had been collected. The reality is usually more complicated. Typically, multiple players can purchase the same ticket (thus having to share the winnings if that ticket is drawn) or the winning ticket might not have been purchased by anyone. In the latter case, the money is rolled over to the next game, which might be better than fair for the players if the jackpot is larger than the total investments for that week. Usually, however, the game is worse than fair for the players, primarily because the state (or whatever organization is hosting the lottery) keeps a portion of the proceeds. The state of Wisconsin, for example, pays out slightly more than half of its lottery revenue as winnings; most of the remaining revenue is used for property tax relief. Other common uses for funds among state-run lotteries include education, transportation, construction, and, ironically, help for compulsive gamblers.

Calculating the Chances

Regardless of the question of fairness, a lottery is clearly disadvantageous to almost every player. Nevertheless, lotteries attract large numbers of players because people are willing to pay a small amount of money for the small chance of winning a fortune. Powerball, operated by the Multi-State Lottery Association, provides a good illustration. There are nine ways to win with a $1 Powerball ticket; in four of these ways, the winnings are less than $10. The probability of winning something is about 1:35, but the probability of winning anything more than $100 is less than 1:700,000. The probability of winning the big jackpot is 1:195,249,054, as can be verified with some basic rules of counting.

Each Powerball ticket consists of five distinct numbers, 1–59, together with a "Powerball" number, 1–39. To determine the winning ticket, five balls are randomly drawn from a drum containing white balls numbered 1–59, and then one ball (the Powerball) is drawn from a drum containing red balls numbered 1–39. The winning ticket must match all five white balls (irrespective of the order in which they are drawn) as well as the red ball. The probability of winning the jackpot is 1 divided by the number of distinct possible tickets (the number of possible outcomes of the drawing). There are 59 possibilities for the first white ball; for each of those there are 58 possibilities for the second white ball. Continuing, there are 57 possibilities for the third, 56 for the fourth, and 55 for the fifth. If the order of drawing these balls were relevant, a total of $59 \times 58 \times 57 \times 56 \times 55 = 600{,}766{,}320$ ways of drawing the white balls would be counted. This number, however, is much larger than the true probability, since the order of the drawings is not relevant. For instance, the possible outcome 2, 4, 8, 16, 32 should be counted once; but among the aforementioned count of 600,766,320, this collection of balls appears $5 \times 4 \times 3 \times 2 \times 1 = 120$ times (because ball 2 could be listed in any one of five positions, and then ball 4 could be listed in any of the remaining four positions, and so on). The earlier count should be divided by 120 in order to correct for this systematic overcounting. Finally, incorporating the possibilities for the red ball, the result should be multiplied by 39. This calculation yields the 195,249,054 possible jackpot tickets.

Winning Strategies?

One way to improve the chances of winning is to buy more tickets. A properly run lottery does not lend itself to winning strategies. For instance, the Powerball drawings are videotaped and audited, and the equipment is stored in a vault and meticulously tested for nonrandom behavior. So bribery would be difficult, and knowledge of historical winning numbers would most likely be pointless. One could ensure a win by purchasing all possible tickets (an attractive option if the jackpot has grown very large because of rollovers), but this would require a huge initial investment, and it would be quite difficult from a practical standpoint to orchestrate the purchase. Further, if multiple people purchased the winning ticket, then the jackpot would be divided among them. Commonly chosen tickets involve previous winning combinations, numbers below 32 (because they could represent birthdays or other significant dates), and simple combinations such as 1, 2, 3, 4, 5, 6. The one bit of control a lottery player does have is to avoid such combinations to reduce the likelihood of splitting the jackpot in the event of a win.

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John Beam

See Also: Betting and Fairness; Expected Values; Military Draft; Probability.

Lovelace, Ada

Category: Communication and Computers.
Fields of Study: Connections.
Summary: Augusta Ada King, Countess of Lovelace, is known as the first computer programmer.

Charles Babbage, the inventor of an early computer, called Augusta Ada King, Countess of Lovelace, "the enchantress of numbers." It is through her association with Babbage and his calculating machines that King influenced the history of mathematics. Ada King is widely regarded as the world's first computer programmer for material in her notes on Babbage's invention, the analytical engine. Her contribution to the development of mathematics was significant in that she created the first set of computer instructions, or algorithm, and anticipated many features of modern computers.

Early Life and Eductation

King was born Augusta Ada Byron on December 10, 1815, in London, England. She was the only child of George Gordon Byron (Lord Byron), the renowned romantic poet, and his wife Annabella Milbanke Byron. Her parents' marriage was stormy and brief; they separated two months after King's birth, and her father left the country for good soon thereafter. Because of the strained association with her mother, Lord Byron never had a relationship with King, though he made several mentions of her in his poetry.

Her parents' difficult relations ultimately influenced King's exposure to mathematics. Lady Byron required her daughter to study mathematics as a way to discipline the passionate side of her nature and to eliminate fanciful tendencies suspected to have been inherited from her poet father. Over the years, King studied mathematics with a variety of tutors and mentors, including mathematicians Mary Somerville and Augustus De Morgan.

In 1835, she married William King. He became the Earl of Lovelace in 1838 and King was thereafter the "Countess of Lovelace." The couple had three children; Byron in 1836, Anne Isabella in 1837, and Ralph in 1839.

Work with Calculating Machines

King's work with Charles Babbage began after the birth of her last child. Charles Babbage was the inventor of two calculating machines. The first, called the difference engine, was created to solve a specific problem involving polynomial equations and was displayed in his drawing room to amaze guests, along with a silver lady automaton. Babbage's other machine, the analytical engine, was never built; however, Babbage created a number of plans and drawings for the machine that excited interest in the scientific community. In 1842, Italian mathematician Luigi Menabrea published a description of the analytical engine. Menabrea's paper was

Margaret Carpenter's portrait of nineteenth-century British mathematician Ada Lovelace.

written in French, and King translated this work into English. Encouraged by Babbage, she compiled a set of seven original notes meant to complement Menabrea's explanations. The translation and notes were published in 1843. These notes constitute King's only scientific publication, and—though brief—they established her modern reputation as the first computer programmer.

King began the notes by contrasting the difference engine, which was well known to the public, and the analytical engine, which would be a much more functional and flexible machine. The most significant difference between the two was that the analytical engine would be programmable, whereas the difference engine was not. In her notes, King anticipated many functions of modern computers and computer programs. She described the machine's projected abilities to utilize symbolic information, to repeat a series of steps multiple times based on a single set of instructions (as in a modern loop), to give intermediate as well as final results, and even to store information meant to explain to the machine's user what was taking place rather than to give instructions to the engine (a form of programming comments). With regard to the engine's capabilities, she said, "The Analytical Engine weaves algebraic patterns, just as the Jacquard loom weaves flowers and leaves." King also predicted that the analytical engine would facilitate faster and more accurate solutions to difficult computing problems, and that it would enable solution of problems theretofore insoluble because of computational complexity. She even foresaw such modern ideas as artificial intelligence and computer music.

King's reputation as the first computer programmer comes from note "G," the final note in the collection. In this section, she gave a detailed example of the functionality of the analytical engine, describing how the engine would compute the Bernoulli numbers—the first computer program.

Ada died from uterine cancer on November 27, 1852, at the age of 36 and was buried beside her father. In reflecting on her own work, she is quoted as saying, "I never am really satisfied that I understand anything; because, understand it well as I may, my comprehension can only be an infinitesimal fraction of all I want to understand about the many connections and relations which occur to me . . ." Her work was largely overlooked for 100 years after her death until she was mentioned in a 1952 book on the history of computing. Since then, many books and articles have been written on her life and work, including a comic in which she and Charles Babbage are portrayed as a crime-fighting team. In 1980, the U.S. department of defense named a computer language "Ada" in her honor. There is also an international day of blogging named for her—Ada Lovelace Day—whose purpose is to draw attention to women in technology.

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Kady Schneiter

See Also: Calculators in Society; Software, Mathematics; Women.

M

Magic

Category: Arts, Music, and Entertainment.
Fields of Study: Algebra; Number and Operations; Representations.
Summary: Many tools of mathematics and mathematical properties lend themselves to tricks.

Mathematical magic may seem to be either redundant or an oxymoron. Many people equate mathematical processes or theorems with magic, such as the magic of logarithms or when mathematicians are thought to have magical powers with numbers floating around their heads in movies and on television. Others view it as a collection of sterile algorithms absent of any signs of magic. However, the realm of mathematical magic counters both of these views, blending together elements from mathematics as a structure with an element of surprise akin to magic. Invoking mathematics of great breadth—arithmetic, number theory, algebra, geometry, and topology—the mathematical magician's "tools" are numbers, cards, string, dice, dominoes, calendars, watches, coins, dollar bills, and rubber bands.

Arithmetic Magic

Arithmetic magic depends on the clever use of divisors, multiples, and basic operations. As an example, ask a friend to write down his or her age. Then, add the age on the friend's next birthday. Add 9 to this sum. Divide that sum by 2. Finally, subtract the friend's current age. Then, magically announce that the answer is 5. It will always be 5, thanks to mathematics. For example, if the friend's age is 24, the friend would calculate: $24 + 25 = 49$; $49 + 9 = 58$; $58 \div 2 = 29$; $29 - 24 = 5$. In fact, with a slight modification of the first calculation (add one more than your starting number), your friend could start with any number, such as 3.5, π, or even -72.3, and the result will still be 5.

Card and Dice Magic

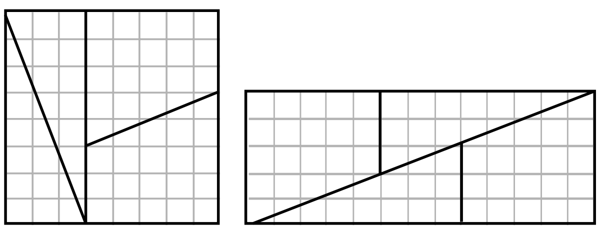
Mathematical magic using playing cards capitalizes on their properties—numerical values 1–13, four suits, two colors, front-back orientation—as well as the fact that a deck of cards can be both ordered and shuffled. As another example with a friend, shuffle a deck of cards, hand it to your friend, and then casually write something on a piece of paper, which is folded and set aside. Ask your friend to deal the top 12 cards face-down on the table and then touch any four cards, which you turn over. Group the other eight dealt cards and return them to the bottom of the card deck. Suppose the four face-up cards are a 3, 5, 7, and King (where all face cards are to be treated as a 10). Taking the deck, deal more cards on top of each card to make 10, counting out loud the sequences (for example, 3, 4, 5, 6, 7, 8, 9, 10 and 5, 6, 7, 8, 9, 10 and 7, 8, 9, 10). Because the

King has the value 10, no cards are dealt on top of it. Hand the deck to your friend, ask him to add the values of the original four cards (3 + 5 + 7 + 10 = 25) and then count out that number of cards (25 cards). When the last card is turned over, reveal that it matches your prediction written on the paper.

Mathematical magic using dice depends on the fact that the pips on the opposite sides sum to 7. As an example of a trick, with your back turned, ask a friend to throw three dice on a table and add the top faces (for example, 2 + 4 + 5 = 11). Then ask the friend to pick up any one of the dice and add its bottom number to the current sum (for example, opposite the 2 on the first dice is a 5, so 11 + 5 = 16). Finally, ask the friend to roll that die again, and add the new top face to the current sum (for example, 16 + 6 = 22). Turn around and announce that you have no way of knowing which die was rolled twice, pick up the 3 dice, shake them in your hand, and magically announce your friend's final sum.

Geometric Magic

Mathematical magic involving geometry or topology is similar to actual tricks performed by magicians, such as the Chinese Linking Rings, Magical Knots, and Houdini Escapes. As a simple example, start with an 8×8 grid square and draw 3 lines to subdivide it as shown. Cut along the 3 lines, producing 4 pieces, which can be rearranged to form the 5×13 solid rectangle. What is the magic? The initial square with an area of 64 square units has been transformed into a rectangle with an area of 65 square units.



The Magic Revealed

Why do the previous four tricks work? The first arithmetic trick is explained using algebra, where N is the starting number, shown as

$$\frac{N+(N+1)+9}{2}-N=5\,.$$

For the second trick, it is important that the card you write on the paper matches the bottom card on the shuffled deck at the start. The trick becomes automatic, since the 4 face-up cards and the 8 cards placed on the bottom as part of the deck essentially force your "secret card" to now be in the 40th position in the original deck. The counting mechanism forces this card to be the card revealed. For the third trick, determine the final sum by adding 7 to the sum of the 3 top faces seen as you pick up the dice. Finally, for the fourth trick, the magical effect is because of the apparent diagonal of the rectangle, as it is not a straight line but is a "thin" parallelogram with an area of 1 square unit. To show this mathematically, the two line segments forming the diagonal have differing slopes of 3/8 and 2/5. As a twist to this trick, note that the square had side length 8 while the rectangle had side lengths 5 and 13, where the numbers 5, 8, 13 are part of the Fibonacci sequence. In fact, any three ordered numbers (different) in this sequence produces this magical effect.

Magic Squares, Cubes, and Circles

In any discussion of mathematical magic, one must mention magic squares, cubes, and circles. First, subdivide a square into smaller squares, each containing a number. The magical effect is that the numbers in each row, each column, and each diagonal all sum to the same constant value.

8	1	6
3	5	7
4	9	2

This common example is the "Lo Shu" magic square with a constant sum of 15, being part of the legend (650 B.C.E.) of the Chinese Emperor Yu finding a turtle with the same square inscribed on its back. Also, the German artist Albrecht Dürer inserted a famous magic square in his painting *Melancholia*, with its constant sum of 34 and the painting's date of 1514 included in the bottom row of cells.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Historically, mathematics and magic are intertwined, back to the Pythagoreans who revered certain numbers with a special mysticism. This "aura" of numbers having special magical effects surfaced often throughout history in the form of special primes, special products, and special properties. For example, one can not dismiss the magic of numbers when considering these number patterns, all evoking a feeling of "Behold!"

$$0 \times 9 + 1 = 1$$
$$1 \times 9 + 2 = 11$$
$$12 \times 9 + 3 = 111$$
$$123 \times 9 + 4 = 1111$$
$$1234 \times 9 + 5 = 11111$$
$$12345 \times 9 + 6 = 111111$$
$$123456 \times 9 + 7 = 1111111$$
$$1234567 \times 9 + 8 = 11111111$$
$$12345678 \times 9 + 9 = 111111111$$
$$123456789 \times 9 + 10 = 1111111111$$

$$1 \times 8 + 1 = 9$$
$$12 \times 8 + 2 = 98$$
$$123 \times 8 + 3 = 987$$
$$1234 \times 8 + 4 = 9876$$
$$12345 \times 8 + 5 = 98765$$
$$123456 \times 8 + 6 = 987654$$
$$1234567 \times 8 + 7 = 9876543$$
$$12345678 \times 8 + 8 = 98765432$$
$$123456789 \times 8 + 8 = 987654321$$

$$9 \times 9 + 7 = 88$$
$$98 \times 9 + 6 = 888$$
$$987 \times 9 + 5 = 8888$$
$$9876 \times 9 + 4 = 88888$$
$$98765 \times 9 + 3 = 888888$$
$$987654 \times 9 + 2 = 8888888$$
$$9876543 \times 9 + 1 = 88888888$$
$$98765432 \times 9 + 0 = 888888888$$

$$12345679 \times 9 = 111111111$$
$$12345679 \times 18 = 222222222$$
$$12345679 \times 27 = 333333333$$
$$12345679 \times 36 = 444444444$$
$$12345679 \times 45 = 555555555$$
$$12345679 \times 54 = 666666666$$
$$12345679 \times 63 = 777777777$$
$$12345679 \times 72 = 888888888$$
$$12345679 \times 81 = 999999999$$

$$1^2 = 1$$
$$11^2 = 121$$
$$111^2 = 12321$$
$$1111^2 = 1234321$$
$$11111^2 = 123454321$$
$$111111^2 = 12345654321$$
$$1111111^2 = 1234567654321$$
$$11111111^2 = 123456787654321$$
$$111111111^2 = 12345678987654321$$

Martin Gardner claimed in his 1956 book *Mathematics, Magic and Mystery* that mathematical magic has a unique but limited audience. In his opinion, mathematicians reject mathematical magic as trivial and dull, while magicians reject it as pseudomagic. The true audience is therefore those who appreciate mathematical recreations implemented in a creative, entertaining context. A master of such presentations is Arthur Benjamin, a combinatorics professor and professional magician, who has appeared on many radio and television programs, such as the widely popular political satire program *The Colbert Report*, and been profiled in entertainment, news, and scientific publications. His popular demonstrations and explanations of methods for rapid mental calculations, which have been enjoyed by audiences of all ages and cultures worldwide, as well as his many popular books on mathematical magic would appear to belie Gardner's claim.

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Jerry Johnson

See Also: Dice Games; Mathematical Puzzles; Optical Illusions; Puzzles.

Mapping Coastlines

Category: Space, Time, and Distance.
Fields of Study: Algebra; Geometry; Measurement; Number and Operations; Representations.
Summary: Fractals can be used to help map coastlines.

A map is an infographic representing an area. Maps use symbols to represent objects or scale renderings of spatial features. The science of mapmaking is called "cartography." The mapping of coastlines is important for navigation and for determining the boundaries of territorial waters, which are measured as fixed distances from coastlines. Coastline cartography presents special mathematics because of connections with several actively developing branches of mathematics, including fractal theory.

Traditional Mapmaking Mathematics and Analytical Cartography

Several mathematical features of maps have been used for centuries. Orientation is the correspondence of the map's coordinate system with directions of the terrain. When three-dimensional objects are depicted in two-dimensional media in the process called "projection," such as maps of Earth on paper, some areas are necessarily distorted. Ratios are used to map objects to scale, including the systematic changes in the ratios in different parts of two-dimensional maps using projections.

With the increasing use of computers in cartography, several new areas of modeling and computation expertise have appeared over the last few decades. These new, mathematics-rich cartography areas include computer-based geographic information systems, interpolation, and photogrammetry. Collectively, these areas of expertise are called "analytic cartography."

Types of data in analytic cartography include numerical data, such as elevation values, images or photographs, and attribute data, like tags identifying features near particular coordinates. All data are dynamically linked and manipulated in a geographic information system; for example, a projection map can be generated from a series of aerial photos, rotated and zoomed. In contrast, paper maps do not allow dynamic data connection and are static, which limits the possibility for mathematical modeling and experimentation with variables. Geographic information systems may also include remote sensing data; for example, displaying changes in coastlines in real time as tides change.

Analytic Cartography and Coastline Changes

Because coastlines change a lot compared to other map features, from tides and floods, analytic cartography that allows for rapid analysis of real-time data is especially valuable in mapping coastlines. Using data from previous events and mathematical models within geographic information systems, cartographers can simulate floods, tsunamis, or effects of rising water levels from global warming on existing coastlines. The same software can be used to predict effects of terrain modification projects over time.

Modeling coastline changes is more complex than simply mapping higher or lower water levels onto the existing coast elevation data. The models also have to take into account erosion, deposits of matter by rivers and rainfall runoff, changes in river basins, and other systemic factors.

Fractal Dimension and the Coastline Paradox

A fractal is a self-similar structure that looks the same at all zoom levels. Coastlines, while not perfectly fractal (not having infinite number of levels), exhibit enough fractal features to make some mathematics of fractals applicable. The famous 1967 paper by Benoît Mandelbrot, "How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension" started this line of thought, though the term "fractal" appeared later.

An important feature of a coast is its fractal dimension (a measure of how long the coast is compared to the area it occupies). Because the area has two coordinate dimensions and the length has one, theoretically, a curve filling a unit of area can have infinite length. Fractal dimension is a way to compare different coasts, from straight coastlines that have the fractal dimension of 1 to increasingly complex, space-filling coastlines that have higher fractal dimensions between 1 and 2. In Mandelbrot's paper, the relatively smooth coast of South Africa has the fractal dimension of 1.02 and the highly irregular (long for its area) coast of Britain has the fractal dimension of 1.25.

The length of the coast and its fractal dimension depend on the units of measure. Because smaller units allow the cartographer to capture more detail of the coastline, measuring with smaller units produces higher total length. This is definitely not true about measuring straight lines, and thus it is called the "coastline paradox."

Randomness and Pattern

Perfectly self-similar fractals created by mathematical models have limited applicability to coastline mapping because real coasts are irregular. Therefore, some mathematical models include the element of randomness in creation of factors and use statistical methods to compute fractal dimensions. For example, one method for generating random fractals is called "random midpoint displacement," produced by using the following cyclic algorithm repeatedly:

- *Step 1*: Start with a straight line.
- *Step 2*: Displace the midpoint randomly, perpendicular to itself, by the distance within the given ratio to its length.
- *Step 3*: Apply Steps 1 and 2 to the segments resulting from the previous steps.

A similar method can be applied to generating elevation of areas. In this case, the algorithm starts with a rectangle, displaces its midpoint, and then is applied to the four rectangles formed by the lines parallel to the original rectangle's sides and crossing at the midpoint.

Because these methods are computationally intensive, as the number of computations at each step grows exponentially with the number of cycles, their development coincides with increases in computing power. In addition to mathematical modeling of existing coasts, these methods are used to generate fictional terrain for computer games, virtual worlds, and digital artworks.

Coast-Mapping Satellites

Several government and private projects connect real-time satellite data to specialized coastline geographic information systems. This connection provides either real-time or within-minutes data for ship navigation charts, environmental hazards (like oil spills in harbors), and natural disaster data (like tracking tsunamis).

Satellite mapping has to use methods beyond optical imagery because data have to come during the night as well as in cloudy conditions. Coast-mapping satellites use radar sensors that do not depend on light. These sensors measure changes in reflected radar pulses. Rougher surfaces reflect differently from water, allowing for relatively precise mapping of the coastline.

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See Also: Curves; Data Analysis and Probability in Society; Maps; Measurements, Area; Measuring Tools; Randomness; Tides and Waves.

Maps

Category: Travel and Transportation.
Fields of Study: Geometry; Measurement.
Summary: Scales and projections are used to display geographic features on maps.

The word "map" is the name given to any representation of the Earth's features—natural and artificial—usually on a plane using a given scale and map projection. In scientific and mathematics applications, the term "map" is more broadly interpreted. The purpose of a map is to register and transmit information about those features and the spatial relations between them.

A common characteristic of all maps is that they are reduced and conventional representations of reality, which makes them significantly different from an aerial photograph. While an aerial photograph depicts all the physical objects that a sensor could detect and register (and only those), a map is a selection of natural and artificial objects, visible and invisible, chosen to fit the cartographer's purpose and the limits imposed by the available space. These objects are represented on maps in a conventional way by means of symbols; this is not the case with photographs, in which they are depicted by the visual image they present when viewed from above by the sensor. The symbols in a map are designed to categorize features by type and to optimize the document's legibility. Very often, their size is not proportional to the size of the objects they represent. For example, roads are symbolized by lines of variable thickness and pattern, often much larger than the corresponding width of the actual roads, since representing them to exact scale would often make them too thin, even invisible. In other cases, such as with cities, features are symbolized by punctual symbols whose color and shape depend on the classification scheme chosen (such as administrative status or population).

Maps are usually classified in three main categories: general reference maps, thematic maps, and charts. A general reference map depicts generic geographic information of various types considered useful to a large spectrum of users. This information may include topography, political and administrative borders, and land cover. The best example of a general reference map is the topographic map. A thematic map, on the other hand, represents the geographic distribution of a specific theme or group of themes such as geological features, population, or air temperature. A chart is a special type of map designed to support navigation, either maritime (with nautical charts) or aerial (with aerial charts).

History

Maps were first made by the ancient civilizations of Europe and the Middle East several centuries before the Common Era. One of the oldest known is a Babylonian clay map of the world c. 600 B.C.E., now kept in the British Museum. Though it is documented in the testimony of Ptolemy of Alexandria (c. 90–169 C.E.) and others that maps were drawn in Greece as early as the seventh century B.C.E., none are known to have survived. However, several medieval manuscript maps have survived that represent the ecumene (the known inhabited part of the world around the Mediterranean basin). Few had any practical purpose, and most were symbolic representations inspired by religion and myth rather than by reality. In his *Geography*, published for the first time in the second century C.E., Ptolemy describes three map projections in detail and presents a list of more than 8000 places in the ecumene, defined by their latitudes and longitudes.

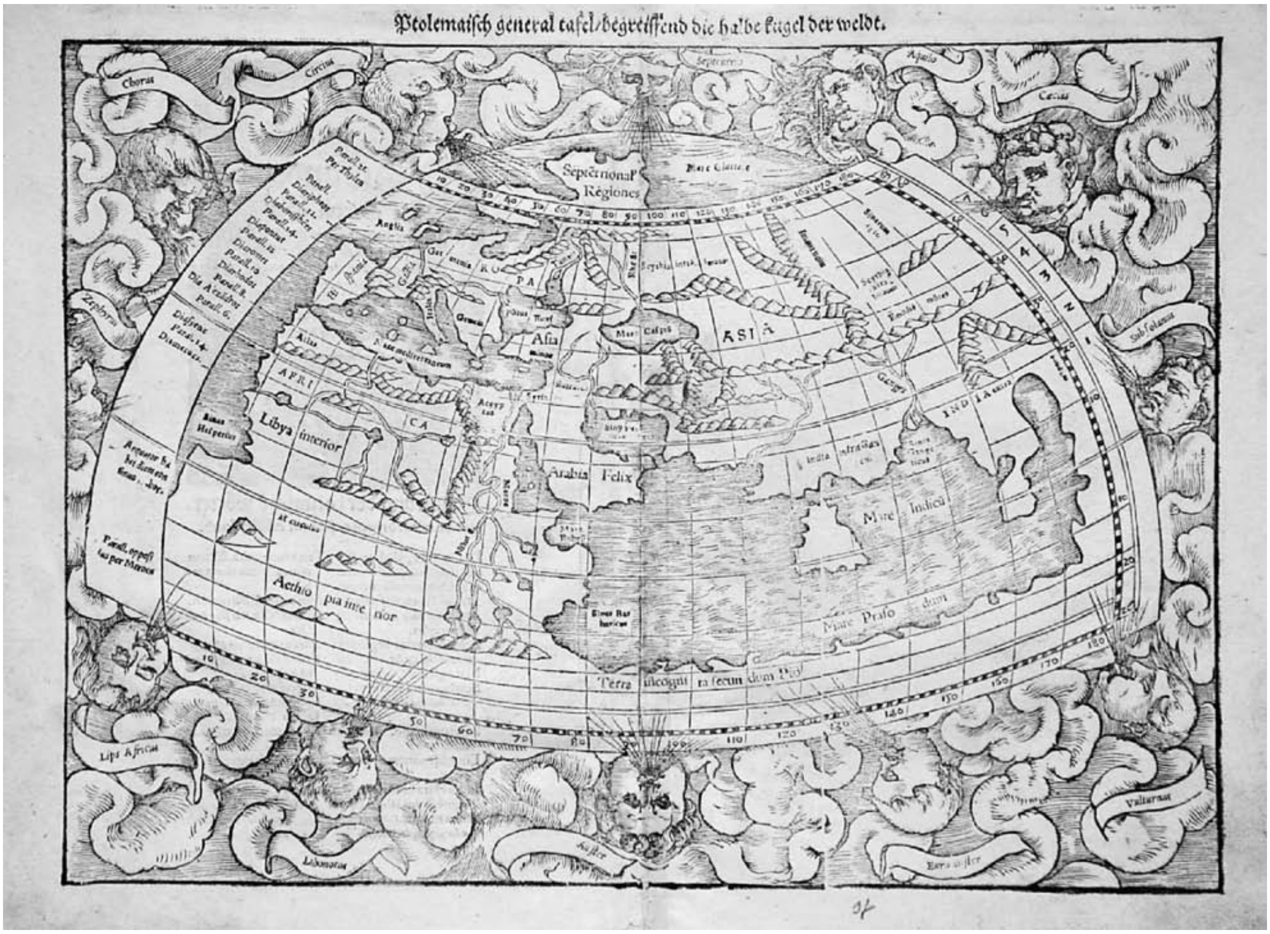
This list permitted others to redraw the maps that may have accompanied the original text once the work was translated into Latin and disseminated throughout Europe during the fifteenth century. The publication of several editions of *Geography* did much to bring about the rebirth of scientific cartography. By this time, nautical charts had already been used to navigate in the Mediterranean for at least two centuries. And while terrestrial cartography quickly adopted the geographic coordinates and map projections proposed by Ptolemy, nautical charts remained based on the magnetic directions and estimated distances observed by pilots at sea. Still, these representations were of astonishing accuracy and detail compared with the traditional maps of the time.

It is now known that the first nautical charts, commonly known as "portolan charts," were constructed in the first half of the thirteenth century, probably in Genoa, after the introduction of the magnetic compass and the adoption of the decimal system in Europe. This basic model continued to be used in nautical cartography for a long time, though much improved by the introduction of astronomical navigation during the fifteenth century. The resulting modality, based on observed latitudes and magnetic directions, became known as the "latitude chart" (or "plane chart") and played a fundamental role in the discoveries and maritime expansion periods. In 1569, an important world map specifically conceived for supporting maritime navigation was constructed by the Flemish cartographer Gerard Kremer (1512–1594), better known by the Latinized name of "Gerardus Mercator." Contrary to traditional portolan charts, this map was based on the latitudes and longitudes of places and represented all rhumb lines (lines of constant course) as straight segments making true angles with the meridians.

Though Mercator did not explain how the planisphere was made, a geometric method was most likely used. The mathematics of the projection is not trivial and its formalization had to wait until after calculus was developed, more than one century later. As for its full adoption as a navigational tool, that did not occur until the middle of the eighteenth century, when the marine chronometer was invented and longitudes could finally be determined at sea.

Mathematical Cartography

Maps may depict only a small part of the whole surface of the Earth. The word "scale" means the quotient between a length measured on a map and the corresponding distance measured on the Earth's surface. Because it is not possible to represent the spherical surface of the Earth in a plane without distorting the



A printed map from Ptolemy's description of the ecumene by Sebastian Münster, Geographia Universalis (1540). The term "ecumene" was used in Greco-Roman times to refer to the known portions of the Earth.

relative position of the places (and thus, the shape of all objects), the scale of a map is not constant, always varying from place to place and, in the generality of cases, also with the direction. In large-scale maps, like the plant of a city or the topographic map of a small region, these distortions can be ignored and the scale considered constant for most practical purposes. That is not the case when a large area of the Earth's surface is represented, like in a planisphere or a map of a whole continent. Here shapes may be strongly deformed and the scale varies significantly from place to place. Measurements made on those maps with the purpose of evaluating distances between places, using their graphical or numerical scale, are only approximations, as the scale strictly applies only to certain parts of the maps (like the central meridian or parallel), and their use in the other regions may lead to very large errors.

"Map projection" refers to any systematic way of representing the surface of the Earth on a plane. The process consists of two independent steps. First, one has to replace the irregular topographic surface, with all its mountains and valleys, with a simpler geometrical model, usually a sphere or an ellipsoid where a system of geographic coordinates (latitude and longitude) is established. Second, one has to project that model onto a plane surface. This step may be accomplished by some geometric construction or by a mathematical function that transforms each pair of geographic coordinates latitude (j) and longitude (l) into a pair of Cartesian coordinates x and y, defined on the plane. Depending on the purpose of the map, there are many different map projections to choose from. Knowing that none of them conserves the relative position of all places on the surface of the Earth, the choice is usually driven by the type of geometric property one wants to preserve. For example, equivalent or equal-area projections conserve the relative areas of all objects and are typically used in political maps. Conformal projections conserve the angles around any point on the map (the scale does not vary with direction), as well as the shape of small objects, and are utilized in nautical charts and topographic maps. Equidistant projections conserve the scale of certain lines and are used whenever one wants to preserve distances measured along those lines. This is the case of the azimuthal equidistant projection, where distances measured from the center of the projection along all great circles are conserved. This property is useful, for example, for quickly determining the distance of any place in the world measured from a chosen location.

However, it is not possible for a map projection to have all these properties at the same time, and the conservation of some properties is usually accompanied by significant distortions of the others. A significant example is the Mercator projection (which is conformal), where all rhumb lines are represented by straight segments making true angles with the meridians. However, the scale increases with latitude in this projection, strongly affecting the proportion of the areas. The branch of cartography dealing with map projections is known as "mathematical cartography." Though some map projections have been well known since remote antiquity, when they were often used for representing the sky, a more formal approach became possible only after the development of calculus. The most important contributions in the formalization of mathematical cartography were those of Johann Heinrich Lambert (1728–1777), Joseph-Louis Lagrange (1736–1813), Carl Friedrich Gauss (1777–1855) and Nicolas Auguste Tissot (1824–1897).

Computers and geographic information systems have made it possible for previously unforeseen numbers of users to produce good-quality maps tailored to their specific needs and at a reasonable cost. They also allow scientists and mathematicians to map increasingly complex systems and concepts, such as the universe and the World Wide Web. They can also often render in three dimensions and beyond. In mathematics, maps can be used to alternatively express functions or connect mathematical objects. In conceiving those systems, as well as in acquiring the geographic data necessary to construct the representations within, mathematics continues to play a fundamental role.

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See Also: Calculus in Society; Functions; Geometry of the Universe; Mapping Coastlines; Marine Navigation; Social Networks.

Marine Navigation

Category: Travel and Transportation.
Fields of Study: Algebra; Geometry; Measurement; Number and Operations.
Summary: With the help of mathematicians, sailors throughout history have been able to devise ingenious methods for navigation.

Marine navigation is the process of conducting a waterborne craft from one point on the surface of the Earth to another, using all the associated science and techniques. The primary activities required for marine navigation may be organized into two closely related components: the planning of the craft's movement, including the determination of the course and speed needed to reach a chosen destination at a specific time, and determining and controlling the craft's position at sea. Many problems in marine navigation are complex because the Earth is spherical. With the help of mathematicians and other scholars, sailors throughout history have been able to devise ingenious methods for approximating workable solutions, resulting in great voyages. Before the 1400s, many cultures sailed in the open ocean, including Pacific Islanders, Persians, Arabs, and inhabitants of some Indian Ocean islands. They used techniques such as poems or visual imagery to remember the positions of the stars, which were their primary guide. Polynesians and Micronesians created the most elaborate star maps and star compasses, as they sailed the longest distances. The Chinese developed a magnetic compass in the eleventh century, which then spread to India and Europe. However, many problems remained numerically and geometrically impractical until the development of modern computers, which are capable of resolving these problems.

Early Developments

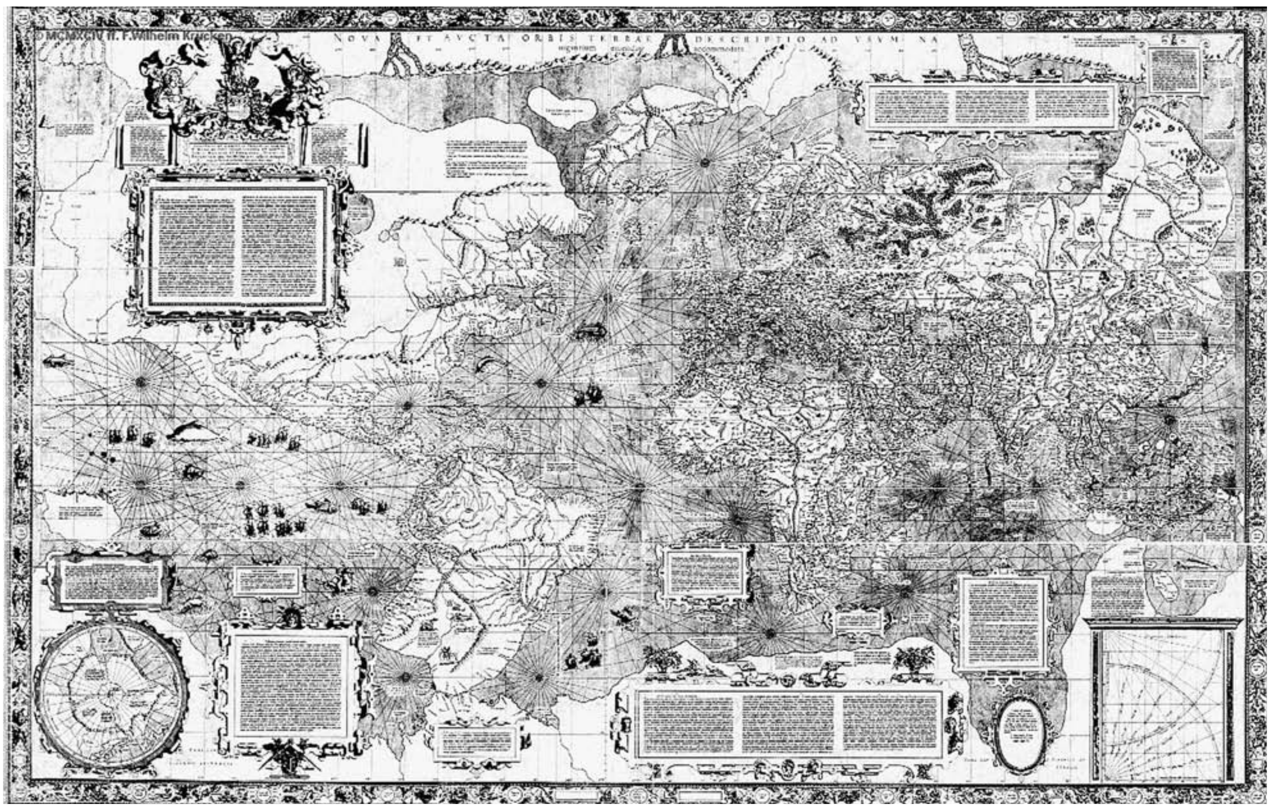
Since the first nautical charts were produced in the Mediterranean approximately 800 years ago, the basic principles of marine navigation have remained unchanged. Still a dramatic improvement of efficiency, accuracy, and safety has occurred during this long period, largely as a result of new navigational techniques. These developments include astronomical methods for measuring latitude (c. 1450), the invention of the maritime chronometer (c. 1750), and the advent of electronic positioning systems (twentieth century). Mathematics has played a fundamental role in this evolution. Scientific navigation in Europe can be traced back to the first quarter of the thirteenth century, following the adoption of the decimal numeral system, the introduction of the magnetic compass in the Mediterranean, and the creation of the first nautical charts. Contrary to traditional medieval maps, which represented the world in some schematic or symbolic way, these charts were drawn to scale using distances and directions measured by pilots at sea. Taking into account the relatively crude methods used for estimating these quantities, the result is astonishingly accurate and detailed when compared to the terrestrial cartography of the time.

The mathematics of navigation is somewhat complicated by the fact that ships move on the spherical surface of the Earth, where the calculation of angles and distances is considerably more complex than on a plane. However, these complications only began to be relevant for the routine practice of navigation when Europe's period of great explorations began in the middle of the fifteenth century and ships started to sail routinely in the open sea.

Although the spherical shape of the Earth was well known to most educated people of the time, including the cosmographers of the Middle Ages and the Renaissance, the fact could be ignored when sailing in the relatively confined waters of the Mediterranean and western Europe. This omission was possible because the geometric errors from assuming a flat Earth were usually smaller than those resulting from the crude navigational methods of the time. In these circumstances, the mathematics of navigation was largely reduced to estimating the distance sailed during a given period, based on simple practical rules and pilots' experience, and determining the ship's position as a function of the course steered and the distance sailed. This determination could be made graphically on a nautical chart using the graphical scale of distance and the mesh of colored lines radiating from chosen spots, representing the directions of the winds, given by the magnetic compass. Because it was not always possible to sail along the straight line connecting the point of departure to the point of destination, tables and abacuses were created to help determine a ship's position relative to that line. These were called the *toleta de marteloio* and gave no more than the solution of the right triangle for some different angles and distances between the planned track and the present track.

By the middle of the fifteenth century, ships started sailing into the open sea on a regular basis and the traditional method for determining their position, based on distances estimated by the pilots and directions given by the compass, was no longer adequate because of the long periods of time ships went without sighting land. This problem was solved with the introduction of astronomical navigation, c. 1450, which permitted sailors to easily determine the latitude by observing the height of the sun and stars above the horizon. Before this important development could be possible, it was first necessary to construct adequate tables with the positions of the heavenly bodies for each day of the year (ephemerides), simplify the instruments of observation used on land (the quadrant and the astrolabe), and devise methods simple and accurate enough to be used on board a ship by uneducated people. The *toleta de marteloio* was then replaced by the *regimento das léguas* (regiment of the leagues), which solved the right triangle formed by the track of the ship along its course (the hypotenuse) and the arcs of meridian and parallel connecting the point of departure to the point of destination. Once again, no allowance was made for the spherical shape of the Earth, since these components were small enough to be considered planar and straight. Soon, though, it was necessary to establish a relation between the degree of latitude and the corresponding arc of meridian on the surface of the Earth so that the length of the degree could be expressed in distance units (leagues). Because this length was directly related to the size of the Earth (a longer degree implied a larger Earth), the problem had significant strategic and political implications. For example, Columbus always defended a degree smaller than the one used by the pilots of his time because this made the distance sailing west to the Indies—what he proposed to the Catholic Monarchs of Spain—considerably shorter. A similar reason was behind the dispute between Portugal and Spain over the location of the spice islands of the Moluccas, in the first quarter of the sixteenth century. The new astronomical methods were soon reflected in the geometry of the charts used for navigation. The new cartographic model, known as the "latitude chart (or "plane chart"), was based on observed latitudes and magnetic courses rather than on estimated courses and estimated distances. It replaced the old portolan chart of the Mediterranean for representing the newly discovered lands.

Though it may contradict common sense, a ship sailing with a constant course between two points on the Earth's surface does not usually follow the shortest track. This paradox is because a line that makes a constant angle with all meridians (called a "rhumb line" or "loxodrome") does not coincide with a great circle arc (or "orthodrome"). This discrepancy was first recognized in 1538 by the Portuguese mathematician Pedro Nunes (1502–1578), who showed that all loxodromes, except the meridians and the equator,



The Mercator chart of the world (1569), Nova et Aucta Orbis Terrae Descriptio ad Usum Navigatium Emendate (New and Augmented Description of Earth Corrected for the Use of Navigation).

are spirals asymptotically approaching the poles without reaching them. This knowledge was later used by Gerardus Mercator (1512–1594) in the construction of a new world map intended to be used in navigation (1569) in which all loxodromes were represented by straight segments making true angles with the meridians. This important map, known as the "Mercator projection," is still used today to support marine navigation, though history has shown that the projection was developed well before it could be consistently put to practical use. Any flat map of the Earth must contain some type of distortion, since it must represent features of a spherical object in a flat surface. Despite its geometric inconsistencies, the latitude chart, based on magnetic courses and observed latitudes, continued in use throughout the seventeenth and eighteenth centuries. This longevity was because determining the longitudes at sea remained impossible, and the spatial distribution of the magnetic declination was still unknown—both necessary for the construction and use of the Mercator projection. Only in the second half of the eighteenth century, following the invention of the maritime chronometer and the development of practical methods for finding the longitude at sea, was the latitude chart finally abandoned by the pilots and replaced with the Mercator projection.

Knowing that the Earth completes one rotation around its axis every 24 hours, the longitude can be expressed as the difference between the local time and the time at the prime meridian (Greenwich Time), from where longitudes are reckoned. Thus, one day corresponds to 360 degrees of longitude, one hour to 15 degrees, and so on. In any method based on this principle, an error in the determination of the time is thus directly reflected as an error in the longitude. Two independent methods for solving the longitude problem were developed in the eighteenth century, encouraged by an important prize offered by the British Admiralty: the lunar distances method, based on measurements of the angular distance between the moon and the sun or a given star, from which the Greenwich Time was determined; and the chronometer method, where the Greenwich Time was given by a very accurate maritime chronometer kept on board. In both methods, local time could be determined by observing the position of the heavenly bodies in the sky. The second method proved to be the more practical of the two and is still used today. However, at the core of both were long and fastidious calculations, done by hand using tables of logarithms and trigonometric functions.

Significant improvements in the accuracy and efficiency of the astronomical navigation methods were made possible in the beginning of the twentieth century by the advent of telecommunications, which permitted ships to receive the exact time on board. The construction of more sophisticated tables in the second half of the century further simplified and significantly shortened the required calculations. Finally, the introduction and dissemination of handheld calculators and computers in the last quarter of the century permitted pilots and other users to easily solve the complex equations governing the heavenly bodies and to determine a ship's position using the time and astronomical observations made on board.

The introduction of radio positioning systems, first relying exclusively on land stations (Loran, Omega, Decca) and later on artificial satellites (GPS, Galileo), represents the latest development in maritime navigation. At the beginning of the twenty-first century, it is possible to find the exact position of a ship in the middle of the ocean and to control its movements with unprecedented accuracy. Although the mathematics involved in all the components of the present navigational systems is vast and complex, the interface is usually transparent enough that the navigator can concentrate his attention on other aspects of the ship's activity and safety.

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See Also: GPS; Mapping Coastlines; Maps; Radio; Satellites; Telescopes.

Market Research

Category: Business, Economics, and Marketing.
Fields of Study: Communication; Data Analysis and Probability; Problem Solving.
Summary: Quantitative and qualitative methods are used to analyze data and guide business decisions.

Market research is a field of study and practice focused on gathering information about markets and customers for the purpose of improving sales or other business outcomes, though similar techniques have been applied to public awareness campaigns designed to change behavior such as smoking and weight loss. Market research draws from a variety of disciplines, with mathematics, statistics, actuarial science, psychology, and business being particularly influential. Careers in market research require strong quantitative skills and market researchers may be required to use concepts from algebra, trigonometry, geometry, calculus, economics, or statistics. Statistical data collection using surveys, experiments, and focus groups is widespread. Both quantitative and qualitative methods are used to analyze these data and guide decisions. Mathematical and statistical models are also developed to try to explain consumer behavior, predict future sales and trends, direct the optimal placement of advertising media or allocation of advertising funds, make consumer recommendations, and simulate market behavior. The availability of enormous consumer databases accumulated from credit cards, store discount cards, and many other sources has spurred the use of data mining techniques, like data fusion and clustering, to merge sometimes-incomplete data sources and then classify subgroups of consumers according to selected criteria.

Types of Market Research

Market research is a broad field and it is important to understand several distinctions about how and why such research is conducted. The first distinction is between marketing intelligence and market research projects: the former is an ongoing, broad-based process of gathering and analyzing information; the latter are focused on a particular question or product and generally have a defined budget and time for completion. A second distinction is between exploratory and confir-

History of Market Research

Formal consumer marketing research got its start in the 1920s with the founding of ACNielsen Corporation in Chicago by engineer Arthur C. Nielsen. Nielsen pioneered many concepts now common in market research, including market share and combined consumer surveys with quantitative audits of sales (both from account books and by observing what was on store shelves) to track sales patterns. Nielsen was also involved in early radio marketing research and later applied the same methods to measure the audiences for different television programs (forerunners of the well-known "Nielsen ratings," which are still used today). In the early days of radio and television it was common for advertisers to sponsor an entire program, rather than to buy a short segment of time to deliver a commercial message, and so the issue of how many people and which particular demographic groups were listening to specific radio programs (or watching specific television programs) became crucial because the sponsor wanted to deliver their message to the right market and be associated with programming that would appeal to that market.

A famous example is the development of "soap operas" on radio and television. These were serial programs about domestic life and were sponsored by soap companies because the programming was developed to appeal to female audiences who presumably were the primary purchasers of household soap products. Marketing research was largely limited to internal departments of mainstream packaged-goods companies until the 1980s but since then has become a major industry as more companies became interested in using market research, and independent consulting firms were developed to answer this need.

matory research: exploratory research is usually conducted early in the decision cycle, and its goal is to discover what options exist; confirmatory research comes into play later in the cycle when the goal is to narrow options and decide which course of action to follow. These distinctions are crucial because the same technique can be used for different purposes; for instance, surveys or focus groups can be part of an ongoing and broad-based data collection effort or may be a one-time effort focused on a particular product or some aspect of a product. Both research techniques may be used either to gather a broad array of data whose purpose may not be known (which might be conceptualized as "seeing what's out there") or as a tightly focused effort at making distinctions to guide decision making among a small set of already-known options.

Another distinction is whether the research will be focused on sales to consumers or to other businesses. The former is sometimes called "business-to-consumer" (B2C), and the latter is called "business-to-business" (B2B) marketing. Most people are familiar with consumer market research and may have taken part in it, whether they were aware of it or not. Consumer market research is focused on the goal of selling a product to a large number of people (or, in a more general sense, of discovering their preferences). For instance, an entrepreneur might want to design a sports sneaker that will appeal to urban young men of high school age. Because of this focus on describing the preferences and judgments of groups, consumer market research often incorporates knowledge and techniques from social sciences, such as psychology, sociology, and anthropology. Techniques include surveys, focus groups, and ethnographic observation (observation of how people make choices or use products without interfering in that process).

B2B refers to commercial transactions between businesses. For instance, a wholesaler may sell goods to a retailer (who will then sell them to the public), or a supplier may provide goods necessary for business operations, such as paper, computers, and other office supplies for business. Although B2B accounts for a high volume of sales, the process of market research is different because the consumers may be assumed to have a high degree of knowledge about the product they will be buying, and usually a single individual or small department can make the decision for large purchases of goods. For these reasons, B2B market research may be focused differently, for instance, on discovering how a corporation views its own brand and how a product may be allied with that effort. However, as with consumer marketing, the goal is still to gain information that will allow businesses to develop and market products that meet the needs and desires of potential purchasers.

Another distinction is between qualitative research, which generally collects verbal data, and quantitative research, which collects information that may be translated into numbers. Qualitative research is often used for exploratory research and to gather information very early in the research process; for example, focus groups and unstructured interviews may be used to gather reactions to a new idea or product. When the research effort has progressed sufficiently that a few questions have been selected for further investigation, more structured quantitative research (for instance, a questionnaire-based survey) may be used to gather precise information relating to these questions.

The Research Process

The process of market research proceeds in a manner similar to much social science research, with the main difference being the ultimate goal. In the social sciences, it is generally to add to human knowledge, while in market research, it is generally to make an optimal business decision. In either case, the first step is to identify the question to be answered or the problem to be solved, a process that is particularly important when the research will be conducted by a separate department or a consulting group. The next step is to elaborate on the problem—exactly what information is required or what questions much be answered in order for a decision to be made? The third step is to identify which research techniques are most appropriate for answering the questions, including consideration of the time and other resources available. Once these steps have been completed, a study can be designed, including specification of a time frame and the data sources to be used.

In research, the distinction is often made between primary and secondary data sources. Primary sources are data that are collected by an individual or organization for its own use, for instance, conducting focus groups to see how people react to several versions of a new product a business is planning to introduce to the market. Secondary sources are those collected by someone else and then made available to others. Examples include government data sets such as the U.S. Census and data collected by private or university researchers for specific projects that are later made available for use by others.

Both primary and secondary data have their advantages and disadvantages. Collecting primary data allows the research team to specify exactly what data they want, for instance, color and design preferences among housewives in a specific urban area. They are generally more expensive because the researchers must collect the data themselves and they are necessarily more limited in scope. It is generally cheaper to use secondary data, and the scope is often much broader (for example, it may have been collected on a national or international basis) than could be collected by a small research team. However, secondary data may be several years out of date by the time it is available and may not focus specifically on the questions of interest for a particular marketing research project. Often, both types of data are combined in the same research project; for instance, U.S. Census data about neighborhoods (racial composition, median household income, etc.) can easily be combined with information from a primary, purpose-designed survey of individuals.

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Sarah Boslaugh

See Also: Budgeting; Careers; Coupons and Rebates; Data Mining; Internet; Social Networks.

Marriage

Category: Friendship, Romance, and Religion.
Fields of Study: Algebra; Data Analysis and Probability.
Summary: Sociologists and others have made many demographic studies of marriage, even modeling it.

Many kinds of arrangements have existed throughout history under the umbrella of marriage, with the expectations and responsibilities of married partners and their rights both to enter into marriage and within the marriage changing considerably over time and across (or within) cultures. It has always included legal and economic dimensions, which have played into the changing demographics of the married.

History of Marriage

The modern concept of "marriage for love" is a relatively recent development in the history of marriage; for several millennia, marriage was an important societal convention fulfilling critical economic, legal, and political functions. Among elite people, marriage was a tool for the control and consolidation of wealth and power by forming strategic alliances between families. Political and military agreements were sometimes forged in the context of a marriage. In middle and lower classes, marriage played a similarly important societal role, especially economically. Marriage's economic role was further reflected in conventions such as illegitimacy, the dowry, and large families of children, which proved a vital source of labor and economic gain for the family. Marriage was also the societal device for conferring a host of legal rights.

The sexual marriage, a marriage that is freely arranged between two people on the basis of love, is a newer development that evolved from cultural changes that occurred during the Enlightenment and were further developed by the Industrial Revolution. The economic and legal changes that grew from this period gradually eroded the historical reasons behind arranged marriages. This gradual change in marriage perhaps culminated with the 1950s concept of the "Leave It To Beaver family"; however, this short-lived paradigm of marriage experienced dramatic shifts in the socially turbulent decades to come.

The legal and political advances for women in the early twentieth century, coupled with important economic and demographic advances in the latter half of that century, paved the way for important changes in the way people approach marriage. Women made significant strides economically and socially that allowed them the possibility of viable, independent lives apart from marriage. The innovation of birth control also played an important role in the evolution of marriage by allowing women to effectively separate sex and child rearing.

Statistically Analyzing Marriage

Marriage in the United States has undergone critical demographic changes that are closely allied with education level and socioeconomic status. Data spanning five decades of the latter twentieth century and early twenty-first century demonstrate a steady decline in marriage rates. In 1970, 84% of adults aged 30–44 years were married compared with only 60% in 2007. The decline in marriage rates mirrors corresponding rises in the divorce rate and a greater tendency of couples to find alternate arrangements, such as short-term relationships and cohabitation.

Table 1. Percentage of married adults 30–44.

1970	84%
1980	77%
1990	69%
2000	65%
2007	60%

Marriage is associated with well-established economic benefits. Most obvious is the economy of scale realized when a couple can share major assets, like a house, a car, or furniture, that they would otherwise each need to purchase individually. This economy of scale is still a significant advantage even when the additional economic cost of raising children is factored in.

However, economic benefits are also realized when a spouse marries someone with a higher income. In 2007, individual income for married men was an average of 12% higher than for single men. Married women outearned their single counterparts even more substantially, with a 53% higher average income. However, this statistic is not a simple causal relationship between being married and accruing greater wealth; these economic gains are closely tied to education level and earning power. Essentially, people with a higher educational level are more likely to be married, more likely to be married to a spouse of a similar educational level, and more likely to realize and compound the economic benefits of marriage. Interestingly, this is a trend not present in the 1970 data, where the marriages rates across the socioeconomic spectrum were nearly identical. The period since the 1970s has seen significant changes in the number of women attending college and their choices in forming relationships.

Research literature also indicates important health and emotional benefits associated with marriage. These benefits stem not only from lifestyle changes (for example, the healthier diet of a married couple or the shared division of household labor); contemporary studies suggest an even more important factor is the mitigation of stress and its effects on health. Married people live longer, experience less illness, and are less prone to many diseases. Importantly, studies clearly indicate that the quality of the marriage is an important factor; poor marriages have been shown to be even unhealthier than being single. There are also clear gender differences in the extent and the way in which spouses realize the health benefits of marriage.

Mathematically Modeling Marriage

Marriage statistics are extensively tabulated like many other social statistics, but researchers also use mathematical modeling to study marriage. The 2003 book *The Mathematics of Marriage: Dynamic Nonlinear Models* was authored by an interdisciplinary team including mathematicians. It used mathematical ideas, such as difference equations, phase space, null clines, influence functions, inertia, and stable steady states (attractors), to model marriage, with applications to other psychological phenomena. In 2009, a team of mathematicians from the United Kingdom and the United States analyzed the behaviors of 700 couples over the course of 12 years to develop a probabilistic model that accurately predicted which marriages would last. It was based on classifying couples into one of five types using behavioral variables. Only one type suggested a long-lasting marriage. In 2010, Spanish economist José-Manuel Rey developed an equation based on optimal control models and the "second thermodynamic law for sentimental interaction," which states a relationship will disintegrate unless it receives input "energy" or effort.

As with the Birthday and Cocktail Party Problems, mathematicians have identified a similar social puzzle in the Stable Marriage Problem. First introduced as a matching problem by D. Gale and L. S. Shapley in 1962, the stable marriage problem consists of equal numbers of single men and women. Every man creates a preference ranking of each woman as a potential match; similarly, every woman ranks each of the men. The goal is to pair the men and women in couples so as to create stable, happy marriages.

The technical challenge is to avoid an "unstable matching," which arises when a man and woman who are not paired under the matching would each prefer to be with each other over their paired spouse. The immediately interesting question—whether there always exists a stable matching given a set of preference rankings for each individual—was answered in the same seminal work. The Stable Matching Algorithm provides a solution to this problem and, furthermore, is guaranteed to always produce a stable matching. Curiously, this algorithm maximizes one gender's happiness while minimizing the other's, depending upon which gender does the proposing and which does the accepting. This same algorithm has other applications, for example, in matching medical school applicants with schools and in pairing roommates for college residence halls.

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Matt Kretchmar

See Also: Birthday Problem; Cocktail Party Problem; Life Expectancy; Mathematical Friendships and Romances; Predicting Divorce; Predicting Preferences.

Martial Arts

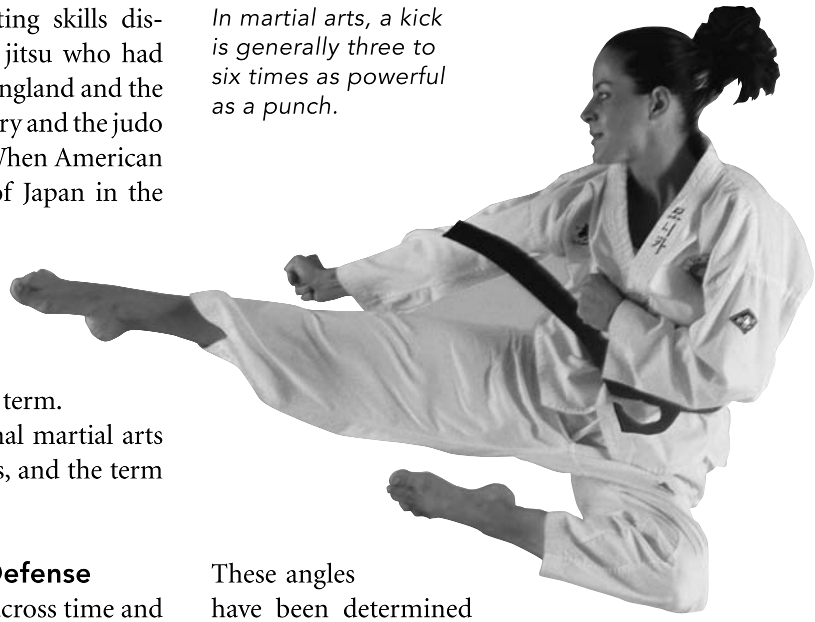
Category: Games, Sport, and Recreation.
Fields of Study: Algebra; Geometry.
Summary: The motions and stances of martial artists can be analyzed for their efficiency and use of force.

In the martial arts, humans use repetitive training to standardize their response to threat. The original barehanded style of ritualized combat training that evolved into the modern martial arts is believed to have developed in China at about the same time as the introduction of bronze, agricultural sciences, and Chinese philosophy, and later spread to Korea and Japan. Many regions of the world have their own native forms of combat training, which are now also called "martial arts" in English, but the English term comes originally from the Japanese.

While techniques and philosophies differ, the underlying goal of all martial arts is the same: that through deliberate physical and mental training, forces can be concentrated or dissipated across time and space in order to either attack or defend. In the modern world, most martial artists train for sport or health promotion. Mathematics can be used to describe and model the stances and movements of martial arts forms and practitioners, such as the geometry of balance and the forces concentrated across time and space in the form of kicks, blocks, and strikes.

Etymology

The term "martial arts" first appeared in English in 1933. The Japanese Railway Ministry released the *Official Guide to Japan*, including a reference to the Butokukai in Kyoto, which they translated as the "Association for Preserving the Martial Arts." "Martial arts" became an umbrella term describing the fighting skills displayed by Japanese practitioners of jiu jitsu who had been invited to give demonstrations in England and the United States in the late nineteenth century and the judo practitioners who followed soon after. When American troops returned from the occupation of Japan in the 1940s and 1950s, they brought along some knowledge of and interest in karate. In the 1960s, the Chinese Martial Arts came to be recognized in the West and were grouped under that increasingly pan-Asian umbrella term. Since then, many modern and traditional martial arts have been recognized to varying degrees, and the term has become international.



In martial arts, a kick is generally three to six times as powerful as a punch.

The Mathematics of Attack and Defense

An attack is the concentration of force across time and space. An ideal blow multiplies the mass of the entire body by the speed at which the striker moves and delivers the resultant force to a precisely determined surface. This may be done to inflict damage directly or to interfere with the opponent's intent by disrupting his or her balance. Defense is the opposite, dissipating the attacking force across both time and space by either absorption, deflection, preemption, or avoidance. The same principles that allow defense against an attack can be used to dissipate an entire conflict.

Variables that affect the force delivered or deflected include center of gravity or mass, kinetic energy, linear and angular (rotational) momentum, velocity, inertia, and acceleration (as governed by Isaac Newton's laws of motion). Mathematicians have studied and modeled many aspects of martial arts. Analysis of data has shown that kicks are typically three to six times as powerful as punches; the speed of a fist during a forward punch is a nonlinear function of arm extension; and a smaller fighter can punch as hard as a larger one by moving faster. Some of these models approximate body parts with geometric forms, such as cylinders for arms, in order to simplify the calculations involved. Geometry is also important for examining the basic stances and movements of all martial arts. Stability for both attack and defense comes from maintaining the correct alignment and balance in three dimensions. The mathematics becomes even more complicated once the practitioner starts moving. Correct form requires a specific angle between body parts when kicking or punching. These angles have been determined through generations of practice and can be measured very specifically by the avid student who enjoys applied mathematics. In this way, experts in many martial arts have learned that correcting the angle of one's foot or knee or wrist by just a few degrees makes all the difference for gaining leverage or applying the maximum amount of force. These small differences, best measured mathematically, can make the difference between a novice and a martial arts master.

There are many martial arts, but they all present both attacker and defender with the challenge of maintaining one's own intent while interfering with the intent of one's opponent. This is like balancing an equation, where the intent of the two or more people involved in a confrontation can be reduced like the terms in an exercise in algebra. A parry on one side negates a strike from the other, and so on. This is why martial artists are sometimes seen standing almost still and looking at each other before a fight begins. In their minds, they are balancing out the equation. Usually this ends when one or the other thinks he or she see a way to make the balance work out in their favor and they start the action. Sometimes, however, the equation is so unbalanced that both sides can see it, and the fight ends without any violence at all.

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See Also: Asia, Eastern; Brain; Strategy and Tactics; Tournaments; World War II.

Math Gene

Category: Mathematics Culture and Identity.
Fields of Study: Communication; Connections; Problem Solving.
Summary: The idea of the so-called math gene is false; mathematical ability is not genetically predetermined.

Some believe that mathematical problem-solving ability is encapsulated in a "math gene" that endows some people with the ability to solve mathematical problems, while those who lack that gene are doomed to mathematical illiteracy. This notion is false; the ability to solve mathematical problems is influenced (but not determined) by many interacting genes, not a particular one. The term "math gene" is often used to indicate an innate facility for mathematics, not a specific gene.

The math gene concept has a negative impact on society; it discourages students from working harder by making failure at mathematics socially acceptable. Because of the many benefits of mathematical literacy, research suggests that the related genes are under a positive selection force, and thus mathematical ability is to a significant degree heritable. However, mathematical ability is also influenced by many nongenetic (environmental) factors. It is this complex web of interactions of genes and environment that is responsible for a person's mathematical ability.

Impact

The concept of a math gene is especially prevalent in places like the United States in the late twentieth and early twenty-first centuries. Research shows that students who believe in the concept may perceive that it is "others" who do mathematics, rather than people with whom they identify. Mathematics educators deem positive attitudes as critical for success in mathematics. "Impostor syndrome" is a well-documented phenomenon in mathematics in which students feel like outsiders, with an accompanying fear of being "found out." However, the notion of a math gene does not proliferate in all areas of the world. For example, it is rare in Japan in the early twenty-first century, where students are expected to work harder if mathematics does not come easily.

Parents, teachers, and students who believe that biology is more influential than education, practice, and effort may find it socially acceptable to use the notion of a math gene as an excuse for poor mathematics performance. Politicians and industry leaders in the United States have stressed the importance of mathematical training and success to the global economy. Some educators have suggested that instead of being complicit in students' failures, society should reject the concept of a math gene and encourage a positive attitude toward learning mathematics.

Genes and Environment

Almost all human cells contain molecules of deoxyribonucleic acids (DNA) that are arranged into functional units called "genes." Information encoded in genes is transmitted from parents to their children, thereby forming the process of heredity. All humans possess approximately the same set of genes, with the exception of genes located on sex chromosomes. This set of genes is the called the "human genome." Individuals carry different variants, called "alleles," of these genes. Different alleles are responsible in part for differences in phenotypes, the observable characteristics of individuals. Characteristics influenced by different alleles include both physical characteristics such as eye color and height and psychological and emotional characteristics such as intelligence and personality.

Despite the important role of genes, the genome should not be viewed as a predetermined recipe in which each gene determines a specific trait or characteristic. There are many nongenetic factors that influence a person's traits; these are broadly termed "environmental factors." These factors include nutrition, societal and cultural influences, education and upbringing, exposure to chemicals and radiation, and other factors of nongenetic origin. Many of these factors exert their influence from the moment of conception.

Genes interact with each other and with their environment, thus forming a remarkably complex network of interdependence, regulation, and feedback. Many of these pathways and networks are still poorly understood. It is therefore practically impossible to point to one gene that controls a particular trait. Instead, the human genome should be viewed as a complex, nondeterministic, flexible blueprint that is influenced by environmental factors.

Although neither genes nor environment alone are responsible for any phenotype, it is nonetheless possible to estimate the relative contributions of genetics and environment to the development of any particular trait. While some traits are influenced largely by genetics, others are due mostly to environment. Determining the relative contribution of genetics and environment is a complicated problem, particularly in the case of mathematical ability.

Environmental Influences

While the ability to excel in mathematics is to a degree inherited, environmental factors play an important role in the development of mathematical ability. Foremost among these environmental factors are education and upbringing. There is substantial evidence that mathematically gifted individuals are often persons who have been exposed to mathematics from a very young age—at home or at school. Support from parents and teachers plays an important role in this process, as is the availability of books and other educational materials.

Other environmental factors are also crucial. These factors include adequate nutrition and the absence of environmental toxins, both of which are necessary to ensure proper brain development and function. Since environmental factors interact with each other as well as with genes, it is difficult to elucidate the precise way in which these factors affect mathematical ability. Further research may reveal other environmental factors that are unknown at present. It may also reveal new ways in which environmental factors interact with each other and with genes.

Genetic and Evolutionary Influences

The process of solving mathematical problems involves the use of abstract reasoning to seek generalizations and relationships and to derive conclusions from given facts and known laws. By its nature, abstract reasoning is nonspecific and can be useful in many situations ranging from solving everyday problems to answering questions arising from complex scientific research projects. Hence, mathematical ability constitutes more than the ability to manipulate numbers and equations; it is the ability to understand and solve problems that arise in many walks of life.

Consequently, mathematically literate individuals are able to solve problems and to adjust to their environment better than those who lack mathematical ability. In modern societies, those who are mathematically literate are also able to pursue careers of high socioeconomic status, thereby adding to the benefits of their mathematical ability. Since adaptation to environment is a major force of evolution, gene variants that influence mathematical ability have enabled their possessors to thrive throughout human history. Thus, mathematical ability is to some degree genetic and hereditary.

Nevertheless, mathematical ability is also influenced by genes that have little or nothing to do with reasoning or with cognition. Since the process of learning and doing mathematics is a long and arduous one, it requires personality traits such as persistence, diligence, patience, and self-discipline. Gene variants that promote these qualities tend to improve mathematical problem-solving ability in those who possess them.

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Or Syd Amit

See Also: Brain; Genetics; Intelligence Quotients; Succeeding in Mathematics; Women.

Mathematical Certainty

Category: History and Development of Curricular Concepts.
Fields of Study: Problem Solving; Reasoning and Proof; Representations.
Summary: Mathematics is arguably the most stable and rigorous source of knowledge; yet any system of mathematical reasoning is incomplete.

At one end of the spectrum in mathematics, it seems as if people can be absolutely certain, using mathematical proofs of concepts. At the other, in statistics and many applied mathematics fields, it is virtually impossible to be certain, but people can make probabilistic statements with regard to degrees of uncertainty inherent in a given calculation or statement.

The Greeks may have been the first to attempt a rational explanation of nature. The crucial tool in their investigations was mathematical reasoning. They assumed that all questions about nature can be answered by reason and that all these answers are knowable and can be discovered, a property known as "completeness." They also assumed that all answers are compatible, which is called "consistency." However, the evolution of mathematical certainty revealed that these assumptions can never be fully realized. The notion of what is certain and what is uncertain is a fundamental component that is threaded in various ways throughout twenty-first-century mathematics curricula. For example, in primary school, students investigate the differences between "likely" and "unlikely" events. Students also develop inductive and deductive reasoning by exploratory investigations and examples as well as by proofs.

Axiomatic Systems

The Pythagoreans (c. 585–500 B.C.E.), a school influenced by Pythagoras of Samos, offered a mathematical plan of nature. The Greeks' goal was to rationally explain why things are the way they are. They confronted a fundamental question: can all knowledge be verified? Aristotle (384–322 B.C.E.) answered "no," since there are self-evident truths (called "axioms") that cannot be explained. Moreover, in geometry, Aristotle said a proposition is proven when it is shown to logically follow from the axioms and other proven propositions. Euclid of Alexandria (323–285 B.C.E.) knew of these developments and incorporated them into his text, *Elements*. It is recognized as the prototype for how mathematics should be done: well-thought out axioms, precise definitions, carefully stated theorems, and logically coherent proofs.

Formulation of the axioms or "postulates" (the Greek term for axioms about geometry) is the crucial step in building an axiomatic system. These statements should be intuitively self-evident, and, from these, it must be possible to deduce the important properties of the objects of study. Later, mathematicians found assumptions used in *Elements* that were not explicitly stated in the axioms. Credit for completely and successfully axiomatizing Euclid's geometry is generally given to David Hilbert.

The Parallel Postulate and Non-Euclidean Geometries

Euclid's fifth (or parallel) postulate states that, "through a given point, not on a given line, only one parallel line can be drawn to the given line." Almost immediately, this postulate was controversial. Many did not find it to be self-evident and thought it required a proof. Over two millennia, countless mathematicians tried to derive the parallel postulate from the others, all with no success. These futile efforts had important consequences in all of mathematics. Beginning in the eighteenth century, some mathematicians began to use indirect methods to "prove" this postulate.

Though unsuccessful, the indirect methods led to the discovery of non-Euclidean geometries—using the other axioms but denying the parallel postulate. Attempts to prove non-Euclidean geometries were invalid were essentially attempts to show that they were inconsistent. Eventually, it was determined that Euclidean and these other geometries were consistent and complete. The discovery of non-Euclidean geometries revealed that mathematics could deal with completely abstract axiomatic systems, which no longer had to correspond to beliefs based on real-world experiences.

The Consistency and Completeness of Mathematics

It became clear that the most important considerations for an axiomatic system were its consistency and completeness, and at the International Congress of Mathematicians in 1900, Hilbert addressed these problems. He felt all mathematics should be put on a sound basis using the axiomatic method. In 1904, Hilbert constructed an arithmetic model of Euclidean geometry, showing that geometry was a subset of arithmetic. Mathematicians then set out to show the consistency of arithmetic, from which it would follow that Euclidean geometry was consistent.

These efforts ended in 1931 with the results of Kurt Gödel. His first Incompleteness Theorem showed that in any axiomatic system rich enough to include the arithmetic of the natural numbers, it is possible to prove some statements that are false, showing the system is inconsistent; or it is not possible to prove some statements that are true, showing the system is incomplete. In his second Incompleteness Theorem, Gödel showed the question of whether an axiomatic system is consistent cannot be determined within the system. Gödel's results revealed that any mathematical reasoning system based on axioms as rich as arithmetic can never be fully realized—such systems must be either incomplete or inconsistent. Modern mathematicians operate under the assumption that mathematics is incomplete and not inconsistent.

Different Axioms Lead to Different Mathematics

One axiom of set theory, the axiom of choice (AC), was used implicitly for years before it was explicitly described. The AC states that for any collection of nonempty mutually exclusive sets, finite or infinite, there is a set that contains exactly one element from each set. The AC with Zermelo–Fraenkel (ZF) axiomatic set theory, named for Ernst Zermelo and Abraham Fraenkel, is the basis of modern mathematics. In 1938, Gödel proved that if ZF set theory without the AC is consistent, then ZF set theory with the AC is also consistent. So, just as it is possible to choose between different acceptable geometries in which the parallel postulate may or may not be true, it is possible to choose between different acceptable ZF set theories in which the AC may or may not be true. On one hand, theorems requiring the AC are fundamental in such areas as modern analysis. Then again, by adopting the AC, results such as the Banach–Tarski paradox, named for Stefan Banach and Alfred Tarski, can be derived, which says a golf ball can be divided into a finite number of pieces and then rearranged to make a solid sphere the size of the Earth. Thus the decision as to which axiomatic system to adopt cannot be made lightly—different mathematics can be derived from these different axiomatic systems.

Valid Proofs

The idea of a valid proof depends on one's philosophical approach to mathematics. A number of schools of thought have evolved, including (1) the logistic school, which holds mathematical proofs derive from logic; (2) the intuitionist school, which maintains mathematics takes place in the human mind and is independent of the real world—it composes truths rather than derives implications of logic; (3) the formalist school in which proofs follow from the application of a system of axioms; and (4) the set theoretical school, which derives proofs from the axioms of set theory.

Mathematics remains our most rigorous form of knowledge. As proofs grow more complicated, mathematicians worry they will have to accept a greater degree of uncertainty in solutions. For example, the entire proof of the Classification Theorem for Finite Simple Groups consists of an aggregate of hundreds of research papers and over 10,000 printed pages. Additionally, the Four-Color Problem solution has been achieved only on the computer and involves checking a prohibitively large number of cases. Some mathematicians believe that since it is not reasonable and possible for any one individual to check all these cases, then a valid proof has not been provided for such problems.

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See Also: Probability; Proof; Randomness; Reasoning and Proof in Society; Risk Management.

Mathematical Friendships and Romances

Category: Friendship, Romance, and Religion.
Fields of Study: Communications; Connections.
Summary: The shared and often highly specialized interests of mathematicians naturally lead to bonding.

While mathematics is often thought of as independent work, collaboration among mathematical peers is evident throughout the history of mathematics and in contemporary research settings. In the twenty-first century, many mathematicians write papers together, and most graduate students share offices and work with an adviser. These types of collaborative interactions may foster a sense of shared bonding that sometimes leads to friendship or romance. In his book *Nicomachean Ethics*, Aristotle posits the widely held view that friendships may be based on pleasure, profit, or similar values. Mathematical partnerships may be rooted in a common desire to produce mathematical results or to discuss the frustrations and difficulties that arise from work. However, successful partnerships do not exist just for pleasure or profit but for the mutual good.

During the seventeenth century, many natural philosophers engaged and developed their mathematical knowledge and shared their theories via letter writing. One prolific example was the Minim monk Marin Mersenne (1588–1648) who had about 200 correspondents, including René Descartes (1596–1650), Pierre de Fermat (1601–1665), and Blaise Pascal (1623–1662). Because of his connections with so many mathematicians and philosophers of the time, and as he lived in Paris near the Place Royale, Mersenne became the hub of a social network that often assembled at his residence. This gathering eventually evolved into the Paris Academy around 1635 and fostered a community of learning.

Mathematical friendships or romances may blossom from a mentorship between professor and student, although this violates the faculty guidelines of many twenty-first-century institutions. There is the potential for abuse because of the power and authority the mentor or instructor has over the student in terms of grades, evaluations, letters of recommendation, and other educational and professional outcomes. Johann Bernoulli (1667–1748) and Leonhard Euler (1707–1783) developed a mutually beneficial relationship that began when Euler was studying at the University of Basel. This friendship spawned another between Euler and Johann's son Daniel Bernoulli (1700–1782), who later encouraged Euler to join him at the St. Petersburg Academy. Eventually, Euler not only joined the academy but, during his early years in Russia, resided with Daniel Bernoulli. Together, these two men engaged in learned discussions of their shared research interests in mathematics and physics, particularly hydrodynamics.

Another friendship between mathematicians that developed from a student–teacher relationship was that of Karl Weierstrass (1815–1897) and Sonia Kovalevsky (1850–1891), who met in Berlin. Because women could not take courses at the University of Berlin, Weierstrauss agreed to privately work with the 20-year-old Russian. Based on her strong independent research and Weierstrass's recommendation, Kovalevsky earned her doctorate

Similar to the Six Degrees of Kevin Bacon game, mathematicians have defined a number that signifies how closely related one is to Paul Erdös (left).

in mathematics from the University of Göttingen in 1874. The mathematical collaboration between these two friends continued even when Kovalevsky returned to Russia, where she also connected with other former students of Weierstrauss.

During the nineteenth century, collaboration or marriage between scientists was one way for women to gain acceptance by the scientific community. However, in the twentieth century, mathematicians like Mary Ellen Rudin (1924–), married to fellow mathematician Walter Rudin (1921–2010), found it difficult to obtain jobs because of antinepotism rules. In 1992, it was widely reported that approximately 80% of female mathematicians were married to other mathematicians. This statistic may be explained in part by the fact that advanced study of any type is time-consuming, and people may move away from home for educational or career opportunities, such that their social circle often overlaps substantially with their work circle. It may also be true that people find personal connections arising from professionally shared interests. Scientific couples refer to the difficulty of finding jobs together as the "two-body problem," which is also a problem in classical mechanics involving the motion of two particles.

Mathematical friendship is famously found in the life of Paul Erdös (1913–1996). The mathematical genius's passion for the subject is illustrated by his more than 1475 academic publications with more than 500 coauthors. Erdös traveled around the globe, arriving at the homes of his friends to work on problems with them. Many such visits resulted in a mathematical research paper authored by Erdös and his host/hostess. Among these collaborators, mathematician husband and wife Ronald Graham (1935–) and Fan Chung (1949–) were particularly close friends who handled many of Erdös's temporal affairs, day-to-day scheduling, and financial matters. Erdös stayed with Graham and Chung regularly, and they even built an addition onto their New Jersey home for Erdös to stay during his annual month-long visits. Similar to the Six Degrees of Kevin Bacon in social network analysis, mathematicians have defined a number that signifies how closely related one is to Erdös. A mathematician has an Erdös number of 1 if he or she has written a paper with Erdös himself; an Erdös number of 2 if he or she published with someone who coauthored a paper with Erdös; an Erdös number of 3 if he or she published with someone who published with someone who coauthored with Erdös; and so on.

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See Also: Clubs and Honor Societies; Mathematics Genealogy Project; Number Theory; Professional Associations; Six Degrees of Kevin Bacon; Social Networks.

Mathematical Modeling

Category: History and Development of Curricular Concepts.
Fields of Study: Communication; Connections; Problem Solving; Representations.
Summary: Modeling reformulates scenarios to mathematical elements for analysis and problem solving.

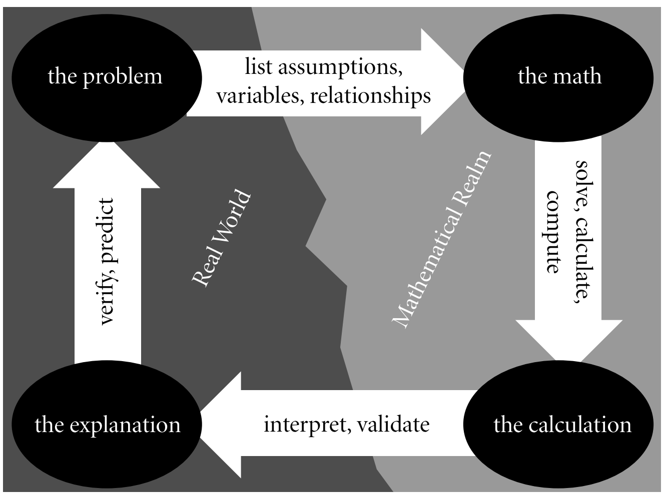
Mathematical modeling has been in use since prehistory and was likely the first kind of mathematics ever employed. Mathematical modeling can be thought of as the activity involved in finding a solution to a real-life problem by working with a mathematical structure that captures the important characteristics of the situation. In the twenty-first century, mathematical modeling is found in many areas of mathematics, engineering, science, social science, and business and has often resulted in the formation of recognized "subdisciplines" within these fields. Research and applications occur in a diverse range of theoretical and real-world problems, and modeling is used in schools starting in the primary grades to help students visualize and solve problems, create alternative representations of various concepts, and make connections between different areas of mathematics. The advent of computers has facilitated mathematical modeling and allowed researchers to conduct simulations or find numerical solutions to problems that may be difficult to solve analytically. However, there are those who argue against overuse of mathematical models, citing issues of faulty data, unwarranted extrapolation, and the inherent error of attempting to quantify many complex or qualitative real-world phenomena. These types of criticisms have been applied to models associated with the financial and housing crises of the early twenty-first century and evidence on both sides of the global warming debate.

Anyone who has ever attempted to solve a story problem has dabbled in modeling. Consider the following story problem:

> I asked my dad for some money. He gave me 24 coins with three times as many dimes as quarters, for a total of $3.30. How many of each coin do I have?

To solve the problem, one converts the verbal statements into equations. The set of equations is the mathematical model, which can be solved to determine an answer. When formulating the mathematical equations, assumptions would be made, such as which denominations of coins to include in the model.

Figure 1. The modeling process.



Process

Modern treatises on the modeling process often portray the steps involved in modeling using a diagram similar to Figure 1.

Starting with a problem statement, the first action is to determine the assumptions that should be made, information in the problem that is extraneous and can be neglected, quantities that are known (parameters) and unknown (variables), and relationships between the quantities. This work may entail using a variety of strategies, including developing or using existing physical laws, proportionality arguments, equations from the current experts in the field, or equations empirically determined from experimental data.

That first step will lead to a mathematical representation of the real-world situation. The mathematics may take the form of equations, inequalities, recursive relations, matrices, graphs, integrals, differential equations, geometric structures, or other mathematical objects.

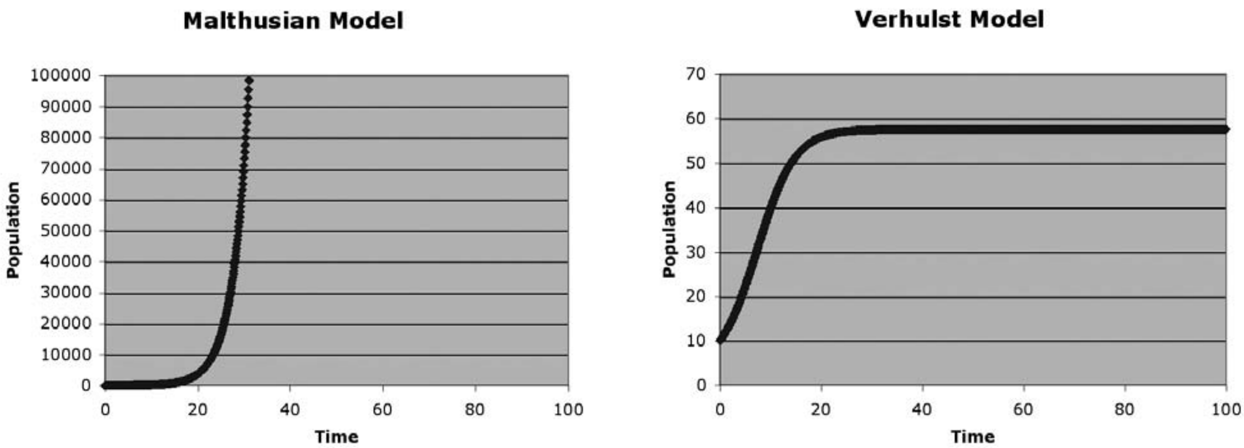
The next action is to "solve" the mathematics, leading to an answer. That answer may be an exact solution, a simulation, or an approximation. The answer must then be interpreted in the context of the problem, and any approximations must be checked (validated) to see if the solution is correct. Lastly, the explanation of the answer in the context of the situation should be used to verify or predict the solution to the problem.

In practice for complex real-world problems, the modeling process is really a cycle that is traversed repeatedly as the model is refined to produce more realistic behavior. It is common to find that these processes involve multidisciplinary teams of professionals, including mathematicians and scientists, who participate in a dialogue to clarify and refine the assumptions based on the success of the last step in the process: analyzing the mathematics in the context of the problem.

History

There is archeological evidence from more than 10,000 years ago of simple mathematical ideas being developed to solve problems related to counting objects, measuring land area and distance, and recording time. More complex mathematical problem solving appears around 3000 B.C.E., when the cultures of Asia, the Middle East, and North Africa began using arithmetic, algebra, and geometry to solve problems in

Figure 2. Malthus's exponential population growth model (left) as compared to the Verhulst model, which incorporates intra-species competition for resources.



astronomy, building construction, and financial situations, such as taxation. The design and construction of complex pyramids and temples and the development of sophisticated astronomical calendars in Central and South America in the first century C.E. point to the development of mathematical ideas to solve problems. It can be argued, in fact, that most of the mathematics developed before 1800 was conceived to help model a situation in the real world.

Before the mid-nineteenth century, many of the real-world problems that were approached using mathematics would be classified today as astronomy, physics, or engineering. For example, Archimedes (287–212 B.C.E.) was instrumental in modeling physical tools, such as levers, and in the development of models for hydrostatics (the properties of water at rest, such as pressure). Eratosthenes (276–194 B.C.E.) used a geometric model and his knowledge of how the sun casts shadows to determine the circumference of the Earth. Abu Ali Hasan Al-Haitham, known more commonly as Alhazan (965–1040), developed the first principles of optics for spherical and parabolic lenses. Blaise Pascal (1623–1662) developed the ideas fundamental to probability while helping a gambling friend by modeling a dice-rolling game. Isaac Newton (1642–1727) is perhaps the best-known "mathematical modeler" who ever lived, famous for his ground-breaking work on the classical laws of motion and gravitation. Building on equations of fluid flow developed by Leonhard Euler (1707–1783), Claude Henry Navier (1785–1836), and George Stokes (1819–1903) produced the Navier–Stokes equations, which model velocity, pressure, temperature, and density of a moving fluid. The Navier–Stokes equations, a set of nonlinear partial differential equations, were truly understood only after the advent of modern digital computers in the 1960s.

The nineteenth century saw an expansion into biological and social science modeling. Thomas Malthus (1766–1834) wrote about population growth and the familiar exponential model for population growth is named after him. Pierre Verhulst (1804–1859) took Malthus's ideas and developed the logistic, limited growth model (see Figure 2).

Late Nineteenth Century Through Twentieth Century

From the late nineteenth century forward, mathematicians have become more concerned with the development of theoretical—sometimes called "pure"—mathematics: abstract structures derived from fundamental axioms and built through proving theorems following logical precepts. However, this interest in mathematics for its own sake did not slow down the development and use of mathematics as a tool to model the real world. The application areas have become increasingly diverse, and the twentieth century saw the process of mathematical modeling adopted in many fields outside physics and engineering.

In the first two decades of the twentieth century, Albert Einstein (1879–1955) developed his theories of relativity, mathematical models that predict gravitational processes on the planetary scale more accurately than Newton's—now called "classical"—mechanics. Alfred Lotka (1880–1949) and Vito Volterra (1860–1940) worked in the 1920s on models of the interaction between predator and prey species, each arriving at the same model using different assumptions and arguments about how variables interact. Population models continue to be explored and refined through the present day. George Danzig (1914–2005) developed the simplex algorithm in 1947 to solve the mixing, supply chain, and other logistical problems that arose in World War II; these problems could be modeled with the well-understood linear programming approach, but the problems had so many variables and constraints that they were too complex to solve without computers. Linear programming is arguably the mathematical model most used in business and agriculture today. Edward Lorenz (1917–2008) developed one of the first nonlinear models for the atmosphere in the early 1960s, a precursor to the sophisticated climate models of today. His model displayed a very interesting sensitivity to initial conditions, and the study of this and similar models led to the field of chaos theory.

Twenty-First Century

In the twenty-first century, the use of mathematical modeling is ubiquitous across many research areas and academic disciplines. The Society for Mathematical Psychology publishes research in mathematical models used to examine psychological problems in neurology and cognition. The Society for Mathematical Biology concerns itself with applications of mathematics to modeling complex ecological systems, genetics, medicine, and cell biology. *The Journal of Mathematical Chemistry* is published by Springer-Verlag to provide a venue for researchers to share results from mathematical models of molecular behavior and chemical reactions. The American Sociological Association Section for Mathematical Sociology meets regularly to share research that uses "the language of mathematics to describe the structure, explain the events, and predict the dynamics of the social world." The American Institute of Physics publishes the *Journal of Mathematical Physics*, which focuses on applications of mathematical modeling to classical mechanics and quantum physics. The field of operations research, also called "management science," has evolved with the goal of solving mathematical models to determine the best business decision (often the maximum profit or minimum cost) given a situation in which there are limited resources. The Institute for Operations Research and the Management Sciences (INFORMS) is one of many organizations that publish results from this discipline.

The research presented through these venues tackles a diverse range of real-world problems. In medicine, mathematical models for physical principles of flow and pressure have been adapted and expanded to model the heart as a double-chambered pump. The flow of blood through the vessels can be examined and the parameters for flexibility of the vessels can be changed to investigate health conditions, such as hardening of the arteries brought on by aging. The ideal gas law and equations governing transport have been used to model how the lungs function to transport oxygen from the air inhaled to the blood in the aveoli in exchange for carbon dioxide. In the social sciences, Markov processes (mathematical matrices of transition probabilities) have been used to model social mobility and vacancy chains (the notion that a vacancy in a company or a house causes a chain reaction as others move in to fill the vacancy) as well as recidivism (the likelihood that a criminal will become a repeat offender). Generalizations of the Navier–Stokes and Lorenz models have been used to model the Earth's atmosphere, including sources of pollution and other greenhouse gases in an effort to prove and to predict the presence (or absence) of global warming. Scientists at NASA's Goddard Institute for Space Studies conduct research in three-dimensional atmospheric circulation models and in coupled atmosphere-ocean models in an effort to understand climate sensitivity.

With the advent and development of computers, increasingly sophisticated situations can be modeled and approximate solutions or simulations obtained using numerical algorithms. With modern computers to do the heavy computational work solving or simulating the mathematics, the most challenging step in the process is often the formulation of the mathematical model.

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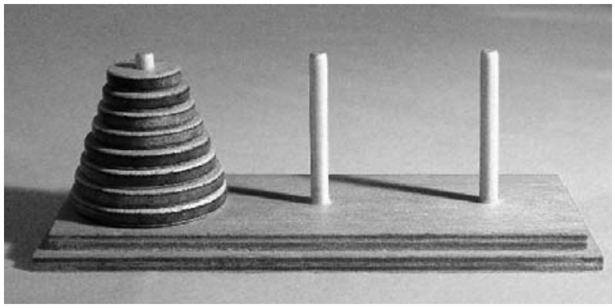
Holly Hirst

See Also: Mathematics, Applied; Mathematics Research, Interdisciplinary; Visualization.

Mathematical Puzzles

Category: Games, Sport, and Recreation.
Fields of Study: Algebra; Geometry; Reasoning and Proof.
Summary: The emphasis on problem solving in mathematics lends itself well to puzzles.

When considering mathematical puzzles, there are really two different types of puzzles available. Some puzzles are mathematical in nature, but require no mathematics to solve—similar to games like checkers, chess, and tic-tac-toe. Other puzzles are mathematical in nature and require a certain level of mathematics to solve—similar to games like cryptograms and Sudoku. Sometimes mathematical puzzles are referred to as "brainteasers."



A model set of the Tower of Hanoi (using only eight disks), among the oldest known mathematical puzzles.

Tower of Hanoi

One of the oldest mathematical puzzles is the Tower of Hanoi. This puzzle was developed in 1883 by French mathematician Édouard Lucas. In the game, the player has several disks of different sizes and three pegs. The object is to move all of the disks from the starting peg to a different peg, according to the rule that a disc can only be placed on an empty peg or on top of a larger disc. In the legend believed to have inspired the game, there is a Vietnamese temple in Hanoi that contains a large room with three posts surrounded by 64 golden disks. The temple priests perpetually move the disks, according to the rules of the puzzle. According to the legend, when they are done, the world will end. If the legend were true, and if the priests moved disks at a rate of one per second, it would take them a minimum of 18,446,744,073,709,551,615 turns to finish—585 billion years. In general, the number of starting disks will determine the minimum number of moves to solve the puzzle.

To move a single disk requires only one move. To move two disks (D_1 and D_2 with the smaller number being the smaller, or topmost, disk) would require three moves: (1) D_1 to an empty, (2) D_2 to an empty, and (3) D_1 onto D_2. Three disks would require seven moves: move the top two disks as described above (three moves), move the last (bottom) disk to the empty, then move the two-disk stack onto the third disk (another three moves). A fourth disk would similarly require $7 + 1 + 7 = 15$ moves. Using this pattern, the minimum number of moves for an additional disk will be double what the previous number of layers took plus one. However, to find the minimum number of moves for 10 disks, one needs to know what the minimum number of moves for nine disks would be. For nine disks, one needs to know the minimum number of moves for eight disks, and so on. Although a working recursive formula exists, it is not helpful for large numbers of disks. However, there is a pattern that can be found looking at the minimum number of moves for a certain number of disks that can be used to determine the minimum number of moves for any number of disks. In general, if there are n disks, the minimum number of moves to solve the tower problem will be $2^n - 1$.

Two-Container Problem

Another old mathematics puzzle that was used in the 1995 movie *Die Hard with a Vengeance* involves two containers of different sizes that are used to measure a different third value. For example, in the movie, the characters were given a 5-gallon and a 3-gallon container and needed to measure exactly 4 gallons of water. It is assumed that there is an unlimited amount of water to pour into either container, and that contents of either container can be poured down a drain. Other versions of this puzzle can be formed by changing the size of the original containers or the quantity needed at the end. If the containers have capacities that are relatively prime to one another (greatest common factor is one), then any number less than the bigger container can be achieved. If the capacities are not relatively prime, then only certain values can be obtained. For this specific version, if x equals the number of times the 5-gallon container is filled and y equals the number of times the 3-gallon container is filled, the problem can be rewritten as an equation in two variables: $5x + 3y = 4$.

Any ordered-pair solution to this equation will be a solution to the problem, although the method would still have to be determined. In the movie, the solution they found was $(2, -2)$. The five-gallon bottle needed to be filled two times and the three-gallon bottle needed to be emptied twice (hence, the negative number). To actually solve the problem, they would have to fill the five-gallon container (first fill) and use it to fill the three-gallon container, leaving two gallons in the five-gallon container. The three-gallon container would then be emptied (first empty) and the remaining two gallons poured into the three-gallon container. The five-gallon container would then be filled again (second fill) and used to pour into the three-gallon container. Since the three-gallon container would have two gallons of water already inside, it would only hold one more gallon, leaving four gallons in the five-gallon container. The three-gallon container would then be emptied (second empty), leaving exactly four gallons. An alternate solution to this equation is $(-1, 3)$.

Cabbage, Goat, Wolf

Another type of mathematical puzzle involves three objects and a keeper. As long as the keeper is present, all objects will remain safe, but if the keeper were to leave certain pairs of objects together unsupervised, at least one would be destroyed. For example, a farmer needs to transport cabbage, a goat, and a wolf across a river. The farmer is the only one who can row the boat and the boat is only large enough to carry the farmer and one other object. The goat and the cabbage cannot be left alone together as the goat would eat the cabbage. Similarly, the wolf and the goat cannot be left together as the wolf would eat the goat. The wolf has no interest in the cabbage, so that pair can be left alone together. The task is to determine how the farmer will get all three objects across the river.

On the initial row, the farmer's only option is to take the goat. If he takes the cabbage, the goat is eaten. If he takes the wolf, the cabbage gets eaten. Once the goat is on the other side, the farmer leaves the goat and returns across the lake alone. The farmer must now choose to take either the cabbage or the wolf to the other side. The farmer returns to the first side with the goat and swaps the goat for the last object on the original side. Upon crossing the river, the farmer now leaves both the cabbage and the wolf on the opposite side of the river and returns to the original side with an empty boat in anticipation of picking up the goat. One final row allows the farmer and all three objects to be on the far side of the river.

Squaring a Double-Digit Number

Some mathematics puzzles take the form of mathematics magic. For example, if a spectator calls out any two-digit number, the mathematician can square the number without a calculator in a short amount of time—with practice, faster than a human verifying it on a calculator. Finding the square of some numbers is easy; for example, any multiple of 10 (such as 10, 20, or 30). All that is needed is to square the 10s digit and concatenate two zeros to the right. For instance, 70 squared would be 4900. A number that has a five in the ones digit is also easy to square; merely take the

10 digit, multiply it by the next-highest integer, and concatenate a 25 to the right. For example, to find 75 squared, take $7 \times 8 = 56$, then append 25 to get 5625. However, there are 90 possible two-digit numbers that could be called out and only 18 that fit one of the patterns above. For the remainder, the mathematician can employ a principle referred to as "squaring a binomial," which is expressed algebraically as

$$(A+B)^2 = A^2 + 2AB + B^2.$$

If one needs to square a different two-digit number, such as 43, mentally rewrite 43 as $(40+3)$. Using the above formula, the square can be found by

$$\begin{aligned} 43^2 &= (40+3)^2 = 40^2 + 2(40)(3) + 3^2 \\ &= 1600 + 240 + 9 = 1849. \end{aligned}$$

As mentioned above, 40 is a multiple of 10 and easy to square; similarly, 3 is easy to square. The more difficult part of the formula to calculate in one's head is the middle—take 40 times 3 and double it. Then, add those three numbers together to get the square of the original number.

Squaring a number that has a 5 in the ones digit is a special case of squaring the binomial. If t equals the tens digit, then $10t+5$ is the original number. Squaring the binomial yields

$$\begin{aligned} (10t+5)^2 &= (10t)^2 + 2(10t)(5) + 5^2 \\ &= 100t^2 + 100t + 25. \end{aligned}$$

Factoring $100t$ from the first two terms yields

$$100t(t+1) + 25.$$

Martin Gardner and Recreational Mathematics

Martin Gardner (1914–2010), an American mathematician, specialized in recreational mathematical games. From 1956 to 1981 he wrote *Scientific American* magazine's Mathematical Games column and is credited by many for almost single-handedly sustaining and nurturing interest in recreational mathematics for much of the twentieth century. The kind of mathematical games Gardner wrote about are still being promoted not only for training children's minds for mathematics, both in and out of school, but also for helping older citizens maintain sharp minds. In addition to paper and pencil books, there are many Web sites aimed at seniors that have mathematical puzzle collections, and popular handheld gaming devices (like the Nintendo DS) are now being targeting at consumers in all age groups for mathematics and memory games.

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Chad T. Lower

See Also: Acrostics and Crosswords; Birthday Problem; Deep Submergence Vehicles; Magic; Puzzles; Sudoku; Tic-Tac-Toe.

Mathematician Defined

Category: Mathematics Culture and Identity.
Fields of Study: Communication; Connections.
Summary: Mathematicians work in a variety of fields and contribute widely to society.

Broadly construed, a mathematician is anyone who actively researches or studies mathematics. Many mathematicians work in academia as professors, involved in teaching, new research, or (most commonly) a combination of both. However, mathematicians are also employed in large numbers by industry, and there are innumerable amateur mathematicians who are drawn to mathematics, pursuing its study and research as an avocation. Some mathematicians, called "applied mathematicians," use mathematical ideas to solve problems arising in other disciplines; others, "pure" or "theoretical mathematicians" focus on furthering mathematics for its own sake. Of course, many mathematicians belong to both categories. The image of the mathematician is somewhat stereotyped in popular culture, but, in fact, mathematicians comprise an extremely diverse group. Mathematicians are women and men, girls and boys, old and young, and come from every country and culture.

If mathematicians are introduced to other mathematicians, they are unlikely to describe themselves as just "mathematicians." More often they would use a more precise term indicating their primary research interests, such as "number theorist," "analyst," "algebraist," "combinatorialist," "probabilists," or "logicians." The degree of specialization varies widely from one mathematician to the next. A mathematician may have only one area of research interest or may work across several. It is now seen as impossible for any single person to be expert in all areas of mathematics, but there are still so-called "generalists" who work in as many branches of mathematics as possible.

Mathematicians, Scientists, and Poets

While there are some overlap and blurred boundaries between the terms "mathematics" and "science" as these terms are used in ordinary discourse, the terms "mathematician" and "scientist" are usually used with more clearly distinct meanings. Scientists apply the scientific method, a continual process of investigating phenomena, collecting empirical data, formulating explanation hypotheses, and testing them by experiment; for scientists, experiments and empirical data provide the ultimate test of a theory. While mathematicians may also use experiments as part of their work, this is chiefly as a source of inspiration, as an aid in formulating conjectures and understanding complex concepts. Some might make the distinction that the scientist is generally an inductive reasoner, while the mathematician is generally a deductive reasoner. However, many applied mathematicians and statisticians may be more like scientists in this regard. To many people, it might seem that mathematicians and poets are polar opposites, or at the very least unrelated. It is remarkable, as such, how often great mathematicians and great poets speak of the vocations as intertwined. For example, Russian mathematician Sonia Kovalevsky (1850–1891) wrote, "It is impossible to be a mathematician without being a poet in soul." Likewise German mathematician Karl Weierstrass (1815–1897) wrote, "A mathematician who is not also something of a poet will never be a complete mathematician." From the other direction, the great English poet John Dryden (1631–1700) wrote, "A man should be learned in several sciences, and should have . . . , in some measure, a mathematical mind, to be a complete poet." Of course,

Rényi's and Erdös's Joke

A well-known joke definition among the mathematics community says that "A mathematician is a device for turning coffee into theorems." This aphorism evokes the image of mathematicians holed up alone in offices, drinking coffee by the pot as they fill blackboards and notebooks with computations and it is often attributed to the famous mathematician Paul Erdös. Though Erdös certainly did popularize this quip, it is more likely that it originated with his friend and colleague Alfréd Rényi. This notion comes from the then-nascent popularity of coffeehouses as gathering places for the European mathematicians. The stimulative effects of caffeine on the mathematician's brain, now well known, were still a relatively recent discovery in the early-to-mid twentieth century.

Paul Erdös (1913–1986) was a legendarily prolific Hungarian mathematician and author of more published mathematical papers than any other mathematician. He collaborated with hundreds of mathematicians in diverse areas of mathematics, including combinatorics, number theory, classical analysis, graph theory, and probability.

Alfréd Rényi (1921–1970) was another Hungarian mathematician, a frequent collaborator and a friend of Erdös. He was primarily a probability theorist, but is also remembered for important contributions to number theory, graph theory, and combinatorics.

Rota and What Mathematicians Do

Mathematician and philosopher Gian-Carlo Rota wrote, "We often hear that mathematics consists mainly in 'proving theorems.' Is a writer's job mainly that of 'writing sentences'? A mathematician's work is mostly a tangle of guesswork, analogy, wishful thinking, and frustration. . . ." Rota goes on to write that the proofs emerge only later, after the mathematician has explored the problem. The proofs allow mathematicians to be sure that they are doing more than guessing and also encapsulate the relationships among different mathematical concepts and objects. It is these relationships among ideas, patterns, and structures that mathematicians are chiefly involved in exploring. Here Rota is writing against the wide gulf between what mathematicians do and what nonmathematicians typically imagine that mathematicians do. There is an also an implicit comparison of mathematicians and other professionals (such as writers) whose role also involves making meanings, exploring patterns, and explaining ideas.

Gian-Carlo Rota (1932–1999) was an Italian-American mathematician and philosopher, unique in holding professorships in both subjects at the Massachusetts Institute of Technology. His mathematical work was chiefly in functional analysis and combinatorics; his main philosophical work was in phenomenology.

there are many major differences between the job of the mathematician and that of the poet. For one, poetry is in some sense purely subjective, while the mathematician is judged on grounds both objective (for example, "is this proof correct?") and subjective (for example, "are these ideas beautiful? Important?"). Let mathematician G. H. Hardy (1877–1947) have the last word here: "A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas."

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Michael "Cap" Khoury

See Also: Mathematics, Applied; Mathematics, Defined; Mathematics Genealogy Project; Mathematics, Theoretical; Movies, Mathematics in.

Mathematicians, Amateur

Category: Mathematics Culture and Identity.
Fields of Study: Communication; Connections; Problem Solving.
Summary: Mathematics has appealed to amateurs as a recreation and even without the rigor of the academy and peer review they have made strong contributions.

Historically, amateurs around the world have made significant contributions to mathematics in amazing and diverse ways. Can anyone now contribute to the development of contemporary mathematics, or can only professionally trained individuals do so? Answering this question requires reflection on both the ways in which mathematical research develops and the nature of the community that defines who is accepted as a mathematician.

The Nature of Mathematics

There are many examples of self-taught mathematicians or part-time mathematicians whose main professions or training was in another field. Some of these are well known in the history of mathematics, like Albert Einstein (1879–1955), who showed an early interest in mathematics by teaching himself geometric concepts at the age of 12. Gottfried Leibniz's (1646–1716) and Pierre de Fermat's (1601–1665) initial formal training was in law, not mathematics. Srinivasa Ramanujan (1887–1920) is cited as a mathematical genius who was self-taught.

People such as these raise the question about the nature of mathematical ability. Experiments on very young children have indicated that all individuals have the innate ability to recognize quantitative differences when the quantities are small. Further, lesions in the angular gyrus within the inferior parietal cortex of the brain can significantly impair mathematical ability, while the inferior parietal lobe region of Einstein's brain was 15 times larger than normal. In approaching the physical world, humans utilize number sense, pattern identification, and spatial awareness. These concepts contribute to mathematical reasoning. Contemporary mathematics as an academic discipline often requires high-level abstraction and complex symbolization.

Mathematicians like Reuben Hersh and Ian Stewart hold that abstract mathematical objects are cultural creations. Although it is true that any contribution to mathematics must take an account of the cultural context, this still leaves open the question whether only individuals professionally trained in these traditions can make contributions to the development of mathematics. For example, Leibniz sought a tutor, and Einstein immersed himself in these cultural traditions. Further, over the course of the twentieth century, mathematics has become increasingly professionalized. Professions consist of individuals with specialized training who, as a result, are granted a large degree of autonomy and self-policing oversight in determining what does and does not constitute acceptable mathematical thought by setting the appropriate standards, methods, and problems of the discipline. These functions are embodied in institutions, such as mathematics departments in universities and mathematical periodicals, societies, and conventions.

Albert Einstein taught himself geometry as a child.

Some conference talks are by invitation only, and, in other cases, a conference or session organizer selects from submissions. Journal editors and reviewers decide what is appropriate for publication. In this way, the people in the mathematical community determine standards and recognition or rejection of ideas and results. Over the twentieth century, mathematics developed through higher levels of generalization and abstraction using the axiomatic method, by cross-fertilization among different mathematical fields, by developing new mathematical theories in an attempt to solve a given mathematical problem, and by examining the foundations of mathematics as a mathematical problem. All of these processes require immersion within the discipline.

Amateur Contributions

It seems that by the mid-twentieth century, barring any new approaches, the mathematical universalist was a thing of the past, which leads to the question of whether this leaves any space for the amateur mathematician. The mathematical profession generally holds that an individual without formal credentials in mathematics could not engage in significant mathematical research or make any meaningful contributions to the discipline. However, there are several areas that are, in theory, still open to amateurs. The development of new forms of mathematics from nonmathematical considerations, the applications of abstract mathematics to real world problems, and discoveries of solutions to specific mathematical problems are three possible ways in which amateurs can make contributions. Further, amateurs can identify mathematical problems, topics, and subject matters that professionals do not recognize. They can conceptualize mathematical problems in ways that the professionals do not with new definitions or proofs. They can develop new symbolic notations that assist in solving existing mathematical problems. Finally, they can develop new methods for solving mathematical problems.

High school students have published their discoveries, such as Ryan Morgan in 1994. Someone outside the profession can have fresh, fruitful insights. Indeed, it has been argued that disciplines go through periods of normal change in which there is development of existing paradigms and revolutionary periods in which basic paradigms change. Often, the revolutionary stage is initiated by individuals outside or at the margin of the discipline. Stock market analyst Robert Prechter, Jr.'s (1949–) love of mathematics, deep belief in the mathematical structure of the universe, and search for innovative ways to apply mathematics to develop an understanding of the real world make him an interesting amateur mathematician. One of Prechter's goals has been to identify Fibonacci growth patterns in the stock markets. Prechter stated, "We would love to see Leonardo Fibonacci (c. 1175–1240) at least make the list of contenders for the real Man of the Millennium."

Obstacles

However, an individual may face an uphill battle to have his or her work understood and accepted. This battle is made even more difficult by the existence of a plethora of what are often called "mathematical cranks" (individuals who claim to be able to solve all sorts of mathematical problems, but often just produce aimless ramblings). English philosopher Thomas Hobbes (1588–1679), who claimed to be able to square the circle, was such a person. Others have submitted proofs of Fermat's Last Theorem. Notable mathematicians are bombarded by such claims, making them less receptive to genuine amateur innovators. For instance, Ramanujan wrote letters about his work to mathematicians outside India. However, his mathematical writing was not the same as the standard communication at the time, and he was ignored until Godfrey "G. H." Hardy (1877–1947) looked beyond the stylistic and notation issues and recognized his genius. This recognition was the beginning of a fruitful and well-known collaboration between them. Hardy noted:

> What was to be done in the way of teaching him modern mathematics? The limitations of his knowledge were as startling as its profundity. Here was a man who could work out modular equations and theorems of complex multiplication, to orders unheard of, whose mastery of continued fractions was, on the formal side at any rate, beyond that of any mathematician in the world, who had found for himself the functional equation of the Zeta-function, and the dominant terms of many of the most famous problems in the analytic theory of numbers; and he had never heard of a doubly periodic function or of Cauchy's theorem, and had indeed but the vaguest idea of what a function of a complex variable was. His ideas as to what constituted a mathematical proof were of the most shadowy description. All his results, new or old, right or wrong, had been arrived at by a process of mingled argument, intuition, and induction, of which he was entirely unable to give any coherent account.

Oliver Heaviside's (1850–1925) operator calculus work was not well received until Thomas Bromwich (1875–1929) justified the theory. Radio engineer and high school teacher Kurt Heegner's (1893–1965) algebraic number theory result was initially dismissed, but number theorist Harold Stark (1939–) filled in the gaps and the Stark–Heegner theorem is named for them. Another notable example is Thomas Fuller (1710–1790), a slave who could perform remarkable mental calculations. In 1788, abolitionists interviewed Fuller in order to demonstrate the superior intellectual abilities of African Americans. Historians do not know exactly how Thomas Fuller performed his calculations. However, they theorize that the algorithms he used were probably based on traditional African counting systems.

Dutch graphic artist M.C. Escher (1898–1972) and a San Diego homemaker, Marjorie Rice (1923–), have been cited as amateur mathematicians. They developed innovative approaches to geometric tiling and tessellations, which were introduced to the mathematical community by mathematicians like Doris Schattschneider (1939–). Some self-taught mathematicians are noted both for their work and for their other contributions to the mathematical community, such as Artemas Martin (1835–1918), who not only published articles but also was cited as having founded journals like the *American Mathematical Monthly* that paved the way for others who followed.

Some mathematicians have given stylistic advice to those who want to be taken seriously. Others identify mathematical puzzles or problems that could be solved by the amateur. For instance, some have noted that the question of whether $P = NP$ in theoretical computer science might be solved by an amateur, and others have noted the Beal Conjecture, named for Andrew Beal (1952–), a self-made billionaire and banker. Mathematicians and historians continue to publicize results from amateurs who might not otherwise be as known to the community, such as Mehmet Nadir (1856–1927), who is noted as an amateur mathematician in Ottoman Turkey, and geometric theorems on Japanese wooden tablets in temples that predate the work of Western mathematicians.

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See Also: Math Gene; Mathematical Puzzles; Mathematician Defined; Mathematics Research, Interdisciplinary; Professional Associations.

Mathematicians, Religious

Category: Friendship, Romance, and Religion.
Fields of Study: Communication; Connections.
Summary: Despite the emphasis in mathematics on logic, empiricism, and proof, many mathematicians have been influenced by religion.

In large part because of writings from the ancient world, cosmological and metaphysical dimensions of mathematical reasoning became closely connected with theological concerns, particularly in the West. Consequently some mathematical practitioners, communicators, and professionals used their knowledge to illuminate religious beliefs and doctrines. Others responded to spiritual convictions in ways that shaped their view of mathematics. Many influential mathematicians are religious, even in the twenty-first century. Noted Islamic mathematician Abu Ja'far Muhammad ibn Musa Al-Khwarizmi's ninth-century algebra treatise *Hisab al-jabr w'al-muqabala* originated the term "algebra," and the pious preface illustrates his Muslim beliefs. Brahmin mathematician Srinivasa Ramanujan (1887–1920) attributed his mathematical ability to the Hindu goddess Namagiri, his family deity. In the twenty-first century, some religious mathematicians have established formal groups, such as the Association of Christians in the Mathematical Sciences. There are also examples of religious leaders like Bharati Krishna Tirthaji, who also wrote mathematical works. Throughout history, there are mathematicians who have been persecuted because of religion. For example, mathematician Ludwig Bieberbach spoke out against Jewish professors in Germany, beginning in 1933. Mathematical historians and others have examined the contributions of people of various religions, such as the Incan religion or the percentage of Jewish mathematicians who have received mathematics' highest awards. Overall, there are numerous cases of those who dedicated themselves to working at, as well as commenting on, the intersection of religion and mathematics.

Roger Bacon

The legacy of Classical thinkers, most notably Plato and Aristotle, significantly influenced perspectives on mathematics through the Early Modern period (through the sixteenth century). One particular concern addressed during this long period involved articulating the appropriate relationships between mathematics and natural philosophy. Roger Bacon (1214–1294) dedicated much of his writing to establishing mathematics as an essential starting point for investigating fundamental areas of knowledge, which included both science and moral philosophy. Making such a claim had important theological implications that Bacon was keen to make explicit. Specifically, he maintained that those dedicated to the promotion of Christianity were obliged to teach mathematics, as this knowledge is prerequisite for the complete and correct interpretation of the scripture. For Bacon, the effective execution of both exegesis and church administration required the development of mathematical skills.

Nicholas Cusanus

Nicholas Cusanus (1401–1464), though primarily remembered for his philosophical and theological treatises, expended considerable effort on the problem of squaring the circle. His dedication went beyond that of many; for him the problem was replete with spiritual significance. For example, he admitted the impossibility of solving the problem exactly, yet continued to develop compass and ruler constructions that could provide a solution within a specific degree of accuracy. Any apparent inconsistency in these attitudes is explained by Cusanus' understanding of the divine. Specifically, humankind has no means for knowing God with certainty, although it can strive for increasingly more exact approximations of such unattainable knowledge. Much of Cusanus' exposition emphasizes practical reasoning based on geometrical figures. It does so as a way of underscoring the limitations of conjectural knowledge that, while inescapable, consistently encourage more fulsome reflection.

Blaise Pascal

His many achievements notwithstanding, Blaise Pascal (1623–1662) claimed that acquiring mathematical knowledge is of lesser significance than attaining spiritual knowledge. Still, his understanding of mathematics supports the positions he adopted on several theological matters. For example, the emerging notion of mathematical probability he helped to develop suggested to him that even though deterministic processes governed human salvation, individual outcomes could not be predicted with certainty. His belief that humankind should seriously consider the difference between seeking pleasure in this life and eternal happiness after death as a wager is indicative of influences that gave rise to probabilistic theorizing.

John Wallis

A theologian by training, John Wallis (1616–1703) was also the third Savilian Professor of Geometry at Oxford. His long-running dispute with philosopher Thomas Hobbes (1588–1679) partly focused on the nature of the infinite—in its potential and actual manifestations—and ranged across the domains of both mathematics and religion. Mathematical considerations also feature, if largely as a source of analogy, in his defense of Trinitarianism within the Anglican tradition. Like other mathematical divines who followed him, Wallis ultimately sought to promote religious doctrine in the face of new developments in mathematics and science that might undermine fundamental tenets.

Evidence of the ways in which Gottfried Leibniz (1646–1716) melded mathematical and religious thinking can be found across various essays and tracts. The essential feature of his position holds that the perfection of mathematics serves to reflect the perfection of God. Moreover, he believed that reason provided the most effective means of promoting true religion. The calculus ratiocinator emerges in relation to this fundamental belief. He maintained that reasoning based on the strict use of rules and symbols could serve religion in its capacity to convince nonbelievers. Additionally, Leibniz considered the binary representation of numbers to be strongly associated with the Creation, in which God created everything from nothing. That the binary representations of numbers exhibit periodic patterns in their digits was further evidence of the harmony embedded within God's creation.

The use of infinitesimals in Leibniz's development of calculus also exemplifies aspects of his theological position. They were essential to attaining knowledge of the infinite complexity of God's creation. For Leibniz, the contingent truths of the world were like irrational numbers insofar as they could only be approximated with finite methods.

Isaac Newton

Isaac Newton (1643–1727) opposed the metaphysical speculation of Leibniz and others, advocating instead the purer considerations associated with natural philosophy. Consequently, disagreements with Leibniz took on theological as well as mathematical dimensions. Newton was also a Unitarian—he did not subscribe to the notion of the Holy Trinity. This theological position bears on tensions he felt as the Lucasian Chair of Mathematics that he held at Cambridge (1669–1702). His heretical view made the idea of ordination in the Church of England, then a requirement of all fellows of Cambridge and Oxford, untenable. Even so, his 1687 text, *Philosophiae Naturalis Principia Mathematica*, reflects Newton's belief in an omnipresent God who created the universe and can intervene in its affairs. The rationalism represented by the text appealed to deists, who took a slightly different view. While sharing Newton's belief in His omnipresence, they denied that God takes an active role in the affairs of His creation.

Maria Agnesi

The contributions to mathematics made by Maria Agnesi (1718–1799) lie primarily in compiling and disseminating its knowledge. Her efforts also served a religious function as part of the Catholic reform movement of the eighteenth century, which sought to incorporate new modes of thought into teaching without jeopardizing church orthodoxy. The movement also called for extending educational opportunities, especially for women. Agnesi's efforts to present a practical account of analytic geometry and calculus are underpinned by these reformist commitments, as well as beliefs she shared with others regarding the power of mathematics, and its distinctive infallibility to religious contemplation. Her decision to develop her popular 1784 textbook *Analytical Institutions*, in ways that privilege geometric reasoning, which contrasts with the Leibnizian approach adopted by many of her Continental contemporaries, reflects these beliefs.

George Boole

Sensitive to the professional expectations of his day, mathematician and logician George Boole (1815–1864) carefully controlled expressions of his contentious and eclectic religious beliefs during his lifetime. There is little doubt, however, that an important aspect of his 1854 work, the *Laws of Thought*, was influenced by particular events and views having spiritual significance for him. Through an acquaintance with a Hebrew scholar during his youth, Boole became familiar with the Judaic tradition of describing the Divine in terms of an all-encompassing, if unknowable, unity. Later revelations, some mystical in natural, regarding this unitary perspective bore on his efforts to recast logic as

Other Religious Philosophies

Like John Wallis before them, George Salmon (1819–1904) and Ernest Barnes (1874–1953) commanded respect in their day as both mathematicians and theologians. Salmon, who long enjoyed a productive association with Arthur Cayley (1821–1895), maintained a tradition of guarding religious faith from speculations that attended new innovations in mathematics and science. He was particularly concerned with threats to the discipline's longstanding commitment to Platonic idealism and Euclidean geometry, as these constituted much of the common ground between mathematics and religion. Consequently, Salmon publicly criticized mathematicians like William Kingdon Clifford (1845–1879), who used new geometric knowledge as a platform for advocating a form of secular humanism. Some years later, Barnes attempted to make investigations into non-Euclidean geometry relevant to religion. In particular, he promoted the geometric understanding of a finite, yet unbounded, universe as part of a spiritual message that reflected the modernist as well as the cynical tendencies of the early twentieth century.

Mathematicians became more reluctant to comment on religious matters as the degree of professionalization within the discipline increased from the nineteenth century onward. Additionally, the failure to identify an uncontested foundation for mathematical certainty presented other philosophical obstacles. Recent research, however, has begun to consider the religious beliefs of mathematicians and the extent to which these relate to their work.

an algebraic system. In particular, the use of the symbol 1 to denote any universe of thought is an essential feature of the Boolean system. According to his wife, the source of much of the reliable bibliographic information on her husband, Boole was working on an unpublished text that was intended to emphasize the spiritual significance of the *Laws of Thought* during the final years of his life.

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See Also: Geometry of the Universe; Mathematical Certainty; Mathematics and Religion; Numbers and God; Probability; Religious Writings.

Mathematics, Applied

Category: Mathematics Culture and Identity.
Fields of Study: Connections; Representations.
Summary: Virtually all human pursuits depend on or were made possible by some application of mathematics, and historically applied mathematics often preceded the study of pure mathematics.

In the 1920s, the German military adapted, from a then recently developed business device, and began using an encryption code device known as the "Enigma" machine. Believing it to be an unbreakable encryption device, they continued to employ it into World War II. A significant effort to crack the Enigma code was undertaken, first by Polish mathematicians prior to the German invasion of Poland and then by British mathematicians at Bletchley Park, culminating in the breaking of the code in 1940. This success was instrumental in the ultimate Allied victory and in shortening the war significantly.

To this day, a key role of many applied mathematicians still involves encryption and cryptography, not just for military and defense purposes, but for a wide range of life activities, including computer and ATM security. In fact, more generally, there is almost no area of science, technology, and culture that is not heavily dependent upon the application of mathematical concepts and techniques. Applied mathematics therefore represents, in many ways, the ultimate multidisciplinary subject.

Historical Context

Although archaeological evidence is spotty and incomplete, it appears that the first mathematical efforts of civilized society involved either commerce, including accounting for transactions and inventories, or the measurement of land holdings for agricultural purposes. For these purposes, ancient Egyptians and Babylonians developed and applied basic concepts and techniques in arithmetic and geometry. Both peoples also used geometry in support of their building efforts and in the placement of monuments.

A large part of what is known about Egyptian mathematics comes from examination of the Rhind Papyrus. This document includes practical mathematical examples and exercises. It is apparent that the early development and purposes of mathematics were in response to, and in support of, practical, real-world problems, often of an engineering nature.

As with so many aspects of human culture and cognition, ancient Greece represented a shift—or, at least, the beginnings of a shift—in its approach to and philosophy regarding mathematics. There were certainly still applied mathematics problems, for example, involving navigation and astronomy. However, apropos of the birth in Greece of philosophical thought and reasoning, there was also some movement toward a reasoned approach to advancing mathematical knowledge. Thus, a divergence between pure and applied mathematics began to emerge.

There were several areas of mathematics in which inroads were made by the Greeks, for example, in geometry, trigonometry, logic and proof, and algebra (although work in algebra began with later Greeks). The Greeks also noted and struggled with irrational numbers (numbers that cannot be expressed as a ratio of integers). One can imagine, in an age where immediate physical needs and practical problems were paramount, that the inability to precisely measure a length (for example, not being able to precisely express the length of the diagonal of a square in terms of the known length of the sides) would have provided a conundrum.

Because of the Greek willingness to consider the theoretical, they were able to deal with, or accept, such situations to a degree. These situations led, however, to certain philosophical problems or paradoxes, such as Zeno's paradox, named for Zeno of Elea, that continue to challenge mathematicians. It can still be difficult for modern people, who initially in life are cognitively dependent upon experience and observation, to make the jump from the world of physically demonstrable, practical, applied mathematics to that of abstract and representative theory.

The Roman Empire had a largely practical, engineering-oriented approach to life, and this was manifested in their approach to mathematics. They were not particularly interested in expanding the horizons of mathematical theory. Instead, they used mathematics for applied engineering purposes from which emerged remarkable achievements that have survived through history.

A key application of mathematics beginning in the seventeenth century involved trying to understand and mathematically model the natural world. Certainly, there had been earlier efforts in that direction going back thousands of years—perhaps most notably by Ptolemy of Alexandria, with his extensive system of cycles and epicycles geared toward explaining, and ultimately predicting, the movements of heavenly bodies. But in the seventeenth century, with mathematicians and physicists such as Galileo Galilei and Isaac Newton, the modern effort to explain the world began in earnest.

Into and through the nineteenth century, a mathematician, like a scientist, was largely capable of understanding and keeping up with mathematical developments. With the explosion of mathematical activity in the twentieth century, it became impossible to do so, leading to a splitting of different specializations and mathematical disciplines and also a split between pure and applied mathematics, particularly in academic institutions. Interestingly, toward the end of the twentieth century and beginning of the twenty-first century, that separation seems to have lessened as each area began to appreciate more the usefulness of the other.

Substance of Applied Mathematics

It is difficult to comprehensively identify the substance of applied mathematics. In part, the difficulty is because of the overlap, which can take several forms, between pure and applied mathematics. First, a mathematical discovery or technique that initially seems without a practical application can, over time, become adopted and embraced by science and technology for practical application. Thus, to complain that an area of mathematics has no current usefulness can be potentially shortsighted; no one knows what future advances in society might be welcoming of—or possibly even made possible by—those pure mathematical excursions.

Another way in which pure and applied mathematics can overlap is simply in how such things are labeled. It is impossible to draw a clear line of demarcation between pure and applied mathematics. A new proof or technique made in a pure mathematics context may have very real practical applications, either now or later. Similarly, a practical, real-world problem may result in the development of a new approach with conceptual implications for theoretical mathematics. Furthermore, while many jobs require mathematical skills and techniques, such as architecture and engineering, they may not be technically classified as "applied mathematics" careers.

For example, a mathematical subject area such as number theory would generally not be considered an area of applied mathematics; and yet, it has significant implications and relevance for certain types of industrial applications, such as encoding. Similarly, abstract algebra would not, on the surface, seem to be applied; nevertheless, physicists now use group theory to better understand the world of elementary particles and quantum physics.

An important organization for applied mathematics is the Society for Industrial and Applied Mathematics (SIAM). According to its Web site, SIAM was organized in 1952 "to convey useful mathematical knowledge to other professionals who could implement mathematical theory for practical, industrial, or scientific use," and its membership in 2011 consisted of some 13,000 individuals and nearly 500 institutions.

A listing of some of the activity groups within SIAM serves to indicate the wide range of mathematics with important applications:

- Computational science and engineering
- Control and systems theory
- Dynamical systems
- Financial mathematics and engineering
- Geosciences
- Imaging science
- Life sciences
- Mathematical aspects of materials science
- Nonlinear waves and coherent structures
- Optimization
- Supercomputing

Example of an Applied Mathematics Discipline: Actuarial Science

Mathematics can be—and is—applicable to most any discipline. An example of an important and well-respected applied mathematics profession (which is generally ranked in the top five and sometimes at the very top of job-ratings surveys) is actuarial science. This applied mathematics career is representative of others and gives a sense of the general nature of applied mathematics work as well as its impact on society.

A well-respected applied mathematics profession is actuarial science—a truly interdisciplinary field.

The ability to manage risk—not necessarily to eradicate or even reduce it, but at least to "manage" its potential impact—is critical in a complex socio-economic environment. Without a way to manage risk, for example via an effective insurance industry, many activities that humans rely on might never happen (bridges might not be built and surgical procedures might not be undertaken) without the protection to society, organizations, and individuals that insurance provides. The ability to offer protection against the impact of risks is based on some key statistical ideas: the Law of Large Numbers and its related concepts. Only with a sophisticated understanding and application of probability and statistics can an effective risk management industry be sustained.

Actuarial science developed as the mathematical discipline underlying the analysis of risk contingencies. There are basically three types of actuaries: (1) life actuaries, who deal primarily with human mortality issues and life insurance; (2) pension actuaries, who focus on pension and retirement systems; and (3) property-casualty actuaries, who deal with other areas of risk and insurance, such as auto, homeowners, workers compensation, and medical malpractice insurance. Actuarial science is, in some ways, the ultimate interdisciplinary field.

Since risk applies to any type of endeavor or situation, an actuary attempting to quantify risk should potentially understand at least the fundamentals associated with almost all topics. One cannot adequately comprehend or evaluate a set of data without understanding where it came from and under what specific conditions it emerged. Thus, being an actuary or a risk analyst involves not only the relevant mathematics but also asking questions and learning about the context of the situation and using the findings to tailor mathematical methods appropriately. Furthermore, as with any quantitative discipline that uses sophisticated techniques, an effective actuary must be a very good communicator—able to translate mathematical concepts and techniques into understandable descriptions for nonmathematicians.

Becoming an actuary is a significant accomplishment. After earning an undergraduate degree (most often in either actuarial science or mathematics), actuaries spend several of their first careers both working at a job and studying for an extensive series of professional exams in an attempt to earn a designation or certification. These exams cover a variety of relevant areas, including specific actuarial techniques, finance and economics, and business processes.

On the job, actuaries use mathematics in an attempt to model real-world stochastic processes, such as the frequency and size of insurance losses, as well as economic and financial variables, such as interest rates, inflation, and investment performance. For example, based largely on historical data, an actuary might estimate that the frequency, or number, of claims that will occur in a given year is well-represented by a certain statistical distribution, such as a Poisson, named for Siméon-Denis Poisson, or a Negative Binomial. Similarly, given that a claim has occurred, historical loss information might suggest that the dollar size of a particular claim probabilistically follows another type of distribution, such as a Normal, Gamma, Lognormal, or Pareto, named for Vilfredo Pareto. Such decisions are largely based upon a thorough analysis of historical data, but other factors are also taken into account, including a qualitative understanding of the nature of the risks and hazards that the insurer is indemnifying and the entire socioeconomic context of the insurance activity. Once a model is developed, it provides a basis for not only prediction and analysis of appropriate future insurance policy rates but also testing the potential impact of making a variety of possible strategic or operational decisions, such as changes to the types of policyholders targeted and changes in policy provisions.

In the last few decades of the twentieth century, actuarial science and risk management became more technically sophisticated and more enterprise-wide in perspective. Part of the actuary's job is to understand the behavior of economic and financial variables and how they may impact the insurance and risk management process. For example, Brownian motion equations and concepts, named for Robert Brown, are frequently used to model the movements of interest rates and equity prices over time. Because insurance companies take in premiums but may not pay out corresponding losses for months or years, it is important to model how the insurer's investments may perform in the future. Insurers may even decide to sell some of their policies at an underwriting loss because they know that they have the opportunity to earn an adequate return on equity from the potential investment earnings on the premiums they take in as well as on their equity. By considering all aspects of an insurer's operations, including the effect of economic and financial conditions, the actuary's job has become much more holistic, or multidisciplinary.

Overall, an actuary's or risk analyst's job is one that is completely predicated upon mathematical techniques and quantitative skills, but it is also a business position. Skills involving communications, problem solving, project management, and teamwork are also essential for success in this environment.

Other Applied Mathematics Fields and Careers

The above description of actuarial science is representative of a variety of areas of applied mathematics. There are several other areas, including the following:

- *Biomathematics and biostatistics*. Applications of mathematics to biology have the potential to advance society and the human condition in substantial ways. Some of that advancement will come from mathematical modeling and analysis of genome and DNA mapping and sequencing. Another important area involves applying network analysis and dynamical systems techniques to the potential spread of infectious disease. Other areas include using geometry, topology, and other mathematical tools to examine and image brain activity; using differential equations and geometry to locate and attack tumors; and modeling human organs to allow testing of new surgical or other medical techniques.
- *Operations research*. Anything involving sequential processes can potentially be made more efficient and effective with the application of mathematical techniques and modeling. A few examples of such processes to be modeled include queue lines to limited resources, such as ATMs or grocery checkout machines; automobile traffic patterns; and Internet traffic. Like much of applied mathematics, the ultimate goal of operations research is to improve operational and strategic decision making.

- *Natural hazard modeling*. Hurricane and earthquake modeling are examples of interdisciplinary applied mathematics areas. Modelers need not only appropriate quantitative skills, such as geometry and systems of differential equations, but also an understanding of the appropriate science or technology, such as atmospheric sciences or geosciences.
- *Software, computers, and data*. Applied mathematics disciplines make use of computers, and some are very heavily dependent upon computational techniques and resources. In addition, numerous areas of mathematics play a role in careers in software engineering, data analysis, digitization, and compression.

Looking Forward

Ian Stewart, in the 2002 book *The Next 50 Years: Science in the First Half of the Twenty-First Century*, offers an essay titled "The Mathematics of 2050." In that chapter, he opines that several areas of mathematical exploration will undergo, and indeed are already undergoing, upswings or even revolutions. Among those he mentions are several areas of applied mathematics, including biomathematics and financial mathematics. Biomathematics certainly seems to be coming of age, and people's lives, and those of their immediate descendants, are being overwhelmingly affected by developments in this area.

One might argue, after the financial and economic crises of the first decade of the twenty-first century, that financial mathematics sustained a "black eye" that will suppress its credibility and potential. However, these same crises certainly made clear the importance of understanding the nature and potential impact of "risk" in the world, perhaps especially economic and financial risks. Being able to identify, quantify, and manage risk is critical to the smooth operation and advancement of society. This ability is simply impossible without a good understanding of the mathematical underpinnings of economics and finance and their attendant risks, as well as the ability to model different approaches and solutions to managing those risks.

It is, of course, difficult to hazard any guesses about long-term societal developments. However, one prospective application from the realm of science fiction is interesting to note. Isaac Asimov, in his *Foundation* series of stories and books, posited a mathematics-based "psychohistory." The stories focus on the legacy of Hari Seldon, a mathematician who built psychohistory into a statistical basis for modeling and predicting how human society will likely respond to various factors and stimuli. In the early twenty-first century, applied mathematicians are far from exhausting the potential of mathematics to change and advance society.

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See Also: Mathematical Modeling; Mathematics, Theoretical; Mathematics Research, Interdisciplinary; Statistics Education.

Mathematics, Arabic/Islamic

See *Arabic/Islamic Mathematics*

Mathematics, Babylonian

See *Babylonian Mathematics*

Mathematics, Defined

Category: Mathematics Culture and Identity.
Fields of Study: Communication; Connections; Reasoning and Proof; Representations.
Summary: Mathematics often begins with definitions; however, it is much more difficult to succinctly describe the whole of mathematics.

For most students, the subject of mathematics is studied on an almost daily basis from kindergarten through high school. But if asked, "What is mathematics?" the majority might struggle to formulate a meaningful answer, perhaps saying that mathematics is "arithmetic" or "algebra." But saying that the essence of mathematics is arithmetic is akin to saying that the essence of chemistry is the periodic table; mathematics and chemistry are both so much more. Not only may students have difficulty describing what mathematics is, but even professional mathematicians struggle to provide a succinct, convincing description of the nature of their subject of expertise.

In his delightful 1940 essay, *A Mathematician's Apology*, G. H. Hardy (1877–1947) spends about 150 pages presenting a passionate case for the meaning, essence, and importance of mathematics and the professional mathematician. Along the way, he offers many keen insights into the nature of mathematics:

> A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas. . . . The mathematician's patterns, like the painter's or the poet's, must be beautiful; the ideas, like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics.

This quote may appear odd; most people do not view mathematics as a creative endeavor, much less one that can be rightly considered "beautiful." Often, students who learn the subject view the discipline as one that is rigidly bound by rules, one in which there is always one right answer, and perhaps even that there is only one right path to follow. But mathematicians have a decidedly contrary viewpoint. Faced with an interesting problem to solve, the mathematician strives to have his full cadre of creativity flowing, perhaps asking: "What unusual approach might I take to solve this problem, one that nobody else has yet considered?" "How might I alter the problem to a new, related one that I might be able to solve first?" and "Is there new language or notation that I might introduce that makes the problem easier to understand or similar to another problem that is already well understood?"

More than this, as Hardy's quote alludes, mathematics is about more than individual problems; rather, it involves the study of patterns. If mathematicians can solve one particular problem, they are next interested in knowing if their methods extend to solving an entire collection of related problems. If a theorem can be proved to explain a wide class of situations, is it possible to extend the result to include even more possible scenarios? In this way, mathematics and mathematicians seek to recognize, understand, and explain patterns. Some of these patterns occur in the world around us; others may be purely theoretical. Once a pattern is understood or explained, mathematicians wonder if they have found the best explanation. What is "best"? While that is somewhat a matter of individual taste, most mathematicians agree that the best mathematics is clear, brief, and elegant. In Hardy's words, "the ideas . . . must fit together in a harmonious way." It usually takes a great deal of creative insight (creative thinking, creative writing, and creative problem solving) to make the ideas fit together in a harmonious way.

Philosophers on Mathematics

Philosophers have argued for centuries, even millennia, over the nature and meaning of mathematics. There are entire schools of thought—referred to with names like "intuitionism," "logicism," and "formalism"—that seek to explain what mathematics is. However, each somehow comes up short. Perhaps mathematics itself is simply too big to describe with a formal philosophical system. Some parts of mathematics do rely on our intuition and understanding of physical happenings in the surrounding world; other aspects of the subject rely considerably on the foundations of logic; and part of mathematics grows from the formal rules that seem to many to lie at its very core. But no one of these perspectives encompasses the entire subject nor satisfactorily describes its essence. Nor does any one of these perspectives fully answer the question of where mathematics exists. Is it embedded in the surrounding world, or is it a mental construct that rightly belongs to humanity's collective brain?

Nobel prize–winning physicist Eugene Wigner (1902–1995) was one scientist who recognized the beauty, power, and harmony of mathematics while still being somewhat mystified by its nature. In his own 1960 essay on the question of what mathematics is, titled "The Unreasonable Effectiveness of Mathematics," Wigner tells the story of a statistician sharing with a friend his work in analyzing population trends. In one of his key formulas, the symbol π arises. The friend asks, "What does that symbol represent?" The statistician notes that, as usual, the symbol is the familiar π associated with circles—the ratio of a circle's circumference to its diameter. The friend is incredulous, for how can the relationship between a circle's circumference and diameter have anything to do with how a population is distributed?

This story that opens the essay illustrates Wigner's broad point: mathematics is unreasonably effective, with abstract mathematical ideas emerging in remarkable and surprising places. As a scientist trying to understand the workings of the physical universe, Wigner was particularly mystified by how well mathematics helped him describe observable phenomena. In his words, "the enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and there is no rational explanation for it." He goes on to argue that somehow the very nature of mathematics, even though it is abstract and a mental construct, leads the way in describing the surrounding world and that somehow this is indicative of a deeper truth. He concludes the essay by observing, "the miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning."

To the pure mathematician, mathematics may be a quest to recognize, understand, and explain abstract patterns that arise in considered ideas. To the applied mathematician, mathematics may be a language that aptly describes patterns that emerge in some sort of physical reality. Somehow, it is the same mathematics in both cases, and the subject seems not to care whether or not it is used for abstract or applied purposes. The history of mathematics is filled with stories that show how mathematics emerges from the mental doodling of interested people, only later to find rich connections with other areas of mathematics itself, and then finally to spectacularly describe some deep physical reality.

As an example, the Greeks (c. 350 B.C.E.) came to know a beautiful number with marvelous abstract properties, today called the "golden ratio"

$$\varphi = \frac{\left(1+\sqrt{5}\right)}{2}.$$

This number, approximately 8/5, arises naturally from considering line segments or rectangles that can be divided in ways that are self-similar and possesses a wide variety of interesting geometric and numeric properties. Roughly 1500 years later, people in India (c. 1150) first encountered the so-called "Fibonacci numbers": (1, 1, 2, 3, 5, 8, 13, 21, 34, . . .), which come from starting with a pair of 1s, and then adding the preceding two numbers to create the next. Spectacular patterns and relationships exist among the Fibonacci numbers, and mathematicians have been fascinated with them since. An early observation showed that the ratios of consecutive Fibonacci numbers (5/3, 8/5, 13/8, 21/13, . . .) forms a sequence of numbers that converges to

$$\varphi = \frac{\left(1+\sqrt{5}\right)}{2}.$$

Much later, near the end of the twentieth century, mathematicians and biologists came to understand the apparent role that both Fibonacci numbers and the golden ratio play in explaining seed distributions in plants, such as coneflowers and sunflowers: the golden ratio appears to be the constant angle at which seeds are "born," and the relationships the golden ratio enjoys with the Fibonacci numbers help explain why this phenomena occurs, which one can better understand when the seeds in the flower are numbered.

What is Mathematics?

There is a great deal of delightful reading one can pursue to learn more about the nature of mathematics. Such investigation will help each person decide individual answers to the question "What is mathematics, really?" For the novice mathematician, Steven Strogatz has written the quintessential modern sequence of essays on the topic, essentially in the form of a blog for the *New York Times*. Strogatz, a prominent applied mathematician who has done groundbreaking work in the field of dynamical systems, begins with the wonderful 2010 essay "From Fish to Infinity," where his overall goal for the series is to be "writing about the elements of mathematics, from pre-school to grad school, for anyone out there who'd like to have a second chance at the subject—but this time from an adult perspective. It's not intended to be remedial." For the reader with a bit more mathematics background, one can consider the American Mathematical Society's Online Feature Column, a monthly column that takes a look at accessible mathematical research and (often) its applications. To begin, the interested reader might read David Austin's immensely popular 2006 explanation of Google's PageRank algorithm, "How Google Finds Your Needle in the Web's Haystack." For a more historical view, it is hard to beat the marvelous writing of Professor William Dunham in his 1990 book *Journey Through Genius*, which surveys some of the great theorems of mathematics.

An encyclopedia entry is a tiny start to describing the essence of mathematics. Each person must read, explore, think, and investigate to seek understanding of what mathematics really is. It is a beautiful example of the depth and complexity of mathematics itself that so many different perspectives on the subject ring true and that each person can find something unique in the subject.

Further Reading

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Matt Boelkins

See Also: Geometry of the Universe; Golden Ratio; Mathematician Defined; Parallel Postulate; Pythagorean School; Pythagorean Theorem.

Mathematics, Egyptian

See *Egyptian Mathematics*

Mathematics, Elegant

Category: Mathematics Culture and Identity.
Fields of Study: Communication; Reasoning and Proof.

Summary: A mathematical accomplishment may be considered elegant because of its conceptual depth, its aesthetic appeal, its importance and implications, its rigorousness, or the surprise of its results.

Elegant mathematics is an elusive idea, often being an aesthetical judgment determined subjectively as a reflection of one's knowledge and understanding of mathematics. That is, an intricate proof in number theory or analysis may be deemed "elegant" by mathematicians, but it would be mere nonsense to a struggling high school student. In turn, the visual "completing of the square" as a proof of the quadratic formula may be deemed elegant by high school students, but it would be too simplistic and inefficient to mathematicians. Thus, it is necessary to dig deeper into the meaning of "elegant mathematics," trying to focus on the many forms of mathematics—its methods, its visual aspects, and its role as a language.

The word "elegance" often is defined as an attribute that is effective and simple. Elegant mathematics can then be defined as mathematics that is effective and simple. However, this definition can be deceptive because one of the primary roles of mathematics is as a language, capitalizing on its ability to effectively record and model ideas and situations using a symbolic notation that is both effective and simple. Thus, since mathematics is broader than a mere language, the view of aspects of mathematics as elegant should include other attributes, such as surprise, nontriviality, consistency, power, conceptual depth, and even beauty.

When discussing elegant mathematics, mathematicians usually refer to proofs as prime examples, shifting the focus from the correctness of the proof's logical structure to its effectiveness and simplicity. Specific elements that suggest elegance are the following:

- Uses a minimal number of necessary assumptions
- Is unusually succinct yet understandable
- Avoids complex or laborious calculations
- Offers a surprising path from assumptions to conclusion
- Models "out-of-the-box" thinking
- Achieves a difficult result with a minimum of work
- Includes original conceptual insights that clarify both the "how" and the "why"
- Can be generalized to a broader context or set of problems
- Displays the power of mathematics as both a method and a language

An example of an elegant mathematical proof is Euclid's proof that an infinite number of primes exist. Using the process of "reductio ad absurdum," assume that the number of primes is finite, which may be written as $p_1, p_2, p_3, \ldots, p_n$. Let

$$N = p_1 p_2 p_3 \cdots p_{n-1} p_n + 1$$

which cannot be prime because $N > p_n$. Thus, N must be composite and have a prime factor. However, all of the known primes $p_1, p_2, p_3, \ldots, p_n$ are not factors of N because on division they leave a remainder of 1. Thus, there must exist another prime $q > p_i$ for all i, such that q is a factor of N. But, this is a contradiction of the original assumption, and the number of primes is infinite. Euclid's proof is mathematically elegant because it is effective, simple, powerful, and surprising. Many other mathematical proofs are regarded as elegant, such as these examples:

- Archimedes's use of mechanical concepts to prove that the volume of a sphere is two-thirds the volume of its circumscribing cylinder
- The Chinese "Behold!" proof of the Pythagorean Theorem
- Fourier's use of series to prove that the number *e* is irrational
- Euler's proofs involving infinite series
- Cantor's diagonal proof of the countability of the rationals, as well as his related proof that the reals are not countable

Paul Erdös, a Hungarian mathematician, often referred to an imaginary book in which God had included all the most beautiful or elegant proofs in mathematics. Then, when he came across a proof that he felt was elegant, Erdös would suggest, "This one's from The Book!" In the 1990s, Martin Aigner and Günter Ziegler capitalized on Erdös's ideas and published *Proofs From THE BOOK*. The most recent edition (2009) includes 30 sections involving elegant proofs from number theory, geometry, combinatorics, analysis, and graph theory.

Inelegant Proofs

A mathematics proof that is not elegant is viewed as ugly, laborious, awkward, or pedantic. Inelegant mathematical proofs often involve computer-based computations that cannot be easily replicated by mathematicians within a reasonable time frame. These inelegant yet effective proofs are akin to proofs by exhaustion involving a great number of cases, thereby disguising any elements of brevity or simplicity. A primary example of such a proof is Kenneth Appel and Wolfgang Haken's proof of the Four Color Theorem in 1976. Despite their use of some clever categorizing techniques, the final steps in the proof required more than 1000 hours of computer time to check 1,936 maps of reducible configurations as possible counterexamples. In fact, some mathematicians do not accept the proof because of its reliance on computers. Yet, the Four Color Theorem as a conceptual statement is itself considered to be elegant.

Elegant Versus Ugly

Famous mathematicians such as Bertrand Russell, G. H. Hardy, Richard Feynman, and Paul Erdös also have shared their opinions relative to the distinctions between elegant and inelegant proofs (or mathematics, in general), often taking strong stands. For example, in a letter to Max Wertheimer, Albert Einstein even discussed the distinctions between elegant and ugly proofs. For him, a proof was ugly if it depended on the artificial introduction of additional elements, such as constructing auxiliary lines, which distracted the reader from the flow and "symmetry" of a proof. In his letter, Einstein provides both elegant and ugly examples of proofs of Menelaus's Theorem on Colinearity.

Authors have jumped on this "elegant versus ugly" bandwagon, extending it by their evaluations of both the proof and the conceptual claims associated with a mathematical theorem. The result is published resources such as *The Most Beautiful Mathematical Formulas* (1992) and *An Introduction to the World's Most Elegant Mathematics* (2006).

Unfortunately, the sorting process is not as straightforward as these authors suggest. Often, mathematicians vacillate, being unsure in the classification of a proof as either elegant or inelegant. A current example of this indecision is Andrew Wiles's proof of Fermat's Last Theorem, which conjectures that no three whole numbers a, b, and c can satisfy the equation $a^n + b^n = c^n$ for any integral value of $n > 2$. On one level, Wiles's approach was ingenious (and thereby elegant) in his use of elliptic curve theory and modular forms to solve this famous extension of the Pythagorean Theorem. And on the other hand, Wiles's final proof is inelegant because it involves more than 100 pages of very difficult mathematics that deters both mathematicians and non-mathematicians. The same can be said for the proof of the Monster Group.

Elegance

Moving the focus beyond proofs alone, mathematicians tend to classify mathematical ideas, such as theorems and concepts, as "elegant" if they establish insightful connections between two areas of mathematics that were assumed to be unrelated. The most famous example perhaps is Leonard Euler's identity that relates special mathematical constants: $e^{i\pi} + 1 = 0$.

Framed copies of this fascinating identity often will be found hanging on the walls of mathematicians' offices. It exudes simplicity and explains unexpected connections of several different mathematical ideas.

The symbolic simplicity of the above identity also illustrates the elegance of mathematics as a language. In fact, combinations of mathematical symbols with words can convey mathematical ideas that are simultaneously complex and powerful. Combined further with mathematical graphics, the elegance of mathematics as a language is enhanced by the ability to convey complex ideas efficiently, consistently, and with economy.

And, partially because of its elegance in form, mathematics also is the preferred language of the sciences and many other disciplines that involve quantitative models. Awareness of this elegance led physicist Eugene Wigner to write his famous essay, "The Unreasonable Effectiveness of Mathematics in the Natural Sciences" in 1960. He concludes his paper with the statement that "The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve."

It is expected that the idea of elegant mathematics will remain an elusive one, because it is a subjective judgment of the aesthetics of proof and ideas within mathematics. Though the quandary will lead to arguments, it should not have any impact on the continuing development of mathematics. That is, because of the nature of both elegance and mathematics, it is not possible to merge them as a thinking strategy. Rather, as history has demonstrated, the mathematics is first developed and proven and only then can the aesthetic judgments (elegant versus ugly) begin. And one cannot ignore the quality of the considerable mathematics that lies between these two extremes.

Further Reading

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Jerry Johnson

See Also: Mathematicians, Amateur; Mathematics, Theoretical; Proof.

Mathematics, Greek

See *Greek Mathematics*

Mathematics, Green

See *Green Mathematics*

Mathematics, Roman

See *Roman Mathematics*

Mathematics, Theoretical

Category: Mathematics Culture and Identity.
Fields of Study: Connections; Reasoning and Proof; Representations.
Summary: The complement to applied mathematics, theoretical mathematics advances the field without necessarily focusing on potential applications.

Often mathematics, as a discipline, is categorized in two general areas: theoretical mathematics and applied mathematics. When this is done, it is common to consider theoretical mathematics (or "pure" mathematics) as the part of mathematics that is carried out for the sheer pleasure of "doing mathematics," for the intrinsic beauty that lies in the study of the logical patterns and abstract relations that can be found when organizing space (geometry and topology), structures (algebra), quantities (number theory), approximations (analysis), and the thought behind these actions (logic and foundations). However, in many historical cases, the results found in theoretical mathematics have had a practical or "applied" value, often not foreseen—or even understood—until many years after their discovery. Such is the case, for example, of the non-Euclidean geometries that were seen to be coherent within their axiomatic systems but were not thought of as representing physical reality. Riemannian geometry, named for mathematician Bernhard Riemann who lived in the first part of the nineteenth century, became the mathematical context for Albert Einstein's General Theory of Relativity and led to other non-Euclidean applications in twentieth-century physics.

In much of the research done in theoretical mathematics, the focus is upon extending the field in which the particular mathematician involved is a specialist. Real world applications are not usually relevant to the activity of the pure mathematician, as these belong to the realm of applied mathematics. However, research in pure mathematics often involves the "application" of results to other mathematical objects. It is also important to emphasize that new knowledge in mathematics does not come about by experimentation but by proof.

Algebra: The Study of Structure

People often associate algebra with their experience in secondary school. Algebra studied at this level is known as "elementary algebra," and, while it is a big step in abstraction for the young student, it still focuses upon real numbers and arithmetic operations in which unknown variables are substituted for numbers. However, the abstract algebra studied and developed by theoretical mathematicians generalizes the structure of the real number system and its arithmetic operations by means of axioms and works with structures that have little to do with the numbers and operations learned in school. Some of the structures most studied in algebra include groups, rings, modules, vector spaces, and fields. These structures are defined by properties and operations. Theoretical mathematicians study the relations that are established between different representations of the same structure, or even between different structures. Once again, even though the exploration, discovery, and development of all these structures is the motivation and a goal in itself, these abstract structures have found applications in areas as diverse as crystallography, computer science, music, and physics.

Geometry and Topology: The Study of Space

The study of symmetries and rigid transformations, such as rotations, reflections, and translations, is associated with the Euclidean geometry that everyone studies in a secondary school program. Euclidean geometry arose from the need to measure and survey as territorial delimitation began to be registered and ancient civilizations developed sophisticated towns and cities. Euclidean geometry, with its study of flat space (where the shortest distance between two points is a straight line), was a faithful representation of "how the world really is." The discovery of the "other geometries" in the 1800s, when some theoretical mathematicians removed the parallel postulate from the axioms of Euclidean geometry, opened a world of possibilities (or possible geometrical worlds) for exploration on the level of theoretical mathematics. However, it was to be seen that the universe, both on the macrolevel (as in, for example, astronomy) and the microlevel (as in, for example, particle physics), could be much more faithfully described with non-Euclidean geometrical properties.

In general, geometry studies the properties that change when an object is deformed, while topology studies the properties that do not change when an object is deformed. For the topologist, a circle and a square are essentially the same, because there exists a continuous function that transforms one into the other. This is the reason that topology is often called "rubber band geometry." Whereas in geometry an object remains the same only under rigid transformations, in topology, as long as adjacent points continue to stay adjacent (which means that the object cannot be cut or twisted), the object is considered the same. The rubber band can be stretched, shaped as a square, ellipse, or circle; but points that are close remain close, and the rubber band itself does not change.

Although the study of topology is very axiomatic and theoretical, its results have had important applications in physics, biology, computer science, and robotics. For example, the study of DNA topology by applied mathematicians, together with biologists and chemists, uses results from the theoretical study of "rubber geometry."

Number Theory: The Study of Quantities

Number theory, also known as "higher arithmetic," studies the properties of the natural numbers and the integers as well as the properties of those structures that are a generalization of natural numbers and integers—those structures that maintain certain fundamental properties that these numbers possess. Some of these properties are as familiar as divisibility, prime factorization, or congruence, while others have arisen through conjectures that theoretical mathematicians have made. Some of these conjectures are extraordinarily easy to understand by any nonmathematician or young student, but they are also extraordinarily difficult to prove.

Such is the case, for example, of the now famous "Last Theorem of Fermat." The theorem states that $x^n + y^n = z^n$ can be true only for $n = 1$ or 2. For over 350 years, some of the best mathematical minds worked on this problem and could not find a proof. In 1995, a proof was presented, but it used some of the most sophisticated and modern mathematical concepts from other areas of pure mathematics to be found.

Number theory has been considered by some mathematicians as a paradigm of pure mathematics. However, since the appearance of computer science, number theory has been applied in a very practical way, especially in cryptography (the encoding of information) and random number generation for statistical analysis; it has even been applied in quantum mechanics.

Analysis: The Study of Approximation

Mathematical analysis began as the process of formalization and axiomatization of calculus, whose dependence on infinitesimally small quantities that "tend" to zero did not have a rigorous foundation. Today, analysis has branched out into different areas of interest. Real analysis is the study of the properties of sequences and functions of real numbers using notions such as limit, continuity, differentiation, and integration. There is also complex analysis, which studies similar notions in the context of the complex numbers, and functional analysis, which studies these notions and others properties of functions that are seen as objects in a "function space." Probability theory is also considered an area of analysis. Indeed, probability theory is a very abstract and axiomatic subject, based on set theory and measure theory.

It is worthwhile mentioning that in the 1960s an alternative axiomatization of the infinitesimal, known as "nonstandard analysis," was developed. There are mathematicians who advocate the use of this formalization as a basis for teaching calculus, given that the concept of "limits" is often difficult to comprehend for the beginning student. As seems to be the rule in theoretical mathematics, although the mathematician does not look for applications and the main goal is to expand the particular field of study, applications of analysis have found their way into science, engineering, and economics.

The Theoretical Mathematician: Training and Workplace

The educational systems in the world are not homogeneous, although once a student is at the level of graduate studies, equivalences are usually recognized. In many countries outside of the United States, an undergraduate program consists of a complete submersion in the field and virtually no courses outside the field are taken. In the United States, the majority of fields, mathematics included, are offered as majors at the undergraduate level; therefore, the number of courses taken in the particular area is less, as there are other general education requirements that need to be fulfilled. It is also common for students to take a minor in another area or even a double major.

However, people with undergraduate degrees in any part of the world will not be formally considered theoretical mathematicians. Theoretical mathematicians will have a graduate degree, almost always a Ph.D., and graduate studies are fairly homogeneous worldwide.

People trained in mathematics will have taken a full calculus sequence (single variable and multivariable), followed by an analysis sequence. As undergraduates, they often will have taken linear algebra, abstract algebra, discrete mathematics, and usually some topology or geometry. Once a student has opted to study pure mathematics, and is in a master's program, the student will orient electives to an area of interest. At the Ph.D. level, students still have to present doctoral comprehensive exams in the subjects of analysis, algebra, and, often, topology as requisites, independently of his or her area of specialization. Students will also present a comprehensive exam in their field of interest, and then they will do doctoral research, culminating in their doctoral dissertation. There are, of course, variants to this process. Some students will specialize in several fields; some will have done research in their master;s program and have produced master's theses.

The University of Cambridge established the Sadleirian Chair in pure mathematics and, since 1863, there have been eight professors who have held it. This position is usually considered a landmark in the recognition of pure mathematics as separate from applied mathematics. Universities, in general, do not have a standard approach to the separation of theoretical and applied mathematics. Some universities have a single mathematics department; some have mathematics and statistics departments in which applied mathematics is considered a concentration in mathematics. Sometimes computer science is part of the mathematics department, although this is not common at research universities.

Foundations of Mathematics: The Study of Thought

The study of the foundations of mathematical knowledge includes areas such as mathematical logic, axiomatic set theory, model theory, and category theory. The quest for understanding the foundations of mathematics is also part of the philosophy of mathematics.

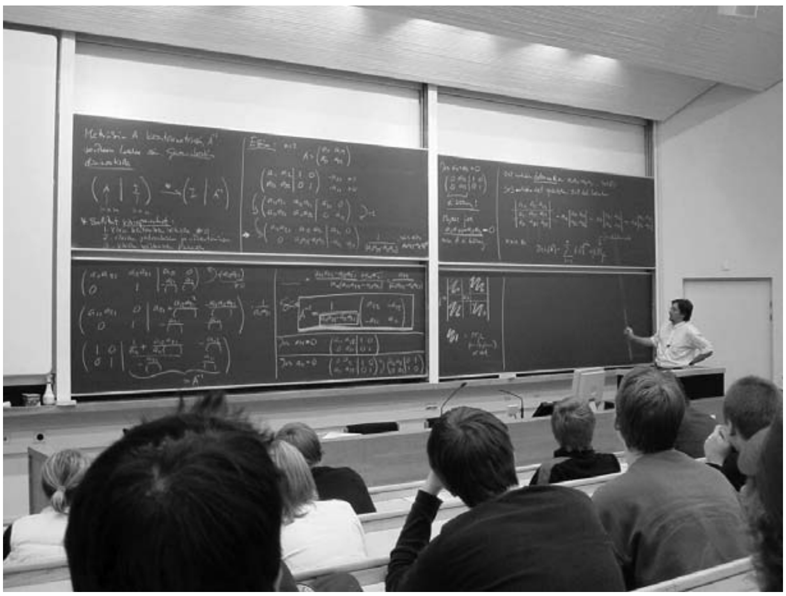
Much of the development of certain areas of theoretical mathematics occurred in the twentieth century; for example, topology has been based on set theory, which was presented in its axiomatic precision in the late nineteenth century. The axiomatization and actual arithmetization of infinite cardinalities was essential to much of the development of theoretical mathematics as well.

On the other hand, category theory, which abstracts the sets and functions from set theory to objects and morphisms and relies heavily on arrow diagrams to model mathematical behavior, has played an important role in pure mathematical areas, such as algebraic topology and algebraic geometry. These areas cross the rather artificial boundaries into which theoretical mathematics has been divided. For this reason, the importance of category theory can be seen, as this theory provides the notions that permit transit between different mathematical structures. At the end, theoretical mathematics is founded on the idea of the demonstration. The importance of Euclid, independently of geometry in itself, rests on the fact that he was the first to formalize the way that, to this day, theoretical mathematics is done and thought about.

The natural ambience of theoretical mathematicians is academia. In this context, they can transmit their knowledge, which is the product of many years of study, reflection, discovery, and creation, to future generations. Academia is also the place where theoretical mathematicians can have the time and resources to dedicate themselves to research. There are also institutions, albeit few, that support theoretical mathematicians to do research, usually at a stage in which they have already produced results and it is clear that they have a big probability of successfully obtaining new ones.

Employment in government and industry is usually reserved for the applied mathematician. However, there are theoretical mathematicians who also have applied knowledge that makes them attractive for these positions. The theoretical mathematician who ends up in an applied context can often provide insights, because of training, that will bring about novel ways of approaching concrete problems.

It is interesting that there is very little difference in the type of work and perspectives of theoretical mathematicians worldwide. The differences have more to do with the size and extension of the university systems in different countries, but the "culture" and daily life of the pure mathematician is remarkably homogeneous.



A mathematics lecture at Helsinki University of Technology in Finland. College students may study theoretical mathematical concepts.

The Germ of Theoretical Mathematics in School Mathematics

In many universities in the world, prospective schoolteachers who will be teaching mathematics must take a course, or courses, that analyze elementary mathematical concepts from an advanced point of view. This requirement is because many of the concepts that are present from the very beginning of mathematical instruction are very deep, although not necessary to understand for the young student who begins the procedure of basic mathematical operations. Felix Klein (1849–1925) is known for his work in geometry, where he demonstrated that the Euclidean and non-Euclidean geometries could be considered as special cases of projective geometry, that algebra (group theory) can be basic to the study of geometry, and other achievements in theoretical mathematics. The "Klein Bottle," a two-dimensional object from topological studies that can only be understood as a whole in a four-dimensional context, is named after him. His book *Elementary Mathematics from an Advanced Standpoint: Arithmetic, Algebra, Analysis* is made up of lectures to future teachers over a 20-year period.

At the elementary school level, for example, the concept of "infinity" is present from the moment that children learn to count with natural numbers. The notion of dimension appears when two- and three-dimensional objects are introduced geometrically, and abstraction is required when these physical objects are represented by formulas, often a first contact with algebra. The notions of number theory are omnipresent, for example, in the concept of "divisibility" and integer numbers. The concepts of "equivalence class" from algebra, and "limit" from analysis are also fundamental to work with both rational numbers and roots and real numbers and approximations. Notions from set theory and logic are implicit in teaching methods and explanations about many of the operations and concepts that are expected to be taught and learned at the school level. For this reason, the schoolteacher who is expected to communicate mathematical ideas should have a basic understanding of many of the concepts of theoretical mathematics. Further, schoolteachers who understand the broader theoretical and applied contexts of the mathematics that they teach can answer student questions and plant the seeds of ideas and connections that will later become important. It is the job of mathematicians at universities' mathematics departments to transmit these ideas to students who, while not pursuing a degree or career in pure mathematics, must understand some of its fundamental components.

Theoretical Mathematicians: Their Work and Their Views

It is usually agreed upon that until the middle 1800s, there was no clear division between theoretical and applied mathematics. Even though, arguably, Euclid's *Elements* could be considered pure mathematics, the majority of mathematicians from ancient times until the 1800s were interested in solving problems. It is also true that some of these problems, such as finding the roots of polynomials of varying degrees (which led to the development of Group Theory), might not seem to have much practical application. However, in general, mathematicians as renowned as Isaac Newton, Gottfried Leibniz, Leonhard Euler, Carl Friedrich Gauss, brothers Jacob and Johann Bernoulli, Joseph Fourier, Joseph-Louis Lagrange, Evariste Galois, or Niels Abel, in their contributions to the ideas now considered part of theoretical mathematics, usually were also involved in research in which direct applications were the central objective.

In the 1800s, the axiomatization of calculus, with its convenient but mysterious infinitesimals, was carried out by Augustin-Louis Cauchy (1789–1857) and Karl Weierstrass (1815–1897). George Boole (1815–1864) tried to formalize the laws of thought using algebra and initiated the algebra of logic, called Boolean algebra, in which algebraic symbols represent logical forms. It is interesting that this theoretical endeavor actually laid the ground for the construction of computers and electric circuits, given that these circuits can represent complex logical operations. These mathematicians would now be considered theoretical mathematicians, as their work was oriented to expanding the mathematical areas in which they worked, not to practical applications.

Although the computer does not play the same role in the work of the theoretical mathematician as it does in that of the applied mathematician, it would be false to think that the theoretical mathematician has remained untouched by the advent of the computer. In number theory, for example, if there is a conjecture about properties of, for example, prime numbers, the numbers can be generated to billons or trillions in a short interval of time, detecting in this way if some counterexample could appear. Before this possibility arose, theoretical mathematicians could sometimes spend a lifetime trying to prove a false conjecture because it would have taken several lifetimes to generate enough numbers to arrive at the counterexample. In purely theoretical areas such as commutative algebra and algebraic geometry, computer programs have been developed that permit the calculation of, for example, Gröbner bases, named for Wolfgang Gröbner, that help to further theoretical results. The proof of the Four Color Theorem, which had been attempted by theoretical mathematicians for over 100 years, was done with the aid of the computer, which carried out the multiple calculations that would not have been possible to do by hand in a lifetime.

Of course, there are those who say this proof (of the Four Color Theorem) does not correspond to pure mathematics. This very interesting debate is a product of the transition at the beginning of the twenty-first century that coexistence with computers has become a reality. A quote from theoretical mathematician David Cox, who has played an important role in bridging this gap, is very illustrative:

> My fascination with algebra led me to algebraic geometry, which was then among the most abstract areas of pure mathematics. At the time, I never would have predicted that 25 years later I would be writing papers with computer scientists, where we use algebraic geometry and commutative algebra to solve problems in geometric modeling.

Often, quotes from actual theoretical mathematicians best give an idea of how they themselves conceive their work. These quotes may illustrate a perception of pure theoretical mathematics, perhaps not so well known to a general public, rather than absolute importance of these mathematicians over any others—an idea that will always be debatable and impossible to define:

> It is not of the essence of mathematics to be occupied with the ideas of number and quantity.
>
> —George Boole (1815–1864)

> No matter how correct a mathematical theorem may appear to be, one ought never to be satisfied that there was not something imperfect about it until it also gives the impression of being beautiful.
>
> —George Boole (1815–1864)

> Mathematics is entirely free in its development, and its concepts are only linked by the necessity of being consistent, and are co-ordinated with concepts introduced previously by means of precise definitions.
>
> —Georg Cantor (1845–1915)

> In mathematics the art of proposing a question must be held of higher value than solving it.
>
> —Georg Cantor (1845–1915)

Often theoretical mathematicians are motivated by the knowledge that their abstract research and discoveries will eventually find their way to applications in technology, medicine, or economics. Theoretical mathematics can very well be conceived of as an art by those who find aesthetic pleasure in its logic and patterns, but there is no doubt, as historically has been seen time and time again, that mathematics is science as well.

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Mariana Montiel

See Also: Connections in Society; Mathematics, Defined; Reasoning and Proof in Society.

Mathematics, Utility of

Category: Mathematics Culture and Identity.
Fields of Study: Communication; Connections; Representations.
Summary: Though doing mathematics does not necessarily require utility as an outcome, there are many examples of applications in various fields.

To discuss the utility of mathematics, there must be some agreement on the definition of the term "mathematics." Many may agree that mathematics is a pure creation of the human mind. It is a body of knowledge at which one arrives by pure reason and does not rely upon any observations of the phenomenal world. This characteristic makes it free from the limitations imposed by the particular way that human minds create experience from their understanding of the underlying phenomena. The argument comes down to the following: is mathematics the complete construction of the human mind or is it universally inherent, only being discovered/uncovered by mathematicians? Many books have been written to discuss this question, and no decision has been (or will be) made on one side or the other.

There are examples of people in the mathematics community, such as G. H. Hardy in *A Mathematician's Apology*, who see a difference between pure and applied mathematics based solely on utility and revel in the fact that nothing that they will do will be useful to humanity. This statement was in part a response to the work of Andrew Littlewood and a group of mathematicians who worked strenuously for the British War Department during World War I.

There are ample examples in the historical record of mathematics, done for its own sake, that were later discovered to be applicable to real-world problems. The theory of tensors by Giovanni Ricci-Curbastro and Tullio Levi-Civita proved to be a cornerstone for some of Albert Einstein's work on relativity. The purely algebraic area of twistor theory in physics, which predicted the existence of certain subatomic particles in the 1980s, started in the area of finite algebraic geometry. Areas that in the 1980s and 1990s were considered pure mathematics now find themselves at the forefront of application: algebraic topology used to study distribution of sensors; hyperbolic geometry used to study the extent and reach of the Internet; number theory used in architecture and cryptography; and category theory used in studying social behavior. Where will the applicable mathematics of the mid-twenty-first century come from?

What sort of mathematics should be taught in schools? This question dates back to at least the early 1800s in the United States and is discussed in the work of Charles Davies. E. R. Hedrick again raised the same question in his address to the New York Section of the Mathematical Association of America in 1933. The question arises with each new generation. Should only the mathematics that is currently known to be applicable be taught to those who will only use the tool, or should they be exposed to the whole of mathematics?

The purpose of asking this question lies in the purpose of mathematics. Should mathematics, or is mathematics, done only for its own sake? If that is the case, then why has mathematics been so useful to science? This is the question raised by Eugene Wigner in his 1960 work *The Unreasonable Effectiveness of Mathematics in the Natural Sciences*. The question and his answer have again brought to the fore this long standing argument.

Historical Context

From the earliest recordings in Babylonian and Egyptian mathematics, historians and archaeologists find problem books with problems created to train the mathematical neophytes—possibly young priests—in the algorithms that were used for building, surveying, and the like. The problems were not all applied problems but did include some examples of mathematics being done for mathematics' sake. This was not the rule, though. Most mathematics of these earlier eras seemed to have been for inherently practical purposes.

From the Western perspective, it was the Greeks under the Pythagoreans who took the idea of mathematics and made it deified. The question of the "utility of mathematics" does not skip the Platonic school. There is a quote, ascribed to Euclid in Stobaeus' *Extracts* "A youth who had begun to read geometry with Euclid, when he had learnt the first proposition, inquired, 'What do I get by learning these things?' So Euclid called a slave and said 'Give him three pence, since he must make a gain out of what he learns.'" Already the teacher has to answer the long-asked question, "What is this good for?" The Platonic school may have been one of the first in which mathematics was studied for its own beauty and internal structure—not being required to have any other purpose. Archimedes saw the utility of mathematics; whether he held the same philosophical beliefs as did the Platonists, we cannot be certain.

The Romans were extremely interested in the utility of mathematics to warfare, navigation, and architecture. It was the Greeks and the Alexandrians, though, that kept mathematics moving forward until it was rescued from the fate of much of the ancient world's science by the Islamic mathematicians. Not only did they need mathematics for navigation and geometry, but they also imbued into the geometry the need to glorify Allah with the perfectness of the geometric form.

In the Renaissance in the late thirteenth century, the early scientist, Roger Bacon, made statements about the utility of mathematics, "Mathematics is the door and key to the sciences," and ". . . mathematics is absolutely necessary and useful to the other sciences."

Perhaps the best summary can be found in various quotes:

> The Universe is a grand book which cannot be read until one first learns to comprehend the language and become familiar with the characters in which it is composed. It is written in the language of mathematics. . . .
>
> — Galileo Galilei (1564–1642)

> Mathematics is a game played according to certain rules with meaningless marks on paper.
>
> — David Hilbert (1862–1943)

> (Cantor's work on set theory) . . . the finest product of mathematical genius and one of the supreme achievements of purely intellectual human activity."
>
> — David Hilbert (1862–1943)

> From the intrinsic evidence of his creation, the Great Architect of the Universe now begins to appear as a pure mathematician.
>
> — James Hopwood Jeans (1877–1946)

> I have never done anything "useful." No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world.
>
> — G. H. Hardy (1877–1947)

> . . . enigma that researchers of all times have worried so much about. How is it possible that mathematics,

a product of human thinking independent of any experience, so excellently fits the objects of physical reality?

— Albert Einstein (1879–1955)

As far as the propositions of mathematics refer to reality, they are not certain; and as far as they are certain, the do not refer to reality.

— Albert Einstein (1879–1955)

The unreasonable effectiveness of mathematics in the natural sciences . . . that the enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and that there is no rational explanation for it.

— Eugene Wigner (1902–1995)

. . . [enumerating cases where structures needed in physics have already been found and developed by mathematicians] . . . long before any thought of physical application arose. It is positively spooky how the physicist finds the mathematician has been there before him or her.

— Steven Weinberg (1933–)

This universality of application [of mathematics] can be traced back to the fact that all aspects of Nature and areas of life are governed by the same principles of order and intelligence that have been discovered subjectively by mathematicians by referring back to the principles of intelligence in their own consciousness.

— Maharishi Mahesh Yogi (1914–2008)

Was it not the Pisan scientist who maintained that God wrote the book of nature in the language of mathematics? Yet the human mind invented mathematics in order to understand creation; but if nature is really structured with a mathematical language and mathematics invented by man can manage to understand it, this demonstrates something extraordinary.

— Benedict XVI (1927–)

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David C. Royster

See Also: Mathematical Modeling; Mathematics, Applied; Mathematics Research, Interdisciplinary; Mathematics, Theoretical; Mathematics: Discovery or Invention.

Mathematics, Vedic

See *Vedic Mathematics*

Mathematics: Discovery or Invention

Category: Mathematics Culture and Identity.
Fields of Study: Communication; Representation.

Summary: One of the central questions of the philosophy of mathematics is that of mathematical realism.

Mathematicians engage in a great many activities, including investigating and extending old and new concepts within the field, as well as developing new techniques to solve problems in mathematics and other disciplines. The question is, when they carry out this activity, do they discover existing laws or do they invent and create? If invention is involved, is it individual or is it social? This question is a polemical topic that has been subject to strong controversy and refers to ideas that have emanated everywhere from ancient Greek personages, such as Plato, up to modern advocates of artificial intelligence (AI).

Platonists

Those who subscribe to the discovery position are usually classified as Platonists. Plato expressed that mathematical ideas are discovered, existing independently of human observation or changes of a physical nature. However, the general trend known as "mathematical realism," which includes formalism and logicism, also catalogued within the discovery perspective. Mathematics is seen as the science of logic with its laws based on enduring truths, whether they have been discovered or not. Those who subscribe to this position cite, for example, the existence of universal constants, such as π, ϕ, Euler's e, or Feigenbaum's α and δ in bifurcation theory. It is put forth that the circumference of a circle has always measured π times diameter, whether or not that fact had been discovered by a particular society or culture.

It is also claimed that the discovery of mathematical laws, objects, and relations occurs simultaneously, or over time, in distant places. The most famous examples include the simultaneous, but independent, discovery of calculus by Isaac Newton and Gottfried Leibniz in the seventeenth century and the independent discovery of the universal constant π by the Babylonians, Greeks, Chinese, and others at different historical moments. Many of the structures from very abstract areas of mathematics are often found to model phenomena in the physical world, such as the case of Cantor's set, originally an abstract construct, which serves as a model for error distribution of the noise in transmission lines (for example, electric power lines or telephone wires). This case is also taken as evidence that mathematics is, apart from a consistent logical system when accepting the axioms, a language that describes the physical universe, whether or not that description was intended by the mathematician who discovered the pattern, technique, theorem, or other relevant mathematical object.

Criticisms of Platonists

This idea adds another element to the discussion. For the realists, it is important to distinguish between mathematics itself, as a timeless science of logic, together with the laws that govern its existence, and the practice of mathematics, which includes many aspects that are language-like and that, they agree, are created, such as particular symbolism, notation, formalization, and nomenclature. Often the Platonists are dismissed by arguments that ridicule or simplify Plato's allegory of the cave to an alleged discovery of an almost physical mathematical realm. This simplification seems because of a literal, instead of a metaphorical, interpretation of the way that many working mathematicians refer to their subject, a way of expression that reflects the actual feeling of "concreteness" that is provoked by daily contact, manipulation, and struggle with their abstract objects. Roger Penrose, for example, who identifies with the Platonist perspective, speaks of the Mandelbrot set as a structure whose constant surprises, within its self-similarity, are waiting to be explored.

Diversities of Non-Platonists

On the other hand, those that challenge Platonism and mathematical realism in general are not a homogeneous group.

One of these positions asserts that the existence of mathematics can be understood only as part of human culture. It is argued that the reality of mathematics is a sociocultural and historical phenomenon and that mathematics exists only because there are human beings who create it. Advocates of this position argue that mathematics is in the same category as law, religion, and money. It is only human consciousness and society with its conventions that makes them real.

Philosopher Ludwig Wittgenstein regarded mathematics as a type of ". . . communication; people play 'language-games' and 'sign-games' to invent, rather than discover, mathematics." The Social Constructivists, supporters of this position, argue that mathematical development is guided by fashions and trends in human societies. They claim that mathematical truth is invented and depends on the sociocultural context.

The term "quasi-empirism" is used for the type of modern mathematical research that relies on computers and other quasi-experimental methods that seem to contradict the deductive nature of mathematics and question the existence of absolute and eternal mathematical truth. The Social Constructivists assert that this activity demonstrates the fallibility of mathematical activity and removes it from the realm of any absolutes, thus supporting their claim that mathematics is "man-made."

The embodied theories consider mathematics as an exclusively human endeavor, invented according to the physical and cognitive human reality. Exponents of this position privilege the biological evolution of the human brain and consider mathematical objects as a reflection of human cognition. Hence, according to this perspective, mathematics is constructed by the human brain, and its apparent truths were created because they actually work efficiently in the universe in which we find ourselves.

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Mariana Montiel

See Also: Axiomatic Systems; Mathematics Defined; Mathematics, Elegant; Mathematics, Theoretical; Mathematics, Utility of; Proof; Reasoning and Proof in Society.

Mathematics and Religion

Category: Friendship, Romance, and Religion.
Fields of Study: Reasoning and Proof; Connections.
Summary: The connection between religion and mathematics is intricate, spanning cultures and centuries, with mathematics itself sometimes manifesting religion-like features.

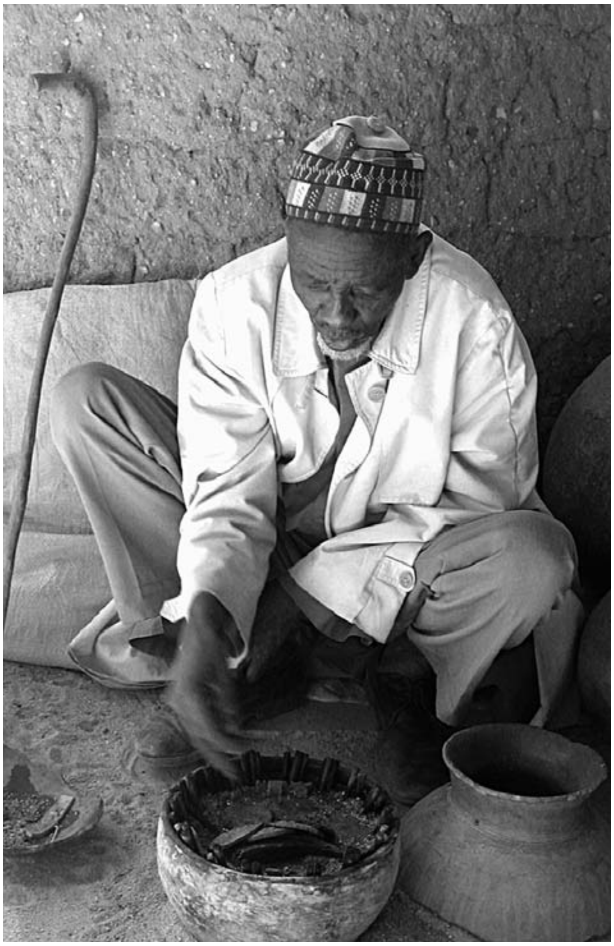
Mathematical knowledge has been intertwined with spiritual or religious contemplation since humans began to develop numerical, spatial, and symbolic reasoning in order to understand the world and humanity's place within it. Both practical and abstract knowledge have been significant to cosmological and theological considerations. Another way that mathematics is linked to religion is by those who suggest that mathematics is a religion.

Mathematics provides tools that underpin computation, prognostication, organization, and design. Consequently, mathematical knowledge—as constituted by practical arithmetical (computational), algebraic (numerical problem solving). and geometric (spatial) knowledge—has been an essential ingredient in divination as well as in ritual constructions and practices. The influences of mathematical knowledge, broadly construed, on cosmology can be found in different times, places, and cultures. They are evident in a variety of contexts that include Pythagorean, Judaic, and Chinese number mysticism; Vedic rituals; Islamic trigonometry; and pattern drawings that some South Pacific Islanders believe are essential to entering the land of the dead.

Beyond skill-based practicality, mathematics as a way of obtaining infallible knowledge of transcendental objects engendered and strengthened spiritual considerations that became more closely aligned with doctrine. It did so to such an extent that the development of new mathematical knowledge often instigated immediate responses from religious authorities. Such symbiotic yet ever-evolving relationships between mathematical epistemology and theological contemplation are a central feature of the Christian tradition across the ages.

Implicit Practices, Divination, and Pattern Drawing

In the oldest cultures it is difficult to separate mathematical and ritual practices. Shamans and priests, from ancient Babylonia to Mesoamerica, used arithmetical and geometrical knowledge as part of their efforts to organize time and space so as to facilitate particular observances. In some cultures, the drawing of geometric patterns was integral to storytelling that conveyed origin myths as well as aspects of the afterlife. For both ancient and contemporary peoples, mathematics is not identifiable as a constituent of an explicitly distinctive knowledge. Rather, mathematics as it is recognized today is seen as implicitly embedded within customs of cultural significance that included spiritual well-being.



A Cameroon sorcerer uses divination— interpreting the positions of various objects—to tell the future.

Divination, as practiced in various times and places, typically involves both randomness and structure. The objects required for the foretelling of events, while specific to custom, are subjected to a process that produces a random outcome. The diviner's skill comes into play when interpreting the result. Doing so involves adhering to rules that apply to the particular procedure. Consequently, divination often involves strictures that can be resolved into numerical or logical systems, systems that often reflect binary considerations. Such can be found today in the methods of divination practiced by the Caroline Islanders of the South Pacific (knot divination), the Yoruba people of Africa (*Ifa*), and the Malagasy (*Sikidy*).

Pattern drawing has often accompanied cultural narratives regarding both ancestors and the afterlife. Such traditions continue into the modern era with the Tshokwe people of Angola and the Malekula of Vanuatu. In each case, intricate patterns are drawn in a continuous, uninterrupted fashion. While modern mathematics conceives of such in terms of graph theory and Eulerian circuits, there is little evidence to suggest that the cultures discussed here have an explicit or external framework within which such patterns are considered. Indeed, Tshokwe have relatively few patterns that accompany their origin myths, and knowledge of their production is limited. The cultural situation for the Malekula is considerably different. Their patterns number in the hundreds and all require that the tracing begin and end at the same point without repetition of any edge. Knowing how such patterns are produced, which constitutes a form of implicit mathematical training insofar as it recognizes various systematic elements within the drawings, is part of what men pass on to their sons. The ability to reconstruct a pattern correctly earns one access to the land of the dead.

Like the Tshokwe and Malekula, Tamil women in southern India draw patterns as a way of marking passages or transitions. The ritual designs produced by them, which are known as *kolam*, are used to decorate the entrance to a house. They vary according to the events being marked, many of which relate to life- or worship-cycles. Recently *kolam* have attracted the attention of computer scientists who are interested in the formulation and formalization of picture languages.

Classical and Judeo-Christian Traditions

The mathematics of Greek antiquity marked a distinctive break with the implicitly integrated practices associated with various cultures across time. Moreover, it laid the foundation for more explicitly considered connections between the mathematical and the spiritual. The perspective held during the earliest portion of the Pythagorean-Platonic period is often simply characterized as follows: number is religion and religion is number. That is, numbers provided the lens through which the Pythagoreans viewed the cosmos. In this way mathematics links the mundane and the sacred in ways exemplified in different ages and cultures. Mathematical reasoning, which at this time might more closely have aligned with numerology or number mysticism, provided a means of bringing order and harmony to the universe.

The realization that certain numbers are irrational (that is, that some measures are incommensurable) represented a serious challenge to Pythagoreans, for it contradicted the assumption of a cosmic harmony. While the need to resolve paradoxes instigated monumental discoveries, it soon became apparent, as it would again in the years to come, that using mathematics as a means of demystifying the world could engender new and even greater mysteries.

According to Plato, arithmetic and geometry constituted areas of study essential to higher education, and thus they became part of the quadrivium of Western education, which included astronomy and music. That philosophical discussions found in his dialogues turn to and on mathematical reasoning underscores the significance of mathematics to Platonic conceptions of the good and true. It represented an "a priori," if for many a latent, body of knowledge through which one accessed eternal and perfect forms rather than transient and imperfect perceptions of these.

While he maintained a distinction between the physical and the otherworldly, Aristotle differed from those who believed that mathematics provided a special conduit to transcendental realms. Rather, his perspective of mathematics as abstraction based on physical reality reverses the mystical point of view. Aristotelian thinking underpins a more humanistic and, in later ages, secular understanding of mathematics. Underscoring the difference between process and object, classical Greek mathematics attempted to distinguish between the potential and the actual when discussing infinity. Powerful analytic arguments and famous paradoxes hinged on the process of infinite subdivision that gave rise to infinitesimal considerations. Amid this conceptual ambiguity, Aristotle maintained that the actual infinite—the infinite as a completed object—is unknowable.

Euclid's *Elements* is especially significant among classical texts that helped to solidify, as well as perpetuate, connections between mathematical and metaphysical reasoning. As a compendium of geometric knowledge of its day, *Elements* is most significant for its presentation of timeless and unassailable conclusions rigorously deduced from self-evident truths. It speaks to absolute certainty and provides geometry as a model for attaining such. Consequently, the influences of the *Elements* on mathematics and Christian theology echo across the centuries.

Aurelius Augustinius (354–430), or Saint Augustine, helped to begin the process of transforming Pythagorean–Platonic conceptions into Christian doctrines during the Middle Ages (fifth through twelfth centuries). His contributions, among many things, served to imbue Christian symbolism, including the Ark of the Covenant with its divinely prescribed dimensions, with numerical and geometric significance. Such symbolism was considered necessary for analogizing and simulating the majesty of God's power in ways comprehensible to a faithful laity. Following classical traditions, numbers represented an ideal conduit for transcendental contemplation. Shapes, on the other hand, could both signify the sacred and convey divine wisdom. The successful adaptation of Hellenistic mathematical cosmology to Christian theology owes much to Saint Augustine and others.

Scholastic theologians of the Early Modern period (twelfth through sixteenth centuries) built upon the connections between mathematics and Christian faith promoted by Saint Augustine. Setting the tone for the age, Giovanni di Fidanza (1221–1274), or Saint Bonaventure, extended Aristotle's prohibition against attempting to understand the infinite by claiming that it existed in God only. Even so, one could aspire to a better appreciation of the divine. To this end Nicholas Cusanus (1401–1464), or de Cusa, believed that mathematics emulates the creative power of God insofar as it is a manifestation of humankind's ability to create knowledge and to completely understand this creation. By virtue of this manifestation, mathematics served as an essential and mutually beneficial component of Cusanus's theology. Specifically, practicing mathematics is a way by which humankind can become closer to the divine. Whereas the platonic dialogues use mathematics to underpin conceptions of the Good, the Neoplatonic theology of Cusanus redirects mathematical attention toward conceptions of the divine.

Rendering perspective in painting by means of a vanishing point is one of the most important markers of Renaissance art. Anticipating the aesthetic significance of this development, Roger Bacon (1214–1294) encouraged the incorporation of geometric innovation in painting, believing it offered a way of better communicating God's majesty through more powerful visual imagery. As such sentiments make clear, the connections between mathematics and religion could be both rendered and read visually, thereby making such concepts accessible to lay audiences who were not necessarily conversant with the particulars of either.

Alongside Neoplatonic scholasticism, the late Middle Ages saw a resurgence of interest in gematria, a practice by which one attempts to reveal and interpret divine secrets through the association of alphabetic characters with numbers. Truth seeking by means of numerically organized systems was not a new development; it has a long history in the Jewish religious tradition and is central to Kabbalism. Among the more shocking identifications established by Michael Stifel (1486–1567) through gematria was Pope Leo X with the Beast of the Apocalypse. Similar ideas underpin recent interest in topics such as the Bible Code.

While breaking with the intellectual traditions of the past, mathematicians associated with the Scientific Revolution (c. sixteenth through eighteenth centuries) and the Modern period (from c. eighteenth century) continued to connect the discipline's reasoning and knowledge with theological concerns. René Descartes (1596–1650) promoted the individual's power of reason through geometry. His rationalism was a reaction against the constraints of scholasticism and, therefore, many considered it a threat to religious authority. Nev-

Mathematics as Religion

Modern mathematics is seen by some people to have features similar to those of religion; for example, that mathematical foundations are accepted on belief rather than logic and comprehension. Analogies between mathematics and religion also include discussions about their omnipresent nature, their pivotal role in society, and their dependence on teaching the next generation. Some people point to aspects of mathematics that may appear unresolved or include contradictions, such as the axiom of choice or Kurt Gödel's incompleteness theorems, which showed that there are limitations to axiomatic systems. For example, in 1999 John Barrow wrote, "If a 'religion' is defined to be a system of ideas that contains unprovable statements, then Gödel has taught us that, not only is mathematics a religion, it is the only religion that can prove itself to be one."

Religious terms have been applied to mathematical theorems or mathematicians. For example, mathematical discoveries are sometimes described in terms of a revelation, epiphany, or heresy. In 1985, mathematician Paul Erdös asserted that it was important to believe in *The Book,* an imagined type of bible containing elegant proofs. This assertion inspired the 1998 work *Proofs from THE BOOK* by Martin Aigner and Günter Ziegler. Some people have referred to mathematicians, including Erdös, as priests of mathematics who share their gospel. Mathematicians, philosophers, and theologians also consider whether a divine force is needed to explain such concepts as how the universe was formed or whether the underlying mathematical and physical principles are sufficient, which is Stephen Hawking's assertion in his 2010 book *The Grand Design.*

ertheless, and like others of the age, including Gottfried Leibniz (1646–1716), he used humankind's ability to reason mathematically as the basis for discussions that ultimately asserted the existence of God.

Unlike some inclined to rationalism and deism, Blaise Pascal (1623–1662) believed that mathematical reasoning could not be applied to prove the existence of God. Another critic of mathematics' influence on theology, George Berkeley (1685–1753) pointed out that accepting the mysterious notion of infinitesimal quantities so essential to the development of calculus was tantamount to an act of faith. Consequently, he contested deism by asserting that mathematical knowledge could not provide a more exact, or more acceptable, model for theological reasoning.

Immanuel Kant (1724–1804) asserted that geometry is a contentful, or synthetic, knowledge that adheres to a universal, "a priori" form of spatial intuition. He did not, however, use this to gird theological speculation. Indeed, he attacked proofs of God's existence in his *Critique of Pure Reason* (1781) and *Critique of Practical Reason* (1788). Rather, Kant posited morality as a distinct form of intuition. The knowledge built upon this intuition leads to an understanding of the divine. Though independent forms of intuition, the geometric and the moral knowledge built upon these exemplified a common epistemological perspective.

The power of Kant's argument is evident in responses to the development of non-Euclidean geometries in the nineteenth century. With this development, the absolute certainty long associated with geometric reasoning gave way to contingent knowledge. Along with more familiar paradigm shifts, most notably Darwinian evolution, new mathematical knowledge contributed significantly to the Victorian crisis in faith. Euclid's *Elements* anchored mathematical and theological speculation for centuries; its promise of eternal and necessary truths was much in doubt.

Considerations outside geometry also exacerbated religious anxieties. Though obsessed with the notion of an all-encompassing infinite informed by the *Ein Sof* of the Jewish religious tradition, Georg Cantor (1845–1918) further destabilized relations between mathematics and spirituality with investigations that sought to establish the cardinality of the real continuum. Correspondences with Pope Leo XIII provide evidence that Cantor himself was concerned with the contentious potential of his work. The distinction between process and object so clearly delineated in antiquity meant that Christianity could safely adjudicate conceptions of the infinite as these pertained to the divine. Cantor's identification of infinite sets as objects of mathematical interest represented a clear threat to this religious privilege.

Some claim that new and contingent perceptions of mathematical certainty evident from nineteenth-century innovations instigated a period of desecularization. Failure to secure mathematics on a firm epistemological foundation through Formalism, Logicism, and Constructivism suggested that its knowledge is the confirmation of intuitions and creative possibilities, even if such cannot be constrained by any particular formal systems. Reminiscent of relationships articulated by Aristotle and Cusanus centuries earlier, modern mathematical thinking provided a new model for theological contemplations attuned to divine immanence inherent in processes and potentialities as much as to transcendental conceptions.

Chinese, Indian, and Modern Esoteric Traditions

Chinese engagements with mathematics have long been intertwined with cosmological and spiritual concerns. Astrology and divination depended on computational abilities. Consequently, one finds strong associations between mathematical practices and number mysticism, relationships not unlike those found in antiquity and throughout Europe during the Middle Ages. Even so, the desire to predict astronomical and calendrical events inspired the need to solve systems of modular congruences. Such solutions date to the thirteenth century and form the basis of the Chinese Remainder Theorem.

Mathematical practices historically associated with the Indian subcontinent also evidence spiritual influences. Ancient Vedic observances required geometric knowledge in the construction of altars that were built in various shapes with fixed areas. Similar mathematical prescriptions eventually extended to the building of temples. Vedic literature also suggests the incorporation of a symbol for zero, which became part of the Hindu–Arabic system later adopted in Europe. The symbol emerged from the considerations of Brahma as universally divine and immanent even in nothingness.

Though distinct traditions, Hinduism and Jainism attended to numerical computations as a way of contemplating the complexity and extent of the universe, including the number of ways that things might be combined. One verse from the Jainaic *sthananga sutra* (c. 300 B.C.E.) identifies algebra, geometry, and combinatorics as constituents of mathematical expertise in a way that reflects the Platonic prescription of mathematics as an essential form of knowledge.

The emergence of modern theosophy in the nineteenth century was precipitated in part by the Victorian crisis in faith and obsession with orientalism. Mathematics occupied a special place in theosophy, particularly in the numerological interests of the ancients. More contemporary concerns, however, also commanded attention within this esoteric movement. The notion of higher dimensional space, which gained credibility and notoriety through the development of algebraic methodologies and non-Euclidean geometries, was a topic of considerable discussion among theosophists. Some appealed to it by way of analogy to support beliefs in a universal present that connected the past with the future. Others made claims of brotherhood based on the notion that all of humankind is the manifestation of a single universal being that could be accommodated in an expanded conception of space. Peter Ouspensky (1878–1947) provided one of the most fulsome accounts of such thinking in his *Tertium Organum*.

Islamic Tradition

Islamic mathematics incorporated and extended ancient Greek and Indian knowledge. More significantly Muslims transmitted this expanding body of knowledge widely during the period that saw their cultural and intellectual influence spread from the Middle East to Spain (c. 700–1500). As with other cultures, astronomical considerations focused attention on geometry and trigonometry. Further, requirements associated with daily prayers, one of the Five Pillars of Islam, served to connect religious and mathematical practices. Interest in accurately establishing the five daily prayer times, which are set according to the Sun's position as determined by shadow length, provides one connection with the trigonometry of astronomical computations. Additionally, the problem of locating the direction of Mecca, toward which the faithful must face when praying, meant the Muslim mathematicians were equally concerned with the trigonometry of geography.

The significant relationships between the offering of prayers and trigonometry notwithstanding, discourses explicitly linking mathematical and theological concerns are not common features of Islamic texts dating from the Middle Ages. Patterns incorporated as architectural ornamentation may reflect natural observations rather than the realization of mathematical knowledge. However, some have suggested that the algorithmic pattern making so prevalent in Islamic architecture may reflect cosmological and theological contemplation. Specifically, it could provide a visual representation of creation that was understood in the context of number, especially in the generation of the many (numbers) from a singular unit (one). The use of multiple geometrical patterns, each integral yet distinct, may also serve a visual invitation to reflect on the parables of the Qur'an. While theological intentions might be difficult to document, mathematical expertise was certainly involved in rendering the elaborate spherical tessellations that adorn many of the domes found in Islamic architecture. Such knowledge is contained in Islamic texts such as *Those Parts of Geometry Needed by Craftsmen* (c. tenth century).

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K. G. Valente

See Also: Arabic/Islamic Mathematics; Chinese Mathematics; Geometry in Society; Graphs; Greek Mathematics; Infinity; Maps; Mathematicians, Religious; Numbers and God; Pythagorean School; Religious Writings; Sacred Geometry; Trigonometry.

Mathematics Genealogy Project

Category: Mathematics Culture and Identity.
Fields of Study: Communications; Connections.
Summary: The Mathematics Genealogy Project maps professional relationships among mathematicians.

Some mathematicians refer to their Ph.D. advisers as their "mathematical parents."

Two fundamental components of the fabric of human societies are family and community. For reasons like innate socialization, sense of responsibility, and loyalty, an individual is compelled to be a part of a larger organization. In a similar way, the desire to distinguish one's place in the community, the wanting to carve out a place in the family, the urge to preserve the past for future generations, and numerous such factors motivate an individual to seek a family history. Consequently, throughout history individuals have spent much time and effort on genealogy in pursuit of their own ancestries and to reconstruct trees of ancestors.

Similarly to an individual's desire of constructing family genealogy, many professionals also have the desire and motivation to pursue their professional history. This desire is particularly the case for the professions or crafts in which some form of "apprentice" and "master" relationships are the main mode of transferring knowledge or skills from one generation to the next. Professional mathematics is a prime example of such a vocation. Particularly since the Renaissance, a prospective mathematician usually studies and conducts research under the supervision or tutelage of a master mathematician whose guidance and knowledge are major factors in obtaining successful certification to become a recognized mathematician—the Ph.D. degree.

Mathematicians usually have very high regard for this type of transfer of knowledge and profession; hence, Ph.D. advisers are given special respect. Indeed, in mathematical events, novice mathematicians' introductions typically include their adviser's name, or novice mathematicians introduce themselves as students of their adviser. Some even go as far as calling their Ph.D. adviser as their "mathematical" parent. In such an environment, it is natural for mathematicians to inquire about their mathematical ancestries. Another factor that contributes to this curiosity is, in the vastness of mathematics, finding the intertwining connections between the various subdisciplines and tracing back the original sources and motivations of the problems or concepts being studied.

Birth of the Project

The Mathematics Genealogy Project is a natural outcome of such curiosity and is the brainchild of Professor Harry B. Coonce. Although several small groups of mathematicians or some individual mathematicians had information on the genealogy of numerous prominent mathematicians, until Coonce's initial work in the late 1990s, no attempt was undertaken to construct a genealogy tree for a large group of mathematicians. In 1997, realizing that there was no central location where the information on mathematics Ph.D. students and their advisers was available, Coonce (whose adviser was Malcolm S. Robertson) started a Web site for this purpose. Upon his retirement in 1999, he devoted all his time to the project and began systematic data collection and formation of a genealogy tree for all mathematicians, which has become the Mathematics Genealogy Project (MGP). In 2003, the MGP moved to North Dakota State University (NDSU) and has been housed there since. The project's primary responsibility rests with the NDSU Department of Mathematics. In late 2009, Coonce retired from being the managing director of the project; and in October 2009, the American Mathematical Society became the sole designated partner of NDSU for MGP.

Construction of a genealogy tree is a complex process that uses historical records and other reliable sources to demonstrate kinship. Because of its unique position and its desire to provide the family tree for all mathematicians, this task is particularly difficult for the MGP. It is essentially a searchable database in which information for each entry contains all relevant professional information about that individual. The project's mission statement, quoting from the project Web site, indicates this ambitious goal clearly:

> The intent of this project is to compile information about ALL the mathematicians of the world. We earnestly solicit information from all schools who participate in the development of research level mathematics and from all individuals who may know desired information. It is our goal to list all individuals who have received a doctorate in mathematics. For each individual we plan to show the following: the complete name of the degree recipient; the name of the university which awarded the degree; the year in which the degree was awarded; the complete title of the dissertation; and the complete name(s) of the advisor(s).

In order to provide all this information as accurately as possible, the project managers gather data from reliable sources. The main sources of data are information provided from the Ph.D.-awarding institutions and the *Dissertation Abstracts*. Another important source is the mathematical community itself; voluntarily, many mathematicians provide valuable information that is not accessible to the project managers. In any case, before any entry is included in the project database, it is scrutinized for possible errors. However, some erroneous information can still be found; some of this is because of changes in the individuals' records, such as name changes because of marriage, revised spellings because of move, and name changes of institutions, and some are genuine errors. These errors are other reasons that the project administrators rely on the mathematical community for monitoring the entries and reporting and correcting the errors found.

Besides providing the information on the genealogy of mathematicians, the MGP aims to be a source of other relevant data and a hub of connections to other related projects. Therefore, the project Web site (http://genealogy.math.ndsu.nodak.edu) also contains interesting features of this kind. It provides links to databases or search tools, like MathSciNet, and links to other institutions that carry relevant information. One can also find some interesting information on the mathematicians who are most prolific and have a large number of descendants.

Mathematicians, naturally, are inclined to seek a mathematical structure within any object on which they cast their eyes. As is seen in the Extrema section of the project Web site, the MGP tree happens to have a special nonplanar graph structure. Researchers are using the data to investigate graph theoretic and visualization issues as well as social issues, like the advisers with the most students or descendants and the role of mentoring in advisee productivity. It is even possible that a new research area of mathematics on the study of structures within the MGP tree may emerge.

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Dogan Comez

See Also: Genealogy; Graphs; Mathematical Friendships and Romances; Mathematician Defined; Social Networks.

Mathematics Literacy and Civil Rights

Category: School and Society.
Fields of Study: Connections; Problem Solving.
Summary: The opportunity to learn mathematical knowledge and problem solving abilities is a right that should not be denied to any social groups.

Mathematical literacy is the conceptual understanding and, especially, the operational skills to deal with mathematical situations encountered in all areas of daily life. At a higher level, it is also the ability to use mathematical knowledge and problem-solving ability in more sophisticated uses of mathematics in careers and technical applications. This mathematical knowledge includes having a "number sense" of comparative sizes of numbers, being able to estimate and to do mental arithmetic, and being able to use technology necessary for modern life and jobs. Beyond basic arithmetic, skills identified by professional organizations also include an understanding of basic statistics—at least enough to read and understand graphs, to interpret statistics reported in the media, and to beware of attempts to mislead with statistics. Similarly, algebra teaches the symbolic and logical sense of problem solving necessary to understand the mathematical issues of modern life.

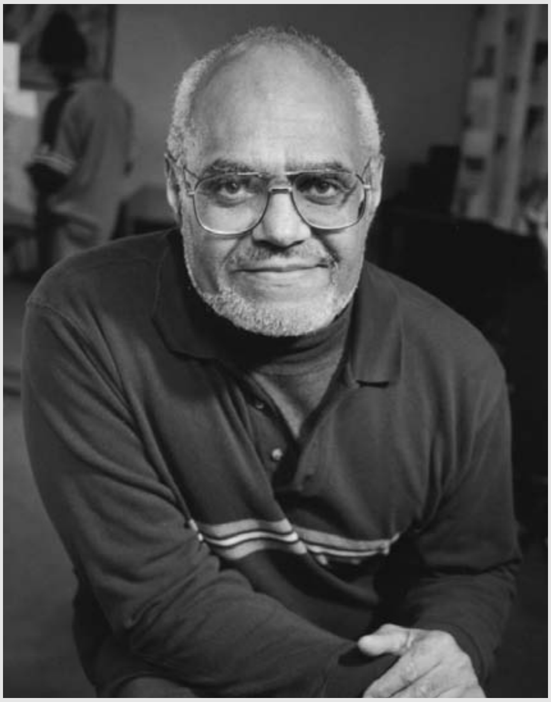
Significance

Some basic mathematical knowledge, such as counting, comparisons of size, and even the fundamentals of arithmetic, may be innate or at least learned easily at an early age from the experience of working with numbers and mathematical concepts. However, beyond the very basic fundamentals, usually mathematical understanding needs to be taught as the processes become more intricate. All people need to use certain mathematical ideas, such as counting and measuring. In the absence of formal schooling, children learn these skills from experience or from older mentors, perhaps even as apprentices. However, modern society usually considers the teaching of basic mathematics as one of the more important tasks of elementary and middle schools.

More than most school subjects, mathematics is cumulative. Each higher level of mathematics content builds on lower levels studied earlier. Even as school mathematics curriculum may spiral, returning to earlier topics, each cycle returns at a higher, more sophisticated level. Consequently, any review that takes place leads to further growth in understanding the content and newer applications. Mathematics is known to open doors for careers in many areas from nursing to accounting to engineering and science. Since no one can predict the future mathematical needs of individual students in elementary or middle school, it is important that all have every opportunity to be adequately prepared for whatever mathematical direction they may go. If a student misses out at understanding a particular topic or has a gap in the coverage of material, he or she may be hindered in the process of learning the next step. Students in high school or college who develop a late interest in scientific or mathematical careers often require additional preparatory coursework, time, and assistance in learning if their academic backgrounds lack the necessary content of the field. This requirement can be seen in the growth of remedial courses in colleges.

These considerations make it important to be watchful for any loss of opportunity that can occur along the path of mathematics learning. A key transition for students is the move from the basic mathematical literacy of elementary school to the start of more specialized mathematics that usually begins in middle school. Sometimes children themselves opt to move away from mathematics. They may be discouraged by a lack of success, pushed by peer pressure, or not fully engaged by the methods of presentation they experienced in their early classrooms. Even for successful students, mathematics classes may not completely capture their interest, especially if much repetition occurs with the intent of filling in content that was missed earlier. Enrichment material and new challenges that address different styles of learning can help show successful students that mathematics is fun and interesting, and mathematics competitions allow them to be cheered and congratulated. Other students may need extra care to learn the concepts and procedures they had missed before, especially if the presentation can be made in new ways to provide extra clarity and interest.

Perhaps of greater concern are the students who feel pressure that mathematics is not for them. Often girls may get the impression that mathematics is only for boys (sometimes from the attitudes of their par-



Robert Moses founded the Algebra Project to help bring math literacy to low-income students.

Civil Rights

Robert Moses was a civil rights activist as a young adult and later became a mathematics educator, but continued his concern for civil rights. He argued that the opportunity to study algebra is a civil right.

As the beginning of the cumulative series of mathematics courses beyond arithmetic, algebra is the gateway to further mathematics. Many advanced mathematics courses, and the resulting career choices and opportunities, depend on the successful completion of a course in algebra; and, of course, completion depends on the opportunity to take algebra at an appropriate point in the school program. If children are being put into tracks that delay or deny their study of algebra, they are being hurt. If tracking occurs based on gender, race, or socioeconomic status—as some believe—then it amounts to actual discrimination.

ents, peers, the media, or society at large). Children of other underrepresented groups may not see people who look like them doing mathematics and therefore come to believe that they are ruled out of these pursuits. Enrichment in school mathematics needs to go beyond additional challenging content, but also to demonstrate that everyone can do mathematics, and that mathematical careers welcome anyone with the interest and motivation to pursue them. Guest speakers from underrepresented groups who have been successful in mathematical careers (or at least stories about their successes) can provide examples of achievement for children of these groups.

Tracking

In many school districts, children as early as the fourth or fifth grades are evaluated, sometimes from one-time tests that may not reflect their overall performance. The evaluations direct or "track" children into various types of mathematics classes as they move into middle schools. Some go immediately into prealgebra or algebra classes, while others remain in arithmetic classes, often recycling content from earlier grades. Once a student is put into the lower-level track, it becomes increasingly unlikely that they will be able to move into faster streams or have the opportunities to take advanced mathematics in high school—even if they are doing well and demonstrating high abilities.

Conclusion

In today's increasingly technical world, ordinary citizens need to understand more mathematics than in the past, just to do the ordinary tasks of daily life. At the same time, the fields of science, technology, engineering, and mathematics need to recruit new workers who can pick up and carry on with this growth. In short, people need more mathematics, and mathematics needs more people. The pipeline of children in mathematics in elementary schools becomes narrower and narrower as one moves through the levels of schools to graduate degrees in mathematics. If today's successes in mathematics are to continue, doors must be opened for all students to study and learn more mathematics.

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Lawrence H. Shirley

See Also: Curriculum, K–12; Minorities; Women.

Mathematics Research, Interdisciplinary

Category: Mathematics Culture and Identity.
Fields of Study: Communication; Connections; Problem Solving; Representations.
Summary: There is a sense in which mathematics is always interdisciplinary but there can be special benefit to approaching it collaboratively with researchers from different disciplines bringing disparate skills, knowledge, and methodologies to bear.

In our increasingly complex society, the problems that need to be solved often lie beyond the scope of a single academic discipline. Interdisciplinary research crosses these traditional boundaries. Frequently, this boundary crossing involves bringing together individuals with a variety of knowledge and skills into collaborative working groups. Interdisciplinary research can also refer to an individual who regularly works in a field that is inherently interdisciplinary, such as mathematical physics. Interdisciplinary research can be both highly productive and truly inspiring, creating connections that lead to new knowledge, both at the intersection of the participating disciplines and within the individual fields involved in the collaboration. Mathematics plays a role in collaborations with a wide variety of disciplines, and many early mathematicians were multidisciplinary researchers and explorers. In the twenty-first century, the sciences, social sciences, businesses, and even the liberal and fine arts are working ever more closely with mathematicians on interdisciplinary problems. Despite the apparent benefits and future promises of interdisciplinary research, those who are interested in pursuing such activities often face obstacles and disincentives.

Sometimes these are barriers of communication or culture, since different disciplines have their own vocabularies and ways of working. Other barriers are related to the tradition of organizing academic institutions into discipline-based departments, which sometimes carries over into support and professional structures like funding organizations, professional societies, and journals. At the same time, mathematicians with interdisciplinary skills and experience are highly sought by employers, resulting in a shift toward creating departments or programs that exist on these interdisciplinary boundaries. There are also interdisciplinary centers and workshops to educate new and current mathematicians in both the rewards and challenges of interdisciplinary research.

Funding and Support

Funding agencies for research are generally supportive of interdisciplinary research because they feel more confident that expert input will be available in all necessary fields and that the results will be usable. They also want to help researchers learn from each other. However, it is not easy to publish interdisciplinary research, as it may not seem sufficiently novel to each discipline. A bigger problem is the lack of academic employment opportunities. One solution is to create a new discipline, for example, mathematical biology, sports science, science policy, or computational science. In times of budgetary constraint, disciplines may be reluctant to share scarce resources in interdisciplinary activities. However, this can be a difficult endeavor, as a new discipline may not immediately be seen as legitimate until it has been established within the peer community—and perhaps in society at large—that its results are valid and important. Enthusiasm for interdisciplinary mathematics research is reflected in a wide range of interdisciplinary societies, Web sites, and emerging venues for interdisciplinary publication and presentation.

Benefits

Interdisciplinary mathematics research can reveal the connections between methods used in different disciplines hidden beneath different representations. The engineer's assessment of smoothness by spatial correlation is basically the same as the economist's assessment of temporal change by autocorrelation. Even within the mathematics community, nomenclature and approaches can differ. The term "normal" means one thing to a statistician and something entirely different to an algebraist, and the use of the term "dimension" within linear algebra somewhat differs from many other applications of the term. Proper communication is essential to clarify and accommodate these linguistic and conceptual differences. However, this is also true for single-discipline research. The extra effort put into ensuring good understanding and communication makes a successful outcome more likely.

All participants in an interdisciplinary group can benefit from the diverse perspectives of the various fields that are represented, but care must be taken to avoid incompatible levels of detail and complexity as well as confusion over discipline-specific use of language or jargon. For example, clarification of confidence intervals can save much misunderstanding in the public sector, and understanding how government works is very useful to mathematicians. In the field of algebraic geometry, algebraic problems may be translated to geometric problems that are more easily solved in that setting, or vice versa.

Lean Six Sigma, a business management strategy that draws heavily on modern quality-improvement techniques, statistical process control, and broader statistical methods, is a good example of interdisciplinary mathematics research. Company staff are trained in a range of statistical methods and have to apply their knowledge in work-based projects. Computational science emerged from the multidisciplinary overlap of computer science, mathematics, and scientific applications. At first it was seen only as the intersection of these disciplines. As it grows in scope, computational science is seen as an independent discipline with unique issues and content. Mathematics and biology have long been intertwined, but the increasing collaboration and interdependence will no doubt enrich not only the interdisciplinary field but also both of its parent disciplines.

Interdisciplinary researchers also influence mathematics by analyzing and forecasting disciplinary trends. For example, technology forecaster Alan Porter and science and technology policy researcher Ismael Rafols examined whether science was becoming more interdisciplinary. They analyzed work between 1975 and 2005 over six research domains using established metrics, a new "index of interdisciplinarity," and a science mapping visualization method.

Their analysis showed large increases in the number of cited disciplines, references, and coauthors per article, but the citations tended to be in close disciplinary areas. This suggested that science has in fact become more interdisciplinary, but incrementally—first to closely related fields and only later to more disparate areas. This is consistent with the fact that close disciplines are more likely to share methods and vocabulary, as well as peer reviewers, conferences, and venues for publication.

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See Also: Careers; Communication in Society; Problem Solving in Society.

Mathematics Software

See *Software, Mathematics*

Matrices

Category: History and Development of Curricular Concepts.
Fields of Study: Number and Operations, Algebra, Communication, Connections, Representations.
Summary: Matrices are useful for a variety of calculations and applications.

Matrices are used throughout modern mathematics and statistics and their applications in the natural and social sciences. Matrix theory and the closely related theory of vector spaces form what is now known as "linear algebra": the study of systems of linear equations and their solutions in n-dimensional space. A matrix is a rectangular array of numbers representing the coefficients of the unknowns in a linear system. The first example of such a system and its solution using matrix operations dates from more than 2000 years ago in China. The closely related concept of "determinants" was introduced independently in Japan and Europe in the seventeenth century. The systematic development of basic matrix theory, in both its algebraic and geometric aspects, took place in the nineteenth and early twentieth centuries. This theory played a major role in the development of quantum mechanics, the branch of physics underlying many of the technological advances of the twentieth and twenty-first centuries. Matrices have been commonly explored in high school since linear algebra became a standard topic in the mathematics curriculum during the middle of the twentieth century. Contemporary applications of matrix theory are cryptography, Internet security, and Internet search engines, such as Google.

Origin of the Term

The word "matrix" comes from Latin, meaning "womb," deriving from *mater* (mother). The mathematical use was introduced by James Joseph Sylvester as "an oblong arrangement of terms consisting, suppose, of m lines and n columns, a Matrix out of which we may form various systems of determinants." At present, the word "matrix" refers to a rectangular array of numbers regarded and manipulated as a single object.

Linear Systems and Row Operations

The first calculation with such an array dates from the Han dynasty in ancient China, in *Nine Chapters of the Mathematical Art*, a practical handbook on surveying, engineering, and finance. One problem posed in the handbook is this:

> There are three types of corn, of which three bundles of the first, two of the second, and one of the third make 39 measures. Two of the first, three of the second, and one of the third make 34 measures. One of the first, two of the second, and three of the third make 26 measures. How many measures of corn are contained in one bundle of each type?

In modern notation, this becomes a system of linear equations, with unknowns x, y, and z representing the three types of corn:

$$\begin{aligned} 3x + 2y + z &= 39 \\ 2x + 3y + z &= 34 \\ x + 2y + 3z &= 26. \end{aligned}$$

The ancient Chinese author writes the coefficients in a rectangular array and solves the system by performing operations on this array. In modern notation, start with the 3-by-4 matrix of coefficients, and then (1) multiply row 2 by 3 and subtract 2 times row 1; multiply row 3 by 3 and subtract row 1; (2) multiply row 3 by 5 and subtract 4 times row 2:

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 39 \\ 2 & 3 & 1 & 34 \\ 1 & 2 & 3 & 26 \end{array}\right] \xrightarrow{(1)} \left[\begin{array}{ccc|c} 3 & 2 & 1 & 39 \\ 0 & 5 & 1 & 24 \\ 0 & 4 & 8 & 39 \end{array}\right]$$
$$\xrightarrow{(2)} \left[\begin{array}{ccc|c} 3 & 2 & 1 & 39 \\ 0 & 5 & 1 & 24 \\ 0 & 0 & 36 & 99 \end{array}\right]$$

The third row of the last array represents the equation $36z = 99$, giving $z = 11/4$. The second row represents $5y + z = 24$, giving $y = 17/4$. The first row represents $3x + 2y + z = 39$, giving $x = 37/4$.

Gaussian and Gauss–Jordan Elimination

This simplification of linear equations by using one variable to cancel another is called "Gaussian elimination." Carl Friedrich Gauss used it systematically in the early nineteenth century in his study of the orbit of the asteroid Pallas. An even more reduced version of a system, called "row echelon form" or "Gauss–Jordan form," was first published in a handbook on geodesy written by Wilhelm Jordan. At that time, elimination methods were considered a tool for geodesy instead of a part of mathematics.

Other Historical Developments

The closely related concept of determinants originated during the late seventeenth century simultaneously in work of Seki Kowa, in Japan, and Gottfried Leibniz, in Germany. From the modern point of view, the determinant is a function of a matrix, so it is remarkable that the study of determinants originated more than a century before the study of matrices. A systematic theory of matrices, determinants, and systems of linear equations was developed by European mathematicians during the nineteenth century: the most important contributors were Augustin Cauchy, Arthur Cayley, Ferdinand Eisenstein, Ferdinand Frobenius, Charles Hermite, Edmond Laguerre, and Karl Weierstrass. Thomas Hawkins, a historian of mathematics, who has done much research on these developments, argues that the most important motivation for this development was the Cayley–Hermite problem of determining all linear substitutions of the variables of a quadratic form, which leave the form invariant.

A famous memoir by Cayley introduced the single-letter notation for matrices together with the operations of matrix addition and multiplication and clarifies the relation between matrices and systems of linear equations: "a set of quantities arranged in the form of a square, for example,

$$\begin{bmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{bmatrix}$$

is said to be a matrix. The notion of such a matrix arises naturally from an abbreviated notation for a set of linear equations, viz. the equations

$$X = ax + by + cz$$
$$Y = a'x + b'y + c'z$$
$$Z = a''x + b''y + c''z\,.$$

Matrix Theory

In the twentieth century, Olga Taussky-Todd became what she later referred to as "a torchbearer for matrix theory." During World War II, she worked on 6-by-6 matrices related to the flutter analysis of aircraft. She used a theorem by Russian mathematician Semyon Aranovich Gershgorin to simplify the amount of calculation and computations. The theory of solving matrix systems continues in the early twenty-first century as numerical analysts search for efficient algorithms. In addition to Taussky-Todd's own theoretical and applied work in the area, she encouraged others to join in its development. Eventually, partly because of her influence, matrix theory became a true branch of mathematics instead of just a tool for applications.

Contemporary Applications

The theory of matrices is an essential part of linear algebra, which is a highly developed branch of mathematics, with many applications to the natural and social sciences. For example, matrix mechanics, the first definition of quantum mechanics, led to the study of infinitely large matrices. Matrices also represent digital images on a computer, and, in musical set theory, matrices are used to analyze or to create compositions. Matrices containing entries other than numbers and the calculus of matrices have found importance in statistics and engineering. Typical applications discussed in modern linear algebra textbooks are network flow, electrical resistance, chemical reactions, economic models, dynamical systems, vector geometry, computer graphics, least squares approximation, correlation and variance, optimization, and cryptography.

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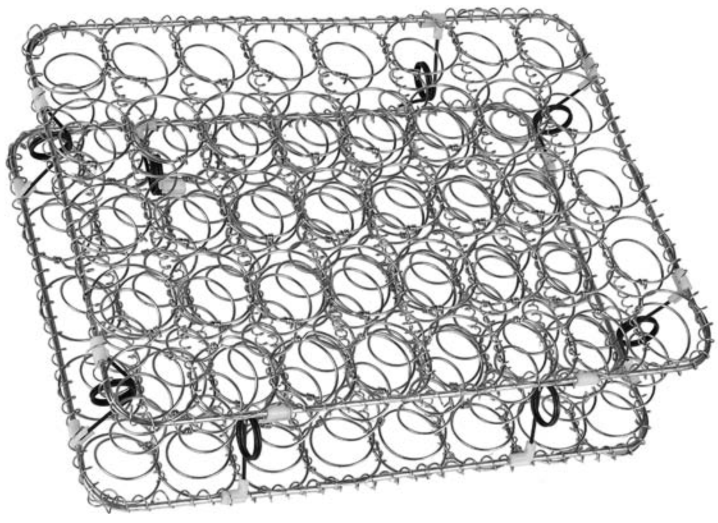
See Also: Algebra in Society; Coding and Encryption; Linear Concepts; Search Engines; Vectors.

Mattresses

Category: Architecture and Engineering.
Fields of Study: Algebra; Geometry; Measurement.
Summary: Modern mattresses are superior to older designs because of the geometry and pressure distribution of the coil springs that define them. Mattresses last longer when rotated through four configurations.

The modern mattress is a cushion for sleeping and sits on top of a box spring that provides support and reduces wear and tear. While straw, coconut fiber, horsehair, feathers, pea shucks, and water have all been used to stuff mattresses in the past, in the twenty-first century most are filled with artificial fibers or foam rubber and derive much of their resilience and support from coil springs, which are either connected by interconnecting wires or encased in fabric. The first innerspring mattress is attributed to Heinrich Westphal in 1871, and its popularity may be because of hygiene and comfort considerations. The geometry of the coils impacts the durability and firmness. Manufacturers use calculations like the average load limit of a floor and the volume and weight of a waterbed. NASA attributes the 1960s invention of soft memory foam with high-energy absorption properties to aeronautical engineer Charles Yost, who was working under a NASA contract. Many studies employ statistics, such as those involving quality of sleep, amount of snoring, and the impact of sleeping positions. Mathematical techniques and models of mattresses have also been useful in studying factors such as pressure distribution, deformation, combustible behavior, and mattress flipping.

The gauge of the coils is one of the factors that impacts the mattress's firmness and, therefore, its support and durability. Counterintuitively for the layman, lower gauges mean larger cross-sectional diameters—the number of passes through the drawing dies that are required to create a wire of a given thickness. Lower gauge means fewer passes, meaning thicker wire. Bonnell coils, named after the inventor, may have been adapted from the coils used for buggy seats in the nineteenth century. The configuration of adjacent hourglass coils connected by helical wire, called "helicals," increases the spring's resistance proportionally to the load. Cylinder pocket springs systems are individual cylinder pieces held together by clips. Continuous coils are rows formed from a single piece of wire. However, while the head-to-toe rows of continuous coils offer good support while still responding to shifts of position and weight, the movement of the coils in response to those shifts is noisier and more noticeable than in other mattresses.



The inner workings of a mattress include coil springs joined together by interconnecting wires.

Wear and tear on a mattress is disproportionate because of the fact that the sleeper and the mattress are not the same shape, and thus some coils will bear more load than others. Compressing and decompressing gradually weakens a mattress; it may show noticeable changes after a few years. Use of a firm box spring helps prevent the sagging that would set in quickly, and rotating and/or flipping a mattress twice a year helps to more equally distribute wear and tear over the course of the mattress's life. The actual technique of mattress flipping has been the subject of some discussion for years because the interval is great enough that it is difficult to remember in which direction the mattress was last flipped or rotated without making some kind of mark on the mattress as a reminder.

A mattress can be rotated along three orthogonal axes (x, y, and z); or to compare a mattress to an airplane, roll, pitch, and yaw. The roll axis parallels the longest dimension, the pitch the next-longest, the yaw the shortest. Because a mattress has two sides suitable for sleeping on, and each of those sides has two possible orientations, this means that there are four possible mattress configurations. One mattress-flipping technique that cycles through these four configurations is called the "Klein 4-group," named for mathematician Felix Klein, which is a group describing the symmetries of a rectangle in three-dimensional space. Absent a mnemonic device to remember the previous and next configuration, random selection may be the best choice to maximize the efficiency of mattress flipping. Over the course of 10 years of random selection every six months, for instance, the most-used orientation will be used about 31% of the time, and the least-used about 19%—a 6% deviation from perfectly distributed usage.

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Bill Kte'pi

See Also: Algebra and Algebra Education; Algebra in Society; Transformations.

Mayan Mathematics

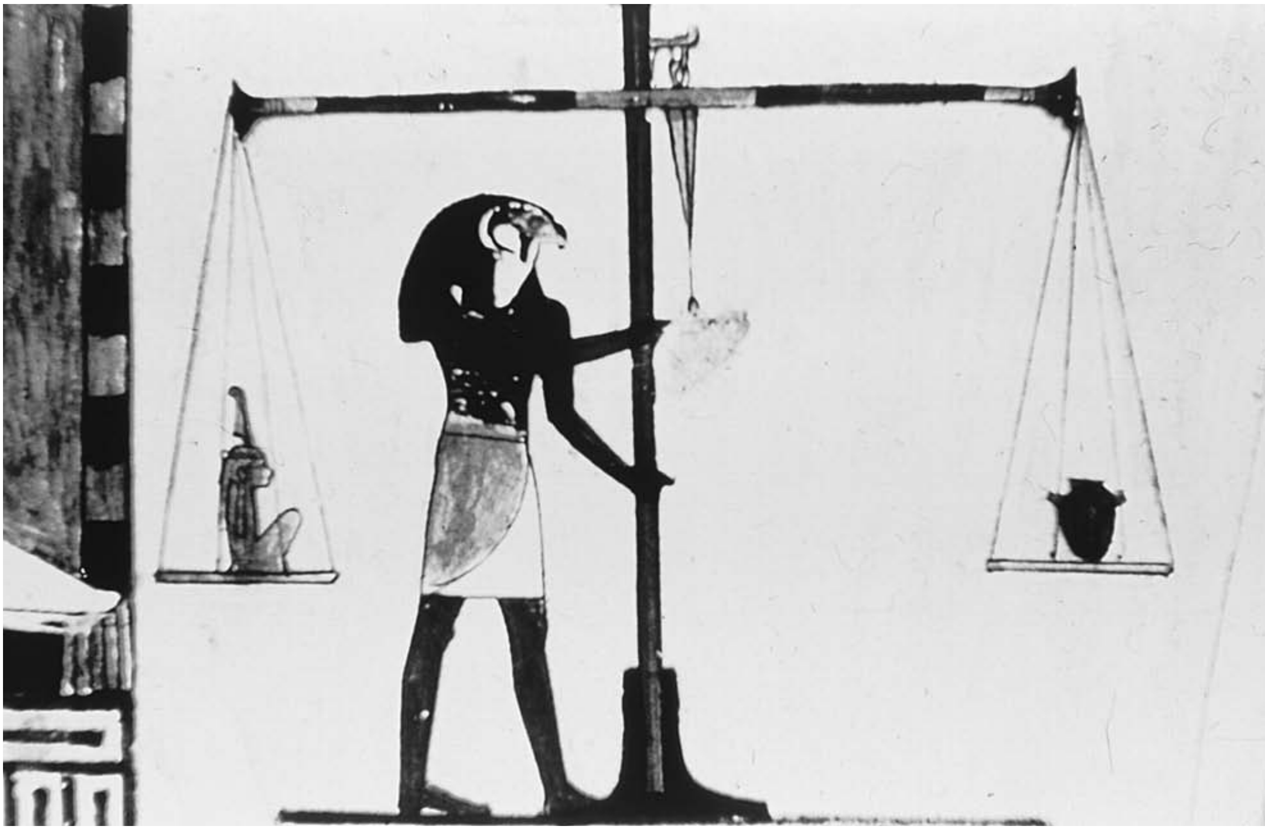
See *Incan and Mayan Mathematics*

Measurement, Systems of

Category: History and Development of Curricular Concepts.
Fields of Study: Communication; Connections; Measurement.
Summary: Various systems of measurement have been used and debated throughout history, with accuracy and precision becoming increasingly important.

Some define "measurement" as the process of determining the magnitude of a quantity. The word comes from ancient Greek *metron*, meaning "proportion," but the process itself is as old as mankind. Long before humans used calculus or algebra, they were measuring length, area, volume, time, and mass. Measures were needed to make furniture, buildings, and ritual places or in landscaping, time keeping, and making skycharts and calendars. Evidence of standardized systems of weights and measures dating back to approximately 3000 b.c.e. has been found, showing that, though measurement systems have become more refined over the centuries, the concept of measurement is an ancient one. By 1600 b.c.e., people offered silver and gold sticks in exchange for products, and in this manner money became a way to measure value.

Mathematicians generalized the notion of measurement in a number of ways, such as in length spaces, metric spaces, and in the field of measure theory. Numerous mathematicians have measures named after them, like the Lebesgue measure, named for Henri Lebesgue, and the Borel measure, named for Emile Borel. Scientists and mathematicians developed systems of measurement in order to quantify objects that were once thought impossible to measure. For instance, they have developed measurements for infinite sets, ways to measure π that are accurate to huge numbers of decimals, measures for hyperbolic geometry in which



Ancient Egyptian art shows a figure measuring weight using a scale. Volume was measured in ancient societies by determining the number of seeds that filled a clay jar or gourd.

the Pythagorean theorem no longer holds, tiny-scale measurements on the quantum level or in nanotechnology, large-scale measurements of the universe, and even measurements of political opinion. Some of these measurements remain controversial, like how to assess educational achievement. Mathematicians and statisticians continue to design, refine, and improve them. In twenty-first-century mathematics classrooms, students in prekindergarten through college investigate measurable attributes, including formulas, models, and processes as well as units and systems of measurement.

Early Measurement Systems

Historical records indicate that the concept of measurement was vital for ancient civilizations, as humans needed to build dwellings, make clothing, and barter for goods. Historical study has indicated that the people of ancient Egypt, Mesopotamia, and the Indus Valley all developed systems of measurement, some of which were remarkably precise. For example, the Indus Valley people used measurements of length where the smallest division was approximately equal to 1/16 inch, as well as "yard sticks" that were exactly 33 inches in length. These measurements, while ancient in origin, were used in traditional Indian architecture and remain in use in the twenty-first century. Ancient humans typically used body parts as instruments for measuring length. The most standard unit of length that developed from ancient cultures is the cubit. The cubit was commonly defined as the length of the forearm from the elbow to the tip of the middle finger.

However, ancient Egyptian culture also defined the Sacred Cubit, which was a common cubit plus an extra hand span. The Sacred Cubit was used for constructing buildings and monuments in ancient Egypt and for surveying land. As ancient civilizations progressed and trade became more vital, standardization of measurement systems became more of a concern. Ancient peoples attempted to solve this problem by creating a rod or bar of a given length (usually a cubit) that was designated as the standard unit of measure. The rod was usually normed on a ruler's dimensions. The original rod was typically kept in a temple or other safe place, and other identical rods were created and distributed throughout the community. The number of seeds that filled a clay jar or gourd served as a measure for volume. Some civilizations also used water instead of grain. Later, stones or sometimes lumps of metal of a certain weight were used for larger units. Like the rods used for length, these stones or lumps of metal were typically kept in temples or other safe places as the official standard of weight. However, duplication of the weight provided opportunities for the deception of customers, as it was fairly easy for merchants to remove weight from a lump of metal. Therefore, inspections of weight measures became common practice, and this practice still continues through the twenty-first century. Some current forms of measurement, such as the carat, were developed out of this ancient tradition.

Standardized Measurement Systems

The English system of measurement was developed from the systems of a variety of cultures, including Babylonian, Egyptian, and Roman. From the Roman culture came the use of the base 12 system (for example, 12 inches in one foot); studies in the etymology of the English measurement units show strong Roman influence. The English system was widely used through the nineteenth century because of royal edicts that helped standardize measurements. For example, King Henry I issued a decree that the distance from the tip of his nose to the end of his outstretched thumb should be designated as one yard. This standardization made the English system very popular in various parts of the world. However, not all areas of the world recognized and utilized the English system, which motivated some to call for a single worldwide standardized system of measurement.

The idea of a single worldwide system of measurement is generally credited to Gabriel Mouton (1670). While several proposals of how such a system might be established were presented at the time, Mouton's proposal used a decimal system based on the length of one minute of arc of a great circle of the Earth. Gottfried Leibniz proposed a similar system in 1673, leading to the concept of a seconds pendulum. However, little was done for more than 100 years to further establish this system of measurement.

The metric system as it is known in the twenty-first century has its origins in the French Revolution, as the National Assembly of France commissioned the French Academy of Sciences to develop a standard of measures and weights. The system that was created was based on establishing a portion of the Earth's circumference as the unit of length. This unit of length was designated a "meter," derived from the Greek word for "a measure." Units of volume and mass were derived from the basic unit of length. The unit of mass, the gram, was found by examining the mass of one cubic centimeter of water at its temperature of maximum density. The unit of volume, the liter, was designated as the amount of water in a cubic decimeter (a cube 10 centimeters on each side). What made the metric system unique was the integral relationship between units of length, mass, and volume. Additionally, the metric system was based on the concept that smaller and larger increments were created by multiplying or dividing the basic units by powers of 10. Working with a base-10 system made the metric system easy to use, as previous measurement systems had used base-12 or base-16 systems.

The first countries that actually used the new system were Belgium, the Netherlands, and Luxembourg, around 1820. France made its use mandatory in 1840. Additionally, the metric system quickly became the standard for scientific and engineering work, which increased its use throughout the world as nations developed technologically. In 1875, 17 countries signed the "Treaty of the Meter," which officially established the standards for metric length and mass. In addition, this agreement established mechanisms for recommending and adopting refinements to the system. The metric system was officially accepted by 35 nations by 1900 and is the standard system of measurement in most nations at the start of the twenty-first century. However, traditional units are still used worldwide and conversion tables and programs ensure successful calculation. Confusion between the two systems can lead to devastating consequences, such as the 1999 crash of the Mars Climate Orbiter, which NASA attributed to the failure to convert from English to metric values.

Modern-day developments in measurement have focused on developing more precise measurement units within the metric system. In 1960, the original nations of the "Treaty of the Meter" convened as the General

Conference on Weights and Measures to develop a revision and simplification of the system. From this convention, seven units of measure were established as the base units of the system: meter for length, kilogram for mass, second for time, ampere for electric current, Kelvin for thermodynamic temperature, mole for substance, and candela for luminous intensity. Also from this convention came the name of Système International d'Unitès (SI), or International Systems of Units, prompting the international abbreviation of SI for the metric system.

Since that time, the General Conference on Weights and Measures has continued to develop more precise and more easily reproducible definitions of measurement units. For instance, the meter was originally a fraction of the distance from the equator to the North Pole, but this measurement was complicated by the fact that the Earth is not a perfect sphere. The late-eighteenth-century expedition of mathematician and astronomer Jean-Baptiste-Joseph Delambre and surveyer Pierre Méchain to calculate the geodesic measurement was fraught with difficulties. Later, a standardized platinum and iridium meter bar was used. In 1983, the meter was redefined to relate to the speed of light, and it became defined as the distance light travels in a vacuum in 1/299,792,458 seconds. Improvements to the metric system have been ratified by the General Conference eight times since the 1960s with the most recent taking place in 1995.

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See Also: Educational Testing; Infinity; Measurements, Area; Measurements, Length; Measurements, Volume; Measuring Time; Measuring Tools; Pi; Temperature; Units Of Area; Units Of Length; Units Of Mass; Units Of Volume.

Measurement in Society

Category: School and Society.
Fields of Study: Connections; Measurement.
Summary: Accuracy and precision are important in the many systems of measurements used in various spheres in society.

Imagine how chaotic the world would be if people could not measure anything! People would not be able to keep track of time, would not know weights or heights of people (or of anything else in the world), could not calculate the distance between any two points, and would not have recipes to cook properly. Indeed, the list of everyday activities that would be impossible to do in the absence of measurement is endless. Thus, measurement is an essential part of everyday life. Measurement is a fundamental part of mathematics research and curricula and there are many types of measurements in society. Some measurements elucidate productivity or change. Others measure large-scale aspects of society, like gross domestic product (GDP). Area measurements have practical applications in areas like surveying and interior design. Measurements are fundamental in drug dosing labels, quality control, missile launches, and in many other applications and fields. Because of its critical and practical importance, measurement is an extensively studied concept in pre-K–12 mathematics education.

Measurement Systems

Numerous measurement systems have been developed and used since ancient times, the earliest of which used body parts as the unit of measurement. The many, diverse measurement systems were a source of confusion, not only among nations, but also among different fields within a nation. To establish common units of measurement and promote their use, a treaty titled the "Convention of the Metre" was signed by 17 countries on May 20, 1875. The Convention of the Metre established three international organizations—the International Bureau of Weights and Measures (BIPM), the General Conference on Weights and Measures, and the International Committee for Weights and Measures—to oversee issues related to measurement in the member nations.

In 1960, the 11th General Conference on Weights and Measures developed and adopted a unified measurement system named International System of Units (SI) to promote a worldwide measurement system. The SI is based on seven dimensionally independent units: meter (the unit of length; abbreviated as m), kilogram (the unit of mass; abbreviated as kg), second (the unit of time; abbreviated as s), ampere (the unit of electric current; abbreviated as A), kelvin (the unit of thermodynamic temperature; abbreviated as K), mole (the unit for amount of substance; abbreviated as mol), and candela (the unit of luminous intensity; abbreviated as cd). Although the spelling of the base units may differ in different languages, the symbols are the same worldwide. The SI is an evolving measurement system to keep up with ever-growing measurement needs. The BIPM, which is comprised of many countries, ensures that measurements throughout the world are traceable to the SI. The BIPM is related to other significant international organizations such as the International Commission on Illumination, the International Atomic Energy Agency, the International Laboratory Accreditation Cooperation, the World Health Organization, and the International Organization of Standardization. Such a worldwide organization to oversee the uniformity of measurements explains clearly the reason for the crucial emphasis on measurement in mathematics curricula.

The United States has its national standards for measurement and measuring devices explained in the U.S. Code. Because measurement and measurement devices are a part of everyday life and are used in various businesses and for commercial purposes, the U.S. Code, published by the Office of the Law Revision of the House of Representatives, includes a chapter titled "Weights and Measures and Standard Time" under the Title 15. The chapter sets standards for weight and measurement devices to enforce accuracy and to ensure equity in the marketplace. In the early twenty-first century, the acknowledgement of measurement in the U.S. Code as a chapter containing 267 sections under nine subchapters is a sound indicator of the importance of measurement in human life.

Accuracy and Precision in Measurement

Any measurement is an approximation to the real value of a quantity. The length of the previous sentence might be 12 centimeters (cm), but in millimeters (mm) it would be 121 mm; 12 cm is not equal to 121 mm. The reason behind the difference between these two measurements is the second measurement is more accurate than the first one. Can it be measured more accurately? This question yields to the need for accurate measurement. One millimeter in this example can be ignored, but an inaccuracy of a mere millimeter in a missile launch may result in a disaster. Improvements in measurement systems are extremely important to make measurements as accurate as possible.

In measurement, the most accurate and precise results are desired. "Accuracy" in measurement refers to the extent to which a measured value matches the correct value. "Precision," on the other hand, refers to the reliability of a measurement and how close individual measurements are to each other. Measurement units and devices in different fields of study are not static; rather, they evolve to improve accuracy and precision. In the United States, the National Institute of Standards and Technology (NIST) is a federal agency that employs mathematicians and scientists, among others, whose main tasks include the advancement of the science of measurement and measurement standards. NIST, together with partners from the government, industry, and academia, also develops measurement tools for different sciences. The services of NIST include verification of the accuracy of measurements, instrument calibration (for example, calibration of dimensional, mechanical, or electromagnetic instruments) to improve measurement quality, and the development of innovative measurement methods.

Although accuracy and precision are always desirable in measurement, in some fields quality of measurement is more crucial. For example, the National Aeronautics and Space Administration (NASA) uses various instruments to measure temperature, pressure, load, and acceleration, and to make other critical measurements for its test programs. The Measurement Standards and Calibration Laboratory of the White Sands Test Facility, which supports an extensive number of test programs, performs instrument calibrations to ensure measurement quality is compatible with recognized national standards that are traceable to NIST. In NASA's test programs, any error in measurements in any equipment may cause not only the deaths of highly trained astronauts but also the loss of millions of dollars. Accurate and precise measurement therefore underpins the success of NASA missions, including launching spaceships and ensuring their safe return to Earth.

Another field where measurement accuracy has critical importance is the health industry. Cancer is one of the most serious diseases that the human race has faced so far. Almost 13% of all deaths in the world were caused by cancer in 2004. Radiotherapy, which uses high-energy radiation to kill cancer cells, is one of the most frequent methods used to treat cancer patients. However, radiotherapy not only kills cancer cells but kills healthy cells as well. Before the start of a cancer treatment, doctors conduct a simulation to locate the patient's tumor and the normal tissues around it. In order to provide effective treatment, the next step is to measure the dose of the radiotherapy required and the safest angles to deliver the radiation to kill cancer cells. Measurements taken for radiotherapy have to be as precise as possible because the amount of radiation required to kill a cancer cell differs by the type of cancer cell and there is a risk of damaging the normal tissue during the radiation delivery. Sophisticated computers capable of making sensitive measurements are used for radiotherapy planning. Thus, human error in measurement is decreased. Advancements in technical equipment used in cancer treatment help to increase the effectiveness of the treatment and decrease the deaths caused by cancer. Monitoring the patient's temperature and thermal dosage in real time provides doctors the opportunity to treat tumors as closely as possible while keeping the adjacent healthy tissues safe.

Measurement in Everyday Life

Measurement is a pervasive mathematical concept in everyday life, so it has many applications to a variety of careers, such as health sciences, architecture and construction, interior design, carpentry, meteorology, and public safety. Precise measurement is crucial in healthcare, as monitoring patient condition has critical importance. Thus, choosing effective measurement devices and obtaining accurate measurements (for example, of weight, blood pressure, or blood sugar) are essential aspects of healthcare professions. Also, healthcare professionals frequently use measurement conversion on the job. Doctors, nurses, and pharmacists convert between English and metric systems, or between Celsius and Fahrenheit, when they collect patient information on weight or temperature or when calculating appropriate medication dosages to administer. Measurement conversion is a particularly important competency for pharmacists, as they convert among different measurement systems such as metric, apothecary, and avoirdupois systems when they calculate medication dosages and fill orders.

Measurement is among the essential mathematics concepts applied in architecture, construction, and related careers. From the design and scale models of a project to its actual construction, precise and accurate measurement is vital. Measurement is also used extensively by interior designers as they improve the aesthetics and function of interior spaces. Interior designers have to determine precise measures of virtually all parts of a space to most effectively utilize the space and to decide the type, size, and placement of furniture or fixtures. Designers need to have precise area measures of walls, floors, or countertops to determine the size and number of tiles needed to cover these surfaces. Indeed, site measure and survey is an essential routine for interior designers in which they get measures of a space and draw an outline of the space, including dimensions.

Carpentry is another occupation for which measurement is substantially important. An old saying emphasizes the significance of measurement in carpentry: "Measure twice, cut once." Because precise measurement is at the heart of good carpentry work, carpenters use various specialized measurement tools, such as a combination square (to accurately measure 45 degree and 90 degree angles), carpenter's square (to plot right angles), and T-bevel (to set and transfer angles), in addition to the regular metal tape measures and folding rulers.

Although most people are familiar with thermometers and their uses, many may not know about various other measurement scales meteorologists use to organize and record weather conditions. Meteorologists use anemometers to measure wind speed or pressure, ceilometers to measure the thickness and height of clouds, barometers to measure atmospheric pressure, and high-tech sensors to measure humidity. People have always been interested in reliable and long-term weather forecasts. Although weather predictions are increasingly accurate and can be made for increasingly longer terms, meteorologists are continuously searching for methods to improve weather predictions. In this effort, innovative measurement devices in meteorology are being developed using the most up-to-date technology to make more accurate and precise weather and climate predictions.

Measurement also has significant applications in public safety. To maintain public travel safety, the Transportation Security Administration (TSA) utilizes the most advanced imaging technology, such as millimeter wave scanners to screen passengers for metallic and nonmetallic threats that might be anywhere on the body without physical contact. Millimeter wave scanners use electromagnetic waves to produce a black-and-white image in seconds. These scanners transmit

Definitions of the Seven Base Units of SI

Length

meter, m: The meter is the length of the path travelled by light in vacuum during a time interval of 1/299,792,458 of a second.

It follows that the speed of light in vacuum, c_0, *is* 299,792,458 m/s *exactly*.

Mass

kilogram, kg: The kilogram is the unit of mass; it is equal to the mass of the international prototype of the kilogram.

It follows that the mass of the international prototype of the kilogram, $m(K)$, *is always* 1 kg *exactly.*

Time

second, s: The second is the duration of 9,192, 631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom.

It follows that the hyperfine splitting in the ground state of the cesium 133 *atom*, ν(hfs Cs), *is* 9,192,631,770 Hz *exactly*.

Electric Current

ampere, A: The ampere is that constant current that, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 meter apart in a vacuum, would produce between these conductors a force equal to 2×10^{-7} newton per meter of length.

It follows that the magnetic constant, μ_0, *also known as the permeability of free space, is* $4\pi \times 10^{-7}$ H/m *exactly*.

Thermodynamic Temperature

kelvin, K: The kelvin, unit of thermodynamic temperature, is the fraction 1/273.16 of the thermodynamic temperature of the triple point of water.

It follows that the thermodynamic temperature of the triple point of water, T_{tpw}, *is* 273.16 K *exactly*.

Amount of Substance

mole, mol:

1. The mole is the amount of substance of a system that contains as many elementary entities as there are atoms in 0.012 kilogram of carbon 12.

2. When the mole is used, the elementary entities must be specified and may be atoms, molecules, ions, electrons, other particles, or specified groups of such particles.

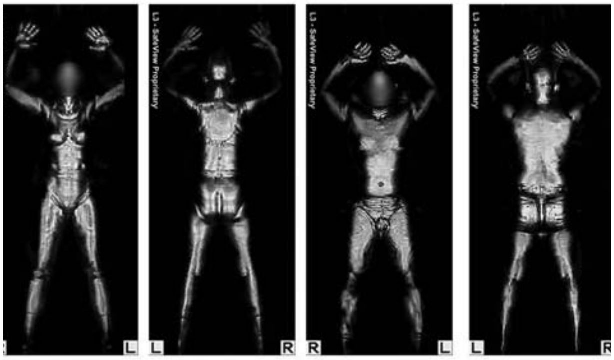
It follows that the molar mass of carbon 12, $M(^{12}C)$, *is* 12 g/mol *exactly*.

Luminous Intensity

candela, cd: The candela is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540×10^{12} hertz and that has a radiant intensity in that direction of 1/683 watt per steradian.

It follows that the spectral luminous efficacy, K, for monochromatic radiation of frequency 540 $\times 10^{12}$ Hz is 683 lm/W *exactly*.

(Adapted from the Bureau International des Poids et Mesures (BIPM) Web site at http://www.bipm.org/utils/common/pdf/si_summary_en.pdf)



The U.S. government claims that millimeter wave scanners emit less energy than a cell phone.

extremely high radio frequencies, a wavelength of 1–10 mm, from two antennas to construct a three-dimensional image of the person scanned. The energy each radio wave reflects back from the passenger's body to the scanner is transmitted to a computer. Then, software measures the energy for each radio wave reflected from the passenger's body to construct an accurate and precise three-dimensional image of the passenger for security check. With the help of such detailed three-dimensional images, any hidden object can easily be identified by security. For such an imaging technology to be used in areas requiring high security needs, like airports, the technology needs to provide fast, accurate, and reliable images. Further, imaging technology developers should consider the amount of radiation emitted by a person who is screened. With more accurate and reliable measurements using advanced imagining technologies, human life can be protected both by eliminating possible threats to public safety and by decreasing side effects of such screening technologies.

Measurement in Pre-K–12 Mathematics Curricula

The study of measurement starts before kindergarten, and most children of pre-K and kindergarten age can acquire considerable knowledge of measurement. Providing young children with motivating opportunities to explore measurable characteristics of objects such as size, weight, and length and engaging them in activities that require comparing and ordering objects by these characteristics can help them develop the concept of measurement. For example, children can order their toys by their size, make short and long (or big and small) animals using clay, or match items of the same size. An activity that can help children start developing an understanding of area might be covering a large flat surface using small sizes of the same surface (such as leaves or cookies) and making comparisons between surface areas (for example, a larger leaf or a smaller cookie). Children can develop a general idea of volume as they pour water from a wider to a narrower container, or from a taller to a shorter container. Parents can also contribute to their children's learning of early measurement concepts and appropriate measurement terms by making comparisons using terms such as "big," "bigger," "small," "smaller," "light," "lighter," "heavy," "heavier," "tall," "taller," "short," and "shorter" when referring to objects or people in their daily conversations. In their daily routines, children encounter various opportunities to develop an understanding of time and its measurement. For example, children can understand the day and night cycle and sequences of their daily activities (washing hands before meals and brushing teeth after meals). The waiting periods for major events that children look forward to, such as special days and holidays, can provide opportunities for children to understand concepts of day, week, month, and year. Young children can learn various measurement devices within daily contexts as they associate money with buying things, clocks and calendars with time concepts, or thermometers with temperature.

In addition to making comparisons and ordering familiar objects, children should experience the process of measurement. Before being introduced to standard units of measure, such as inches or feet (or equivalent units in the metric system), children typically start measuring using nonstandard measurement units. For linear measurement children can measure the length of a table using their hands, the height of a chair using paper clips, or the distance between two points using their feet. Children can explore measuring area as they cover different sizes of flat objects with uniform blocks. An activity for children to learn about volume measurement is placing uniform cubes in a box and counting the number of cubes used to fill up the box. Balance scales can be used to provide children with comparisons of weights of different objects, such as comparing an eraser's weight to a pencil's weight. Children can also weigh objects with nonstandard units using balance scales. They can weigh a book using unifix blocks or a pencil using paper clips. The Illuminations Web site of the National Council of Teachers of Mathematics (NCTM) provides various lesson samples that can be used in preschool classrooms or at home to teach children measuring with nonstandard units. When measuring with nonstandard units, students can conceptualize that they determine the total length, area, volume, or any attribute of interest as they repeatedly measure using the same measurement unit.

After children experience the measurement process using nonstandard units, they will be better prepared to explore measuring with standard measurement tools and units. Measuring with standard units as well as nonstandard units is among NCTM's measurement standards for grades pre-K–12. According to NCTM standards, pre-K–12 students also should be able to choose appropriate measurement units and tools to measure different attributes. Students can be introduced to standard measurement tools, such as tape measures, scales, or rulers, with activities that allow them to experiment with the measurement process. For example, to learn measuring weight using a scale and to gain an idea about weights of different objects, students can weigh themselves and various items such as a bag, a book, or fruit on a scale and record the weights. As students weigh using the scale, they will recognize the units of measurement. After students gain some experience with measuring weights, an enjoyable activity might be to ask students to estimate weights of things that they identify in the classroom.

Throughout elementary and middle school, students learn conversions within a measurement system; measure time, area, volume, temperature, and angle size using appropriate measurement units and tools; find the areas of rectangles, triangles, parallelograms, circles, and irregular shapes; and calculate volumes and surface areas of rectangular solids, cylinders, and trapezoids. In later grades, students are expected to analyze measurement precision and accuracy and approximate measurement error.

An important concept that students need to learn when they study measurement is "estimation." Students in early grades can determine common or personal referents (for example, the width of an index finger is 1 centimeter) as they estimate different attributes, such as length and weight of common objects. As students move on to higher grades, they should be prompted to estimate perimeters, areas, and volumes using benchmarks. Students in college explore the theory of measurement. For instance, they use techniques from calculus to represent the length of a curve and the notion of a metric space is defined in topology. Mathematicians measure hard-to-define quantities, like the length of a coastline, refine and improve systems of measurement, and also research related concepts in the field of measure theory.

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See Also: Hunt, Fern; Measurement, Systems of; Measuring Tools; Representations in Society.

Measurements, Area

Category: History and Development of Curricular Concepts.
Fields of Study: Communication; Connections; Geometry.
Summary: Measuring area is an important mathematical calculation that has been studied for thousands of years.

"Area" is often thought of as the amount of a plane that a two-dimensional figure occupies. The name comes from the Latin word *area*, which means a vacant piece of level ground, reflecting the fact that formulas to calculate area were often developed to facilitate surveying and the calculation of the size of land plots for tax or other purposes. Some formulas to perform area calculations for simple geometric shapes were known in ancient times, while other calculations like the area of curved figures could only be approximated before the development of calculus and a greater depth of understanding regarding the constant π.

In the twenty-first century, primary school students explore attributes such as area and how it changes when the shape of an object changes. They also use formulas to find the areas of rectangles, triangles, and parallelograms and investigate the surface areas of rectangular solids. In the middle grades, the surface areas of prisms, pyramids, and cylinders are also a focus. In high school, students calculate the area and surface area of various geometric figures such as cones, spheres, and cylinders. In calculus classes, students develop a deeper understanding of area through integration techniques.

Ancient History: Egypt, Babylon, and India

The Moscow Mathematical Papyrus (c. 1850 B.C.E.) and the Ahmes, or Rhind, Papyrus (c. 1650 B.C.E.) provide evidence that Egyptians of this period had systems for calculating the areas of different geometric shapes, including triangles, rectangles, and circles. The approach to these calculations is frequently expressed using methods based on the interrelationship between different geometric figures. For instance, one problem in the Ahmes Papyrus notes that the area of a circular field of diameter 9 is the same as the area of a square field with a side of length 8. Historians of mathematics have converted these calculations to an estimate of π that is about 3 1/6 (compared to the correct value of 3.141592 . . .). No differentiation is made between exact and approximate formulas, and there is nothing resembling a proof or theorem in the modern sense—the papyrus presents ways to perform calculations.

The ancient Babylonians also had methods, preserved on clay tablets written in cuneiform, for calculating area. Historians have these tablets and inferred from the calculations values of π, such as 3 and 3.125. As with the Egyptians, methods to calculate area were often expressed by the relation between different geometric figures, and there is no evidence of proofs.

Practical needs also motivated the mathematics presented in the *Sulbasutras*, appendices to the *Vedas* (Hindu scriptures), which explain how to construct sacrificial altars. These scriptures include methods for constructing circles from squares and vice versa, indicating 577/408 as an approximation of $\sqrt{2}$.

Ancient Greeks

The ancient Greeks were able to estimate or derive many areas, in some cases building upon earlier work done by people from other cultures and civilizations. Antiphon the Sophist (480–411 B.C.E.), Eudoxus of Cnidus (408–355 B.C.E.), Archimedes of Syracuse (287–212 B.C.E.), and others approximated the area of figures like the circle by using inscribed and circumscribed polygons, a technique referred to as the "method of exhaustion." Using polygons, Archimedes was also able to show that the surface area of a sphere is four times the area of "the greatest circle in it." In mathematics classrooms, students may discover this relationship by peeling an orange and fitting the peel pieces into four circles that have the same diameter as the equator of the orange. Expressed in modern terminology, each circle has area πr^2 where r is the radius, so the surface area is $4\pi r^2$. In ancient Greece, the Pythagorean theorem, named for Pythagoras of Samos, was expressed in terms of areas of squares rather than the lengths of the sides of a right triangle; the square figure on the hypotenuse had the same area as the sum of the other two squares. Euclid of Alexandria (325–265 B.C.E.) collected the theorems of Pythagoras and other predecessors into the treatise now known as the *Elements*, which has proven to be one of the most influential mathematical textbooks in history. Heron of Alexandria (10–70 C.E.) published the *Metrica*, a treatise that collected formulas for calculating the area and volume of many different geometric figures and also presented what is known today as "Heron's formula" for expressing the area of a triangle in terms of its sides:

$$K = \sqrt{s(s-a)(s-b)(s-c)}$$

where K is the area of the triangle, a, b, and c are the length of the sides, and s is the semiperimeter (half the sum of the lengths of the sides).

Seventeenth Century

In the seventeenth century, German astrologer and mathematician Johannes Kepler applied the ideas of calculus to compute the area and volume of conic sections and casks. Reportedly, his interest was sparked while at his own wedding reception; Kepler became interested in finding a method for calculating the volume of wine in barrels that were not perfect cylinders (they were wider in the middle than at the top and bottom), meaning that the simple formula for the volume of a cylinder could not be applied. Development of differential and integral calculus, necessary to find the area of curved figures, is attributed to both German mathematician Gottfried Leibniz and English mathematician Sir Isaac Newton, in the seventeenth century. Also in the seventeenth century, the French mathematician Albert Girard published a treatise that was the first to use the abbreviations "sin," "cos," and "tan" for the sine, cosine, and tangent and demonstrated that the area of a spherical triangle depends on its interior angles, which is known as "Girard's Theorem."

Recent Developments

In the nineteenth century, the ancient area problem of squaring the circle was finally resolved. The challenge had been to construct a square that had the same area as a circle using only a ruler and compass. In 1882, German mathematician Ferdinand von Lindemann proved that π is a transcendental number, meaning that it is not equal to any finite sequence of algebraic operations on integers. This characteristic also meant that ruler-and-compass methods of constructing a square with area the same as the area of a circle of radius 1 were also doomed to failure. However, methods other than ruler and compass constructions can be used to obtain such a square.

In 1917, the Japanese mathematician Soichi Kakeya posed what is known as the "Kakeya problem," which asks whether there is a minimum region in a plane in which a needle (line segment) can be freely rotated. This area minimization problem was solved in 1927 by Russian mathematician Abram Samoilovitch Besicovitch and also in 1928 by German mathematician Oskar Perron. In the 1930s, American mathematician Jesse Douglas and Hungarian Tibor Rado published solutions to "Plateau's Problem," which requires finding the area of a minimal surface bounded by a curve. The problem is named for nineteenth-century Belgian physicist Joseph Plateau, although it was first posed in the eighteenth century by Joseph-Louis Lagrange.

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See Also: Measurement, Systems of; Measurements, Volume; Measuring Tools; Polygons and Units of Area.

Measurements, Length

Category: History and Development of Curricular Concepts.
Fields of Study: Communication; Connections; Measurement; Number and Operations; Representations.
Summary: The challenge of measuring lengths spurred numerous mathematical developments.

The origin of length measurements certainly predates any recorded history. One can imagine a hunter in the Pleistocene making arrows whose length only marginally exceeds the draw length of his bow, or perhaps measuring spear-throwing distance so that when hunting, throws are not wasted on animals out of range. The introduction of new technologies invariably increased the demand on the range and precision of measuring abilities. To build a house, beams need to be cut to specific lengths and notched at nearly exact positions. To build a cart or any wheeled object, lengths need to be gauged with remarkable precision in order for the wheel to have the freedom to rotate while still having weight-bearing support in all directions.

As older technologies were improved and new inventions arose, the terms and mathematics of length measurement were forced to keep pace. In order to convey the perception of length without having to give an example—indicating, for example, the width of a farming field to a friend, or the height of a horse to a potential buyer—it quickly became useful to adopt certain units (agreed-upon conventions for fixed lengths that could be used for reference when desired). Many of these units originated from roughly constant measurements of parts of the body, simply because this turned every person into a walking measuring stick. The foot and the hand are perhaps the most obvious examples of body-related units of measure. In fact, the width of the palm of the hand is roughly 4 inches (including the thumb when closed against the palm), and is still used today to indicate the height of horses. The inch was originally the width of a thumb. The cubit was perhaps the first standardized unit of length and is defined to be the length from the elbow to the tip of ones longest finger. There is some evidence indicating that the yard was defined by King Henry I to be the distance from the tip of the king's nose to the end of his outstretched thumb.

History of Standardized Measures

The adoption of widespread and official standardization began, as far as is known, in Europe during the reign of Richard the Lion-Hearted in the late twelfth century. At this time it was decreed that, "Throughout the realm there shall be the same yard of the same size and it should be of iron." During the reign of Edward I, in the late thirteenth century, additional terms were created:

> It is remembered that the Iron Ulna of our Lord the King contains three feet and no more; and the foot must contain 12 inches, measured by the correct measure of this kind of ulna; that is to say, one thirty-sixth part [of] the said ulna makes one inch, neither more nor less . . . . It is ordained that three grains of barley, dry and round make an inch, twelve inches make a foot; three feet make an ulna; five and a half ulna makes a perch (rod); and forty perches in length and four perches in breadth make an acre.

This quest for standardization lasted through multiple revisions of terms and new techniques for representing the meter or the yard. In fact, measurements of weight and time evolved in very similar ways with similar revisions. These efforts occasionally reached giant proportions. In 1791, after a protracted debate over the most natural and elegant way to define these units of length, the French National Assembly decided that the meter should be defined as one ten-millionth of one-quarter of the circumference of the Earth. Using geometric techniques, they had already been able to estimate this distance to be very similar to the previously held definition of the meter. France then sent surveyors all over the globe to more exactly measure this distance. Although the surveyors often encountered hostility, occasionally being arrested as French spies, in 1799 the project was completed and a platinum bar representing the definition of the meter was created and stored in a safe location.

As technology improved, so did the definitions of units of length. In the mid-twentieth century, the meter was redefined using the wavelength of light emitted by fluorescing krypton atoms. This definition, although much more complicated, had the enormous advantage that meter could now be reproduced almost exactly by any laboratory that had sufficiently advanced equipment. No longer was the definition for the meter something that lived in isolation, requiring careful guarding. Once the laser was invented in 1960, it became practical to redefine the meter in terms of the speed of light, often considered the ultimate physical constant. Thus the meter became, precisely, the distance traveled by light in a vacuum during 1/299,792,458 seconds—a definition that continues to be used at the beginning of the twenty-first century. This definition, of course, gives rise to the question of exactly how a second is defined.

Other Considerations

During this time, however, there were many more complicated considerations than simply how to define a unit of length. Once the units of length were defined, it was invaluable to have the ability to calculate the lengths of objects that seemed difficult to measure or to predict the lengths of objects that did not yet exist.

Once units of length were defined, humans gained the ability to calculate the lengths of objects that seemed difficult to measure.

When building a house, the builder must decide first how wide and how deep and how high the house is to be, and then the builder will cut down trees of the right size, and trim them down to obtain the needed logs. But how can the builder be sure that a tree is of the right size? If a builder is planning to cut down a tall tree, it is usually impractical to climb it just for the sake of measurement. Because of this impracticality, people adopted several clever methods for estimating the height of a tree without having to leave the ground.

Native Americans had a particularly clever tool-free method. They would bend over and look through their legs at the tree. Then they would walk away from the tree and repeat this process until they found a point where they could just barely see the top of the tree. It turns out that the distance from this point to the base of the tree is almost exactly the height of the tree (provided the measurer is an average-sized person). The reason for this measurement technique is that a normal person looking between one's legs sees at about a 45-degree angle upward. Geometric principles indicate that since a tree makes roughly a 90-degree angle with the ground, then the measurer, the base of the tree, and the top of the tree make a 45-45-90 triangle. Such a triangle has equally long legs, which means that the height of the tree is equal to the distance from the base to where the measurer is standing. The Native Americans probably did not think about it in these exact terms, of course; they most likely discovered this trick by trial and error. Nonetheless, the ability to make these calculations is extremely important when it comes to building large structures.

Measuring Triangles

Ancient civilizations have long known that when building structures that need to hold weight, triangular supports are very effective. A natural question, then, is how long to make the triangular piece. Say a person is building a simple box to stand on. The box will be 1 meter wide, 1 meter deep, and 1 meter tall. If the person builds just the box, there is a danger it will collapse when stood upon, so triangular supports are included. Specifically, this person decides to build each of the four "wall" sides to be a square with a single piece added in diagonally to form two triangles. Since each of the squares is 1 meter tall and 1 meter wide, how long should the single piece of wood be so that it can join opposite edges of the square?

Pythagoras, a Greek philosopher and mathematician who lived around 500 B.C.E., developed a simple formula, the Pythagorean Theorem, that can be used to answer the question. His formula states that for a triangle where one of the angles measures to be 90 degrees, and *a*, *b*, and *c* are the side lengths where *c* represents the hypotenuse (the side across from the 90-degree angle) then $a^2 + b^2 = c^2$. Using this formula, the square can be imagined as two triangles, each of which has a 90-degree angle, and it can be seen that c^2, the length of the hypotenuse squared, must be equal to $1^2 + 1^2 = 2$. Therefore, the length of the piece of wood needed is 1/7, which is approximately 1.4 meters.

When Pythagoras answered this question, it provided a huge boost to the ability to manufacture precisely engineered constructs. But practically and mathematically, it also raised additional questions. How can one determine the length of the triangular piece if the triangle does not possess a 90-degree angle? What are the properties of triangles that are best at supporting weight? The mathematical field of trigonometry (from the Greek words *trigonon*, meaning "triangle" and *metron*, meaning "measure") was invented to answer these questions.

Measuring Curves

The ancient Greeks were obsessed with geometry and incorporated it into many nonmathematical aspects of their lives. This incorporation led to the aesthetic, if intellectually dubious, concept of sacred geometry that infuses some spiritual movements. Perhaps the single most prevalent concept in geometry is that of a circle. The concept of relating various geometric shapes to circles can be seen anywhere, from images showing triangles (or more complicated shapes) connecting to a circle in various ways to Leonardo DaVinci's drawing of the Vitruvian Man. Although the circle relates heavily to the measurements computed via trigonometry, it is the source of a different and altogether more elusive measurement of length—the length of curves.

Of course when one thinks of the measurement of length, one generally considers the length of straight line segments. But even thousands of years ago, it was obvious to some mathematicians that it made sense to ask about the length of a curve. It is easy enough to draw a circle, take a piece of string, mold it to nearly the same shape as the circle, cut it off at the right point, and then remove the string, lay it in a straight line, and measure it. This measurement gives, roughly, what is called the "arc length" of the circle. To attempt to compute this length using pure mathematics, mathematicians would undergo a tedious process where they would approximate the curve using dozens, or hundreds, of small line segments. Then they would measure each tiny line segment and add up the results to get, again, roughly, the arc length of the curve. It was through this process that mathematicians discovered the remarkable fact that—although circles could be made with very large or very small arc lengths—for any circle, its arc length (also called "circumference" when refering to a circle) divided by its diameter was always a fixed number. This fixed number is π, which is approximately equal to 3.14. This fact was known to the ancient Egyptians, who, like the Greeks, had a penchant for incorporating mathematical references into culture, literature, and architecture. In fact, the Great Pyramid at Giza was built with a perimeter of 1760 cubits and a height of 280 cubits. $1760 \div 280$ is almost exactly equal to 2π.

Calculus

In part because of the appeal of discoveries made about the arc length of circles and in part because of the practical application, there was more research done into calculating arc lengths of curves, and in general calculating other abstruse quantities, such as the area enclosed by a curve, or how wind would change the velocity of a balloon. The bulk of the theory necessary to make these computations was the mathematical field of calculus, coinvented by Isaac Newton and Gottfried Wilhelm Liebniz.

The fundamental concept inherent in calculus is to break up an object into a very large number of very small pieces and to put those pieces back together again. The advantage calculus has over the cumbersome approach used by earlier mathematicians (breaking up a curve into many individual line segments, measured individually) is twofold. First, instead of using large numbers of small pieces, they actually used infinitely many infinitely small pieces. This means that instead of getting only an approximation, the error was infinitely small, and so the methods of calculus would actually yield exactly the correct answer. The second advantage is that calculus incorporates many methods to simplify these calculations involving infinity. These techniques are so simple that many high school and college students routinely master the subject. However, a lingering flaw in calculus after Newton and Leibniz's development was the fact that the notion of "infinitely small" and "infinitely big" was vague and never precisely defined.

Mathematician Augustin-Louis Cauchy, in the mid-1800s, created the precise definition of these elusive concepts. An infinitely small quantity was defined to be a sequence of numbers that got arbitrarily close to zero; for example, 1, 1/2, 1/3, 1/4, 1/5, ... To make this even more precise, Cauchy pointed out that this sequence had a special property. To illustrate, pick as small a positive number as you can, for example, 1/1,000.000. Draw a circle around the point 0 of that small radius. At some point along the sequence, all the terms past that point will lie inside that circle—in other words, all the terms past that point will have distance from the point 0 less than the specified number. Even if you chose a new number, much smaller than the first one you picked, that would still be true—you would simply have to traverse farther along the sequence before you would find that special point. This property is called convergence—and clearly relies heavily on the notion of distance for its definition.

In the late 1800s and early 1900s, there was a large amount of work done in improving the techniques and perfecting the details of calculus. The mathematical operation that allowed one to find the area enclosed by a curve was called an "integral." One of the more important improvements to calculus, created by Henri Lebesgue in 1901, was the Lebesgue integral, a concept that extended and strengthened the original idea and allowed the development of more robust mathematical machinery. An interesting feature of this new integral was that it was so general it could be applied to curves that in some sense couldn't even be graphed. Soon after the development of the Lebesgue integral, a mathematician named Maurice Frechet, impressed by the generality of Lebesgue, invented metric spaces.

A metric space is a very general idea. It is the concept that one begins with a group of objects about which absolutely nothing is known, except that the distances between them are measurable. This idea turned out to be enormously powerful because it was able to capture the precise definitions made by Cauchy while at the same time being so general that they could apply to almost any mathematical system that people wished to study. The genius of the idea was in the realization that in so much of the complicated mathematics that was now being done, the one idea always relied on was that of measuring distance. Cauchy defined an infinitely small quantity to be a collection of numbers that becomes arbitrarily small. But this definition can be generalized to a collection of these objects, where the distance between them becomes arbitrarily small. Metric spaces quickly permeated all areas of mathematics, and metric space theory remains one of the foundational components of the mathematical area of analysis, the branch of mathematics used most heavily by scientists.

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See Also: Carpentry; Mapping Coastlines; Measurement, Systems of; Measurement in Society; Ruler and Compass Constructions; Units of Length.

Measurements, Volume

Category: History and Development of Curricular Concepts.

Fields of Study: Communication; Connections; Geometry; Measurement.

Summary: Volume has been measured in numerous ways throughout history, with calculus playing an integral role.

Volume is the amount of space that is occupied by an object. In other words, volume is a three-dimensional analogue of the area. Volume is important in construction, engineering, and physics. Early on, weight was easier to measure than volume, especially because crops and other real-life objects often had irregular shapes. One early volume calculation can be found in an ancient Egyptian mathematical work called the "Moscow Papyrus," named for the country where it resided in the twentieth century. It dates back to almost 2000 B.C.E., and its author is unknown.

One problem provides a method for calculating the volume of a truncated pyramid. However, mathematicians developed a variety of methods to calculate volume, including via the displacement of water, the method of exhaustion, and connections to determinant and integration methods. In twenty-first-century mathematics classrooms, students investigate volume relationships and formulas. Students calculate the volume of geometric objects like cylinders, cones, and spheres. The volume of a cylinder ($\pi r^2 h$), is obtained by multiplying the area of the base, πr^2 where r is the radius of the given circle, by the height (h). The volume of a cone is one-third that amount, and the volume of a sphere is

$$\frac{4}{3}\pi r^3.$$

However, these formulas took a long time to develop and could only be approximated before the development of calculus and a fuller understanding of π. Students also compare quantities of water or sand-filled relational geosolids and examine volume integrations and the volume interpretation of determinants.

Early Methods

The method of exhaustion has long been used to estimate volumes. Democritus of Abdera is noted as the first to state that the volume of a cone is one-third that of a cylinder of the same height and radius and that the volume of a pyramid is one-third of the corresponding prism. Eudoxus of Cnidus developed the method of exhaustion that uses what would now be referred to as "limits" of sums of well-known areas or volumes. He justified Democritus' relationships and explored other areas and volumes. Some of Eudoxus's work appears in Euclid of Alexandria's *Elements*. In ancient China, volume calculations were published in the Nine Chapters on the Mathematical Art. In his commentary of 263 C.E., Liu Hui calculated the volume of figures like a tetrahedron and the frustum of a cone. The volume of the sphere was challenging and he noted: "Let us leave the problem to whoever can tell the truth." Archimedes of Syracuse researched the volume of various figures, including surfaces of revolution. He showed that the volume of a cylinder equals the sum of the volume of a cone of the same height and the volume of a sphere of the same height. Here, the height can be expressed as twice the radius. Archimedes is reported to have considered this as one of his greatest achievements because a related inscription appeared on his tombstone. In twenty-first-century classrooms, students understand Archimedes' statement by pouring sand or water from a cone and a sphere into a cylinder to fill it up. They also investigate the related formulas. Archimedes also reportedly noticed that water displacement could be used to measure volume while famously expressing: "Eureka!"

Using Calculus and Integration

Computing volume by integration allowed for the calculation of the volume of irregular objects. In 1615, Johannes Kepler published *Nova Stereometria Doliorum Vinarorum* (*New Solid Geometry of a Wine Barrel*). He apparently became interested in the volume of casks on his wedding day. Methods of integration and volume calculations developed along with calculus before the related analysis was well understood. Cavalieri's principle is named for seventeenth-century mathematician Bonaventura Cavalieri. Cavalieri combined the method of exhaustion with Kepler's work and computed volumes by comparing cross-sectional areas. This method predates the analysis that was needed to put it on a sound footing, and Cavalieri was criticized for his ideas. Some results seemed counterintuitive and provided additional fodder for critics. For example, the surface of revolution obtained from revolving the region under $1/x$ between 1 and infinity has finite volume. In the seventeenth century, mathematician Thomas Hobbes is noted as having remarked about this result: "To understand this for sense it is not required that a man should be a geometrician or a logician, but that he should be mad."

Mathematicians eventually developed the analysis rigorously. The Riemann integral is named for nineteenth-century mathematician Bernhard Riemann. Measuring the area below a graph of the function is accomplished by dividing the region under the graph into extremely small rectangles and adding these rectangles up. Roughly, the volume of a region in space would be computed with a similar idea. The given space would be divided into small rectangular boxes. Each piece would have the volume $(dx \times dy \times dz)$, and the volume of the whole space would be computed by a triple integral. However, this method supposes that one understands the functions that make up the surface. Many mathematical theories about approximation of the boundary surface have been developed for a long time, and they have played an important role since they are indeed extremely useful in actual computation.

Other Methods

Another method of computing volume that is explored in linear algebra and physics classes is by the determinant. In a 1773 paper on mechanics, Joseph Lagrange calculated the volume of a tetrahedron in terms of the locations of the coordinates. In modern terms one would recognize the connection of the expression to a determinant calculation of a 3-by-3 matrix.

Integration formulas such as Green's theorem and the divergence theorem, which are studied in a multivariable calculus course, connect volume to other calculations. Mathematician and physicist George Green worked on vector calculus integral theorems, and Green's theorem is named after him. Green's theorem relates surface and volume integrals. Mathematician Carl Friedrich Gauss contributed to the geometry of surfaces as well as the divergence theorem. The divergence theory relates the volume integral of the divergence inside a surface to the flux of a vector field on the closed surface. A well-studied question related to volume measurements dates back to ancient Greece. Archimedes and Zenodorus examined the sphere as the surface that would enclose a given volume with the least amount of surface area. Mathematician Hermann Schwarz proved that the sphere maximized volume with minimal surface area in 1884. In the twentieth century, mathematicians solved the Double Bubble Problem, showing that a standard configuration is the most efficient way to enclose two regions.

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See Also: Calculus and Calculus Education; Measurements, Area; Pi; Units of Mass; Units of Volume.

Measures of Center

Category: History and Development of Curricular Concepts.

Fields of Study: Communication; Connections; Data Analysis and Probability; Geometry.

Summary: Mode, median, and various averages (including the arithmetic mean and weighted average) are all examples of deriving a central value.

Mathematician and anthropometry pioneer Adolphe Quetelet is sometimes called the "father of the average man." His nineteenth-century work *A Treatise on Man and the Development of His Faculties* outlined theories regarding distributions of human traits. Whereas others before him had applied the normal distribution to describe measurement errors, Quetelet asserted that human traits, both physical and intellectual, were normally distributed around some central value. In his later work, the "average man" was sometimes presented as an ideal human, a concept that mathematicians, such as Antoine Cournot, disputed. Nonetheless, notions of a central value representing a typical case within a set of observations became very influential in research and statistical data analysis. There are many ways to think about center or typical values.

One of the most common measures is the arithmetic mean, often simply known as "average," which some trace back to Pythagorean writings on properties of music. Other types of averages include harmonic, geometric, trimmed, weighted, and moving or rolling means. Measures of center besides the mean include median and mode. Statistician Frank Zizek stated in his 1913 book *Statistical Averages*:

> An average may be computed for its own sake, merely to obtain a comprehensive characteristic expression for a series of divergent values, but it is often found as a means to another end, mainly for purposes of comparison. . . .

Students in twenty-first-century classrooms may use measures of center in the primary grades, focusing on mode and median, while mean is introduced in the middle grades. Expected value, which is the long-term average for a random variable or process, is a probability concept most commonly addressed in high school or college.

Mode

In the nineteenth century, psychologist and physicist Gustav Fechner studied the nonlinear relationships between subjective psychological sensations and the actual physical intensity of different stimuli, a field now known as "psychophysics." Use of the mode as a measure of center occurs in Fechner's work, which appears to be the first mention in print. He defined it as the value "around which the items . . . collect most densely, so that equal intervals contain more items the nearer the intervals lie to this value." Later in the same century, Karl Pearson would use a probabilistic and graphical approach to the definition, stating that the mode was the "abscissa corresponding to the ordinate of maximum frequency." Consistent with the probabilistic approaches used by both Fechner and Pearson, mode has come to be defined as the most frequently occurring value in a probability distribution or set of data. It is the only measure of center that is appropriate for both categorical and numerical variables because it does not require the data to be ordered in any way.

Median

Fechner's work also contained reference to medians, which he called the "middlemost ordinate" or "centralwerth" of an ordered series of values or data points. Some credit Carl Friedrich Gauss for "inventing" the median earlier as part of his work on the normal distribution. The name "median" is attributed to Francis Galton in the late nineteenth century. Inspired by Quetelet, Galton researched ways to measure and express center and variation in data, both numerically and graphically. He devised the "ogive graph," named after a curve common in architecture and ballistics, which graphed data versus ranks. His method of "statistics by intercomparison" used quantiles and percentiles, including the median, to consider deviations. Galton's median represented a typical value, which he termed "mediocrity," often assigning it a standardized value of zero as a point of reference for comparisons. Subsequently, many nonparametric (also called "distribution-free") statistical methods based on medians were developed by mathematicians and statisticians. Some of these procedures are named for them, including Henry Mann, William Kruskal, W. Allen Wallis, Donald Whitney, and Frank Wilcoxon.

Mean

Though the exact age of either mode or median is unknown, available evidence suggests that the mean may be older. In the Pythagorean treatise, *On Music*, from the school named for Pythagoras of Samos, there is some discussion of finding the middle value of two data points, such that the value exceeds the lower value by the same amount that the upper value exceeds the middle. While this basic description could be either the mean or the median for the case of two points, some historians consider this to be evidence of Pythagorean use of the mean. Statistician and historian Robin Plackett examined evidence from Babylonia, Egypt, and Greece and concluded that, while the mean may have been used in selected cases, it did not appear to be standard practice among astronomers and others who were typically collecting data. He credits sixteenth-century astronomer Tycho Brahe with introducing the mean into scientific methods of the times.

Mathematician Thomas Simpson showed in the eighteenth century that an average was a better measure than a single observation in a very limited set of cases and astronomers often used probability and means to quantify errors of deviations in observations. Other mathematicians, such as Joseph Lagrange, Abraham de Moivre, Pierre-Simon Laplace, and Carl Friedrich Gauss, contributed to mathematical developments that addressed the mean of a probability distribution or data set in the eighteenth and nineteenth centuries, while Quetelet, Galton, and others sought novel applications of measures of center. Statisticians in the twentieth century continued work on means, including George Box and Gwilym Jenkins, whose research about moving averages is the basis for many time-series forecasting models, and new research is ongoing into the twenty-first century. The mean has many mathematical properties that make it more desirable for widespread use than the median, such as connections to the least squares criterion and the method of moments. Many statistical techniques are concerned with estimating and comparing means.

Rules and generalizations have been devised and taught over the years regarding which measure of center is best to use for any given set of data, particularly with regard to choosing between the mean and median. Mathematically, the arithmetic mean minimizes the sum of squared distances of all points from the center, while the median minimizes the sum of absolute distances. For data with perfect symmetry, these are equivalent. Data with skew or outliers may yield very different outcomes. Mode is also less clear as a measure of center if there are no repeated values or if there are two or more values that occur most frequently. In the twenty-first century, educators like sociologist and statistician Paul von Hippel continue to investigate methods to teach concepts and relationships between mean, median, mode, and skew.

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See Also: Expected Values; Forecasting; Harmonics; Statistics Education.

Measuring Time

Category: Space, Time, and Distance.
Fields of Study: Measurement; Number and Operations; Representations.
Summary: A variety of mathematical calculations are used to define, measure, and apply measurements of time.

Time measurement (chronometry) serves two tasks: (1) indication of temporal instants, which are events occurring in time and (2) determination of temporal extensions (durations), which are amounts of time between events. Both types of tasks are essential for practical purposes, such as organization of life in civilized societies or intersubjective synchronization of various activities, as well as for the scientific study of nature and society. Time measurement relies upon the arithmetic model of time: events are mapped onto a numerical continuum so that if event A precedes event B, the relation $t_A < t_B$ holds between their time indices; such a mapping is called a "timescale." Given a timescale t, durations can be calculated as differences between time indices; and conversely, time indices can be defined by durations elapsed from a certain reference event (epoch).

Clocks and Timescales

Theoretical chronometry studies mathematical properties of timescales, while practical chronometry (called "horology") is concerned with devices realizing timescales, such as clocks. Any physical system, natural or artificial, producing a series of distinct and observable—thus countabl—events can serve as a clock. Periodic processes in our lifeworlds, such as the day/night cycle, the lunar cycle, or seasonal changes, provide natural bases for timekeeping and measurement. Counting recurrent observable events yields a measure of durations longer than the clock's basic period; measurement of shorter times than the clock's base period enforces a subdivision of the period into equal subunits—a refinement of the timescale. Therefore, time measurement spanning several orders of magnitude

A modern sundial. Predecessors of clocks also included hourglasses and burning candles.

requires alignment of timescales defined by different physical processes, periods of which are in constant arithmetic relations and thus define together a common timescale. For this purpose, timescales generated by artificial clocks are used.

Predecessors of clocks were devices used to indicate one standard time period, for example, outflow water clocks (clepsydrae) or sand glasses or burning candles. A clock in the proper sense of the word generates series of events at equal periods between them. In history, different physical principles have been used to ascertain isoperiodic clock action, including mechanical oscillations of a pendulum or a balance wheel, vibrations of quartz crystals or molecules in electromagnetic fields, and electromagnetic radiation emitted/absorbed by atoms, providing high-precision frequency standards. Base periods of clocks vary from a few seconds to fractions of a second in mechanical clocks, down to an accuracy 10^{-10} seconds per day for atomic clocks, as established by national standards agencies.

The Second

The second (*s*) is the fundamental unit of temporal duration. Originally, the second was defined as a constant fraction (1/86,400) of the solar day. Hence its name: 1 day is 24 hours, 1 hour is 60 minutes (*pars minuta prima*, which means "first minor part"), 1 minute is 60 seconds (*pars minuta secunda*, which means "secondary minor part"). With increasing precision of time measurement, fluctuations of the Earth's rotation period had become evident: thus the second was redefined in the 1950s as a constant fraction (1/31,556,926) of the period of the Earth's orbital motion around the sun (ephemeris time). In 1967, a new definition of the second was adopted; one second is defined as a constant multiple (9,192,631,770) of the period of electromagnetic radiation emitted by cesium atoms in transition between two defined energetic states under precisely specified conditions. Thereby astronomic definitions were abandoned in favor of standards derived from the inner structure of matter, which is considered constant throughout the Universe.

Time Measurement Technologies

Advanced time measurement and synchronization technologies allow people to define a unique timescale to be used all over the world. Historically, the first universal timescale was Greenwich Mean Time (GMT), based on telescopic observations at the Royal Observatory in Greenwich, England, which was later replaced by the international Universal Time (UT). At present, the most precise basis for timekeeping is International Atomic Time (TAI), based on a worldwide network of atomic clocks. Coordinated Universal Time (UTC) is the basis for international timekeeping. UTC differs from TAI by an integer number of seconds to approximate UT. Irregularities in Earth's rotation are compensated by adding or subtracting a leap second to/from the UT-TAI offset, if necessary; the corrections take place on June 30 or December 31. In this way, uniformity of UTC with respect to the atomic time unit definition is maintained, and continuity with the astronomical timescale is preserved.

Time Calculations

Irreversible natural processes, laws of which are well known, can be used to calculate time extensions, particularly those escaping direct observation and measurement. Of special importance are estimations of geological or archaeological age based on radioactive decay of certain elements or particular isotopes of otherwise stable elements. Estimates of time extensions in astrophysics are, to a large extent, theory based (for example, dependent on stars' evolution models). This dependency applies a fortiori to time magnitudes discussed in cosmology. Any time measurement implies observational (or at least conceptual) separation between the measured process and the reference process, defining the timescale. If the universe as a whole is considered, such separation is no more possible, so that the notion of the universal "cosmic" time meets logical difficulties.

Finally, there is no direct evidence that timescales defined by different classes of physical processes (inertial motions, light radiation, or radioactive decay) are really equivalent: the "unity of time" in physics is a convenient hypothesis, not an empirically secured fact. Since precise time measurements are available only for a short historical period—negligibly short relative to cosmological orders of magnitude—the alignment of radiation-based and motion-based timescales is merely temporally "local." Some cosmologists suggested that different classes of physical phenomena may define different timescales between which a nonlinear relation may hold. Consequently, two or more different time measures might be needed for adequate description of cosmic processes on a large scale.

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See Also: Calendars; Clocks; Relativity.

Measuring Tools

Category: History and Development of Curricular Concepts.

Fields of Study: Communication; Connections; Measurement.

Summary: Mathematicians have developed a number of tools to make accurate measurements.

Body parts, including the thumb, hand, and foot, have long been used to measure distance. Some of the oldest known mathematical measuring tools were notched bones, such as the Lochango and Ishango bones, which may have been designed for use in counting or multiplication. However, many concepts and objects in astronomy, navigation, surveying, optics, medicine, and other fields cannot be directly or accurately measured with body parts or tools like marked bones. Indirect measurements require advances in engineering and instrumentation, sometimes using sophisticated mathematical transformations.

Scientists, mathematicians, and inventors have created many ingenious tools to accurately quantify concepts such as distance, area, temperature, mass, and time. Measuring devices are used to collect data, create mathematical models, verify mathematical relationships, and make predictions. They also have widespread applications in everyday life, including household thermometers, rulers, and watches. Teachers bring measuring devices into the classroom in order to help their students learn mathematics. Some, such as rulers and compasses, form the basis for an object of mathematical study. Others, such as yardsticks, are used to discover or verify relationships. In the twenty-first century, measuring devices continue to be refined and improved for greater precision and accuracy, as well as to develop theories and to solve new problems.

Direct Comparison Tools

In many cases, it is possible to physically measure an object or event directly by making comparisons. For example, rulers and tape measures directly compare lengths of objects to standard units of length. Protractors directly measure angles, balance scales are used for weights, and measuring cups and graduated cylinders and pipettes are used for volumes. Hourglasses and water clocks compare known units of time, measured out by the device, to the time people try to measure. Many such measuring tools that use direct comparisons of units were developed relatively early in the history of humanity, with different versions built by many different cultures.

Indirect Measurement

Other measurements are indirect. While people can directly experience temperature and pressure and are sensitive to relatively small variations in them, the physical properties and the measurements of temperature and pressure are less directly observable and comparable than length or weight. Because of this fact, units of temperature and pressure, as well as tools for measuring them, were developed several thousand years later than units of length and weight.

Planimeters are tools for measuring area and provide an interesting example of relatively sophisticated use of mathematics in measurement. They use a mechanical arm that traces the perimeter of an object, while its other end moves along a straight axis. The principle of the device, designed through calculus, is that the distance the end of the arm traces on the axis is proportional to the area of the object. Units of area were used in ancient times, but area was always separated into rectangles or right triangles for direct comparison with units. Planimeters do not depend on such direct comparison.

Calculating Measures

While there are tools for direct comparison of lengths of certain magnitudes, it is impossible to use these tools for very large lengths, such as the distance to the moon, or very small lengths, such as wavelengths of different colors. For these cases, various computational tools are more appropriate. For example, large distances can be measured by sending radio, light, or other wave pulses to distant objects and measuring the return time. The distance is equal to the product of velocity and time. An interferometer is a tool for observing changes in wave frequencies when there is wave interference. Known frequencies can be used to compute wavelength, inversely proportional to them.

Antiquated Measuring Tools

Some measuring inventions are no longer in use because of advances in mathematics and technology that also leads to changes in educational emphases. For instance, the astrolabe is an ancient measuring device that was once very popular. In the tenth century, Abd al-Rahmân al-Sufî detailed the flexibility of the astrolabe with reportedly 1000 applications. In the twenty-first century, it has mostly been relegated to collections and astronomy history and education. The sextant, which replaced the astrolabe for navigation in the eighteenth century, has mostly been replaced by global positioning system (GPS) devices.

Measuring in the Classroom

In the twenty-first century, students in mathematics classrooms use a variety of tools and systems of measurements. Ruler and compass constructions were a focus of ancient investigations and students in mathematics classroom continue to explore them using physical instruments and dynamic software programs. In the late eighteenth century, Dr. Buxton obtained a patent for printed graph paper. In the early nineteenth century, mathematicians such as E. H. Moore advocated that graph paper be used to help students in algebra and it took on an increasingly important role in schools. Cartographers were using protractors to measure angles in the late sixteenth century. Mathematician Alexis Clairaut described protractors in his 1741 book *Elements de géometrie*, and protractors appeared in some geometry and trigonometry textbooks in the nineteenth century. However, they were not common in mathematics classrooms in the United States until the early twentieth century. Representations and measurements of geometric solids have been the focus in mathematics since antiquity. Teachers and mathematics departments in the nineteenth and twentieth century showcased models made of a variety of different materials, including wood and string. These physical models became rarer because of the software that can perform measurement calculations and present interactive three-dimensional models. However, young children continue to fill plastic geometry shapes with water or sand to measure volume.

Measurement systems are also explored in mathematics classrooms. Those that have high-enough accuracy and precision for the given purpose are called "valid." Precision and accuracy are established using statistical calculations such as mean and standard deviation and statistical laws such as the central limit theorem. Accuracy and precision are expressed using significant figures of numbers, with the error margin being half of the last significant place value. For example, the weight of 3.0×10^4 g means the last significant place value is thousands and the error margin is $1000\text{ g} \div 2 = 500\text{ g}$. On the other hand, 3.00×10^4g means the last significant place value is hundreds and the error margin is $100\text{ g} \div 2 = 50\text{ g}$, which is more precise.

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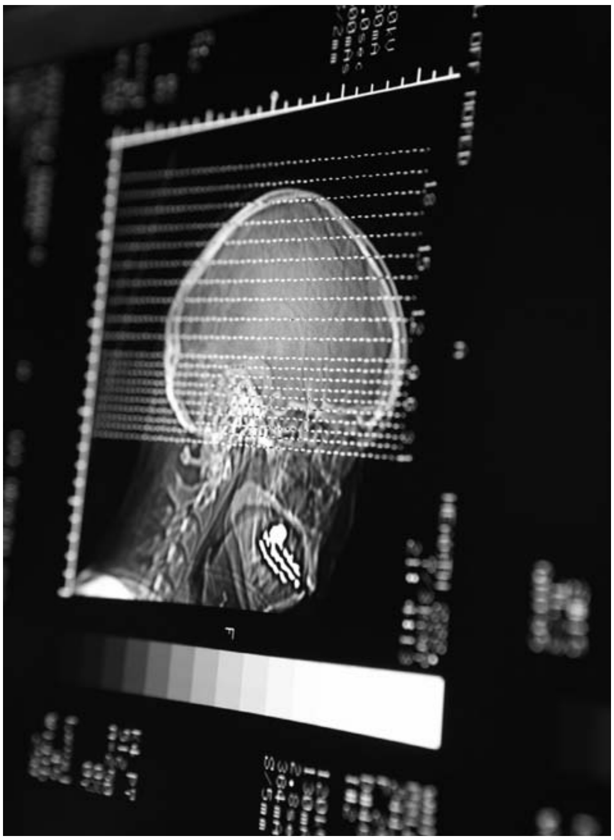
See Also: Educational Manipulatives; Measurement, Systems of; Measures of Center; Ruler and Compass Constructions; Units of Area; Units of Length; Units of Volume.

Medical Imaging

Category: Medicine and Health.
Fields of Study: Algebra; Geometry; Representations.
Summary: Mathematical models interpret measurements, and algorithms construct images used in the health industry.

Until the late nineteenth century, the structures of the human body were represented only by illustrations found in medical books. However, in 1885, Wilhelm Conrad Röntgen introduced the humanity into a new path in the world of images: the access to visual information from inside the human body. He used X-rays, which pass through objects with different densities producing images on photographic plates. Since the insertion of radiographic diagnosis, new technologies have brought great progress for medical diagnosis, such as ultrasound, computed tomography, and magnetic resonance imaging (MRI). Furthermore, medical images are currently used for navigation systems that guide surgeons during surgical interventions or aid in surgical planning, for example, in minimally invasive operations. Mathematical models interpret measurements and algorithms reconstruct images. Signal processing and noise analyses, as well as geometric, statistical, and algebraic techniques, are fundamental in this area. Mathematicians have also been the subjects of medical imaging studies. For example, one study found that mathematicians had an increased gray matter density in the cortical regions.



A study found that mathematicians had an increased gray matter density in the cortical regions.

X-Rays

When a physician performs a radiograph on an arm, the image is obtained in different shades of gray, aiding the identification of different anatomical structures. This identification is possible only because the arm is a structure formed by tissues of different densities, such as muscles, bones, and cartilage. The possibility of differentiation of these tissues occurs because of attenuations caused by a partial or total absorption of the rays before the formation of the image. Since X-rays are a type of ionizing radiation, they can cause damage to the human body, such as cancer, if used in excess. Modern equipment has been developed to minimize this risk. On the other hand, it is necessary to manipulate parameters that affect image quality and at the same time to control the amount and the dose distribution of this material on the patient.

Other Medical Imaging Devices

While X-rays detail the morphology of bone structures, bone densitometry provides the mineral content of the bone. This technique is used to control and to prevent osteoporotic fractures. With the advent of computerized tomography and magnetic resonance imaging, the human body is being studied in a segmented way. These advanced imaging techniques are especially useful in the study of central nervous system disturbances. Ultrasound is a diagnostic tool that, like magnetic resonance, does not use ionizing radiation. It is used to investigate soft tissues and is based on reflection of high-frequency sound waves to form two- and three-dimensional images, for example, in monitoring fetal development. Some diagnostic imaging techniques require the use of tracer substances. Scintigraphy, for instance, is a technique used for the evaluation of the cardiovascular system. This procedure uses the injection of radioactive substances to provide a two-dimensional image through the use of radioisotopes.

Resolution and Software

The spatial resolution of a digital image refers to the amount of points per unit of measure that allows the perception of details of a structure. Each point or constituent element of the array image is called a "pixel" (an abbreviation of "picture element"). The pixel is the smallest unit that can conduct operations. Colored or gray levels inform the size and location of the structure analyzed. Image processing is used to reduce interference and to increase the contrast to aid the analysis of the structures. It is possible to use mathematical techniques to manipulate the pattern of gray pixels. The interaction with neighboring pixels highlights structures of interest.

The mathematical definition of the images provides important clinical information, such as the size of lesions or fetal structure length, as well as morphology of structures, gland volume, blood supply area, and the monitoring of prostheses. Without these appropriate tools to analyze medical data, the images could be devoid of concrete meaning and require the use of complex computing resources to process the data. To achieve a medical image in real time, complex mathematical algorithms are needed. Diverse software has been appearing to meet the growing demands in the medical field, as well as needs concerning the storage and handling of patient data. Innovations continue to meet the growing challenges in this dynamic field.

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See Also: Diagnostic Testing; Digital Images; EEG/EKG; Ultrasound.

Medical Simulations

Category: Medicine and Health.
Fields of Study: Algebra; Geometry; Representations.
Summary: Virtual simulations of medical procedures are used in medical training and are possible because of advanced numerical simulation techniques and software.

Initially, virtual simulations were used only by the aviation and military industries. In the first decade of the twenty-first century, they have become important tools for teaching and research in almost all fields of medicine. It is now possible to model a physical system and to express it in the language of mathematics, enabling realistic simulations of several clinical and surgical procedures as well as the testing of medical implants. Thanks to the advances of computer science, many simulations once deemed impossible have become routine. This progress is because of the continued advance of numerical simulation techniques and software packages that allow the creation of numerical models with sufficient detail and complexity. During the twenty-first century, the use of computerized simulators is expected to develop considerably and to spread quickly into the very important domain of medical schools throughout the world. A computer simulation is nothing more than a computer program that runs a mathematical model of a physical situation. To do this, first a geometrical model is created and then a mathematic algorithm describes the behavior of the model under influences of external agents. A simulation is effective only if the physical situation is accurately modeled, providing a convincing user experience.

Benefits

Virtual medical simulation is an important tool for medical training in cases of high-risk, unusual, or difficult surgical procedures and for predicting the interaction of medical devices (such as implants or prostheses) with biological tissues. The main advantage to using these simulations is to provide a safe environment for both patients and students during training in risky procedures, as well as the opportunity to repeat several medical performances with lower costs. Furthermore, the number of animals used in medical experimentation can be reduced through the use of virtual simulations.

Applications

Medical simulators based on the finite element method are used in almost all fields of medicine. The creation of an accurate mathematical model of a given anatomical structure includes a three-dimensional reconstruction from medical images, a description of the material properties of the biological tissues that form this structure, and a description of the limits and interfaces between the adjacent structures, besides the external loading that actuates in the physical system.

The finite element method is a numerical procedure that reduces an anatomical structure, such as a kidney, to a mesh of nodes. During a simulation, a set of discretized partial differential equations defines the movements of the nodal points as a result of external force, for example, because of the contact of a medical instrument. Therefore, the deformation of this structure is a function of the acting forces applied at discrete points of the mesh as well as the elastic properties and geometry of the structure. Many arithmetic operations that require fast computer processing are necessary to find the solution for the system of equations to provide the detailed behavior of structures under particular conditions. Furthermore, the modeling process requires an interdisciplinary team of people from a wide range of disciplines, including computer science, electronics, mechanical engineering, clinical specialties, medical training, mathematics, and physics.

For example, there is a standard surgical procedure for the treatment of chronic sinusitis, an inflammation of the airspaces within facial bones. A robotic arm can be used to hold and to guide the endoscope. This method can help the surgeon in the procedure and decrease the time to perform the surgery. A proper mathematical modeling of the inner nose structures followed by a realistic simulation of this surgical procedure can predict the risk of using the robotic arm in this surgery. It can be used to define the range of movement and forces used by the robotic arm close to the vital structures, such as the optic nerve, the carotid arteries, and the brain. Moreover, it can be a virtual environment for surgical training ensuring a safe robotic endoscopic guidance for the patient.

Accuracy and Validation

Without an accurate model, it is not possible to obtain an accurate simulation. Therefore, a very important aspect of a medical simulation is validation to be sure that the model is correct and that the simulation corresponds to the reality. To validate a medical model and the respective simulation, some experiments are performed and the results of these experiments are compared with the results of the simulations. A difference in this comparison can indicate that the numerical code is not accurate enough or that the theoretical predictions do not agree with the experiments, which means that the mathematical model is not satisfactory.

Significant research has been conducted to model the deformation behavior of biological tissues. Accurate simulation, in the sense that one can confidently control the numerical error compared to real subjects, is very difficult to obtain because of the difficulties in building mathematical models of real biological tissues. The development of appropriate mathematical models is dependent on the knowledge of the tissues' elastic properties. In some cases, because of the limitations of measurement technology, some models have not been rigorously validated.

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See Also: Medical Imaging; Surgery; Transplantation.

Microwave Ovens

Category: Architecture and Engineering.
Fields of Study: Algebra; Geometry; Measurement.
Summary: An accidental discovery led to the use of microwave ovens for cooking, a process that continues to be studied.

In 1873, James Clerk Maxwell, using only mathematical considerations, formulated the electromagnetic theory. Maxwell's equations are fundamental to physics and engineering and describe light as a form of electric and magnetic energy. Fifteen years later, experiments carried out by Heinrich Hertz validated Maxwell's theory of electromagnetic waves. This development is a good example of mathematics as a creative medium for the development of science and technology. One of the technological products of Maxwell's theory can be found in most homes in developed countries. Domestic microwave ovens have become increasingly popular since the 1960s, as the device offers a quick method for heating food compared to conventional heating methods. The discovery of electromagnetic waves by Maxwell shows how pure abstract mathematics can generate new technologies. Applied mathematicians also learn new mathematics from problems motivated by this type of application.

Electromagnetic Waves

Electromagnetic waves are a form of radiation represented by their frequency and wavelength. Frequency is the number of cycles that occur in a second and is measured in Hertz (Hz). Wavelength is the measure of the distance over which the wave's shape repeats (λ). The electromagnetic spectrum consists of all possible frequencies and wavelengths of electromagnetic radiation, for example, radio waves, microwaves, infrared, visible, ultraviolet, X-rays, and gamma rays. Microwaves are electromagnetic waves with high frequencies (between 300 MHz and 300 GHz and short wavelengths (from as long as one meter to as short as one millimeter). Besides microwave ovens, practical applications of microwave technology can be found in cellular telephones, radar, satellites, and medical systems.

Discovery

The discovery that microwaves could be used for heating food is one of the accidental cases in the history of science. It occurred in 1945 when Percy Spencer, an American self-taught engineer, was working with microwaves in a radar system and a peanut chocolate bar that was in his pocket started to melt. In the same year, after some experiments with popcorn and eggs, Spencer created the microwave oven. It consisted of a metal box with a high-density electromagnetic field to heat food quickly and efficiently. Twenty years later, microwave ovens were adapted for domestic use as the typical consumer microwave ovens that are known today.

Microwave technology can also be used for melting metal, saving energy and reducing cycle time.

How it Works

The physical and operating principles of microwave ovens are quite simple. Most foods are composed of polarized molecules that are bound together in different ways. When microwave radiation is exposed to food, the molecules within the food are forced to align themselves with a rapidly changing alternating electrical field. Charged molecules oscillate and gain thermal energy via friction. Therefore, microwave radiation can heat food when the radiation is absorbed. This process is dependent on the time of radiation exposition, type of food, and the way the radiation is distributed (scattered, reflected, or transmitted).

In the early twenty-first century, mathematicians are working in universities and industries where interesting problems can be solved using a mathematical approach. Industrial mathematicians at the University of Bath have been working on the microwave cooking process.

A problem with this process is that it can result in localized points inside a food where the radiant electromagnetic field is relatively weak—the temperature in this point may be lower—and the food will be poorly cooked. Theoretically, it is possible using a combination of both analytical and numerical calculation to create a three-dimensional field simulation of this process.

Through a mathematical simulation, an averaged electromagnetic field can be calculated, and it will be possible to determine how it penetrated a moist foodstuff. This example from applied mathematics shows us how mathematics can be used to help us create and enjoy the benefits of technology.

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See Also: Cooking; Mathematics, Applied; Radiation.

Middle Ages

Category: Government, Politics, and History.
Fields of Study: Algebra; Geometry; Measurement.
Summary: Medieval mathematics developments included Scholasticism and the emergence of secular universities.

The European Middle Ages, or the "medieval period," lasted from the fall of Rome to the Renaissance and was identified by Renaissance thinkers as separating their own period from that of classical civilization. The Middle Ages were construed as a time of backwardness, but in fact progressed in spite of economic, medical, and political difficulties. Mathematicians made original contributions to such areas as algebra and astronomy and commentaries on historic texts preserved Greek works. Mathematics historians have studied Arabic, Persian, Turkish, Indian, Islamic, and European contributions during the Middle Ages. For example, Adolf Yushkevich wrote a seminal work on the history of mathematics in the Middle Ages. He highlighted similar features of medieval mathematics based on the cultures in Europe and Asia and, along with Boris Rozenfeld, studied Arabic contributions.

Early Middle Ages

The transfer of western Europe from the Roman Empire to the Goths occurred gradually through the fourth and fifth centuries, partly by conquest and partly by migration and assimilation. The old travel and trade network decayed and scholarship retreated mostly into monasteries. The philosopher Boethius straddled the Roman and Goth eras. He valued mathematics highly, endeavoring to translate several important mathematical works from Greek to Latin and dividing the seven liberal arts into two tiers: a lower tier, the trivium—containing logic, grammar and rhetoric—and an upper tier, the quadrivium—containing the four mathematical arts of arithmetic, geometry, astronomy, and music theory. Boethius is remembered primarily for his work *Consolation of Philosophy,* written while he was imprisoned before execution. Christianity became a primary supporter of higher learning, music, and art in Europe, and also a strong participant in government owing to the high levels of literacy among Church officials. Monasticism also gained momentum during the early middle ages, inspired by the isolated communities in Syria and Egypt. Owing to the importance of study in religious life, many monasteries functioned also as schools and libraries.

Carolingian Renascence

Around the ninth century, Charlemagne and his successor, Louis the Pious, enacted various reforms to effect uniform standards in a renascence of the Roman Empire. Charlemagne had schools created to restore education across Europe, reunifying the dialectized Latin and creating a script for it, the Carolingian minuscule. The standard curriculum saw Boethius's trivium and quadrivium become the foundations for the bachelor and master of arts degrees. A standard currency facilitated reformation of the economy and long-distance trade and taxation. The Roman influence is evident in monumental architecture, which incorporates elements from classical styles in clear, relatively simple arrangements. Circles, squares, cubes, and cones feature prominently, as does symmetry. Carolingian architecture and painting became the basis for the more ornate Romanesque style and, ultimately, the Gothic.

Byzantium and the East

The Greek-speaking part of the Roman Empire, also called Byzantium, survived the Latin half's decline. In the sixth century, Byzantium extended around the eastern Mediterranean from Egypt to Greece, expanded across all of north Africa, and even took Carthage and Italy from the Goths. Then, severely weakened by epidemics thought to be the Black Death, the Byzantine Empire shrank to what is now Turkey and Greece, plus Carthage and some parts of Italy. Even after this decline, Byzantine culture stood as the standard for both western Europe and the Near East. Owing to increasing influence from Christianity, art and monumental architecture tended to manifest in churches (such as Hagia Sophia), and philosophy intertwined with Christianity on many topics, including ethics, existence, governance, and death.

Hellenistic knowledge percolated gradually eastward from Byzantium, first in translation into Syriac and then into Arabic, which fueled a philosophical community in Damascus. By the seventh century, Neoplatonism, which had been Christianized in late antiquity, had been accommodated into the Islamic framework. This set the backdrop against which Aristotelianism, and all of its disagreements with Platonism, had to be accommodated next.

In the eighth century, Baghdad became the cultural focus of the East. The scholarly community there attracted scholars of diverse races and religions. The Islamic Golden Age continued into the eleventh century, with many advances of significance to western Europe, including those by al-Khwarizmi in algebra, by Ibn al-Haytham (Alhazen) in optics and scientific method, by al-Battani (Albategnius) in astronomy, by Jabir ibn Hayyan (Geber) in alchemy, and by Ibn Sina (Avicenna) in medicine. A rich tradition of poetry and calligraphy also emerged.

Al-Andalus

In the eighth century, the Moors of north Africa took most of the Iberian Peninsula that ultimately became the Umayyad caliphate based at Córdoba after the Abbasids came to power in Baghdad. While the Abbasid caliphate suffered from political fragmentation, the Umayyad territories in the Iberian Peninsula thrived.

Astronomy and botany were especially active in al-Andalus, both for intellectual interest and for applications in timekeeping, astrology, and medicine. While the societal framework was predominantly Islamic, numerous Jews and Christians participated in high culture during extended periods of cosmopolitanism. Al Zarqali (Arzarchel) discovered the ellipticity of planetary orbits in the eleventh century, and ibn Baija (Avempace) deduced that the Milky Way was not a continuous cloud but numerous stars. Studies of Aristotle by ibn Rushd (Averroës) shaped philosophy and religion for centuries later.

High Middle Ages

From the eleventh to the thirteenth centuries, western Europe was peaceful enough to entertain a high degree of cultural development. Windmill- and waterwheel-powered industries developed, economies flourished, and urban populations grew quickly, spreading into formerly Moorish Iberia, into southern Italy, and even into the Baltic and the Near East. The Arabic heritage was absorbed and then reacted against in a philosophical movement called "Scholasticism."

Scholasticism emerged from the works of Aristotle. They were translated from Arabic into Latin and provided a basis for a worldview based on empiricism and logic. Although the philosophy was secular, it was pursued largely for its power to support Christian doctrine. The Arabic writers had already weighed Platonist versus Aristotelian views and largely harmonized the philosophy with religious givens. Much of the result was hence incompatible with new movements in Christianity, and the Scholastics sought to rebuild it by returning to the original sources. The scientific content was developed notably by Robert Grosseteste and Roger Bacon in England and Albertus Magnus, Thomas Aquinas, and Duns Scotus in France. These five also ranked highly in the Church, illustrating the continuing need that religion had for higher education and the support for intellectuals that the Church provided.

Early in the Middle Ages, higher learning had been concentrated in monasteries and Church schools. With the new secular engagement, universities appeared, beginning with Bologna in 1088, then Paris in 1150, then Oxford in 1167, then others. Learning emerged from the monasteries into urban surroundings and engaged more with secular needs, such as commerce and industry. Gothic architecture replaced the hefty, solid Romanesque, with height and lightness built from thin stone ribs reaching up and out to become the ribs of vaulted ceilings. Acute arches and vaults replaced the Romanesque semicircle, and walls gave way to large glass windows. Gothic designs manifest Euclidean geometry problems, including constructing regular polygons, dividing arbitrary angles into equal parts, dividing lines into equal parts, fitting circles through points, tangent to lines or tangent to other circles.

In the fourteenth century, frequent plagues and crop failures decimated the population, undermining social structure, industry, and economies. From the turmoil sprang new outlooks on all fronts. Among the more famous literary achievements, Dante wrote his *Commedia* and other tracts (including some scientific ones), Chaucer wrote his *Canterbury Tales,* Bocaccio wrote the *Decameron.* Such fresh thoughts ultimately gave rise to the Renaissance in fifteenth-century Italy.

A number of European mathematicians were important in helping to introduce eastern mathematics into Europe. Many Greek works were unknown in Europe and were found only in Arabic. Adelard de Bada translated the Arabic texts of Arabic and Greek mathematicians into Latin. Leonardo Pisano Fibonacci was educated in north Africa and traveled extensively. In Pisa he introduced the Hindu–Arabic place-valued decimal system and the use of Arabic numerals into Europe, while also making fundamental contributions of his own.

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See Also: Arabic/Islamic Mathematics; Greek Mathematics; Renaissance; Roman Mathematics.

Military Draft

Category: Government, Politics, and History.
Fields of Study: Data Analysis and Probability; Number and Operations; Problem Solving.
Summary: Military drafts must make use of probabilities to ensure the draft is equitable.

The U.S. military is made up of volunteers. However, if more people are needed than the number of people who volunteer, there needs to be a method for procuring enlistment. The method used is called a “military draft.” It is the law that all male citizens ages 18–25 are to register with the Selective Service. If a need arises, the U.S. Congress would have to pass legislation instituting a draft. The U.S. president would have to sign the bill into law.

When a draft occurs, there is a lottery of the registered men that is intended to be fair. Each registered man of the same age should be as likely as every other registered man to be selected. Once selections are made, some men are excused if they are not fit to serve. A military draft has not been used since 1973.

The Current Lottery

The current lottery method that would be employed if there were to be a draft is to place a capsule with dates for every possible day of the year (month and day) into a barrel. For example, December 1, January 27, and March 13 would be three such capsules. A second barrel will contain the numbers 1 through 365. These barrels are well mixed. In fact, one way to mix the barrels is to not place the capsules into the barrels in order. Rather,

Future soldiers being sworn into the army. The U.S. military is made up of volunteers; however, if more people are needed, the U.S. government would have to pass legislation instituting a military draft.

the capsules are placed into the barrels in a random manner. One capsule is drawn from each barrel, one at a time, and paired. For example, if November 4 is drawn from one barrel and 78 is drawn from the other barrel, then November 4 and 78 are paired. This continues until all 365 days have a number. The number becomes the day's rank. This process forms 365 ranked groups. Each group consists of those registered men whose birthday is the corresponding date pulled from the barrel and who will turn 20 in the current year.

For example, assume that each date is paired with the following number:

November 4 paired with 78,
December 28 paired with 1, and
January 12 paired with 25.

Then all men who turn 20 in the current year of the draft and have a birthday on November 4 will be the 78th group to be called to serve. Before they are drafted, groups 1–77 would be exhausted of possibilities (that is, all fit to serve in the previous 77 groups would be called to serve first). All registered men who turn 20 in the current year and have a birthday on December 28 are in the first group. All men who turn 20 in the current year and have a birthday on January 12 are in the 25th group. Again, these groups are made up of men who will turn 20 in the year of the draft. Once all 365 groups are used, then the rankings are followed again, calling all men turning 21, then 22, 23, 24, 25, 18, and 19.

What it Means to be Random

A selection process of this nature is random only if any person is as likely as any other person to be selected to serve. Thus, each of the 365 birthdays must be as likely as each other birthday to be ranked first. Each of the remaining birthdays must be as likely as any to be ranked second. A man's birthday should not allow one to predict the likelihood of his being drafted.

Vietnam Draft

The 1969 lottery drawing for the Vietnam War was demonstrated not to be random. A barrel with 366 plastic capsules was used, where each capsule had a birth date on it (month and day); one capsule was for those who were born on leap day. One at a time, the capsules were drawn by hand. The first to be drawn was ranked first. The second to be drawn ranked second. Thus, if September 21 was drawn first, then all men aged 18–26 with a birthday on September 21 would be the first group called to service.

The procedure that was followed to order the men with the shared birthday depended on each man's initials. A separate lottery was held in which the 26 letters of the alphabet were ranked. This followed the same process as the birthdays, in that 26 letters were placed in a barrel and one by one were drawn. Using the resulting ranking, each man within a shared birthday was ranked according to the permutation of the first letter of his last name, the first letter of his middle name, and the first letter of his first name. Overall, this should have been a fair method for selection, as it was based on randomized birthdays and letter permutations.

Why It Was Not Random

The above-mentioned method would be random if implemented properly. However, it turned out that men with birthdays later in the year (for example, December birthdays) were much more likely to be drafted than those with birthdays in the beginning of the year. What happened is quite simple. The capsules were placed in the barrel month-by-month beginning with January, and the barrel was not well mixed. The December capsules were on top and they had a higher probability of being pulled out first, resulting in lower draft numbers for those men.

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See Also: Lotteries; Probability; Randomness; Vietnam War.

Minorities

Category: Mathematics Culture and Identity.
Fields of Study: Communication; Connections.
Summary: Minorities are historically underrepresented in American mathematics and efforts have been made to rectify this.

Mathematics is a vital tool in modern life and mastery of mathematical subjects is a requirement to enter many professions, including medicine, engineering, and the sciences. For this reason, observed trends in mathematical achievement in school and representation in mathematics-oriented professions, both dominated by whites and Asians with other minorities lagging behind, give cause for concern. At the end of the twentieth century and into the twenty-first century, the media publicized information about the performance and underrepresentation of minorities in mathematics, many authors published works about minority individuals in mathematics, and mathematicians and mathematics educators designed and implemented successful educational initiatives and programs.

The United States is a racially and ethnically diverse country with a history of reporting extensive statistics about school and professional accomplishment by race and ethnicity. Few in the twenty-first century would argue that observed differences are because of inherited differences in ability; instead, several other explanations have been offered.

One is that minority students have fewer opportunities to master mathematics because they may be more likely to attend low-achieving schools, which may have more inexperienced and uncertified teachers and fewer teachers with graduate degrees. A second explanation is the lack of role models, since many mathematics faculty and prize winners are white or Asian, so students of color (or their teachers) may incorrectly believe that mathematics ability is somehow linked to race or ethnicity. In addition, students may not feel comfortable taking advanced mathematics classes in which they are the only person of color. A third factor is that some minority students report being actively discouraged from pursuing careers in mathematics and science. Racial and ethnic categories used for collecting data are not consistent across all organizations and some have changed over time, somewhat complicating comparisons. The terms "minority" and "person of color"

Eighteenth-Century Minority Mathematicians

In the eighteenth century, Benjamin Banneker created astronomical almanacs, solved mathematical puzzles, and wrote to Thomas Jefferson to plead against slavery. Other eighteenth-century individuals include ex-slave Thomas Fuller, who was known for his calculating abilities, and Muhammad ibn Muhammad al-Fullani al-Kishnawi, a mathematician, astronomer, astrologer, and mystic who constructed magic squares.

Mathematicians and historians have also written ethnomathematics works on African mathematics, Native American mathematics, and Incan and Mayan mathematics.

are themselves controversial; for instance in the United States, persons of Asian descent would qualify on both scores and yet are not usually classified as such. A better formulation in this case might be "members of ethnic groups with traditionally lower representation in mathematics," but the terms "minority" and "person of color" will be retained, since those terms are commonly used and understood.

Minority Mathematicians in History

In part because of research that suggested the importance of role models, the known benefits of humanizing mathematics, and a desire to provide counterexamples to noted racist comments, historians and mathematicians have detailed the lives and work of many outstanding mathematically talented minority individuals.

Minority mathematicians in the nineteenth and twentieth centuries faced many barriers, including restricted educational, employment, and publishing opportunities; derogatory comments and intimidation; and Jim Crow treatment that barred minorities from attending conferences. Despite these conditions, many minority mathematicians succeeded in making great contributions to the mathematics community. Elbert Cox was the first minority American to obtain a Ph.D. in mathematics. He attended a segregated primary school with what has been noted as inadequate educational resources. In high school he became a talented violinist, and he also enjoyed and excelled in mathematics and physics. He graduated from Indiana University with a degree in mathematics and his transcript listed "COLORED" across it. His 1925 Cornell University Ph.D. thesis was "Polynomial Solutions of Difference Equations." He was recognized as an outstanding teacher and effective master's thesis adviser during his career at Howard University, a historically black institution.

Other early minority Ph.D.s in mathematics include dozens of mathematicians whose contributions to mathematics and mathematics education have been broad and varied. One name that often appears on lists of prominent minority mathematicians is that of David Blackwell, a noted statistician and game theorist who earned his Ph.D. in 1941. He stated, "[Racial discrimination] never bothered me. I'll put it that way. It surely shaped my expectations from the very beginning. It never occurred to me to think about teaching in a major university since it wasn't in my horizon at all." Joaquin Diaz is noted as the first Hispanic to obtain his Ph.D. in mathematics from an American institution. His 1945 thesis at Brown University was titled "On a Class of Partial Differential Equations of Even Order." He worked at a number of different institutions, including as a research associate at the Institute for Fluid Dynamics and Applied Mathematics at the University of Maryland and as a professor at Rensselaer Polytechnic Institute.

Until the twenty-first century, it was thought that Evelyn Boyd Granville, who received her Ph.D. in 1949 from Yale University in functional analysis, and Marjorie Lee Browne, who received her Ph.D. in 1950 from the University of Michigan in topological and matrix groups, were the first minority women Ph.D.s in mathematics. They both remained active in the mathematical community. Earlier in the 1940s Martha Euphemia Lofton Haynes obtained her Ph.D. from

Catholic University of America by writing a thesis on the "Determination of Sets of Independent Conditions Characterizing Certain Special Cases of Symmetric Correspondences." While she had a very distinguished teaching career in the Washington, D.C. public school system, her divergence from the research community may explain why mathematicians were not aware that she was the first woman minority Ph.D. in mathematics. In addition, histories and statistics on minority mathematicians were not common until later in the twentieth century, so it is difficult to identify some of the early mathematicians.

In 1964, when Thomas Storer graduated from the University of Southern California with a thesis on "A Family of Generalized Difference Sets," he may have been the first Native American to obtain a Ph.D. in mathematics, although some historians refer to the possibility of an earlier Ph.D. in mathematics education. Storer's research was primarily in combinatorics, although he was also known for his teaching, advising of honors students, and as a leading authority on string tricks and figures. Another notable minority mathematician who obtained his Ph.D. before 1970 is Hispanic mathematician Richard Tapia, who graduated from the University of California, Los Angeles, in 1967. He has received many honors and awards and his research in computational mathematics and educational outreach programs are known nationwide. He explained:

> Some of my job duties include teaching mathematics and science to college students, writing books, doing research, and working with the community. When I made my career choice, I knew I wanted to reach out to underrepresented groups, especially Hispanics. I wanted to show minority students that if they really want to do something, they can. I believe I can improve minorities' participation in science and mathematics. However, in order to do this, I have to serve as a role model by first being an excellent scientist.

Recent Developments

Despite the climbing cumulative numbers of minority mathematicians and improving conditions and opportunities for minority students, during the latter part of the twentieth century authors noted that the traditional stereotypes of mathematicians conflicted with the cultural identities of minority groups. In 1997, mathematician Scott Williams created the Mathematicians of the African Diaspora Web site, "to suggest modern mathematicians and scientists as images of success to present to the African American community." The site grew to thousands of Web pages filled with history, statistics, articles, and reference lists. The Society for Advancement of Chicanos and Native Americans in Science and the Mathematical Association of America program on Strengthening Underrepresented Minority Mathematics Achievement also host biography Web pages. In addition, there are a number of published articles and books on minorities in mathematics.

Many researchers have conducted studies exploring factors relating to the continued underrepresentation of minorities in mathematics. For example, some researchers noted that differences in mathematics achievement may begin at the elementary school level. The Early Childhood Longitudinal Survey (ECLS), which followed a cohort of children from kindergarten in fall 1998 to grade 5 in spring 2004, found that in kindergarten there were already noticeable gaps in achievement by race and ethnicity. At the high school level, the National Assessment of Educational Progress reported that 12th graders in all racial and ethnic groups showed similar improvement in mathematics achievement scores from 1990 to 2000, but that minority groups still had lower achievement. Scores on the Scholastic Aptitude Test (SAT), a nationally administered exam often taken by college-bound students, over the period 1990–2008 show a similar pattern with most racial and ethnic groups showing improvement but Asian and white students consistently having the highest scores. Recently the numbers of African-American and Hispanic students taking Advanced Placement (AP) exams, specialized subject exams offered in some high schools and which may gain students college credit, has increased.

According to the National Center for Education Statistics, mathematics teaching staff tended to be primarily white in U.S. public schools. Data from the National Center for Education Statistics also gives credence to the argument that some of the achievement gap may be because of minority students being more likely to have been taught by teachers with inferior qualifications. In 2007–2008, 12% of high school mathematics teachers had neither a college major nor standard certification in mathematics, but in schools with at least 50% African-American enrollment this was true of 25% of people teaching mathematics. Schools with a majority of African-American students were also likely to have less experienced teachers.

The millennial mathematics major consists of diverse students pursuing diverse careers and yet there are concerns about the percentages of minorities, including Asians/Pacific Islanders, African Americans, Hispanics, and American Indians/Alaskan Natives. For instance, in the late twentieth century and early twenty-first century, the percentage of undergraduate degrees in mathematics and statistics awarded to such minorities was approximately 20%, which was below the percentages of the resident college population. Historically, in the United States, Asian and white students have comprised the bulk of enrollment in graduate programs in mathematics and have received a disproportionate share of advanced mathematics degrees.

Minorities are also underrepresented among scientists and engineers in the United States. For instance African Americans, Hispanics, and American Indians as a group constituted about 24% of the U.S. population in 1999 but only 7% of the science and engineering workforce, while Asians constituted about 4% of the population but 11% of the science and engineering workforce. Some evidence suggested that choice of career fields also differed by race. Salaries in science and engineering fields also differed by race.

Researchers continue to study factors related to the underrepresentation of minorities in mathematics. There have been many successful programs that increased the participation of minorities in mathematics, including the Meyerhoff Scholars Program, the Tensor-SUMMA Grants, and the Enhancing Diversity in Graduate Education Program. Organizations, and conferences, such as the National Association of Mathematicians, the Society for the Advancement of Chicanos and Native Americans in Science, the Conference for African American Researchers in the Mathematical Sciences, and the Mathematical Association of America through its Strengthening Underrepresented Minority Mathematics Achievement (SUMMA) program, have been dedicated to supporting and promoting minorities in the mathematical sciences.

The International Study Group on Ethnomathematics has focused on the cultural diversity in mathematics and its applications to mathematics education. The Benjamin Banneker Association has been dedicated to the mathematics education of minority children. These professional associations have sponsored mathematics talks, sessions, and awards, published newsletters, and provided opportunities for social interaction and support.

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See Also: African Mathematics; Castillo-Chávez, Carlos; Hunt, Fern; Incan and Mayan Mathematics; Jackson, Shirley Ann; Mathematics Literacy and Civil Rights; Native American Mathematics; Ross, Mary G.; Succeeding In Mathematics.

Missiles

Category: Government, Politics, and History.
Fields of Study: Algebra; Calculus.
Summary: Mathematicians have long worked on improving missile accuracy and performance.

Stone or arrow missiles have been used for thousands of years. Missiles with explosives can be traced back to China following the Song dynasty. Mathematical and technological advances have led to countless improvements in missile design, trajectory, range, and accuracy and have continually revolutionized warfare. Aristotle theorized on laws governing projectile motion, as did mathematicians like Leonhard Euler and Daniel Bernoulli, who derived or refined mathematical principles of projectile motion using geometry, calculus, and differential equations. In the nineteenth century, mathematicians Alfred Freenhill and Percy MacMahon worked on a missile trajectory model that related resistance to the cube of the velocity, suggested from experimental data. During World War I, mathematics took on an increasingly important role. John Littlewood created techniques to reduce the work required for accurate missile trajectory calculations, and Gilbert Bliss used the calculus of variations to account for variables like wind and the rotation of the Earth. During the 1950s, mathematician John von Neumann headed the committee that led to the development of U.S. intercontinental ballistic missiles. During the space age, mathematicians made a breadth of contributions, like Evelyn Boyd Granville, who worked on the development of missile fuses at the National Bureau of Standards.

Mathematician Peter Swerling, known for his theory of radar, also researched optimal estimation of satellite and missile orbits and trajectories. Missiles of the twenty-first century can be defined as weapons that follow a trajectory for the purpose of delivering explosive warheads to targets by means of lift and rocket propulsion. They may be launched from ground, submarines, and airplanes to nearly any target on the face of the Earth. Mathematicians working in government, industry, and academia continue to contribute to the development of all types of missiles and missile defense systems.

Trajectory and Guidance

The basic flight path of a missile is a parabolic arc. Sixteenth-century mathematician Niccolo Tartaglia described cannonball flight paths. Seventeenth-century mathematician Evangelista Torricelli published a geometric method for computing projectile range. Benjamin Robins, an eighteenth-century mathematician, invented the ballistic pendulum. His experiments, later expanded by Euler, demonstrated that air resistance could not be ignored in calculating trajectories. Scientist Heinrich Magnus showed that other forces could affect spinning spheres and cylinders; this effect is now known as the Magnus Effect. The importance of higher mathematics, like calculus, in computing trajectories contributed to the inclusion of these topics in many military school curricula in the nineteenth century.

In the early twenty-first century, mathematics continues to play a key role in missile accuracy.

Surplus Phoenix missiles like this one mounted on NASA's F-15B research aircraft can be used to gather hypersonic flight test data.

Most modern guidance systems use mathematical methods to determine the trajectory needed, such as angular coordinates between the missile and the target or the distance between the target and the missile. Sometimes computations are done ahead of time and the missile follows a predetermined path. Other times, the missile can make adjustments to the flight path in order to correct the trajectory as needed and may follow a path that is very different from the basic parabola. Some systems utilize astronomy—the accuracy of a missile is determined by examining the relationship of the missile to a fixed start position. Others employ altitude maps and compute the missile's distance from the ground to determine the path of the missile. These systems, however, are subject to error. Navigation systems that utilize a path calculated prior to launch may be influenced by instrument errors, while systems that utilize flight path data are more accurate but are subject to the effects of countermeasures such as radar decoys or infrared flares.

Advanced missiles are propelled by an internal combustion mechanism and guided by radiation, lasers, radio waves, or computers. Guidance often involves the use of mathematical techniques, like Kalman filtering, named after Rudolf Kalman, which allows a missile's course to be manipulated. Many of these latest-generation weapons come complete with cameras that record visual and spatial location information to aid human operators in their direction. Other missiles are guided by locations systems, such as INS, TERCOM, or GPS, which are programmed to recognize the weapon's global positioning at its origin and use it to calculate the distance, trajectory, and course to the target. These modern flight systems use positioning, targeting, and guidance data, along with thrust and aerodynamics, to maneuver missiles while they are in flight, even allowing them to seek and destroy moving targets.

Defensive Systems

With the development of more advanced missiles has come the need for more advanced defense systems. For example, satellites could measure the missile's trajectory and speed to determine a probable impact point and relay this information to an interceptor vehicle. The interceptor might initially utilize celestial guidance to track the incoming missile, and then use preset guidance to collide with the incoming missile. The U.S. Missile Defense Agency employs many engineers, scientists, and mathematicians to work collaboratively on defense solutions.

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See Also: Artillery; Satellites; Weightless Flight.

Molecular Structure

Category: Medicine and Health.
Fields of Study: Algebra; Geometry; Representations.
Summary: The geometry of molecules can be an important property, as in the shape of a protein molecule or the double-helix of DNA.

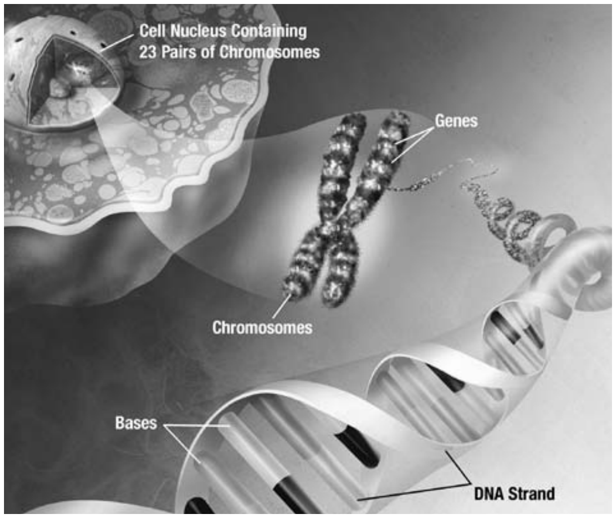
The physical structure of molecules is important in chemistry, biology, physics, and engineering. The precise structure can influence the chemical reactivity of a molecule as well as its response to other physical interactions, such as how it can absorb energy in the form of photons (light particles or X-ray particles). These interactions can have important implications in biology, medicine, health, and engineering. For instance, how proteins fold determines their function, and the shapes of certain protein molecules influence the existence of diseases. For example, shape is important in the normal function of the hemoglobin molecule, the molecule crucial for absorbing oxygen in red blood cells so that they can transport it throughout the body.

Hemoglobin consists of four protein subunits, associated with four heme subunits (ring-like structures containing an iron atom). As one oxygen molecule (O_2) binds to one of the heme units, the molecule distorts so as to allow another oxygen molecule to more readily bind in a cooperative way to another heme unit. This in turn distorts the molecule so that another O_2 finds it even more readily. Altogether, four O_2 molecules can ordinarily bind to one hemoglobin molecule. In sickle-cell anemia, two mutations in two of the four protein units distort the hemoglobin molecule so that the misshapen units form long chains. These in turn cause the red blood cell to become misshapen and lose its elasticity so that it can no longer readily move through small capillaries. Besides being painful, the misshapen red blood cells are destroyed by the spleen, resulting in anemia.

Another example of how the shape of a protein can cause disease is that of prions, which are misshapen proteins that enter (or "infect") cells and cause the cells' proteins to become misshapen. Prions are probably best known as the cause of bovine spongiform encephalopathy (commonly called "mad cow disease") in cattle. Finally, protein folding is also implicated in Alzheimer's disease. Thus, there is natural interest in understanding how these molecules fold. Knowing precisely how any particular protein folds in a particular chemical environment generally requires intensive mathematical computations that implement various equations from the area of physics known as "quantum mechanics." It is interesting that while supercomputers are usually used for this work, dozens of scientific articles have been written that instead relied on computations performed by harnessing millions of ordinary PCs, volunteered by millions of individuals—over 5 million CPUs as of September 2010.

DNA

Besides proteins, another important molecule studied extensively for its structure is DNA. While the double helix structure has been known for over 50 years, precisely how DNA is used in the cells of the body is still a source of research in the twenty-first century. In order to fit inside a cell nucleus, DNA must be very tightly coiled. How the appropriate sequence of DNA that a cell might need at a particular time can be rapidly located and then rapidly transcribed into messenger RNA for making a particular enzyme of interest is a complex process. Simply understanding how unknotting the knotted DNA takes place within the nucleus is nontrivial, and the mathematical discipline known as "topology" (and its subdiscipline, knot theory) has helped to elucidate how the cell handles the knotted DNA. One key equation to help understand the process of DNA supercoiling is $Lk = Tw + Wr$, where Lk is the linking number, Tw is the twist, and Wr is the writhe. This equation, attributed to G. Calugareanu, J. H. White, and F. B. Fuller, relates the linking number of the DNA (which essentially describes how the two backbones of the double-stranded DNA are linked) to the twist (the twisting of either backbone relative to the central axis of the DNA) and the writhe (which relates how the central axis of the DNA is oriented in three-dimensional space).



An illustration created for a National Institutes of Health study on DNA and Alzheimer's disease.

Other Structures

Besides proteins and DNA, molecular modeling is important in other areas. In the past, a scientist looking for a chemical that would have a certain effect in a certain situation, given a compound that reacts in a slightly different way in a slightly different situation, would likely have changed one part of the molecule and tested the new product; changed another feature and tested that product; and so on. Combinatorial chemistry is devoted to trying to automate the synthesis—and efficacy studies—of a huge number of different permutations of some basic chemical structure, somewhat in parallel. Interest in combinatorial chemistry is widespread among pharmaceutical companies.

Determining the molecular structure of molecules often relies on the general area of spectroscopy, which involves examining the spectrum that results when visible, ultraviolet, or infrared light or X-ray radiation, is applied to molecules. Mathematics that can categorize the different types of symmetry that molecules can assume can be used to help spectroscopy determine what shape the molecule must have. As one example, analysis of DNA in crystalline form by X-ray crystallography led to James D. Watson and Francis Crick's determination of the double-helix structure of DNA in 1953.

More recently, a form of pure carbon was found to be created from an electric arc between graphite electrodes (or from high-temperature burning of gaseous hydrocarbons). The carbon compounds created are known as "fullerenes," which are cage-like in appearance. The first fullerene to be discovered and have the results scientifically published is now known as Buckministerfullerene or C_{60}. Discovered in 1985 by Richard Buckminster "Bucky" Fuller, it was determined to essentially look like a soccer ball in appearance (a truncated icosahedron). How the precise polyhedral cagelike structure was determined from spectroscopy relied heavily on mathematics, specifically the area of abstract algebra known as "group theory," applied to quantum mechanics. Whereas fullerenes like C_{60} and C_{70} are cage-like, other pure forms of carbon obtained from graphite that do not fully close up include nanotubes. While fullerenes and nanotubes may have health applications, they are also of interest purely as nanotechnological objects. Indeed, some nanotubes are extremely strong and one day may make superstrong fibers; some, when other atoms such as potassium are added, are superconductors. For instance, the orientation of carbon atoms in nanotubes affects electrical conductivity (whether the molecules are conducting or semiconducting).

Another approach to determining molecular structure, particularly to surfaces, is to use instrumentation such as the scanning tunneling microscope. This tool relies heavily on physics (quantum mechanical tunneling) principles.

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Rick Kreminski

See Also: Crystallography; Genetics; Knots; Nanotechnology; Polyhedra.

Money

Category: Business, Economics, and Marketing.
Fields of Study: Measurement; Number and Operations; Representations.
Summary: Money has always been one of the subjects of applied mathematics, from interest to currency exchange.

When the earliest people wanted to acquire goods they could not make, grow, or hunt themselves, they exchanged other goods for them. Later, civilizations began to use smaller and more portable objects to represent value: shells, beads, pieces of leather, or shapes made from metal such as iron, among other things. Precious metals and printed paper currency supplanted most of these forms of money, and in the twenty-first century, intangible "digital cash" is exchanged electronically for goods and services.

Money is also a representation of wealth or value and is a basis for measuring economic and financial activity. Whether it is balancing a checkbook, analyzing a complex financial derivative, or anything in between, the mathematics of money is an indispensable tool for understanding and evaluating economic or financial transactions. Money is also multidimensional: value or wealth must be specified not only with respect to its amount, but also according to its time frame and to its country or currency framework. Translation of money and monetary transactions across these dimensions involves mathematical processes and an understanding of financial context, and mathematicians are actively involved in virtually all aspects of its production, management, and study. The first director of the U.S. Mint was David Rittenhouse, a well-known mathematician, inventor, astronomer, and surveyor. Mathematician

Marc Fusaro is a research assistant in the Research and Statistics division of the Federal Reserve System. Regarding his career, he has said:

> The mathematics in economics, where it is not explicit, is implicit. It underlies the economics everywhere. I can not always identify when I am doing mathematics. However, the thought processes learned in doing mathematics are crucial to economics and help at every step.

Time Value of Money

Money may differ or change in value across one or more dimensions, including, in particular, time. A dollar today is generally not worth the same as a dollar one year from now. The familiar effect of inflation over time is to decrease the value of a unit of money—a dollar bill will typically buy less one year from now than it will buy today. Also, a dollar today can often be invested so that it grows to a greater value a year from now. Another way of looking at this is to ask the question, "How much needs to be invested now, so that an investment account will be worth one dollar one year from now?" If the investment environment involves, as it generally does, positive interest rates or rates of return, the answer would be that an amount less than one dollar would need to be invested now in order to grow to a full dollar one year from now.

The Babylonians appear to have used interest on loans to model time doubling. Clay tablets dating back to about 2000 B.C.E. contain the following example: "given an interest rate of 1/60 per month (no compounding), compute the doubling time." This situation corresponds to annual interest rate of $12/60 = 20\%$. The money would double in five years, which is 100% (growth) divided by 20% (growth per year). Some also cite the Babylonians as the first civilization to use formal banking.

Interest and Interest Rates: The Cost of Money

One of the key issues associated with money is interest, which can be viewed as the cost associated with using money. Interest can be looked at from either side of a financial transaction. An individual earns interest on a savings or money market account or by lending money to someone else; these are examples of asset positions. On the other hand, when someone takes out a loan or otherwise borrows money, that person pays interest to the lender; this is an example of a liability or debt position. So, regardless of the side of the financial transaction, the interest involved in the transaction is the cost, or reward, associated with the use or employment of money.

Interest is the dollar amount of the cost or reward associated with a monetary position. However, it is

Definition and Function of Money

By definition, money is something that is acceptable in trade and transactions. More specifically, money is typically identified as having three functions:

- It is a "unit of account." Money is a mathematical representation of value and provides a measure of value or wealth.
- It is a "medium of exchange." Money facilitates efficient trades and transactions between parties.
- It is a "store of value." Money is an easily transportable and liquid entity that maintains its usefulness for exchange over time.

Historically, a variety of forms of money have been adopted in different eras, including barter, commodity money, and representative money. The United States and many other countries formerly operated on the gold standard (a type of representative money), where currency value is tied to a fixed weight of gold. The modern economy uses fiat money, which means that the value of a piece of money as a medium of exchange is based on governmental decree and is not related to its inherent value as a material object.

not really the dollar amount but rather the amount of interest as a proportion of the base or principal money amount that more clearly indicates the cost or reward associated with the transaction. This proportion is expressed as an "interest rate," and can be represented, for example, by i.

The value of money can change over time. However, time is just one of the dimensions over which the value and cost of money can change. For example, consider the following questions regarding interest rates:

- A lender is considering loaning money to someone for one year. What interest rate might the lender charge for loaning $1? For loaning $100? For loaning $100,000?
- A lender is considering loaning $10,000 to someone. What interest rate might the lender charge for a loan of term one month? For a term of one year? For a term of five years?
- A lender is considering loaning $10,000 for one year to one of three different people. What interest rate might the lender charge to the person who is perceived as the least risky (the one most likely to pay back the loan completely and in a timely fashion)? To the person of middle risk? To the person perceived as most risky?

Precise answers to these questions are not necessary to imagine that, within each set of questions, the answers may potentially be very different. For example, one may require a higher interest when lending a greater quantity of money; one may charge a higher interest rate when lending over a longer term; and one may insist on a higher interest rate when the borrower represents a greater risk. Thus, there are numerous dimensions and contexts in which the cost of money and its use can differ.

Money and Investments

Examining further the phenomenon of the time value of money, it is worth exploring more deeply how and why money can have a different value at one time compared with another. Consider a typical investment situation, which can be characterized as having four parameters: (1) the amount of money initially invested; (2) the interest rate, or the rate of return, which will be earned on the money invested; (3) the period of time over which the money will be invested; and (4) the future, or accumulated, value of the money at the end of the investment period.

As an example, suppose one invests $100 for one year at an effective annual interest rate of 10%. The future value (one year after the initial investment) is then calculated as $\$100 + (\$100 \times 0.10) = \$100 \times 0.10 = \$110$.

This example could also be done in reverse. One could ask what amount, invested now, would yield $110 one year from now, if money can be invested at an effective annual interest rate of 10%. A minor algebraic adjustment to the prior solution yields the answer:

$$\frac{\$110}{1.10} = \$100 .$$

The result of $100 can be referred to as the "present value" (PV) of $110 one year from now—it is the amount obtained when the future value is "discounted back" one year.

The concept of "present value" is one of the most important in all of finance and economics. The present-day equivalent of any set of future cash flows can be determined by "discounting back" each individual future cash flow and summing all of the discounted cash flows together. This discounted sum is the present value of the future cash flows, and—assuming that the interest rate used for discounting is correct—it is essentially the amount of money that, invested now, would replicate those future cash flows. In that sense, a person could be described as being "indifferent" between receiving the future cash flows or receiving an amount now that is equal to the sum of the present values of those future cash flows.

Mathematically, present value can be determined as

$$PV = \sum_t \frac{CF_t}{(1+i_t)^t}$$

where CF_t is the cash flow that will occur t periods from now, and i_t is the annual effective interest rate appropriate for an investment of t periods.

The above reference to i_t (an interest rate appropriate for an investment of t periods) suggests that cash flows over different time periods, or with different characteristics, might be associated with different levels of interest rates. Indeed, this is true, and in fact the cost of using money can be different in accordance with the period of time over which an investment is made. Typically, annual interest rates associated with relatively longer term investments are relatively larger than those associated with shorter term investments. This relationship is described formally by the yield curve, or the term structure of interest rates.

Similarly, one of the most critical factors in determining an appropriate interest rate is the level of riskiness inherent in the investment process (or uncertainty in the amount and timing of the future cash flows, if discounting is being performed for present value purposes). In general, riskier cash flows or investment opportunities are associated with higher interest rates. This association is a manifestation of the risk-return relationship, which suggests that taking on greater risk should be compensated by a relatively greater reward.

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Rick Gorvett

See Also: Accounting; Loans; Pensions, IRAs, and Social Security; Stock Market Indices.

Moon

Category: Space, Time, and Distance.
Fields of Study: Data Analysis and Probability; Geometry.
Summary: Though mankind has always looked up at the moon and even visited, most of the body of lunar knowledge is actually contributed by mathematics, which continues to attempt to model its motion.

The moon is the sole natural satellite of the Earth. Specific astronomical searches have established positively that the Earth has no other satellites larger than a few meters. The lunar body is nearly a sphere with a mean radius of 1738 kilometers (km) or 1000 miles—only 3.7 times less than the Earth. The mean distance of the moon from the Earth is 384,400 km (238,855 miles). The moon is the fifth largest satellite in the solar system and the largest one relative to the size of its planet. The moon is so near and so large in comparison with its "host" that the entire system is often dubbed the "double planet."

Viewed from above the North Pole of the Earth, the moon travels around it counterclockwise in a slightly elliptical path. The sideric month (one orbit around the Earth with respect to the stars) is 27.3217 days. The synodic month (the cycle of phases visible from the Earth; for example, the time interval between two successive "new moon phases") is 29.5306 days.

The period of one spin of the moon around its axis (a "lunar day") is exactly equal to the sideric month because of tidal breaking. This phenomenon is also known as "synchronous rotation," or tidal coupling." As a result, from the Earth, people can observe only half of the lunar surface (called the "near," or "visible," "side"). The "far" *(*called "invisible"*)* hemisphere was photographed for the first time in 1959 by the Soviet robotic spacecraft *Luna-3*, an episode of the space race between the United States and the Soviet Union. On the moon, the disk of the Earth does not rise and set. It is observable only from the near side in an almost permanent point of the lunar sky (fluctuating a little from a small phenomenon called "libration").

The face of the moon was influenced by both internal and external factors. On the surface, observers distinguish so-called darker "maria" (flat "seas" without water) and brighter highlands. All of them are covered with numerous craters, the highlands more so than the seas. The far side of the moon has practically no seas. Because of constant bombardment by various small interplanetary particles, the entire surface is enveloped with thin fractured material called "regolith." There is no atmosphere on the moon. As a result, the difference in temperatures between a lunar day and a lunar night is very high: between –170 degrees Celsius and +130 degrees Celsius (–274 degrees Fahrenheit to +266 degrees Fahrenheit). Water in the form of subsurface ice exists in polar regions. There are no traces of modern tectonics on the surface.

From the Earth, the visible angular diameter of the moon is 0.5 degrees and fairly close to the angular diameter of the sun. This property is essential

Astronaut Harrison H. Schmitt standing next to a huge boulder during the Apollo 17 mission to the moon.

because sometimes the three bodies, the sun, the Earth and the moon, align along a straight line. In this case, humans observe either a total lunar (if the moon is farther from the sun than the Earth) or a total solar (if the moon is between the sun and the Earth) eclipse. The latter is visible only within narrow strips on the Earth. Such observations are important for solar physics. To see these phenomena, astronomers regularly organize special expeditions. Eclipses often held great religious significance. Scholar Anaxagoras of Clazomenae explained the phenomenon using mathematics. He was imprisoned for asserting that the sun was not a god and that the moon reflected the sun's light.

The age of the moon is about 4.5 billion years, which is close to the age of the sun and the entire solar system. Of the various concepts of the moon's origin, the prevailing hypothesis is that the Earth-moon system was formed by a giant impact: a planet-sized body hit the nearly formed proto-Earth, ejecting material into orbit around the proto-Earth, which accreted to form the moon.

The mean density of the moon is just 3.34 grams per centimeter^3 and, as a result, the mass of the moon is 81 times less than that of the Earth. The interior of the moon is geochemically differentiated: it has a distinct crust, mantle and core. Surface gravity on the moon is six times less than on the Earth. The general magnetic field of the moon is practically absent.

The moon has always played a significant role in religion, science, art, and culture. Since the Paleolithic, the lunar orb in the sky has been utilized for calendar purposes. That is why the similarity of the terms "moon" and "month" is not coincidental. For the philosopher Aristotle, the moon marked a great border between a mortal and corruptible sublunar (terrestrial) world and an immortal world of ideal heavenly bodies. It became a significant symbol for Islam. For Isaac Newton, the moon was the prime test body to demonstrate mathematically that the fall of an apple and the orbiting of a celestial body are ruled by a single natural law of universal gravity.

Mathematical Modeling

Many mathematicians have developed theoretical models for the motion of the moon. The exact path of the moon around the Earth is affected by many perturbations and is extremely complicated. That is why, after Newton, research of lunar motion (lunar theory) became the central problem of celestial mechanics. Consequently, it appeared among the most critical and difficult tasks for applied mathematics. The moon's gravitational influence on the Earth produces the ocean tides and the tiny lengthening of the calendar year. Most of what we know about the moon's size, shape, and other properties has been derived largely through mathematical computations, using mathematical theory and data from Earth-based observations, satellite imagery, and direct measurements made by astronauts.

Human Exploration

Starting at least from Roman times, science fiction authors were the forerunners for delivering terrestrials to the moon. In reality, the first space robots to the moon were launched by the Soviets in 1959. But they failed in the space race with the United States to realize manned expeditions. The first terrestrials to visit the moon were the American astronauts of the Apollo program. After preliminary robotic programs (Ranger, Lunar Orbiter, and Surveyor) and Apollo flybys, American manned landings on the moon occurred in 1969–1972. Among seven planned landings (from Apollo-11 up to Apollo-17), six missions were tremendously successful. Twelve crewmembers stepped down on the near side of the moon, and six more orbited it. Astronauts performed a number of experiments and returned to the labs about 382 kg of lunar matter. Since 2004, Japan, China, India, the United States, and the European Space Agency have each sent successful automatic lunar orbiters.

Among the many thousands of contributors to lunar programs, mathematicians often played outstanding roles. One significant individual was mathematician Richard Arenstorf, who solved a special case of the three-body problem with figure-eight trajectories now called "Arenstorf periodic orbits." In 1966, he was awarded a NASA medal for exceptional scientific achievement for this work. Another was Evelyn Boyd Granville, who used numerical analysis to aid in the design of missile fuses. She later worked on trajectory and orbit analyses for several space missions, including Apollo. She said, "I can say without a doubt that this was the most interesting job of my lifetime—to be a member of a group responsible for writing computer programs to track the paths of vehicles in space." In fact, mathematicians occupied many seats in the first row of the Mission Control center. Their work was critical for calculating trajectories and for maneuvers that involved the meeting of two objects in space, including landing on the moon. They also played a significant role in determining a rapid and feasible solution that would safely return the damaged Apollo 13 manned spacecraft to Earth.

Among mathematicians in Russia, the most noticeable contribution to flights to the moon was made by Efraim L. Akim of the Keldysh Institute for Applied Mathematics at the Russian Academy of Sciences in Moscow. He was the principal investigator for special lunar orbiters to create a mathematical model of the lunar gravitational field and the leader of a team to calculate trajectories of the Russian lunar robotic spacecraft.

Several international treaties regulate mutual relations of various states with respect to modern space explorations of the moon. The most important among them are the Outer Space Treaty (1967) and the Agreement Governing the Activities of States on the Moon and Other Celestial Bodies (1979).

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Alexander A. Gurshtein

See Also: Interplanetary Travel; Measuring Time; Planetary Orbits; Ride, Sally; Spaceships; Weightless Flight.

Movies, Making of

Category: Arts, Music, and Entertainment.
Fields of Study: Algebra; Geometry; Representations.
Summary: A variety of mathematics, including signal processing, geometry, and lighting, are required for making movies.

It takes many people with many different talents to make a movie. Some of these required talents are very technical, so these filmmakers must have a working knowledge of various mathematical principles to employ the tools of their trade. A sampling of these areas includes camerawork, sound recording, and special effects. Signal processing, a branch of applied mathematics, is necessary both during the production of a film (for selection of filters and set dressings of acceptable visual frequency) and in postproduction, where dialogue must be made understandable in the sound track. In addition, often the shooting of a scene itself, with its restrictions on space and desired camera angles as well as satisfying lighting needs, becomes a problem in geometry. Physical phenomena and their interactions can increasingly be modeled using mathematics. Mathematicians such as Tony DeRose, who won a 2006 scientific and technical Academy Award for his work on surface representations, play an increasingly important role in producing modern special effects.

Camera Work

In the shooting of a scene, the number of variables is considerable, and those directing the operation of a camera have many decisions to make. Considerations include viewing angles, shutter speeds, lens selection, current lighting, and the format of the film. These considerations become considerably more complicated when working with miniatures in which an attempt is made to fool the eye of the viewer into believing the miniature is a real, full-sized object. The choice of the camera itself, which has many parameters, has a significant effect on the look of the film.

The operation of the camera depends a great deal on the lighting of the set. An f-stop, which has been used for many years on cameras, is the ratio of the focal length of the lens to the diameter of the entrance pupil. This unit was used to control the quantity of light reaching the film. However, because of the fact that much of the light reaching the film plane is lost to diffraction, reflection, and refraction, more modern cameras use the T-stop calibration, which is a measure of the actual amount of light reaching the film plane. If no light were lost to optical factors, these two values would be identical. Both of these measures are used extensively: the f-stop for depth of field calculations and the T-stop for light transmission.

The gaffer (crew boss responsible for planning the lighting) uses a variety of tools to light a scene so that the film can be recorded with the desired viewing window, shutter speed, and camera angles, as well as various aesthetic considerations. One such tool is the inverse square law. This law states that the intensity of a single source of light decreases in proportion to the square of its distance from the subject. Using this law, a small light puts less light on the background, if desired, or a larger light farther away creates a larger area with a similar light level. The light used will also affect the T-stop to be used on the camera, so the light placements must be planned carefully and light output levels must be known exactly.

One calculation the camera operator must constantly make is to determine the depth of field. A lens can focus on only one distance at a time. Therefore, technically, both the foreground and the background of a scene are never in focus simultaneously; in fact, only one point on an actor is in focus at any one time. However, objects close to this distance will not appear blurry to the human eye, which does not perceive imperfection within a certain distance of the point of focus. The distance interval in which all objects are acceptably focused is called the "depth of field." To determine the depth of field, one must first determine the hyperfocal distance, the smallest distance such that all objects from half this distance through infinity are in acceptable focus. This distance can be approximated algebraically, with a parameter known as the "circle of confusion" determining what is considered to be acceptable focus dependent on the focal length and f-stop setting of the lens. Finally, the near and far limits of the depth of field can be determined with the equations

$$\frac{1}{D_n} = \frac{1}{S} + \frac{1}{H} \text{ and } \frac{1}{D_f} = \frac{1}{S} - \frac{1}{H}$$

where D_n and D_f are the near and far limits of the depth of field, S is the distance from the camera to the subject, and H is the hyperfocal distance. These formulas are simplified versions of the normal depth of field equations, which have an interesting geometric derivation.

Audio and Visual Signal Processing

The production sound mixer is in charge of recording the sound and dialogue for a film. Typically, crewmembers are hired to operate microphones, often using long poles with a microphone on the end. These microphones are used to record various sounds on the set, with wireless microphones attached to the actors to record dialogue. Sound effects are recorded separately, as is the score. During post-production, unwanted noise must be filtered out of the recordings, the dialogue must be made understandable, and the effects, score, and dialogue must be mixed together meaningfully.

To remove background noise, the audio signal (composed of sound waves) is decomposed using a Fourier transform, so that the model of the audio signal is divided into simpler, trigonometric components. These components are then analyzed, isolating frequencies corresponding to unwanted artifacts in the recording, such as the sound of the wind on the microphone. Background noise is removed by removing the Fourier components of amplitude below a certain level. Finally, by reversing the transform, a more filmworthy audio signal is obtained.

Processing must also be done to the visual signal. Video cameras record at a "frame rate," the frequency with which the camera produces images. These images are recorded as discrete signals, which are then reconstructed on film. If objects of a high visual frequency are used in a scene, a loud tie for example, then the image on film will experience aliasing, causing visual distortion or artifacts. To avoid this, the set designer or costumer needs to avoid objects above a certain visual frequency. This frequency, called the "Nyquist frequency," is half the frame rate. If images of high-visual-frequency objects are desired, then an antialiasing filter must be used, such as a lowpass filter, which will pass the low-frequency objects but reduce the amplitude of the high-frequency objects. Filmmakers have many filters that can be used to capture a wide variety of objects in a scene, depending on the mix of visual frequencies present.

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William Griffiths

See Also: Animation and CGI; Optical Illusions; Televisions.

Movies, Mathematics in

Category: Arts, Music, and Entertainment.
Fields of Study: Communications; Connections.
Summary: Mathematics and mathematicians often appear in movies, helping to shape the public's image of mathematics

Mathematics has been showcased in a number of movies. Often, the lives of mathematicians, both real and fictional, have been dramatized for use in film. Mathematics in movies can reveal, reflect, and shape how society views mathematics. Images of nerds and geniuses are very common in movies, and the mathematical powers of geniuses are sometimes equated with mental illness. There are also examples in which talented women deny their mathematical ability. Some of the films that focus heavily on the lives of mathematicians include *A Beautiful Mind* (2001), *Good Will Hunting* (1997), *Pi* (1998), *Proof* (2005), *I.Q.* (1994), *Infinity* (1996), and *Agora* (2009). Other movies, such as *21* (2008), *Contact* (1997), *Cube* (1997), *Jurassic Park* (1993), *Fermat's Room* (2007), and *Mean Girls* (2004), use mathematics as the basis of key plot points. Some films, such as *Stand and Deliver* (1988), dramatize the teaching of mathematics. There are also numerous documentaries, including *N is a Number: A Portrait of Paul Erdös* (1993) and *Julia Robinson and Hilbert's Tenth Problem* (2008). Such films are often the public's only connection to mathematics. As such, it important to point out how accurately these films communicate these ideas.

Mathematics and Character Development

A Beautiful Mind is a 2001 film directed by Ron Howard based on the life of Nobel Prize–winner John Nash. The film chronicles the life of Nash (Russell Crowe) beginning with his graduate studies at Princeton. While there, Nash discovers the principle of governing dynamics, a central principle in game theory and modern economics. The film simplifies the principle by showing an attractive blonde woman and her four friends entering a bar. If Nash's friends all flirt with the blonde, they impede each other. Further, her friends will also spurn the would-be suitors, as they do not want to be the second choice. Hence, the best strategy for Nash's friends is to avoid the blonde and instead approach her friends. The film later shows Nash's romance with Alicia Larde (Jennifer Connelly) and his problems because of schizophrenia. The film omits many details of Nash's life, including both his previous marriage and his divorce from and later remarriage to Alicia. Nonetheless, the film received four Academy Awards, including Best Picture and Best Director.

Good Will Hunting is a 1997 film directed by Gus Van Sant starring Matt Damon and Robin Williams. In this film, Will Hunting (Damon) is a janitor working at MIT. One night, he solves a difficult mathematics problem left on a chalkboard in the hall. When none of the students admit to solving the problem, Professor Lambeau (Stellan Skarsgard) places a second problem on the chalkboard, which Will promptly solves. Unfortunately, Will is plagued by antisocial behavior, causing him to be arrested after a fight in a bar. Lambeau has Will released into his custody provided that Will works with him on mathematics and Will sees a therapist (Williams). While the film does present what appear to be well-posed problems in graph theory and combinatorics, it is unlikely the solution to such difficult problems could be placed on a single chalkboard in a short amount of time. The film won two Academy Awards.

The 1998 film *Pi* was directed by Darren Aronofsky. In *Pi*, Max Cohen (Sean Gullette) is a number theorist who believes in three basic principles: mathematics is the language of the universe; the universe can be understood through numbers; and if one graphs these numbers correctly, patterns will emerge. In particular, he studies the stock market looking for patterns. If one were able to find patterns in the stock market, one could accurately predict the future. After a conversation with Lenny Meyer (Ben Shenkman) a Hasidic Jew, Cohen begins to do research on the numbers in the Torah. Meyer tells Cohen that the Torah is composed entirely of numbers, and there are relationships between the numbers. Thus, Cohen searches for a pattern in the Torah in the hope of revealing the true name of God. There has been mathematical research on finding patterns to the stock market. However, the level presented in this film is impractical. For instance, there are literally thousands of variables involved in the stock market. A slight change in any of these variables can lead to a large change in the market.

The 2005 movie *Proof* was directed by John Madden, based on the play by David Auburn. In *Proof*, Gweneth Paltrow plays Catherine, a woman who has been looking after her father (Anthony Hopkins). Her father was a brilliant mathematician who later became incapacitated because of mental illness. During this time, he wrote numerous notebooks full of his delusional ramblings. After his death, his former student Hal (Jake Gyllenhaal) finds one of these notebooks containing what appears to be an important result. While Catherine claims the result is hers, some question both her authorship and her sanity. Catherine's father seems based in part on real life mathematician Kurt Gödel. Like her father, Gödel was an important mathematician who descended into insanity, writing endlessly. Like most research professions, it is important to establish priority. However, the film does not accurately depict suitable ways to establish priority, such as presenting at research conferences or publishing a preliminary technical report.

Romance and Mathematicians

Two films, *I.Q.* (1994) and *Infinity* (1996), focus more on the romantic sides of mathematicians lives. In *I.Q.*, Albert Einstein (Walter Matthau) and his friends help local mechanic Ed Walters (Tim Robbins) to romance Einstein's niece, Catherine Boyd (Meg Ryan), a doctoral candidate in mathematics. In order to accomplish this, Einstein makes it appear that Ed is a genius in physics. In typical romantic comedy fashion, Catherine falls for Ed, only to discover the deception. The film takes many liberties in portraying the personalities of Einstein and his friends. Further, the film does not discuss the science of Einstein or his colleagues.

Science and Mathematics Themes

Infinity is a 1996 film based on the books *Surely You're Joking, Mr. Feynman,* and *What Do You Care What Other People Think* by Richard Feynman. Feynman (Matthew Broderick) was a Nobel Prize-winning physicist who worked at Los Alamos and later investigated the *Challenger* shuttle disaster. Like *I.Q.*, the film does nothing to discuss Feynman's scientific discoveries. Instead, the film focuses on the romance of Feynman and his first wife Arline (Patricia Arquette). In one of the film's more memorable scenes, Feynman impresses Arline with his ability to do mental computations faster than a shop owner with an abacus. Such anecdotes form the cornerstones of Feynman's biographies, however, they are largely omitted in the film.

Agora is a 2009 Spanish drama that presents a semi-fictionalized account of the life of Hypatia of Alexandria. This film has elicited a variety of reactions among members of the mathematics community, including concerns about its focus on certain aspects of her personality and private life as well as her mathematical investigations and achievements.

The 2008 film *21* is based on the book *Bringing Down the House* by Ben Mezrich. In this film, Ben Campbell (Jim Sturgess) and the mathematics club (Kate Bosworth, et al.) are coached by Professor Micky Rosa (Kevin Spacey) to count cards in blackjack. Using simple counting techniques and signals, they act as a team to bring in the big player when the deck begins to favor the player. By doing so, they are able to bring in much greater returns than an individual counting cards on their own. Unfortunately, their success is hampered by a security chief (Lawrence Fishburne) who begins to realize their system.

Jurassic Park is a 1993 movie directed by Steven Spielberg starring Sam Neill, Laura Dern, Jeff Goldblum, and Richard Attenborough. Jurassic Park was constructed by John Hammond (Attenborough) to feature genetically recreated dinosaurs. To gain support for the park, he recruits paleontologists (Neill and Dern) as well as mathematician Ian Malcolm (Goldblum). Malcolm specializes in chaos theory, preferring to be called a "chaotician." Chaos theory deals with unpredictability in a complex system. The film accurately illustrates this phenomenon with an experiment involving placing drops of water on the back of a still hand. Despite the best attempts to achieve predictable results by placing the drops of water at the same place, subtle differences on the hand will cause the drops to roll off in different places. Malcolm argues that, despite the best intentions of the engineers and geneticists, Hammond and his associates do not have control of Jurassic Park. For instance, they do not predict the actions of a saboteur or that the dinosaurs would begin to breed. These oversights lead to the failure of the park. Jeff Goldblum returns as Ian Malcolm in the 1997 sequel *The Lost World: Jurassic Park*.

Prime Numbers

The search for extraterrestrial intelligence is the basis of the 1997 film *Contact*, based on the book by Carl Sagan. In this film, Dr. Eleanor Arroway (Jodie Foster) discovers a signal that appears to be from the star Vega using the telescope array in New Mexico. The signal consists of a long string of prime numbers. Prime numbers are positive numbers divisible only by themselves and the number 1. Moreover, the prime numbers are not associated with any random natural phenomenon. Hence, the broadcast of the primes from an extraterrestrial source suggests the work of an alien intelligence. As various layers of the message are decoded, her team discovers the plans for a machine that will hopefully transport one individual to meet these aliens. The film won the Hugo Award for Best Dramatic Presentation.

The 1997 psychological thriller *Cube* also uses prime numbers as a key plot point. In *Cube*, a group of five individuals (Nicole de Boer, David Hewlett, et al.) find themselves in a cube-shaped room. The room they are in is surrounded on all six sides by other cube-shaped rooms. They quickly discover that many of the rooms contain deadly traps. One of the group is a mathematician and realizes that if the rooms are labeled with prime numbers, then the room is trapped. They also discover that the numbers give the position of the room within the larger cube structure. Finding an autistic savant in the maze allows them to factor the numbers more quickly and thus navigate the cube structure.

The 2007 Spanish film *Fermat's Room* uses a famous conjecture about prime numbers as a catalyst. The Goldbach Conjecture states that every even number greater than two can be expressed as the sum of two primes. Despite being conjectured in 1742, this problem has remained unsolved as of 2010. A young mathematician known only as "Galois" claims that he has a proof of the Goldbach Conjecture to impress a young woman called Olivia. They are soon invited to dinner by the enigmatic Fermat along with a middle-aged mathematician, Pascal, and an older mathematician, Hilbert. Coincidently, Hilbert has been working on Goldbach for 30 years. When their host, Fermat, leaves, they soon find themselves locked in the room. They are presented with riddles one after another. If they are unable to solve the riddles within their time limit, the walls of the room close in until they solve the riddle. While the names of the characters are based on famous mathematicians and the Goldbach Conjecture is presented accurately, the riddles in the film are quite elementary.

Social Life and Mathematics

In the 2004 movie *Mean Girls*, the main character, high school student Cady Heron, struggles to balance her mathematical talent with social pressures and attempts to be popular. She pretends to struggle at mathematics in her calculus class in order to impress a boy she likes.

Finally, the 1988 movie *Stand and Deliver* illustrates the difficulty of teaching mathematics in an inner-city school. Jaime Escalante (Edward James Olmos) teaches basic arithmetic in a Los Angeles high school. He senses that his students are capable of more, and using props, humor, and examples from their lives, he motivates them to learn calculus. At the end of their senior year, his students all pass the AP calculus exam.

Some producers hire mathematical consultants to ensure the accuracy of the content. It is likely that mathematics and mathematicians, both real and fictional, will continue to remain sources of dramatic material in feature films. Mathematicians also analyze these representations in the classroom and in publications, such as the media column in the *Association for Women in Mathematics Newsletter*.

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See Also: Betting and Fairness; Coding and Encryption; Curriculum K–12; Einstein, Albert; Game Theory; Mathematical Friendships and Romances; Mathematical Puzzles; Number Theory; Numbers and God; Pi; Proof; Puzzles; Television, Mathematics in.

MP3 Players

Category: Communication and Computers.
Fields of Study: Algebra; Communication; Data Analysis and Probability; Number and Operations.
Summary: Mathematics and mathematical data compression algorithms make MP3 players possible.

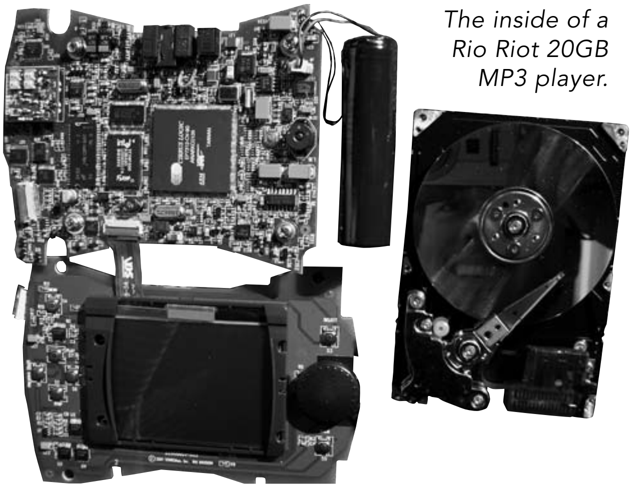
MP3 players have revolutionized the way people listen to music. MPEG Audio Layer III (MP3) is an audio compression standard that reduces music files with little perceptible loss of quality. It is one of the Motion Pictures Expert Group standards for lossy compression. The inventors of MP3, according to the United States MP3 patent, are engineers Bernhard Grill, Karl-Heinz Brandenburg, and Bernd Kurten; computer scientist Thomas Sporer; and mathematician Ernst Eberlein. The development was mathematically and technically challenging according to Brandenburg, who is sometimes called a specialist in mathematics and electronics. He stated, "in 1991, the project almost died. During modification tests, the encoding simply did not want to work properly. Two days before submission of the first version . . . we found the compiler error." Scientists at Fraunhofer-Gesellshaft developed an MP3 player in the early 1990s.

In 1997, engineer Tomislav Uzelac invented the AMP MP3 Playback Engine, which is regarded as the first successful MP3 player. Computer science student Justin Frankel, who also helped develop the peer-to-peer Gnutella network, and fellow student Dmitry Boldyrev created the free MP3 player Winamp in 1998. Inventor Briton Kramer contributed to the first mass-produced player MPMan. The ability to share files over the Internet, legally and illegally, for free or for purchase, was a significant factor in the rapid spread of the MP3 format. By the twenty-first century, iPods became one of the most popular MP3 players, in part because of the availability of music and video via the iTunes store. The ability to hold thousands of songs, videos, and other types of files is one of the benefits of MP3 players, all of which would not be possible without mathematics and mathematical data compression algorithms.

Compression and Encoding

Data compression is either "lossy" or "lossless," referring to whether any data is discarded in the process of creating a smaller file. Huffman coding, developed by mathematician David Huffman, is used for MP3 compression. It employs a mathematical idea called a "frequency-sorted binary tree" to look for recurring strings of binary information in the digital file. These strings are replaced by shorter binary codes. The most frequently occurring strings are assigned the shortest replacement codes, optimizing compression. In lossless compression, all original information is preserved in some way. In lossy compression, some information is discarded to decrease file size. MP3 compression relies, in part, on perceptual coding.

In a human ear, certain waveforms are indistinguishable. Psychoacoustic models prioritize data according to the ear's ability to distinguish the sounds the data produce. Mathematical models of auditory processing yield encoding information and algorithms, such as frequency threshold curves, masking functions, and critical bandwidths. Signal processing typically relies on Fourier transforms, named for mathematician Joseph Fourier, to enable coding and decoding. Ultimately, an MP3 music file consists of a series of short, dependent



The inside of a Rio Riot 20GB MP3 player.

frames, like a filmstrip. Each frame has a header with information about the data in the frame. Inside the frame is audio information in frequencies and amplitudes. Sometimes, at the beginning or end, there is an ID3 data block, which stores the artist name, track title, album name, recording year, or other information.

Sound

Optimization of audio playback depends not just on the human ear but on the equipment used. Speakers are common on computers, while most MP3 players use some form of over-ear headphones or earbuds that fit into the ear canal. Empirical studies suggest that noise from internal earbuds may be damaging to hearing because the decibel level experienced by listeners is higher on average than with external earphones, and long-life batteries players allow people to listen longer. Some researchers have reported average listening levels of about 110–120 decibels, equivalent to a rock concert. Based on findings of such studies, many audiologists recommend using noise-canceling headphones rather than turning up the volume. Engineer Lawrence Fogel first explored noise-canceling headphones for aviation in the 1950s. Noise cancellation uses the mathematical properties of waves to create a signal with the same amplitude but with an inverted phase to unwanted noise, creating a combined wave inaudible to the human ear.

Shuffle

One other interesting mathematical problem related to MP3 players is the shuffle function. Various mathematical algorithms are used to permute the play order of songs in an MP3 player's library. In the early twenty-first century, the iPod's default shuffle system reorders songs much like someone shuffling a deck of cards, giving each song an equal chance to end up in any position in the shuffle and resulting in no repeats. However, many factors can affect perceived randomness and equal likelihood of orderings. For example, users can request higher chances of play for songs with high user ratings. Songs can also be marked "Skip When Shuffling" so that they are completely excluded. Most people frequently reshuffle, generating new random orderings before completing the library, and so some tracks appear to repeat or group in nonrandom ways.

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BILL KTE'PI

See Also: Cocktail Party Problem; File Downloading and Sharing; Randomness.

Multiplication and Division

Category: History and Development of Curricular Concepts.
Fields of Study: Algebra; Communication; Connections; Number and Operations.
Summary: Scholars throughout history have developed a variety of algorithms to compute multiplication and division.

Multiplication and division of numbers are useful for scaling a quantity, which is fundamental in any quantitative society. For example, to determine the correct cost for purchasing more than one unit of some item, the buyer should multiply the number of units by the unit price. One of the most common exposures people have to the concept of multiplication is the idea of repeated addition, which is frequently taught in elementary school. Some mathematicians object to this analogy, since it fails in specific instances, such as the case of fractions. Multiplication in this context can be thought of as a scaling process. Multiplication of mathematical objects is an operation that combines those objects in some representative manner. For example, matrix multiplication is the composition of the corresponding linear transformations.

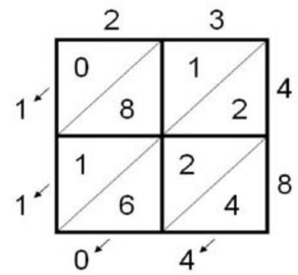
Some theorize that the first evidence of multiplication is the Ishango Bone, a tool from Upper Paleolithic era, which may demonstrate multiplication by two. The ancient Chinese as early as the Warring States period (475–221 B.C.E.) developed a system of multiplication using a place-value system and counting boards for calculations. Multiplication using a place-value system of Hindu–Arabic numerals dates back to Indian mathematician and astronomer Brahmagupta in the seventh century.

Division is an operation that is generally the inverse of multiplication. For whole numbers, division can be thought of as finding the number of identically sized groups into which a number of individuals can be divided or partitioned. The remaining individuals, after all identically sized groups have been removed, are called the "remainder." People in ancient Egypt commonly divided food and other supplies and formalized the notion of division. Concepts of division and multiplication are generalized and studied in the fields of number theory, algebra, and numerical analysis.

History of Multiplication Algorithms

One of the earliest methods for multiplying whole numbers is called the "Russian Peasant algorithm," but a similar procedure was described thousands of years earlier in ancient Egyptian papyri and is based on doubling the multiplicand. Assume that an Egyptian scribe wanted to compute the product of 13 and 23. The scribe would compute the products $1 \times 23 = 23$, $2 \times 23 = 46$, $4 \times 23 = 92$, and $8 \times 23 = 184$. By doubling each previous result, the scribe would, realizing the sum of the multipliers 1, 4, and 8 equals the multiplier 13, find the sum $23 + 92 + 184 = 299$, the required product. Thus, multiplication is reduced to being able to both double a number and add.

Another method used to multiply numbers is called "gelosia" or "lattice multiplication," which is still taught in some elementary schools. This method probably originated in India before the twelfth century and eventually became the inspiration for Napier's bones, an instrument created by John Napier, which was used to accomplish multiplication after its invention in the early 1600s. To multiply, for example, 23×48, one draws a 2-by-2 lattice of squares where each square is bisected by a diagonal from upper right to lower left. The rows are labeled, top to bottom, 4 and 8, while the columns are labeled, left to right, 2 and 3. The product of each of the single digits is written in the corresponding square as shown below. Once numbers in the lattice have been written, the final product, 1104, is formed by adding along the diagonals from upper right to lower left, being careful to remember to carry.



Another method still taught in the early twenty-first century has been called "cross-multiplication." It was described by Leonardo Fibonacci in the twelfth century but was certainly known earlier in India and the Middle East. To multiply 23×48, one starts from the right and multiplies 3×8 to get 24. The 4 is written down and the 2 is remembered. Then the cross multiplication is performed, $2 \times 8 + 3 \times 4$ to get 28, which is added to the remembered 2, obtaining 30. The 0 is written down and the 3 is remembered. Finally 2×4 is computed obtaining 8, which is added to the remembered 3, resulting in 11, which is written down. The final result is thus 1104. This method can be generalized and, by keeping various "remembered" digits using finger numbers, it is possible to multiply many two-digit numbers without writing down any intermediate results.

Probably the most popular method for multiplying that is taught in the early twenty-first century computes the products of multiplicand with each of the digits of the multiplier, working from right to left. Each of these successive products is shifted one more digit to the left. These partial products are then summed to obtain the final result as the following example shows:

```
    4 8
  x 2 3
  1 4 4
+ 9 6
1 1 0 4
```

This and similar methods were implemented on various counting boards, the abacus, and dust boards.

Division Algorithms

An ancient method for finding the quotient of two large whole numbers that is most appropriate for either the dust board or the Chinese counting board was adapted to pen and paper and became the "scratch method" that was used in Europe up into the nineteenth century. Two methods of computing long division using pencil and paper eventually replaced the scratch method.

One method is popular in Italy, England, and the United States and will be referred to as the "Italian method." The other is popular in Spain, France, Latin America, Austria, and Germany. This second approach will be referred to as the "Spanish method." Both methods date from at least the dawn of the sixteenth century. Both methods are shown by demonstrating how to find 2456/57, which is 43 with a remainder of 5.

Italian Method

```
          4 3
5 7 | 2 4 5 6
     -2 2 8
        1 7 6
       -1 7 1
            5
```

Spanish Method

$$\begin{array}{r|l} 2\ 4\ {}_3 5\ 6 & 5\ 7 \\ \hline 1\ 7\ {}_2 6 & 4\ 3 \\ 5 & \end{array}$$

In the Italian method, the dividend is written under a horizontal line above which the quotient will be written as it is found. The divisor is written to the left of the dividend with a vertical line drawn between them. It is determined that 57 will go into 245 at least (but no more than) 4 times, and 4—the first digit of the quotient—is written above the 5 of the dividend. The product $4 \times 57 = 248$ is computed and written below the 245. Subtraction is performed, obtaining 17 and the 6 from the dividend is brought down to the right of the 17. Then, it is determined that 57 will go into 176 at least (but no more than) 3 times. The product $3 \times 57 = 171$ is computed and subtracted from 176 to get the remainder of 5.

The Spanish method is cosmetically different and is characterized by many fewer digits being written down because the multiples $4 \times 57 = 248$ and $3 \times 57 = 171$ are never explicitly computed. To start, the dividend is written down followed by a long vertical line and the divisor. A horizontal line is then drawn under the dividend and divisor. The quotient will be developed to the right of the vertical line below the horizontal line one digit at a time. First, it is determined that the first digit of the quotient is 4. Then $4 \times 7 = 28$ is computed, which must be subtracted from 5. This operation cannot be done, and so a little 3 is written between the 4 and the 5 of the dividend. Now 28 can be subtracted from 35 obtaining 7, which is written under the 5 of the dividend. The 4×5 is found and added to the little 3 obtaining 23. This number is subtracted from 24 obtaining 1, which is written below the 4 of the dividend. To complete this phase, the 6 from the dividend is brought down so that the problem is now to divide 176 by 57. It is determined that 57 will go into 176 at least (but no more than) 3 times, and so 3 is written down as the next digit of the quotient. Computing $3 \times 7 = 21$, try to subtract 21 from 6, which cannot be done, and so a small 2 is written between the 7 and 6. Now, subtract 21 from 26, obtaining 5, which is written below the 6. Now, 3×5 is computed and added to the little 2, obtaining 17, which when subtracted from 17 is 0. As such, nothing needs to be written to the left of the 5. By the time a student is out of elementary school, it is expected that the annotations like the little 2 and 3 will no longer need to be written.

Checking Results

Because the algorithms for computing products and quotients of whole numbers are complex, methods to check the results have existed since antiquity. Since division and multiplication can be viewed as the inverse operations of each other, a division can be checked by multiplying the quotient by the divisor and adding the remainder. If the result is the dividend, then the division has been performed correctly. Another approach to checking the result of an arithmetic operation is to perform the operation using modular arithmetic, typically with respect to 7, 9, or 11. A number a modulo b is defined to be the remainder of $a \div b$. Thus, 267 modulo 9 is 6. There are easy tricks for computing numbers modulo 7, 9, and 11. For example, the method of "casting out 9s" can be used to compute 267 modulo 9 by simply summing the digits and subtracting 9 whenever the sum is over 9. For example, $2 + 6 + 7 = 15 - 9 = 6$. In order to check the correctness of the multiplication $23 \times 48 = 1104$, check to see if $[(23 \text{ modulo } 9) \times (48 \text{ modulo } 9)]$ modulo $9 = 1104$ modulo 9.

In this case, $5 \times 3 = 15$ and 15 modulo 9 is 6, which is equal 1104 modulo 9. One can conclude that the product, 1104, is probably correct. Although it is possible that such a check will confirm an incorrectly performed multiplication, this is unlikely.

Multiplication by Addition

Because multiplication and division are time consuming and tedious to perform compared to addition and subtraction, there has been much work to simplify the finding of products and quotients. Simplification results from using logarithms, invented in the seventeeth century, since

$$\log(A \times B) = \log(A) + \log(B).$$

Thus, it is easy to find the product of two numbers using a table of logarithms by looking up the logarithm of both the multiplier and the multiplicand in the table, adding these two logarithms and then using the table to find the number that has that sum as its logarithm.

It is also possible to convert multiplication to addition (and halving) using a table of cosines along with the trigonometric identity

$$\cos(A)\cos(B) = \frac{\cos(A+B) + \cos(A-B)}{2}.$$

This method was used by some astronomers before the invention of logarithms.

Rapid Multiplication and Division

Throughout history, people have developed the ability to multiply large digits in their head. Thomas Fuller, a slave, was one such person. For example, he calculated the number of seconds a man who is 70 years, 17 days, and 12 hours old has lived. He correctly answered 2,210,500,800 in only a minute and a half, and historians hypothesize that the algorithms he used were probably based on traditional African counting systems. In the twenty-first century, mathematician and magician Art Benjamin has turned his rapid mental calculations into educational entertainment.

Multiplication of whole numbers that can be represented as single binary words in a computer can typically be done with a single hardware instruction that combines addition and shifting to find the product. To multiply numbers with thousands of digits, other methods are possible that make use of fast methods for computing Fourier transforms, named for Joseph Fourier. The Schönhage–Strassen algorithm, developed by Arnold Schönhage and Volker Strassen, and the Fürer algorithm, developed by Martin Fürer, are two such methods. These complicated algorithms, however, have limited practicality when implemented on a conventional computer. Mathematicians in the field of numerical analysis consider issues of speed and error in computer algorithms.

Early computers multiplied or divided using repeated addition, subtraction, and shifting algorithms. Depending on the computing system, the amount of time required for multiplication or division on a computer is approximately the same. However, factoring a number into its unknown divisors is significantly harder. The RSA (which stands for the people who first described the system: R. Rivest, A. Shamir, and L. Adleman) cryptosystem takes advantage of this characteristic.

Generalizing Multiplication and Division

Multiplication and division can be used for computational purposes and to help understand other mathematical principles. The product of two rational numbers a/b and c/d is defined to be the rational number

$$\frac{ac}{bd}$$

whereas the quotient, $a/b \div c/d$ is defined be the product of a/b and d/c. The reciprocal of a rational number, a/b, is the number b/a, since the product of a/b and a/b is 1. The product of two irrational numbers, which cannot be represented as a fraction of whole numbers, is approximated by taking the product of their approximating rational numbers. The product of irrational measurements can be found exactly with geometry using similar triangles.

Multiplication can be generalized to other mathematical objects, such as complex numbers and matrices. One of these objects, typically called the "identity" and denoted by "1," has the property such that if a is any object, then the product of 1 and a is a. The reciprocal of an object a, if it exists, is denoted by a^{-1} and is defined to the object so that the product, $a \times a^{-1}$ is 1. When the reciprocal of an object b exists, then the quotient of objects a and b is defined to be $a \times b^{-1}$.

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See Also: Addition and Subtraction; Magic; Mathematicians, Amateur; Number and Operations; Number and Operations in Society.

Music, Geometry of

See *Geometry of Music*

Music, Popular

See *Popular Music*

Musical Theater

Category: Arts, Music, and Entertainment.
Fields of Study: Communication; Geometry; Number and Operations; Representations.

Summary: Mathematical concepts and mathematicians have become interesting subjects of musical theater.

After the popular and critical success of Tom Stoppard's *Arcadia*, first performed in London in 1993, playwrights began making regular use of mathematics as source material for new scripts. This interdisciplinary collaboration, however, has largely been confined to stage plays and in the early twenty-first century has not found its way into musical theater—with one glaring and quite remarkable exception. In 2000, the husband-and-wife team of Joanne Sydney Lessner and Joshua Rosenblum created *Fermat's Last Tango*, a comic musical inspired by Princeton mathematician Andrew Wiles and his successful proof of Fermat's Last Theorem.

Fermat's Last Tango

Fermat's Last Theorem (FLT) is arguably the most famous mathematical problem in history. When Pierre de Fermat left his tantalizing note in the margin of his copy of Diophantus' *Arithmetica* in 1637, the result was a challenge that resisted the efforts of mathematicians for the next 350 years. By the twentieth century, FLT had acquired such a daunting reputation that when Princeton mathematician Andrew Wiles decided to take it on around 1986, he did not tell anyone what he was doing until seven years later, when he emerged from the office in his attic with what he thought was a proper proof of the Taniyama–Shimura conjecture. A proof of Taniyama–Shimura was known to imply FLT, and the unassuming Wiles was propelled to unprecedented stardom far beyond the mathematical community.

This event is the jumping-off point for *Fermat's Last Tango*. Because of the fictional liberties they take with the story, Lessner and Rosenblum have changed the name of their protagonist from Andrew Wiles to Daniel Keane, and the first major piece of revisionism we experience is when Keane is visited by a devilish and vindictive Fermat and whisked off to "the Aftermath" to fraternize with Pythagorus of Samos, Euclid of Alexandria, Isaac Newton, and Carl Friedrich Gauss. The fantasy is enjoyable, but what is really striking is how few liberties are taken with the mathematics. That the authors have done their homework is clear early on when Fermat rhymes "Shimura–Taniyama" with "algebraic melodrama." In the Aftermath, Fermat reveals that Keane has made some incorrect assumptions about the

Galois representations he used in his argument—which is indeed a mistake Wiles had made—and Keane retreats to his attic to try to repair the "big fat hole" in his proof.

Wiles, like Keane, was deeply uncomfortable trying to fill the gap in his proof under the glare of public scrutiny. The writers also keep the touching anecdote that Wiles promised his wife a corrected proof by her birthday, although it is unlikely that the real Ms. Wiles tried to lure her husband away from his research by crooning "Check out my modular form." Taken in the lighthearted spirit in which it was intended, *Fermat's Last Tango* is roundly successful entertainment. Beyond this achievement, it also comes as close as any other piece of science theater to effectively staging the "moment of discovery," creating a genuinely breathless moment when a defeated Keane finally realizes how to repair the hole in his proof using the Iwasawa theory approach he had abandoned several years earlier.

For those who do not have an opportunity to view a live production, a performance of *Fermat's Last Tango* was recorded and is available through the Clay Mathematics Institute. Others have staged *Fermat's Last Tango* specifically as a teaching experience. A 2007 article in *PRIMUS*, a publication dedicated to teaching undergraduate mathematics, describes a fully student-mounted production, along with suggestions for related educational activities. The play is cited as a good introduction to not only mathematics products but also the personalities of people and the processes involved in mathematics research.

The Natural Sciences

There does not seem to be any other piece of widely disseminated musical theater devoted to a mathematical topic, though certainly there are mentions of mathematics in various popular scores. For instance, in *Pirates of Penzance*, first performed in 1879, W. S. Gilbert and Arthur Sullivan include the following stanza in the famously tongue-twisting Major-General's song:

I'm very well acquainted, too, with matters mathematical
I understand equations, both the simple and quadratical
About binomial theorem I'm teeming with a lot o' news
With many cheerful facts about the square of the hypotenuse

Broadening the net to include the mathematical sciences brings into play the work of American composer Philip Glass. In 1976, Glass scored and wrote *Einstein on the Beach*, which was viewed as groundbreaking in several ways—one being that it was nearly five hours long with no intermission. The implication here was that audience members were expected to come and go as they so desired. In a similar vein, it was not plot driven but did contain many references to Einstein, including a musical event meant to suggest a nuclear explosion.

In 2001, Glass wrote the music for the opera *Galileo Galilei*, which tells the life story of Galileo in reverse. The opera opens with Galileo blind and on his deathbed, follows him back through his trial and astronomical discoveries, and ends with Galileo as a child attending an opera written by his father. Glass returned to the natural sciences a third time in 2010 when he wrote the music for *Kepler*, an opera that features Johannes Kepler as the only named character, although there are six other soloists and a chorus.

Glass did study mathematics early in his education before devoting himself wholly to music, and he readily admits to seeing mathematics and music as being linked—not just technically but artistically. "The beauty of mathematics is something that mathematicians talk about all the time," Glass said in a November 2009 feature for the *Wall Street Journal*. "And the elegance of a mathematical theorem is almost as good as its proof. Not only is it true, but it's elegant. So you get into almost aesthetic questions."

Kepler and Galileo are also the featured characters in a 2001 musical called *Star Messengers*, written by Paul Zimet with music composed by Ellen Maddow. A much more widely toured musical production was *Dr. Atomic*, written by John Adams with libretto by Peter Sellars. This opera tells the story of the Manhattan Project largely through the eyes of physicist J. Robert Oppenheimer, in part by borrowing text from government documents and interviews with scientists who worked on the bomb. First produced in San Francisco in 2005, *Dr. Atomic* has since been performed at multiple locations in Europe and the United States, including a live broadcast from the Metropolitan Opera in 2008.

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Stephen Abbott

See Also: Movies, Mathematics in; Plays; Television, Mathematics in; Wiles, Andrew.

Mutual Funds

Category: Business, Economics, and Marketing.
Fields of Study: Algebra; Data Analysis and Probability; Measurement; Number and Operations.
Summary: Many mathematicians attempt to develop mathematical models that forecast the future direction of the stock market and thus to produce better investment results for mutual funds.

Mutual funds are a type of investment in which large numbers of people pool their money and a fund manager invests these funds in one or more types of security. Investors own shares in the fund, and the value of those shares is determined by the total value of all the securities owned by the fund. Mutual funds are a popular investment vehicle because they allow people to achieve a varied investment portfolio with a relatively small investment, thus limiting their risk in comparison to buying individual stocks, bonds, or other assets.

Many types of mutual funds are available, depending on the desires of the investor. For instance, are they more interested in a riskier fund that may produce a higher yield for their investment, or a safer fund that is more likely to preserve the value of their capital? Some mutual funds specialize in a single type of investment—for instance, international stocks, health sector stocks, U.S. government bonds, or real estate—while others invest in a variety of securities in order to achieve a desired balance between yield and risk. Although mutual funds are often perceived as a safe investment, they are not guaranteed by the Federal Deposit Insurance Corporation (FDIC) as are bank deposits, and it is possible to lose money by investing in mutual funds. Economists, statisticians, actuaries, and others frequently try to predict the stock market using time series analyses and other mathematical methods. Prediction has historically proven to be quite challenging because of the complexities of time series data and the different socioeconomic variables and human psychological factors that appear to influence the stock market.

History and Growth

Although the first mutual funds were offered in the United States in the 1920s, the modern mutual fund industry dates from 1940 when the Investment Company Act established a body of rules regarding financial investments. In 1949, less than $2 billion were invested in mutual funds, but they became a more popular investment vehicle in the 1960s. By 1973, $47 billion was invested in mutual funds. By 1987, this amount had grown to $4 trillion, and by 2000, to $6 trillion, representing the investments of over 83 million investors. One factor in the growth of individual investments in mutual funds is the shift in the United States from guaranteed pension plans to retirement savings plans like the 401(k) in which an individual worker is responsible for choosing how to invest his or her retirement funds.

In 2008, there were over 8000 mutual funds in the United States versus about 3000 stocks listed on the NASDAQ stock exchange and a similar number on the New York Stock Exchange. It may at first be counter intuitive that there should be more funds than stocks, but this fact is not surprising if one considers any mutual fund as a composite made up of individual stocks or a subset of the total number of stocks (although, of course, a mutual fund may also include bonds and other components). Any set of n elements has 2^n possible subsets, so a set of 10 elements has 1024 subsets and a set of 25 elements has over 33 million.

Risk Minimization

One appeal of mutual funds is that they allow people to reduce their risk through diversification. Modern portfolio theory attempts to select assets to minimize risk, maximize return, or some combination of those two (in general, higher risk is associated with higher return, although this does not hold absolutely). The basic concept behind the theory is that stocks or other assets, such as bonds, in the fund are evaluated in the context of other assets, and the goal is to maximize return or minimize risk for the total collection of assets, called a "portfolio." American economist Harry Markowitz developed portfolio theory beginning in the 1950s, and in 1990, was awarded the Nobel Prize in Economics for this achievement.

Management

Because the performance of a mutual fund is often related to that of the economy as a whole, the performance of specific mutual funds as well as mutual funds as a class is often evaluated against the performance of indices such as the Dow Jones Industrial Average (a scaled average of the stocks of 30 large, publicly owned companies) or the S&P 500 (a weighted index of 500 large-cap common stocks). There are always pitfalls in making these types of comparisons; for instance, the return of mutual funds as a whole appears larger than it really is because funds that do poorly often go out of existence and are thus dropped from the average (survivorship bias). Interestingly, over time most individual funds produce somewhat worse results than a large index, such as the S&P 500, suggesting that the talent of individual managers (who choose when to buy and sell the stocks or other investments that comprise a mutual fund) are less efficient than the stock market as a whole. For this reason there are mutual funds today that are not "actively managed" in the sense that an individual manager makes buying and selling decisions. Instead, such funds simply own the stocks that comprise some index, such as the S&P 500, with buying and selling decisions motivated by changes in the makeup of the index (for instance, because of mergers or to new stocks joining or leaving the index).

This method is not a criticism of mutual funds per se but simply an argument for the efficiency of the market. Studies of the stock picks of professional analysts also tend to perform only marginally better than those selected randomly—most famously by throwing darts at a dartboard. Despite this well-known result, many individuals and investment firms have developed complex mathematical models that attempt to forecast the future direction of the stock market and thus produce better investment results. In addition, people have tried to predict the movement of the stock market with other types of data; for instance, in 2010, two graduate students found that the emotional content of tweets (messages sent on Twitter, a social networking Web site that can receive and send text messages from mobile devices, such as mobile phones) from the general public could be used to predict movement of the Dow Jones Industrial Average several days in advance.

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See Also: Forecasting; Mathematical Modeling; Money; Pensions, IRAs, and Social Security; Stock Market Indices.

N

Nanotechnology

Category: Architecture and Engineering.
Fields of Study: Geometry; Measurement; Number and Operations.
Summary: Nanoscience relies on mathematical modeling to predict the behavior of substances at the nanoscale.

Nanotechnology is a relatively new field of scientific study, the conceptual origins of which are typically credited to a presentation by physicist Richard Feynman in the late 1950s. A nanometer is one-billionth of a meter, and nanoscience focuses on matter with dimensions between 1 and 100 nanometers. For comparison, an ordinary sheet of paper is about 100,000 nanometers thick, a human hair is between 60 and 120 nanometers thick, and the diameter of one atom of gold is about 1/3 of a nanometer. Thus, nanotechnology is concerned with studying materials at a very small scale, ranging from roughly larger than a single atom at the lower end to objects that can be seen with a high-quality optical microscope at the upper end.

Physicists, mathematicians, and other nanotechnologists are often particularly interested in how the physical, chemical, and biological properties of materials may differ at this scale as opposed to properties of the same materials in bulk or at the scale of single atoms or molecules. Feynman discussed the notion that human beings would someday be able to create increasingly smaller and smaller machines, in part through directed, precision arrangement of atoms and molecules. He also introduced the idea that change in scale would affect the mathematical and physical properties of technology and processes. For example, relatively large-scale forces like gravity would begin to diminish in importance as machines grew smaller, while molecular-level van der Waals attractive forces, named for chemist Johannes van der Waals, and other properties would take on more important roles. However, he did not call his own ideas "nanotechnology." Instead, the first use of the term as it is typically meant in the early twenty-first century is credited to engineer K. Eric Drexler in the 1980s. He also helped spread nanotechnology and molecular manufacturing ideas to a broader audience. There are types of technology that are already being created at nanoscales. Some visions about the future of molecular manufacturing are much like the replicator device in the science fiction franchise *Star Trek*: human-scale or even larger objects, even complex devices like computers, quickly assembled atom by atom.

Any word with the prefix "nano" means at a nanometer scale (for example, the word "nanofilter" would refer to a filter at the nanometer scale), but there are also some basic classifications that are in common use. Nanomaterials are furthered classified as nanoparticles (if all three dimensions are nanosized), nanotubes (which have a nanosized diameter but greater length), and nanofilms or nanosheets (the thickness is nanosized, but the width and height may be much greater). Nanostructured materials have an internal structure that is nanosized, but the pieces of material may be much larger.

Principles

Nanotechnology draws on many scientific fields, including chemistry, physics, and biology, as well as engineering and materials science, and one common thread among them all is mathematics. Interestingly, the extreme difference in size between usual applications and applications at the nanoscale means that some of the most fundamental laws describing natural processes do not apply. For instance Ohm's law describes the flow of electrical current as

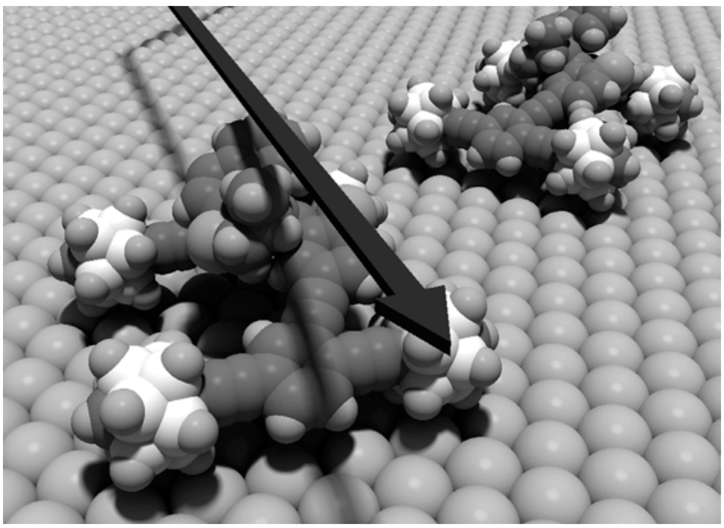
$$I = \frac{V}{R}$$

where I is the current in amps, V is the potential difference in volts, and R is the resistance of a conductor in ohms. This law is based on the free flow of electrons and hence does not describe the movement of electrons through nanowires, which may be so narrow as to allow only one electron to pass through at a time. To take another example, at the nanoscale, heat flow is no longer governed by standard continuity boundary conditions and different assumptions that allow for discontinuities must be used instead. Identifying and quantifying how such fundamental laws and expectations change at the nanoscale is one important field of study within nanotechnology.

Construction of systems at the nanoscale allows researchers great control over the form of the nanoparticles developed as well as the ways they form three-dimensional wholes. One line of research involves devising structures that require the minimum number of molecules for a given construct, while another involves developing self-assembling structures, such as cubes and buckyballs. Nanotechnology also adds new complications to issues of dimensionality. From elementary geometry, humans are accustomed to thinking in terms of one dimension (a line), two dimensions (a plane), and three dimensions (a cube, or any object in space). However, at the nanoscale the picture is not so clear. For instance, quantum dots or "artificial atoms" that contain only one or a few electrons with discrete energy states are zero-dimensional solids, which can function in quantum computers as a binary switch. Fractals, which are described by noninteger dimensionality (for example, a two-and-a-half-dimensional object) are also used to model nanoscale systems.

Applications

Medicine is one of the most promising fields for nanotechnology because many internal processes of the human body take place at nanoscale dimensions. Drug delivery is one promising field: nanoparticles can be used to deliver drugs directly to particular cells, for instance, for chemotherapy that targets cancerous cells but not healthy cells and thus reduces tissue damage. Nanotechnology has also developed ways to use nanoshells to concentrate heat from infrared light to destroy cancer cells with minimal damage to adjacent healthy cells. Nanotechnology promises to allow some drugs now delivered by injection to be taken orally, encapsulated in a nanoparticle, which would help it pass into the bloodstream from the stomach. Nanofibers have been used to repair damaged joints by stimu-



The world's first motorized light-powered nanocars made from only 169 atoms were built at Rice University in 2006.

lating the body's production of cartilage; nanoparticles have been used to increase the speed of blood clotting to prevent blood loss in trauma patients; and nanocrystalline silver is already being used as an antimicrobial agent for wound treatment. Nanocrystal technology is being developed to improve medical imaging, and in the future it may be possible to develop cell repair nanorobots, which could be programmed to repair diseased or damaged cells in a person's body.

Nanotechnology has many applications in the fields of energy production and pollution control. Nanotechnology has made it possible to create more efficient solar cells at a lower cost (making the technology more likely to be adopted) and provided new forms that make solar technology more convenient. For instance, solar cells created by embedding nanoparticles in plastic film can be incorporated into mobile phones and portable computers. Batteries created using nanotechnology can be made lighter and more powerful and can also be charged more quickly than conventional batteries, increasing the efficiency of hybrid automobiles. Nanofilters are increasingly being applied in food production, water filtration, and air pollution control, and nanoparticles are also used in some applications to absorb contaminants.

In manufacturing and construction, nanotechnology has led to the development of new materials that are lighter, stronger, and possess more desirable properties than their conventional analogues. For instance, nanomolecular structures are already being used to make concrete and asphalt more resistant to water, and nanomaterials added to light-emitting diode (LED) lighting makes them more resemble standard lighting, allowing the incorporation of more efficient LED lights in home and industrial use while retaining the look of traditional lighting. Nanocoatings are commercially available that resist corrosion, offer insulation and UV protection, and can remove pollutants from a building's atmosphere.

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Sarah Boslaugh

See Also: Chemotherapy; Molecular Structure; Personal Computers; Water Quality.

National Debt

Category: Government, Politics, and History.
Fields of Study: Data Analysis and Probability; Measurement; Number and Operations.
Summary: The accumulation of federal government budget deficits over time is the national debt, which is best considered relative to GDP or other factors.

Mathematician Richard Feynman once said, "There are 10^{11} stars in the galaxy. That used to be a huge number. But it's only a hundred billion. It's less than the national deficit! We used to call them astronomical numbers. Now we should call them economical numbers." In modern society, entities from individuals through governments need money to function. Most government funds are generated by taxing individuals, businesses, goods, and services.

At the same time, governments must spend money for various purposes. If a government has more income than expenditures in a given fiscal period, usually one year, the excess of income over expenditures is called a "surplus"; if a government has more expenditures than income, the excess of expenditures over income is called a "deficit." The sum of all of these single-year surpluses and deficits over the entire history of the federal government is called the "national debt." Mathematics has long been used to quantify expenditures, deficits, and debts. Taxation and deficiency problems were mentioned in the Chinese mathematical text *The Jiuzhang suanshu* (Nine Chapters on the Mathematical Art), and Indian mathematician Brahmagupta referred to debts to mean what are now called "negative numbers." William

Playfair created some of the earliest graphical representations of social and economic data around the time of the American Revolution, such as trade balances between England and other countries and the English national debt. By the twentieth century, mathematical measurement, estimation, and modeling were increasingly used. Standard economic measures like gross domestic product (GDP) were common, and there were theories and research on principles like return on capital, interest rates, and exchange rates, many of which cannot be known with certainty. Stochastic modeling, random walks, particle theory, and Brownian motion, named for botanist Robert Brown, have been used extensively in the mathematical modeling of financial processes. After events like the Wall Street crash of 1929, there was also interest in forecasting models that could warn of debt crises. Mathematicians continue to research and create models to address both historic and new financial concerns, and many people have created representations such as the national debt clock and deficit calculators to extrapolate trends. Others argue against too much aggregation or extrapolation in mathematical models, citing inherent data collection errors in large-scale indices, like the consumer price index and gross national product, as well as subjectivity in individual perception and often-complex interactions between variables such as debt, deficit, production of good and services, and allocation of consumer resources.

Inflation

National debt differs uniquely from individual debt in the fact that governments usually have the power to print more money to pay debts. However, doing so often leads to undesirable economic consequences. More money in circulation can lead to increased demand for goods and services, which in turn may lead to inflation. Mathematicians and economists study inflation trends and cycles, as well as the reciprocal impacts of inflation on factors such as labor costs. While most economies function reasonably well with some level of inflation, too high a level of inflation leads to a host of problems, including hoarding of goods, increases in interest rates for credit and loans, and trade deficits with other countries. For this reason, most governments borrow rather than print the money to finance debt. In the United States, borrowing is accomplished primarily by selling government bonds. The purchaser, who may be an individual or another country, pays for a bond at the time of sale, and in return is promised a future amount of money, sometimes with interest payments made before the end of the bond period. The United States pays interest on its national debt bonds, which can be significant. For example, in 2009, interest on the national debt was $260 billion, approximately 8.5% of that year's federal budget and the fourth-largest single expense.

Federal Reserve System

The Federal Reserve System (sometimes called the "Fed") is the national bank of the United States and is independent of other United States institutions, including the Treasury Department. While it is not directly related to administering the deficit or to making decisions on government spending, it helps to manage the money supply in the United States by facilitating the lending of money between banks and by lending money to banks directly, which in part determines the interest rates that banks charge for borrowing money. These interest rates in turn influence the rates on the Treasury bonds that finance deficits. Many mathematicians and actuaries work for the Federal Reserve. For example, mathematician and Federal Reserve board member (as of 2010) Gary Anderson and economist George Moore developed the Anderson–Moore algorithm for solving linear saddle point models, which are used in economic modeling.

Intentional Debt

Having a large national debt poses many risks to an economy. A large national debt can help contribute to inflation and can lead to tax increases. Economists have also determined that GDP tends to grow more in an economy with a moderate level of national debt than in one with a high level of debt. Many variables affect spending, deficit, and debt. For example, governments often run deficits during economic recessions or depressions, spending money to attempt to stimulate the economy, partly under the notion that future gains will compensate and yield a positive long-term average or expected value. In 1900, the national debt in the United States was $2.6 billion and experienced overall nonlinear growth approaching the twenty-first century. Mathematical analyses have shown that debt increased sharply during World War I, while in the 1920s national debt decreased due to surpluses. It increased sharply again during the 1930s because of the Great Depression. Another increase occurred with spending for World War II. By 1950, the U.S. national debt had grown to $256.8 billion. After several relatively small increases, the national debt grew quickly beginning in the mid-1970s. Using exponential regression, mathematicians and economists have estimated that the national debt was doubling approximately every six years during this latter period up to nearly the end of the twentieth century. Projective models extrapolate such trends to estimate debt, often based on other estimated values, like the future population.

Debt Compared to GDP

In the same way that individuals can afford to spend more money when they receive a raise in salary, it can be misleading to look at the dollar amount of the federal debt without considering the overall size of the economy and the time value of money. For this reason, economists often evaluate the economic health of governments by considering national debt as a percentage of the country's GDP. In the United States in 1940, the national debt was 52.4% of the GDP. This number increased during World War II to 121.7% in 1946, meaning that national debt was actually larger than the GDP, but fell below 100% again in subsequent years. Mathematicians and economists have created models to forecast this index, with some predicting that factors like the housing and financial crises will cause the United States to once again pass the 100% threshold in the twenty-first century.

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Pete Johnson

See Also: Forecasting; Gross Domestic Product (GDP); Pensions, IRAs, and Social Security; Sales Taxes and Shipping Fees.

Native American Mathematics

Category: Government, Politics, and History.
Fields of Study: Connections; Geometry; Measurement; Number and Operations; Representations.
Summary: Native Americans developed numbering systems and had a clear sense of dimension, geometry, and probability.

The term "Native American mathematics" is deceptive because there is no single culture for all Native Americans. Rather, each of more than 400 Native American tribes has its own distinct culture, with each mathematical element being specific to that culture. Nonetheless, in an examination of mathematical aspects, it is possible to discuss some commonalities across the many tribes, producing evidence of multiple number systems, arithmetic operations, geometry, and probability.

Number Systems

Native American numbering systems often used a simple grouping system that corresponded to different parts of the human body. For example, the idea of "tens" is contained in the numbering system of the San Gabriel Indians in California, where "all my-hand finished" represented the number 10, "all my-hand finished and one my-foot" represented the number 15, and "another finished my-foot the side" represented the number 20. It is inferred that they used single fingers on each hand to represent any number less than 10. Often, a Native American tribe would have names for large numbers, but had little use for such in their daily lives. For example, Michael Closs, a cultural historian, describes a Copper Eskimo elder while relating a story about two men who, trying to settle an argument, begins to count the hairs on a wolf and a caribou. The story ends with the count unfinished, as both men die of starvation. And, the story concludes with the phrase: "That is what happens when one starts to do useless and idle things that can never lead to anything."

Though using groupings of 5 and 10 as the structure for their number systems, the idea of a number base is not always evident. Also, some evidence exists for the use of 2, 4, and 20 as the structuring element. For example, the Yukis tribe in northern California used a combination of the quaternary (base four) and octal (base eight) systems. In turn, their counting mechanism depended on referring to the four spaces between the fingers on both hands, not the fingers themselves.

In a study of North American Native Americans, researchers documented the use of 307 different number systems; 33% were base 10, 33% were base 5, 23% were base 2, 10% were base 20, and the remaining 1% were base 3.

In any discussion of Native American mathematics, it is necessary to include the Aztecs, Incas, and Mayans. For example, the Aztecs' number system was based on the number 20, with the numbers 400 and 8000 given special significance. In contrast, the Incas used a slight variation of the base 10 system, and even had specific words for the numbers 1–10, 100, 1000, and 1,000,000. Finally, the Mayans, the most mathematically sophisticated of the three, had a vigesimal system using the number 20 as its base. The Mayan system also included special notations for multiples of numbers and used a special symbol glyph to represent zero.

Arithmetic

The idea or use of arithmetic operations was not something needed by early Native Americans, who depended on a hunting-gathering culture. Historians suggest that any signs of significant arithmetic are due to a tribe's interactions and trade with the early fur traders or buffalo hunters. For example, the language of the Navajo does not include words for "multiply" or "divide," yet that should not imply their inability to perform either process computationally or using real items.

Evidence of addition is found in words used to denote different numbers, using a process of addition by juxtaposition. For example, Alaskan Natives living near the Yukon River essentially used the words "five one" and "five four" to denote the numbers 6 and 9, respectively. In direct contrast, the Miluk Coos, an Oregon tribe, used subtraction by juxtaposition, where "four ten" and "one ten" denoted the numbers 6 and 9, respectively. Some historians claim that 40% of the Native American tribes used some version of this subtraction process, especially for numbers close to multiples of 10.

Evidence of multiplication is found among Pawnee tribes, in a very creative fashion. Their term "50 persons" represented the number 1000, based on their use of the word "man" for the number 20, knowing that "man" had 10 fingers and 10 toes. Thus, "50 persons" was equivalent to 50 sets of 20 fingers and toes, or a total of 1000.

Measurements

The measurements invoked by Native Americans were context-sensitive and personal in nature. No standard units were established and used widely either within a tribe or across tribes. In most instances, the measurements used were specific to the context and informal. The Ojibwa tribe is a good example. For short lengths, their units were finger widths, hand spans, forearm lengths, and arm spans, while their longer lengths might reflect a changing position of the sun or even mention the unit "number of sleeps" involved in traversing a long distance.

Geometry

Native American geometry is evident in the colorful decoration and intricate patterns found on knife cases, moccasins, blankets, pouches, baskets, and pottery. At first, many of these patterns were created using porcupine quills but eventually the shift was made to using glass beads.

When creating a pattern, the different Native American tribes differed in their use of geometrical structures. In some instances, a tribe's members created irregular floral patterns, while other tribes used a geometry based only on straight lines, allowing them to create blocks, crosses, and triangles. The types of triangles ranged from isosceles to equilateral to right, with common traits being tall isosceles triangles or pairs of reflecting congruent triangles. Occasionally, circles and

Figure 1. Native American pattern.

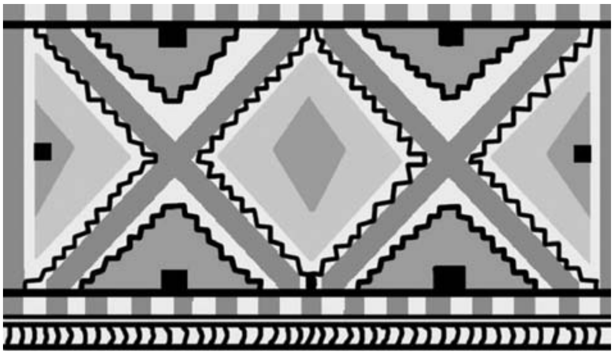

Figure 2. Navajo blanket.

spirals appear as part of a design.

Many studies have focused on Native Americans' use of symmetry in strip patterns using beads. Of the seven possible symmetry groups, the most popular pattern is labeled "pmm2" in standard transformational schemes, which means that the pattern has horizontal, vertical, and rotational symmetry. Figure 1 shows this pattern.

Also, in the process, the creator of the visual pattern possibly used counting or even some computing skills (for example, skip counting by threes to form a border). It is possible that creators of some of the patterns included elements of measurement (perimeter or area), number theory (multiples and divisors), and fractions (common, decimal, and ratios).

Tiling patterns are evident in the creation of blankets, going beyond strip patterns. An example is the section from a Navajo blanket in Figure 2.

Some historians claim tiling elements are also found in some of the Native American petroglyphs carved on the surfaces of caves, cliffs, and large stones.

Finally, Native Americans had a clear sense of dimension, using objects to represent the three possibilities. A stick represented dimension one, an animal skin represented dimension two, and an apple or walnut represented dimension three. However, in their paintings on flat surfaces, the idea of dimensional perspective is not utilized.

Probability

Elements of probability are found in some of the children's games played by various Native American tribes. For example, consider the Apaches' "Throw Sticks" game involving two or more people. In one version, three sticks are decorated with colorful designs on one side only, called the "face." The sticks are held in one hand and then dropped on the ground. The scoring is as follows: 10 points for three faces up, 5 points for two faces up, 2 points for 1 face up, and 1 point for no faces up. The score is kept by moving small sticks or "horses" around a circle of 30 stones. Play continues until someone travels the full circle. Elements of probability, such as likelihood, events, and dice-like actions, are all evident in this game.

Native Americans also played dice games, using dice made from bone, peach stones, deer horn, beaver teeth, or walnut shells. As most of these dice were two-sided, one side was colored to distinguish the two sides. When sets of dice were thrown, the scoring was based on the number of a given side appearing. Because the "dice" were crudely made, the chances of each side appearing are not equal. This observation actually validates the claim that Native Americans had a good sense of probability, because the higher score values were assigned to the least probable events.

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Jerry Johnson

See Also: Babylonian Mathematics; Basketry; Incan and Mayan Mathematics; Measurements, Length; Measuring Tools.

Nervous System

Category: Medicine and Health.
Fields of Study: Algebra; Number and Operations.
Summary: Mathematicians use a variety of mathematical modeling techniques to map and analyze nervous systems.

Human beings and many animals have two systems that are responsible for regulating and coordinating the activities of the body: the nervous system and the endocrine system. The first provides extremely fast responses, like reacting to touching a hot stove. The second responds more slowly and continuously, such as regulating blood sugar after a meal. Both systems work by detecting internal and external variations, such as shapes, odors, or temperature, to maintain the balance of body functions. Neuroscience, which is the study of the nervous system (including the brain) and its functions, is an interdisciplinary field that draws concepts and methods from many fields such as mathematics, psychology, biology, physics, and medicine. The Hodgkin–Huxley equations, named for Alan Hodgkin and Andrew Huxley, are fundamental to the development of mathematical models and simulations that have long been the basis of many experiments to study the nervous system. Many neuroscience researchers and teachers use the open source NEURON computer simulation system, which incorporates systems of equations and computational algorithms to mathematically model and display the behavior of individual neurons or networks of neurons in a dynamic way that is often difficult or impossible to achieve in traditional laboratory experiments.

Nervous System Processes

Everyday situations can highlight the complex action of the nervous system. For example, in a soccer match, players anticipate the opportunity to act. At the exact moment the ball is thrown in a player's direction, thousands of nerve connections start to become active. In milliseconds, the player begins to use sensory memories and visual information to immediately decide the best course of action, such as to kick the ball to another player or directly to the goal. The central nervous system consists of the brain and spinal cord. The brain is the control central of the nervous system. The spinal cord conducts electrical signals between the brain and various nerves throughout the body, and controls some reflex functions. Neurons are cells that propagate the electrical impulses in the nervous system, and glial cells help maintain parts of the nervous system. For example, they produce myelin, which coats many neurons like insulation in electrical wiring. The neurons have important properties, such as excitability and conductivity, and act similarly to an electric current transmitted along a wire. This phenomenon occurs because of permeation of ions, such as sodium and potassium, through the neural membrane, which generate an electrical signal that propagates between neurons via its branched structure, consisting of thousands of small extensions.

Early Research

Nerve impulse propagation and the nervous system processes have been researched for many years using theories and techniques from genetics, molecular biology, physiology, psychology, and mathematics, among others. In the 1950s, physiologists and biophysicists Alan Hodgkin and Andrew Huxley experimented on the nervous systems of squids, specifically on a structure known as the "giant axon." An axon transmits electrical impulses in the nervous system, and a squid's giant axon can be up to 1 millimeter in diameter, much larger than most axons and visible to the naked eye. These experiments led to the development of the Hodgkin–Huxley equations, which are nonlinear ordinary differential equations that describe or approximate the electrical characteristics of neurons and other electrically excitable cells, such as those in the heart. They involve concepts like gates (channels that allow the ions to flow), voltage thresholds, and conductances, which act together to determine if and when a neuron "fires" an electrical burst. They are very similar to electric circuit theory, and some models of nervous systems look very much like electrical circuit diagrams.

Other Mathematical Connections

Hodgkin and Huxley won a Nobel Prize for their experimental and mathematical work, which has since led to other mathematical explorations of the nervous system. The nervous system in mammals is a very complex dynamic system, with many interconnected components. Periodic rhythms are found in some types of movement-related behaviors that are governed by the nervous system, like walking and breathing. They are also related to sensation and cognition.

Studies of all these various substructures involve not only understanding how each structure behaves on its own but also how they interconnect and communicate with one another. Because of the vast degree of intercorrelation among various nervous system structures, from individual neurons to larger structures like the brain and spinal cord, one challenge facing mathematical modelers is creating systems of equations that optimize the ability of the equations to realistically represent neuronal systems and their behaviors while making them tractable for computation and interpretation. One of the interesting mathematical phenomena that researchers study is called "gamma and beta rhythms." These brain waves have been connected to so-called "higher" mental activity, like perception and consciousness, as well as to synchronous activity that may help link various sensory inputs into a single mental construction of an object. However, many questions remain. Techniques such as graphs, circuits, networks, clustering, geometry, and simulation all play a role in investigation of nervous system properties and functions.

One additional important advance in neuroscience is the neurochip. It can be used to help link biological neurons and semiconductor materials, which may one day help to create prosthetics that integrate fully into the body's own neural system. They may also facilitate treatments for neurological diseases like Alzheimer's and Parkinson's.

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Maria Elizabeth S. Rodrigues

See Also: Brain; Mathematical Modeling; Medical Simulations; Neural Networks.

Neural Networks

Category: Communication and Computers.
Fields of Study: Data Analysis and Probability; Number and Operations; Representations.
Summary: Artificial neural networks use sophisticated mathematical algorithms and computational functions to simulate biological neural networks.

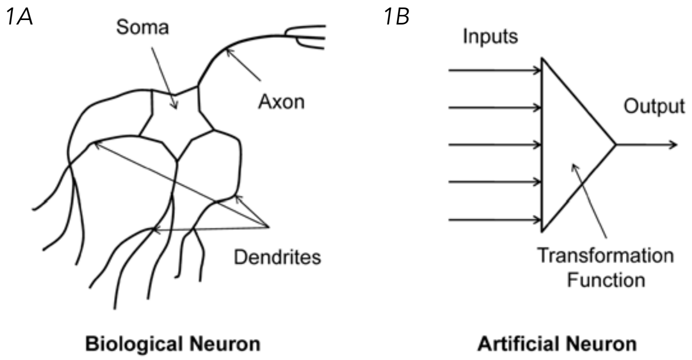
The term "neural networks" is generally applied to the systems of biological or artificial neurons. More often it is used in application to artificial neural networks that are designed to reproduce some human brain functions, such as information processing, memory, and pattern recognition. However, this term is also used for biological neural networks, for which the term "neural system" is more common. The beginning of modern neural network research is credited to neuroscientist Warren McCulloch and mathematician Walter Pitts in 1943. McCulloch had spent decades pondering the logic of the nervous system (for example, what allows people to think or feel) before beginning his collaboration with Pitts. He specifically credited Pitts's knowledge of modular arithmetic for the success of their joint work, which produced the McCulloch–Pitts Theory of Formal Neural Networks. Their research suggests that any computable function can be completely realized by a McCulloch–Pitts artificial neural network, though some such networks would be impractically large.

Artificial Neural Networks

Artificial neural networks are mathematical tools or physical devices that function similarly to biological neural systems. They consist of building blocks, called "artificial neurons," which resemble the structure of real neurons. Each biological neuron includes three major parts: dendrites, soma, and axon (see Figure 1A). Correspondingly, each artificial neuron also consists of three major parts: inputs (or "dendrites"), transformation function ("soma"), and output ("axon") (see Figure 1B). The terminology that is generally used for biological neurons is also often applied to artificial neurons.

Modern neural networks use data analysis and nonlinear statistical methods to model complex relationships between inputs and outputs or to find patterns. Bayesian methods of inference, named for Thomas Bayes, are increasingly employed. Graph theory and

Figures 1A and 1B. Biological and Artificial Neurons.



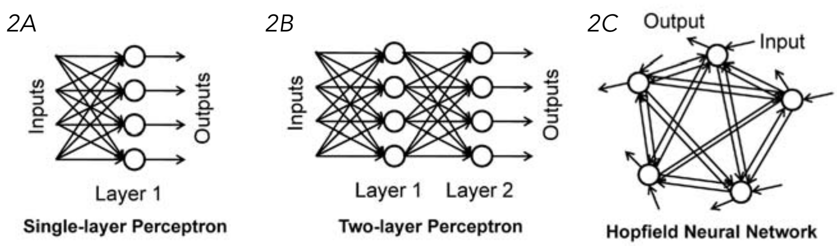
geometry are also very useful for mapping neural networks, assessing their capabilities, and studying pattern classification. Artificial neural networks are applied to a variety of problems in science, industry, and finance in which people must draw conclusions and make decisions from noisy and incomplete data. They can perform pattern recognition and pattern classification, time series analysis and prediction, function approximation, and signal processing. Several types of artificial neural networks were developed for the specific problems for which they can find the best solution. The most famous of them are single- and multi-layer perceptrons; Hopfield neural networks, named for John Hopfield; self-organizing Kohonen maps, named for Tuevo Kohonen; and Boltzmann machines, named for the Ludwig Boltzmann distribution. Regardless of the type of neural network or the problem it is designed to solve, the output is some mathematical function of the inputs, often involving probability distributions. As examples, consider functions of the three types of the artificial neural networks represented in Figure 2.

The first, single-layer perceptron consists of one layer of artificial neurons and was designed for pattern recognition and classification problems (see Figure 2A). The input pattern of signals s_i is fed to each neuron in the perceptron with different weights, w_{ij}. Then the signals are added in each *j*th neuron to form a weighted sum $\Sigma_i w_{ij} s_i$, which is processed by a transformation (nonlinear) function, resulting in a pattern of the output signals o_j. Thus, the pattern of output signals o_j is determined by the set of weights w_{ij}, and this set of weights forms a memory in the neural network. To obtain desired response pattern d_j to a given input pattern s_i, the perceptron is required to be "trained." Training (or learning) procedure consists of the method that adjusts neural network weights w_{ij} that form desired output pattern d_j.

Because of limited capability of single-layer perceptrons (for example, they cannot reproduce "exclusive OR" logical operations), the multilayer perceptrons (see Figure 2B) became very popular for different problems in pattern recognition and classification. Inclusion of one or more "hidden" layers into the neural networks increased their learning capability and performance. Multilayer perceptrons are learned by so-called backpropagation algorithm that changes weights w_{ij} in all layers to ensure desired output in the last layer.

Both single-layer and multilayer perceptrons belong to a class of feedforward neural networks, as connections between the neurons do not form closed loops (see Figures 2A and 2B), and information transfers only in one direction, from the input to the output. A Hopfield neural network is a representative of another class, recurrent artificial neural networks, with bi-directional flow of information (see Figure 2C). Each neuron in this network is connected to the others with symmetric bidirectional connections, and its output is calculated in a way similar to that for perceptrons. A Hopfield neu-

Figures 2A, 2B and 2C. Artificial Neural Networks.



ral network runs by cycles. During one cycle, the output of each neuron is calculated using external inputs and neural outputs from the previous cycle. These neuronal outputs become their inputs, with corresponding weights and transformation function, during the next cycle. Neural outputs are recalculated for each cycle until the system reaches a steady state. This steady state pattern of neural outputs represents a stored pattern in the Hopfield neural network. Information in Hopfield neural networks, as in perceptrons, is stored in the weights, w_{ij}.

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See Also: Nervous System; Parallel Processing; Robots.

Newman, Ryan

Category: Games, Sport, and Recreation.
Fields of Study: Connections.
Summary: Ryan Newman is a NASCAR race car driver who credits his success, in part, to his engineering background.

Ryan Joseph Newman, National Association of Stock Car Auto Racing (NASCAR) great and structural engineer, was born on December 8, 1977, in South Bend, Indiana. He attended Purdue University and in 2001 earned a bachelor's degree in vehicle structure engineering, which has since benefitted him throughout his illustrious auto racing career and pursuit of a NASCAR Sprint Cup Championship.

Early Career

In 1993, at the age of only 16, Newman made his auto-racing debut in the United Midget Auto Racing Association (UMARA) and the All-American Midget Series (AAMS). Success came quickly to young Newman as he won not only Rookie of the Year honors but the AAMS Championship. During these early years, Newman amassed over 100 wins in these and other divisions. His rapid rise and prolific success behind the wheel earned him a step up to the nationally acclaimed United States Auto Club (USAC) Series in 1995, where he competed in racing competition at various levels. Again, he was recognized with Rookie of the Year honors. In 1999, he became the first driver in USAC history to win races in a midget, sprint car, and Silver Crown car. He also won the Silver Bullet Series Championship that year.

Engineering Skills

It was also during this time that Newman was studying vehicle structure engineering at Purdue University in West Lafayette, Indiana, and in 2001 he earned his bachelor's of science degree in this field. His engineering skills have been useful in fuel management, understanding the geometry and physics of each race track, the design of race cars, and more generally in time management and problem solving. He stated, "I've always said that an engineer, every time he gets one answer, he gets two additional questions, which is easily 'How' and 'Why.' I think that for me, it has made my career more successful being an engineer."

This training proved valuable in Newman's chosen profession, and as he continued to win on the track, people took notice. One observer, in particular, was racing legend Roger Penske, who asked Newman to drive his cars in NASCAR's ARCA Series and Busch Series. Newman experienced immediate success, winning three of his first five races, and in a matter of weeks it became clear he was well on his way to NASCAR's premier division: the Sprint Cup Series. He made his Sprint Cup Series debut at Phoenix International Raceway in 2000. By 2002, Newman had won six pole positions and his first race in the Sprint Cup Series, also at Phoenix International Raceway, and was well on his way to yet another Rookie of the Year Award. It was during this

Ryan Newman stands beside his Number 39 Army Chevrolet Impala in New Hampshire in 2009.

time, in part because of his rapid rise through the NASCAR ranks and race-qualifying prowess, that he earned the nickname "Rocket Man."

Sprint Cup Series Success

Newman has been among the most popular and consistent competitors in NASCAR's Sprint Cup Series each season since 2002, winning 14 races as of 2010, the most notable being his 2008 Daytona 500 victory. In 2009, he joined Stewart-Haas Racing, driving the No. 39 U.S. Army Chevrolet. Since making the move to Stewart-Haas, Newman has won two pole positions as of 2010 and has finished a race in the top 10 no less than 15 times. His success has kept him in the hunt for the series championship. In his nearly 300 starts in the Sprint Cup Series, Newman has qualified for the pole position 46 times, earning at least one pole position each year since 2001. At this pace, he is well on his way to breaking into NASCAR's top 10 pole position winners of all time, placing him among many of auto racing's elite and Hall of Fame drivers.

Newman has brought a new perspective to NASCAR racing, showing fans that scientific knowledge can play a major role in success on the track. Newman is often asked about engineering in interviews, and in this context, he regularly critiques and analyzes the pros and cons of changes in racing. Some of his comments have been controversial, such as those related to the original moon landing. Newman's crew chief, Matt Borland, also possesses an engineering degree, and mathematical conversations are commonplace with the crew. As Newman told *Sports Illustrated*, "It's created a common language for me and the crew because there's at least three other guys on our travel team that are engineers alongside of the engineers that we have in a group back in the shop. So we have that common language." Newman's success opened NASCAR to engineering specialists, which has brought significant changes to the world of NASCAR. In 2005, Newman and wife Krissie founded the Ryan Newman Foundation, where its mission, in part, is to provide college scholarship funding to students interested in auto racing careers.

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See Also: Arenas, Sports; Auto Racing; Engineering Design; Extreme Sports.

Nielsen Ratings

Category: Arts, Music, and Entertainment.
Fields of Study: Communication; Data Analysis and Probability; Measurement.
Summary: Television viewing data are estimated using metrics collection and statistical modeling.

The Nielsen Ratings are a measure of how many people are watching certain television programs. When

Arthur Nielsen began measuring television viewing in 1950, there were three networks and about 9% of U.S. households had a television. In the twenty-first century, homes often have multiple televisions receiving scores of channels. Even when the number of television sets was small, it was not possible to gather complete viewing data for every single person who owns a television. Instead, Nielsen Media Research uses statistical sampling methods to take a representative subset of viewers and then extrapolates from the sample's viewing activities to the whole population of viewers. The statistical methods Nielsen uses to collect its data have been refined several times in response to changes in viewer behavior. People who develop and analyze ratings typically have expertise in analytics, metrics, and statistical modeling.

Advanced statistical methodologies, like data mining and software such as Mathematica, are used to extract patterns from Nielsen data that help explain which segments of the population view particular shows. Networks make decisions about whether to cancel or renew programs based on Nielsen ratings. Companies also use Nielsen audience estimates to allocate tens of billions of television advertising dollars each year.

Statistical Sampling and Data Collection

No one knows exactly how many households have televisions, but 2010 estimates suggest that the average U.S. household has just fewer than three televisions. Using statistical sampling, Nielsen can obtain representative data using small a proportion of households: approximately 9000 in its national sample, another 1000 in its Hispanic sample, and various smaller amounts in selected local markets. For the national television sample and major local markets, "people meters" record what television shows household members watch using an electronic set meter, along with a remote control that distinguishes each individual member of the household. Set meters are also used to collect data in mid-sized local markets, but with paper diaries for individual demographics. Meters transmit data to Nielsen every night, where it is checked mathematically for transmission or recording errors before analysis. In the smallest markets, viewers record programs in paper diaries and mail them to Nielsen. Historically, Nielsen tracked only television programs that were viewed live at the time they aired. However, people are increasingly using digital video recorders (DVRs), streaming video, and other delayed viewing technologies, which biases live ratings and affects both programming and advertising decisions. Nielsen began adding DVR households to its sample in 2006 and now regularly reports same-day and seven-day DVR playback ratings as well as its traditional live viewer ratings. People's failure to return paper diaries is also a growing source of bias, and research methodologists are working on revising this method to make completing the diaries easier to encourage greater response.

Television Metrics

Nielsen's primary metrics for television viewing are rating, share, and projected audience. A program's rating is a percentage that represents the number of households that watched the program out of the total number of households that could have watched the program. In this case, the denominator of the fraction is fixed according to the Nielsen sample size. The 1983 finale of the television show *M*A*S*H* holds the record for highest Nielsen rating, 60.2, which means slightly more than 60% of possible sample households tuned in to watch. At the time there were about 83 million television households, so one sample rating point represented 1% or 830,000 households in the population. However, it would be very unusual for every household to be watching television at the same time.

Share adjusts for this fact by computing the percentage of households that watched a specific program out of the number of households that were actually watching television during that time. This is a more complicated calculation, since the number of televisions being used at any given moment changes constantly. Shares are often used to measure how competitive a program is in its particular time slot. The *M*A*S*H* finale had a 77 share, which means 77% of households watching television at all were tuned to that program. Ratings and shares are also computed for several age, race, and other subgroups, as these are very important to advertisers. Since data are recorded at the household level—and many people may watch the same program in one house, or outside the home in places like sports bars or dorms—the number of individual viewers in a subgroup or population can only be estimated from the demographic data recorded by people meters and diaries. In 2007, Nielsen also began to measure college students' viewing habits by treating them as if they were watching an additional television set at home.

Projected audience is the estimated number of people reached in the overall population, which is calculated using statistical modeling. The *M*A*S*H* finale had a projected audience of 106 million viewers. The number of television households grows every year, as does the number of channel choices, so it can be difficult to compare ratings from across years, especially over large stretches of time. For example, although Super Bowl XLIV surpassed the *M*A*S*H* finale in terms of estimated viewers (106.5 million), it had a lower rating (46.4). Another reason that the numbers may be difficult to compare is that Nielsen produces rapid overnight ratings for many media outlets and these values are later adjusted. Further, only selected numbers are made public, such as the daily or weekly top 20 shows, which vary from week to week. Comprehensive data is generally available only to Nielsen's clients, and networks may advertise only the statistics that are most favorable.

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See Also: Randomness; Rankings; Sample Surveys.

Normal Distribution

Category: History and Development of Curricular Concepts.
Fields of Study: Calculus; Communication; Connections; Data Analysis and Probability.
Summary: Better known to laymen as the bell curve, there are many applications for normal distribution.

The normal distribution is one of the most useful and important probability distributions, with a wide range of theoretical and real-world applications. Many people know the normal distribution primarily by its colloquial name, the "bell curve," which comes from its characteristic shape: a symmetric curve with a pronounced peak in the middle and diminishing tails. Mathematically, normal distributions are a family of continuous probability distributions. The normal function has no closed-form integral, but areas under the curve, which correspond to probabilities, can be accurately approximated with methods like numerical integration. All normal probability distributions display the same symmetric bell shape, but can have any real-valued mean (μ) and positive real-valued standard deviation (σ). The standard normal distribution is a special case with a mean of zero and standard deviation of one. All normal distributions can be transformed or standardized to the standard normal, which is theoretically important and extensively tabulated. Computers and calculators also allow direct calculation of normal probabilities. Students often use both technology and tables when they study the normal distribution in high school and beyond.

Many naturally occurring phenomena are normally or approximately normally distributed, like the heights of adult human beings. In other cases, such as intelligence tests, the measurements are purposely structured or scaled according to this distribution. Several other probability distributions converge to the normal distribution or are well approximated by it. The central limit theorem, based on normal approximations, is the foundation for a wide range of commonly used statistical procedures, particularly for estimation and inference. Another common name for the normal distribution is the "Gaussian distribution," after Carl Friedrich Gauss, whose work significantly advanced many statistical theories and concepts. Occasionally it is referred to as the "Laplace distribution," after Pierre-Simon Laplace. The variety of names for the normal distribution likely reflects the debate on the origins of the term "normal distribution" and the breadth of people who influenced its development.

History

The first appearance of the term "normal distribution" in a published document is often credited to a seminal paper from Karl Pearson in 1895. However, there are some who say the first use corresponds to

Galileo and Bernoulli

The early origins of the normal distribution can be traced in part to Galileo Galilei and his work in astronomy. In 1610, Galileo noticed that the measurement errors in astronomical tables were distributed symmetrically (in an unbiased fashion) around the correct value.

A century later, Jacob Bernoulli made two critical advances toward the development and characterization of the normal distribution. The first was the "law of large numbers" (named as such by Simeon Poisson in 1835). The second was the development of the binomial distribution. The law of large numbers predicts the convergence of sample means to the true population mean as sample size approaches infinity. The binomial distribution models probability in situations in which there are sequences of independent random events with two equally likely outcomes for each event, such as flipping a coin.

Charles Peirce in 1783, to Francis Galton in 1889, or to Henri Poincaré in 1893. Statistician and historian Stephen Stigler believes that it might have been used much earlier, and there is certainly evidence to support that assertion.

Abraham DeMoivre is credited with the first mathematical derivation of the normal distribution in his 1733 work *Approximatio ad summam terminorum binomii (a+b)^n in seriem expansi*. Using sums of Bernoulli´s binomial random variables, he approximated a continuous distribution to the discrete binomial using integral calculus, which resulted in a bell-shaped continuous distribution. Continuing this idea, Pierre-Simon Laplace presented the central limit theorem in 1778, which is also sometimes called the "DeMoivre–Laplace theorem." In fact, the name "central limit theorem" is credited to George Pólya's 1920 work on the normal distribution. Since the central limit theorum is the limit of a summation of binary variables, it is applicable to both discrete and continuous random variables. It has many real world applications along with its theoretical importance, and it is fundamental to statistical inference.

Robert Adrain, an American, and Carl Friedrich Gauss, a German, worked simultaneously on similar notions at the start of the nineteenth century without being aware of each other's work. In 1808, Adrain presented arguments regarding the validity of the normal distribution for describing distributions of measurement errors, inspired by a real-world problem in surveying. He used this initial work to further develop and prove Adrien-Marie Legendre's method of least squares. Gauss published his *Theory of Celestial Movement* in 1809. This work included several critical contributions to mathematics and statistics, including the maximum likelihood parameter estimation, the method of least squares, and the normal distribution. This is perhaps part of the reason that Gauss tends to be given credit over Adrain for their similar contributions regarding the normal distribution.

In 1829, Adolphe Quetelet brought the concept of the normal distribution of error terms into the analysis of social data. He wanted to discover the underlying laws of society in the same way other researchers were exploring scientific and mathematical laws. Quetelet invented the term "social physics" and empirically developed the first notions of the measure now called "body mass index." He analyzed several data sets of human biological and social data, such as the heights and weights of conscripted soldiers, and by inductively using the central limit theorem, he concluded that the normal error distribution described these measures quite well. Galton also contributed to the application and development of the normal distribution in the biological and social sciences. He produced the first known index of correlation as well as regression analysis, and he proved that a normal mixture of normal distributions is itself normal. His colleagues Walter Weldon and Karl Pearson also contributed to normal theory and applications, and the three of them cofounded the journal *Biometrika*. The field of biometrics is generally traced back to Weldon's seminal papers. Pearson used the method of moments to estimate mixtures of normal distributions and further developed correlation and regression methods based on the normal distribution. However, part of his motivation for developing methods like chi-square analyses was apparently to try to decrease the growing reliance on the normal distribution as a foundation of statistical theory and analytic methods.

Pearson's efforts to diminish the role of the normal distribution in statistics failed. Many other mathematicians and statisticians, including Pearson's son Egon, continued to develop theory and applications in a variety of areas. For example, William Gossett and Ronald Fisher derived and refined the closely related Student's *t* distribution in the early twentieth century. The distribution is not called Gosset's *t* because he worked for Guinness Brewery and he could not publish his work in his own name because of proprietary issues, so he adopted the pseudonym "Student." Starting in the 1930s, Samuel Wilks explored many aspects of normal distributions. These included deriving sampling distributions for parameter estimates in bivariate normal distributions as well as for covariances in multivariate normal distributions, which led to important advances in multivariate statistical methods. The American Statistical Association's Wilks Award is one of the most prestigious in the field of statistics. Miroslaw Romanowski published a generalized theory of modified normal distributions in 1968 that help characterize errors that do not seem to be well-described by the normal distribution. Another such generalization is the skew normal. Other related distributions include the "lognormal distribution" or "Galton distribution," which describes a variable whose log is normally distributed, and the "folded normal," which is based on taking the absolute value of a normal distribution.

Recent Developments

The term "bell curve" became even more widely known in 1994 when psychologist Richard Herrnstein and political scientist Charles Murray wrote *The Bell Curve*, which took its name from the distribution of IQ scores and included a picture of the normal distribution on its front cover. Herrnstein and Murray correlated intelligence scores with social outcomes and asserted that social stratification based on intelligence was on the rise. The book remains highly controversial for the authors' inclusion of discussions regarding supposed relationships between race and intelligence and has spurred many debates on both social and statistical matters.

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Carlos J. Vilalta

See Also: Expected Values; Probability; Randomness; Sample Surveys.

North America

Category: Mathematics Around the World.
Fields of Study: All.
Summary: Mathematics has a long history in North America, including a twentieth and twenty-first-century focus on improving mathematics education.

North America, as defined by the United Nations, includes the United States, Canada, the Danish autonomous country of Greenland, the British overseas territory of Bermuda, and the French overseas territory of Saint Pierre and Miquelon. The United States and Canada have been especially active in the field of mathematics. By the mid-twentieth century, people from around the world were increasingly coming to North America to study and to work in mathematical disciplines. At the beginning of the twenty-first century, mathematicians and mathematics educators continue to explore ways to improve and advance research and teaching. Research and other work done by mathematics organizations in Canada and the United States show that mathematics education is a concern in North America, in part because of international comparisons of student performance. These efforts are also driven in part by the increasingly technical demands of society and the resulting economic and social needs.

Brief Early History

Mathematics played a role in the societies of the earliest native peoples as well as those of settlers from around the world. The prehistoric serpent burial mounds in what is now the state of Ohio have mathematical elements and interpretations.

In the seventeenth century, the first North American colleges began to teach a variety of subjects, including mathematics. North American mathematicians made advances in mathematical theory and contributed to a wide range of inventions.

Canada

One way to explore mathematical efforts and priorities in the twenty-first century is to examine the activities of professional associations like the Canadian Mathematical Society (CMS). The purpose of the CMS is to promote and advance the discovery, learning, and application of mathematics in Canada. According to the CMS Web site, the CMS is currently seeking to "more aggressively reach out to and form new partnerships with the users of mathematics in business, governments, and universities, educators in the school and college systems as well as other mathematical associations; and in doing so, share experiences, work on collaborative projects and generally enhance the perception and strengthen the profile of mathematics in Canada."

The mathematical skills of Canadian students have been a primary concern for Canadian educators and business owners alike. The CMS is particularly interested in reaching out to students who are interested in mathematics and in working with the educational system to improve mathematics education. To that end, the CMS sponsors a variety of educational activities, including national and regional mathematics camps, the Sun Life Financial Canadian Open Mathematics Challenge, and the Canadian Mathematical Olympiad. Additionally, the CMS publishes a journal dedicated to unique and challenging mathematics problems that can be used in secondary and collegiate mathematics classes. The CMS also provides funding for a Public Lecture Series with the goal of promoting public awareness of mathematics. The CMS strongly promotes collaboration between mathematics education and business in an effort to align the education of students with the needs of the business community, and it has developed workshops and publications to broaden participation in mathematics,

United States

World wars, especially World War II, had a notable influence on the evolution of twenty-first-century mathematics, especially in the United States. Many European mathematicians fled their native countries because of violence or oppression and settled in the United States. Military and industrial needs spurred a great deal of mathematics research and applications, which further escalated during the Cold War, spurred by advances like the Soviet Union's Sputnik satellite. The growth of universities in the wake of this boom, along with the relative isolation of the Soviet Union, were contributing factors to the rising numbers of students from other countries studying mathematics in the United States. By the beginning of the twenty-first century, the influx of foreign nationals into the United States educational system and workforce had slowed, in part because of change in political policies, including caps on visas; the rising prominence of universities in many other parts of the world; and the efforts of many nations to stem the "brain drain" or emigration of educated individuals.

Within the United States, many mathematical organizations have had a strong impact on the field of mathematics, including the Mathematical Association

The Mathematical Association of America headquarters (since 1979) in Washington, D.C.

of America (MAA) and the American Mathematical Society (AMS). Many of the concerns in the United States are similar to those in Canada. There has also been a great deal of concern and discussion regarding the perception that only some students are capable of succeeding at mathematics. Some assert that the No Child Left Behind Act of 2001 was designed to challenge this perception by ensuring that all students could demonstrate grade-level mathematics proficiency. However, this measure was negatively received by many, in part because increased demands on teachers and schools were not always fully funded and criteria used to measure success and improvement were not universally agreed upon as appropriate. A primary focus is on improving the mathematics achievement of public school students in an effort to ensure that more students are "college-ready."

In an effort to address this need, the National Council of Teachers of Mathematics (NCTM) released a series of publications that focus on the idea that mathematics education at every grade level needs to center on in-depth development of a few key mathematical concepts. The MAA and AMS both have made resources available to teachers to aid in this endeavor. Like Canada, the United States also works to recruit a wider demographic of students into mathematical fields.

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See Also: Algebra and Algebra Education; Calculus and Calculus Education; Curriculum, College; Geometry and Geometry Education; Statistics Education.

Number and Operations

Category: History and Development of Curricular Concepts.

Fields of Study: Communication; Connections; Number and Operations.

Summary: Numerous civilizations throughout history developed unique number systems and number operation methods, some principles of which survive into the twenty-first century.

The properties of numbers and operations are among the first concepts that most people learn about mathematics and they were also among the earliest type of mathematical knowledge developed historically. Number and operations are pervasive in school curricula. There are many types of numbers (for example, integers, irrational numbers, and imaginary numbers), each with their own properties. Learning how to work with different types of numbers is basic to the work of learning mathematics. The term "operations" refers to the practice of applying some rule on a set of numbers; the four basic operations are addition, subtraction, multiplication, and division. However, there are a wide variety of other mathematical operations or operation-like procedures on many types of mathematical objects, such as modular arithmetic, that may be explored at many levels.

In the twenty-first century, students in the earliest grades start to investigate whole numbers and common fractions along with addition and subtraction. In the later primary grades, they may study base-10 decimals, a broader range of fractions, negative numbers, and equivalent forms for fractions, decimals, and percentages. Operations extend to include addition and subtraction of common fractions or decimals; multiplication and division of whole numbers; and relationships between operations. Concepts like ratios and proportions, integers, factorization, prime numbers, and some alternative methods of notation for very large numbers begin to be introduced in middle school. Students learn more arithmetic procedures with fractions, decimals, or integers, as well as to simplify computations using addition and multiplication properties. They also investigate squares and square roots. In high school, students may study very large and small numbers; properties of numbers and various number systems; vectors and matrices with real number properties; and number theory. Operations begin to include addition and multiplication of vectors and matrices, as well as permutations and combinations. These concepts continue to be extended into college with new systems of numbers and operations or operation-like procedures.

Early Number Systems

The first type of numbers people generally learn about are called the "natural" or "counting numbers": 1, 2, 3, 4, 5, 6 . . . The historical record shows that the use of counting numbers is an ancient practice. As with measurement, body parts may have been used, and archaeologists have found other evidence, such as notched bones, that support the idea of tallying or counting. The Egyptians used a base-10 system, written either with hieroglyphs or hieratic (cursive) script, and using special symbols for powers of 10 (10, 100, 1000, and so on). The ancient Egyptians were also aware of fractions, which were primarily written as unit fractions of the form $1/n$, such as 1/2 or 1/4, although some Egyptian texts contain fractions of the form $2/n$ or $3/n$, and other quantities were expressed as combinations of unit fractions.

The Babylonians used a numerical system with a base of 60, a practice that survives into the twenty-first century in the convention of dividing a circle into 360 degrees and in units of time, such as 60 seconds in a minute and 60 minutes in an hour. They used only two symbols, one signifying 1 and the other 10, to write all the values 1–60, and used the place system so that the meaning of a symbol depended on its place within a number—a major advance that was crucial to the development of modern mathematics. In a decimal or base-10 system,

$$111 = \left(1 \times 10^2\right) + \left(1 \times 10^1\right) + \left(1 \times 10^0\right).$$

The same digits in the base-60 system would mean

$$111 = \left(1 \times 60^2\right) + \left(1 \times 60^1\right) + \left(1 \times 60^0\right)$$

or the same quantity as 3661 in a base-10 system.

The ancient Mayan developed a number system with a base of 20 and a place system, using dots (with a value of 1) and bars (with a value of 5) to write the numbers 1–19, with powers of 20 indicated vertically. The Mayans also understood the concept of zero as a placeholder and had a special symbol for it, which they used in their calendar system.

An acrophonic number system was used in Greece by the first millennium B.C.E. Acrophonic means that numbers are signified by the first letter of the word used for that number, with symbols for 1, 5, 10, 100, and so on. As with the more familiar Roman numerals, this system was an additive system (rather than place), so the value of a number was found by adding up the value of all the symbols that comprised it. A competing system also used in Greece was one in which each letter of the alphabet was assigned a numeric value reflecting its order in the alphabet. In this system, the first 10 letters (*alpha* through *iota*) correspond to the numbers 1–10, then the next letter (*kappa*) stands for 20, the next (*lambda*) for 30, until *rho*, which signifies 100.The next letter (*sigma*) signifies 200, and so on. This system was also an additive system, so that 12 was written as *iota beta* or $10 + 2$ and 211 as *epsilon iota alpha*. Numbers 1000–9000 were written by adding a superscript or subscription to the letters *alpha* through *theta*, while larger numbers were written with the symbol *M* (meaning "myriad") for 10,000, with multiples indicated by writing other numbers above the *M*.

Roman Numerals

The familiar system of Roman numerals was developed from about the third century B.C.E. It was used throughout the Roman Empire and in Europe into the Middle Ages and was eventually replaced by the more efficient Hindu–Arabic number system. The Roman number system has the benefit of using only a few symbols, but does not include the concepts of zero or of place, so the value of a number is calculated by adding together all the values of its elements. The symbols used include M for 1000, D for 500, C for 100, L for 50, X for 10, V for 5, and I for 1, with the later refinement that a smaller number could be placed next to a larger number to indicate subtraction. Roman numerals translate to Hindu–Arabic numerals as the following:

LXXIII = 73
CDXXXII = 432
MCMLXXXV = 1985
MMX = 2010

Roman numerals are still in use in the twenty-first century to indicate succession (for example, King Richard III of England) and sometimes in film release dates. The inefficiency of the Roman system compared to the modern system of Hindu–Arabic numerals can be illustrated by trying to quickly determine which of the following three dates is most recent: MCMXCIX, MCMLXXXVII, and MMVII. Now try again with the same values in Hindu-Arabic numerals: 1999, 1987, and 2007.

Indian or Hindu Numerals

Indian or Hindu numerals and the concept of zero (written as a dot or small circle and referred to by the Sanskrit term *sunya*, which means "empty") also appear to date to the third century B.C.E. Historians have cited Brahmi numerals, which share a name with a family of alphabets or scripts; which evolved into Gupta numerals, named for the fourth to sixth century C.E. Gupta dynasty; and then Nagari or Devanagari numerals, also named for alphabet systems, beginning in about the ninth century; and finally symbols that looked very much like the familiar numerals 0–9 somewhere around the fourteenth century. There are many origin theories for Hindu numerals, which fall into two general classes: they came from an alphabet (as did the Greek system) or they came from some other earlier number system (as did Roman numerals). Hindu number systems were predominantly base 10, and documents suggest that Indians were using a place value system by the sixth century C.E. Mathematician Pierre-Simon Laplace said, "The ingenious method of expressing every possible number using a set of ten symbols (each symbol having a place value and an absolute value) emerged in India . . . Its simplicity lies in the way it facilitated calculation and placed arithmetic foremost amongst useful inventions."

Hindu systems of numerals appear to have made their way into Arabic and Islamic cultures in the latter half of the first millennium C.E. These Hindu numerals, along with a base-60 system using Arabic letters to represent numbers (common among astronomers) and a "finger arithmetic" system (widely used in business), coexisted for some time in the Arabic world. In the ninth century, mathematician Abu Ja'far Muhammad ibn Musa Al-Khwarizmi wrote *On the Calculation with Hindu Numerals*. His contemporary Abu Yusuf Ya'qub ibn Ishaq al-Kindi also wrote *On the Use of the Indian Numerals* (c. 830 C.E.). Several Arabic and Islamic scholars studied Hindu numerals in the tenth century. Abu Ali al-Husain ibn Abdallah ibn Sina (also known as Avicenna) was purportedly taught by Egyptians, and Abu'l Hasan Ahmad ibn Ibrahim Al-Uqlidisi is credited with helping to modify Hindu numerals to replace the traditional "finger arithmetic." Mathematician Abu Arrayhan Muhammad ibn Ahmad al-Biruni visited India in the eleventh century C.E., though even before his first travels he had examined Arabic translations of Indian mathematics texts.

Hindu–Arabic Number System

The Hindu–Arabic number system was adopted in Europe a few centuries later, replacing Roman numerals, as Europeans became familiar with Arabic manuscripts. The first known example of Hindu numerals in a European document are in the tenth-century *Codex Vigilanus*, but the beginnings of widespread use appear to date closer to the fifteenth century. The symbols used in this system are similar to those used in Europe in the twenty-first-century (0–9), while a different set of symbols is used with the same number system in the Middle East and in parts of India (thus many Arabic speakers do not use what in the United States are commonly called "Arabic numerals"). The Moroccan mathematician Abu Bakr Al-Hassar is credited with developing the modern method of notating fractions (two numbers separated by a horizontal bar) in the twelfth century. *Liber Abaci,* written by Italian mathematician Leonardo Fibonacci in the early thirteenth century, was also influential in spreading the use of Hindu–Arabic numerals (and the place system) throughout Europe.

Any number may be used as a base when numbers are written using the place system. For instance, the binary (base 2) and hexadecimal (base 16) systems are used in work with computers. In the binary system, there are two digits (0 and 1), and each successive place is a greater power of 2. In the hexadecimal system, there are 16 digits (letters are used to express the extra digits required, so A = 10, B = 11, C = 12, D = 13, E = 14, and F = 15). If necessary to avoid confusion, the base of the number system may be included as a subscript; for example, 17_{10} would be 17 in the base-10 system.

Modular arithmetic was developed by German mathematician Leonhard Euler in the eighteenth century and was advanced by others, including German astronomer and mathematician Carl Friedrich Gauss, whose 1801 *Disquisitiones Arithmeticae* established many of the rules of number theory. Modular arithmetic is sometimes called "clock arithmetic" because the concept is similar to that of a 12-hour clock. If it is currently 9 o'clock, 6 hours later it will be 3 o'clock, not 15 o'clock, because the clock starts over with 0 as soon as it reaches 12.

Aids to Computation

Systems to aid computation are almost as old as number systems themselves. For instance, Egyptian scribes used tables to help them perform arithmetic with fractions, and the abacus or counting frame was used in several ancient cultures, including those of Mesopotamia, Egypt, Persia, and Rome. However, the abacus is most strongly identified today with Asia, in particular China, where it was used at least as early as the second century B.C.E. A Chinese abacus consists of a number of rods divided by a beam into two regions or decks: the upper deck of each rod has two beads, and the lower deck has five. Mathematical operations are carried out by sliding the beads toward or away from the deck, and expert abacus operators can rapidly solve problems involving not only the four basic functions (addition, subtraction, multiplication, and division) but also square and cube roots. The simplicity and efficiency of the abacus encouraged its spread to other Asian countries, including India, Japan, and Korea. Use of the modern Japanese abacus, which uses one bead in the upper deck and four in the lower, is still taught in primary schools in Japan today as it is believed to aid students in forming a mental representation of numbers.

Arabic mathematicians developed a system of lattice multiplication, which involves using a lattice or grid of boxes divided into diagonal halves. To perform lattice multiplication, the two numbers to be multiplied are written across the top and the side of the grid, the digits are multiplied separately and then added along the diagonals to produce the result. This system was introduced to Europe by Fibonacci in 1202. It was improved by Scottish mathematician John Napier in the early seventeenth century through a type of abacus referred to as "Napier's bones," which consists of a tray and a set of 10 rods, one for each digit 0–9. Each rod is divided into nine squares, with each but the top divided by a diagonal line. Each square contains the product of its own digit multiplied by each other digit; for instance, the rod for 5 contains the values 5, 1/0, 1/5, 2/0, and so on (the / indicating the diagonal of the square). Napier's bones are used to multiply, divide, and extract square roots. For example, to multiply, the rods for one number are placed in the tray, and the values from the rows comprising the digits of the second number are read off, adding together the pairs of values on the diagonals.

Logarithms are another important aid to calculation. A logarithm is an exponent such that when the base of a number system is raised to that power, the result will be the number. For instance in base 10, the logarithm of 100 is 2 because $10^2 = 100$. In the system of natural logs, the base is e (sometimes called "Euler's number" after the Swiss eighteenth-century mathematician Leonhard Euler), the irrational constant 2.718281. . . . The natural log of 100 is 4.6051 because $e^{4.6051} \approx 100$.

One common use of logarithms before the advent of electronic calculators and computers was to simplify multiplication, division, and the calculation of powers and roots. As such, logarithms played an important role in the development of astronomy and other mathematically based sciences.

Napier is usually credited as the inventor of the logarithm due to his 1614 publication *Mirifici Logarithmorum Canonis Descriptio*, which included tables of natural logarithms and explanations of their use. Important tables of base 10 logarithms were published in 1617 and 1624 by English mathematician Henry Briggs.

Multiplication using logarithms rests on the following rule. For any base b

$$c \times d = b^{\left(\log_b c + \log_b d\right)}.$$

For instance, if the base is 10, c is 108, and d is 379:

$$108 \times 379 \approx 10^{(2.033424 + 2.578639)} \approx 10^{4.612063} \approx 40{,}932$$

$$\text{because } 108 \approx 10^{2.033424} \text{ and } 379 \approx 10^{2.578639}.$$

Conducting multiplication in this way requires only looking up the two logarithms in the table, adding them, and looking up the antilogarithm (the base 10 raised to a power) in another table, which for large numbers is much quicker than doing the multiplication by hand. The slide rule, also developed in the seventeenth century, made the process even quicker and remained in common use well into the twentieth century.

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See Also: Addition and Subtraction; Multiplication and Division; Number and Operations in Society; Number Theory; Numbers, Real.

Number and Operations in Society

Category: School and Society.
Fields of Study: Connections; Number and Operations.
Summary: Number, arithmetic, and estimation are parts of daily life.

In early human societies, wealth was often measured in terms of physical possessions. Commerce depended on finding an "even" or fair exchange of goods. Counting and arithmetic were fundamental skills for enumerating goods and engaging in trade. Later agrarian societies also needed these skills to plan for activities like planting crops and storing or dispersing the harvest in an equitable way. When humans began to travel farther from home, they needed to be able to measure and calculate distances and directions.

The introduction of money and more advanced tools and technology did not change the need to count and calculate; in most cases, they merely altered what was counted and the way in which the counting and arithmetic were done. In modern society, numbers and their basic operations are pervasive. Numeracy (or quantitative literacy) is a primary concern of government and educators in the twenty-first century. The increasingly quantitative nature of society requires some level of basic proficiency in all its citizens, not merely from mathematicians, engineers, scientists, and others in traditionally quantitative professions. In modern society, studies have shown that lack of basic number and operations skills can be associated with negative outcomes, such as financial mismanagement, consumer debt, poor risk assessment, and limited job prospects. Some individuals who experience difficulty with arithmetic operations have a condition called "dyscalculia," which may be caused by neurological lesions.

Early Number History

Rows of tally marks have been found in many archaeological sites, indicating that people not only counted, but also recorded their counts. However, it is difficult to quickly know a total merely by looking at a long row of tally marks. Recognizing a quantity without counting one by one is called "subitizing." Psychologists note that humans usually can subitize accurately only up to quantities of about 5, 6, or 7 without making some combinations, so a line of 23 tally marks would allow only a guess of its number. Because of this limitation of subitizing ability, ancient humans arranged the marks into groups—usually equal groups—and counted the groups to find the total (many people also commonly do this in the twenty-first century by drawing every fifth tally mark over the first 4 to make groups of 5 for easier counting).

This system appears to exist in many parts of the world and led to both the idea of place value and the operation of multiplication. The counting system of nearly every language uses terms of grouping; many, including English, group by 10s and then 10s of 10s (100s) and continue with higher powers of 10. Probably the group size of 10 was chosen for physiological reasons—humans have 10 fingers—since other choices, such as dozens, might have made for more convenience (especially for fractions) and mathematical efficiency. Some languages do use other bases, including 4, 5, and 12, and the English words "dozen" and "score" indicate an earlier use of groupings of 12s and 20s in old English. Similarly, multiplication in objective terms amounts to finding a total by counting groups of equal quantities (even when used for area or combination calculations). Learning the "times tables" is simply learning how those groupings come together and grow into such totals.

The groupings were originally oral linguistic terms, but the idea also translated into written numerals—the ancient Egyptian, Babylonian, Mayan, and Chinese symbols (and others) were fitted into various types of place-value frameworks, some more structured than others. It is thought that the current widespread system of written numerals, the Hindu–Arabic system, originally grew out of earlier place-value grouping systems. It developed in India, early in the common era, where repeated marks were replaced by a single cipher (for example, "7" instead of "///////"), allowing more efficient writing and calculation. The Islamic mathematicians added more convenient algorithms—smoother techniques for doing arithmetic calculations and handling rational numbers.

Operations

Although people in all parts of the world needed to do arithmetic and developed their own methods, the standard algorithms most widely used in the twenty-first century developed from the Islamic algorithms (the word "algorithm" even comes from the name of Al-Khowarizmi, a mathematician who worked in Baghdad around the year 800). These algorithms were modified and refined over the centuries as the techniques were carried into Europe. Usually the adding and multiplying methods were straightforward, mostly collecting and regrouping symbols, but subtracting and dividing were more complicated and led to a greater variety of algorithms—especially different ways and sequences to regroup numbers, both in conceptual terms and in the written expositions. An important difference involved either starting with the units and moving to the higher groupings, often called a "right-to-left method," or the reverse of working first with the larger groups and then taking care of the smaller unit details.

The standard symbols of the numerals stabilized in medieval Europe (c. sixteenth century), as did most of the algorithmic methods. Along with this standardization came the symbols for the operations (+, −, ×, ÷, and =) and later other notations such as exponents for powers, the square root symbol, various kinds of brackets for groupings, and symbols for fractions and decimals. These symbols are not completely standardized; for example, Americans use a dot (.) between the whole number and a decimal fraction, while many Europeans use a comma (,). Also, there are some remaining discrepancies in the terminology of large numbers, as Americans say a thousand millions is a "billion," but British usage is that a "billion" is a million millions.

Counting and arithmetic are fundamental skills for many occupations, including the pharmacy trade.

Types of Numbers

Languages usually differentiate number usage according to the purpose of the number. If the number is an adjective that tells the numbers of members in a set or collection, is it called a "cardinal number." Thus, "three houses" describes the quantity of houses being discussed. In higher mathematics, especially in number theory, the quantity of the members of a set is called the set's "cardinality," which, significantly, can be infinite or even different infinities. In many languages, especially in eastern Asia, an extra word is inserted after the number to describe the category of the item being counted. For example, a certain word for the category of flat items might be used to describe quantities of paper, boards, or leaves, but a different category word would be used for quantities of round objects.

Another everyday use of numbers is to describe the place of something in a sequence. If there is a line of houses along a street, the third house may be noted—counting from the beginning of the line to position number 3. Since the houses are considered in order, this is called the "ordinal number": first, second, third, fourth, and so on, with most of the higher values using a "-th" at the end of the number word.

A third kind of number is as a name, so it can be called a "nominal number." These are used when counting in general or referring to the number itself, as in, "I am writing a five." In English, this is usually the same word as the cardinal form, though in other languages, this is not always the case. Also, the nominal form is often used in nonmathematical language, where a number term is used to name a person or something else. Examples include numbers of Social Security accounts, house street addresses, bank accounts, telephones, routes of highways, buses, or planes, and car license plates. Sometimes they are arranged in a numerical order for convenience, but usually do not represent true ordinal usage. These are only a convenience (for example, they work well in computers) and usually have no mathematical meaning—one would not think of adding two phone numbers! Distinctions of cardinal, ordinal, and nominal usage are (1) taking three buses with the necessary changes from one to another (cardinal), (2) waiting as two buses pass and then taking the next one—the third bus (ordinal), and (3) looking for a sign on an approaching bus that says it is bus #3 (nominal) and then getting aboard. Often, when people complain of modern society reducing everyone to a number, they are in fact referring to the nominal usage. Since nominal numbers are so pervasive, it is important for children to learn the distinctions, so they will understand that these nominal usages are not mathematical.

Economics and Demographics

Beyond the nominal names, actual quantities are used in nearly all aspects of society. At the heart of economic activity is the need to quantify money and compare this quantity with measurements of value, which may also be quantified. Accountants and bankers may not be mathematicians, but they constantly use numbers and carry out operations that may be based on simple arithmetic but used in very complex applications. These users may range from high-level financial managers to retailers to children selling lemonade. Some have suggested that, especially in the modern world, economic uses of numbers may be the biggest application of mathematics in society. Another important subject of counting is people—for records of population, attendance at schools and events, families, public health, television viewing, and many others.

Measurement

When numbers are applied to comparisons, measurement is happening. Measurements of length, weight, volume, and many of the technical quantities—such as electrical conductivity, strength of magnetism, blood pressure, engine power, and acoustic properties—are used by scientists, engineers, architects, medical workers, mechanics, and even artists and musicians to deal with properties essential to their work. Numbers are not needed for sophisticated technicalities but may may help with shopping for shoes, getting a first down in football, and giving directions to the library. Operations include totaling a shopping bill, converting currency, checking the movement of a comet, and building an oil rig. Few people may calculate comet orbits, but nearly everyone needs to check their shopping and bank account calculations. Over time, many systems of units of measurement developed, showing the importance of this use of numbers. Many measurements, especially linear measures, were comparisons with human body parts, such as the length of a handspan, the distance from the elbow to the fingertips (called a "cubit"), or the distance of a walking pace. It became clear that standardized measurements were needed for fair comparisons, especially in trade, so governments as early as the ancient Egyptians and Romans developed standardized systems. Many traditional measures were converted to standard systems, but often the units did not fit well into an organized system. In the late 1700s in France, the metric system was devised to serve as a well-organized standard system for world use. In the two centuries since then, that goal has almost been achieved.

Statistics

The tools of statistics are used to analyze and report results of counting and measuring. Tables arrange data in columns and rows for easier comparisons as well as summations, averaging, and other calculations. In the twenty-first century, computerized spreadsheets have given new power to the calculation and manipulation of data in tabular form. Graphical displays make the information visible for quicker comprehension. Bar graphs and histograms sort data into comparative categories, while line graphs are especially useful to show changes over time. Circle graphs show comparative portions of a total. Newer displays include bar-and-whisker charts, which show the distribution of a collection of data, and stem-and-leaf charts, which are used to assemble data for bar charts. Statistics educators often warn that the ease of display of statistical graphs can also be misused to offer misleading implications, so a familiarity with statistics is considered important in evaluating displays in advertising and reports.

Arithmetic is considered part of the basic foundation of the school curriculum because the need to deal with numbers and arithmetic is central to so many aspects of daily life and is the starting point of all higher mathematics and applications of mathematics in science, engineering, and technology. Usually instruction in counting begins even before formal schooling, the basic arithmetic operations are taught in the early grades, and work with fractions, percentages, and ratios in the upper grades of elementary school. Even in areas where few children may have the opportunity to attend higher levels of school, it is considered essential that they learn this foundational material—in school or perhaps on the job—because of the central role of number and operations in so much of life activity.

Estimation

Often, the use of numbers implies that precision and accuracy are required—particularly true in scientific and technical applications. However, for many applications of daily life, and even some economic and technical uses, the specific exact quantity may not be necessary. The time (and expense) it takes to find the exact quantity may not be justified, and such quantification may not even be possible. For example, populations of countries are often quoted down to the exact number of individuals, but in reality, people are born and are dying every day, causing the number to vary constantly. Similarly, if one inquires about the distance between two cities, one does not need the answer to the nearest meter—even an error of a few kilometers would be tolerated.

Mental Arithmetic

Mental arithmetic is the operational counterpart to estimation, in which calculations are done without writing or using other calculation tools. A variety of techniques for mental arithmetic have been developed. Sometimes, it simply means using rounded off estimates to make the calculation easier. In addition and subtraction, the technique might mean ignoring the ones column or even more. Also, using factors can often simplify multiplication and division. Sometimes there are special “tricks” to using specific numbers in calculations, such as adjusting numbers to fit together to make 10s, adding a reciprocal to carry out subtraction, or applying algebraic techniques to simplify the numerical work.

Calculation Tools

Even though mental arithmetic is fast and convenient, many mathematical calculations are too complex for such methods. Very early in history, people realized that they needed various tools to assist their computational work. It might even be argued that the process of writing numerals and using written algorithms is the most fundamental tool—though perhaps counting on one’s fingers is an even earlier tool. More than 2000 years ago, tools were developed to handle basic arithmetic. Romans made shallow grooves in the ground to represent the place-value positions and moved stones within the grooves to represent the value of each position. Adding and removing stones from the grooves carried out addition and subtraction operations, often requiring regrouping or exchanging 10 of a smaller position for 1 of the next larger position in order to have enough stones for the results or to reduce an overloaded position. The abacus uses the same principles of mechanizing arithmetic, but does it with beads strung onto wires in a frame instead of with stones in grooves. Since the beads cannot be physically added or removed from the wires, various new techniques were developed to handle the regrouping, often involving reciprocal adding or mental regrouping.

As early as the 1600s, more sophisticated mechanical devices were being developed to make arithmetic even more automated. Two famous mathematicians, Blaise Pascal (1623–1662) and Gottfried Leibniz (1646–1716), both made mechanical devices with gear wheels and a ratchet mechanism to handle regrouping. John Napier (1550–1617) invented two very different tools, one for ordinary people and one for scientists. For ordinary people, Napier took the idea of lattice multiplication, which had come from Islamic mathematicians, and used small four-sided rods of multiplication tables to arrange like lattices to ease the multiplication of multidigit numbers. They were called "Napier's Rods" or "Napier's bones," since he sometimes made the rods from bones. More significantly, he (and others) introduced the concept of logarithms, which are actually representations of powers of a common base (usually 10 or *e*). Since multiplication of two numbers written as exponents of the same base can be done by addition (and handling powers can be done with multiplication), logarithms simplified multiplication to addition and exponentiation to multiplication and thus allowed scientists to deal with much more complicated powers and roots than other techniques allowed, greatly speeding their calculations.

Charles Babbage (1791–1871) is sometimes called the "father of computers," but also he was a very frustrated man, since he was trying to invent devices one century too early—in the first half of the nineteenth century. Noting that calculations by hand often had errors (even errors in transcription), he wanted to avoid errors by substituting the handwork with complicated machinery. His inventions—the "difference engine," followed by plans (which he could never completely carry out) for the "analytical engine"—had the same basic parts as modern computers: input/output, a storage memory, and a central processor. He used a system of programming to input data and to instruct the machine on what to do and then had the results printed out—all aiming to keep the work away from human error. Unfortunately, his plans were beyond the technical capability of his day. His support from the British government was used up as he struggled unsuccessfully to overcome technical problems.

Later in the nineteenth century, Herman Hollerith (1860–1929) also worked to mechanize data handling. He noticed the "programming" of Jacquard textile weaving looms was implemented by wooden plates with holes arranged in particular patterns to control the movement and alignment of the threads. He realized that paper cards similarly punched with holes could be used to direct the movement and combination of data. He convinced the U.S. Census Bureau to use the idea in tabulating its data, and later he joined Thomas Watson in starting the company that became IBM. His punchcards were a staple of data processing and, later, computing for many decades.

Computers

Computers finally came on the scene from considerable theoretical work in the 1930s, the pressures of war needs in the 1940s, and the growth of technology in general in the 1950s and 1960s. Technical developments, such as transistors, integrated circuits, and interactive interfaces, moved the development toward enabling the common person to compute. Interactivity opened the door for word processing and publishing, e-mail and other communications, and, eventually, the Internet. Meanwhile, fitting greater power into smaller and smaller devices allowed cell phones, thin television sets, laptops, and the explosion of handheld devices with thousands of applications.

Computers have become such a central part of modern life that some concern has been raised about their role and their power. Even as computers may seem cold and inhuman, programming and merging of data files allow many more individualized responses than humans would be able to handle efficiently. Mathematicians and mathematics teachers sometimes debate the use of calculators and computers in both school mathematics and mathematical research. In both cases, the main argument is the efficiency and accuracy of using electronic tools against the sense that doing mathematics should be a human, mental activity.

In a broader sense, this same question comes to the role of numbers and operations in society: quantitative versus qualitative. Certainly, numbers and operations are essential to science, business, and in fact all of modern life (and were quite essential even in ancient times). Some would argue, however, that the essence of humanity is found in the arts, philosophy, and religion. The division of the two worlds has long been debated. However, a convergence may have been found as quantitative measures are increasingly applied to the humanities and the sciences have researched the mysteries of the brain and cognition, quantum mechanics and cosmology, and multiple dimensions and infinities.

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See Also: Algebra in Society; Calculators in Society; Calculus in Society; Connections in Society; Geometry in Society; Learning Exceptionalities; Measurement in Society; Number and Operations.

Number Theory

Category: History and Development of Curricular Concepts.
Fields of Study: Communication; Connections; Number and Operations.
Summary: Number theory has captured the imagination through numerous famous problems, many still unsolved.

The legendary mathematician Carl Friedrich Gauss (1777–1855) famously described number theory as "the queen of mathematics." The core of number theory is the study of the integers, but number theory includes a much wider class of concepts and problems that arise in the study of the integers. Number theory is extremely popular as recreational mathematics and is explored in twenty-first-century high school classrooms. Early mathematicians in Greece, India, and the Islamic world investigated and developed number theory, making enormous and widely varied contributions. The field has continued to blossom through the twenty-first century. Some mathematicians view number theory as a branch of pure mathematics, but others view it as applied mathematics because of its utility to so many fields, such as physics, chemistry, biology, computing, engineering, coding, cryptography, random number generation, acoustics, communications, graphic design, music, and business.

Prime Numbers

A large portion of number theory is related directly or indirectly to the study of prime numbers and divisibility. An integer a is divisible by integer b if there is an integer c such that $a = bc$. A prime number is an integer $p > 1$ that is divisible only by itself and 1, and a composite number is a positive integer with more than two factors. It is technically most convenient to consider 1 to be neither prime nor composite. The so-called Fundamental Theorem of Arithmetic, investigated by Carl Friedrich Gauss in the nineteenth century, states that every positive integer $n > 1$ can be written as a product of prime numbers, and furthermore, that this prime factorization is unique (except for the order in which the factors are written). The theorem was partially proved by Euclid of Alexandria in ancient Greece. The recognition that the theorem does not hold in more general number systems by mathematicians such as Ernst Kummer led to the development of the field of algebraic number theory.

It is well known that there are infinitely many prime numbers, so it would not be possible to obtain a complete list of all prime numbers. The search for ever-larger prime numbers is ongoing, and testing numbers for primality is sometimes used as a test of the computational power of supercomputers.

Modular Arithmetic and Cryptography

An important component of elementary number theory is modular arithmetic. In modular arithmetic, two numbers are treated to be the same if they have the same remainder when divided by some given number, the "modulus." One writes $a \equiv b \pmod{m}$ if m divides the difference $a - b$. One reason why this concept is so useful is that it is compatible with the operations of addition, subtraction, and multiplication. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $a + c \equiv b + d \pmod{m}$, $a - c \equiv b - d \pmod{m}$, and $ac \equiv bd \pmod{m}$. The situation is complicated for division, unless m is a prime. Modular arithmetic is sometimes called "clock arithmetic" because of the similarity between arithmetic mod 12 and the system for counting hours. One related theorem that is frequently studied in classrooms in the Chinese remainder theorem, named so because the theorem originates in Chinese texts by Sun Tzu and Qin Jiushao.

If there was ever a time when number theory was studied *only* for its elegance and beauty and not for any application, that time is past now. Modern cryptography, essential for the security of the Internet, is based heavily on number theory. The widely used RSA (Rivest, Shamir, and Adleman) encryption methods are based on modular arithmetic and rely for their security on number theorists' understanding of how difficult it is to find large prime factors of a large number. Other encryption systems based on more exotic number theoretic objects, such as elliptic curves, are under active development by cryptographers and number theorists.

Algebraic and Analytic Number Theory

Number theorists study in a diverse range of fields related to number theory, including probabilistic number theory, diophantine approximations, the geometry of numbers, and combinatorial number theory. Number theorists do not exclusively study the integers, nor are the integers the only system that admits something like prime numbers. Number theorists also extensively study other systems of algebraic numbers. An algebraic number is a number that is the root of a polynomial with integer coefficients; the role of "integer" is played by numbers that are roots of polynomials with integer coefficients and leading coefficient 1. For example, the square root of 10 is an integer in this extended sense. Much of algebraic number theory concerns how the concepts of prime number and divisibility as applied to other number fields compare and contrast with the familiar situation for the standard integers.

For example, the Gaussian integers are those complex numbers $a+bi$ with a and b both integers. The Gaussian integers form a number system in which the concepts of prime number and divisibility above apply almost exactly as described above. However, the set of prime numbers here is very different. The number 7, for example, is still prime in the Gaussian integers, but 13, which is prime in the integers, factors here as the product of the two primes, $3+2i$ and $3-2i$.

Though the arithmetic integer is apparently part of discrete mathematics, there is a large branch of number theory, called "analytic number theory," which applies extremely sophisticated techniques from calculus and complex analysis to the problems of number theory.

A basic fact of analytic number theory is that the sum of the reciprocals of all primes,

$$\sum \frac{1}{p}, \text{ is infinite.}$$

Note that it can be recovered from this that the set of prime numbers must itself be infinite. With care, one can estimate the sum of the reciprocals of all primes up to some number x, which turns out to grow at the same rate as $\log(x)$. With more refinement along these lines, one obtains the celebrated Prime Number Theorem, which says that the number of prime numbers less than some large integer x is well-approximated by

$$\frac{x}{\log(x)}.$$

Famous Problems in Number Theory

One feature of number theory is a large number of intriguing problems that are very simply stated but require unexpectedly advanced and specialized techniques to solve; many remain unsolved in the early twenty-first century. Indeed, many or most of the major problems in mathematics that are known to the general public have origins in number theory.

One major recent mathematical breakthrough, which received mainstream media coverage, was the proof of Fermat's Last Theorem. Pierre de Fermat (c. 1601–1665) wrote a note in the margin of a book he was reading to the effect that there are no integral solutions to the equation $x^n + y^n = z^n$ with $n > 2$. There are infinitely many solutions with $n = 2$, and these have been well studied; by the Pythagorean Theorem they correspond to right triangles with integer-length sides. Fermat never wrote down a proof, writing instead that the margin of his book was too small to contain it. In the intervening centuries, mathematicians tried to supply the missing proof. Much of algebraic and analytic number theory was developed as part of the effort to prove this theorem. The problem was finally solved in 1995 by Sir Andrew Wiles (1953–), using extremely sophisticated number-theoretic objects involving elliptic curves and modular form. It is not now generally believed that Fermat ever possessed a valid proof.

Because all primes other than 2 are odd, the smallest possible difference between consecutive primes (other than 2 and 3) is 2. Prime numbers that differ by 2 (those that are as close as possible) are called "twin primes." The Twin Prime Conjecture asserts that there should be an infinite number of twin primes, but mathematicians are very far away from being able to prove this. In a triumph of analytic number theory, Viggo Brun (1885–1978) showed that the sum of the reciprocals of all twin primes is finite, though mathematicians still do not know whether there are finitely or infinitely many of them!

Another famous problem about the additive distribution of the set of prime numbers is Goldbach's Conjecture, named for Prussian mathematician Christian Goldbach (1690–1764). The conjecture asserts that every even number larger than 2 can be written as the sum of two (not necessarily different) prime numbers. Computer searches have verified that there are no counterexamples smaller than one quintillion. This problem has occupied the attention of many recreational mathematicians and has been featured in several novels and television shows. Again, mathematicians are very far from proving such a theorem.

The Riemann Hypothesis is the most important open problem in number theory, and arguably in all of mathematics. Named for Bernhard Riemann (1826–1866), this conjecture concerns a particular function of the complex numbers called the *zeta function*. Let $\zeta(s)$ be the value of the infinite sum

$$1^s + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots .$$

This is apparently defined only for real numbers $s > 1$, but it turns out that there is a uniquely meaningful way to extend this to allow any complex number as input. It is relatively easy to show that

$$\zeta(-2) = \zeta(-4) = \zeta(-6) = \cdots = 0$$

and there are infinitely many other "nontrivial zeroes" s such that $\zeta(s) = 0$. The standing conjecture is that all the nontrivial zeroes of ζ lie on a particular line in the complex plane. Though a tremendous amount of effort has gone into trying to prove this and though mathematicians have much corroborating evidence, it is still open. The statement might seem esoteric and arcane; surprising as it may seem, this statement, if true, would have profound implications about the distribution of prime numbers, which would have ramifications throughout all mathematics. Mathematicians have found dozens of very different-looking statements that are known to be equivalent to the Riemann hypothesis, and there are hundreds of statements that have been proven contingent on a future proof of the Riemann hypothesis.

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See Also: Coding and Encryption; Mathematics, Theoretical; Proof.

Numbers, Complex

Category: History and Development of Curricular Concepts.
Fields of Study: Algebra; Communication; Connections; Number and Operations.
Summary: Complex numbers inevitably arise in many situations, but may be difficult to accept.

Complex numbers are ubiquitous in modern science, yet it took mathematicians a long time to accept their existence. They are numbers of the form $z = a + bi$ where a and b are real numbers, and i is a symbol called the "imaginary unit," which satisfies the seemingly impossible equation $i^2 = -1$. The numbers a and b are called the "real" and "imaginary parts" of z, respectively. The imaginary unit can be thought of as the square root of -1 and is also written $i = \sqrt{-1}$. In fact, any negative number has a complex square root; for example, the square root of −15 is the complex number

$$\sqrt{-15} = \sqrt{15} \cdot i .$$

In the twenty-first century, science students routinely encounter complex numbers, for instance, as solutions to quadratic equations.

In mathematics, complex numbers form an independent area of research and are also used to prove theorems in other areas of mathematics; examples are Machin's formula and the Prime Number theorem. In the natural sciences, complex numbers often simplify calculations, for example, in the theory of relativity where the distance between points in space-time can be computed using imaginary time coordinates. The complex exponential function is used in electrical engineering as a convenient way of simultaneously describing the amplitude and phase of an alternating current, and in chaos theory, the complex plane is the scene of computer-generated fractals, such as the "Mandelbrot set," named after Benoît Mandelbrot.

Unlike natural numbers, which are used for counting, and real numbers, which are used for measuring distances, complex numbers have no obvious real-life interpretation. For this reason, the questions of what complex numbers really are remained a controversial topic for three centuries after their discovery in the sixteenth century. The term "imaginary numbers" for nonreal complex numbers was coined by René Descartes in 1637 to indicate that they do not really exist, a view later shared by Isaac Newton. About 1765, Leonhard Euler characterized square roots of negative numbers as impossible quantities, and as late as 1831, Augustus De Morgan objected to the absurd nature of complex as well as negative numbers. Only in 1837 did William Rowan Hamilton give a proper construction of the complex numbers, thereby indisputably proving their inner consistency. Nevertheless, it was their usefulness, the beauty of their simplicity, and the ability to visualize them rather than Hamilton's proof that eventually outweighed the objections against complex numbers and led to their universal acceptance by the end of the nineteenth century.

Algebra and Geometry of Complex Numbers

Complex numbers appeared for the first time in Gerolamo Cardano's *Ars Magna* from 1545. In this famous book containing the formulas for solving cubic and quartic equations, Cardano also showed that the equations $x + y = 10$ and $x \cdot y = 40$ have the common solution

$$x = 5 + \sqrt{-15} \text{ and } y = 5 - \sqrt{-15}.$$

Cardano, however, dismissed these complex numbers as useless and did not pursue the matter further. Rafael Bombelli undertook a more systematic investigation in *L'Algebra* from 1572, where he demonstrated how complex numbers can be added, subtracted, multiplied, and divided using the usual rules of algebra and the equation $i^2 = -1$. For example,

$$(1+3i)+(2+i) = 3+4i$$

$$(1+3i)-(2+i) = -1+2i$$

$$\text{and } (1+3i)\cdot(2+i) = 2+i+6i+3i^2 = -1+7i.$$

Division is slightly more complicated; it is most easily performed by multiplying both numerator and denominator by the conjugate of the denominator:

$$\frac{1+3i}{2+i} = \frac{(1+3i)\cdot(2-i)}{(2+i)\cdot(2-i)} = \frac{5+5i}{5} = 1+i.$$

Using these operations, Bombelli showed how real solutions to cubic equations can be found even when square roots of negative numbers appear in Cardano's formula for cubic equations. Bombelli's brilliant use of complex numbers for solving polynomial equations eventually led to the Fundamental Theorem of Algebra, according to which every polynomial equation of positive degree has a complex solution. The first essentially correct proof of this result, which had been anticipated already in the seventeenth century, was given by Carl Friedrich Gauss in 1799 in his doctoral dissertation.

Complex numbers can be represented geometrically as points in the complex plane, invented in 1797 by Caspar Wessel. Shortly afterward, it was independently conceived and popularized by Gauss, who used it implicitly in his proof of the Fundamental Theorem of Algebra. This concrete geometric interpretation of complex numbers was instrumental in the struggle to come to terms with their nature. In the complex plane, points on the x-axis correspond to real numbers, points on the y-

axis to so-called purely imaginary numbers, and in general, the point with coordinates (a, b) corresponds to the complex number $a+bi$. Viewing complex numbers as points in the complex plane gives a new geometric understanding of Bombelli's rules of addition and multiplication. Also, the "numerical value" $|z|$ of a complex number $z=a+bi$ is defined geometrically as the distance between the points (0, 0) and (a, b), or

$$|z|=\sqrt{a^2+b^2}\,.$$

Machin's Formula and the Computation of π

John Machin, in 1706, discovered the formula

$$\frac{\pi}{4}=4\cdot\arctan\left(\frac{1}{5}\right)-\arctan\left(\frac{1}{239}\right)$$

and used it together with the Taylor series

$$\arctan(x)=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\cdots$$

to compute π to 100 decimal places, a world record at the time. Although Machin's formula involves only real numbers, it has a surprisingly simple and elegant proof using the following identity of complex numbers, thus illustrating their utility in other areas of mathematics:

$$\frac{(5+i)^4}{239+i}=2\cdot(1+i)\,.$$

Exponential and Trigonometric Functions

The exponential function e^x and the trigonometric functions cos(x) and sin(x) are well-known functions of a real variable x. They can be expressed as Taylor series as follows:

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$$

$$\cos(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\cdots \text{ and}$$

$$\sin(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\cdots.$$

Using these expressions, the complex functions e^z, cos(z), and sin(z) are defined for a complex variable z. With these definitions and the fundamental equation $i^2=-1$, Euler, in 1748, proved a formula that reveals a surprising kinship between these seemingly unrelated functions:

$$e^{ix}=\cos(x)+i\cdot\sin(x)\,.$$

This result, known as "Euler's formula," generalizes a formula found by Abraham de Moivre in 1730:

$$(\cos(x)+i\cdot\sin(x))^n=\cos(nx)+i\cdot\sin(nx)\,.$$

Inserting $x=\pi$ into Euler's formula gives *Euler's identity*: $e^{i\pi}=-1$.

This identity combines the three most important mathematical constants—π, e, and i—into one single expression of striking simplicity and beauty. A 1988 poll of readers of *Mathematical Intelligencer* voted Euler's identity "the most beautiful theorem in mathematics," ahead of the infinitude of primes, the transcendence of π, and the Four-Color Theorem.

Complex Analysis

Complex analysis is the study of complex functions—functions $f(z)$ defined on some subset U of the set of complex numbers and with complex values. After initial contributions by Euler and Gauss, complex analysis was systematically investigated by Augustin-Louis Cauchy in the 1820s. Later in the nineteenth century, the theory was further developed by Bernhard Riemann and Karl Weierstrass. The set of definition U is called a "domain" if it is open and connected, and $f(z)$ is called "holomorphic" if it satisfies the condition of complex differentiability. Contrary to what the name suggests, complex analysis is in many ways simpler than real analysis, since complex differentiability is a much-stricter property than real differentiability. For example, every holomorphic function satisfies the so-called Cauchy-Riemann equations, which have no analogue in the realm of real functions. Also, the "Identity theorem" states that two holomorphic functions $f(z)$ and $g(z)$ defined on the same domain U are identical only if they agree on a line segment, a result very far from being true in real analysis. Other theorems and conjectures in complex analysis are concerned with other types of complex functions, such as entire and meromorphic functions.

An "entire" function is a holomorphic function defined on the entire complex plane. The complex exponential function e^z and the complex trigonometric functions cos(z) and sin(z) are examples of entire functions. Liouville's theorem, named after Joseph Liouville, states that every nonconstant entire function is unbounded. This theorem is considerably strengthened by Picard's little theorem, named after Charles Picard, which states that every nonconstant entire function takes every complex value with at most one exception. For example, cos(z) and sin(z) both take every complex value, whereas e^z takes every complex value except 0.

A meromorphic function is a quotient of two holomorphic functions defined on a domain U where the denominator is not identically zero. The zeros of the denominator are called "singularities"; they can be either "removable singularities" or "poles." The complex tangent function

$$\tan(z) = \frac{\sin(z)}{\cos(z)}$$

is an example of a meromorphic function; it has zeros at 0, $\pm\pi$, $\pm 2\pi$, and so on, and poles at $\pm\pi/2$, $\pm 3\pi/2$, $\pm 5\pi/2$, and so on. The mysterious Riemann zeta function $\zeta(z)$ is another example; it has a single pole at $z = 1$. The Riemann conjecture, arguably the most important unsolved problem in all of mathematics, states that all nonreal zeros of $\zeta(z)$ have real part equal to one-half. The Riemann conjecture is one of the seven Millennium Prize Problems for whose solution the Clay Mathematics Institute has offered a prize of $1 million.

Hamilton's Quaternions as Extensions of Complex Numbers

The complex numbers form an extension of the real numbers, just as the real numbers form an extension of the rational, integral, and natural numbers. It is therefore natural to ask if there are further numbers extending the complex numbers. This question was answered in the affirmative by Hamilton in 1843 when he discovered the quaternions. A quaternion is a number of the form $q = a + bi + cj + dk$ where a, b, c, and d are real numbers, and i, j, and k are symbols satisfying

$$i^2 = j^2 = k^2 = i \cdot j \cdot k = -1.$$

Quaternions, however, do not satisfy the commutative law of multiplication. For example, the product of i and j depends on the order of the factors: $i \times j \neq j \times i$. The numerical value of a quaternion q is defined as

$$|q| = \sqrt{a^2 + b^2 + c^2 + d^2}\ .$$

A quaternion with numerical value $|q| = 1$ is called a "unit quaternion." Each unit quaternion corresponds in a certain way to a rotation of three-dimensional space. For this reason, quaternions have important applications in computer graphics.

It happens that each unit quaternion q corresponds to the same rotation as its negative, $-q$. This mathematical subtlety explains one of the most surprising phenomena in quantum mechanics, namely, that the state of an electron is changed if the electron is rotated 360 degrees; only a rotation of 720 degrees leaves the electron unchanged.

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See Also: Mathematics: Discovery or Invention; Pi; Renaissance; Trigonometry; Vectors.

Numbers, Rational and Irrational

Category: History and Development of Curricular Concepts.

Fields of Study: Communication; Connections; Number and Operations; Representations.

Summary: While the concept of rational numbers is easily understood, mathematicians has struggled with the concept of irrational numbers since antiquity.

Early philosophers and mathematicians explored whether real-life lengths were made up of whole numbers. The discovery of irrational numbers caused great concern and led to the development of number theory and real analysis. A rational number is a real number that can be written as a ratio of two integers. Real numbers that cannot be so written are called "irrational numbers." So, for example, 17/47 is a rational number, while π or $\sqrt{2}$ are irrational numbers. Rational and irrational numbers can also be represented using the decimal notation. The rational numbers are precisely those numbers whose decimal representation either terminates after a finite number of digits or is repeating. The decimal representation of irrational numbers does not have a repeating pattern. So, for example, 1/8 corresponds to 0.125 (a terminating decimal), while 1/7 corresponds to the repeating decimal 0.142857142857. . . . On the other hand, the irrational number π has a decimal expansion that begins with 3.14159265358979323846 . . . and continues indefinitely without any patterns.

Students in twenty-first-century classrooms explore rational numbers in middle school and irrational numbers in high school, and these numbers appear in nature and in daily calculations. For example, e, which is also irrational, is needed to calculate the interest compounded continually on a loan, and π appears in circular or spherical objects. In fact, as a consequence of Georg Cantor's work, given any real number, there is a higher probability of it being irrational. There are still open problems to explore, such as whether $\pi + e$ is irrational.

Definition

Irrational numbers are numbers that are not rational; in other words, any number that is not the ratio of two integers is an irrational number. This definition by itself, however, is circular. To be able to use it, one first has to know what a number is.

What is a number? This question is harder to answer than one might expect at first, and, in fact, has been contentious for most of the history of mathematics. The positive integers (or the counting numbers) 1, 2, 3, . . . directly arise from the daily experience of humans, and it is impossible to trace how long ago humans went from the concrete ideas of three cows, three stones, and three trees and abstracted out the number 3 as a stand-alone concept. The advantage of this abstraction is that people could study operations on numbers and apply them to a large number of settings. If one knows that $3 + 5 = 8$, this indicates simultaneously that 3 cows together with 5 cows are 8 cows, and that 3 trees and 5 trees are 8 trees. If one knows that $3 \times 5 = 15$, then, while 3 trees and 5 trees cannot be multiplied, this can be used to model many situations. Three boys each having five apples have a total of 15 apples, and a 3-by-5 piece of land has an area of 15. Mathematicians can now concentrate on finding better algorithms and methods for doing number operations. The concept of number was first enlarged to also encompass rational numbers—the ratios of positive integers.

Some 3500 years ago, Egyptians used unit fractions (reciprocals of positive integers) and 2/3 to pose and solve problems. For example, the third problem in the Rhind Mathematical Papyrus is about dividing 6 loaves among 10 men, and the answer given is

$$\frac{1}{2} + \frac{1}{10} .$$

Around the same time, Babylonian scribes in Mesopotamia used a base-60 place value system for fractions, but confined themselves to those rational numbers that have a finite sexagesimal representation. In any case, for a very long time, the term "number" meant positive integers and ratios of positive integers—what are now called the "positive rational numbers."

If the only numbers are rational numbers, then by definition there are no irrational numbers. To enlarge the definition of "number" beyond rational numbers, one has to somehow construct these other numbers, which can be done by proposing that the length of any line segment is a number.

It is believed that in early mathematics it was assumed that given any two line segments, it is possible to find a third line segment—maybe a very small one—that measures both lines a whole number of times. In other words, the length of each of the original line segments is an integer multiple of the third smaller line segment. The third line segment is called a "common unit of measure," and the original two line segments are called "commensurable." On the face of it, this assumption may seem reasonable, but, if true, it would mean that the length of any line segment is a rational number. Given an arbitrary line segment of length a, find a common unit of measure for it and a line segment of unit length. If the common unit of measure has length b, then $a = mb$ and $1 = nb$ for some integers m and n. But this means that

$$\frac{a}{m} = b = \frac{1}{n}$$

from which it can be determined that

$$a = \frac{m}{n}$$

is a rational number.

The Pythagorean Theorem, which was known at least 1000 years before Pythagoras, states that given a right triangle whose sides are of unit length, then the length of its hypotenuse will be such that yields 2 if multiplied by itself. Since it has been decided that all lengths are numbers, the length of this hypotenuse must be a number called the "square root of 2" and denoted by $\sqrt{2}$. One can prove that $\sqrt{2}$ is not a rational number, thereby proving that not all pairs of line segments are commensurable and that irrational numbers exist.

There are many proofs of the irrationality of $\sqrt{2}$, but the most common one is as follows: Assume by way of contradiction that $\sqrt{2}$ is rational and equal to n/m, where n and m are integers. One has many choices for n and m; for example, one could multiply both by 47 and get a new pair of integers with the same ratio—and values of n and m are chosen such that they do not have any common factors. This is, of course, possible. From $\sqrt{2} = n/m$, one obtains $2m^2 = n^2$, which means that n^2 and therefore n is an even number. If $n = 2k$, then $2m^2 = 4k^2$ and so $m^2 = 2k^2$, which means that m is also an even number. But it had been assumed that n and m have no common factors. A contradiction was reached but, since the logic along the way was impeccable, it must have been that the original assumption that $\sqrt{2}$ is rational must have been wrong.

Implications

The discovery of irrational numbers led to a crisis in geometry and a need to revisit all the results that depended on the commensurability assumption. Following Eudoxus, Euclid in his very influential book *Elements* makes a distinction between a number and a magnitude. Roughly, one can think of numbers as the rational numbers and magnitudes as the lengths of line segments—Euclid had an elaborate classification of magnitudes. In an attempt to be rigorous, Euclid treats number and magnitude differently, and hence he does not regard irrational numbers as numbers. For example, he develops the theory of proportions once for magnitudes and once for numbers.

It took the effort of many mathematicians in the middle ages—and most notably mathematicians living in Islamic lands and writing in Arabic—to expand the notion of number to include Euclid's magnitudes and to have a single treatment of all numbers, rational and irrational. During this period, the decimal number system—first developed in India and crucial in understanding irrational numbers—became widespread. Ninth-century Persian mathematician Al-Mahani gave a definition of irrational numbers (as opposed to Euclidean magnitudes), and ninth-century Egyptian mathematician Abu Kamil used irrational numbers as coefficients in algebraic equations. By the fifteenth century, Persian mathematician Jamshid Kashani (also referred to as al-Kashi) was able to comfortably work with real numbers and their decimal expansions. He treated both rational and irrational numbers as numbers.

In the West, sixteenth-century Flemish mathematician Simon Stevin played an important role in advocating the use of decimal fractions, in eliminating the Euclidean distinction between numbers and magnitudes, and in the understanding of real numbers as numbers. Finally, a modern rigorous construction and definition of real numbers (rational and irrational) was given by nineteenth-century German mathematician Richard Dedekind. He started with rational numbers and defined irrational numbers using the rational numbers.

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See Also: Arabic/Islamic Mathematics; Babylonian Mathematics; Greek Mathematics; Measurement, Systems of; Number and Operations; Numbers, Real; Pythagorean Theorem; Squares and Square Roots; Zero.

Numbers, Real

Category: History and Development of Curricular Concepts.
Fields of Study: Communication; Connections; Number and Operations.
Summary: The real number system is commonplace, but required centuries before it came to be understood in its modern form.

The real number system is often thought of as a number line, with each point on the line corresponding to a number. The set of real numbers includes all the integers and fractions (rational numbers); algebraic irrational numbers, such as the square root of 3 and the cube root of 19; and transcendental numbers such as π, $\log(2)$, and *e*, which do not satisfy any polynomial equation.

In the twenty-first century, students begin to explore whole numbers beginning in the earliest grades, as well as common fractions like 1/2 and 1/4. In the later primary grades they develop knowledge of base-10 decimal places, fractions as portions or divisions of a whole, negative numbers, and equivalent forms for fractions, decimals, and percentages. These notions are further expanded and applied in middle school, including concepts like ratios and proportions, integers, factorization, prime numbers, and exponential and scientific notation for very large numbers. Very large and small numbers, properties of numbers and various number systems, vectors and matrices with real number properties, and number theory may be studied in high school.

The real number system is the principal number system used in calculus, geometry, and measurement. In particular, when one uses coordinate (Cartesian) geometry to describe the plane or space, one labels points by pairs or triples of "real" numbers. In mathematics and the sciences, the word "number" without qualification is generally used to mean "real number."

Development of the Real Numbers

The ancient theory of length and measurement was very different from current understanding. The ancient Greeks (the civilization about which exists the most complete mathematical history) believed that any set of lengths were commensurable; in modern language, they believed that the ratio of any two lengths (or areas, or volumes) was a rational number. This was not a totally unreasonable belief, since indeed all lengths can be approximated very well by commensurable ones. It is not correct, though; for example, the ratio of the diagonal of a square to its side is the square root of 2.

Greek mathematician and numerologist Pythagoras knew this (it is a simple consequence of what is now called the "Pythagorean Theorem") and was further able to prove, contrary to the notion of commensurability, that no rational number, when squared, could equal 2. According to some stories, probably apocryphal, this discovery was so contrary to the belief system of Pythagoras and his followers that a discoverer was murdered or committed suicide. Ultimately, geometers were forced to accept the existence of irrational numbers.

The Greek mathematician and astronomer Eudoxus (c. 400–350 B.C.E.) wrote about the theory of proportions in a way that did not assume all lengths were commensurable and is generally credited with laying the groundwork for irrational numbers as legitimate mathematical objects.

Even after mathematicians realized that irrational numbers were required for practical purposes, the understanding of the real number line was somewhat vague and confused. Real numbers were understood, if at all, as things that could be approximated well by rational numbers or by decimal approximations. The major modern contribution to the understanding of real numbers was made by Richard Dedekind (1831–1916), who described the real numbers in terms of so-called "Dedekind cuts." In addition to its significance for abstract mathematics, Dedekind's insight also helped to explain some important phenomena in geometry (for example, why a line with points inside and outside a circle must intersect the circle).

This resistance to advancements in the understanding of number, this tendency for even very intelligent people to oppose enlarging the number system, even when doing so enables scientific and technological progress, is not unique to the ancient Greeks. A similar story unfolded much more recently with the development of the complex number system.

Decimal Representations

Every real number has a base-10, or decimal, representation. This consists of three components: a dot (called a "decimal point" in this context), a finite sequence of digits to the left of the decimal point (the integer part), and an infinite sequence of digits to the right (the fractional part). A digit can be 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9. Working from the decimal point left, the digits occupy the ones place, the tens place, the hundreds place, the thousands place, and so on; from the decimal point right, the digits occupy the tenths place, the hundredths place, the thousandths place, and so on. In symbols, if the a_k are digits, then the decimal expansion

$$a_k a_{k-1} \cdots a_2 a_1 a_0 \cdot a_{-1} a_{-2} \cdots a_{-k} \cdots$$

represents the real number

$$a_k 10^k + \cdots + a_1 10 + a_0 + a_{-1} 10^{-1} + \cdots + a_{-k} 10^{-k} + \cdots.$$

If from some point rightward, all the digits in the decimal representation of a number are zero, that expression is said to terminate, and the trailing zeroes are typically not written; for example, "7.24" instead of "7.24000. . . ." Integers have only zeroes to the right of the decimal point, and in such cases even the decimal point is often omitted.

Relying exclusively on decimal expansions as a way to understand real numbers can be problematic. Specifying a real number in this way requires an infinite sequence of digits. Unless there is a pattern, this requires specifying an infinite amount of information. For example, there is no known digit-by-digit description for important numbers like π and *e*. Dealing with infinite expressions is confusing for many people. For example, some people find it difficult to accept that 0.33333 . . . = 1/3, and even more people find it uncomfortable that 0.99999 . . . = 1.

Almost all real numbers have a unique decimal expansion, but some have two. As 0.99999 . . . = 1 illustrates, every number that can be written so that it ends in an infinite string of 0s also has an expansion that ends in an infinite string of 9s.

Structural Properties of the Real Number System

The real numbers form a field, which means that real numbers can be added, subtracted, multiplied, and divided (except by 0), and that the operations satisfy certain properties (for example, commutative, associative, and distributive laws). The real numbers are actually an ordered field, which means that there is a notion of what it means for one number to be less or greater than another that is compatible with the operations.

There is a natural way to measure distance between two numbers: the distance between numbers *a* and *b* is $|a-b|$, where $|\cdot|$ is the absolute value function. Loosely speaking, this means that one can talk about "closeness" of real numbers to each other; in technical language, the number line has a metric and a topology.

Unlike the set of integers (which is discrete), the real number line is continuous. The discrete/continuous distinction in mathematics is analogous to the digital/analog distinction in science and technology. A digital thermometer has discrete output, moving from 24 degrees to 25. An analog thermometer, on the other hand, can register 24 degrees or 24.65474 degrees or any other number. Unlike both the integers and the rational number system, the real number line is what called "topologically complete." Because of the ordering on the reals, this can be summarized as: "Any set of real numbers which has an upper bound has a least upper bound."

In mathematics history, adopting a continuous number system made it possible to develop "limits," the focal concept of calculus. The development of calculus, in turn, made possible numerous advances in sciences, especially physics and engineering.

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See Also: Calculus and Calculus Education; Coordinate Geometry; Numbers, Complex; Numbers, Rational and Irrational.

Numbers and God

Category: Friendship, Romance, and Religion.
Fields of Study: Number and Operations; Representations.
Summary: Many numbers and mathematical ratios are associated with religion and the notion of deity.

Numbers and religion have been linked since at least the beginning of recorded history. Many societies throughout the world have associated numbers with their spiritual beliefs. Some of these numbers still play a role in the fabric of society's belief systems, religious rituals, artistic renderings, and symbolisms. They continue to be explored, evaluated, and recognized in the religious teachings and traditions of many of the world's religions. As early as 1150 B.C.E., Indian mathematician Bhaskaracharya attributed the creation of the base-10 numeration system and zero to the Hindu god Brahma. Many ancient cultures and societies believed that certain numbers had spiritual significance. Historians, mathematicians, religious scholars, and others interested in such connections have found evidence of such beliefs in civilizations and religions like ancient Babylonia, the Society of Pythagoreans, Greece, Hellenistic Alexandria, Judaism, Christianity, and Islam. Many of these same beliefs continue into the twenty-first century.

Numbers of Pythagoras

Pythagoras of Samos (570–495 B.C.E.), who is often called the first pure mathematician, and his followers, the Pythagoreans, are well-known for their mathematical, philosophical, and religious beliefs. In antiquity, philosophy was believed by many to encompass the very essence of mathematics and religion. The perceived link between mathematics and the spiritual or divine world is succinctly stated by the Pythagoras maxim "All is Number." Among the legacies associated with Pythagoras are the theorem that bears his name; the creation and study of musical harmonies, which may have originated in Babylon; and concepts of sacred geometry, such as the divine proportion.

The "Divine Proportion" (or "Golden Ratio") is often seen by mathematicians and other scholars in nature's designs and natural phenomena. The Greeks widely used the principle in sculptures and architecture. Phidias (490–432 B.C.E.), who is counted among the best Greek sculptors, used the Divine Proportion in designing the Parthenon, a temple to the goddess Athena. In honor of Phidias, the Divine Proportion is usually symbolized by the Greek letter φ representing the first letter of his name. To understand the Divine Proportion, consider a rectangle. The rectangle is said to be in Divine Proportion if the ratio of its length to its width has the following value

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618\,.$$

Rectangles with these proportions are called "Golden Rectangles." This proportion (about 8/5) continues to be used by artists and architects in designing structures for aesthetic appeal.

The Number "12"

The Greeks considered the number "12" to be significant since it represented the number of gods on Mt. Olympus: Zeus, Hera, Athena, Poseidon, Apollo, Artemis, Demeter, Hermes, Aphrodite, Ares, Hephaestus, and Hestia. The significance of "12" probably originated with the Sumerians in Mesopotamia. Later, the Babylonians used the number "12" in developing their calendars and their clocks. They developed the zodiac by dividing the heavens into 12 equal sections named for constellations, one for each calendar month. These sections continue to be the 12 signs of the zodiac, an idea that was passed down from society to society throughout the ages. The Babylonian zodiac impacted many societies in the Western world. In Christianity, the 12 disciples of Jesus are usually considered to be symbolic of the 12 tribes of Israel, which may have been influenced by the 12 signs of the zodiac. The number "12" also has significance in Buddhism. For example, the Buddhist Wheel of Life, which depicts the world and the human condition, has 12 stages. In this tradition, life is composed of 12 stages, which keep the wheel of life turning.

The Number "7"

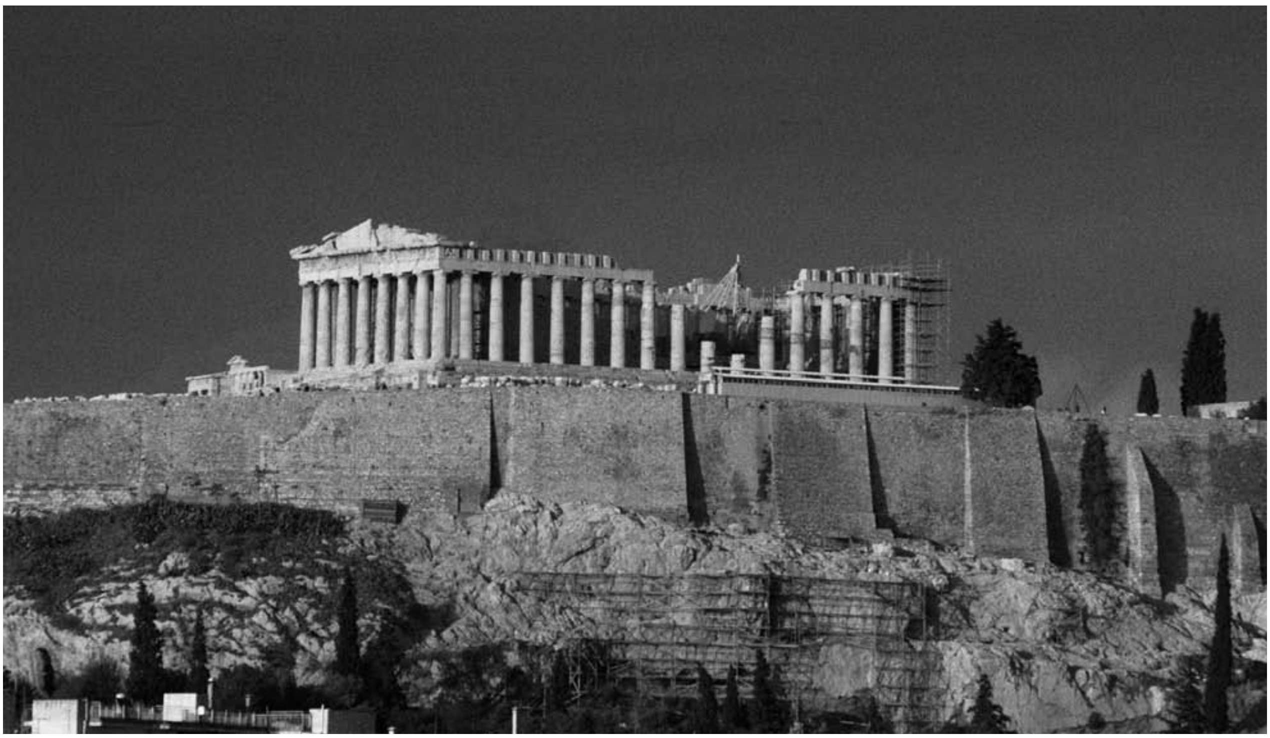
The number "7" is a significant number in Judeo-Christian and Islamic religious traditions. The creation story in the book of Genesis states that God made the heavens and the Earth in six days and rested on the seventh day. The number "7" is associated with divine completion and perfection. There are also references to 7 spirits, 7 churches, 7 stars, 7 seals, 7 trumpets, 7 vials, 7 thunders, 7 plagues, 7 mountains, and 7 kings, and many more references. The number "7" occurs frequently in Muslim architecture, art, and literature. The Qur'an often couples the number "7" with references to Allah as the all-powerful creator as well as with concepts like the 7 heavens, the 7 Sleepers of Ephesus, and the 7 periods of creation.

The Number "19"

In 1974, Rashad Khalifa used a computer to explore the structure of the Qur'an. He discovered that the number 19 occurred with unusual frequency. This occurrence was unexpected since 19 had never before been recognized as a significant number in the Islamic religion. Khalifa published his discovery in his 1981 work, *The Computer Speaks: God's Message to the World*. These findings were called the "Qur'an Code." The first verse of the Qur'an states: "In the name of Allah, the compassionate, the merciful." In Arabic, the letters that make up this verse total 19. Khalifa discovered that every word in this verse is mentioned a number of times throughout the Qu'ran, and these numbers are all multiples of 19. Consequently, Khalifa's conclusion was that the number "19" was divinely selected as a number of significance in the Islamic religion.

Bible Codes

What are often now known as "bible codes" were popularized in the twentieth century, but numerical symbolism dates back to much earlier times. The Jewish book *Sefer Yetzira* (Book of Creation) contained sacred numbers. As writer and scientist Clifford Pickover has explained: "Kabala is based on a complicated number mysticism whereby the primordial One divides itself into 10 sephiroth [numbers] which are mysteriously connected with each other and work together. 22 letters of the Hebrew alphabet are bridges between them." In gematria methods of analysis, each letter was assigned a number. The values of a word or phrase were added and the then values were analyzed for spiritual implications. For example, the word for "life" in Hebrew is *chai*, which is made up of two letters, a *chet* (8) and a *yud* (10). When added together, they sum to 18. The number 18 then took



The Parthenon was built in Athenian Acropolis, Greece, to honor the Greek goddess Athena. It is thought to be the perfect example of a Doric temple and was designed using the Divine Proportion.

on symbolic meaning, which also translated to daily life. It was considered good form to give monetary gifts in 18 and its multiples. The number 18 has been also considered prosperous in certain parts of China, and it also took on spiritual importance in India, such as in the 18 chapters of the sacred Hindu text Bhagavad Gita.

Researchers have mathematically examined the Bible using methods such as two-dimensional arrays, which have been tested for what are known as "equidistant letter sequences." Some found what seem to be words meaningfully related to adjacent portions of the text, and they claimed that their results were statistically unlikely to be due to chance alone. Author Michael Drosnin reported on some mathematical and computer analyses, referring to them as the "Bible code," in order to highlight apparent predictions and to compare to twentieth-century knowledge. Some of the advocates of Bible code analyses point to apparent prediction of the dates of major world events as proof of the existence of such codes. Computations on the age of the universe are also sometimes cited as evidence, such as when first century rabbi Nechunya ben Hakanah used the Bible to compute the age as 15.3 billion years, which is relatively close to some twenty-first-century estimates. Critics, however, have countered these assertions by citing flaws in the statistical methodology and noting that any sufficiently long text may produce seemingly nonrandom patterns or clusters.

Numerical Defense of the Resurrection

During the twenty-first century, associations of numbers with religion continue to evolve. In 2002, Richard Swinburne, philosophy professor at Oxford University, applied Bayesian statistical methods, named for mathematician Thomas Bayes, in his defense of the Christian tenet of Jesus's resurrection from the dead. He noted that it was extremely improbable, based on the laws of nature, for someone who had been dead for 36 hours to come back to life. Swinburne asserted that if there is a God, only God would be able to defy the laws of nature and make the dead come alive. In proving his point, Swinburne assigned probability values to the existence of God and some of the events described in the New Testament, such as the credibility of witness testimony. After mathematical analysis, Swinburne concluded that Jesus's resurrection was extremely probable, namely, 97 percent. Swinburne's use of mathematical logic and statistical methods to answer questions of faith is another step in a long tradition of connections between numbers and religion.

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See Also: Infinity; Numbers, Rational and Irrational; Religious Symbolism; Religious Writings; Sacred Geometry.

Nutrition

Category: Medicine and Health.
Fields of Study: Algebra; Data Analysis and Probability; Number and Operations.
Summary: Mathematicians and nutrition scientists model and analyze numerous aspects of nutrition and diet.

Nutrition is the system of providing food to an organism as well as the science of food and eating. Nutrition science is an interdisciplinary field that involves a wide variety of disciplines, including mathematics, statistics, culinary science, physiology, genetics, biochemistry, psychology, medical sciences, sociology, anthropology, and ethnography. Mathematical and statistical methods are widely used to describe and analyze different nutrients in food, determine their impact on nutrition and health, develop eating plans, assess public opinion, and inform public policy.

Meal-planning and nutritional labels are often used in classroom mathematical problems.

Taxonomies of Human Nutrients

The understanding of what balanced nutrition means is a difficult and controversial subject, with many schools of scientific thought, cultural traditions, and governing bodies proposing different ways of eating. Many people may be familiar with the U.S. Department of Agriculture's food pyramid, which was introduced in the 1990s to replace the older four food groups model. There are similar guidelines produced in many countries, using a mix of scientific research and expert opinion. One measurement scale for a nutrient is the time it takes for its lack to manifest itself in health problems. Lack of energy-providing nutrients, such as fats, proteins, and carbohydrates, is felt within hours as hunger and causes symptoms within days. A disbalance of macrominerals, such as potassium and calcium, as well as some vitamins has been shown to lead to specific diseases and is felt within weeks to months.

For example, physician James Lind published an influential treatise on scurvy in 1753, based in part on his controlled experiments with British sailors. Lack of some other vitamins and minerals can take years to manifest. Flavonoids are found in plants, changing coloration, smell, and taste. Researchers hypothesize that flavonoids regulate organism responses, such as inflammations and allergies, as well as reactions to carcinogens, bacteria, and viruses. Most studies of flavonoids in human nutrition are only decades old. Probiotics are live microorganisms most frequently eaten with fermented food. They may affect the immune system, blood pressure, inflammation, and cancer. Prebiotics are food items, such as inulin in chicory roots, that promote the growth of microorganisms in the digestive tract. Water participates in most systems and processes in the body, playing roles such as a solvent for other substances and entering chemical reactions.

Researching and Modeling Nutrition

Data collection and quantitative analysis are used for a variety of purposes in nutrition science. For example, they are used to investigate the effects of nutritional deficiencies, optimize diets for long-term health and longevity, study the effectiveness of weight-loss plans, and establish causal links between political policy changes and nutrition and health effects. Qualitative methods like case studies and ethnographic studies are insufficient to establish cause, though they may highlight key variables. One critical principle of scientific studies that seek to make causal connections is isolating a small number of variables to systematically manipulate, while controlling the rest. Because nutrients interact with all systems in the body—with other organisms living in the body, with each other, and with behaviors other than eating—the complexity of the resulting system can make this approach difficult. Further, the effects of some types of nutrients take years or even decades to uncover, or they may occur in only a small number of people. Studying these would require extensive longitudinal studies or very large sample sizes to be statistically valid, which may have significant practical and ethical barriers. Finally, individual differences in reactions to nutrition changes may be large and nonrandom, depending on genetics, culture, and daily habits, which means that averaging the effects of nutritional interventions may overlook important effects on small minorities, such as allergic reactions.

Mathematicians and nutrition scientists use mathematical modeling and simulation to investigate the functions of systems and to experiment with the conditional responses of multiple variables. Increases in computing power have made complex modeling a feasible alternative to traditional scientific experimentation. Problems are drawn from areas of concern, such as obesity, diabetes, cancer, and toxicology. Many models rely on collection of kinetic body data to develop accurate models of physiological processes, such as bioperiodicity and membrane transport, which is also possible because of advances in medical imaging and other technology. Computational approaches are used to estimate distributions of parameters, evaluate linear integrators and other functions, manipulate multiple variables in stochastic models, and create visualizations. Mathematical or statistical approaches, such as neural networks, graph theory, and cluster analysis, have also been used to model data or systems and to make connections.

Genetically modified foods are a controversial subject in nutrition. Typical reasons for altering food are for resistance to pests or disease or for nutritional benefits. The Swiss-developed "golden rice" has higher levels of vitamin A than standard rice strains, which would theoretically benefit third world countries where rice is a staple food and vitamin A deficiencies

The Japanese traditionally eat more fish, vegetables, grains, and fruit and consume smaller portions than most Western diets.

are common. Some support the use of such foods to combat hunger in areas of the world with chronic shortages and endemic malnutrition. Others cite the unknown long-term effects, such as spontaneous cross-pollination with unmodified organisms, as well as the ethical implications. Mathematicians and scientists have helped to create genetically modified foods and have investigated many questions related to them. For example, informaticists have used combinatorial reduction rules to create a model to detect unknown, genetically modified organisms. Others research and model aspects such as the likelihoods of positive and negative ecological outcomes, pathogenicity, public acceptance, and impacts on international trade using probabilistic and statistical methods, simulation, differential equations, and a wide variety of computerized modeling techniques.

Diets and Meal Planning

A diet is the description of types and quantities of nutrients consumed. Because organisms vary in ways other than food intake, dietary variables are typically studied in their relationships with other variables—either direct proportionality or more complex functions. Different cultures have varied proportions of nutrients in their diets, as well as certain prohibitions. For example, Aleuts traditionally eat a large amount of meat, consuming about eight times more protein than South American agricultural tribes. Japanese and Mediterranean diets are often cited for emphasis on certain fats, fruits, vegetables, and carbohydrates. Both Jewish and Muslim traditions forbid certain types of foods. People may also choose diets for specific goals, such as weight loss or control of medical conditions like diabetes or high blood pressure, often with little scientific evidence of effectiveness—though scientists are seeking ways to validate or refute such claims. Globalization has made different types of diets and foods increasingly known and accessible to people everywhere.

Software for planning least-cost nutritional meals was developed for mainframe computers in the early 1960s and evolved during the 1970s to include food preferences options. Later research in the 1980s and the evolution of personal computing led to new software that used mathematical programming to optimize and maximize menu planning for different variables, including nutrition, allergies, and preferences. Internet-based software and algorithms, such as that used by the weight-loss company Weight Watchers with their Weight Watchers Online program, now allows people to track and plan menus based on a variety of criteria, often dynamically linked to databases with recipes, past behavior, and weight or measurement tracking. Large institutions, such as schools and hospitals, may use software that includes inventory and other supply variables.

Nutrition and Mathematical Problem Solving

There are studies directly linking nutrition and success in mathematics. One group of researchers found that providing a balanced breakfast before the morning mathematics class raised test scores more than any other variable they analyzed, such as changes in teaching methods. Different cultures have different beliefs of what constitutes "brain food." Certain types of fat, vitamins B and C, and monosaccharides have been shown to increase memory and speed of computation within time periods from minutes to days from increased consumption. More complex cognitive effects of food, such as connections between gluten or lactose sensitivity and attention, are being investigated.

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See Also: Cooking; Drug Dosing; Genetics; Measurement in Society.

O

Ocean Tides and Waves

See *Tides and Waves*

Oceania, Australia and New Zealand

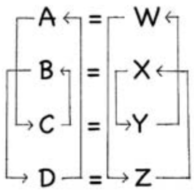
Category: Mathematics Around the World.
Fields of Study: All.
Summary: The indigenous cultures of Oceania are mathematically interesting.

The United Nations classification for Oceania includes Australia and New Zealand as well as the hundreds of Pacific Islands groups under the headings Melanesia, Micronesia, and Polynesia. The Australian Mathematical Society was founded in 1956 and promotes mathematics and its applications. The New Zealand Mathematical Society was found in 1974 and promotes research and the dissemination of mathematics. Mathematicians born in Australia and New Zealand include Field's Medal winners Terence Tao (2006) from Australia and Vaughan Jones (1990) from New Zealand. High school students participate in the International Mathematical Olympiad. Australia began its participation in 1981 and hosted the contest in 1988, while New Zealand first participated in 1988. Mathematics historians and ethnomathematicians have researched the mathematics of the indigenous inhabitants of Australia and New Zealand. For example, the structures of Australian Aboriginal kinship systems can be modeled by the algebraic theory of groups, while the wood carving and tattooing done by the Maori of New Zealand embody geometrical principles of symmetry. These cultural achievements interest mathematicians and teachers of mathematics and also have influenced the humanities, the social sciences, and popular culture.

Australia

Studying Aboriginal kinship systems has greatly influenced anthropology and can be mathematically modeled. To give just one important example, Claude Lévi-Strauss, in support of his ideas on structural anthropology, cited what he called "the Australian facts" to help argue that a system of exchange (as illustrated by marriage partners reciprocally chosen from paired sections) underlies the origin of marriage rules.

Many Aboriginal societies are divided into two halves, with four sections in each half, for the purpose of determining kinship. The example best known to mathematicians, because of the classic work of Marcia Ascher, is that of the Warlpiri of Australia's Northern Territory. A schematic diagram is shown below.



The equal sign designates allowed marriages. That is, for members of a section in either half, there is one section in the other half from which marriage partners come. For instance, women in section A marry men from section W, and men from section A marry women from section W. Children's sections are determined by their mother's; the directed arrows show how. For instance, if a mother is in section A, her children are in section C; mothers in C have children in B; mothers in B have children in D; and mothers in D have children in A—completing a cycle. Similarly, mothers in W have children in Z, and so on. Thus, the matrilineal cycle has a length of 4. For fathers, if a man is in A, for instance, following the arrow backward shows that his mother is in D, so his father is in Z. Then his father's mother is in W, so his father's father is in A again. Thus the complete patrilineal cycle has a length of 2.

If one writes I for one's own section, m for one's mother's section, m^2 for one's mother's mother's section, f for one's father's section, and so on, the cyclic relationships can be expressed by $m^4 = I$ and $f^2 = I$. Other algebraic relationships, like $(mf)(mf) = I$, can be verified from the diagram. The resulting algebraic structure is that of the dihedral group of order 8. The Warlpiri, of course, do not have the concept of group, but those learning the system are asked to solve word problems like, "If someone's mother is in a particular section, then in what section is such-and-such a relative?" The Warlpiri abstract from the personal relationships to conceptualize the system itself. General terms of address reflect the individual's place in the structure. Kin relationships determine a person's behavior, obligations, place to live, and relationships to plants, animals, and landscape; they also link past, present, and future generations.

The Aboriginal view of the origin of their kinship system in the journeys of their ancestors during the ancestral past (known as the "dreamtime") is reflected in Aboriginal paintings. Such paintings are noted for their symmetry, and particular geometric elements indicate individual places, ancestral beings, or clans. The current interest in Aboriginal art has brought these geometric forms to a worldwide audience.

New Zealand

Geometric art pervades Maori culture in dance, song, music, weaving, painting, latticework, carving, and tattooing. Wood carving is the most prominent, though facial and body tattoos also continue to be symbols of Maori identity. Traditional Maori carving uses a small number of design forms and motifs, combined according to well-established rules. Rafters and ridgepoles of the Maori meetinghouse are decorated with carvings

A Maori carving representing Marupo, a warrior ancestor of the Maori tribe in New Zealand.

that embody tribal history. These carvings employ all seven of the symmetry groups that characterize strip patterns. They are often colored in ways that complement, rather than echo, the symmetries. Maori art also uses bilateral symmetry, but the symmetry is often broken by the nonsymmetrical use of colors or by the addition of small figures that vary. Maori tattoos use many of the same themes and motifs as does carving. Also, individuals' tattoos serve to identify family, tribe, community, birthplace, and inherited or achieved authority.

Maori symmetric forms are united by their near identity while differing in their asymmetries. This aspect reflects the way the Maori characterize reality by pairs of things existing in a tension between union and separation. Understanding the formal geometric patterns thus gives insight into Maori culture. Maori geometric art has become part of global culture. For example, Maori carved wooden bowls appear in Paul Gaugin's paintings. In Herman Melville's *Moby Dick*, the tattooed harpooner Queequeg possesses—and sells—Maori tattooed ancestral heads. Enlightenment philosopher Immanuel Kant felt that he had to discuss Maori tattoos in examining the nature of beauty, though he concluded that Maori tattoo designs could be beautiful only if they were not on a human face. Additionally, Maori tattooing plays a key role in the acclaimed 1994 film *Once Were Warriors*.

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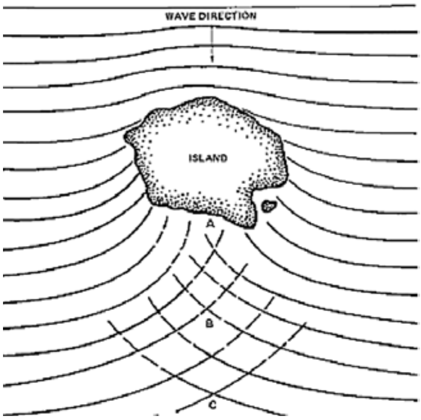
See Also: Asia, Eastern; Europe, Northern; Oceania, Pacific Islands; Tao, Terence.

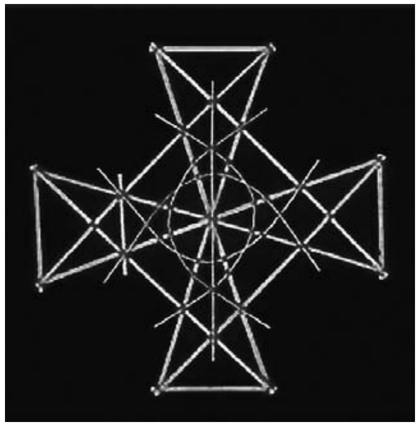
Oceania, Pacific Islands

Category: Mathematics Around the World.
Fields of Study: All.
Summary: The people of the Pacific Islands historically used sophisticated mathematics, including a unique method of navigation.

The Pacific Ocean covers more than one-fifth of the Earth's surface and includes hundreds of islands. In the nineteenth century, few visitors to the Pacific Islands were able to match the skill of Pacific Islanders in solving arithmetic and algebra problems. The people of the Marshall Islands, scattered over dozens of atolls across the central Pacific, were master navigators who tracked their way over huge expanses of ocean without mechanical aids. The compass, sextant, and chronometer, which their European contemporaries were reliant upon for safe and successful voyaging, were completely unknown to them. What they possessed instead were a set of aids that relied upon an extremely complex type of knowledge related to what they could observe and even feel about the ocean around them. These aids were called *Mattangs*, *Meddos*, and *Rebbelibs* by their users and are known today as "stick charts."

Some other instances of mathematics in Pacific Island culture and in the Pacific Islands region include the often-complex geometric patterns found in basket weaving, such as the design named "stars," which has a tessellation pattern that is mathematically sophisticated and is reminiscent of Dutch graphic artist M.C. Escher's drawings. These patterns can also be found in traditional tattoos, where the type of pattern had great cultural significance and represented the rank and bravery of the tattooed person. Scientists have also modeled the number of species on islands as a mathematical power function that depends on land area, and they continue to study island populations of birds and other species in this context. Researchers have explored barriers to success in mathematics in the Pacific Islands





Figures 1 and 2.

and recommended that teachers include culturally relevant content in their classrooms. Professional development programs and consortiums offer training for teachers and explore mathematics education for Pacific-region children.

Stick Charts

Stick charts were made from strips of the midrib of a coconut frond or pandanus root bound together with coconut sennit in geometric patterns meant to represent currents flowing around their low-lying atolls. Small shells or coral pebbles were attached to indicate the location of islands, and curved sticks were used to represent wave patterns.

The first of these charts, the Mattang, was a small square chart used to teach how waves reflect and refract, or bend, around a single island or atoll (see Figures 1 and 2). By detecting a change in the direction of the prevailing swell, a navigator could discern the presence of an island or atoll over the horizon. The Meddo was an actual chart covering a small set of atolls and used for voyages to nearby atolls. Meddo charts also showed the direction of the main ocean swell and how it curves around specific islands and the distance from a canoe at which an island could be detected. The Rebbelib was a more complex version of the Meddo and was used to represent an entire chain of islands or even the whole of the Marshall Islands. It showed the complex relationship between the islands and the major ocean swell.

Stick charts were not made and used by all Marshall Islanders. Only a select few knew the method for making and reading the charts, and the knowledge passed only from father to son. However, so that others could utilize the expertise of the navigator, 15 or more canoes sailed together in a squadron, accompanied by a lead navigator skilled in use of the charts. Because the knowledge contained in each chart was a closely guarded secret, they were not normally carried on a voyage. Instead, the navigator memorized the chart and gauged the wave patterns entirely by his sense of touch. Crouching in the bow of his canoe, he would literally feel the motion of his vessel.

It was not until 1862 that this unique navigational system was revealed in a public notice prepared by a resident missionary. It was an additional 30 years before it was comprehensively described by Captain Johann Winkler of the German Navy. He became so intrigued by the stick charts that he made a major effort to determine the navigational principles behind them and convinced the navigators to share how the stick charts were used. He recognized that the stick charts represented a significant contribution to the history of both navigation and cartography because they symbolized something that had never before been accomplished—a system of mapping and navigating by ocean swells. They are an indication that ancient maps may have looked far different, encoding different aspects from the natural world, than the maps commonly used today. The use of stick charts and navigation by swells apparently came to an end shortly after World War II. The venerable stick chart and ocean-going canoe were no match for large motorized vessels with modern navigational devices. They do, however, continue to be made in the Marshall Islands, though very few people are able to use them as navigation aids. They are primarily made and sold instead as tourist souvenirs.

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Thomas W. Hair

See Also: Escher, M.C.; Mapping Coastlines; Marine Navigation; Oceania, Australia and New Zealand.

Operations

See *Number and Operations; Number and Operations in Society*

Optical Illusions

Category: Arts, Music, and Entertainment.
Fields of Study: Geometry; Measurement; Problem Solving.
Summary: Optical illusions are predictable illusory phenomena that are not yet fully understood.

Optics, generally, is the science of the visible. Physical optics is the study of the nature and propagation of light. Physiological optics is the study of neurophysiological processes of light reception and image forming as conditions of vision and merges with psychology and cognitive science into a unitary "vision science." Optical illusions likely have been observed as long as mankind has existed. Some optical illusions arise from people's ability to see in three dimensions, even though retinal images are flat representations on a curved surface. Extracting three-dimensional information from ambiguous two-dimensional images requires interpretive rules in the brain. Many optical illusions have mathematical connections, especially in the perception of geometry within the illusion. They are popular as entertainment, and mathematics teachers sometimes use optical illusions in the classroom in order to engage students and to develop visualization skills.

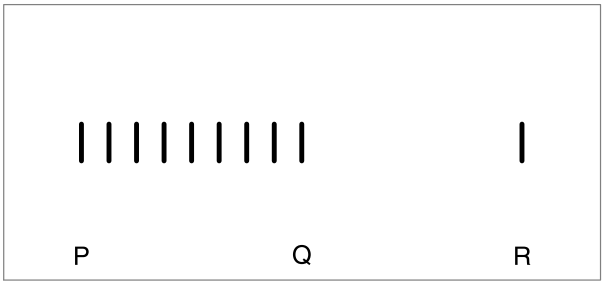
Examples

Physical phenomena leading to seeing unreal things, or to seeing real things in a distorted way (for example, phenomena due to special atmospheric conditions: halos, coronas, and sightings of distant objects caused by reflections between air layers of different density) are now well understood, and not usually named "illusions." The apparent flattening of the sun disc at the sunset is in accord with the laws of light propagation (differential refraction), but it is not an illusion. Illusions of perception are situations "when perception goes wrong" and where a central (neurophysiological or psychological) cause must be supposed—something is perceived as something else (error of identification) or is perceived differently than it is (error of quality or quantity). For example, the moon at the horizon is often reported to appear larger than if seen high in the sky, although the angular size of the moon disc is in both instances the same (approximately 30 arc minutes)—this is the famous "moon illusion."

Illusory phenomena have been observed since ancient times, for example, the "moon illusion" was known to Ptolemy, and an illusory "motion aftereffect" caused by watching a waterfall was mentioned by Aristotle. However, the proper scientific study of visual illusions began in the middle of the nineteenth century with the discovery of geometric-optical illusions, (distortions of perceived lengths, sizes, and shapes observed in simple drawings or in real-world situations). For example, a path in the visual field subdivided into a series of segments usually appears longer than the same path that is empty (see the Oppel–Kundt illusion, Figure 1). Lengths of linear segments may be overestimated or underestimated, depending on added elements (for example, the popular Müller–Lyer illusion). Geometric figures drawn over linear or curvilinear rasters often appear deformed (see the Hering, Zöllner, or Ehrenstein–Orbison illusion Figure 2). Other instances of optical illusions involve judgments of brightness (see Figure 3), and particularly illusory "contrast phenomena," such as well-known Mach bands, or the Hermann grid (see Figure 4). More recently, dynamic phenomena, such as illusory motion seen in static pictures, or the famous "scintillating grid," have been described.

Generally, all these phenomena demonstrate the universal principle of context-dependence in visual (and any) perception: a stimulus, *S*, is perceived

Figures 1 and 2



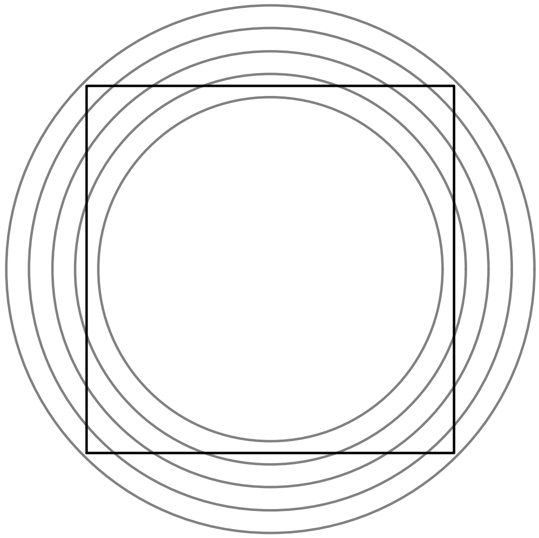
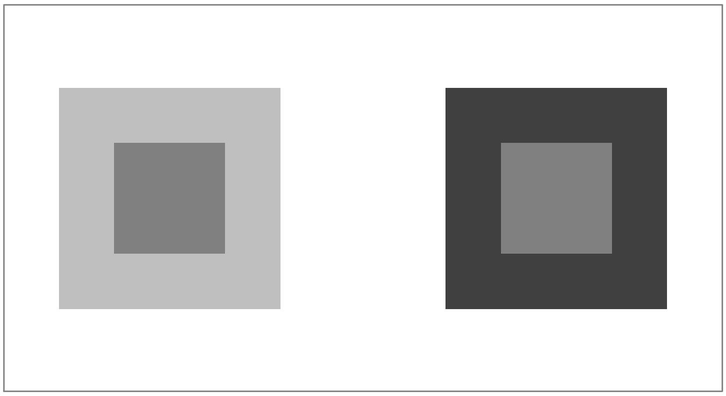


Figure 1. Oppel-Kundt illusion: the length of the PQ segment appears greater than the segment QR, although PQ = QR.

Figure 2. Ehrenstein-Orbison illusion: sides of the square drawn in an array of concentric circles appear inward-bent, although they are really segments of straight lines.

Figures 3 and 4



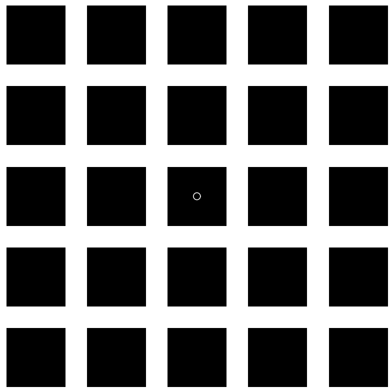


Figure 3. Context effects on perceived brightness: the inner square of the left-hand-side figure appears darker than the inner square of the right-hand-side figure, although they are printed with exactly the same gray-shade level. Figure 4. Hermann grid: illusory grayish shadows are seen at the crossings of white stripes, although objectively the background is uniformly white. (Fixate on the figure center marked by the circle.)

differently if presented together with a context stimulus *S′* than if presented alone. In other words, a purely attentional separation of *S* from *S′* is impossible in spite of the observer's effort.

What differentiates all these phenomena from incidental errors of perception is that they occur regularly and predictably in most or all observers. After more than 150 years since their discovery, there is still no satisfactory theory of these phenomena, although a great variety of explanations have been proposed.

Explanations

The two main directions of explanatory approaches have traditionally been the empiricist and the nativist theories. The empiricist theories, going back to Herman Helmholtz's theory of unconscious inferences, emphasized the role of the subject's past experience and of cognitive factors forming the perception. By contrast, the nativist theories searched for explanations in the structure and the functional principles of the sensory organ itself.

Empiricist theories, in spite of their speculative character, have been revitalized by cognitive psychologists and are still influential; for example, a popular theory sought to explain a group of optical illusions as results of inappropriate constancy scaling due to erroneous perspectival interpretation of the illusion-inducing figure. However, these theories ignore much of empirical counter evidence, such as tactile analogies of certain optical illusions, or geometrico-optical distortions observed in contexts not suggesting any perspectival interpretation. Neonativist theories integrating approaches of Gestalt psychology and neurophysiology and searching for interactions within higher levels of the visual system are arguably more promising, although they are usually limited to circumscribed groups of illusory phenomena. The general opinion in the early twenty-first century is that the broad variety of optical illusions cannot be explained by a single cause; therefore, a unitary theory of optical illusions is rather unlikely.

Optical illusions are neither deceptions of the eye nor errors of the cognitive processing of sensory data. They are facts of vision, presumably manifestations of the functional principles of the visual system in its entirety. The same functional principles, or the "laws of seeing," are at work in visual arts, or in visualization technologies such as virtual reality. The study of optical illusions in laboratory as well as in natural environments importantly contributes to the understanding of the process of vision and of the nature of the visual life world.

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Jiri Wackermann

See Also: Magic; Mathematical Puzzles; Puzzles; Vision Correction.

Orbits, Planetary

See *Planetary Orbits*

Origami

Category: Arts, Music, and Entertainment.
Fields of Study: Geometry; Representations.
Summary: People explore many interesting mathematical questions using the art and principles of paper folding.

Origami is the famous Japanese art of paper folding. Historically, it has been used for a variety of purposes, including document certification and as a way to represent religious symbols. In traditional origami, a single piece of paper is folded to construct one of a variety of objects. The art has grown to include compound forms that involve connecting several individual origami pieces together, with modular origami specifying geometrically equal pieces.

Origami art, mathematics, and science have many explicit interconnections, and in the 1990s and 2000s there have been several conferences specifically devoted to these links. In the twenty-first century, computational origami is an emerging discipline that applies mathematical theory and computational algorithms to formulate and solve complex folding problems, many of which have applications in engineering, industrial design, and a variety of sciences. Such solutions are often called "origami technology." For example, engineers and mathematicians explored origami lenses for use in space telescopes, and precision folding technology is already being used to optimize manufacturing processes.

Origami forms are inherently mathematical. Their geometry can be identified as reflections with respect to the folding line. The possible operations for points and lines in origami, using a single fold, are described by seven axioms generally known as the "Huzita–Hatori axioms," named for mathematicians Humiaki Huzita and Koshiro Hatori. However, mathematician Jacques Justin may have been the first to enumerate these seven axioms. The axioms allow mathematicians to answer interesting questions, such as the classic problems of trisecting an angle and doubling the cube, which are impossible using only ruler and compass constructions. More generally, it is possible to solve any equation up to degree three with origami geometry. Further, although origami forms are usually produced using finite sheets of paper, origami folding can theoretically be extended to the infinite plane.

Use of Origami in Modern Mathematics

In the late twentieth century, mathematicians got interested in the foundations of this art. For this community of scientists, the creation of models in origami is not a matter of inspiration; it is spurred by the search for understanding of the concepts and limitations of Euclidean geometry, properties of geometric figures, symmetry, angles, lines, and mathematical communication, among others.

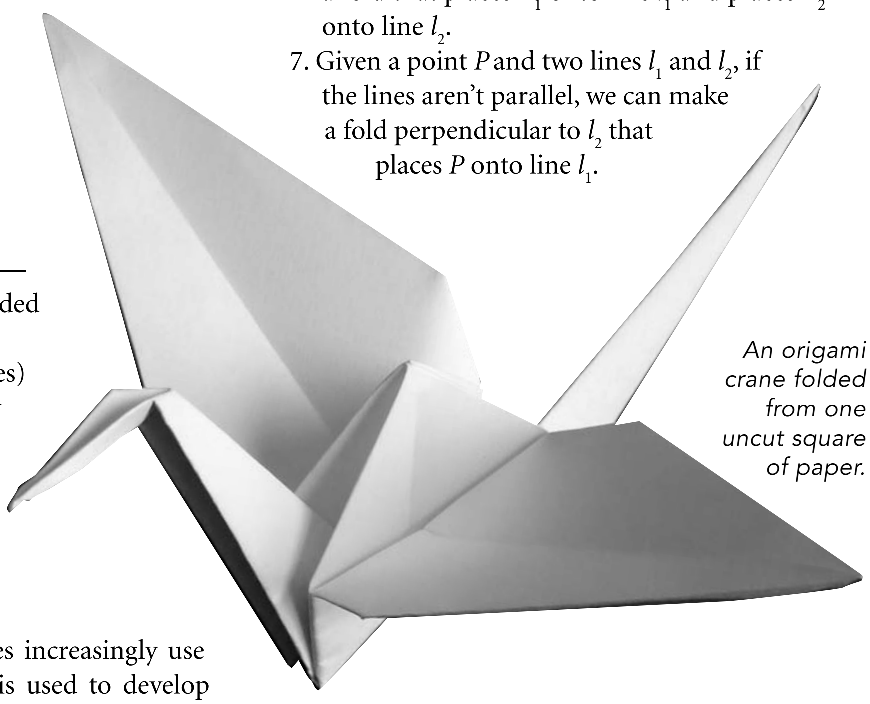
There are several major topics in the practice and study of origami, including the following:

- Its geometry and relationship between this and other geometries, in particular, Euclidean geometry
- The straightening of the bend—whether a model can be unfolded (which has been studied by Marshall Bern and Barry Hayes)
- Rigid origami—the possibility of constructing models if the paper were replaced by metal (which has already been used for solar panels of satellites in space)

Mathematics teaching techniques increasingly use origami. Moreover, paper folding is used to develop manual dexterity, as well as to teach aesthetics appreciation and topics such as proportions, foundations of geometry, and measurements. Origami is also a handy resource for other areas, like mathematical communication, problem solving, and investigation of three-dimensional objects and spatial relationships.

Huzita–Hatori Axioms

1. Given two points P_1 and P_2, we can fold a line connecting them.
2. Given two points P_1 and P_2, we can fold P_1 onto P_2.
3. Given two lines l_1 and l_2, we can fold line l_1 onto l_2.
4. Given a point P and a line l, we can make a fold perpendicular to l passing through the point P.
5. Given two points P_1 and P_2 and a line l, if the distance between P_1 and P_2 is equal to or larger than the distance between P_2 and l, we can make a fold that places P_1 onto l and passes through the point P_2.
6. Given two points P_1 and P_2 and two lines l_1 and l_2, if the lines aren't parallel and if the distance between the lines isn't larger than the distance between the points, we can make a fold that places P_1 onto line l_1 and places P_2 onto line l_2.
7. Given a point P and two lines l_1 and l_2, if the lines aren't parallel, we can make a fold perpendicular to l_2 that places P onto line l_1.



An origami crane folded from one uncut square of paper.

Robert Lang proved that this list of axioms covers all possible cases for a single folding. If one of them is removed from the list, it is no longer complete.

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Liliana Monteiro

See Also: Axiomatic Systems; Geometry and Geometry Education; Greek Mathematics; Symmetry.

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Pacemakers

Category: Medicine and Health.
Fields of Study: Algebra; Geometry.
Summary: Artificial pacemakers send a signal to the heart to keep it pumping and mathematicians develop models to determine when and how often to do so.

While a pacemaker is often thought of as a regulator for the heart, a variety of natural pacemakers are responsible for regulating numerous bodily functions including circadian rhythms and menstruation. The actions of natural pacemakers can be modeled as coupled oscillators, where, for example, the behavior of the natural pacemaker influences the function of the heart and vice versa. Square waves or sine waves are often useful in understanding the theory of coupled oscillators, which dates back to 1665 when Christiaan Huygens noticed synchronization in pendulum clocks.

Scientists and mathematicians have shown that chaotic oscillation or amplitude death can also occur in coupled scenarios. A change in the rhythm or in the way they are coupled can result in a change in function, such as in irregular menstrual periods or menopause. Dynamical systems model the interactions between coupled oscillators and allow for theoretical predictions. Using these models, mathematicians, biologists, and medical professionals have made significant advances in understanding natural pacemakers and in designing effective artificial pacemakers. Some of the related mathematical theory is taught to undergraduate mathematics students.

Heart Rhythms and Pacemakers

The sinoatrial node (SA node) is thought to act as the heart's natural pacemaker via electrical impulses. The typical rate for a resting heart is 60 to 70 beats per minute. The pacemaker cells keep the heart pumping at a steady rate, but medical problems can lead to chaotic behavior and cardiac arrest.

Defibrillation may reset the rhythm in some cases but an artificial cardiac pacemaker may be required if the rhythm remains chaotic. Wavelet transforms have been used to effectively model cardiac signals but implementation is difficult because of high power consumption. Australian anesthesiologist Mark Lidwell and physicist Edgar Booth are believed to have designed the first artificial pacemaker in 1928.

American physiologist Albert Hyman also developed an early pacemaker. Many designers of artificial pacemakers have assumed that regular impulses from a pacemaker should be used to stabilize the heartbeat. However, a periodic signal may lead to chaos in some mathematical models, so scientists are developing pacemakers that send impulses based on chaos control theory.

Body Clock and Jet Lag

Jet lag is thought to to result from a desynchronization of the suprachiasmatic nucleus (SCN) pacemaker cells in the hypothalamus of mammals. Experimental studies suggest that the SCN may synchronize within one week. Scientists and mathematicians have mathematically modeled the system as a network with connections between the cells, which are called *nodes* in the language of graph theory.

For example, mathematicians Channa Navaratna and Menaka Navaratna have adapted a model of neuroscientist Peter Achermann and bioinformaticist Hanspeter Kunz. The hypothalamus is thought to have 16,000 pacemaker cells, so they analyzed computer data from a model with this many pacemaker cells and found that the number of long-distance connections in the network determined the synchronicity time. They examined the types of network connections that are needed between the nodes in order to make the model synchronize in a week, and they designed a model that consistently synchronized in close to seven days.

Scientists and mathematicians have also studied many other issues related to pacemakers, such as interference and power issues. There is controversy and conflicting evidence on whether devices such as cell phones or iPods affect pacemakers. Many medical professionals presume an association until clearer evidence to the contrary is found and recommend keeping the devices at least a few inches away from a pacemaker to err on the side of caution. Scientists have developed what some call "origami batteries" made of carbon nanotubes and cellulose that may power the next generation of pacemakers. The batteries can be cut into many shapes.

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Sarah J. Greenwald
Jill E. Thomley

See Also: Clocks; EEG/EKG; Function Rate of Change; Medical Simulations; Origami.

Packing Problems

Category: Architecture and Engineering.
Fields of Study: Algebra; Data Analysis and Probability; Geometry.
Summary: Packing problems challenge the solver to optimally fill a given space or determine how many objects of some type may fit in a space.

The name "packing problems" has been given to a variety of mathematical problems in both "serious" and "recreational" mathematics. Packing problems are mainly geometric but the term is sometimes also applied to certain numerical problems important to computer science. The distinguishing feature of a geometric packing problem is the objective to position a family of shapes with no overlap and a minimal amount of leftover space.

If a baker has rolled out a certain sheet of dough and has a certain size and shape of cookie cutter, how should the shapes be cut to leave as little wasted dough as possible? How many tennis balls will fit in a very large box? These are packing problems of the most fundamental type. The most thoroughly studied case is that of packing identical spheres (circles in the two-dimensional case or hyperspheres in the four-dimensional—or more—space) in Euclidean space. The most efficient way to pack circles in the plane is to surround each circle with six others in a honeycomb formation, filling about 90.7% of the plane. It was conjectured by Johannes Kepler that a similarly symmetric arrangement of spheres (filling about 74% of space) is optimal in three-dimensions. This conjecture was generally considered to have been proved by Thomas Hales in 1998. Hales's proof of Kepler's conjecture relies on large-scale computer calculation and represents an important example of a well-studied mathematical problem that is "solved," but for which no hand-checked proof is known.

There is much potential for generalization. There is already much unknown for hyperspheres in four dimensions; other important variations include considering spheres of varying sizes, considering non-Euclidean geometry, and using different shapes instead of spheres. There is significant, active research along all of these lines.

Packing Puzzles

The above type of problem has been studied both by professional mathematicians and recreational mathematicians, but there is another category of packing problem particular to the recreational mathematician (and to the puzzle enthusiast). In these packing puzzles, the solver tries to fit a collection of shapes into a larger shape; typically the pieces are of a sort that fits together exactly (for example, packing rectangles into a rectangle rather than packing circles into a square). An old puzzle is to determine, for each *n*, how large a square is needed to accommodate a 1×1 square, a 2×2 square, a 3×3 square, . . . , and an $n\times n$ square. The oldest known problem of this type is the Tangram puzzle that originated in ancient China, a set of seven simple shapes that can be rearranged to perfectly fill a square (and many other shapes). In the twentieth century, a wide range of packing puzzles involving polyominos (shapes made by gluing together unit squares along their edges) have enjoyed considerable popularity. Packing puzzles can also be posed in three dimensions. Three particularly popular and interesting examples involving blocks illustrate the concept well. The Slothouber–Graatsma puzzle, named after architects Jan Slothouber and William Graatsma, is to pack a $3\times3\times3$ cube with six $1\times2\times2$ blocks (leaving three $1\times1\times1$ holes). The Conway puzzle, named after mathematician John Conway, is to fill a $5\times5\times5$ cube with 13 $1\times2\times4$ blocks, one $2\times2\times2$ block, one $1\times2\times2$ block, and three $1\times1\times3$ blocks. A harder puzzle is to pack 41 $1\times2\times4$ blocks into a $7\times7\times7$ cube (leaving 15 $1\times1\times1$ holes).

Covering Problems

A class of problems closely related to the first type of packing problem discussed are so-called covering problems. Covering problems are "dual" to packing problems; instead of positioning non-overlapping copies of a shape in a region with minimal leftover space (as in a packing problem), the solver positions overlapping copies of a shape so that they completely cover a region with a minimal overlap. For example, how many circles of radius 1 does the solver need to completely cover a circle of radius 10? Covering problems have historically received less attention than packing problems, perhaps because packing problems correspond more obviously to physical situations. Nonetheless, covering problems have applications: for example, to designing satellite or cellular networks. Covering a large region as efficiently as possible with circles corresponds to placing security guards in a large area as efficiently as possible so that each point is within a fixed distance of at least one guard.

Packing Problems in Computer Science

Another class of packing problem, the so-called knapsack problems, is numerical (rather than geometric) in nature. A typical example is sometimes called the "Aladdin's saddlebag problem."

Aladdin is in a cave full of a variety of treasures: gold, silver, rubies, diamonds, rare books, and other valuable objects. Each type of object takes up a certain amount of space in Aladdin's saddlebags, weighs a certain amount, and has a certain value. The problem is to decide how to get the most valuable hoard possible, if there is a limited amount of space and if Aladdin's mule can carry only a limited amount of weight. If the solver thinks of the quantities as continuous (if it makes sense in context to take exactly as much gold as is wanted), then this is a classical instance of linear programming, a powerful and efficient technique in applied mathematics.

On the other hand, if the quantities are discrete (for example, if the gold is in large bars, and Aladdin cannot take more than two bars but less than three), then the problem is in general very difficult. Indeed, the simplest version of the discrete knapsack problem is already believed to be computationally quite hard. In this problem, a list of integers is given as well as a large target integer. The goal is to achieve the target as a sum of integer multiples of the given numbers. Any progress toward finding more efficient solving methods or toward showing that the current methods are optimal would be extremely significant in the field of computer science.

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Michael "Cap" Khoury

See Also: Polygons; Polyhedra; Puzzles; Shipping.

Painting

Category: Arts, Music, and Entertainment.
Fields of Study: Geometry; Representations.
Summary: Painting incorporates many mathematical concepts, and mathematics is also used to analyze paintings.

Human beings strive to comprehend their reality in a number of ways, including artistic expression and mathematics. Examples can be found in many cultures, such as the long history of interesting mathematical patterns in Islamic art and in the cave paintings of Paleolithic people. Many artists throughout history also have been mathematicians, such as fifteenth-century painter Piero della Francesca. Modern painter Michael Schultheis also worked as a software engineer. He and Mary Lesser, a painter and printmaker, both explicitly include mathematical elements like numbers, equations, and geometric objects in their work. Mathematical concepts, especially geometry, are embedded throughout the art of painting. Some that are most commonly used for analyzing paintings involve symmetry, perspective, golden ratios and rectangles, and fractals, as well as fundamental geometric forms, shapes, fractals, and abstraction. Mathematicians and scientists also use mathematical methods to determine whether or not unidentified paintings belong to a particular artist.

Symmetry

M.C. Escher, a graphic artist, used transformational geometry to create a variety of works that explored symmetry. His classic work *Day and Night*, a 1938 woodcut, transforms rectangular fields into flying geese and uses a black and white color scheme to emphasize the transition of a setting from day to night. While many artists explore symmetry and transformational geometry, Escher took it further by exploring and emphasizing mathematical concepts including *Convex and Concave*, a 1955 lithograph, *Two Intersecting Planes*, a 1952 woodcut, and *Moebius Strip II*, a 1963 woodcut. The use of symmetry as the catalyst for transforming the plane is one of the more pleasing aspects of his work. Navajo sand painting also offers many good examples of various types of symmetry. Four-fold symmetry is widely found in Native-American painting and other art forms, and it plays a role in some spiritual and healing ceremonies.

Perspective

Early paintings did not use perspective to show a three-dimensional world on a two-dimensional canvas. Giotto di Bondone, a thirteenth-century painter began to develop depth of field in some of his work; but the first artist credited with a correct representation of linear perspective is Filippo Brunelleschi (1377–1446), who was able to devise a method using a single vanishing point. An architect and sculptor, he shared his method with fellow artist Battista Alberti, who wrote about the mechanics of mathematical perspective in painting. Leonardo da Vinci used perspective in his paintings and explored artificial, natural, and compound perspective in his work. He examined how the viewer's observation point changed the perspective, and how the perspective could be perceived by changing where the viewer was observing the painting. Notably, while perspective and the illusion of depth were widely used in Western painting from the 1300s onward, it was not universal. Painters from India rarely used this technique; rather, they tended to focus more on patterns and geometric relationships.

Golden Ratio and Golden Rectangles

Consider a rectangle with short side a and a long side that is $a+b$. A golden rectangle would be where the ratio a/b is equal to

$$\frac{a+b}{a}.$$

In other words, the large rectangle is proportional to the smaller rectangle formed by side b and side a—this is the golden ratio. Some claim that this proportion influenced many artists and early Greek architecture, while others note the variability of picking points in a painting to have golden rectangles superimposed. It is, however, a way of considering the proportionality of a work.

Fundamental Geometric Forms or Shapes

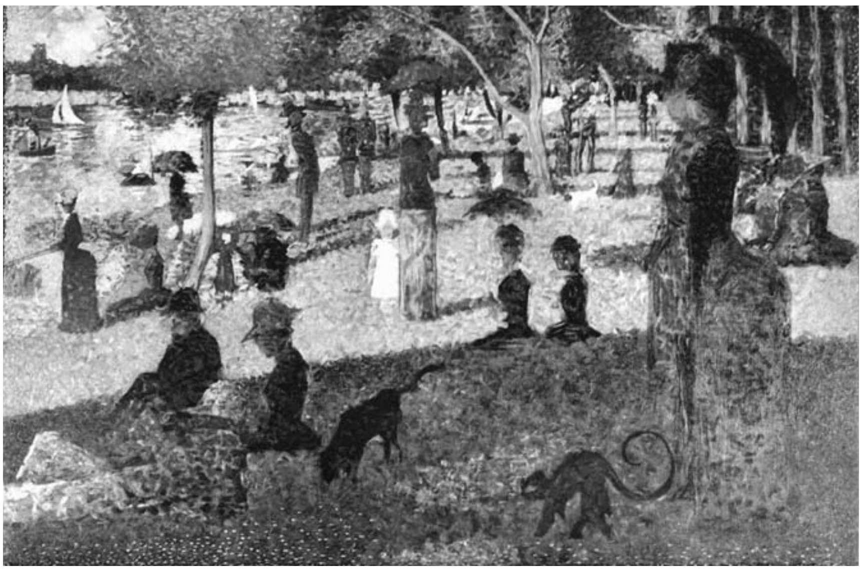
Geometric forms and shapes are the basis for drawing and painting. For example, Piet Mondrian (1872–1944) explored cubism in his work from black and white lines and blocks of primary colors that divided the plane. Other cubists, such as Pablo Picasso, broke with the Renaissance use of perspective to provide an alternative conception of form. Cubists made it possible for the viewer to see multiple points of view simultaneously.

Paul Cézanne ignored perspective in some of his work to construct color on the two-dimensional surface. Pointillism was used by Georges Seurat (1859–1891) to create *Sunday Afternoon on the Island of La Grande Jatte*. In pointillism, a series of small, distinct points of color are used to create a painting that relies on the viewer's eye to blend them into a cohesive form. The brain uses the dots to create a solid space. The primary colors are used to create secondary colors for shading and create the impression of a rich palate of secondary colors.

Art deco is characterized by the use of strong geometric forms that are symmetrical. This style of painting was popular in the 1920s and 1930s.

Abstraction and Fractals

Abstraction is an important tenet of mathematics. In mathematical abstraction, the underlying essence of a mathematical concept is removed from dependence on any specific, real-world object and generalized so that it has wider applications. In abstract expressionism, the artist is expressing purely through color and form, with no explicit representation intended. However, that does not mean that abstract art is entirely unstructured. Fractals are one tool used to quantitatively analyze and explain what makes some paintings more pleasing than others. The argument is that, even in an apparently random abstract work, there is an underlying logic or structure that the human brain recognizes as fractal patterns and that it inherently prefers over other works that do not have these patterns. This preference is perhaps because such works are more reflective of the geometry of naturally occurring spaces. For example, physicists Richard Taylor, Adam Micolich, and David Jonas analyzed Jackson Pollock's paintings and found two different fractal dimensions in his work that are mathematically and structurally similar to naturally occurring phenomena, like snow-covered vegetation and forest canopies. In addition to the application of fractals, mathematical concepts like open and closed sets have been used to compare and contrast the work of abstract expressionist artists like Pollock and Wassily Kandinsky to artists like Joseph Turner and Vincent van Gogh, whose works are among those credited with inspiring the expressionist movement.



French painter Georges Seurat used the painting technique of pointillism to create Sunday Afternoon on the Island of La Grande Jatte.

Mathematical Analysis to Determine Authenticity

Sometimes, the painter of a particular artwork is unknown or disputed, which affects the study of art and the monetary valuation of paintings. Hany Farid and his team created a computer program that uses wavelets to analyze digital images of paintings and map the stroke patterns—some too small to be seen with the naked eye—that characterize an artist's unique style. In one case, known drawings by Pieter Bruegel the Elder were compared to five drawings originally attributed to him. The analysis determined that the five drawings were different from the original eight and also from each other, suggesting multiple creators. Chinese ink paintings are an example in which brush strokes are critical to identification, since they do not have colors or tones to distinguish style. One successful method, tested on the work of some of China's most renowned artists, used a mixture of stochastic models. In another case, fractal geometry was used to question the authenticity of some newly discovered Pollock works, based on his earlier patterns. Radioactive scans and X-ray analysis help to authenticate works by well-known and highly valued masters, such as Johannes Vermeer.

Additional Parallels in Painting and Mathematics

There are many natural parallels in the work of painters and mathematicians. In the same way that painters of different traditions and schools may represent the same scene in drastically different ways, mathematicians may approach the same problem from a variety of disciplines or perspectives. There are also varying degrees of connection to reality in both mathematics and painting. Applied mathematicians and realist painters may be primarily concerned with detailed and faithful representations of the real world in their work, while abstract painters and theoretical mathematicians often work in ways that are logically coherent and consistent, but that do not immediately or obviously connect to the real world. As with art, there is also subjective appreciation of the beauty of mathematics and arguments over what is or is not mathematically valid. Artist Michael Schultheis reported that he was often inspired by mathematical and scientific writing on whiteboards from his days as an engineer, and said, "I constantly revise equations with the Japanese calligraphy brush, rubbing out an area and thus creating a window into the equations. I draw and re-draw new ideas. All of these ideas are analytical. But they also live in the realm of beauty."

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See Also: Escher, M.C.; Geometry in Society; Golden Ratio; Greek Mathematics; Renaissance; Sculpture; Symmetry; Transformations.

Parallel Postulate

Category: History and Development of Curricular Concepts.
Fields of Study: Communication; Connections; Geometry.
Summary: The parallel postulate led to thousands of years of investigation and debate.

One of humanity's greatest intellectual achievements occurred in approximately 300 B.C.E. when the axiomatic method was born. The classic text *Elements*, written by the great Greek geometer Euclid of Alexandria, is a work that shaped the nature of mathematics and stands to this day as an example of the beauty and elegance of reasoning and proof.

Euclid was among the first people to understand that abstract mathematics is based on reasoning, from assumptions to general conclusions. From a very modest set of assumptions—his five postulates (called "axioms")—Euclid set out to argue the truth of a large number of propositions (called "theorems") in geometry.

The first four of Euclid's postulates appear reasonable enough: (1) any two points determine a unique line; (2) any line segment can be extended to an infinite line; (3) given any center and radius, a circle can be constructed; and (4) all right angles are congruent. But the fifth postulate stands out for its comparative complexity:

If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

This fifth postulate has come to be known as the "parallel postulate," in part for its very content, but also for the key role it plays in proving certain propositions about parallel lines.

From an historical perspective, Euclid himself seemed a bit uncomfortable with his fifth postulate. This discomfort is evidenced by the order of his work in Book I of *Elements*, where, on his way to eventually proving 48 propositions, he waited until proposition 29 to use the parallel postulate. The first 28 results rely only on the first four postulates and theorems that can be proven using those assumptions.

Attempts to Prove the Parallel Postulate as a Theorem

As subsequent mathematicians studied the *Elements*, most were troubled in some way by the parallel postulate. Because of its complexity, as well as its "if-then" format, it struck most mathematicians that Euclid's fifth postulate really ought to be a theorem. In other words, the parallel postulate ought to be a consequence of the first four postulates, and this fact ought to be provable, using only those four postulates and any theorems that could be derived from them.

Thus, many mathematicians set out to prove the parallel postulate as a theorem. It is one of the great tales of the history of mathematics that every single mathematician who attempted to prove the parallel postulate failed. Early on, many of these esteemed intellects made a common error that the rules of logic forbid—they assumed precisely what they were attempting to prove. Clearly, if the goal is to prove a statement *S*, one should never be allowed to simply assume that *S* is true. While certainly no mathematician was so dull as to say, "To prove the parallel postulate, I will assume the parallel postulate," many people did make the mistake of making the assumption that certain "obvious" statements were true. For example, they may have assumed statements such as the following:

- Parallel lines are everywhere equidistant.
- The sum of the measures of the interior angles of a triangle is 180 degrees.
- If a line intersects one of two parallel lines, then it must also intersect the other.
- There exists a rectangle (a quadrilateral having four right angles).

Remarkably, each of the above statements (along with many others) is equivalent to the parallel postulate. Said differently, if one of the above statements is called *P* and the statement of the parallel postulate is called *S*, then it turns out that *P* is true if and only if *S* is true—the truth of one implies the truth of the other, and vice versa.

Hence, when a mathematician said, "Using the fact that any triangle's angle sum is 180 degrees," and then went on to "prove" the parallel postulate, this argument was like saying "the parallel postulate is true because the parallel postulate is true." These errors came to be well understood by the end of the eighteenth century, perhaps most prominently in G. S. Klugel's 1763 doctoral dissertation in which he debunked 43 flawed "proofs" of the parallel postulate.

Girolamo Saccheri's Developments

Of course, even though nobody had found a valid proof of the parallel postulate did not mean that one could not be found, and many continued the search. Around the turn of the eighteenth century, a Jesuit priest named Girolamo Saccheri (1677–1733) made a lasting contribution to the study of the parallel postulate in particular, and to the history of mathematics in general. Saccheri considered the unthinkable, as part of his effort to prove the parallel postulate through a contradiction argument: what if the parallel postulate is false?

It was well understood by Saccheri's time that an equivalent statement of the parallel postulate was Playfair's Postulate, which states that

> For any line *l* and any point *P* not on *l*, there exists a unique line through *P* parallel to *l*.

A contradiction argument works by assuming that the statement one wants to prove true is actually false and showing that some contradiction follows. Thus, it is natural to consider Playfair's Postulate and suppose that there is not be a unique line through *P* parallel to *l*. That is, one would assume that either there is not *any* line through *P* parallel to *l*, or there is *more than one* line through *P* parallel to *l*. Saccheri considered a similar scenario where he had transformed the problem about parallels to an equivalent one about quadrilaterals (now called "Saccheri quadrilaterals") in which the quadrilateral has two congruent sides perpendicular to the base. Fundamentally, Saccheri was trying to prove that a rectangle existed by showing that the summit angles of his quadrilateral were also right angles. After proving that the summit angles were congruent, he realized that there were three possibilities: the summit angles were each right angles, each was less than a right angle, or each was more than a right angle.

While Saccheri was able to rule out the possibility that the summit angles were obtuse by assuming that they were obtuse and finding a contradiction, when he

Modern Conceptions

Today, mathematicians understand a great deal about the role of Euclid's parallel postulate. Euclid's parallel postulate really is an axiom, and not a theorem. The parallel postulate is independent of the first four postulates. One can assume that the parallel postulate is true, or one can assume that the parallel postulate is false. Either leads to a perfectly valid geometry, with the truth of the parallel postulate leading to Euclidean geometry. Considering Playfair's postulate, named for John Playfair, if one assumes there are no parallel lines through a point *P* not on a line *l*, then this leads to so-called elliptic geometry, which is like the geometry of the sphere. If instead one assumes that there is more than one parallel line through a point *P* not on *l*, then this leads to "hyperbolic geometry," a geometry that some believe may help describe the shape of the universe.

It took approximately 2000 years for humankind to fully appreciate the work of Euclid and to reconcile the fact that Euclid was right—his fifth postulate really is an axiom, and not a theorem that can be derived. More than this, the parallel postulate is like a door that opens the world to one geometry—Euclidean—while there are other similar postulates that open doors to different universes, those of elliptic and hyperbolic geometries.

assumed that the summit angles were acute, he could not find a contradiction. From this assumption, he went on to prove many strange and unusual theorems. Unknowingly, Saccheri had discovered a whole new geometry, one that another mathematician named Janos Bolyai would call "a strange, new universe" in his own investigations. What both of these mathematicians, along with others such as Carl Gauss, started to realize is that there actually exists a geometry in which there is more than one line through a point *P* not on line *l* such that each is parallel to *l*. This realization stands as one of the greatest accidental discoveries in the history of the human intellect: Saccheri did not find what he set out to prove, but instead developed a collection of ideas that would radically change mathematics.

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See Also: Axiomatic Systems; Geometry of the Universe; Proof.

Parallel Processing

Category: Communication and Computers.
Fields of Study: Algebra; Number and Operations.
Summary: Parallel processing speeds up the run-time of computing through the use of mathematical algorithms.

In computing, parallel processing is the action of performing multiple operations or tasks simultaneously by two or more processing cores. Ideally, this arrangement reduces the overall run-time of a computer program because the workload is shared among a number of engines—central processing units (CPUs) or cores. In practice, it is often difficult to distribute the instructions of a program in such a way that each CPU core operates continuously and efficiently, and without interfering with other cores. It should be noted that parallel processing differs from multitasking, in which a single CPU core provides the effect of simultaneously executing instructions from several different programs by rapidly switching between them, or interleaving their instructions. Modern computers typically include multi-core processor chips with two or four cores. The most advanced supercomputers in the early twenty-first century may have thousands of multi-core CPU nodes organized as a cluster of single processor computers and connected using a special-purpose, high-speed, fiber communication network. Although it is also possible to perform parallel processing by connecting computers together using a local area network, or even across the Internet, this type of parallel processing requires the individual processing elements to work predominantly in isolation because of the comparatively slow communication between nodes. Parallel processing requires data to be shared among processors and thus leads to the concept of "shared memory" where multiple processing cores work with the same physical memory. In large computer clusters, the memory is usually distributed across the nodes, with each node storing its own part of the full problem. Data are exchanged between nodes using message-passing software, such as Message Passing Interface (MPI).

Amdahl's Law and Gustafson's Law

The speed-up gained through parallelization of a program would ideally be linear; for example, doubling the number of processing elements should halve the runtime. However, very few parallel algorithms achieve this target. The majority of parallel programs attain a near-linear speed-up for small numbers of processing elements but for large numbers of processors the addition of further cores provides negligible benefits.

The potential speed-up of an algorithm on a parallel computing platform is given by Amdahl's law, originally formulated by Gene Amdahl in the 1960s. A large mathematical or engineering problem will typically consist of several parallelizable parts and several non-parallelizable parts. The overall speed-up attainable through parallelization is proportional to the size of the non-parallelizable portion of the program and is given by the equation

$$S = \frac{1}{1-P}$$

where S is the speed-up of the program (as a factor of its original sequential runtime), and P is the fraction that is parallelizable. Amdahl's law assumes the size of the problem is fixed and that the relative proportion of the sequential section is independent of the number of processors. For example, if the sequential portion of a program is 10% of the run-time ($P = 0.9$), no more than a 10-times speed-up could be obtained, regardless of how many processors are added. This characteristic puts an upper limit on the usefulness of adding more parallel execution units.

Gustafson's law is closely related to Amdahl's law, but is not so restrictive on the assumptions made about the problem. It can be formulated algebraically as

$$S(P) = P - a\,(P-1)$$

where P is the number of processors, S is the speed-up, and a is the non-parallelizable proportion of the process.

Applications

Parallel computing is used in a broad range of fields, including mathematics, engineering, meteorology, bioinformatics, economics, and finance. However, all of these applications usually involve performing one or more of a small set of highly parallelizable operations, such as sparse or dense linear algebra, spectral methods, *n*-body problems, or Monte Carlo simulations. Frequently, the first step to exploiting the power of parallel processing is to express a problem in terms of these basic parallelizable building blocks.

Parallel processing plays a large part in many aspects of everyday life, such as weather prediction, stock market prediction, and the design of cars and aircraft. As parallel computers become larger and faster, it becomes feasible to solve larger problems that previously took too long to run on a single computer.

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See Also: Mathematical Modeling; Mathematics Research, Interdisciplinary; Software, Mathematics; Weather Forecasting.

Payroll

Category: Business, Economics, and Marketing.
Fields of Study: Algebra; Number and Operations.
Summary: Various payroll systems employ different mathematical calculations.

A variety of pay practices date back to ancient times, including compensation for services in the form of food, commodities, land, or livestock. Payroll systems are connected with the history of bookkeeping, which can be traced back to 4000 B.C.E. Paymasters were responsible for paying workers. Governments kept financial records called "pipe rolls" at least as early as the eleventh century. In 1494, Franciscan friar and mathematician Luca Pacioli published the book *Summa de Arithmetica, Geometria, Proportioni et Proportionalita*, which contained double-entry bookkeeping. The term *payroll* dates back to the seventeenth century, and compensation gradually changed from goods to money. In the mid-twentieth century, mathematician Grace Murray Hopper developed a compiler, later known as the FLOW-MATIC, which could be used for payroll calculations. When the U.S. Navy could not develop a working payroll plan, they called Hopper back to active duty. In the early twenty-first century, a payroll specialist is listed by some schools as a career option for mathematics majors. Accountants and actuaries calculate quantitative measures and predictions based on historic payroll information and salary increases. For example, the pensionable payroll is calculated as an integral that takes salary increases into account. In payroll analysis, the impact of changing salary expenses is compared to other factors, such as sales or profit.

Frequency

Some employees are paid each day they work; however, in many cases, an employer will withhold daily earnings and pay the cumulative amount earned at a later time as a lump sum. Common payroll frequencies include weekly, bi-weekly (every other week), semimonthly (twice a month), and monthly. Each of these frequencies would correspond to receiving 52, 26, 24, and 12 paychecks each year, respectively, assuming a full year of work. Some seasonal jobs pay only for part of the year, but still use the standard payroll frequencies. For example, teachers often receive pay for only nine months. Some schools offer for that pay to be spread over a full year to guarantee consistent income during the summer months when teachers are not actually working.

On payday, the employee will receive earned wages for the previous pay period. Rather than receiving cash, sometimes an employee will receive a check that can be exchanged for an equivalent amount of cash. Other times, an employee will receive income as a "direct deposit" where the income is automatically deposited into the employee's checking or savings account.

Earning Money

Some employees work for an hourly wage—for every hour of work they perform, they get paid a specified amount of money. Suppose that a worker had an hourly wage of $10 and worked for 20 hours. To find the total amount of the paycheck, the worker would multiply the hourly wage by the number of hours worked. For example, $\$10 \times 20 = \$200$.

Sometimes, contracts or laws dictate the number of hours a person can work per week and—should they work more than that amount—his or her income increases. For example, in the United States, 40 hours is a common workweek. A person working over 40 hours often gets paid "time and a half" or "wage and a half" for the number of hours over 40 that he or she works (called "overtime"). Again, assuming an hourly wage of $10, an employee who worked 48 hours in one week would earn $\$10 \times 40 = \$400$ for the first 40 hours they worked. The eight hours he or she worked beyond 40 hours would earn him or her extra money. If the employee earns "time and a half," the time would be multiplied by 1.5 before being multiplied by his or her hourly wage. If he or she earns "wage and a half," the wage would be multiplied by 1.5 before being multiplied by the number of hours worked beyond 40. In reality, the method of calculating overtime earnings is irrelevant since multiplication is associative. Time and a half would be calculated as $\$10 \times (1.5 \times 8) = \$10 \times 12 = \$120$, and wage and a half would be calculated as $(\$10 \times 1.5) \times 8 = \$15 \times 8 = \$120$. The total earnings for that week would be found by taking the sum of these wages: $\$400 + \$120 = \$520$.

Another method for earning money is a salary. Unlike the hourly wage, a salary is a predetermined amount of money that the worker earns regardless of how long (or how short) it takes the worker to accomplish those tasks. Often, salary is determined based on how much a person will make over a year's time. However, rarely does a person only receive one paycheck a year. The amount of money earned on each paycheck is calculated by taking the salary and dividing it by the number of pay periods in a year. That number will vary depending on how often a person gets paid. Suppose an employee agreed to work for a salary of $31,200 each year. Looking at the common pay periods, weekly, bi-weekly, semi-monthly, and monthly, this employee would earn $600, $1,200, $1,300, or $2,600, respectively, for each paycheck during the year.

A worker earning commission does not actually get paid based on how long it takes to do the job, but by how productive the worker is (oftentimes based on the amount of items the worker sells). Sometimes, commission is a flat fee per item sold, other times, it is a percentage of sales. For example, if an employee earned 7% commission on sales and sold $1,250 worth of merchandise on a given day, then pay would be calculated $\$1{,}250 \times 7\% = \$1{,}250 \times 0.07 = \$87.50$. Some jobs combine an hourly rate and commission—the employee earns a certain amount of money for every hour they are at the job, but then also earns commission on top of that wage to determine the total money earned.

Payroll Withholdings

Upon receipt of a paycheck or notice of direct deposit, usually the amount paid to the employee (the net pay) is less than what is calculated as his or her earnings for the pay period (the gross pay). Before being issued money, an employee may have his or her income reduced by certain amounts—some voluntary, others involuntary. In order to pay for various levels of government (and the benefits they offer), income and payroll taxes are frequently withheld from earnings. Some employees pay premiums for different insurances (such as medical, life, or disability) from their pay. Sometimes, money is withheld as a long-term savings for eventual retirement of the employee. Job-related expenses can also be withheld, such as for dues or charges for employee uniforms.

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See Also: Accounting; Budgeting; Income Tax; Money.

Pearl Harbor, Attack on

Category: Government, Politics, and History.
Fields of Study: Geometry; Measurement; Number and Operations; Problem Solving.
Summary: Mathematicians were involved in both the planning of and the response to Pearl Harbor.

The attack on Pearl Harbor, a major engagement of World War II and the impetus for the United States' entry into the war, took place early Sunday morning, December 7, 1941, on the island of Oahu, Hawaii. The Japanese Navy, commanded by Admiral Isoroku Yamamoto, planned and executed the surprise attack against the U.S. naval base and nearby army air fields. As a result, the United States declared war on Japan. In his address to Congress, President Franklin D. Roosevelt famously proclaimed December 7 "a date which will live in infamy."

Both leading up to and as a result of the attack on Pearl Harbor, mathematicians in Japan and the United States mobilized for the war. For instance, after Pearl Harbor, the American Mathematical Society and the Mathematical Association of America converted their War Preparedness Committee to a War Policy Committee to increase research on "mathematical problems for military or naval science, or rearmament" and to strengthen mathematics education in order to prepare undergraduate students for military service. The attack has also been surrounded by speculation as to how the United States could have been caught off guard so easily. The naval base had been designed as nearly impenetrable to surprise attack because of the geography and geometry of the island. However, new

A photo taken from a Japanese plane during the Pearl Harbor torpedo attack on ships moored on both sides of Ford Island.

technologies made the attack possible: the aircraft carrier could bring low-flying aircraft within attack range, and the Japanese development of shallow-running torpedoes could skim the surface of the harbor's relatively shallow water. One of the largest controversies involves U.S. efforts to decode Japanese communications that may have given forewarning of the attack.

Japanese Mathematicians

Leading up to Pearl Harbor, the number of Japanese graduate students increased and several studied in Germany. Mathematicians applied lattice theory and logic to the design of circuits. In the 1930s, both the United States and Japan successfully built a cyclotron, an early particle accelerator. Mathematics was also important in electrical engineering and airplane design. With a focus on aerodynamics and science and technology policy, the Japanese Technology Board was founded in 1941. A statistical institute contributed to war production. Japanese cryptologists also created many variations of military codes that were in use prior to Pearl Harbor, such as Kaigun Ango—Sho D, later referred to as "JN-25B" by cryptanalysts in the United States. Before committing to the attack on Pearl Harbor, the Japanese Navy conducted feasibility studies that included calculations and considerations of their current military resources; the need for a longer, circuitous route outside the customary naval traffic lanes to avoid detection by both military and civilian ships,; the probability of encountering severe winter storms and critical data obtained from spies in Hawaii, such as the patterns of military activity at Pearl Harbor. They concluded that the attack was possible, if dangerous, and they originally intended to specifically target U.S. aircraft carriers to optimize the long-term effects of the attack. Experimentation and simulated training attacks yielded a satisfactory plan only a few weeks before the event.

U.S. Mathematicians

In the United States, mathematicians conducted ballistics research at Aberdeen Proving Ground. Max Munk used the calculus of variations in airfoil design at the National Advisory Committee for Aeronautics, a precursor to the National Aeronautics and Space Administration (NASA). Technology such as radar, developed by scientists and mathematicians including Christian Doppler and Luis Alvarez, served military uses, though it was still in its infancy at the time of Pearl Harbor. Responsibility for compiling codes for military use and using cryptology to decipher codes shifted from Military Intelligence to the Army Signal Corps in 1929. William Friedman was the chief civilian cryptologist at the Signal Intelligence Service. The U.S. Army at that time realized the importance of mathematics in deciphering, and the first three civilian cryptanalysts hired by the U.S. Army were mathematics teachers.

Forewarning

Many wonder how the United States could not have known of the impending Japanese attack, which had been planned and practiced months in advance. The new radar installation on Opana Point did, in fact, detect the incoming Japanese attack planes, but they were ultimately mistaken for a group of U.S. planes that were due to arrive from the mainland that morning. A U.S. destroyer also spotted a Japanese submarine attempting to enter the harbor, which it reported, but the information was not acted upon immediately. Both would have given at least short-term warnings to the ships and personnel. However, much of the accountability is assigned to the U.S. and Japanese intelligence and counter-intelligence efforts. Correspondence declassified many years after the war suggests that the United States could at least partially understand the codes needed to monitor Japanese naval movements on the eve of Pearl Harbor. While U.S. and British cryptanalysts had successfully broken some Japanese codes, such as the MAGIC code, the United States was not able to determine from those messages that the attack was about to happen. The broken codes were the ones used primarily for diplomatic messages sent by the Japanese Foreign Office and military strategy was rarely shared with the Japanese Foreign Office. The U.S. Navy had three cryptanalysis centers devoted to breaking Japanese naval codes. Prior to the attack, American cryptanalysts had been using traffic analysis to follow Japanese naval movements. Traffic analysis is the process of looking for patterns in communications to infer if an attack is about to occur. According to the National Security Agency, the Japanese, aware that their communications were being monitored, issued "dummy traffic to mislead the eavesdroppers into thinking that some of the ships sailing through the North Pacific were still in home waters." Additionally, as Japanese forces were preparing for the attack, radio traffic was limited, greatly reducing the ability of American intelligence to determine a pattern. These efforts to stymie cryptologists were effective in keeping the impending attack a secret from the United States. Mathematicians and historians continue to analyze whether signal intelligence techniques could have revealed Japan's intentions.

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See Also: Aircraft Design; Artillery; Coding and Encryption; Predicting Attacks; World War II.

Pensions, IRAs, and Social Security

Category: Business, Economics, and Marketing.
Fields of Study: Algebra; Data Analysis and Probability; Measurement; Number and Operations.
Summary: The development and allocation of retirement income can involve significant mathematical analysis.

Planning for retirement is one of the most important financial responsibilities a person faces. Ideally, after working for several decades, a person will be in a financial position to sustain a desired lifestyle during retirement. In the United States, the source of retirement income can be from a combination of one or more of the following: Social Security; an employer-sponsored pension plan; individual savings, including individual retirement accounts (IRAs); or other mechanisms. The U.S. government has provided military pensions to disabled veterans and widows since the Revolutionary War. This benefit expanded after the U.S. Civil War to include nearly any veteran who had served honorably for some minimum time. Southern states also paid Confederate veterans.

By the early twentieth century, state, municipal, and city governments were paying pensions to their employees, especially firemen and policemen. Teachers were the next large group to receive benefits. Private pensions started in the late nineteenth century with the American Express Company and several railroads. When the 1926 Revenue Act exempted pension trust income from taxes, companies had a new incentive to provide employee pensions, which became commonplace by the 1930s. Social Security was designed in 1935 to extend pension benefits to those not covered by a private pension plan. In the early twenty-first century, Social Security benefits are the main source of retirement income for most retirees, though this varies greatly depending on income from earnings, assets, and private pensions.

Each of these potential sources of retirement income can involve significant mathematical and financial analysis to estimate an individual's retirement needs, determine necessary pre-retirement financial planning, and evaluate the potential uncertainty associated with personal and economic factors. Specialized mathematicians known as "actuaries" work for governments and industries to design financially sound insurance and pension programs that help meet people's retirement needs.

At the same time, as the professionals at the American Pension Corporation (a major pension administrator) assert, good actuaries "are more than just mathematicians—[they] take great pride in [their] ability to dissect and communicate the intricacies of pension administration in layman's language."

Pensions

A pension provides a stream of income during retirement. It is typically sponsored by a person's employer—either a corporation or governmental entity. The amount and timing of the retirement income stream provided by a pension are a function of several factors, such as the worker's salary, the proportion of that salary invested into the pension plan, any matching funds or contributions to the pension fund provided by the employer, the length of the worker's tenure with the employer, and the investment performance of the pension fund. There are two types of pension plans: defined-benefit (DB), and defined-contribution (DC). DB plans, which have to some extent been phased out in the private sector but are still common in the public sector, define the benefits that will be paid to the worker during retirement. Assuming the solvency of the pension plan—a significant issue in itself—a worker covered under a DB plan is guaranteed to receive the benefits defined by the plan.

Because of potential difficulties in adequately funding DB plans, many (particularly private sector) employers converted to DC plans during the last several decades of the twentieth century. With DC plans, the retirement benefits are not specified; rather, the plan defines the periodic contributions to be invested during the worker's life, and then the retiree receives an income stream based on the actual accumulated amount of the investment fund. Relative to DB plans, this means that the employer's risk of inadequate retirement benefit funding is reduced, and that some risk has been transferred to the employee, who faces an uncertain pension income stream.

Mathematics of Pensions

The mathematics associated with pensions involves both "future (or accumulated) value" and "present value" concepts. The general idea is that a worker (or the sponsoring employer) accumulates a retirement fund by setting aside and investing periodic amounts during the working years. Then, upon retirement, this accumulated amount ideally represents sufficient funds with which to provide the retiree an adequate stream of income until death. This retirement income stream may be obtained by leaving the funds invested and withdrawing a certain amount per year, or through the purchase of an annuity, which provides the payment stream. In most cases, these two approaches are mathematically equivalent.

While somewhat straightforward conceptually, achieving an adequately funded and effective retirement plan (especially DB plans, which generally involve more sophisticated and extensive mathematical and financial considerations than DC plans) is a challenging mathematical and actuarial problem. Some of the parameters involved in a pension analysis, and for which assumptions must be made, include the following:

1. Periodic contributions to the pension investment fund—usually expressed as a percentage of worker salary during employment.
2. Size of the retirement income stream needed or desired—generally estimated as a percentage of projected salary immediately prior to retirement.
3. Rate of return on the invested retirement funds, both before and after retirement.
4. Changes in worker salary throughout employment.
5. Impact of inflation on the worker's buying power.
6. Taxation rules and regulations, both during employment and in retirement.
7. Longevity and mortality.

Along with these assumptions, actuaries use mathematics and computer modeling to determine potential answers to questions such as how much must a worker (or employer) invest every month (or year) into a retirement plan in order to successfully achieve that worker's financial goals in retirement?

IRAs and Social Security

In addition to having an employer-sponsored pension plan, a worker can supplement retiree income with personal savings. One such mechanism is one or more types of IRA. While the rules surrounding IRAs are extensive, they can have potential advantages for some people, including certain tax-advantaged properties.

Social Security is a particularly contentious issue in the twenty-first century. Some have compared Social Security to a type of scam called a "Ponzi scheme" in which a growing pool of new investors' money is used to pay the promised returns to previous investors. Despite superficial resemblances (for example, current taxpayer money is used to pay variable benefits to others), Social Security is not a savings plan or investment account, but rather a tax, which nullifies the comparison. However, there have been proposals to replace Social Security with an investment program, using a variety of calculations and probabilistic mathematical models to try to demonstrate its cost-effectiveness and the likelihood of the system's impending failure.

Another major financial issue related to social security is the potential misuse of Social Security numbers. Initially issued to track workers for taxation and benefits, these nine-digit numbers are now assigned routinely at birth and have grown over time into the role of a unique identifier for creditors, schools, employers, and others who want to assign codes to individuals. Modern identity theft, which usually involves a person using a fake or stolen social security number to obtain credit or other benefits, has been on the rise as a result of Internet growth and the widespread collection of personal data. Mathematicians have calculated that a person making up a false social security number in 2010 has about a 50% chance of matching a real number. Faking multiple numbers results in an almost-guaranteed match very quickly.

These calculations have been used to counter thieves' assertions that they did not know numbers they were using were real. Social Security numbers themselves are not random (for example, the first three digits are a numerical code for geographic location), and mathematical and computer methods have used publicly available data, like date and place of birth, to successfully predict most or all of a person's social security number. There are also concerns that the government will run out of Social Security numbers, which are not reused after a person dies. Some calculations suggest that the supply will be exhausted early in first half of the twenty-second century. Alternative proposals include using alphanumeric or hexadecimal strings, which offer more permutations for a series of nine "digits." Others suggest including a security checksum in the number to decrease fraudulent use.

Stochastic Variables

What makes such a quantitative analysis particularly challenging is that many of these parameters are stochastic rather than deterministic—their future values are uncertain, and they can (and do) change value over time. Data analysis and probability concepts are used to account for this uncertainty. For example, inflation and investment rate of return are both stochastic variables, with considerable uncertainty regarding their values in both the short- and long-term.

Estimates of possible future values and the relative probabilities or likelihoods of those values can be made by analyzing historical data. These estimates can then be used to project future scenarios and quantify the potential impact of possible future values on the retirement funding process.

Another critical stochastic variable in retirement planning is the age at death of the retiree. The number of years that a retiree lives beyond the date of retirement is an essential factor in determining the total amount of income needed during the retirement years. Actuaries research and analyze historical mortality data for people with various identifiable attributes. From these analyses, a probability distribution of possible ages at death, with their relative likelihoods, can be developed.

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Rick Gorvett

See Also: Budgeting; Forecasting; Loans; Money; Mutual Funds.

Percussion Instruments

Category: Arts, Music, and Entertainment.
Fields of Study: Geometry; Number and Operations; Representations.
Summary: The vibrations that emanate from percussion instruments vary mathematically based on the type of instrument.

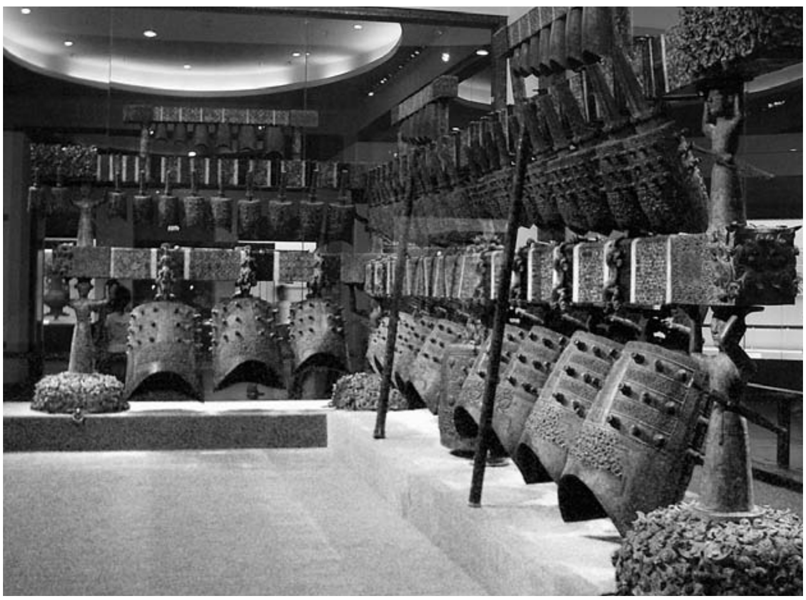
Percussion instruments are characterized by vibrations initiated by striking a tube, rod, membrane, bell, or similar object. Percussion instruments are almost certainly the oldest form of musical instrument in human history. The archeological record of percussion instruments, in particular the *bianzhong* bells of ancient China, give clues to the history of music theory. From a mathematical point of view, percussion instruments are of special interest because—unlike other types of instruments, such as string and wind instruments—the resonant overtones typically do not follow the harmonic series. In the last half of the twentieth century, a question of great interest in applied mathematics has been the famous inverse problem: can one hear the shape of a drum?

Rods and Bars

Some percussion instruments produce a distinct pitch by the vibration of a rod or bar. Examples included the tuning fork (a U-shaped metal rod suspended at its center), a music box (a metal bar suspended at one end), and the melodic percussion instruments such as the xylophone and marimba (suspended at two non-vibrating points or "nodes" along the length of metal or wooden bars). Like vibrating strings, the frequency of the bar's vibration and the pitch of the musical sound it produces are determined by its physical dimensions. In contrast to the string in which the frequency varies inversely with the length, the vibrating bar has a frequency that varies with the square of the length. The resonant overtone frequencies f_n of the vibrating bar are related to the fundamental frequency f_1 by the formula

$$f_n = \alpha \left(n + \frac{1}{2} \right)^2 f_1$$

where the constant α is determined by the shape and material of the bar. In contrast with the harmonic overtone series of vibrating strings, $f_n = n(f_1)$, these inharmonic overtones give percussion instruments their distinct metallic timbre. The overtones of vibrating bars decay at different rates, with rapid dissipation of the higher overtones responsible for the sharp, metallic attack, while the lower overtones persist longer. The bars of the marimba are often thinned at the center,



The ancient bianzhong *bell set on display at the Hubei Provincial Museum. Each bell can produce two pitches when struck.*

effectively lowering the pitch of the certain overtones, in accord with the harmonic series.

Bells

Like the vibrating bar instruments, the classic church bell possesses highly non-harmonic overtones. These are typically tuned by thinning the walls of the bell along the circumference at certain heights. A distinctive feature in the sound comes from the fact that apart from the fundamental pitch, the predominant overtone of the church bell sounds as the minor third above the prevailing tone. This feature accounts for the somber nature of the sound.

The *bianzhong* bells of ancient China were constructed in a manner that produced two pitches for each bell, depending on the location at which it was struck. In the 1970s, a set of 65 such bells were discovered during the excavation of the tomb of Marquis Yi in the Hubei Provence. The inscriptions on the bells make it clear that octave equivalence and scale theory were known in China as early as 460 B.C.E.

Membranes

Drums are perhaps the most common percussion instrument. Consisting of vibrating membranes (called the "drum heads") stretched over one or both ends of a circular cylinder, drums exhibit a unique mode of vibration, which accounts for their characteristic sound. Mathematical models of vibrating drumheads provide a fascinating application of partial differential equations. The inharmonic overtone frequencies are distributed more densely than for vibrating strings or rods. Further, each overtone is associated with a particular vibration pattern of the drum head. These regions can be characterized by the non-vibrating curves (called "nodes") that arrange themselves in concentric circles and diameters of the drum head.

An important question in the study of spectral geometry asks: "Can one hear the shape of a drum?" In other words, can mathematical techniques be used to work backwards from the overtone frequencies to determine the shape of the drumhead that caused the vibration? The answer, as it turns out, is "not always."

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ERIC BARTH

See Also: Geometry of Music; Harmonics; Scales; Wind Instruments.

Perimeter and Circumference

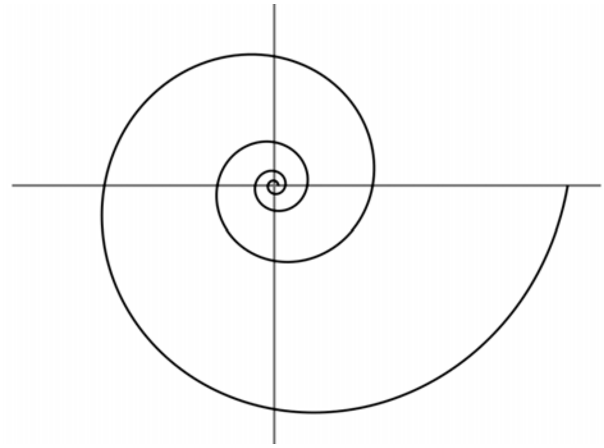
Category: History and Development of Curricular Concepts.

Fields of Study: Communication; Connections; Geometry; Measurement.

Summary: Measuring perimeter and circumference is a geometric task with a long history of methods.

Measurements of length and distance abound in daily life, from the height of a child to the distance from home to the store. Perimeter and circumference are types of length measurements. The perimeter of a geometric entity is the path that surrounds its area. The word derives from its Greek roots *peri* (meaning "around") and from *meter* (meaning "measure"). In stricter mathematical sense, perimeter is defined as the length of the curve constituting the boundary of a two-dimensional, planar closed surface.

For example, the perimeter of a square whose side measures length a is $4a$. Perimeter is important for applications such as landscaping projects, construction, and building fences. Circumference is defined as the perimeter of a circle. The circumference of a circle of radius (r) is $2\pi r$. The circumference of a circle has played a very important role throughout history in the approximation of the mathematical constant π, which was defined as the ratio of the circumference (C) of the



The curve that forms the shape of a nautilus shell is known as a logarithmic spiral *or* equiangular spiral.

circle to its diameter (*d*). Perhaps the most common reference to circumference that most people encounter regularly is the circumference of one's waist—the size of their waist.

The waist circumference is used as a measurement for some clothing and is also associated with type II diabetes, dyslipidemia, hypertension, and other cardiovascular diseases. Students investigate perimeter beginning in primary school, and middle grade students explore circumference. Students formulate the length of general curves, referred to as the "arc length" or "rectification," as integrals in calculus courses.

History

There is a long history of computations involving the perimeter or circumference of figures. One way was to measure length was with ropes. For instance, statements about rope measurements and the Pythagorean theorem can be found in Katyayana's *Sulbasutra*. Another way was to compare the length of two figures. A Babylonian clay tablet was discovered in 1936 and was noted as relating the hexagon perimeter to 0;57,36 (in base 60), or 24/25 times the circumference of a circumscribed circle. Mathematicians like Archimedes of Syracuse estimated the circumference or a value for π by using the perimeters of inscribed and circumscribed polygons with many sides. For example, Archimedes was known to have used 96-sided polygons. Mahavira estimated the circumference of an ellipse. In ancient times, the semiperimeter, or half the perimeter, was useful in computing many geometrical properties of polygons such as altitude, exradius, and inradius of a triangle. The semiperimeter also appears in Heron of Alexandria's formula for the area of a triangle. The semiperimeter of a rectangle is the sum of the length plus the height and is noted as appearing on Babylonian clay tablets. Brahmagupta used the semiperimeter of a quadrilaterial in the computation of its area.

The circle is a special geometric figure, for it is the curve, given a fixed perimeter, which encompasses the maximum surface area. This is known as the isoperimetric problem. Proclus commented that, "a misconception is held by geographers who infer the size of a city from the length of its walls." The Babylonians may have worked on related problems in their investigations of solutions to quadratic equations generated by the setting of the semiperimeter and area to constants. The isoperimetric problem was partially solved by the Greek mathematician Zenodorus.

Pappus of Alexandria compared the areas of figures with a fixed perimeter. In the tenth century, Abu Jafar al-Khazin proved that an equilateral triangle has greater area than isosceles or scalene triangles of the same fixed perimeter. Many mathematicians worked on the isoperimetric problem using a variety of techniques including methods from geometry, analysis, vectors, and calculus. In 1842, a German mathematician named Jakob Steiner used geometric arguments to present five proofs of the theorem. However, Steiner had assumed that a solution was possible, which was a subtle flaw to otherwise creative arguments. Karl Weierstrass proved the existence of such solutions in 1879. Other mathematicians proved the results in a variety of other ways.

Historical Applications and Computations

One application of circumference of a circle is the computation of the Earth's circumference. Eratosthenes of Cyrene, in 240 B.C.E., computed the Earth's circumference using trigonometry and the angle of elevation of the sun at noon in Alexandria and Syene. He made an assumption that the Earth and the sun were perfect spheres and that the sun was so far away that its rays hitting the Earth could be considered parallel. By measuring the shadows thrown by sticks on the summer solstice, Eratosthenes derived a formula to measure the circumference of the Earth and determined it to be 252,000 stadia. Teachers in mathematics classrooms share Eratosthenes's calculation in order to highlight his ingenuity and showcase the power of setting up proportions and applying the congruence of alternate interior angles of parallel lines. There is debate about the value of a stadia, but historians estimate that Eratosthenes was correct within a 2% to 15% margin of error. The Indian mathematician Aryabhata made revolutionary contributions toward the understanding of astronomy at the turn of the fifth century. His calculations on π, the circumference of Earth, and the length of the solar day were remarkably close approximations.

The middle of the seventeenth century marked a fruitful time in the history of calculating the length of general curves. For instance, the curve that forms the shape of a nautilus shell is called the "logarithmic spiral" or "equiangular spiral." Evangelista Torricelli described its length using geometric methods. Christopher Wren published the rectification of the cycloid curve. Hendrik van Heuraet and Pierre de Fermat independently explored ideas that would eventually lead to the integral formula of arc length.

In the twentieth century, methods from fractals, popularized by Benoit Mandelbrot, have proven useful in modeling objects like a coastline. One example that is regularly examined in mathematics classrooms is the Koch snowflake, named for Helge von Koch, an example of a curve that bounds a region with finite area yet has infinite perimeter.

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See Also: Curves; Mapping Coastlines; Measurements, Length; Pi; Polygons.

Permutations and Combinations

Category: History and Development of Curricular Concepts.

Fields of Study: Communication; Connections; Data Analysis and Probability; Number and Operations.

Summary: For centuries, mathematicians have posed and studied problems that involve various arrangements or groupings of sets of objects, which are known as permutations and combinations.

In a very broad sense, combinatorics is about counting. The mathematical discipline of combinatorics addresses the enumeration, permutation, and combination of sets of objects, as well as their relations and properties. Combinatorial problems can be found in many areas of pure and applied mathematics, including algebra, topology, geometry, probability, graph theory, optimization, computer science, and statistical physics. One of the earliest problems in combinatorics is found in the work of Greek biographer Plutarch, who described mathematician Xenocrates of Chalcedon's work on calculating how many syllables could be produced by taking combinations of the letters of the alphabet. This occurred between 400 and 300 B.C.E. Millennia later, mathematician William Gowers won the 1998 Fields Medal, widely regarded as the most prestigious prize in mathematics, for his "contributions to functional analysis, making extensive use of methods from combination theory." In twenty-first-century school curricula, primary school children study number combinations to facilitate learning basic operations like addition, subtraction, multiplication, and division. High school students often study permutations and combinations as counting techniques. Permutations and combinations were fundamental for cracking the World War II Enigma code and continue to remain vital in cryptography, among other fields.

Definitions

In mathematical fields like algebraic group theory, combinatorics, or probability, the term "permutation" has several meanings that are all essentially related to the idea of rearranging, ordering, or permuting some kind of mathematical object. When paired with combinations, particularly in primary and secondary curricula, a permutation is usually thought of as an ordered arrangement of some set or subset of objects. For example, for the set of objects A, B, and C, there are six permutations of the set: {A, B, C}, {A, C, B}, {B, A, C}, {B, C, A}, {C, A, B}, and {C, B, A}. A combination is then a subset of objects selected from a larger set, where order does not matter. For example, for the set A, B, and C, one combination of two objects is {A, B}. In some applications, the objects in a combination are thought of as being chosen sequentially. However, since order does not matter, the selection {B, A} would represent the same combination as {A, B}. All possible two-object combinations are {A, B}, {B, C}, {A, C}. Mathematicians Blaise Pascal and Gottfried Leibniz used the specific term "combinations" beginning in the seventeenth century, while Jacob Bernoulli is often credited with introducing the term "permutations" a short while later. Some alternatively trace it to Thomas Strode in the seventeenth century.

History and Early Applications

The real-world motivation for many early problems involving what are now called "combinations" and "permutations" was religion. For example, Jaina, Christian, and Jewish scholars were interested in letter permutations, which some believed had spiritual power. In the ninth century, the Jaina mathematician Mahavira discussed rules for using permutations and combinations. In the tenth century, Rabbi Abraham ben Meir ibn Ezra used combinations to study the conjunction of planets. Another motivator was games of chance, which also drove probability theory. Archaeological evidence suggests that gambling has been around since the dawn of humankind, and many games rely on players achieving special combinations of symbols or objects like knucklebones, sticks, or polyhedral dice. Surviving writings show that Egyptian, Greek, Hindu, Islamic, and perhaps Chinese scholars and mathematicians studied permutations and combinations.

The Egyptian game "Hounds and Jackals" used a set of "throw" sticks that resulted in combinations of outcomes that determined how far a player might move. In the sixth century B.C.E., Hindus discussed combinations of six tastes: sweet, acid, saline, pungent, bitter, and astringent. Some consider the Chinese divination text *I-Ching* to be part of the literature on combinations and permutations since it discussed arrangements sets of trigram and hexagram symbols. Versions date to at least 400–300 B.C.E. In the sixth century, Roman philosopher Anicius Manlius Severinus Boëthius presented a rule for finding the possible combinations of objects taken two at a time from some set.

In the tenth and eleventh centuries, mathematicians like Acharya Hemachandra explored the how many combinations of short and long syllables were possible in a line of text with a fixed length, and Bhaskara's treatise *Bhaskaracharyai* contained an entire chapter devoted to combinations, among other chapters on topics like arithmetic, geometry, and progressions. Both al-Marrakushi ibn Al-Banna and Kamal al-Din Abu'l Hasan Muhammad Al-Farisi explored the relationship between polygonal numbers, the binomial theorem, and combinations. Al-Farisi used what historians consider a form of induction to show the relationship between triangular numbers (numbers that can be represented by an equilateral triangular grid of points such as, 1, 3, 6, 10), and the combinations of subsets of objects drawn from a larger set. Mi'yar al-'aqul ibn Sina (Avicenna) developed a system of combinations of "simple" machines to classify complex mechanisms.

The original concept of simple machines is attributed to mathematician Archimedes of Syracuse. A group might be machines containing rollers and levers, chosen from a larger set of possibilities that included windlasses, pulleys, rollers, levers, and other components. Starting in the Renaissance, the most commonly recognized set of six simple machines was the lever, inclined plane, wheel and axle, screw, wedge, and pulley. Students continue to discuss more complex machines as combinations of simple machines.

In Europe, beginning around the twelfth century and up through the nineteenth century, many mathematicians such as Levi ben Gerson, Bernoulli, Leibniz, Pascal, Pierre Fermat, Abraham de Moivre, George Boole, and John Venn worked on the development of combinations and permutations, frequently in the context of probability theory. For example, Johann Buteo (or Jean Borell) discussed the possible throws of four dice as well as locks with movable combination cylinders in his sixteenth-century work *Logistica*.

Bernoulli's *Ars Conjectandi* collected knowledge of permutations and combinations through the seventeenth century and was a popular combinatorics book in the eighteenth century. However, standard notation for permutations and combinations was still emerging.

Factorials

A mathematical function called a *factorial* is used to compute the number of possible permutations and combinations. Let $n!$ equal

$$n\times(n-1)\times(n-2)\times(n-3)\times\ldots\times3\times2\times1.$$

For example, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$. Further, $0!$ is defined to be 1. Bernoulli had proved many factorial results, like the fact that $n!$ gives the number of permutations of n objects. The use of the exclamation point to indicate a factorial, which was more convenient for printers of the day than some older notations, has been attributed to mathematician Christian Kramp. He worked in the late eighteenth and early nineteenth centuries. The general rule for finding permutations and combinations is sometimes attributed to Bernoulli and sometimes to sixteenth and seventeenth century mathematician Pierre Hérigone, who is also famed for introducing a variety of mathematical and logical notations. However, mathematicians used their own methods for indicating permutations and combinations well into the nineteenth century. For example, Thomas Harriot's seventeenth-century work *Ars Analyticae Praxis* contained unique symbolism for displaying the combinatorial process of finding binomial products.

In the notation common in the twentieth and twenty-first centuries, the number of permutations is stated as $n\mathrm{P}r$ where n is the total number of objects in a set and r is the number of objects selected from n and permuted,

$$n\mathrm{P}r=\frac{n!}{(n-r)!}.$$

The number of combinations is $n\mathrm{C}r$, which is read as "n choose r,"

$$n\mathrm{C}r=\binom{n}{r}=\frac{n!}{r!(n-r)!}.$$

The partial origins of this approach may perhaps be traced to nineteenth-century amateur mathematician Jean Argand, who used (m, n) to represent combinations of n objects chosen from a set of m objects.

Modern Developments

In the early twentieth century, mathematicians and others continued to develop theories and applications of combinatorial concepts. For example, statistician Ronald Fisher applied combinations to the design of factorial experiments, while artist Maurits Cornelius (M.C.) Escher developed his own system for categorizing combinations of shape, color, and symmetrical properties, which can be found in his 1941 notebook later referred to as a paper, *Regular Division of the Plane with Asymmetric Congruent Polygons*. Historians discuss that the sketchbooks of a typical artist contain preliminary versions of final works. Escher's book, on the other hand, appeared to form a theoretical mathematical basis for his tiling work. These combinatorial categories also influenced the field of crystallography.

Circular permutations are also common. One could think of lining up six people in a straight line to take their picture versus seating them at a round table. There are $n!$ permutations of the people lined up. However, once all six people are seated, even if they were all asked to move over one seat, they would all still be seated in the same overall order. There are therefore $(n-1)!$ ways of putting objects in a circle. Another possibility is that all items in the set are not unique, like the letters in "Mississippi," which reduces the number of unique permutations and combinations versus a set of the same length with unique components.

Permutation Groups

In a field like modern algebra, permutations can be viewed as maps that relate a set to itself. The set of permutations is then collected into an algebraic structure called a "group." One example is the various possible transformations of a Rubik's Cube puzzle, named for Erno Rubik. There are 43,252,003,274,489,856,000 permutations in the group for a 3-by-3-by-3 Rubik's Cube. Mathematicians often use software like the Groups, Algorithms, Programming (GAP) system to model and understand the transformations. Theories about permutation groups have been traced by historians to at least as far back as Joseph Lagrange's 1770 work *Réflexions sur la résolution algébrique des équations*, in which he discussed the permutations of the roots of equations and considered those roots as abstract structures. Paolo Ruffini used what would now be called *group theory* in his work, including permutation groups, and proved many fundamental theorems. In the nineteenth century, Augustin-Louis Cauchy generalized some of Ruffini's results. He studied permutation groups and proved what is now known as Cauchy's theorem. High school mathematics teacher Peter Sylow wrote his book *Théorèmes sur les groupes de substitutions* in the latter half of the nineteenth century, and it contained what are now known as the three Sylow theorems, which he proved for permutation groups. Arthur Cayley wrote about the connections between his work on permutations and Cauchy's, extended the notion of permutation groups into the broader idea of algebraic groups, and ultimately proposed that matrices and quaternions were types of groups. Some of his work served as one foundation for physicist Werner Heisenberg's development of quantum mechanics.

In the early twentieth century, George Pólya used permutation groups and other methods to enumerate isomers (compounds that have the same molecular components but different structural arrangements, or permutations) in organic chemistry. He also influenced Escher's studies of combinations. The George Pólya Prize is given every two years by the Society for Industrial and Applied Mathematics. One criterion for winning is "a notable application of combinatorial theory." Mathematicians continue to explore permutations and combination concepts in algebra and many other areas of mathematics.

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See Also: Binomial Theorem; Data Analysis and Probability in Society; Dice Games; Lotteries; Probability; Transformations.

Perry, William J.

Category: Government, Politics, and History.
Fields of Study: Connections.

Summary: Influential secretary of defense William J. Perry earned a Ph.D. in mathematics.

William J. Perry (1927–) is an American businessman, mathematician, engineer, and former secretary of defense under President Bill Clinton. William Perry received many honors and recognition for his work. In 1997, he was awarded the Presidential Medal of Freedom, and he has been decorated twice each with the Department of Defense Service Medal and the Defense Intelligence Agency's Outstanding Civilian Service Medal. He has also received many awards from foreign governments. William J. Perry's academic degrees, all of which are in pure mathematics, may seem an unlikely preparation for a successful businessman and secretary of defense. The logical mindset and the steps of analytic problem solving he learned as a student of mathematics helped Perry make rational and objective decisions about complicated situations for which only partial information was available. This connection is not an unusual; people who have been trained in mathematical reasoning before going on to careers in nonmathematical fields often cite the utility of mathematical thinking as a way to approach difficult and complex problems.

Early Life and Education

Perry was born in Vandergrift, Pennsylvania. After graduating from high school in 1945, Perry enlisted in the U.S. Army and served in Japan before attending Stanford University. Perry was always interested in both mathematics and English, but he finally settled on mathematics as a major because, "I simply had more flexibility by going into mathematics." He attributed his interest in advanced mathematics to George Polya, his advisor at Stanford, saying: "He just pushed me and gave me interesting problems to work on. And he exposed me to parts of mathematics that I had never seen before. And he was just a warm human being." He later earned a Ph.D. in mathematics from Pennsylvania State University, where his research was in the field of partial differential equations. While working on his doctorate, Perry loved teaching mathematics, and he imagined that he might become a mathematics professor. He took a part-time job as an applied mathematician at an electronics company in order to support his family and decided to concentrate on the applied side of mathematics.

Career

Perry enjoyed a successful career as an engineer and businessman. He spent 10 years as director of the Electronic Defense Laboratories of Sylvania/GTE, followed by 13 years as founding president of ESL Inc. In 1977, he became President Jimmy Carter's undersecretary of defense for research and engineering. In this position, he played an important role in developing stealth aircraft technology. In 1981, he returned to industry as the managing director of an investment bank that focused on high-technology companies. In 1993, William Perry was appointed as deputy secretary of defense under then-secretary of defense Les Aspin. The following year, he was promoted to secretary of defense, a position he would hold until 1997. He stated: "Quite clearly, knowing how to solve a differential equation is not a useful tool for me. I've never been asked to solve one since becoming the Secretary of Defense. But, the discipline of thinking, systematically approaching problems, of rigorous thinking is a useful—I would say even an indispensable tool—for a job of this sort."

"Preventive defense" was the watchword of Perry's strategy as secretary of defense: prevent threats before they happen, deter threats that are realized, and respond with decisive military force to threats that cannot be deterred. He noted that: "Analytical thinking is a good framework, a good foundation for which to approach problems." This strategy manifested itself as threat reduction programs, including the START II treaty (for which Perry advocated strongly), active opposition to nuclear proliferation, and expansion of the North Atlantic Treaty Organization. He worked hard to maintain an effective and modern military in spite of defense budget shortfalls. One of his priorities was to establish relationships with members of the military at all levels. Unlike many other secretaries of defense, William Perry was an active participant in foreign policy, traveling often to foreign countries as part of his response to the many global challenges during his tenure as secretary of defense, which included the Bosnian War, conflict in Somalia, the aftermath of the first Gulf War, North Korean nuclear aspirations, and the crisis in Haiti.

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Michael "Cap" Khoury

See Also: Careers; Mathematics, Applied; Strategy and Tactics.

Personal Computers

Category: Communication and Computers.
Field of Study: Algebra; Communication; Data Analysis and Probability; Number and Operations; Representations.
Summary: Advances in computing have made mathematical processing power so inexpensive that it has become more practical to do many tasks on the computer.

A computer is a device that manipulates raw data into potentially useful information. Computers may be analog or electronic. Analog computers use mechanical elements to perform functions. For example, Stonehenge in England is believed by some to be an analog computer. It allegedly uses the stones along with the positions of the sun and moon to predict celestial events like the solstices and eclipses. Electronic computers use electrical components like transistors for computations.

Many consider the first personal computer to be Sphere 1, created by Michael Wise in the mid-1970s. The Apple II was introduced in 1977, and Apple Inc. offered the Macintosh, which had the first mass-marketed graphical user interface, by 1984. IBM debuted its personal computer in 1981. "Macs" and PCs quickly became common in businesses and schools for a variety of purposes. Processing speed, size, memory capacity, and other functional components have become faster, smaller, lighter, and cheaper over time, and personal computers have evolved into a multitude of forms designed to be customizable to each user's needs.

At the beginning of the twenty-first century, desktops, laptops, netbooks, tablet PCs, palm-sized smartphones, handheld programmable calculators, digital book readers, and devices like Apple's iPad offer access to computing, the Internet, and other functions.

Mathematical History of Computers

Modern computing can be traced to nineteenth century mathematician Charles Babbage's analytical engine. Boolean algebra, devised by mathematician George Boole later in the same century, provided a logical basis for digital electronics. Lambda calculus, developed by mathematician Alonso Church in the early twentieth century, also laid the foundations for computer science, while the Turing machine, a theoretical representation of computing developed by mathematician Alan Turing, essentially modeled computers before they could be built. In the 1940s, mathematicians Norbert Wiener and Claude Shannon researched information control theory, further advancing the design of digital circuits. The Electrical Numerical Integrator and Calculator (ENIAC) was the first general purpose electronic computer. It was created shortly after World War II by physicist-engineer John Mauchly and engineer J. Presper Eckert. They also developed the Binary Automatic Computer (BINAC), the first dual-processor computer, which stored information on magnetic tape rather than punch cards, and the first commercial computer, Universal Automatic Computer (UNIVAC). Mathematician John Von Neumann made important modifications to ENIAC, including serial operations to facilitate mathematical calculations. Scientists William Bradford Shockley, John Bardeen, and Walter Brattain won the 1956 Nobel Prize in Physics for transistor and semiconductor research, which influenced the development of most subsequent electronic devices, including personal computers. During the latter half of the twentieth century, countless mathematicians, computer scientists, engineers, and others advanced the science and technology of personal computers, and research has continued into the twenty-first century. For example, Microsoft co-founder Bill Gates published a paper on sorting pancakes, which has extensions in the area of computer algorithms. Personal computers have facilitated mathematics teaching and research in many areas such as simulation, visualization, and random number generation, though the use of calculators and software like Maple for teaching mathematics generated controversy.

Devices, Memory, and Processor Speeds

The typical personal computer has devices for the input and output of information and a means of retaining programs and data in memory. It also has the means of interacting with programs, data, memory, and devices attached to the computer's central processing unit (CPU). Input devices have historically included a keyboard and a mouse, while newer systems frequently use

Many consider the Sphere to be first personal computer. Sphere had difficulty providing the product and shut down after two years.

touch technology, either in the form of a special pad or directly on the screen. Other devices include scanners, digital cameras, and digital recorders. Memory storage devices are classified as "primary memory" or "secondary" devices. The primary memory is comprised of the chips on the board inside the case of the computer. Primary memory comes in two types: read only memory (ROM) and random access memory (RAM). ROM contains the rudimentary part of the operating system, which controls the interaction of the computer components. RAM holds the programs and data while the computer is in use. The most popular types of secondary memory used for desktop computers include magnetic disk drives, optical CD and DVD drives, and USB flash memory.

The speed of the computer operation is an important factor. Computers use a set clock cycle to send the voltage pulses throughout the computer from one component to another. Faster processing enables computers to run larger, more complex programs. The disadvantage is that heat builds up around the processor, caused by electrical resistance. ENIAC was 1000 times faster than the electromechanical computers that preceded it because it relied on vacuum tubes rather than physical switches. Turing made predictions regarding computer speeds in the 1950s, while Moore's law, named for Intel co-founder Gordon Moore, quantified the doubling rate for transistors per square inch on integrated circuits. The number doubled every year from 1958 into the 1960s, according to Moore's data. The rate slowed through the end of the twentieth century to roughly a doubling every 18 months. Some scientists predict more slowdowns because of the heat problem. Others, like mathematician Vernor Vinge, have asserted that exponential technology growth will produce a singularity, or essentially instantaneous progress. Processing speed, memory capacity, pixels in digital images, and other computer capabilities have been limited by this effect. There has also been a disparity in the growth rates of processor speed and memory capacity, known as *memory latency*, which has been addressed in part by mathematical programming techniques, like caching and dynamic optimization.

Carbon nanotubes and magnetic tunnels might be used to produce memory chips that retain data even when a computer is powered down. At the start of the twenty-first century, this approach was being developed with extensive mathematical modeling and physical testing. Other proposed solutions involved biological, optical, or quantum technology. Much of the physics needed for quantum computers exists only in theory, but mathematicians like Peter Shor are already working on the mathematics of quantum programming, which involves ideas like Fourier transforms, periodic sequences, prime numbers, and factorization. Fourier transforms are named for mathematician Jean Fourier.

The Digital Divide

The digital divide is the technology gap between groups that have differential access to personal computers and related technology. The gap is measured both in social metrics, such as soft skills required to participate in online communities, and infrastructure metrics, such as ownership of digital devices. Mathematical methods are used to quantify the digital divide. Comparisons may be made using probability distributions and Lorenz curves, developed by economist Max Lorenz, and measures of dispersion such as the Gini coefficient, developed by statistician Corrado Gini. Researchers have found digital divides among different countries, and within countries, among people of different ages, between genders, and among socioeconomic strata.

The global digital divide quantifies the digital divides among countries and is typically given as the differences among the average numbers of computers per 100 citizens. In the early twenty-first century, this metric varied widely. Several concerted private and government efforts, such as One Laptop Per Child, were directed at reducing the global digital divide by providing computers to poor countries. The breakthroughs connected to these efforts, such as mesh Internet access architecture, benefited all users. The Digital Opportunity Index (DOI) is computed by the United Nations based on 11 metrics of information and communication technologies, such as proportion of households with access to the Internet. It has been found to be positively associated with a country's wealth.

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See Also: Cerf, Vinton; Internet; Lovelace, Ada; Servers; Software, Mathematics.

Pi

Category: History and Development of Curricular Concepts.
Fields of Study: Communication; Connections; Measurement; Geometry.
Summary: The ratio of a circle's circumference to it's diameter, π, is one of the most important constants and the first irrational number encountered by most students.

By definition, pi (π) is the ratio of a circle's circumference to the diameter. This definition holds for any circle, with the value of π being the constant value 3.14159265358979. . . . This decimal neither terminates nor repeats, making π irrational. Mathematicians and non-mathematicians alike are intrigued by the many appearances of π in diverse situations. Capturing this apparent mysticism in the 1800s, the mathematician Augustus de Morgan wrote, "This mysterious 3.14159… which comes in at every door and window, and down every chimney."

Values Used for Pi

Since the beginning of written mathematics, people have tried to calculate π's value. Around 2000 B.C.E., the Babylonians and Egyptians assigned values equal to 3 1/8 (3.125) and $4(8/9)^2$ (3.1605). In 1100 B.C.E., the Chinese used $\pi = 3$, a value which also appears in the Bible (I Kings 5:23). In 300 B.C.E., Archimedes of Syracuse produced the first "accurate" value, using inscribed and circumscribed 96-sided polygons to produce the approximation $3\ 10/71 < \pi < 3\ 1/7$ (or $3.140845. . . < \pi < 3.142857. . .$). Since that time, multiple methods and formulas have been created to determine more exact values of π. Today, powerful computers use similar formulas to calculate values of π to extreme precision, with the current value exceeding 2.7 trillion digits (the record as of January 2010). Two examples of these formulas involving infinite series are

$$\frac{\pi}{2} = \frac{2\times 2\times 4\times 4\times 6\times 6\times 8\cdots}{1\times 3\times 3\times 5\times 5\times 7\times 7\cdots}$$

$$\text{or}\quad \frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots.$$

Students in the twenty-first century learn about π in elementary school, and exposure to π continues in later courses in mathematics and physics. Since spherical coordinates are used in many applications, π is found in physical formulas such as Einstein's field equations, the Heisenberg uncertainty principle, and Coulomb's law for electric force, which are named after Albert Einstein, Werner Heisenberg, and Charles-Augustin de Coulomb, respectively. Mathematicians and computer scientists describe π as a great stress test for computers because of the seemingly random aspects of its digits.

Algorithms to compute the digits of π are regarded as more important than the digits themselves. Mathematicians continue to investigate other unsolved problems related to π, including attempts to determine how random the digits are.

Applications

The number π has played important roles in multiple situations. In 1767, Johann Lambert proved that π was irrational (it could not be written as the ratio of two integers). Then, in 1882, Ferdinand von Lindemann proved that π was transcendental (it could not be constructed using geometric tools and was not a root of a non-constant polynomial equation with rational coefficients). These two discoveries provided the key to proving the impossibilities of the Greeks' three problems of antiquity—squaring a circle, trisecting an angle, and duplicating a square.

Considered by many to be a ubiquitous number, π shows up in odd situations. First, in 1777, the naturalist Georges Buffon approximated the value of π experimentally by tossing a needle (length L) on a ruled surface (parallel lines spaced at distance D). If the tossed needle touches a line S times on N tosses, then

$$\pi \approx \frac{2SL}{DN}.$$

Second, the probability that two random integers are relatively prime (they have no common divisor) is

$$\frac{6}{\pi^2}.$$

Anyone can try these experiments, either by dropping needles or taking ratios of random integers; many are surprised that both produce good approximations for π. However, complex mathematics is needed to explain "why."

In 1743, Swiss mathematician Leonhard Euler published the formula $e^{ix} = \cos(x) + i\sin(x)$, linking exponentials, trigonometric functions, and complex numbers. Substituting $x = \pi$, the result becomes the most beautiful formula in mathematics: $e^{i\pi} + 1 = 0$.

Popular Culture

The fascination with decimal expressions of π has led to competitive memorization contests. The Guinness World Records officially recognized Lu Chao as the most recent record holder in the early twenty-first century, but others have claimed more digits. Some people use piems (mnemonic poems); for example, "How I need a drink, alcoholic of course, after the heavy lectures involving quantum mechanics." In this piem, replace each word with its number of letters, producing $\pi \approx 3.14159265358979$. Hideaki Tomoyori, who held the world record of 40,000 digits memorized from 1987–1995, used a pictorial mnemonic system and explained, "I want to go on with the challenge of memorizing π, for just the same reason that people climb high mountains. I think it's a wonderful thing to challenge the limits of what we can do. . . . the more one memorizes of it, the closer one comes to the real value of the circle—closer to perfection." Researchers compared his cognitive abilities with a control group and concluded that he was not superior; they attributed his achievement to extensive practice.

The number π also is connected to some odd events. In 1897, the Indiana State Legislature almost passed a mathematically incorrect bill relating to π and squaring the circle. By its definition, the value of π changes if the circle shifts out of the Euclidean world. That is, in taxicab geometry, or metric geometry on a rectangular lattice structure, the value of π is 4.

The number π is an amazing number, both in its interesting properties and the obsessive attention given it by both mathematicians and non-mathematicians. How else could one explain why on March 14 at 1:59, many people shout, "Happy Pi Day!"

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Jerry Johnson

See Also: Archimedes; Numbers, Rational and Irrational; Sequences and Series; Universal Constants.

Planetary Orbits

Category: Space, Time, and Distance.
Fields of Study: Algebra; Geometry.
Summary: It took mathematicians thousands of years to accurately describe planetary motion.

For millennia, the shape of the paths in which the planets orbited was dominated by metaphysical concerns and assumed, almost without question, to be circular. It was not until the seventeenth century that science discovered the actual shape of planetary orbits, the ellipse.

Early Conceptions

In ancient Greek astronomy, it was assumed that the Earth was the center of the universe, and all of the known planets (including the sun and the moon) as well as the stars revolved around it. Furthermore, at least from the time of Pythagoras (c. 569–475 B.C.E.), these orbits were assumed to be circular. This assumption was a metaphysical one.

The Pythagoreans believed in the perfection of mathematics and held the view that the circle was perfect because of its symmetry and continuity. Therefore, the universe must surely be constructed to reflect this perfection by requiring the planets to revolve around the Earth in perfect circular motion. That influential philosophers such as Plato and Aristotle accepted the perfection of circular motion contributed to the fact that the idea went almost unchallenged for nearly 2000 years.

With the increasing ability to make accurate observations of the movements of the heavens and mathematical calculations to predict those movements, the simple assumption of perfect circular motion became more problematic. The predictions of the planetary positions did not match the actual observed locations. Eudoxus (408–355 B.C.E.) addressed this discrepancy by devising a complicated system of nested spheres in which each planet moved, maintaining circular motion of each sphere while more accurately predicting the location of the planets.

For many centuries, one man's work dominated European thinking on planetary motions. The Greek mathematician and astronomer Ptolemy (85–165 C.E.) compiled all that was known about the movements of heavenly bodies into one work that came to be known as *The Almagest*. This book employed an array of very complex geometric and trigonometric theories to describe the movement of the planets, with the Earth remaining at the center. In order for the observations to be as close as possible to the calculations, Ptolemy used epicycles (small circles revolving upon bigger circles as they revolve around the Earth) and moved the Earth away from the center of revolution of the planets.

The new center of revolution was an imaginary point some distance away from the Earth. Ptolemy's influence on Western astronomy was partially because of its general agreement with Christian doctrine. As the center of God's creation, the Earth must rest at the center of the cosmos. Furthermore, a perfect Creator would use the perfect circle to put His creation in motion.

Challenges

The most serious challenge to Ptolemaic cosmology came from the Polish church official, Nicolaus Copernicus (1473–1543), whose revolutionary work *De Revolutionibus* placed the sun, not the Earth, at the center of the universe, relegating the Earth to mere planethood. Copernicus, however, remained adamant in his belief that the planets orbited the sun in a composite of perfect circular motions. The doctrine of perfect circular motion in the heavens was finally challenged by the German astronomer Johannes Kepler (1571–1630). Kepler, after many years of tedious and painstaking calculations involving the orbit of Mars, finally determined that Mars actually orbited the sun in an elliptical orbit, not a circular one. This revolutionary idea was based in part on another discovery by Kepler that the speed of the planets varied as they orbited the sun. Later, the great British mathematician and scientist, Isaac Newton (1643–1727), used his universal law of gravitation and laws of motion to provide a mathematical explanation for Kepler's claim of elliptical orbits, finally putting an end to the ancient doctrine of circular motion in the heavens.

Mathematics continues to play an important role in modeling planetary orbits. For example, Mercury's orbit is more accurately represented with hyperbolic geometry than with Euclidian geometry. Further, the orbit of Mercury allows researchers to see the impact of the sun's gravitational field on the curvature of space.

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Todd Timmons

See Also: Astronomy; Conic Sections; Geometry of the Universe; Greek Mathematics.

Plate Tectonics

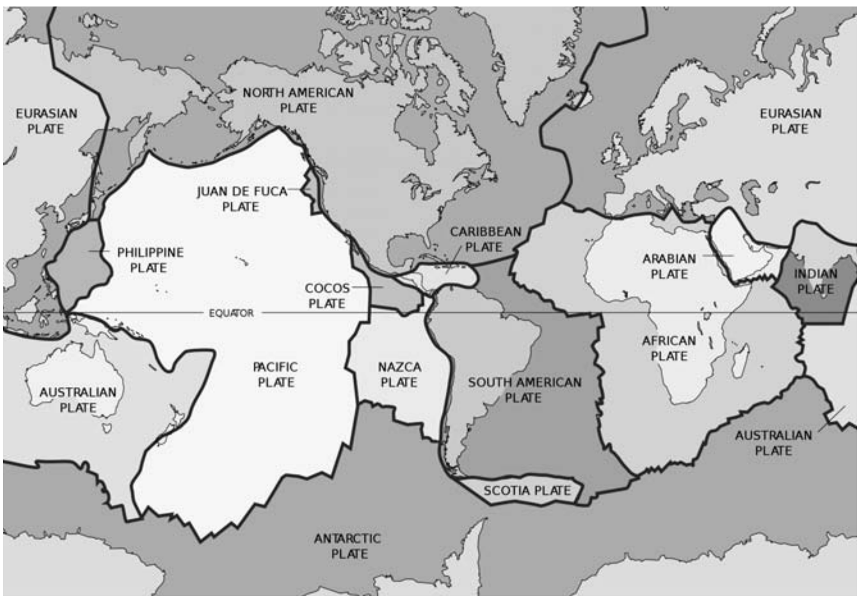
Category: Weather, Nature, and Environment.
Fields of Study: Data Analysis and Probability.
Summary: Tectonic plate movement is measured and analyzed using mathematics.

The ideas of plate tectonics and continental drift have been theorized by many scientists over the years. For example, in the early twentieth century, Alfred Wegener publicly presented theories regarding the existence of a supercontinent called "Pangea" that eventually formed all the known continents. He supported centrifugal force as an explanation for drift. A few years later, Arthur Holmes supported thermal convection as an explanation. At the time, there was insufficient mathematical and scientific evidence to support these theories and they were largely dismissed, in part because seeing into the depths of the oceans and into the Earth itself is often a more difficult venture than seeing galaxies at the far reaches of the universe. By the latter half of the twentieth century, discoveries such as mid-Atlantic underwater volcanic chains and the mapping and mathematical analysis of seismic activity suggested the existence of large, mobile plates in the Earth's crust.

In the twenty-first century, scientists and mathematicians are still developing new and innovative ways to collect data, model, visualize, and simulate the Earth's inner structure. For example, geophysicist Robert van der Hilst and mathematician Maarten Van de Hoop have used a mathematical technique known as "micro-local analysis," as well as statistical methods, such as confidence intervals, to explore the geometry of the layers near the boundary of the Earth's core and mantle. This technique extends existing methods for analyzing noisy seismic data. It produces not only an image, but also an estimate of the probability that a true layer has been discovered. Ongoing collaboration between mathematicians and geophysical scientists is crucial to address the massively scaled problems that arise in geoscience, such as continental drift. This is true not only for data collection in the field, but also for computer simulation, which is increasingly an avenue of exploration and cross-validation for theories and data. These simulations often require combining many scales of data, both macro and micro, as well as observations collected over different periods of time. Further, much of the data is noisy, incomplete, or difficult to directly measure. Mathematics is also involved in the increasingly sophisticated tools that allow scientists to visit the depths of the oceans and begin to look at some previously impenetrable layers of the Earth.

The Spreading Sea Floor

As an officer in the U.S. Navy, Harry Hess's curiosity led him to measure the ocean floor using sounding gear and magnetometers during World War II. Once the war ended, Hess developed the theory of sea floor spreading to explain his data. He proposed that magma oozed up



A U.S. Geological Survey illustration of Earth's rigid slabs (called tectonic plates) that are moving relative to one another.

between the plates along the ridges in the ocean floor, pushing them apart and causing the plates to move.

Strips of rock parallel to the ridges provide evidence for sea floor spreading. Strips closest to the ridge have the same polarity as the Earth (magnetic north pointing to the north pole); however, the strips moving out away from the ridge on opposite sides mirror each other and alternate between current polarity and reversed polarity as the Earth's magnetic field reversed over time. These alternating strips suggest that new rock is created along the ridges over geologic time.

Continents Adrift

Until 1912, scientists assumed that the continents were fixed in place. In that year, Alfred Wenger suggested that the continents were adrift, originally part of one large landmass. Wegner cited evidence such as matching geological formations and fossils from South America and Africa. It was not until the late 1960s that discoveries were made and measuring techniques improved to the extent that the theory of plate tectonics emerged and became widely accepted. Scientists now recognize that the continents are attached to plates and move with them rather than moving independently. Scientists also now know that the plates that make up Earth's crust and the continents attached to them are moving several centimeters per year on average as they collide, move apart, and brush up against each other.

Plate Movement

Muawia Barazangi and James Dorman (1969) charted the locations of all earthquakes occurring from 1961 to 1967 and found that most occurred in a narrow band of seismic activity. This band of high earthquake and volcanic activity, commonly called the "Pacific Ring of Fire," defines many plate boundaries around the Pacific Ocean.

Most plate movement occurs along the edges of the plates. Scientists can measure the velocity (speed and direction) of plate movement and determine how that relates to earthquake and volcanic activity. For historical information, scientists turn to ocean floor magnetic striping data and geological dating of rock formations.

Measurement techniques have improved greatly since Hess's measurements. The most common technique for measuring plate movement in the early twenty-first century is the Global Positioning System (GPS). As satellites continuously transmit radio signals to Earth, each GPS ground site simultaneously receives signals from at least four satellites. By recording the exact time and location of each satellite when its signal was received, it is possible to determine the precise position of the GPS ground site on Earth (longitude, latitude, and elevation). Regularly measuring distances between specific points allows scientists to determine if there has been active movement between plates on a scale of millimeters. Using time-series graphs and plotting vectors, it is possible to learn how the plates move.

While scientists know that most earthquakes and volcanoes occur along plate boundaries, they still cannot predict exactly when and where they will occur. By monitoring plate movement, scientists hope to learn more about the events building up to earthquakes and volcanic eruptions.

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Christine Klein

See Also: Earthquakes; Geothermal Energy; GPS; Tides and Waves; Volcanoes.

Plays

Category: Arts, Music, and Entertainment.
Fields of Study: Communication; Geometry; Representations.
Summary: Numerous plays explore mathematical concepts and mathematicians.

The genre of "mathematical theater" is a relatively recent phenomenon. A smattering of earlier examples of mathematics appeared on stage, but the turning point was Tom Stoppard's 1993 play *Arcadia*, which opened the door to an entirely new realm of collaborative possibilities between theater and the mathematical sciences. Following on the heels of *Arcadia* was the award-winning *Copenhagen* (1998), a play by Michael Frayn about the fraught relationship between physicists Neils Bohr and Werner Heisenberg. If there were any lingering doubts as to whether mathematics was a relatable theme for theater audiences, David Auburn's Pulitzer Prize–winning play *Proof* (2000) laid them firmly to rest. The ensuing years have produced successful dramas, comedies, and biographical scripts that are marked not just by the inclusion of mathematical references, but also by the wholesale incorporation of mathematics into the content and structure of the play. Some even turn a critical lens back on traditional mathematics education and related gender issues.

Stoppard and Science

Bertold Brecht's *The Life of Galileo* (1939) gives a cursory acknowledgment of the protagonist's training as a mathematician. *The Physicists* (1962), by Friedrich Durenmatt, features Isaac Newton as a character—or rather, it features a spy who is posing as a patient in a mental institution, pretending to believe he is Newton. Terry Johnson's play *Insignificance* (1982) contains a scene where Marilyn Monroe explains special relativity to Albert Einstein.

But the best place to look for a forerunner for the substantial and explicit role of mathematics in *Arcadia* is in Tom Stoppard's earlier writing. His first major success was *Rosencrantz and Guildenstern are Dead* (1966), a dark comedy, which opens with a scene of the two Shakespearean characters trying to rectify the laws of probability with the fact that they have just witnessed nearly 100 occurrences of heads in as many flips of a coin. Zeno's paradoxes appear in *Jumpers* (1972), and there is a cameo appearance of Leonhard Euler's famous Bridges of Königsburg problem in *Hapgood* (1988), a play that also contains significant discussions of quantum mechanics. *Hapgood* comes closest to *Arcadia* in its attempt to fully integrate mathematics and science into the mechanics of the play, but this was confusing for some audiences and the reviews for *Hapgood* tended to be rather harsh. *Arcadia*, in contrast, was greeted as something of a marvel and an instant classic when the play opened in London in 1993.

The opening scene of *Arcadia* is set in 1809, where 13-year-old Thomasina Coverly is growing frustrated with her tutor, who has asked her to find a proof for Fermat's Last Theorem. Thomasina has more romantic issues on her mind ("Septimus, what is carnal embrace?" is the first line of the play), and her restlessness—and her genius—eventually lead her to discover the core principals of fractal geometry and chaos theory 150 years before their time. *Arcadia* also contains a second set of characters living in the present day in the same house, and among them is a mathematician whose expertise in dynamical systems allows him to decipher Thomansina's notebooks for the other characters—and the audience. In a clever homage to Fermat, Thomasina writes in one of her notebooks that "I, Thomasina Coverly, have found a truly wonderful method whereby all the forms of nature must give up their numerical secrets and draw themselves through number alone. This margin being too mean for my purpose, the reader must look elsewhere for the New Geometry of Irregular forms discovered by Thomasina Coverly."

A recurring theme in *Arcadia* is the juxtaposition of reasoned, classical thinking with untamed, romantic expression. With respect to the mathematics in the play, the Euclidean geometry of circles and spheres is contrasted with the fractal geometry of leaves and clouds. In a related way, the determinism inherent in Newton's Laws of Motion is challenged by the unpredictability of chaotic systems and ultimately by the Second Law of Thermodynamics. These scientific ideas provide a compelling metaphorical backdrop for the interpersonal tensions that drive the emotional arc of the script. The result is a play where the science and the storytelling work in a mutually enriching collaboration.

Copenhagen

Whereas *Arcadia* is a hybrid of mathematics and science, Frayn's *Copenhagen* is very much a "physics play," but its influence is too significant to ignore. The play is inspired by a real historical event. Werner Heisenberg had been put in charge of the Nazi nuclear program, and in 1941, he paid a visit to his mentor Neils Bohr, whose hometown of Copenhagen was under German occupation. The visit ended abruptly, and the deep friendship between these two pioneers of atomic physics ended with no clear resolution ever agreed upon as to what exactly was discussed. Frayn's play explores this question by recreating the experiment of Heisenberg's visit multiple times and, in the spirit of quantum mechanics, each

The Most Well-Known Mathematics Play: *Proof*

Even if it had not been turned into a popular film, *Proof* would still likely be the most well-known mathematics play and most frequently performed. It should be pointed out, however, that unlike *Arcadia* and *Copenhagen*, there is virtually no technical material written into the script. The central relationship in the play is between a father and daughter. The father is a brilliant mathematician who, the audience learns, has become debilitated by serious mental illness. His daughter Katherine has given up on her own education to care for her father, and upon his death, Katherine is plagued by the question of whether this was the right decision, as well as whether she has inherited her father's mental instability. A major plot twist comes when Katherine discloses the existence of a mathematical proof hidden in her father's desk, and a central issue is to determine its rightful author. The audience is never told what the theorem actually is, but is made to understand that it is a monumental result on the order of the Riemann Hypothesis.

A debate among theater critics is whether the mathematics in *Proof* is crucial to the workings of the play, or whether it is intellectual window dressing that could be replaced by some other creative art form; for example, the father might be a composer and put a symphony score in the desk drawer. Although the discussions of explicit mathematics in *Proof* are confined to a few imaginary number jokes and some witty banter about primes, there are several compelling conversations about the aesthetic beauty of mathematics and the discipline is sympathetically portrayed. Because the main questions in the play deal with degrees of certainty, there is an argument that the rigorous standard for what constitutes a mathematical proof provides a valuable point of contrast for the various investigations by the characters in the play.

run of the experiment results in a different outcome. Along the way, the fundamental ideas behind Bohr's Theory of Complementarity and Heisenberg's Uncertainty Principle are given enough explication for the audience to apply these ideas to the process of human introspection as well as to the play itself.

Hardy, Ramanujan, Turing, and Beyond

The most high-profile play about mathematics since *Proof* is *A Disappearing Number*, created and produced by a London-based company called Complicite under the leadership of Simon McBurney. *A Disappearing Number* won the 2007 Olivier Award for Best New Play, among many others, and eventually it toured internationally. The starting point for *A Disappearing Number* is G. H. Hardy's famous essay, *A Mathematician's Apology*. Hardy appears as a character as does the Indian genius Srinivasa Ramanujan. The celebrated collaboration between Hardy and Ramanujan is also the subject matter for a less well-known play called *Partition* (2003), written by Ira Hauptman, and in a less direct way it served as inspiration for *The Five Hysterical Girls Theorem* (2000) written by Rinne Groff. Whereas *Partition* is a fanciful account of a real historical friendship, *The Five Hysterical Girls Theorem* is a purely fictitious comedy about an international mathematics conference that features a protagonist loosely based on Hungarian mathematician Paul Erdös.

Biography and historical fiction are the dominant forms for most new mathematical theater. Isaac Newton is the central subject of *Leap* (2004), by Lauren Gunderson as well as *Calculus* (2003) by Carl Djerassi. *Seventeenth Night* (2004), by Doxiadis Apostolos, tells the story of the final days of logician Kurt Gödel's life in a way that is meant to illustrate the actual content of Gödel's revolutionary Incompleteness Theorems. Georg Cantor's bouts with mental illness are the subject of *Count* (2009), by John Martin and Timothy Craig, and Cantor also appears alongside his philosophical nemesis Leopold Kronecker in a scene in the experimental play *Infinities* (2002), written by John Barrow. *Infinities* actually consists of five scenes or scenarios—one features the Hilbert Hotel introduced by mathematician David Hilbert—each of which explores some paradoxical aspect of infinity.

The drama, and ultimate tragedy, of Alan Turing's life is the subject of at least four plays. The most well-known of these is *Breaking the Code* (1986) by Hugh Whitmore, which is available as an episode of *Masterpiece Theater*. The most ambitious play about Turing in terms of engaging the essence of his mathematical work is probably *Lovesong of the Electric Bear* (2003) by British playwright Snoo Wilson, which received a string of productions in the United States.

Plays By and About Women

Lauren Gunderson, who has been writing plays since she was 16 years old and is known for her interpretations of feminism, science, and history, has spoken widely on the rich intersection of science and theater. She cites *Arcadia* as a good example of the idea that "Science, like any theoretical idea, should lead to a deeper kind of play—a more layered, woven play where the science permeates the form of the play as well as the content." She also encourages playwrights to explore these themes, noting that the fundamental questions of mathematics and science do not exist in some inaccessible other world, but rather are deep and universal. One of her most well-known plays is *Emilie: Le Marquise Du Chatelet Defends Her Life Tonight*, which is about eighteenth-century woman mathematician Gabrielle Émilie Le Tonnelier de Breteuil, Marquise du Châtelet, whose many achievements include a translation and commentary on Isaac Newton's *Principia*. In 2010, Gunderson was the first Playwright in Residence at The Kavli Institute for Theoretical Physics.

Emilie du Chatelet was known for passionately pursuing mathematics in a time when many women were barely literate. Kathryn Wallet's *Victoria Martin: Math Team Queen* examines the modern-day tug of war between popularity and mathematics talent that girls often face as they move into middle school and high school. This theme is also critically explored in Gioia De Cari's autobiographical play *Truth Values: One Girl's Romp Through M.I.T.'s Male Math Maze*. The author uses her personal experiences, such as being asked to serve cookies at a seminar, for comic effect. However, the play is a serious exploration of traditional mathematics in higher education and the role of women in science and mathematics.

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Stephen Abbott

See Also: Literature; Movies, Mathematics in; Musical Theater.

Poetry

Category: Arts, Music, and Entertainment.
Fields of Study: Geometry; Measurement; Representations.
Summary: Rhyme schemes and meter in poetry can be mathematically analyzed and some new forms of poetry are based on mathematical priniciples.

A popular sentiment is that mathematics and poetry lie on opposite ends of some spectrum. However, both are the works of pure intellect and they share many similarities. Whether considering rhyme, rhythm, or visual layout, effective poetry is rich with patterns that may be analyzed with a mathematical eye. At the same time, succinct mathematics has often been compared to poetry. In the modern era, the connections have become explicit, as mathematics has been co-opted by poets to create new poems, while poetry has been analyzed (and occasionally written) by mathematicians.

Meter and Rhyme

Poetic meter is a formalized version of rhythm. When considering rhythm in spoken language, one can focus on syllable stresses, pitch, tone, or *morae*. *Mora* (plural *morae*) is a term used by linguists to denote an individual unit of sound; a long syllable (such as "math") consists of two morae, while a short syllable consists of a single mora. A poetic cadence of length n is a pattern of long and short syllables whose total number of morae is n. Cadences play an especially important role in Indian and Japanese poetry, as well as in modern free verse.

Traditional English meter, however, is usually based on stressed syllables (denoted —) versus unstressed syllables (denoted ˘). The most well-known English meters are iambic pentameter and dactylic hexameter, used extensively by William Shakespeare and Henry Wadsworth Longfellow, respectively. In each of these meters, the first word denotes the metrical foot, and the second word denotes the number of feet per line. A metrical foot is a particular pattern of stressed and unstressed syllables. It usually consists of two, three, or four syllables. For example, an iamb consists of an unstressed syllable followed by a stressed syllable. So a line of iambic pentameter is $2 \times 5 = 10$ syllables in length, and the pattern is ˘—˘—˘—˘—˘—. A dactyl consists of stressed syllable followed by two unstressed syllables. A line of dactylic hexameter is $3 \times 6 = 18$ syllables, and the pattern is —˘˘—˘˘—˘˘—˘˘—˘˘—˘˘. Simple counting shows that there are four possible disyllabic feet (pyrrhus is ˘˘, iamb is ˘—, trochee is —˘, and spondee is ——), eight possible trisyllabic feet, and 16 possible tetrasyllabic feet.

There are further formal devices used by poets, often with the aim of producing euphony, which is beautiful sound combinations: assonance (the same sound repeating within a line), alliteration (multiple words beginning with the same consonant), or specific rhyme schemes. Two examples of rhyme schemes are *ababcdcdefefgg* for a Shakespearean sonnet and *abbaabbacdecde* for an Italian sonnet. The initial lines of Shakespeare's "Sonnet 30":

When to the sessions of sweet silent thought
I summon up remembrance of things past,
I sigh the lack of many a thing I sought,
And with old woes new wail my dear time's waste:

illustrate alliteration and an "abab" rhyme in iambic pentameter—though "past/waste" is only a near rhyme.

Classical Poetic Traditions and Forms

History is rich with individuals such as Omar Khayyám, who excelled in poetry and mathematics separately without drawing a strong connection between the two. However, in at least one culture, the two disciplines were intimately interwoven. In the Indian Vedic civilization, poetic chants and hymns were utilized to pass down a vast body of knowledge. A portion of this knowledge was mathematical, including theorems in arithmetic and geometry. The method of transmission was mathematical: a single text would be recited in up to 11 different ways. Each way emphasized a different poetic approach, such as applying devices of euphony, pausing every other word, or repeating groups of words forward, backward, and in even more complicated permutations. This method is reminiscent of the error-correcting codes employed in twenty-first century CD audio discs. Just as a scratched CD will often still play seamlessly, the redundancy of the Vedic poetic chants allowed for an uncorrupted oral transmission year after year.

A poetic form offers the writer a set of constraints to which the work has to conform. There are many such prescribed forms, some very strict, and others quite open. Perhaps the best-known forms are the sonnet, ode, and haiku. The traditional Japanese haiku, for instance, comprises three lines of 5, 7, and 5 morae, respectively. In English, the syllable is used as counter instead of the mora.

A sestina is a 39-line poem consisting of six 6-line stanzas followed by a 3-line envoy. The six words ending the lines in the first stanza must end the lines in each of the subsequent stanzas, but in a fixed new order. The permutation of the words may be denoted

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 5 & 3 & 1 \end{pmatrix}.$$

This notation indicates that the word ending the first line must end the second line of the next stanza, the word ending the second line must next end the fourth line, and so forth. This permutation is then repeated from one stanza to the next. Mathematicians Anton Geraschenko and Richard Dore have investigated a generalized notion of a sestina to an (n-line-per-stanza) n-tina where n can be any whole number. They prove that if the n-tina is to be interesting—in the sense that the pattern does not repeat before the poem ends—then $2n + 1$ must be a prime number.

Modern Directions

In the modern era, poetry is more often read on a page than spoken aloud, and the two-dimensional geometry of the text is visible. For example, a poem in traditional meter naturally takes on a ragged-on-the-right rectangular shape. The diamond shape of a diamante poem, introduced in 1969 by Iris Tiedt, naturally results from the prescribed construction of its seven lines: one noun, two adjectives, three gerunds, four nouns, three gerunds, two adjectives, and one noun. When poetry purposefully forms a recognizable shape it is called "shape poetry," "griphi," "carmen figuratum," or "concrete poetry." The idea of a shape poem is nothing new: around 300 B.C.E. the Greek poet Simias of Rhodes wrote *Pteryges, Oon,* and *Pelekys* (*Wings, Egg,* and *Hatchet*, respectively) poems whose shape mirrored their subject. Recently, shape poetry has flourished: Lewis Carroll gave a mouse's tail; Guillaume Apollinaire, the Eiffel Tower; e e cummings, a snowflake; John Hollander, a swan with reflection; and Mary Ellen Solt, a forsythia bush. In the 1990s, Eduardo Kac moved poetry into the third dimension with his holopoetry: poetry that floats above a surface as a hologram and takes different meanings when viewed from different angles.

The group Ouvroir de Littérature Potentielle (Workshop of Potential Literature), or Oulipo for short, originated in 1960 with 10 writers, mathematicians, and philosophers. The group has the twin goals of elucidating old and creating new rigid forms for potential literature. A prototypical example of their oeuvre may be seen in Raymond Queneau's *Cent Mille Milliards de poèmes* (*One Hundred Thousand Billion Poems*). This work appears at first glance to consist of 10 sonnets. However, it also includes the instruction that the reader should consider all poems that may be formed by choosing a first line from among the 10 given, then a second line, and so forth. At each stage, the reader has 10 lines from which to choose, and there are 14 lines, so this work encompasses $10^{14} = 100,000,000,000,000$ complete sonnets.

Many forms of poetry have emerged that are very consciously mathematical. The "pioem" is a poem whose words are of length determined by the digits of π in order: 3, 1, 4, 1, 5, 9, ... The number of words in a pioem is not predetermined; it may be as long or short as the author desires. The "Fib" is a poetry form that, like the haiku, prescribes the number of syllables to appear in each line. This prescription is based upon the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, . . . in which each number is the sum of the previous two numbers. Interestingly, the Fibonacci sequence not only gives a form for poetry, but also arises in the mathematical study of poetic cadence. If C_n denotes the number of poetic cadences of length n, Indian polymath Acarya Hemacandra showed

$$C_n = C_{n-1} + C_{n-2}.$$

This equation, known as a *recurrence relation*, generates the Fibonacci sequence. Hemacandra's observance was about 50 years prior to Leonardo of Pisa's 1202 treatise *Liber Abaci*, from which the Fibonacci sequence derives its name.

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Caleb Emmons

See Also: Literature; Permutations and Combinations; Stylometry; Vedic Mathematics.

Polygons

Category: History and Development of Curricular Concepts.
Fields of Study: Communication; Connections; Geometry.
Summary: Polygons have properties making them important in engineering, architecture, and elsewhere.

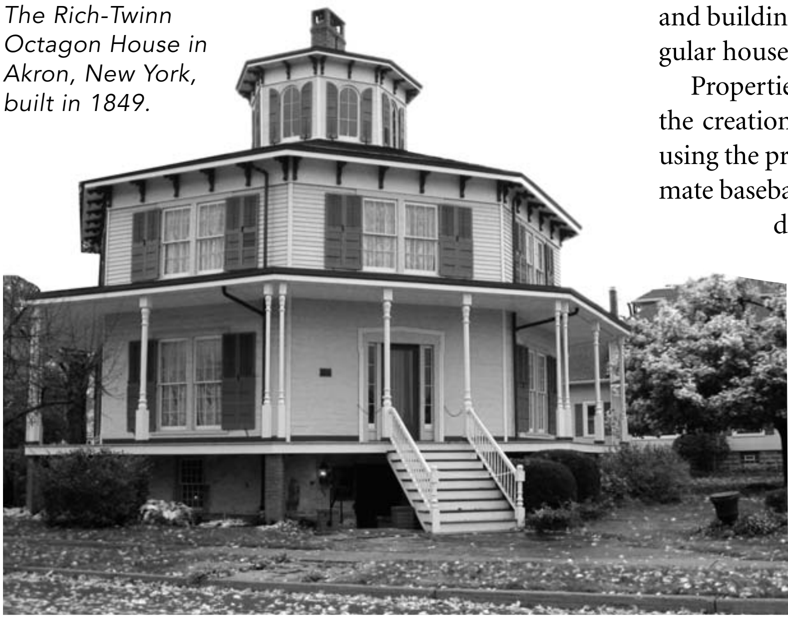
Shapes and figures define how people view the world. Polygons are special figures whose properties and relationships are prevalent in nature and are used extensively by architects, engineers, scientists, landscapers, and artists. Specifically, polygons are traditionally planar (two-dimensional) figures that are closed and comprised of line segments that do not cross. These line segments are called "edges" or "sides," and the points where the edges meet are called "vertices." Planar polygons are very important in engineering, computer graphics, and analysis because they are rigid, they work well with functions, and they are easy to transform. Other types of polygons are also useful, such as spherical, hyperbolic, complex, or near polygons.

Properties of Polygons

Polygons are named by the number of their sides. Typically, polygons with more than 10 sides are called *n*-gons.

Calculating angle sums, areas, and perimeters of polygons is important in architecture, landscaping, and interior design. Understanding properties of triangles and parallelograms facilitates these kinds of calculations. For instance, the sum of the measures of the interior angles of a polygon can be determined by realizing that a polygon with n sides can be divided into $n - 2$ triangles, and that the sum of the measures of the interior angles of any triangle is 180 degrees. Using these ideas, a carpenter could easily determine the angles at which, for example, the sides of a hexagonal window frame should meet. Furthermore, the ability to create polygons from triangles and the ability to rearrange or duplicate some polygons to form parallelograms allow the derivation of area formulas. Michael Serra describes in his 2008 book, *Discovering Geometry: An Investigative Approach*, how the area of a parallelogram can be derived from a rectangle, and the area of a triangle can be derived from a parallelogram.

The Rich-Twinn Octagon House in Akron, New York, built in 1849.



Real World Examples

Polygons are prevalent in the world. Even traffic signs come in the shapes of triangles, rectangles, squares, kites, and octagons. The properties of polygons make them useful in many areas including architecture, structural engineering, nature, and art.

Polygons are sometimes used in architecture for their structural benefits. Trusses formed from triangles provide support for bridges and roofs because, unlike other polygons, triangles do not tend to deform when force is exerted on a vertex. Fences are often formed into polygons because they can be built by linking together straight segments of material that are of equal size and shape. The buildings that comprise the Pentagon building in Washington, D.C., are arranged in a pentagonal shape because, according to Stephen Vogel, walking distances between buildings are less than in a rectangle, straight sides are easier to build, and the symmetrical shape is appealing. In the 1850s, Orsen Fowler popularized octagonal-shaped houses because octagons have larger areas than rectangles with the same perimeter. Thus, octagonal houses provided maximal living space while keeping heating, cooling, and building costs similar to that of the smaller rectangular house with the same outer wall space.

Properties of quadrilaterals and triangles facilitate the creation of squares and right angles. For example, using the properties of a square's diagonals, an approximate baseball diamond could be constructed by cutting diagonals of equal length from string or rope. To form the square, the diagonals would be positioned to bisect (halve) each other at right angles. The ends of each string would then mark the square's four corners. The same format could be used to create a rectangular play area, except the diagonals would not be perpendicular. According to Sidney Kolpas, although unaware of the Pythagorean theorem, ancient Egyptians used right triangles to reconstruct property boundaries after the annual flooding of the Nile River. To create a 90 degree angle, Egyptians would create a 3-4-5 right triangle by tying 13 equally spaced knots in a rope, placing stakes at knots 4 and 8, then drawing the ends of the rope at knots 1 and 13 to meet.

Polygons are prevalent in nature. Mineral crystals often have faces that are triangular, square, or hexagonal. The cross section of the Starfruit is shaped like a pentagonal star. Katrena Wells describes practical applications of hexagons, such as the often hexagonal shape of snowflakes and the hexagonal markings on many turtles' backs.

Tessellations of polygons are arrangements of polygons on a plane with no gaps or overlaps. These are also seen frequently in nature. Marvin Harrell and Linda Fosnaugh discuss many examples, including the facts that bees use a hexagonal tessellation for their honeycomb, some plant cell structures form hexagonal tessellations, and cooling lava may have formed the tessellating hexagonal columns of basalt rock at the Giant's Causeway in Ireland. Interestingly, a giraffe's skin is covered with a tessellation of various approximate polygons.

When creating sketches of objects or animals, artists often use polygons as the basis of their work by breaking the figure down into polygons and circles, then smoothing and filling in the details of the drawing after the rough polygonal sketch is created. Michael Serra explains how artist M.C. Escher used tessellations of triangles, squares, and hexagons as a framework, then rotated or translated various drawings along the sides of each polygon in the tessellation to create marvelous patterns of reptiles, birds, and fish. Islamic artists covered their buildings with ornate tessellations of polygons. A prime example is the Alhambra Palace in Grenada, Spain.

Investigating polygons as they exist in the world is one method of introducing geometry and instilling a value of geometry to people of all ages. Examining polygons with hands-on learning activities and real-world examples provides students with opportunities to investigate the characteristics and properties among polynomial shapes and helps them grasp an understanding of geometry at a higher level.

Development of Polygons

Planar polygons have been important since ancient times. Up until the seventeenth century, polygons that inscribed and circumscribed a circle were used by Archimedes and many others to estimate values of π. In 1796, at the age of 19, Carl Friedrich Gauss constructed a 17-sided polygon using a compass and straight edge. A year earlier, he had described the area of a polygon, which is often referred to as the "Surveyor's formula," although this concept also is attributed to A. L. F. Meister in 1769. The concept of a tiling or tessellation also requires polygons, and these have a long history of represention in art, weaving, architecture, and mathematics. Johannes Kepler studied the coverings of a plane with regular polygons, and in 1891, crystallographer E. S. "Yevgraf" Fedorov proved that there are 17 different types of symmetries that can be used to tile the plane. Planar polygons also star as main characters in Edwin Abbott's 1884 novel *Flatland* and the subsequent twenty-first-century movies. In the early twenty-first century, young children investigate the mathematical properties of planar polygons in primary school.

Other types of polygons are also interesting and useful. Non-convex polygons like a star polygon, where line segments connecting pairs of points no longer have to remain inside the polygon, were studied systematically by Thomas Bredwardine in the fourteenth century. Generalized polygons in the twentieth century include complex polygons investigated by Geoffrey Shephard and H. S. M "Donald" Coxeter; Moufang polygons, named after Ruth Moufang; and near polygons. In 1797, Norwegian surveyor Caspar Wessel explored planar and spherical polygons in his theoretical investigation of geodesy. M. C. Escher represented hyperbolic polygons in his tessellated artwork. Some twenty-first-century college geometry texts contain spherical and hyperbolic polygons.

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Linda Reichwein Zientek
Beth Cory

See Also: Archimedes; Bees; Pi; Polyhedra; Ruler and Compass Constructions.

Polyhedra

Category: History and Development of Curricular Concepts.
Field of Study: Communication; Connections; Geometry.
Summary: Regular solid shapes play important roles in nature and geometry.

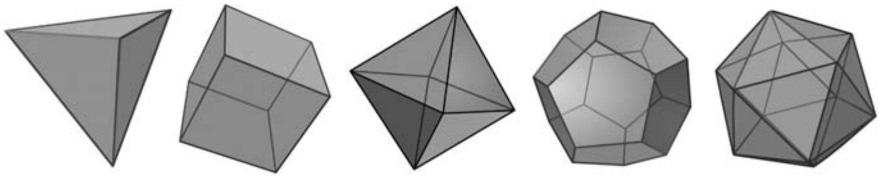
People frequently encounter objects in polyhedral shapes, such as buildings that have cubic or prismatic shapes and geodesic domes or dice that are shaped like polyhedra. This prevalence is partly because of their aesthetic appeal and partly because of their practical properties. Polyhedra also appear in nature; many crystals have the shapes of regular solids, particularly of tetrahedron, cube, and octahedron, and virus capsids can be icosahedral. Furthermore, carbon atoms can form a type of molecule known as "fullerenes," which are in the form of a triangulated truncated icosahedron. A polyhedron is a solid in space with polygonal faces that are joined along their edges. If the faces consist of regular polygons, then it is called a "regular polyhedron." A polyhedron is convex if the line segment joining any two points lies on or inside it. Regular convex polyhedra are particularly important for their aesthetic value, symmetry, and simplicity. There are only five of them: the tetrahedron, cube or hexahedron, octahedron, dodecahedron, and icosahedron. Beginning in primary school, students investigate and classify geometric shapes, including polyhedra. In middle school and high school, students explore area and volume measurements as well as transformations and cross-sections.

History

Some of the earliest known polyhedra are the Egyptian pyramids. The five regular solids appear as decorations on Scottish Neolithic carved stone balls, which date to 2000 b.c.e. There are also examples of cuboctahedra worn by east-African women around the ankle and a variety of polyhedral earrings in medieval Europe. The Greeks are thought to have first studied the mathematical properties of regular solids, particularly the Platonic solids, named for Plato. The last book of Euclid of Alexandria's *Elements* is devoted to the study of the properties of these solids, including detailed descriptions of their construction. The book is based on the work of Theaetetus of Athens. There is some evidence that Hippasus of Metapontum may have been the first to describe the dodecahedron. Hypsicles of Alexandria inscribed regular polyhedra in a sphere in his treatise. The Platonic solids also represented physical aspects: Earth was associated with the cube, air with the octahedron, water with the icosahedron, fire with the tetrahedron, and the dodecahedron with the universe. Plato noted: "So their combinations with themselves and with each other give rise to endless complexities, which anyone who is to give a likely account of reality must survey."

The Kepler–Poinsot polyhedra are named for the 1619 work of Johannes Kepler and the 1809 work of Louis Poinsot. They constructed four regular "stellated" polyhedra. These new solids were obtained by extending the faces. In the twentieth century, Donald Coxeter classified and studied the stellation process and described many stellated polyhedra.

Figure 1. Five Platonic solids.



and rotations. One can also generate new polyhedra from old by truncating the vertices of polyhedra, a process known and studied since antiquity. Some of the truncated polyhedra are also known as the "Archimedean solids," named for Archimedes of Alexandria, whose faces consist of two or more types of regular polygons.

There are 13 Archimedean solids, and there are 53 other semiregular, non-convex polyhedra, which are non-Archimedean. The collection of all Platonic, Kepler–Poinsot, Archimedean, and semiregular, non-convex polyhedra together with prisms form the family of polyhedra called "uniform polyhedra."

Non-Euclidean polyhedra took on a prominent role in some theories of a spherical dodecahedral universe at the beginning of the twenty-first century. There are also non-Euclidean polyhedra with no flat equivalents. For instance, a spherical hosohedron with Schläfli symbol {2, n} is shaped like a segmented orange or beach ball with lune faces. The name "hosohedron" is attributed to Coxeter.

There have been many artistic and physical models of polyhedra in mathematics classrooms. With the advent of perspective, polyhedra were easier to draw and mathematicians and artists designed and collected polyhedral models. Albrecht Dürer introduced polyhedral nets in his 1525 book. Students continue to use nets to build models. In 1966, Magnus Wenninger published a work on polyhedral models for the classroom through the National Council of Teachers of Mathematics. Wenninger noted that the popularity of the book reflected the continued interest in polyhedra. In the twenty-first century, origami polyhedra have also become important in mathematics and computer science classrooms and research.

Properties

One common classroom investigation that relates to polyhedra is the Euler characteristic χ, named for Leonhard Euler. It is an equation that combines the number of vertices (V), edges (E), and faces (F) of a polyhedron as $\gamma = V - E + F$. All convex polyhedra have the same Euler characteristic: 2. René Descartes discovered the polyhedral formula in 1635, and Euler discovered it in 1752. In the nineteenth century, Ludwig Schläfli generalized the formula to polytopes and Henri Poincaré proved the result.

The shape of a polyhedron lends itself to a very convenient symbolic or combinatorial description, called the "Schläfli symbol" of the polyhedron. Let {n, p} represent a regular polygon with n-gon faces, p of them meeting at each vertex. For example {4, 3} would represent a cube because three squares meet at each vertex This symbolic representation is particularly useful if one would like to express various quantities like the dihedral angle, angular deficiency, radii of inscribed and circumscribed spheres, and surface area. For instance, the surface area of a Platonic solid {n, p} can be expressed by

$$S = nF\left(\frac{a^2}{2}\right)\cot\left(\frac{\pi}{n}\right)$$

where F is the number of faces and a is the side length.

Mathematically, polyhedra are very appealing for their fine properties such as duality, symmetry, and versatile constructability. The dual of a polyhedron is constructed by taking the vertices of the dual to be the centers of the faces of the original figure by interchanging faces and vertices. For instance, the dodecahedron and the icosahedron are duals. Many polyhedra are highly symmetrical, and in the nineteenth century, Felix Klein investigated them. The groups of symmetries are algebraic structures consisting of reflections

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See Also: Crystallography; Greek Mathematics; Polygons; Symmetry; Transformations.

Polynomials

Category: History and Development of Curricular Concepts.
Fields of Study: Algebra; Communication; Connections.
Summary: Polynomial functions have long been studied by mathematicians and have interesting and important applications.

Polynomials have a broad array of theoretical and real-world applications and are widely used by mathematicians, scientists, and engineers to mathematically model data and explore many mathematical and scientific concepts. Technologies that transmit electronic signals, ranging from deep space probes communicating with Earth, to home DVD players, commonly use polynomial error-correcting codes, like the Reed–Solomon codes, named for mathematicians Irving Reed and Gustave Solomon. Cryptographic algorithms that help ensure secure data transmission also rely on polynomials to represent and manipulate data. Calculators may use approximations called "Taylor polynomials," named for mathematician Brook Taylor, for functions like square roots. Civil engineers model and estimate properties, such as volume for lakes and other irregular natural features, with polynomials. Orthogonal polynomials provide the foundation for many multivariate statistical procedures. In twenty-first-century classrooms, polynomials are typically part of advanced middle school or high school curriculums, though linear functions and comparisons of linearity versus nonlinearity are common in middle school, and some of the basic concepts of functions are introduced in the elementary grades.

Early in their mathematical studies, students learn that the graph of the squaring function is a parabola, and that the plot of $y = p(x) = x^2$ is shown in Figure 1, which is the first natural function to consider beyond ones that generate straight lines.

There is an entire family of functions like the squaring function, the cubing function, the fourth power function, and more. If indexed, one could call

the squaring function $p_2(x) = x^2$,

the cubing function $p_3(x) = x^3$,

the fourth power function $p_4(x) = x^4$,

and, in general, the nth power function $p_n(x) = x^n$.

The family of power functions also includes the zero power function $p_0(x) = 1$ and the first $p_1(x) = x^1$. These power functions are the building blocks of "polynomial functions," functions that are made from taking sums and constant multiples of power functions. As such, these functions are especially simple because

Figure 1.

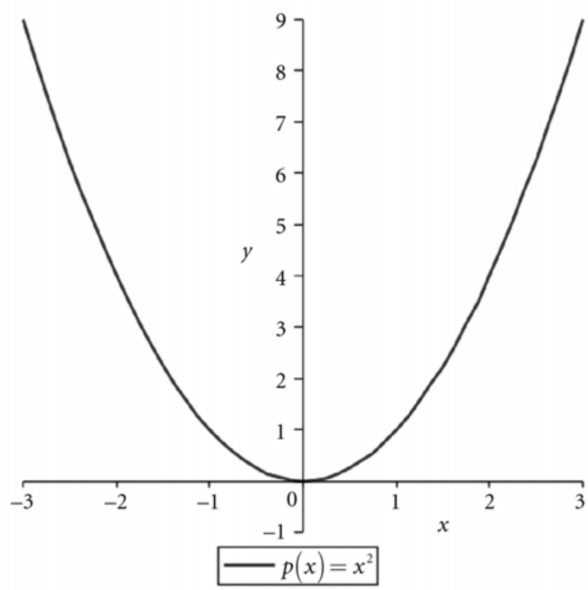
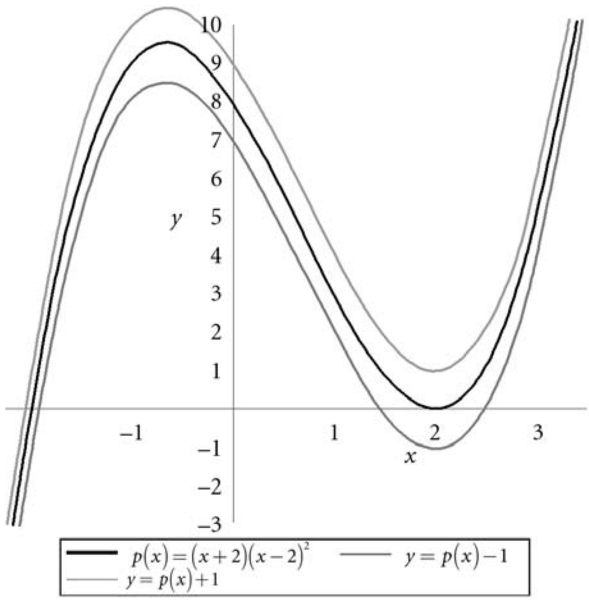


Figure 2.



their formulas only involve addition and multiplication. The first understanding of these power functions is generally credited to Abu Bekr ibn Muhammad ibn al-Husayn Al-Karaji, who lived c. 1000 C.E. in what is now Iraq. In particular, he made advances in the use of variables and humankind's ability to think of arithmetic operations on "placeholders," instead of simply on individual numbers.

Finding the Zeros

Consider this example, $p(x) = x^3 - 2x^2 - 4x + 8$: this function is obtained by taking the cubing function, subtracting twice the squaring function, subtracting 4 times the first power function, and finally adding 8. Regardless of the power functions chosen and the constants multiply by, a polynomial is built. That is, polynomials are functions that have the form

$$p(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are real numbers. Provided that a_n is not zero, it is stated that p is a degree n polynomial; the degree represents the highest power of x that is present. Much of the modern notational perspective on these functions is due to the work of René Descartes, who in the early 1600s did important work that popularized not only the notation above using subscripts and superscripts but also offered a visual perspective on polynomial functions through their graphs.

Going back to the first polynomial example, $p(x) = x^3 - 2x^2 - 4x + 8$, one can rewrite this sum of multiples of power functions in the formula as a product of even simpler functions. Specifically, it is possible to show that

$$p(x) = x^3 - 2x^2 - 4x + 8 = (x+2)(x-2)(x-2).$$

One can easily observe that $p(-2) = 0$ and $p(2) = 0$. Mathematicians call −2 and 2 the *zeros* or *roots* of $p(x)$; since the $(x-2)$ factor, which leads to the zero 2, appears twice, mathematicians say that "2 is a double root" or "2 is a zero of multiplicity two." The graph of the polynomial in Figure 2 is also enlightening as it shows that the zeros of the function lie where the function crosses or touches the horizontal axis:

If one shifts the graph of the degree 3 polynomial $p(x)$ (in black) slightly up, the new graph (top line in light gray) will have just one real zero, while if one shifts the graph slightly down, the new function (bottom line in medium gray) will have three distinct real zeros. This illustration demonstrates an important fact about degree 3 polynomials: every degree 3 polynomial has 1, 2, or 3 distinct real zeros. Indeed, the Fundamental Theorem of Algebra, which was proved in its earliest form in 1799 by the great mathematician Carl Friedrich Gauss, states that every polynomial of degree n has at most n distinct real zeros.

If one is willing to permit zeros to be complex numbers and count zeros by their multiplicity, a much stronger version of the Fundamental Theorem of Algebra (which was also known to Gauss) can be proved: every polynomial of degree n has exactly n zeros, provided one counts them according to their multiplicity and allows zeros to be complex. The Fundamental Theorem of Algebra asserts only that n roots of a polynomial function of degree n exist; it does not tell what those roots are.

Quadratic, Cubic, and Quartic Formulas

The search for the zeros of polynomial functions attracted many great minds. The quadratic formula, which calculates the zeros of any degree 2 polynomial, was understood in certain forms by Babylonian mathematicians as early as 2000 B.C.E. The quadratic formula asserts that in order for $ax^2 + bx + c = 0$, it must be the case that

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

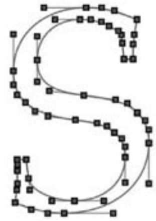
For cubic equations and their roots—finding where a polynomial of degree 3 is zero—it took another 3500 years for mathematicians to fully understand the situation. Following contributions from ancient Greeks, Indians, and Babylonians, as well as Persians in the eleventh and twelfth centuries, a group of Italian mathematicians in the 1500s (Scipione del Ferro, Niccolo Tartaglia, and Gerolamo Cardano) proved that there is a cubic formula. In other words, based on the coefficients of a degree 3 polynomial, there is a very complicated formula involving cube roots that calculates the locations of the polynomial's zeros.

Mathematicians were able to take these discoveries a step further. Near the mid-1500s, Ludovico Ferrari found a way to solve quartic equations. This quartic formula is incredibly complicated and represents a major feat in the understanding of polynomial functions. Interestingly, these general formulas cease to exist beyond polynomials of degree 4. In 1824, Neils Abel and Paolo Ruffini published a theorem, based on the work of Evariste Galois, proving that there was no general formula for the roots of a degree 5 polynomial or higher. This latter work on polynomials ended up founding an entire new branch of mathematics called *modern algebra*. Sometimes in mathematics, the quest to solve one problem leads to a whole host of other interesting problems or even a new collection of coherent ideas.

Applications

Polynomial functions demonstrate all sorts of interesting patterns and properties and have long been studied because they are interesting in their own right. But even more than this, polynomials play important roles in other areas of mathematics and in applications. For example, polynomial functions spawned the subject of modern algebra, and key ideas in modern algebra are used in the field of public key cryptography—the science of keeping important information private in such essential settings as Internet commerce.

A more direct application of polynomial functions comes in the design of fonts that appear on computer screens. So-called Bezier curves, named for mathematician Pierre Bezier, are degree 3 polynomial functions that can be easily spliced together to form elegant shapes. For instance, at right is the letter S in the Palatino font.



Each piece of the S—the portion of the curve between consecutive squares that represent points on the curve—consists of a degree 3 parametric polynomial. There is deep and elegant mathematics behind why Bezier curves work so well and why they are particularly suited to computer graphics. This is just one example of how substantial ideas and applications in mathematics often emerge from simple beginnings.

Polynomial functions are the simplest of all functions, can be used to approximate more complicated functions that are not polynomials, and often emerge in important applications. They are indeed some of the key building blocks of mathematics.

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Matt Boelkins

See Also: Coding and Encryption; Exponentials and Logarithms; Functions.

Popular Music

Category: Arts, Music, and Entertainment.
Fields of Study: Algebra; Measurement; Number and Operations; Representations.
Summary: Popular music can be analyzed and enhanced by mathematical techniques and to some degree the popularity of music can be predicted mathematically.

The interaction between mathematics and popular music goes far beyond the popularity of numbers in song titles, like Tennessee Ernie Ford's "16 Tons" or 2gether's "U + Me = Us (Calculus)." Mathematics is fundamental to musical theory and composition. The twentieth-century subgenres math rock and mathcore are perhaps the most explicitly mathematical compositions, but there are also songs about mathematics concepts. These are usually intended to be humorous or educational, such as "That's Mathematics" by mathematician and musician Thomas Lehrer. Mathematics is also increasingly important to recording and analyzing popular music, including its potential effects on learning. Experimental electronic artist Jamal Moss, founder of record label Mathematics, notes: "Mathematics is the body of sound knowledge centered on such concepts as quantity, music, structure, space, and change—and also the academic discipline that studies them."

Popular Artists

Mathematics in popular music reflects society's often polarized opinions on mathematics. For example, Jimmy Buffet's song "Math Suks" expressed the singer's feelings about the difficulty of mathematical concepts like fractions, algebra, and geometry. Other singers and groups embrace mathematics, like the Texas indie rock band named "I Love Math." Mathematics is often found in album cover art. British band Coldplay's 2005 *X & Y* album featured a cover with colored blocks that spell out "X and Y" in the binary code developed in 1870 by Emile Baudot for use with telegraph systems. Coldplay's lead guitarist Jonny Buckland studied astronomy and mathematics at University College London. Some artists have been criticized for incorrectly using mathematics. Pink Floyd's very popular 1973 album *Dark Side of the Moon* features cover art showing a prism and spectrum. It is correct in depicting some facts, like violet light refracting the most and red the least, but some other aspects are not accurate, such as the relative dispersion of the different colors. Mariah Carey's 2009 album $E=MC^2$, borrowed from Albert Einstein's well-known theory of relativity.

Mathematical Subgenres of Popular Music

Avant-garde composer Iannis Xenakis and post-rock subgenres math rock and mathcore are prominent examples of popular music that relies heavily on mathematics. Xenakis was one of the most significant avant-garde composers of the twentieth century and a grandfather of modern electronic music. His work incorporated mathematical models, such as probability theory, stochastic processes, group theory, set theory, game theory, and Markov chains. He developed algorithms to produce computer-generated music using probability theory and stochastic functions in the 1960s. In his 1966 cello solo "Nomos Alpha," he divided the 24 sections of the piece into two layers. The first layer, consisting of every section not divisible by four, is determined by the 24 orientation-preserving elements of the octahedral group, while the second layer is a more traditional structure. The work has been compared to a musical kaleidoscope, and its structure likened to a fractal.

In the 1990s, post-rock like Slint's *Spiderland* became a dominant genre in experimental rock. Critic Simon Reynolds coined the term "math rock" to describe music that "uses rock instrumentation for non-rock purposes, using guitars as facilitators of timbre and textures rather than riffs and power chords." Math rock bands began to explore the use of dramatically alternating dynamic shifts and unusual time signatures and dissonance, and songs tend to avoid the verse-chorus-verse structure of pop songs. Mathcore developed largely independently of math rock, growing out of hardcore punk and extreme metal, with a huge debt to hardcore pioneers Black Flag.

Mathematics Songs

As of 2010, the Web site M A S S I V E: Math And Science Song Information, Viewable Everywhere is part of the National Science Digital Library and contains over 2,800 mathematical and scientific songs. Popular YouTube songs include mathematical raps and parodies, like "I Will Derive." Hard 'n Phirm's song "Π" rose in popularity because of the 2005 music video by award winning director Keith Schofield. Some songs help students learn mathematics concepts, like multiplication. Other songs showcase the mathematicians who love to sing. The Klein Four Group is a Northwestern University a cappella group who sing about undergraduate and graduate level mathematics. They are most known for their song "Finite Simple Group (of Order Two)."

Self-proclaimed "mathemusician" Lawrence Lesser writes educational songs in order to increase mathematics awareness. Educators often incorporate mathematics songs into their classrooms to enhance student learning of specific concepts and many students use music of various kinds to help them focus while they study mathematical concepts, but these effects are not yet definitively supported or refuted. One study that investigated using jingles to teach statistics concepts found that students who sung several jingles versus reading aloud definitions for the same concepts performed better as a group on a follow-up test. On the other hand, a study that compared classical, popular, and no music to enhance learning found that the students in the three groups performed no differently on a mathematics placement test. This matched findings regarding the effect of music on other academic areas.

Audio Processing

While music production techniques have always allowed a certain amount of alteration and error correction by adjusting the relative levels and balance of the recorded elements, twenty-first century software capabilities have progressed to the point where lower-quality vocals can be processed to professional-sounding quality.

The software package most associated with this is Auto-Tune, released in 1997, and developed by Exxon engineer Harold "Dr. Andy" Hildebrand, who applied seismic data interpretation methods to the analysis and modification of musical pitch. Auto-Tune is an enhancement of existing phase vocoder technology, which uses short-time Fourier transforms, named after mathematician Jean Fourier, to convert time domain representations of sound into time-frequency representations that can be modified before being converted back. Extreme changes can leave tell-tale artifacts in recordings, in the form of a warble like a degenerating audiocassette tape. Audio processing has become standard in many pop albums and on television shows, such as *Glee*. Some well-established singers regularly use Auto-Tune for both albums and in live performances. Other musicians have refused to do so out of fear that it will change the sound enough to make them unrecognizable.

Predicting Popular Song Success

In 2010, Platinum Blue and Music Intelligence Solutions specialize in mathematically predicting hit songs, while services like iTunes and Music IP create suggested playlists or make recommendations. Platinum Blue CEO Mike McCready explained that he and others discovered mathematical patterns in hit songs while trying to build an automated recommendation platform. The algorithm his company uses is based on roughly 30 song traits that are quantified mathematically, such as melody, harmony, beat, tempo, and rhythm. These traits are analyzed for patterns, resulting in groups of songs that are ranked according to probability of success. Hit songs tend to have identifiable similarities, but falling into a particular category is not a guarantee of success. For example, lyrics are an influential song component that are not reliably quantifiable, and aggressive marketing can have an effect not captured by the algorithm. McCready noted: "We figured out that having these optimal mathematical patterns seemed to be a necessary, but not sufficient, condition for having a hit song."

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Bill Kte'pi

See Also: Composing; Geometry of Music; Harmonics; Pythagorean and Fibonacci Tuning; Scales.

Predator–Prey Models

Category: Weather, Nature, and Environment.
Fields of Study: Algebra; Data Analysis and Probability; Number and Operations.
Summary: The interaction between the population sizes of a predator species and a prey species can be modeled using systems of equations.

Predator–prey models are systems of mathematical equations that are used to predict the populations of interacting species, one of which—the prey—is the primary food source for the other—the predator. One famous example that has been extensively studied is the relationship between the wolves and moose on Isle Royale in Lake Superior.

The Isle Royale populations are well suited for modeling the predator–prey relationship because there is little food for the wolves other than the moose and there are no other predators for the moose. In addition, the geographic isolation limits other factors that would complicate the mathematics in the equations, such as hunting or migration. This predator–prey interaction has been carefully studied since the 1950s and continues to be investigated into the twenty-first century.

Modeling Predator–Prey Populations

Most predator–prey models are composed of two equations, the first representing the change in the prey population, and the second the change in the predator population. Each equation has the following form: birth function minus death function.

If $X(t)$ represents the quantity of prey at time *t*, and $Y(t)$ represents the quantity of predators at time *t*, then the instantaneous rate of change in prey is

$$\frac{dX}{dt} = f_1 - f_2$$

and the instantaneous rate of change in predators is

$$\frac{dY}{dt} = f_3 - f_4$$

where f_1 is the mathematical term that describes the births in the prey population, f_2 describes the deaths in the prey population, f_3 describes the births in the predator population, and f_4 describes the deaths in the predator population.

There have been many predator–prey models proposed since the beginning of the twentieth century. The most famous and the earliest known is the Lotka–Volterra system, named for the two scientists who developed the same mathematical model independently, American Alfred Lotka (1880–1949) publishing the equations in 1925 and Italian Vito Volterra (1860–1940) publishing them in 1926. Lotka had degrees in physics and chemistry, and he believed that one could apply physical principles to biological systems. His work on predator–prey interactions is just part of extensive work he published in 1925 in the text titled *Elements of Physical Biology*. Lotka used a chemical reaction analogy to justify the terms in the model.

In the absence of predators, the prey should increase at a rate proportional to the current quantity of prey, *X*. In other words, more moose around to mate without being hunted means more calves would be born. Likewise, in the absence of prey, the predators should die off at a rate proportional to the current predator population, *Y*. In other words, with many wolves and no moose for food, more wolves would starve.

Lotka used a chemical reaction analogy to explain prey deaths and predator births: when a reaction occurs by mixing chemicals, the rate of the reaction is proportional to the product of the quantities of the reactants. Lotka argued that prey should decrease and predators should increase at rates proportional to the product of the quantity of prey and predators, *XY*. In other words, the moose deaths should be closely related to the rate of interaction of wolves and moose, and the wolf births should be as well because wolves need the moose for food to be healthy and have pups. The equations can be written as

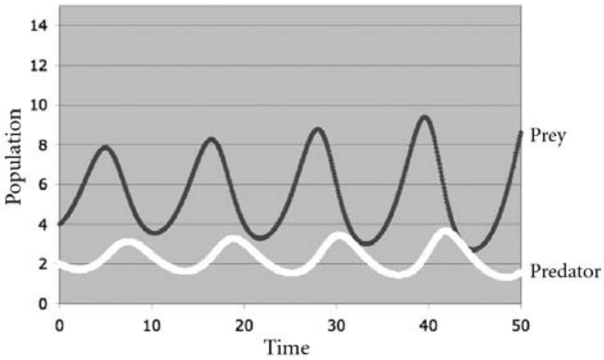
$$\frac{dX}{dt} = aX - bXY \text{ and } \frac{dY}{dt} = cXY - dY$$

for non-negative proportionality constants *a*, *b*, *c*, and *d*.

Volterra arrived at the same model using different reasoning. Volterra was a physicist whose daughter and son-in-law were biologists. While looking for a mathematical explanation for a problem his son-in-law was working on, Volterra became very interested in interactions of species and spent the rest of his professional life looking for a mathematical theory of evolution.

The Lotka–Volterra predator–prey model can be solved without a computer and yields a graph that makes sense. The population of the predator oscillates as does that of the prey, with the predator population trailing slightly behind. Too many prey results in more predators, who swamp the prey causing a decrease in prey. As the prey become scarce, the predators also start to die out, and the cycle begins again (see Figure 1).

Figure 1. Predator–prey interaction.



While this result has reasonable qualitative behavior, many scientists have objected to the equations in this form. Some of the concerns about the model have included the following:

- If there are no predators, the prey population would grow arbitrarily large
- A reduction in the number of prey should cause more predator deaths rather than fewer predator births
- For a fixed number of predators, the number of prey eaten is proportional to the number of prey present, implying that predators are always hungry and eat the same proportion of the prey no matter how large the number of prey gets
- The food for the prey plays a role in the births and deaths of the prey, and should be included in the model
- No spatial considerations are incorporated in the model, so factors such as migration or seeking safety in herds are ignored
- These equations do not take into account gestation periods and seasonal changes in birth rates
- The constants a, b, c, and d are difficult to estimate for a given situation without a large amount of data collected from field observations

Much work has been done since the 1930s to modify the equations to address these concerns and to apply the equations to data from specific situations, such as the moose and wolves of Isle Royale. In the twenty-first century, scientists use sophisticated computer models to model predator–prey interactions using increasingly intricate equations to incorporate more realistic assumptions in the mathematics.

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Holly Hirst

See Also: Animals; Fertility; Social Networks.

Predicting Attacks

Category: Government, Politics, and History.
Fields of Study: Algebra; Data Analysis and Probability.

Summary: Predictive mathematical models can be used to attempt to foresee and counter various types of attacks.

An increasing area of interest in mathematics is the use of algorithms and computer models to predict attacks—military attacks, terrorist attacks, and even attacks on Web servers. As with meteorology, a model is a probabilistic statement; the future cannot be predicted with absolute certainty but probable causes, patterns, and outcomes can be quantified and mathematically modeled to extrapolate the likelihood of new events. Humankind has been trying to predict attacks ever since one group first fought another using some combination of observation and subjective judgment. However, formal prediction of attacks using mathematical methods appears to have originated only within the last two centuries and has escalated with advances in technology and data gathering.

Mathematician Lewis Richardson made contributions to many areas within and outside mathematics, such as numerical weather prediction, in the first half of the twentieth century. The Richardson iteration is one method for solving systems of linear equations, while the Richardson effect refers to the apparently infinite limit of coastline lengths as the unit of measure decreases, a precursor to the modern study of fractals. Richardson spent many years analyzing data on wars from the early nineteenth century onward, using mathematical methods such as probability theory and differential equations, often quantifying psychological variables, such as mood. He identified several patterns in war and identified some variables likely to prevent conflict. He is often credited with first introducing the notion of power laws to relate conflict size, frequency, and death toll. At the start of the twenty-first century, models had grown in complexity. In 2009, a University of Maryland team developed a model that uses 150 variables and data accumulated from the activity of 100 insurgent groups in the Middle East in order to model their reactions to Israeli activities. Other models have been developed to attempt to predict violence and attacks in Iraq and continue to be refined. Statistical methods like data mining and power law functions are prevalent in modern predictive modeling.

Data Mining

Data mining is the process of extracting patterns from large to enormous bodies of data. Isaac Asimov's *Foundation* stories, the first of which was published in 1942, depicted a future where "psychohistory" was the study of the future using the body of history as data from which to extrapolate the future. Modern data mining is quite similar to Asimov's predictions and may be accomplished by many mathematical methods. For example, many use artificial neural networks, which are computational models that mimic neuron behavior. Genetic algorithms, credited to scientist John Holland, are search heuristics inspired by the processes of gene recombination and evolution. Decision trees may be used to determine conditional probabilities. In the 1980s, support vector machines (SVMs) were developed to analyze data to find patterns for statistical classification. All of these developments greatly advanced the state and potential of machine learning and facilitated rapid processing of increasingly larger and frequently interlinked databases from sources such as credit card companies, telecommunications businesses, and government intelligence agencies. Within the U.S. government, the Department of Defense began using data mining in the late 1990s in its Able Danger program, which gathered counterterrorism data, including data about the Al Qaeda terrorist group. Some asserted that the program uncovered the names of four of the alleged September 11, 2001, hijackers a year before the attacks. In February 2002, the U.S. Office of Science and Technology Policy convened a panel of government and industry leaders to discuss data mining as a counterterrorism tool. While it is now widely used, some criticize it because the sparsity of some information and the relative infrequency of terrorist attacks make identifying statistically significant patterns, which are critical to finding the anomalies that signal an attack, prone to unacceptable levels of error.

Cyber Security

Mathematicians, computer scientists, and others are continually working on new methods to predict and counter attacks on Web servers, e-mail, and digital records of all kinds. The Internet is filled with malicious activity, from phishing and identity theft to distributed denial of service attacks. Electronic attacks are facilitated by the same computer technology that is used to predict attacks. The traditional guard has been to block a source of malice after the attack, by e-mail as spam or blocking an IP address after harmful activity originates from it. These methods are commonly known as *blacklists* and are now widely compiled and shared. However, they are by definition reactive measures to attacks. Just as e-mail spam filters have become preemptive, marking mail as "spam" automatically based on a number of factors, IP-blocking can also be conducted preemptively.

The method of predictive blacklisting uses shared attack logs as the basis for a predictive system, like the customer recommendation systems employed by Amazon or Netflix. Computer scientists Fabio Soldo, Anh Le, and Athina Markopoulou developed what is known as an "implicit recommendation system"—implicit because ratings are inferred rather than given directly by the subjects of the model. Their multilevel prediction model uses mathematical methods, such as time series analysis and neighborhood models, adjusted specifically for attack forecasting. Inputs to the model include factors such as attacker-victim history and interactions between pairs or groups of attackers and victims. Similar models—using different types of data—can be built to predict terrorist attacks and the behavior of enemy forces, and such models are included in the standard order of battle intelligence reports used by the U.S. Army.

The data needed to predict attacks are not restricted to private databases. Information is widely available from the Internet or the scrolling news banners of 24-hour news networks. Neil Johnson used a variety of sources to investigate insurgent wars, employing some of the same mathematical techniques as Richardson in his analyses and modeling. After gathering and analyzing data for almost 60,000 insurgent attacks occurring in multiple conflicts around the world, he and his collaborators discovered similarities between the frequency and intensity of attacks in all conflicts. Further, they found that the statistical distribution for insurgency attacks was significantly different from the distribution of attacks in traditional war. The model quantifies connection between insurgency, global terrorism, and ecology, and counters the common theory of rigid hierarchies and networks in insurgencies. Johnson notes:

> Despite the many different discussions of various wars, different historical features, tribes, geography and cause, we find that the way humans fight modern (present and probably future) wars is the same, just like traffic patterns in Tokyo, London, and Miami are pretty much the same.

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Bill Kte'pi

See Also: Intelligence and Counterintelligence; Predicting Preferences; Spam Filters; Vietnam War.

Predicting Divorce

Category: Friendship, Romance, and Religion.
Fields of Study: Algebra; Communication: Data Analysis and Probability.
Summary: Statistical data analysis and mathematical models can be used to predict the likelihood of divorce.

There is a common misconception that one out of every two marriages ends in divorce. The 50% number comes from dividing the number of divorces in a given year (about 1.3 million) by the number of marriages in that same year (about 2.6 million). The mistake is failing to realize that, in any given year, the people getting divorced are probably not the same as those getting married, because the average length of a marriage before a divorce is about eight years (the overall length of marriage, on average, is about 24 years). Hence, those getting married in any given year have an eight-year lag in their projections for divorce. This lag means that the numerator and denominator of the above ratio are not comparable. Instead, experts suggest that about two out of every five marriages end in divorce (or about 40%).

Because of the propensity for some to remain married, for some to divorce more than once, and for some to never marry, only about one out of every five people are predicted to experience a divorce in their lifetime. However, these figures mask the distribution of divorce rates by category—40% of all first marriages end in divorce, 60% of second marriages end in divorce, and 73% of all third marriages end in divorce. There are also some differences by age group, with divorce rates highest for those in their early 20s and declines steadily in subsequent age groups.

There are two main ways to predict divorce: empirical (or statistical) methods that take advantage of data gathered on married and divorced couples; and mathematical models that try to make a priori predictions of future divorce using features of existing marriages or theoretical assumptions based on extensive work in the area.

Empirical Methodology

Empirical work suggests that indicators predicting divorce can be separated into two groups: factors present before marriage and factors that occur within the marriage. Some of the more common risk factors brought into a marriage include parental history of divorce (children of divorced parents are more likely to divorce), educational attainment (those with lower levels of education are more likely to divorce), and age (those who marry younger are more likely to get divorced). The risk factors that arise within the marriage include communication styles (couples with poor or destructive communication have a greater chance of divorce), finances (couples with financial problems, including a large disparity in spending habits, disposable income, and wealth goals, are at a greater risk for divorce), infidelity, commitment to the marriage (a lack of commitment or a dissimilarity in the amount of commitment often leads to divorce), and dramatic change in life events.

Mathematical Models

Mathematical models seek to discover features of current relationships that will put a couple at risk for future divorce. Professor John Gottman argues that the way couples communicate can often predict divorce. His research, which is based on analyzing hundreds of videotaped conversations between married couples, claims a 94% accuracy rate. The work also monitors pulse rates and other physiological data that, when combined with the observations, leads to what he calls the "bitterness rating." The rating is based on six signs. The first sign posits that when a conversation starts with accusations, criticisms, or negativity, the discussion is likely to end badly. However, he argues that the opposite is also true. The second sign encompasses four patterns of negative interaction that can be deleterious to a marriage: criticism, contempt, defensiveness, and stonewalling. The third sign is "flooding," in which negativity of one partner overwhelms the positive feelings of the spouse until there is virtually nothing left but discontent. The fourth sign recognizes that physiological changes, such as increases in adrenaline and blood pressure, often lead to feelings of entrapment and serve to poison an otherwise benign conversation. The fifth sign identifies the fact that some marital discord is unchanged by the repeated attempt by one partner to repair the damage done to the relationship. Finally, the sixth sign involves one or both people rewriting the history of their relationship to be largely negative. Once people reach the sixth sign, Gottman argues, divorce is likely.

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See Also: Mathematics, Applied; Measurement in Society; Psychological Testing.

Predicting Preferences

Category: Business, Economics, and Marketing.
Fields of Study: Data Analysis and Probability; Geometry; Measurement.
Summary: Psychology of choice and predictive models of preferences are exciting areas of mathematics blending social science, economics, and commerce.

Mathematically, preference is an ordering of alternative possibilities. It can refer to conscious choices based on ideas and beliefs, positive emotional responses or liking, or biologically mandated behaviors. Preferences are usually determined statistically: for individuals, based on multiple instances of decisions over time; and for groups, based on aggregated data of members. In 2009, the Netlix Prize contest awarded a team called BellKor's Pragmatic Chaos $1 million for their preference-predicting algorithm.

Theoretical and Behavioral Economics

Among all sciences that deal with predicting preferences, such as social psychology and education theory, the most developed mathematical apparatus can be found in economics. As any branch of mathematics, theories of economic preferences start with axiomatic assumptions. These abstract axioms do not always apply to all real situations. Economic theories that take into account psychological factors, such as cognitive limitations and emotions, are developed within an interdisciplinary area called "behavioral economics."

Most abstract theories of preference prediction assume most parts of the so-called total order, which is a group of mathematical axioms and properties from set theory. Let A, B, and C be different choices. Total order assumes that either $A \leq B$ or $B \leq A$. In real life, this assumption is a statistical statement at best: today a person can prefer apples, but might prefer bananas tomorrow. The property of transitivity says that if $A \leq B$ and $B \leq C$, then $A \leq C$. This property works in some situations; for example, if one prefers $20 over $10, and $100 over $20, it is likely the person will prefer $100 over $10. However, in complex situations with multiple choices, such as elections, transitivity fails to describe real human behavior. Experiments show that, given a choice between one pair of candidates at a time, people may prefer Beth over Alice, Carol over Beth, and Alice over Carol. One axiom of total order, called "antisymmetry," that almost never makes sense in preference theories is that if $D \leq E$ and $E \leq D$, then $E = D$. For example, when group data shows that people think diesels are worse or the same than electric cars, and electric cars are worse or the same as diesels, it does not mean that diesel cars and electric cars are the same entity. It means that people prefer them about the same. Economic theories call this situation "indifference" and use a separate symbol for it: $E \sim D$.

Another assumption frequently made in economic preference predictions comes from topology and is called "continuity." It is the assumption that if A is preferred over B, then an option that is very similar (close) to A will also be preferred over at option that is very similar to B. Many complex phenomena, including preferences, are discontinuous. They exhibit various "tipping points," near which minute differences cause radical changes in preferences. These non-continuous phenomena are studied using models from calculus or chaos theory, a branch of differential equations. One frequent example of noncontinuous preference is price near powers of 10: many people choose to buy an object that costs $999 over a similar object that costs $1,001 even though the difference in prices is minuscule compared to the total. Behavioral economics explains this by cognitive limitations: people see 1001 as thousands and 999 as hundreds, which is technically correct but makes less of a difference in this case than intuition leads one to believe.

Psychology of Choices

Statistical analysis of real situations, such as elections, as well as results of experiments and questionnaires, allow scientists to aggregate increasingly sophisticated knowledge of human mechanisms of choice and preference. For example, from the purely mathematical viewpoint, gaining an amount and avoiding loss of the same amount are equivalent. However, most people regret loss more strongly than they regret missed opportunity—a fact extensively used in advertisements of savings and discounts.

Preferences are very strongly influenced by power over the situation. Most people accept much higher risks for given gains if they enter the situation of their free will, compared to risks of mandated behaviors. This phenomenon comes up, for example, when mandatory immunizations are proposed—the fact that people would not have a choice makes very small risks unacceptable.

Paradoxical Preferences

A paradox is a false or contradictory statement that logically follows from a set of true statements. Preference prediction leads to several types of paradoxes.

A very frequent type is the situation when an initial model describes the reality well, but its mathematical corollaries do not. Another type, a true logical paradox, occurs when mathematical corollaries contradict one another.

For example, the expected value is the sum of products of probabilities and payoffs. Suppose a fair coin is flipped in a hypothetical game and the player is paid \$10 if the coin lands on heads and \$20 if it lands on tails. The expected value of winning is \$15 because $0.5(10)+0.5(20)=15$. When the same game is played many times, it is rational to prefer options with higher expected values. Under this assumption, it is better to play the game where the player is paid nothing for heads and \$40 for tails than the first game, because the expected value of winning is higher: $0.5(0)+0.5(40)=20$. However, in real life, risk aversion will make many people choose the first game.

To resolve this and other related paradoxes, many preference models account for risk aversion as a separate variable. A utility function is the measure of relative satisfaction of a range of choices. An assumption that people will only want to maximize utility is not realistic, because it does not account for risk aversion. Because marginal choices usually come with higher risks, the utility function that accounts for risk aversion will look like a hump, being concave.

Bounded rationality principle is commonly used to explain paradoxical preferences by taking into account limited information, time, and cognitive abilities of people. Models based on bounded rationality include human limitations, such as computational capacity, and are based on computer science, statistics, and psychology.

Information Theory and Aesthetic Preferences

Information theory is a mathematical science that studies storing, compressing, and processing of data. In the 1990s, its branch called "algorithmic information theory," which deals with the complexity of algorithms, was applied to explain some aspects of the human sense of beauty and of aesthetic preferences. According to this theory, objects that have shorter algorithmic descriptions in terms of observer's knowledge will seem more beautiful, compared to objects with longer algorithmic descriptions. For example, it is easier to remember an object with mirror symmetry because only half of the information is original—symmetry provides information compressibility. Therefore, symmetric objects, as well as objects with patterns or fractal self-similarity, are seen as more beautiful.

Algorithmic information theory also models preferences by interest, which are separate from preferences based on beauty. Within these models, interest can be compared to the first derivative of beauty, showing the observer's perception of change in understanding. People prefer an experience on the basis of interest when it involves better compressibility or predictability of information than before. For example, noticing a new pattern (and therefore better organizing an image) is preferred because it is interesting.

Preferences, Desires, and Motivation

Many preferences and choices are based on needs, wants, and desires, which are explained in theories of motivation. Researching motivation is challenging because of individual differences among people, as well as language ambiguity. There are disagreements among researchers even over relatively straightforward terminology, such as intrinsic and extrinsic motivation. Many motivation theories include taxonomies of needs and desires. For example, in Maslow's hierarchy, named after Abraham Maslow, unsatisfied physiological needs, such as hunger or thirst, have higher priority than unsatisfied self-esteem needs, such as recognition. Some theories identify long lists of motivators, such as curiosity, tranquility, order, and independence. Other theories only define a few broad classes of needs.

Each category of need can be considered a variable. Graphs of values of these variables versus levels of motivation often demonstrate the characteristic "mirrored C" shape called a "backward bending curve." For example, as activities provide more order, they first become more motivating (and preferred), but beyond a certain point, more order becomes less motivating. This curve is famously described in the baseball manager Lawrence "Yogi" Berra's joke about a restaurant: "Nobody goes there anymore. It's too crowded." People usually prefer restaurants that are not too empty or too full.

Preferences and Demographics

A number of statistical studies find significant differences in preferences of different demographics within populations, such as males and females, socioeconomic classes, ages, and political affiliations. Because statistical packages make many types of mathematical and statistical analyses of databases very easy, there are many results that demonstrate significant differences in preferences among different demographics. However, determining meanings of these differences is a significantly more difficult research problem. Demographic differences in preferences can also vary from culture to culture. In some cultures, for example, more females than males prefer bright colors in clothes, and in other cultures, it is reversed.

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See Also: Data Mining; Expected Values; Mathematical Modeling, Predicting Attacks; Predicting Divorce; Raghavan, Prabhakar.

Pregnancy

Category: Medicine and Health.
Fields of Study: Algebra; Data Analysis and Probability.
Summary: Various mathematical models help describe issues related to conception, diseases associated with pregnancy, and population dynamics.

Much of the conclusions drawn in medicine, in particular in obstetrics and gynecology, are often based on heuristics, limited observations, and sometimes even biased data. Mathematicians and statisticians have recently attempted to develop general theoretical models that can be adapted to specific situations in order to facilitate the understanding of various aspects of human pregnancy. Specifically, more recent studies have been conducted regarding conception time, disease prediction related to pregnancy, and the effect of pregnancy on population growth.

Modeling the Most Efficient Time to Conceive

One of the most fundamental and important research topics in the study of human pregnancy is the so-called time-to-pregnancy (TTP). TTP can be defined scientifically as the number of menstrual cycles it takes a couple engaging in regular sexual intercourse with no contraception usage to conceive a child. Fittingly, statisticians attempt to generate as much data as possible from various couples regarding their personal TTP experiences. The data are collected in a way that is as unbiased as possible—it is intended to accurately represent couples in the general population attempting to conceive a child. From the data, both qualitative and quantitative statistical methods are implemented in order to ascertain the most efficient method to achieve conception.

For example, some social trends increase the age at which a woman attempts to become pregnant. When this situation arises, women are often concerned about achieving conception before the onset of infertility, which proceeds menopause. In fact, couples that are unsuccessful in conceiving within one year are clinically classified as *infertile*. When this condition occurs, medical doctors often recommend that the couple engage in assisted reproductive therapy (ART). However, ART can be very expensive and often increases the risk of adverse outcomes for the offspring, including various birth defects. Therefore, statistical models have been developed that pose an alternative to ART. These models are developed using Bayesian decision theory, named for Thomas Bayes, and search for optimal approaches for a couple to time intercourse in order to achieve conception naturally, without the potentially disadvantageous ART. These models quantitatively incorporate various biological aspects, including menstrual cycles and basal body temperature, as well as the monitoring of electrolytes—among other phenomena—in order to be as efficient as possible.

Predicting Diseases Associated With Pregnancy

Medical evidence supports the notion that women often repeat reproductive outcomes. In particular, women with a history of bearing children with adverse outcomes often have up to a two-fold increase in sub-

Mathematical epidemiology is used to predict conditions like pre-eclampsia during pregnancy.

sequent risk. Therefore, researchers in the mathematical and statistical sciences realized the necessity for statistical analyses that address this issue. In fact, statistical research has been conducted in order to promote a consistent strategy that assesses the risks each woman may face in a subsequent pregnancy. The goal is for these types of models to become increasingly more accurate, as they incorporate statistical data regarding the recent reproductive history of the woman, among other biological factors, which were not fully taken into account in previous studies.

Mathematical epidemiology (the study of the incidence, distribution, and control of diseases in a population) attempts to better comprehend, diagnose, and predict various diseases incorporated with pregnancy, and this field is ever-expanding. By designing and implementing various statistical approaches and mathematical models to better predict realistic outcomes, mathematicians and statisticians have studied congenital defects and growth restrictions, as well as preterm delivery, pre-eclampsia, and eclampsia.

For example, pre-eclampsia is a pregnancy condition in which high blood pressure and high levels of protein in urine develop toward the end of the second trimester or in the third trimester of pregnancy. The symptoms of this condition may include excessive weight gain, swelling, headaches, and vision loss. In some cases this condition can be fatal to the expectant mother or the child. The exact causes of pre-eclampsia are unknown at the beginning of the twenty-first century, and the only cure for the disease is the delivery of the child. Therefore, it is apparent that determining which women are prone to develop pre-eclampsia is an exceedingly important area of research.

Empirical evidence indicates that a woman's heart rate is a deterministic factor in the prediction of pre-eclampsia. In recent times, statisticians have therefore developed a novel and non-invasive approach to detect abnormalities in pre-eclamptic women that distinguishes from women with non-pre-eclamptic pregnancies. This approach is accomplished by comparing the dynamical complexity of the heart rates of women that are pre-eclamptic with those that are non-pre-eclamptic. The analysis revealed that the heart rate of pre-eclamptic women demonstrated a more regular dynamic behavior than those women that were not pre-eclamptic, which substantiates the empirical notion that diseased states may be associated with regular heart rate patterns.

Population Dynamics

Mathematicians have long developed models to analyze population dynamics. One contemporary model also incorporates how pregnant women directly influence such dynamics. This model consists of an equation that describes the evolution of the entire population and an equation that analyzes the evolution of pregnant women. These equations are coupled—they are studied simultaneously. Moreover, this particular system of equations can be analyzed as a linear model (not sensitive to initial data), with or without diffusion (permitting members of the population to travel large distances), or as a nonlinear model (sensitive to initial data) without diffusion. The asymptotic behavior of the solutions to this system (the long-term behavior of the population) was also addressed.

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See Also: Disease Survival Rates; Drug Dosing; Fertility; Genetics; Mathematical Modeling; Ultrasound; Viruses.

Prehistory

Category: Government, Politics and History.
Fields of Study: Measurement; Number and Operations.
Summary: Historians believe that even the earliest people used mathematics.

Many books on the history of mathematics begin with the ancient Egyptians and Babylonians, but those civilizations did not begin until about 5000 years ago. Although historians do not know many details, human life had been progressing for several millennia prior to that time. Even archeology offers little detail on the earliest mathematics, so most knowledge comes from speculation. However, from what is known about human beings in general, and especially about prehistoric life, even the earliest people must have known and used some mathematics.

The use of "mathematics" probably even precedes the development of modern human beings. Studies of animal behavior have shown that animals, and especially birds, seem to possess limited number sense, recognizing the difference between groups of two and three and even larger sets. Bees can recognize and even communicate information about the location of orchards and fields for pollination, displaying a sense of space that could be called "geometry." Even more spectacular are the long migratory trips of herd animals, flocks of birds, and groups of butterflies, often traveling thousands of miles to return to the same fields every year. These examples certainly do not represent a sophisticated concept of mathematics and are instinctual, but they show a mathematical organization in the brain.

Language, Counting, and Quantities

The earliest humans (wherever the line is drawn between pre-human and human) continued the mathematical thinking shown in animals. As their brains developed, their mathematics also grew stronger and more sophisticated. This progression continued as early grunts become proto-languages, for a key part of mathematics is not only having the concepts in one's head, but also representing and communicating the concepts to others. Hence, language was a key ingredient in prehistoric mathematics (as it remains today).

A concept of counting must have come early, as people began to distinguish quantity. Even if they did not have linguistic terms for numbers beyond three or four, they would at least be able to make rough comparisons of large quantities and much larger quantities—consider that even modern humans often need notations, pictures, or concrete examples to handle specific large quantities, but certainly can tell the difference between a dozen and a hundred and a million. Many aspects of life require at least limited counting—to make sure all one's goats (or children) are present, to share items fairly in a group or to calculate the size of a load to be carried, and many other applications.

It is only a small jump of abstraction to begin to record quantities with tally marks. It is likely that people first collected stones or other small objects to represent quantities and later began to "write" them as tallies. Tally marks have been found in many parts of the world scratched on cave walls or carved onto wooden sticks and were also likely written in sand or clay, which shifted to destroy the writing. Probably the most famous prehistoric mathematical object is the Ishango bone, found in south-central Africa, and thought to be at least 15,000 years old. The bone has several sets of tallies scratched onto it—some have pointed out that they are mostly prime numbers, but that is probably a coincidence. Using tallies quickly leads to a problem: a long line of marks is hard to deal with, even if one had some limited counting words. Probably, many people around the world recognized that some structure helped handle large quantities of tally marks—especially collecting them into groups of the same size. Not only does this make counting more efficient but it also leads to the concept of multiplication. In nearly all modern languages—most derived from ancient or even prehistoric languages—the higher counting words use a system of groups and groups of groups, now called "place-value," but they reach back to the prehistoric convenience of putting tally marks together.

Measurement and Geometry

Closely tied to counting was the use of comparative relationships—especially large and small, tall (or long) and short, and even old and young. These may have come when exact counts were difficult, but the comparisons were obvious and usually visual. A tall stack of blocks would easily be seen to have more items than a short stack; a long line of tally marks (grouped or ungrouped) was a greater quantity than a short line. As actual counting developed and numbers were applied to comparisons, the beginnings of measurement occurred—measurement is really just comparisons of quantities where one side of the comparison is a defined unit. To make comparisons easier, certain items of specific size or quantity became units, and as people reached farther to wider audiences, units became at least roughly standardized. Often, body parts were used both for counting tabulations and as "standard" units. For example, the distance from the elbow to the fingertips was approximately the same for most adults, so in the Middle East, this length became the "cubit."

Geometry also has deep roots in the human story. Circles must have been recognized in the shape of the sun and full moon and the apparent edge of the horizon. Efficiency caused people to arrange objects to fit together well in patterns—often circular but sometimes rectangular. The first tools used sharp angles, heavy weights, and tall, thin cylinders. The beginnings of farms led to more organized geometrical arrangements in the shapes of fields and structures. Often, the "invention" of the wheel is considered one of the big milestones of the start of civilization, and this represents a practical understanding of the geometry of circles. As objects became more sophisticated—woven mats, farming tools, larger structures, and even bridges—many more geometrical relationships and properties were discovered. These might be considered the beginnings of engineering—using mathematical properties in practical applications.

Pure Mathematics

Archeologists have also noted some prehistoric mathematics that may have been closer to pure mathematics. Cave paintings, carved sculptures, and textile patterns show contemporary mathematical objects such as circles, triangles, parallel lines, quadrangles, symmetric patterns, and the crosshatch. However, no one has yet deciphered what the geometric signs meant to prehistoric peoples. Some symbols appeared repeatedly in various parts of the world. They may have served practical or religious values, but they also were art—perhaps art for its own sake, for beauty. Certain numbers may have had mystical meanings that were seemingly less useful for day-to-day activity but important for esthetics and spirituality.

The overlap between this pure mathematics and the practical needs of early farmers was the use of mathematics in astronomy and calendars. Could the gods show the times for planting and harvesting? Could humans discern the plans of these gods and use them in practice? Most of the spectacular prehistoric structures, from Stonehenge in England to the huge geometrical patterns of Nazca in Peru, have been linked to measures of the sun's movement and the seasons. Mathematics led prehistoric peoples in solving their daily problems and to thinking of the universe and infinity. Mathematics still serves modern humans in the same ways.

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See Also: Animals; Basketry; Bees; Calendars; Number and Operations in Society

Probability

Category: History and Development of Curricular Concepts.
Fields of Study: Communication; Connections; Data Analysis and Probability.
Summary: Humans have implicitly understood concepts of probability and randomness since antiquity, but these concepts have been more formally studied since the seventeenth century.

Throughout history, humans have used many methods to try to predict the future. Some believed that the future was already laid out for them by a divine power or fate, while others seem to have believed that the future was uncertain. There are still debates on the extent to which people were able to speculate on the future prior to the development of statistics in the seventeenth and eighteenth centuries. Some assert that such speculations were impossible, yet other historical evidence suggests that at least some people must have been able to perceive the world in terms of risks or chances, even if it was not in quite the same way as later mathematicians and statisticians. The Greek philosopher Aristotle proposed that events could be divided into three groups: deterministic or certain events, chance or probable events, and unknowable events. The idea of "randomness" is often used to indicate completely unknowable events that cannot be predicted. In mathematics, the long-term outcomes of random systems are, in fact, "knowable" or describable using various rules of probability. Probability distributions, expressed as tables, graphs, or functions, show the relationship between all possible outcomes of some experiment or process, like rolling a die, and the chance that those outcomes will happen. For example, lotteries state the chances of winning various prizes, and people seeking medical treatment might be told the odds of success. Random processes and probability can run counter to human intuition and the way in which human brains perceive and organize information, which is perhaps another reason that quantifying ideas of probability is still an ongoing endeavor. Students are often introduced to probability concepts in the earliest elementary grades, such as basic binary classifications of outcomes as "likely" or "unlikely" and the notion of probabilities as experimental frequencies. More formal axioms of probability may be introduced in the middle grades. Probability theory and probability-based mathematical statistics are typically studied in college, though they may be included in advanced high school classes. Some elements of probability theory and applications are also taught in other academic disciplines, like business, genetics, and quantum mechanics.

Early History

Archaeological evidence, such as astragalus bones found at ancient sites, suggests that games of chance have been around for several millennia or longer. Egyptian tomb paintings show astragali being used for games like Hounds and Jackals, much like the way twenty-first-century game players use dice. The ideas of randomness that underlie probability were often closely tied to philosophy and religion. Many ancient cultures embraced the notion of a deterministic fate. The Greek pantheon was among those that included deities associated with determinism, literally known as the Fates. The popular goddess Fortuna in the Roman pantheon suggests a recognition of the role of chance in the world. Jainism is an Indian religion with ancient roots, whose organized form appears to have originated sometime between about the ninth and sixth centuries B.C.E. The Jainist logic system known as *syadvada* includes concepts related to probability; its sanskrit root word *syat* translates variously as "may be" or "is possible." Probability is also a component of the body of Talmudic scholarship; for example, the notion of casting lots, used in some temple functions. Babylonians had a type of insurance to protect against the risk of loss for sea voyages, called "bottomry," as did the Romans and Venetians.

Origins of Study in the Seventeenth Century

Given the near omnipresence of probability in the ancient world, it seems reasonable to think that there were some efforts to estimate or calculate probabilities, at least on a case-specific basis; for example, those who issued maritime insurance would have assigned some type of monetary values for cost and payoff. There is relatively little evidence of broad mathematical research on probability before about the fifteenth century, though some analyses for specific cases survive. For example, a Latin poem by an unknown author called "De Ventula" describes all the ways that three dice can fall. Mathematician and friar Luca Paccioli wrote *Summa de arithmetica, geometria, proportioni e proportionalita* in 1494, which contains some discussion of probability. A few other works address dice rolls and related ideas. Historians tend to agree that the systematic mathematical study of probability as it is now known originated in the seventeenth century. At the time, considerable tensions still existed between the philosophies of religion, science, determinism, and randomness. Determinists asserted that the universe was the perfect work of a divine creator, ruled by mathematical functions waiting to be discovered, and that any apparent randomness was because of faults in human perception. Many emerging scientific theories, like the heliocentric model of the universe advocated by mathematician and astronomer Nicolaus Copernicus, challenged this view by explicitly exploring and quantifying variation and deviations in observations. Astronomy and other sciences, along with the rise of combinatorial algebra and calculus, would ultimately prove to be very influential in the development of probability theory. Changes in business practices also challenged notions of risk, requiring new methods by which likelihood and payoffs could be determined. Harkening back to ancient human activities, however, the most popular story for the origin of probability theory concerns gambling questions posed to mathematician Blaise Pascal by Antoine Gombaud, Chevalier de Méré.

The cube design of dice allows for each of their sides to have an equal probability of being rolled.

In 1654, the Chevalier de Méré presented two problems. One concerned a game where a pair of six-sided dice was thrown 24 times, betting that at least one pair of sixes would occur. Méré's attempts at calculation contradicted the conventional wisdom of the time and purportedly led him to lose as great deal of money. The second problem, now called the Problem of Points or Problem of Stakes, concerned fair division for a pot of money for a prematurely terminated game between equally skilled players where the winner of a completed game would normally take the whole pot. Spurred by de Méré's queries, Pascal and Pierre Fermat exchanged a series of letters in which they formulated the fundamental principles of general probability theory.

At the time of its development, Pascal and Fermat's burgeoning theory was commonly referred to as "the doctrine of chances." Inspired by their work, mathematician and astronomer Christian Huygens published *De Ratiociniis in Ludo Aleae* in 1657, which discussed probability issues for gambling problems. Jakob (also known as James) Bernoulli explored probability theory beyond gambling into areas like demography, insurance, and meteorology and he composed an extensive commentary on Huygen's book. One of his most significant contributions was the Law of Large Numbers for the binomial distribution, which stated that observed relative frequencies of events become more stable, approaching the true value, as the number of observations increases. Prior definitions based on gambling games tended to assume that all outcomes were equally likely, which was generally true for games with inherent symmetry like throwing dice. This extension allowed for empirical inference of unequal chances for many real-world applications. Bernoulli also wrote *Ars Conjectandi*. Influenced by this work, mathematician Abraham de Moivre derived approximations to the binomial probability distribution, including what many consider to be the first occurrence of the normal probability distribution, and his *The Doctrine of*

Chances was the primary probability textbook for many years.

Objective and Subjective Approaches

Historically and philosophically, many people have asserted that to be objective, science must be based on empirical observations rather than subjective opinion. Estimating probabilities through direct observations is usually called the "frequentist approach." The method of inverse or inductive probability, which allows for subjective input into the estimation of probabilities, is traced back to the posthumously published work of eighteenth-century minister and mathematician Thomas Bayes. Conditional probabilities had already been explored by de Moivre, providing the basis for what is known as "Bayes theorem" (or "Bayes rule"). In Bayes's inductive framework, there is some probability that a binary event occurs. A frequentist would make no assumptions about the probability and carry out experiments to attempt to determine the true probability value. Using Bayes's approach, some probability value can be arbitrarily chosen, and then experiments conducted to ascertain the likelihood that the value is in fact the correct one. In later interpretations and applications of the method, the initial value might be chosen according to experience or subjective criteria. His work also produced the Beta probability distribution. Bayes's writings contained no data or examples, though they were extended upon and presented by minister Richard Price. At the time, they were relatively less influential than frequentist works, though Bayesian methods have generated much discussion and saw a great resurgence in the latter twentieth century.

Applications

Like Bernoulli, Pierre de Laplace extended probability to many scientific and practical problems, and his probability work led to research in other mathematical areas such as difference equations, generating functions, characteristic functions, asymptotic expansions of integrals, and what are called "Laplace transforms." Some call his 1812 book, *Théorie Analytique des Probabilités*, the single most influential work in the history of probability. The Central Limit Theorem, named for George Pólya's 1920 work and sometimes called the DeMoivre–Laplace theorem, was critical to the development of statistical methods and partly validated the common practice at the time (still used in the twenty-first century) of calculating averages or arithmetic means of observations to estimate location parameters. Error estimates were usually assumed to follow some symmetric probability distribution, such as rectangular, quadratic, or double exponential. While they had many useful properties, they were mathematically problematic when it came to deriving the sampling distributions of means for parameter estimation. Laplace's work, which he proved for both direct and inverse paradigms, rectified the problem for large-sample cases and formed the foundation for large sample theory.

Normal Distribution

The normal distribution is among the most central concepts in probability theory and statistics. Many other probability distributions may be approximated by the normal because they converge to the normal as the number of trials or sample sizes approach infinity. Some of these include the binomial and Poisson distributions, the latter named for mathematician Simeon Poisson. The Central Limit Theorem depends on this principle. Mathematician Karl Friedrich Gauss is often credited with "inventing" the normal (or Gaussian) distribution, though others had researched it and Gauss's own notes refer to "the elegant theorem first discovered by Laplace." He can fairly be credited with the derivation of the parameterization of the distribution, which relied in part on inverse probability. Mathematician Robert Adrain, who was apparently unaware of Gauss's work, discussed the validity of the normal distribution for describing measurement errors in 1808. His work was inspired by a real-world surveying problem. However, Gauss tends to be credited over Adrain, perhaps because of his many publications and the overall breadth of his mathematical contributions.

The fact that Laplace and Gauss worked on both direct and inverse probability was unusual from some perspectives, given the philosophical divide between frequentist and Bayesian practitioners even at the start of the twenty-first century. Later, both would gravitate toward frequentist approaches for minimum variance estimation, which is seen by some as a criticism of inverse probability. Other mathematicians, such as Poisson and Antoine Cournot, criticized inverse methods, while Robert Ellis and John Venn proposed defining probability as the limit of the relative frequency in an indefinite series of independent trials—essentially, the frequentist approach. The maximum likelihood estimation method proposed by Ronald Fisher in the early twentieth century was interpreted by some as melding aspects of frequentist and inverse methods, though he adamantly denied the notion, saying, "The theory of inverse probability is founded upon an error, and must be wholly rejected." This may explain the essential absence of inverse or Bayesian probability concepts in the body of early statistical inferential methods, which were heavily influenced by Fisher.

Mathematician and anthropometry pioneer Adolphe Quetelet brought the concept of the normal distribution of error terms into the analysis of social data in the early nineteenth century, while others like Francis Galton advanced the development of the normal distribution in biological and social science applications in the latter half of the same century. Many mathematicians, statisticians, scientists, and others have contributed to the development of probability theories, far too many to exhaustively list, though recognized probability distributions are named for many of them, such as Augustin Cauchy, Ludwig von Mises, Waloddi Weibul, and John Wishart. Pafnuty Chebyshev, considered by many to be a founder of Russian mathematics, proved the important principle of convergence in probability, also called the Weak Law of Large Numbers. Andrei Markov's work on stochastic processes and Markov chains would lead to a broad range of probabilistic modeling techniques and assist with the resurgence of Bayesian methods in the twentieth century.

Some historians have suggested that one difficulty in developing a comprehensive mathematical theory of probability, despite such a long history and so many broad contributions, was difficulty agreeing upon one definition of probability. For example, noted economist John Keynes asserted that probabilities were a subjective value or "degree of rational belief" between complete truth and falsity. In the first half of the twentieth century, mathematician Andrey Kolmogorov outlined the axiomatic approach that formed the basis for much of subsequent mathematical theory and development. Later, Cox's theorem, named for physicist Richard Cox, would assert that any measure of belief is isomorphic to a probability measure under certain assumptions. It is used as a justification for subjectivist interpretations of probability theory, such as Bayesian methods. There are variations or extensions on probability with many applications. Shannon entropy, named for mathematician and information theorist Claude Shannon and drawn in part from thermodynamics, is used in the lossless compression of data. Martingale stochastic (random) processes, introduced by mathematicians such as Paul Lévy, recall the kinds of betting problems that challenged de Méré and inspired the development of probability theory. Chaos theories, investigated by mathematicians including Kolmogorov and Henri Poincaré, sometimes offer alternative explanations for seemingly probabilistic phenomena. Fuzzy logic, derived from mathematician and computer scientist Lotfali Zadeh's fuzzy sets, has been referred to as "probability in disguise" by Zadeh himself. He has proposed that theories of probability in the age of computers should move away from the binary logic of "true" and "false" toward more flexible, perceptual degrees of certainty that more closely match human thinking.

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See Also: Closed-Box Collecting; Data Analysis and Probability in Society; Game Theory; Normal Distribution; Randomness; Sample Surveys.

Probability in Society

See *Data Analysis and Probability in Society*

Problem Solving in Society

Category: School and Society.
Fields of Study: Connections; Problem Solving.
Summary: Mathematics is used to find and solve problems, often spurring new mathematical investigations.

Problem solving is fundamental not only to the learning and application of mathematics as a student, but to all walks of life. Many people consider mathematics and problem solving synonymous. However, there are many mathematicians who do not solve problems or who do more than solve problems. Some work to build new theories or advance the language of mathematics. Others unify or explain previous results, sometimes from many fields of mathematics. Yet others consider the very nature and philosophy of mathematics as a discipline. In twenty-first-century society, mathematics teaching at all levels seeks to develop students' abilities to effectively address a wide variety of mathematics problems, including proving theorems; reducing new problems to previously solved problems; formulating and solving both real-life and abstract word problems; finding and creating patterns; interpreting figures, graphs, and data; developing geometric constructions; and doing appropriate computations or simulations, often with computers or calculators.

Problem solving is also an instructional approach in which students actively learn fundamental concepts through their contextualization within problems rather than from a passive lecture. What fundamentally connects these activities, beyond the mathematics techniques and skills necessary to solve them, is the framework of "how to think." Students must have the necessary tools and techniques at their disposal through a solid education in the fundamentals. They must also be able to either analyze the characteristics and requirements of a problem in order to decide which tools to apply, or know that they do not have the appropriate tool at their disposal. Further, students must practice with these mathematical tools in order to become skilled and flexible problem solvers, in the same way that athletes or craftsmen practice their trades. As Hungarian mathematician George Pólya expressed, "If you wish to become a problem solver, you have to solve problems." This idea extends to the notion that problem solving is by its nature cyclic and dynamic. In many cases, the solution to a problem results in one or more new problems or opens the path to solving an older problem for which a solution has previously proven elusive. Sometimes, mathematics problems have real and immediate applications, and many new mathematical disciplines, like operations research or statistical quality control, have developed from these sorts of problems. In contrast, there are many issues in theoretical mathematics that do not appear to have any immediate benefit to society. In some cases, people question the need to explore such abstract problems when there are more immediate needs. Often, these abstract problems turn out to have very concrete applications decades or even centuries after their initial introduction. Even if that is not the case, theoretical problem solving adds to the growing body of mathematics knowledge and, just as importantly, shows people yet another way to think about the world.

History

The mathematics body of knowledge is not static; it has been evolving with humans. As soon as humans organized themselves into communities attached to the land, benefits rapidly emerged. Certainly, an advantage was an increase in agricultural and livestock productivity. As a result, part of the harvest and the cattle was accumulated for worse times. Accumulation demanded certain mechanisms to identify the ownership and use of the land (the process of land surveying) and to record who contributed to what was collected (the system of counting). The success of such social structure allowed skilled individuals to take advantage of their abilities to exchange the resultant products for food surplus (the beginnings of commerce). The development of commerce demanded a new tool to register the commercial operations in order to recognize who was implicated and the amount involved. This tool was based in a new kind of language (mathematics) able to do operations such as additions, subtractions, iterative sums, and partitions that natural languages were unable to support. As with any language, it consisted of two elements: notation to represent ideas (numbers) and syntax to manipulate these ideas (calculation).

After the accumulation of goods came the capability to organize collective efforts. It was possible to build massive public works. Warehouses, markets, fortresses, temples, aqueducts, and even pyramids were constructed in urban centers and their surroundings. Construction presented a new problem related to the manipulation and combination of forms. Early exercises were based on rules used for land surveying; for instance, to calculate areas and volumes. Additional difficulties arose when public works increased their complexity; hence, the application of forms and their interactions to develop better habitats gave rise to the development of architecture as an independent discipline. The Greeks separated land surveying from the study of spatial relations and forms; as a result, geometry was born. This discipline was used to solve abstract mathematical problems. For instance, Pythagoras recognized the relation between the sides of a right triangle as $a^2 + b^2 = c^2$ (the Pythagorean Theorem), and Archimedes studied the relation between the circle's circumference and its diameter. The latter is known as *pi* (π), an irrational number with the value of 3.141592653589793238462643383279502 . . . .

The Problem of Representation and the Dynamics of Change

With the accelerated increase of richness and variety in social interactions, intractable problems of representation appeared. Operations were required to record social experiences from an ever growing dynamism. This endeavor made limitations in the notation systems available at that moment evident. Hindis and, afterwards, Arabs and Muslims developed the positional decimal system still in use in the twenty-first century. The decimal system allows the representation of arithmetic operations without the need to use an abacus. Changes in quantities demanded introducing a general notation for variable and constant amounts, which were linked by operators to form different sentences, called "equations." The study of these relations is known as "algebra."

The capability to represent abstract ideas and their relations allowed mathematicians at the beginning of the sixteenth century to discuss problems related to the dynamics of change. In fact, the field of astronomy proposed new challenges to mathematics. Between 1507 and 1532, Copernicus presented a series of works where he substituted the traditional viewpoint, which located the Earth at the center of the universe (the geocentric view), with another where the sun was at the focus (the heliocentric view). This view helped to explain inconsistencies in the stellar movement, such as the retrograde displacement of planets. Around 1605, Johannes Kepler empirically discovered the elliptic orbit of planets around the sun. He also noticed that the line that joins each planet with the sun (called the "radius vector") sweeps the same area in the same period of time. Galileo focused his telescope to Jupiter, and, in 1610, he posited that the lights surrounding that planet were, in fact, satellites. To demonstrate all of this in mathematical terms demanded the study of change in relation to time, something impossible to solve at that moment. Isaac Newton and Gottfried Leibnitz simultaneously developed a useful procedure known as "calculus." When it is used to represent the change of a certain quantity in relation to another in terms of infinitesimal moments, it is called "differential calculus." Interestingly enough, this procedure can be reversed to reckon space sections bounded by different functions. The general procedure consists on dividing them into additive infinitesimal blocks—a process named "integral calculus." Both procedures operate in an inverse manner through the fundamental theorem of calculus.

The Problem of Estimation

In the seventeenth century, additional problems appeared when the practical world confronted an impossible question. How can one characterize something that is not stable enough to be counted? For instance, in order to establish public policies, politicians need to know what resources are at their disposal—the demographic and economic capabilities, which can be determined in a census. The main problem with exhaustive counting of populations is that they change. There are births and deaths. In order to solve this issue, one method is to select a fraction (called a "sample") of the object of study (called the "population"), to identify the sample characteristics and to generalize them to the population. Advantages for this sampling procedure are lower costs and faster data collection than following a comprehensive census. But there is an important difficulty: how to guarantee that the characteristics of the sample are the same as those of the entire population. One needs to estimate the sampling error because of selecting a sample that does not represent the population and to define a confidence interval by identifying the reliability of the estimate. The part of mathematics interested in this kind of problems is known as "statistics."

Statistics helps to solve many technical problems. Statisticians may need to (1) estimate the size of a population, as Laplace did in 1786 for France, by using a sample; (2) describe a population in terms of different numerical relations, such as its expected value (called "average"), its most frequent value (called "mode"), the limits of the data series (called "range"), the value that separates the higher half of the data series from the lower half (called "median"), and the data dispersion (called "standard deviation"); (3) test a hypothesis as J. H. Jagger did in 1873 at the Beaux-Arts Casino at Monte Carlo, when he collected results from a roulette wheel to prove that it was fraudulent; (4) estimate if a process needs products of a certain quality or it requires to be fixed, as in statistical quality control; (5) identify if changes in a process result in a positive outcome (called "correlation"), such as the Hawthorne study done in a working line to correlate the increase in illumination with workers' productivity; (5) predict and forecast future outcomes by means of recognizing patterns of behavior, what is known as "regression"; (6) extrapolate future data through the analysis of previous results; (7) reconstruct incomplete series data by means of that which is known and available, through interpolation; or (8) model the behavior of an entity in order to transform data into valuable information (called "data mining").

The Problem of Decision Making

The Industrial Revolution introduced a massive change in the social order. Early stages of the period witnessed the substitution of agricultural workers with machines by the thousands. It represented an increase in the productivity for many industries and services, mainly textiles and transportation, to levels never before seen. It surpassed the previous cumulative capacity of mankind. It also implied a surplus of energy with the use of internal combustion engines and electrical power generation. However, finding the equilibrium in this new social order was not an easy endeavor. Two world wars witnessed this planetary enterprise, and the postwar era during two different visions of the best way to organize the global society developed into a mortal conflict: capitalism versus communism.

At the beginning, the Industrial Revolution promised benefits with no end, although it made the medieval work system based on guilds inoperative. Groups of artisans loyal to a closed system of hierarchical progression were substituted by interchangeable clusters of men and machines located at industrial centers with short-term economic success as its main performance criterion. These were operationalized in terms of effec-

The Problem of Distributions

Although knowing certain characteristics from the population allows one to make more informed decisions, it does not solve particular cases. For instance, if 80% of people in a community prefer vanilla flavor rather than chocolate ice cream, will Ms. X like it? If the identification of general preferences does not ensure that individual expectations will be fulfilled, how can one propose the best offer to an individual in particular? How can one quantify the chance of an event happening? The study of the individual behavior from a collective characterization is known as *probability*. It is important to note that probability has to do with descriptions from populations and not from individuals.

Probability studies began with Blaise Pascal and Pierre de Fermat (1654), when the former was approached by a gamester, the Chevalier de Méré, to solve a game problem—how to divide the stakes between two players who want to leave the table before finishing their game. It was not until the nineteenth century, again in the field of astronomy, that the potential use of probability was recognized. In 1801, Giuseppe Piazzi discovered the first asteroid, Ceres, but he had so few observations that he was unable to determine its orbit. A mathematician, Carl Friederich Gauss, analyzed the data available and, in order to correct the observational errors, he supposed that they would follow a normal distribution. This distribution is one of the most well-known among probability users. Probability has been used for hypothesis testing according to different probability distributions, statistical mechanics, probabilistic processes, the random movement of particles suspended in a fluid, and options' valuation.

tiveness and efficiency, and optimization was the prime improving activity. Methods based on empiricism and not on tradition acquired a new value. For instance, in 1840, Charles Babbage realized a study about mail classification and transportation; the result was the institution of the Uniform Penny Post; a taxation procedure by which a letter not exceeding half an ounce in weight could be sent from any part of the United Kingdom to any other part for one penny. In 1911, Taylor proposed a series of managerial principles that were the foundations of what is currently known as "management science" or "operational research." This Science of the Better consists in the application of advanced analytical methods to help make better decisions.

Operational Research took shape just prior to World War II. At the beginning, exercises were focused on solving problems of fighter direction and control in the British air defense system. The new radar system acted as an early warning system that was able to identify German aircraft before they would bomb air bases, ports, industrial areas, and cities. Success demanded, later during the war, to extend these exercises to the Atlantic Ocean. Massive ship losses because of the attacks of U-boats (German submarines) put Allied supplies to Europe and North Africa at risk. Accordingly, different analyses were conducted to increase the U-boat sinking rate. Different criteria were mathematically explored and solutions were implemented, including (1) identifying which kind of aircraft was the best suited to chase German submarines; (2) reckoning the time at which depth charges should explode, and (3) defining the size of merchant fleets that minimizes Allied losses when crossing the Atlantic.

From the success of analyzing the performance of military operations, this field of mathematics was extended to other industrial and social activities. Many different problems have been studied and alleviated by this approach, including (1) community development, in order to organize collectives, support strategies that deal with social dissatisfactions, help groups in rural communities and developing countries, and create the social conditions for effective public policies; (2) criminal justice, to maintain a safe society by optimizing the use of resources allocation that enforce the law and reduce spaces for organized crime and to assess policy impact; (3) education, to evaluate teaching quality, students learning experiences, and assessment procedures; (4) efficiency and productivity analysis; (5) healthcare services; (6) logistics and supply chains; (7) quality control; (8) security and defense; (9) scheduling; (10) strategic management; and (11) transport.

The Problem of Prediction in a Complex World

The acquisition, distribution, and use of knowledge are key factors for the development of individuals and society, an idea that has shifted social structures to more complex levels of organization. The introduction of concepts such as "entrepreneurship" (a wild spirit who causes creative destruction by innovation and disruption) or "leadership" (a process of social influence and emotional contagion) are the result of recognizing that people's actions affect many others in non-evident ways. Economy, ecology, management, and politics require new approaches as these phenomena develop with intensities never before expected. The limitation of resources demands humans to use them responsibly and to make decisions for a better future. The main difficulty consists of predicting the future from the present. How can a person predict future concequences of actions to recognize good actions from bad ones?

Advising people on how to act is an age-old business. For a long time, the unique sources at disposal were divinely inspired or supported by powerful collectives. However, since the 1800s, the emphasis shifted toward scientific study of the environment regarding which actions take place. Prediction was focused on learning from the past and expecting the future to behave similarly, what is known as "time-series procedures." These can be useful where individual decisions have little impact on the overall behavior; for example, the results of the lottery or the weather conditions for the next few days.

Accordingly, different patterns can be found in the data (such as horizontal, seasonal, cyclic, or trend), but no explanations for the phenomenon under study have been developed. Explanatory models require assuming a relationship between what one wants to forecast (called the "dependent variable") and something one knows or controls (called the "independent variable"). Through a regression analysis one may minimize differences between observations and the points from an expected trend, linear or not, which can be adjusted to indicate certain seasonality. For more complex phenomena, one may introduce additional independent variables in order to conduct multiple regression analysis. In certain conditions, this approach enhances information for a better decision-making process but assumes the non-evolutionary viewpoint that the best model for the future is the one which better fits historical data. This approach also reduces the size of phenomena under scrutiny because modeling a real complex phenomenon such as the world's climate goes easily beyond twenty-first-century computers' capabilities and human understanding.

Complexity is related to many things such as size, difficulty, variety, order, or disorder. However, it has nothing to do with complication. Anything complicated can be solved, usually by introducing more resources to crack current problems. Conversely, complexity is associated with the impossibility of guaranteeing future behaviors based on current ones. The mathematical treatment of complexity introduced a discipline known as "chaos theory." It is a collection of mathematical, numerical, and geometrical techniques that allow mathematicians to deal with non-linear problems that do not have explicit general solutions. It is based in the use of differential equations to analyze dynamic behaviors extremely sensitive to initial conditions. In this context, predicting the future has to do with recognizing stable equilibrium points (called "fixed point attractors"), those that appear when dynamic systems stop. An attractor indicates the natural tendency of a system to behave in a certain way in the long-term future, if nothing else disturbs it. Common physical examples of this kind of behavior are pendulums and springs. Attractors are used for decision making in different fields, such as finance, where investors try to identify stock market tendencies. Some major applications related to its origins are weather prediction, solar weather prediction models, and predicting fisheries dynamics.

The increase of computing power allows mathematicians to run mathematical models based in little pieces of code that represent specific behaviors (called "intelligent agents"). Agent-based models can be used to study complex behaviors to simulate individual behaviors, such as people's movements inside stadiums or automobiles avoiding traffic jams. Other studies related to self-organized and self-organizing behaviors can also be conducted as they can represent phenomena from economy and financial markets; opinion dynamics; emergency of social rules and institutions; creation or disappearance of companies; and technology innovation, adoption, and diffusion.

To recognize stability areas and patterns in complex behaviors resulting from a multiplicity of agents interacting is then at the basis of the next social challenge, and procedures to deal with this are at the edge of twenty-first-century capabilities. The study of elements and their interactions have developed new viewpoints to observe reality. To visualize problems as a myriad of elements richly interconnected with unseen behaviors and consequences has introduced notions such as "systems" and "networks" in discourse. In 1950, Ludwig von Bertalanffy, a biologist, recognized similar fundamental conceptions in different disciplines of science, irrespective of the object of study. He tried to represent those rules through a language to describe such entities, which he named the "General System Theory."

A year before, Werner introduced the notion of communicative control in machines and living beings by looking at the effects of feedback on future behaviors. He named it "cybernetics." Based on this, in 1956, Ashby provided a single vocabulary and a single set of concepts suitable for representing the most diverse types of systems. Since then, different researchers have developed alternative methodologies to describe phenomena not in terms of problems and solutions, but in terms of satisfaction and alleviation. This has been used to deal with non-technical problems—those considered impossible to solve only through analytical tools, as they include humans' interactions. In this context, relations between individuals are diagramed and studied in terms of bunches of nodes interconnected by links. From this viewpoint the image of a "network" emerges. This notion has been developed, for instance to measure the "distance" between two persons from different places and contexts and reckoned that the average number of intermediate people between them is 5.5, hence the phrase "six degrees of separation." Network analysis is important as it can be used to model and study phenomena such as the Internet and its vulnerability to hackers, viruses and their uncontrollable expansion, or technology innovation and its diffusion. Future developments on this area are expected.

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Eliseo Vilalta-Perdomo

See Also: Mathematics, Applied; Mathematics, Defined; Mathematics, Elegant; Reasoning and Proof in Society.

Producers

See *Writers, Producers, and Actors*

Professional Associations

Category: Mathematics Culture and Identity.
Fields of Study: Communication; Connections.
Summary: Professional mathematical associations help mathematicians advocate, share ideas, and organize.

Organizations are a fundamental component of society, in part because of the human need to connect around similar interests. Professional associations form in response to individual and societal needs and concerns, and in turn impact society. Mathematics students, teachers, and researchers may join professional mathematics organizations to feel like a part of the larger mathematics community and make a difference beyond their school or university. There are international associations with worldwide memberships, like the International Mathematical Union, as well as associations that are organized by geographical region. National and regional associations in countries around the world address many of the same issues as mathematics associations in the United States. These issues include teaching, research, service, and the mathematics profession.

Mathematical associations may advocate for the mathematical sciences, engage in public policy discussions, and promote collaboration among specialized subgroups. They may provide professional development to mathematicians and engage in public mathematics outreach. Professional associations organize regional, national, or international conferences; fund professional development and outreach; publish a diverse array of books and journals on mathematical topics; and facilitate peer review and curricular changes. Philosophers and mathematicians like Paul Ernest and Reuben Hersh have written about the social and ethical responsibility of mathematicians, and mathematicians may work toward the greater good within the structure of professional organizations. As officers and committee members, mathematicians also run these associations.

Mathematical Organizations

The American Statistical Association (ASA) was formed in Boston, in 1839, by members with diverse interests. ASA's Web site states the following:

> Present at the organizing meeting were William Cogswell, teacher, fund-raiser for the ministry, and genealogist; Richard Fletcher, lawyer and U.S. Congressman; John Dix Fisher, physician and pioneer in medical reform; Oliver Peabody, lawyer, clergyman, poet, and editor; and Lemuel Shattuck, statistician, genealogist, publisher, and author of perhaps the most significant single document in the history of public health to that date.

From the beginning, the ASA had close ties with the government on statistical issues like those surrounding the census. ASA is international and comprised of professionals from industry, government, and academia in fields ranging from pharmaceuticals, health policy, agriculture, business, education, to technology. It promotes statistical knowledge through meetings, publications, membership services, education, accreditation, and advocacy.

In the United States, two well-known mathematics organizations are the American Mathematical Society (AMS) and the Mathematical Association of America (MAA). Both publish research journals, host professional conferences, and engage in student, community, and public policy outreach, although they have different focuses. The AMS originated as the New York Mathematical Society in 1888, and, in 1894, it became a national organization that concentrated on research. Teacher Benjamin Finkel created the *American Mathematical Monthly* in 1894, stating the following:

> Most of our existing journals deal almost exclusively with subjects beyond the reach of the average student or teacher of mathematics or at least with subjects with which they are familiar, and little, if any, space, is devoted to the solution of problems.

In 1915, when managing editor H. E. Slaught unsuccessfully tried to bring the *Monthly* to the AMS, the society instead recommended that there should be a different organization devoted to the journal. The AMS continues to focus primarily on mathematics research and scholarship, while the MAA promotes communication, teaching, learning, and research in mathematics and its applications, especially at the collegiate level. The mission of the MAA incorporates five core interests of education, research, professional development, public policy, and public appreciation. The MAA sponsors the highly regarded William Lowell Putnam Competition for undergraduate students and the American Math Olympiad mathematics competitions. The AMS and MAA join at the Joint Mathematics Meetings each January.

The National Council of Teachers of Mathematics (NCTM) was created in 1920, in part, to counter the efforts of social efficiency experts who believed that school curricula should emphasize fostering job-related skills and knowledge. Its membership includes mathematics teachers, mathematics teacher educators, and mathematics education researchers. It is perhaps most well-known for publishing one of the earliest sets of K–12 mathematics standards. NCTM's stated objectives are to develop effective curriculum and instruction, ensure equity in mathematics education, shape public policy, produce high quality mathematics education research, and provide professional development opportunities for mathematics educators. NCTM publishes works like the *Principles and Standards for School Mathematics* and *Curriculum Focal Points*, in addition to journals such as the *Mathematics Teacher, Teaching Children Mathematics, Mathematics Teaching in the Middle School*, and the *Journal for Research in Mathematics Education.* State, regional, and local affiliates also work to carry out NCTM's mission through annual conferences and other professional development opportunities. Similarly, trainers of mathematics teachers assemble in the Association of Mathematics Teacher Educators, and supervisors assemble in the National Council of Supervisors of Mathematics.

The Society for Industrial and Applied Mathematics (SIAM) originated in the early 1950s to represent mathematicians working in industry. Their numbers had grown as a result of the importance of mathematics in military research during World War II and the evolution of computers. SIAM seeks to advance applied mathematics, promote practical research, and encourage the exchange of applied mathematical ideas. Annual meetings, subject-specific workshops and conferences, and discipline-specific activity groups allow members to develop new applied mathematical ideas and techniques.

Organizations designed to promote minorities in mathematics include what is now known as the National Association of Mathematicians (NAM), which started as an informal group at the Annual Meeting of the American Mathematical Society in 1969. Lee Lorch recalled:

> In 1960, when A. Shabazz and S.C. Saxena, both on the faculty of Atlanta University (now Clark-Atlanta), and their graduate student W.E. Brodie were subjected yet again to Jim Crow treatment at the spring meeting of the Southeastern Section of MAA. . . . This, it should be noted, was several years after AMS and MAA commitments to the contrary. They had not been warned in advance that such discourtesy would be in store. The three left in protest. And so in 1969 the National Association of Mathematicians (NAM) came into being to address the needs of the Black mathematical community. This

> was a turbulent period. A group of more or less left-oriented mathematicians established the Mathematicians Action Group (MAG) that same year. We were motivated largely by concern over the Vietnam war, the militarization of mathematics, the lack of democracy in the AMS, the existence of racism and sexism, and related social issues as they impinged on mathematicians and vice versa.

NAM focuses on education, career development, research, student development, and databases. NAM also publishes a newsletter and organizes a lecture series. The Benjamin Banneker Association was founded in 1986 to concentrate on the mathematics education of African Americans. There are also many associations that focus on science, like the Society for the Advancement of Chicanos and Native Americans in Science.

Organizations like Association for Women in Mathematics, European Women in Mathematics, and Korean Women in the Mathematical Sciences were created to support and promote female students, teachers, and researchers via social events, sponsored talks or conferences, workshops, and contests. Many attribute the beginning of the Association for Women in Mathematics to events in Boston and Atlantic City. In the late 1960s, Alice Shafer and Linda Rothschild organized a mathematics women's group in the Boston area. At a 1971 conference in Atlantic City, Joanne Darken suggested that women already at the Mathematics Action Group remain to form a caucus. As noted by president Lenore Blum:

> What I remember hearing about Mary Gray and the Atlantic City Meetings, indeed what perked my curiosity, was an entirely different event, one that was also to alter dramatically the character of the mathematics community. In those years the AMS was governed by what could only be called an 'old boys network,' closed to all but those in the inner circle. Mary challenged that by sitting in on the Council meeting in Atlantic City. When she was told she had to leave . . . she responded she could find no rules in the by-laws restricting attendance at Council meetings. She was then told it was by 'gentlemen's agreement.' Naturally Mary replied 'Well, obviously I'm no gentleman.' After that time, Council meetings were open to observers and the process of democratization of the Society had begun.

Mary Gray placed an official announcement about the organization in the Notices of the American Mathematical Association and created its first newsletter in 1971.

Other notable mathematical organizations include the American Mathematical Association of Two Year Colleges (AMATYC), which was founded in 1974. AMATYC organizes conferences and workshops and publishes books and proceedings related to mathematics education in the first two years of college.

Mathematicians create other professional organizations under the umbrella of a wide variety of interests and themes. They assemble in national and international subject-specific societies that focus on areas such as linear algebra, mathematical physics, or mathematics and art, including the International Linear Algebra Society and the Association for Symbolic Logic, or through special interest groups at the Mathematical Association of America. Mathematical organizations that are related to religion or sexual orientation include the Association of Christians in the Mathematical Sciences and the Association of Lesbian, Gay, Bisexual and Transgendered Mathematicians. National and international mathematics honor societies include Kappa Mu Epsilon and Pi Mu Epsilon. Mathematicians interested in the advancement of science policy participate in advocacy groups such as the Triangle Coalition for Science and Technology Education and the American Association for the Advancement of Science.

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Christopher J. Stapel

See Also: Clubs and Honor Societies; Curricula, International; Ethics; Government and State Legislation; North America.

Proof

Category: History and Development of Curricular Concepts.
Fields of Study: Communication; Connections; Problem Solving; Reasoning and Proof.
Summary: The product of deductive reasoning, the nature of proof has long been fundamental.

Deductive proofs have been an essential part of mathematics for over 2000 years, and some equate proof ability with competence in mathematics. Mathematician David Henderson defines an effective proof as a convincing communication that answers "why." Thus, a proof may connect ideas within a mathematical system or illuminate both the "how" and the "why" underlying the conjecture. Since a proof depends on the accepted standards of the audience or society through some type of peer review, there is also a long history of concerns about the nature of proof. For example, Galileo Galilei and Christoph Clavius debated the legitimacy of pictures, and Leopold Kronecker criticized the use of nonconstructivist methods. In the twentieth and twenty-first centuries, philosophical concerns about proofs continued as mathematicians considered the role of computers or empirical aspects and the implications of Kurt Gödel's groundbreaking work on consistency. While reasoning and proof have long been a part of mathematics curricula, the concepts took on an increased importance in the United States during the era of Sputnik and the space race, when many different types of proofs were emphasized. In the early twenty-first century, proofs remain fundamental in education, beginning in primary school. In pure mathematics, new research depends on proofs. The notion of proof has been clarified by mathematicians in the field of logic, who explore the foundations of proof.

Brief History

Early civilizations developed sophisticated notions of mathematical argumentation, as documented by evidence such as cuneiform tablets, papyri, and mathematical texts from ancient Babylon, China, and Egypt. The idea of a formal deductive proof arose as a distinct part of ancient Greek mathematics. Greek mathematicians studied and generalized mathematical ideas, using proofs to justify their claims. Mathematics historians theorize that the prevalence of debate in Greek society provided a conducive environment for the development of axiomatic argumentation. Euclid's *Elements* became "the" model for using a small set of axioms to deduce a large system of theorems and knowledge, now known as "Euclidean geometry."

Logic

Logical systems become the foundational structures necessary to create a proof. First, mathematicians use logic tools to argue that one mathematical statement follows as a logical consequence from other mathematical statements, and then they use logic tools to establish a formal proof by building a chain of consequent statements from initial assumptions. These logic tools include connectives (negation, conjunction, disjunction, conditional implications, and equivalence), quantifiers, truth statements, tautologies, and inferential structures (such as *modus ponens* and *reductio ad absurdum*).

Direct Proofs

Mathematical proofs can be done in diverse ways, all reflecting different inferential structures. Starting with an initial conjecture (such as $H \rightarrow C$) involving a hypothesis H and a conclusion C, Direct Proofs build logical chains of compound statements, using conditional implications as the links. They are usually written in the traditional two-column format using high school geometry. The following illustrates the use of a Direct Proof, though it is not entirely rigorous.

Conjecture: If a and b are prime numbers greater than 2, then their sum $a + b$ is composite.

Proof:

Statement	Justification
1. a and b are prime numbers > 2	1. Given
2. a and b are odd numbers	2. The only even prime is 2
3. Let $a = 2m + 1$ and $b = 2n + 1$ for $m, n > 1$	3. Definition of an odd number
4. Sum $a + b = (2m + 1) + (2n + 1)$	4. Substitution
5. Sum $a + b = 2(m + n + 1)$	5. Properties of arithmetic operations
6. Sum $a + b$ is even	6. Definition of an even number
7. Sum $a + b > 2$	7. $m + n + 1 > 1$
8. Sum $a + b$ is composite	8. The only even prime is 2

A Proof by Contrapositive is very similar to a Direct Proof, with the difference being the format of the conjecture itself. While a Direct Proof proves the conjecture $H \rightarrow C$, a Proof by Contrapositive uses a Direct Proof to prove the contrapositive, $\sim C \rightarrow \sim H$. In the previous example, a Proof by Contrapositive would prove the statement "If the sum $a + b$ is prime, then a and b are not both prime numbers greater than 2."

Indirect Proofs

In contrast to Direct Proofs, an Indirect Proof assumes the negation of the conclusion $\sim C$ to be true and then uses a Direct Proof to prove the truth of the negation $\sim S$ for some true statement S. By the Law of Logic Contradiction (S and $\sim S$ cannot both be true), which implies that the original conclusion C must be true. The following illustrates the use of an Indirect Proof.

Conjecture: The $\sqrt{2}$ is an irrational number.

Proof:

Statement	Justification
1. Assume $\sqrt{2}$ is a rational number	1. Negation of conclusion
2. $\sqrt{2} = a/b$ where $\gcd(a, b) = 1$ and a,b positive integers	2. Definition of the rationals and greatest common divisor
3. $2 = a^2/b^2$	3. Squaring both sides
4. $2b^2 = a^2$	4. Multiplying both sides by b^2
5. a^2 is even	5. Definition of an even number
6. a is even	6. Squares of odd integers are odd
7. $a = 2m$ for $m > 1$	7. Definition of an even number
8. $2b^2 = (2m)^2 = 4m^2$	8. Substitution
9. $b^2 = 2m^2$	9. Multiplying both sides by 1/2
10. b^2 is even	10. Definition of an even number
11. b is even	11. Squares of odd integers are odd
12. $\gcd(a,b) \geq 2$	12. Definition of gcd
13. Original assumption is false	13. Contradicting assumption $\gcd(a,b) = 1$

Using the idea of infinite descent, this Indirect Proof is considered to be one of the most "beautiful" proofs by the mathematical community. Though not as elegant, it would be possible to prove this same conjecture using a Direct Proof. Also, it is important to note that this Indirect Proof uses some outside knowledge from number theory (such as, squares of odd integers are odd), which would have to be proved prior to its use as justification within the Indirect Proof.

Deduction and induction constantly play important roles in the proof process. For example, in the previous Direct Proof, a considerable number of examples could be systematically examined: 3 + 5 = 8, 3 + 7 = 10, 5 + 11 = 16, 23 + 47 = 70, and so on. These examples provide inductive evidence that the conjecture is true, nothing more. That is, the cumulative contribution of the examples is only increased confidence that the conjecture is true and that a formal deductive proof is needed. And, a Proof by Exhaustion of All Cases is not possible because the number of pairs of primes to consider is infinite. Nonetheless, a Proof by Induction is often possible in situations involving an infinite number of examples, as illustrated by the following proof.

Conjecture: $1+2+3+4+\cdots+n=\dfrac{n(n+1)}{2}$.

Proof:
Case $n=1$: Substituting, $1=\dfrac{1(1+1)}{2}=1$.

Assume case for k is true, need to show case for $(k+1)$ is true: Given the assumption

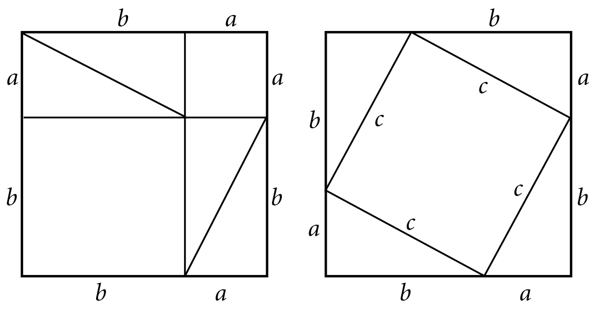
$$1+2+3+\cdots+k=\frac{k(k+1)}{2}. \text{ Then,}$$

$$\begin{aligned}1+2+3+\cdots+k+(k+1)&=\frac{k(k+1)}{2}+(k+1)\\&=(k+1)\left(\frac{k}{2}+1\right)\\&=(k+1)\left(\frac{k}{2}+\frac{2}{2}\right)\\&=\frac{(k+1)(k+2)}{2}\\&=\frac{(k+1)\left[(k+1)+1\right]}{2}.\end{aligned}$$

For some conjectures, a visual "Behold!" Proof is possible. A common example is this proof of the Pythagorean Theorem.

Conjecture: In any right triangle, the square of the hypotenuse c (side opposite the right angle) equals the sum of the squares of the other two sides a and b: $c^2 = a^2 + b^2$.

Proof: Behold!



As in most proofs, effort is needed to understand a "Behold!" Proof. In this example, focus on the common areas and rearrangement of the four triangles. The first large square involves two smaller squares (areas a^2 and b^2) and four triangles, while the second large square involves one small square (area c^2) and the same four triangles. Noting that this proof has been traced back to early Chinese mathematics, it is important to add that more than 360 different proofs of the Pythagorean Theorem are known.

Despite their connection to truth, proofs can create mathematical fallacies. Examples include the use of Mathematical Induction to prove that, "All horses are of the same color," the misleading dependence on a geometrical diagram to prove that all triangles are isosceles or even proofs that disguise computational errors, such as the following:

Conjecture: $1 = 2$
Proof:

Statement	Justification
1. Let $n = m > 0$	1. Assumption
2. $n^2 = mn$	2. Multiplying both sides by n
3. $n^2 - m^2 = mn - m^2$	3. Subtracting m^2 from both sides
4. $(n+m)(n-m)=m(n-m)$	4. Factoring both sides
5. $(n+m)=m$	5. Dividing both sides by $(n-m)$
6. $(m+m)=m$	6. Substitution as $n = m$

7. $2m = m$	7. Simplification
8. $2 = 1$	8. Dividing both sides by m

In supporting this obviously wrong conclusion, this proof relies on the reader's literal acceptance of each statement and its justification. That is, the proof seems "true" unless the reader notices that statement five involves division by zero, which is not possible.

When constructing proofs of mathematical conjectures within a system, mathematicians are concerned with many issues related to the logical structure. Is the system "consistent," in that no proven theorem contradicts another? Is the system "valid," in that no mathematical fallacies or false inferences will be created? Is the system based on underlying axioms or initial assumptions that are reasonable? And, is the system "complete," in that every conjecture can be proven either true or false? In the 1930s, logician Kurt Gödel shocked the mathematical world when he proved that a "powerful" mathematical system cannot be both complete and consistent at the same time. For some mathematicians, Gödel's theorems weakened the foundational structure of mathematics, while others felt that it strengthened it. Mathematicians also debate about the role of computers in proofs. In the seventeenth century, Gottfried Leibniz predicted an automatic counting machine that would vastly improve reasoning. In the twentieth century, Herbert Gelernter wrote a program to prove theorems from Euclid's *Elements*, but critics noted the dependence on programmer-supplied rules. Some mathematicians do not accept proofs such as the first proof of the Four-Color Theorem in 1977, which depended on an analysis of many cases by a computer.

Formal proof is a special technique within the realm of mathematics, which is why the public views mathematics as the prime model for establishing truth via argumentation. The idea of proof is invoked in other fields, but with a more limited meaning. For example, in courtrooms, the element of truth is replaced with the phrase "beyond reasonable doubt" given the available evidence. In the sciences, proof is desired but cannot be established by experimental data; at best, the data can support the creation of hypotheses and theories, which will be either further verified or discounted by new experiments and new data.

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See Also: Mathematics, Elegant; Reasoning and Proof in Society.

Proof in Society

See *Reasoning and Proof in Society*

Psychological Testing

Category: Medicine and Health.
Fields of Study: Algebra; Data Analysis and Probability; Representations.
Summary: Though they often require a subjective element, psychological tests make every effort to generate useful quantitative data.

Testing is used for many different purposes within psychology—among them to evaluate intelligence, diagnose psychiatric illness, and identify aptitudes and interests. Although the results of testing are rarely used as the sole criterion to make a diagnosis or other decision about an individual, they are often used in conjunction with information gained from other sources, such as interviews and observations of behavior. There are many types of psychological tests, but most share the goal of expressing an essentially unobservable quality such as intelligence or anxiety in terms of numbers. The numbers themselves are not meant to be taken literally—no one seriously believes that a person's intelligence is equivalent to their IQ score, for instance. Instead the numbers are useful tools that help evaluate a person's situation; for instance, how does the intellectual development of one particular child relate to that of other children of his age? Of course, the results of psychological testing should be evaluated with the social context of the individual in mind and with full respect for human diversity.

Most psychological tests try to translate unobservable qualities such as intelligence or anxiety in terms of numbers.

Psychometrics

Psychometrics is a field of study that applies mathematical and statistical principles to devise new psychological tests and evaluate the properties of current tests. Psychologist Anne Anastasi was often known as the "test guru" for her pioneering work in psychometrics. In her 1954 book *Psychological Testing*, she discussed the ways in which trait development is influenced by education and heredity as well as how differences in training, culture, and language affect measurement. The two most common approaches to psychometrics in the twenty-first century are classical test theory and item response theory (IRT).

Classical test theory is the older approach and the calculations required can be performed with a pencil and paper, although twenty-first-century computer software is often used. Classical test theory assumes that all measurements are imperfect and thus contain error: the goal is to evaluate the amount of error in a measurement and develop ways to minimize it. Any observed measurement (for instance, a child's score on an intelligence test) is made up of two components: true score and error. This may be written as an equation: $X = T + E$, where X is the observed score, T is the true score (the score representing the child's true intelligence), and E is the error component (resulting from imperfect testing). Classical test theory assumes that that error is random and thus will sometimes be positive (resulting in a higher observed score than true score) and sometimes negative (resulting in a lower observed score than true score) so that over an infinite number of testing occasions, the mean of the observed scores will equal the true score. Although normally a test is administered only once to a given individual, this is a useful model that facilitates evaluation of the reliability and validity of different tests.

Item response theory (IRT) is a different approach to psychological testing and assumes that observed performance on any given test item can be explained by a latent (unobservable) trait or ability so that individuals may be evaluated in terms of the amount of that trait they contain, and items may be evaluated in terms of the amount of the trait required to answer them positively. For an item on an intelligence test (intelligence being the latent trait), persons with higher intelligence should be more likely to answer the question correctly. The same principle applies to IRT-based tests evaluating other psychological characteristics; for instance, if an item in a psychological screening test is meant to diagnose depression, a person with more depressive symptoms should be more likely to answer it positively. IRT is a mathematically complex method of analysis that depends on the use of specialized computer software and has become a popular means to evaluate psychological tests as computers have become more affordable. Although the mathematical models of IRT differ from that of classical test theory, the goals are the same: to devise tests that measure characteristics of individuals with a minimum of error.

Reliability and Validity

The term "reliability" refers to the consistency of a test score: if a test is reliable it will yield consistent results over time and without regard to the irrelevant conditions such as the person administering the test. Internal consistency is considered an aspect of reliability: it means that all the items in a test measure the same thing. Temporal reliability is also called "test-retest reliability" because it is typically evaluated by having groups of individuals take the same test on several occasions and seeing how their scores compare Some differences are expected because of the random nature of the error component, but there should be a strong relationship between the observed scores of individuals on multiple occasions.

The term "inter-rater reliability" refers to the consistency of a test or scale regardless of who administers it. For instance, psychiatric conditions are often evaluated by having an observer rate an individual's behavior using a scale, and the results for different observers evaluating the same individual at the same time should be similar. For instance, three psychologists using a scale to evaluate the same child for hyperactivity should reach similar conclusions. Both types of reliability are typically evaluated by correlating test results on different occasions (temporal) or the scores returned by different raters (inter-rater).

Internal consistency can be measured in several ways. The split-half method involves having a group of individuals take a test, then splitting the items into two groups (for instance, odd numbered items in one group and even in the other) and calculating the correlation between the total scores of the two groups. Cronbach's alpha (or coefficient alpha) is a refinement of the split-half method: it is the mean of all possible split-half coefficients. The measure was developed and named "alpha" by Lee Chronbach, an educational psychologist and measure theorist who began his career as a high school mathematics and chemistry teacher.

The term "validity" refers to whether a test measures what it claims to be measuring. Three types of validity are typically discussed: content, predictive, and construct. Content validity refers to whether the test includes a reasonable sample of the subject or quality it is intended to measure (for instance, mathematical aptitude or quality of life) and is usually established by having a panel of experts evaluate the test in relation to its purpose. Predictive validity means that test scores correlate highly with measures of similar outcomes in the future; for instance, a test of mechanical aptitude should correlate with a new hire's success working as an auto repairman. Construct validity refers to a pattern of correlations predicted by the theory behind the quantity being measured: the scores on a test should correlate highly with scores on other tests that measure similar qualities and less highly with those that measure different qualities.

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See Also: Diagnostic Testing; Educational Testing; Intelligence Quotients.

Pulleys

Category: Architecture and Engineering.
Fields of Study: Algebra; Geometry.
Summary: Pulleys provide mechanical advantage and help people do work.

A pulley is a simple machine consisting of a cylinder, called a "drum," "wheel," or "sheave," rotating on an axle, and a rope, chain, or belt running over the cylinder without sliding. Pulley drums often have grooves and ribs that prevent their ropes from sliding over the edge. People use pulleys in three ways: to change directions of forces, to change magnitude of forces, and to transmit power. Pulleys are used in building and construction, ship rigging, and within belt-driven mechanisms.

Mathematicians have investigated many aspects of pulleys. There is evidence that Archimedes of Syracuse used a compound pulley to move a ship and studied the related theories. He famously expressed: "Give me a place to stand and I will move the Earth." While his mechanical inventions brought him recognition among his contemporaries, he seems to have preferred pure mathematics. Guidobaldo del Monte reduced systems of pulleys to levers. Guillaume de l'Hôpital investigated the equilibrium of a pulley system, and mathematicians continue to explore his pulley problem using algebra, geometry, trigonometry, and calculus. A mechanical tide-predicting machine, which incorporated pulleys, is attributed to William Thomson, who later became Lord Kelvin.

Changing Directions of Forces

In an example of this use of pulleys, construction workers often attach pulleys to roofs of buildings. A builder standing on the ground can pull down on one end of the pulley's rope and a weight on the other end will move up as the drum rotates.

The vectors of input and output forces always go along the two ends of the pulley's rope. This means that a pulley can change the direction of a force within the plane that is perpendicular to the pulley's axle but not sideways from that plane. The builder can also stand inside the building, pulling the rope through a window, or on the roof pulling horizontally, as long as the triangle formed by the worker, the weight, and the pulley's drum is perpendicular to the pulley's axle.

Changing Magnitudes of Forces

When a pulley is used to change the magnitude of a force, its axle is attached to the weight, and the pulley moves up together with the weight. For example, a sailor can attach one end of a line to a yardarm, string it around a pulley's drum attached to a weight, and pull the other end up, standing on the yardarm. The sailor will only have to apply the force equal to one-half of the weight.

Does the other half of the force disappear, breaking the conservation of energy law and the work-energy theorem? No, it is distributed to the other, attached end of the rope. Moreover, the sailor will use half the force, but pull enough line to cover twice the distance the weight is lifted. The total work, which is equal to the product of the force and the distance, will be the same as in the fixed pulley case:

$$W = F \times d = \frac{1}{2} F \times 2d.$$

Changing Directions and Magnitudes of Forces: Blocks and Tackles

Because it is much easier to work for longer than to increase one's force, movable pulleys are widely used. A block and tackle is a pulley system where the rope zigzags through movable and fixed pulleys. Depending on the way the tackle is rigged, it can provide a force advantage with the factor of two, as in the example above, or 3, 4, 5 and so on. At first sight, it would seem that a block and tackle can reduce the force required to lift weights by any factor. However, friction interferes increasingly with more pulleys used.

Marine cadets memorize rigging of common block and tackle systems, and the names of tackles corresponding to force advantage factors: factor 2: "gun"; factor 3: "luff"; factor 4: "double"; factor 5: "gyn."

Drums for tackles may have multiple grooves to reduce rope friction. When tackles are combined, for example, a double tackle upon a luff tackle, their force advantage factors multiply, in this case, creating the force advantage of $3 \times 4 = 12$.

Transmitting Power

A belt or a chain going in a loop over two or more pulley drums makes all of them rotate when one is rotated. For example, a bicyclist rotates the special pulley drum, called a "crank," to which pedals are attached. The rotation of this crank is transmitted to the rotation of the rear wheel's crank, which makes the bicycle move. Using drums of different diameters, such as cranks on a sports bicycle drivetrain, can produce a force advantage.

Until the mid-twentieth century, factories typically used belts distributing power to individual machines from one central rotating drum, connected to a large steam, turbine, or animal-powered capstan engine. This power transmission system is called "line shaft." Because most industries have switched to compact electric motors, one is currently more likely to meet this type of a pulley in a museum or a history book. A human-powered capstan is also a popular science or historical fiction trope, used to demonstrate oppression, for example, in *Conan the Barbarian* and *Captain Blood*.

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Maria Droujkova

See Also: Archimedes; Bicycles; Elevators.

Puzzles

Category: Games, Sport, and Recreation.
Fields of Study: Algebra; Geometry; Number and Operations.
Summary: Because problem solving is a core activity of mathematics, it lends itself well to puzzles.

A puzzle is a question, problem, or contrivance designed to challenge and expand the mind and perhaps test ingenuity. Puzzles have been found in virtually all cultures and all historic periods, even in mythology. According to legend, the Sphinx prevented anyone from entering Thebes who failed to find the correct answer to the question: What is it that has four feet in the morning, two at noon, and three at twilight?

Mathematicians have long created puzzles and explored their solutions for research and applications. They have also created puzzles for purely recreational purposes. Teachers in many subjects within and outside mathematics use puzzles in the classroom.

There are a number of ways in which words and arrangements of letters or objects are used to create puzzles. Some problems in the Rhind Mathematical Papyrus (1650 B.C.E.) are seen as puzzles. One example is a rhyme that also appears in Leonardo Pisano Fibonacci's 1202 work *Liber Abaci* and is still popular today. Here is a modern version:

As I was going to St. Ives,
I met a man with seven wives.
Each wife had seven sacks,
Each sack had seven cats,
Each cat had seven kits.
Kits, cats, sacks, wives,
How many were going to St. Ives?

One may only assume that the narrator was going to St. Ives, not necessarily the other travellers. Mathematically, logic, branching diagrams, multiplication, and addition can be used to determine the final solution.

Traditional in several cultures, namely in Africa, is the Crossing Problem. The following is a version from Alcuin of York (735–804):

> A man wishes to ferry a wolf, a goat, and a cabbage across a river in a boat that can carry only the man and one of the others at a time. He cannot leave the goat alone with the wolf nor leave the goat alone with the cabbage on either bank. How will he safely manage to carry all of them across the river?

To solve this problem, one must recognize that the man may carry an item back and forth across the river as many times as needed and ultimately find appropriate combinations and sequencing. Dynamic versions of this game appear online, adding visual and tactile components to the solving process. Extensions of this problem include adding more items to the list, increasing the size of the boat to carry more items, and adding an island in the middle of the river where objects may be placed. Mathematicians such as Luca Pacioli, Niccolo Tartaglia, Claude-Gaspar Bachet, and Edouard Lucas investigated this problem. A well-known medieval task consisted of arranging men in a circle so that when every *k*-th man is removed, the remainder shall be a certain specified man. Several authors commented on this, from Girolano Cardano in the sixteenth century to Donald Coxeter in the twentieth century.

Word Puzzles

Anagrams have a long and mysterious history, being seen as source of ludic pleasure but are also believed by some to possess mystic powers. Inside a word or phrase, another one is hiding that one can get by permuting the letters in a different order. For instance, the letters in the word "schoolmaster" may be rearranged to form the related phrase "the classroom."

Lewis Carroll (1832–1898) invented a forerunner of the crossword: the "doublet." There are two words presented to the solver, who is required to change one word to the other by replacing only one letter at a time, forming a legitimate word with each transformation. One of his examples is to change "HEAD" into "TAIL," which can be done via the following sequence: "HEAL," "TEAL," "TELL," "TALL," and "TAIL."

Visual Puzzles

Visual puzzles are also popular, such as optical illusions, which have long been investigated by mathematicians. Some of these address mathematical questions in disciplines like geometry and visualization, including figures that appear to be impossible.

Figure 1. Are the two dark lines parallel?

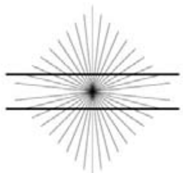
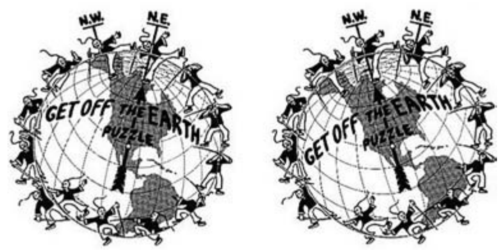


Figure 2. An illustration of an "impossible" object.

Samuel Loyd (1841–1911) is referred to by some as "America's greatest puzzlist." He reputedly created thousands of puzzles. Some of his inventions were very original, like the Get Off the Earth puzzle. There are 13 men in the figure on the left. Rotating the puzzle, as shown in the figure on the right, produces a drawing that has 12 men. What happened to the 13th man?

Figure 3. The Get Off The Earth puzzle.



Arithmetic Puzzles

Numerical relations and arithmetical principles are often found in puzzles. "Magic squares," which are square arrays of consecutive numbers with constant sum in columns, rows, and diagonals, illustrate this clearly. One of the oldest, the Chinese *lo-shu*, dates back thousands of years. Leonhard Euler's (1707–1783) work on Latin Squares, which are arrays of symbols with no repetitions in rows or columns, is one of the foundations of *Sudoku* puzzles, which appeared in a U.S. magazine in the 1970s but became famous first in Japan and then in the world. Tartaglia (1500–1557) presented the following numerical problem: A dying man leaves 17 horses to be divided among his three sons in the proportion 1/2 : 1/3 : 1/9. Can the brothers carry out their father's will? Since 17 is not a multiple of 2, 3, or 9, there is no solution that would give all of the sons a whole number of horses.

Some authors shared problems, even if they lived in different centuries. Fibonacci (1170–1250), Tartaglia, and Bachet (1581–1638) all investigated the question:

> If you have a balance, what is the least number of weights necessary to weigh any integer number of pounds from 1 to 40? (Assume you can put weights in either side of the balance.)

"Cryptarithms," created for training the calculating mind in 1913, were very popular in the twentieth century. In a cryptarithm, one is asked to find the digits erased from a valid calculation. Later, prolific English puzzle inventor, Henry Dudeney (1857–1930), substituted letters for the unknown numbers to create another layer of meaning. In his first example of an "alphametic" is the equation: SEND + MORE = MONEY, where each letter represents a different digit, and the addition is correct.

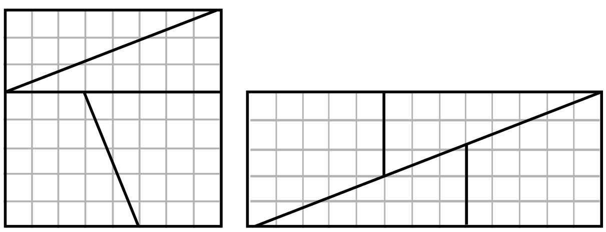
Rearrangement Puzzles

Some dissection and rearrangement puzzles are based on mathematical principles. Archimedes of Syracuse (287–212 B.C.E.) may have created a 14-piece puzzle, the "Stomachion," as part of his research. It resembles a version of a "Tangram," a Chinese puzzle that became very popular in the nineteenth century in the West and is often used in mathematics classrooms in the twenty-first century to investigate dissections and concepts like the Pythagorean Theorem, named for Pythagoras of Samos. The Fibonacci sequence relation

$$\left(F_n\right)^2 = F_{n-1}F_{n+1} + \left(-1\right)^{n-1}$$

with $n = 6$ can be used to create a dissection puzzle. Larger values of n generate similar, more impressive puzzles, where the difference of area between a large square and a large rectangle is always included. Some dissection puzzles may lead to optical illusions when the pieces do not fit exactly together, leading to two figures composed of the same pieces that have different areas.

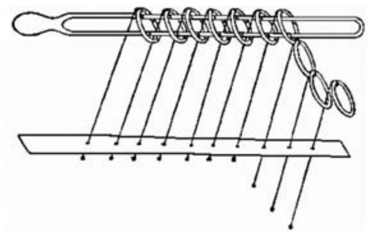
Figure 4. An 8-by-8 square and 5-by-13 rectangle made with the same pieces?



Topological Puzzles

Ring and string puzzles as well as knotted puzzles are examples of topological puzzles, where no discontinuous deformations like cutting the string are allowed. In his *De Viribus Quantitatis* (c. 1500), cited as the oldest book in recreational mathematics, Luca Pacioli (1445–1517) describes the Chinese Rings, a topological puzzle still popular in the twenty-first century.

Figure 5: A modern version of the Chinese Rings puzzle



Euler's name is linked to several puzzles. He solved the Bridges of Konigsberg Problem, and this work of his is usually seen as the starting point of topology and graph theory.

Figure 6: The Bridges of Konigsberg Problem: is it possible to cross all the bridges only once?

The concept of the Eulerian graph is rooted in Euler's resolution of the Bridges of Koenigsberg problem.

Movement Puzzles

Numerous puzzles involve patterned movement within some type of framework, and solutions sometimes involve mathematical techniques like numbering, recursion, group theory, and determinants. Peg Solitaire traces its origins from seventeenth-century France. It is a game where a board has all its holes occupied with pegs except for the central one. The objective is, making valid moves (small jump capture), to empty the entire board but for a solitary peg in the central hole (see Figure 7).

The Towers of Hanoi is a puzzle invented in 1883 by N. Claus, a pseudonym of the mathematician Edouard Lucas (1842–1891). A pile of discs of decreasing radius lays on one of three poles. Moving one disc at a time, without letting a bigger disc rest on a smaller one, the solver is asked to change the pile from one pole to another (see Figure 8).

Figure 7. Peg Solitaire: starting and target position.

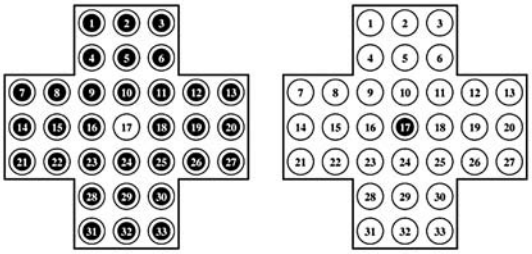
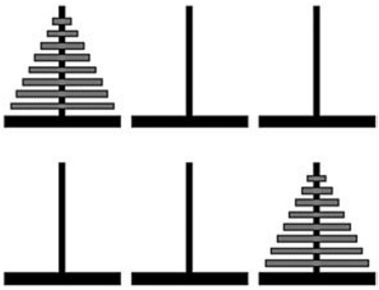


Figure 8. Towers of Hanoi: starting and target positions.

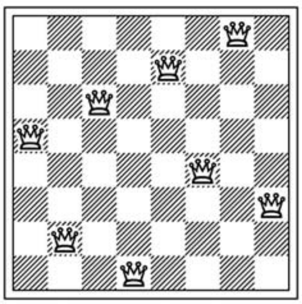


The recursive character of the solution to this puzzle makes it somewhat similar to the Chinese Rings.

Other Puzzles

The chessboard is a rich source of puzzles that attracted many mathematicians. In the Knight Tour problem, a knight must visit all the squares of the board just once. Euler is one mathematician who published a solution. Mathematician Johann Carl Friedrich Gauss (1777–1855) was attracted by the 8-Queen Problem, in which eight queens must be placed on a chessboard so they cannot capture any other queen. Some mathematicians have used determinants to solve this problem.

Figure 9. The 8-Queen Problem: one solution.



The nineteenth century produced a popular puzzle named "15." It consists of a sliding device, a 4-by-4 array with the numbers one through 15 and an empty cell. The puzzle was scrambled and the solver was required to transform the scrambled order back to the natural order with the empty cell in the last position. Sam Loyd offered $1,000 to whoever could reorder a scrambled 14 and 15 in an otherwise solved puzzle. The prize was never claimed. The impossibility of this challenge can be understood when phrased in the language of group theory.

Figure 10: The "impossible" task.

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

Another very mathematical puzzle that captivated the world was Rubik's Cube, created by Hungarian architect Erno Rubik in the 1970s that became the best selling puzzle in history. A 3-by-3-by-3 cube, with differently colored faces, moves by slices, getting scrambled with just a few moves. To find the way back to the starting position is an incredible challenge. This toy puzzle is used to illustrate many group theory concepts. On the other hand, knowledge of group theory facilitates the understanding of the puzzle itself.

Since ancient times, descriptions of "mazes" that must be traversed in a particular pattern of moves have abounded in legend and literature. The Minotaur–Theseus tale is one such example. Stone and hedge labyrinths may still be found in places like Europe and many puzzle books contain paper mazes. Some mazes can be understood using what is known as "level sequences."

The "jigsaw puzzle" was invented in England in the mid-1870s as a pedagogical device. Children were asked to rebuild maps. In the twentieth and twenty-first centuries, jigsaw puzzles expanded to include three-dimensional jigsaw puzzles, including spherical three-dimensional puzzles, and two-dimensional jigsaw puzzles that are all one color that have all the pieces cut to the same shape. This last style of puzzle is related to tiling. Another mathematical question is how to optimally and efficiently design and cut out puzzle pieces according to certain specifications.

Puzzle designer Scott Kim is considered by some to be a master of symmetry. He has diverse interests in many fields, including mathematics, computer science, puzzles, and education. When discussing these interests, he emphasizes the ties between them rather than their differences. One of his creations is an ambigram to honor of the great Martin Gardner (1914–2010), who invented many puzzles and is known for his recreational mathematics works. An ambigram is a figure that appears the same when rotated 180 degrees or viewed upsidedown.

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See Also: Acrostics, Word Squares, and Crosswords; Board Games; Coding and Encryption; Dice Games; Mathematical Puzzles; Optical Illusions; Sudoku.

Puzzles, Mathematical

See *Mathematical Puzzles*

Pythagorean and Fibonacci Tuning

Category: Arts, Music, and Entertainment.
Fields of Study: Algebra; Measurement; Representations.
Summary: The relationship between mathematics and music led to several tuning systems.

A musical scale is a sequence of ordered notes used to construct music compositions. Scales can be classified according to their starting point, the intervals between their notes, or the number of notes they contain. Instruments may be tuned according to many possible systems. There are close mathematical connections between musical scales, tuning systems, and number theory, as well as dynamical systems. Mathematics also plays a critical role in designing playable and efficient keyboards for instruments that will be tuned to something other than the standard eight-note Western scale.

Most Western music uses an eight-note "octave" scale (do, re, mi, fa, sol, la, ti, do), where the two "do" notes have the same tone but different pitches. The piano keyboard is set up in the C major key, where the white keys starting with C correspond to the eight notes in the octave.

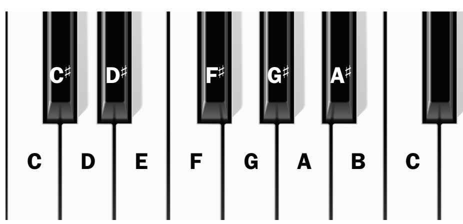

Figure 1.

F♯	G	G♯	A	A♯	B	**C**	**C♯**	**D**	**D♯**	**E**	**F**	**F♯**	**G**	**G♯**	**A**	**A♯**	**B**	**C**
11	3	14	6	17	9	**1**	**12**	**4**	**15**	**7**	**18**	**10**	**2**	**13**	**5**	**16**	**8**	**19**

There are also tones between some of the notes on the scale, represented on the piano by the black keys. Counting from C to B, there are 12 equal semitones in the chromatic scale of Western music.

To tune an instrument with strings, the lengths of the strings are adjusted to produce the correct pitch. Pythagoras of Samos (570–495 B.C.E.) is credited with realizing two things that allowed him to calculate the string lengths for the 12 semitones of the chromatic scale:

1. A string that is half as long produces the tone that is one octave higher. A string that is twice as long produces a tone that is one octave lower.
2. A string that is two-thirds as long produces a tone that is up five notes (called a *fifth*, or *do-sol* interval), seven semitones higher in the 12-tone chromatic scale.

Pythagoras saw that seven and 12 share no common factors and that he could use this fact to generate the lengths of all 12 strings in the chromatic scale.

1. Start with a string that sounds like a C note.
2. Cut a string that is two-thirds of the C string to give G.
3. Cut a string that is twice as long as G, yielding the same tone down an octave.
4. Cut a string two-thirds of this new lower G to give D.
5. Cut a string two-thirds as long as D to give A.
6. Cut a string twice as long as A, yielding A down an octave.
7. Cut a string two-thirds of the lower A to give E.
8. Cut a string two-thirds of E to give B.
9. Cut a string twice as long as B, yielding B down an octave.

Continue in this pattern, shortening a string to two-thirds to produce new higher notes and doubling the string when needed to avoid going past the top of the octave. After 19 steps, all of the strings of the C to C octave are determined, as well as a few extra notes below C (see Figure 1).

Called the "circle of fifths," this method of tuning by shortening the string to move up seven semitones (and back 12 when needed) would not work if the two numbers involved shared a common factor, such as four and 12. Not all of the semitones would be "hit" in that case.

Equal Tuning

Pythagoras was a little off when he assumed that a string two-thirds as long would produce the seventh semitone. In actuality, using irrational numbers (something Pythagoras did not believe in), the lengths of string needed to produce all of the semitones can be found more precisely. Starting with a string of length two, one can factor two into 12 equal parts or "twelfth roots." This method of tuning, used in the twenty-first century for most music, is called "equal tuning" (see Figure 2). The values of these irrational numbers to three decimal places show that the fifth note (or seventh semitone) string, G, is actually slightly more than two-thirds of the C string: two-thirds of a string of

Figure 2.

C	**C♯**	**D**	**D♯**	**E**	**F**	**F♯**	**G**	**G♯**	**A**	**A♯**	**B**	**C**
$\left(\sqrt[12]{2}\right)^{12}$	$\left(\sqrt[12]{2}\right)^{11}$	$\left(\sqrt[12]{2}\right)^{10}$	$\left(\sqrt[12]{2}\right)^{9}$	$\left(\sqrt[12]{2}\right)^{8}$	$\left(\sqrt[12]{2}\right)^{7}$	$\left(\sqrt[12]{2}\right)^{6}$	$\left(\sqrt[12]{2}\right)^{5}$	$\left(\sqrt[12]{2}\right)^{4}$	$\left(\sqrt[12]{2}\right)^{3}$	$\left(\sqrt[12]{2}\right)^{2}$	$\sqrt[12]{2}$	1
2	1.888	1.782	1.682	1.587	1.498	1.414	1.335	1.260	1.189	1.122	1.059	1

length 2 would yield a G string of length 1.333 rather than the equal tuning length of approximately 1.335. This little bit of difference is magnified when the circle of fifths technique is used to tune the strings, yielding notes that sound flat.

Other Tuning Systems

Between Pythagoras's time and the twenty-first century, a number of other tuning strategies were developed as music and mathematics knowledge grew. Popular in the medieval age, for example, was "just" tuning, which differs from both Pythagorean and equal tuning. To use equal tuning in the twenty-first century, one does not have to physically measure strings precisely; equipment can be used to measure the fundamental frequency (related to the pitch) of the sound wave generated by the string in order to tighten the string to the correct length.

There is also a method of tuning based on the Fibonacci series of Leonardo Pisano Fibonacci, which has been analyzed by English mathematician Sir James Jeans. The numbers in the musical Fibonacci series (2, 5, 7, 12, 19, . . .) can be generated by increasingly long series of musical fourths and fifths from the octave scale. An interval of two tones that are a fifth apart, such as F and C, have a frequency ratio of three-halves. The next fifth is a G, which is musically very close to the original F, but an octave higher, so the two-tone scale is left as F and C. Extending the fifths to a five-tone scale gives F, C, G, D, and A. This would be followed by E, which is again almost the initial F. A slight modification made by slightly raising all the tones (after the initial F) would create a five-note equal tuning scale. Increasingly larger scales can be made by continuing this pattern.

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Holly Hirst

See Also: Geometry of Music; Harmonics; Popular Music; Scales.

Pythagorean School

Category: Friendship, Romance, and Religion.
Fields of Study: Algebra; Communication; Geometry; Number and Operations.
Summary: Religious devotees of mathematics, the Pythagoreans could not accept irrational numbers but made lasting contributions.

The Pythagorean School is the name given to a number of mathematicians and followers of Pythagoras. Pythagoras founded the school in the sixth century B.C.E. in what is now southern Italy. It appears to have been a religious sect built around the proposition that reality was revealed through numbers. It was one of the earliest philosophical schools, and at the time there were no rigid boundaries between philosophy, religion, and politics.

The school had aspects of all three and was a major political force in some Greek cities. To some extent, it was thought of as a secret society. Initiates are said to have taken a vow of silence. This fact and many others about the school are difficult to verify because of a lack of sources from this time. Most of what is known about the Pythagoreans comes in fragments from later philosophers like Plato or Aristotle. Much of the detail about Pythagoras's life is revealed from even later sources, in the works of Diogenes Laertius, Iamblichus, and Porphyry, who wrote many centuries after his death. As a result, much information about the school is ancient hearsay that embellishes what was already a peculiar belief system. The Pythagorean habit of attributing discoveries to Pythagoras, as well as the silence, also makes it hard to distinguish the discoveries of the man from his school. Nevertheless, the influence of his school and mathematical philosophy can still be felt in the twenty-first century concept of "the liberal arts."

Pythagoras and the Foundation of the School

Pythagoras lived from around 580 to 500 B.C.E., but the exact dates are uncertain. He was the son of a leading citizen of Samos (an island in the Aegean), and it is possible that his political significance led Pythagoras to leave the city during the rule of Polykrates the Tyrant. He does not seem to have become prominent until around 530 B.C.E. in the city of Croton, on the southern shore of Italy. The ancient authors account for his life before then by a journey gathering the wisdom of other cultures, such as the Egyptians and Babylonians.

The wisdom he gained is said to have given him many powers. For example, he claimed to recall his previous incarnations, such as his life as the Trojan hero Euphorbus. He is also said to have appeared talking to friends at Metapontium in southern Italy and Tauromenium, on Sicily, on the same day, despite this being impossible with the transport of the day. The same chapter of Porphyry's *Life of Pythagoras* also recounts that a river spoke very clearly to say "Hail Pythagoras!" as he crossed it. Some ancient authors, such as the philosopher Heraclitus, were unconvinced. These and similar tales show not only that he was seen as a divine figure in the ancient world but also that the ancient sources are not wholly reliable. Some modern historians go so far as to discount any mathematical achievements being the work of the actual Pythagoras. Instead, they argue that the achievements of Pythagoreans were attributed to Pythagoras to add luster to his memory.

The school was extraordinarily egalitarian for its era. It admitted both men and women at a time when women were not considered citizens and were usually treated in the same manner as children. The school spread as a society throughout southern Italy and seems to have become a potent political force. Eventually, the power of the school was challenged by the non-Pythagoreans, and violence ensued. Polybius, writing in the second century B.C.E., described the chaos as being a maelstrom of murder, sedition, and "every kind of disturbance." There are several conflicting stories of the death of Pythagoras, but the oddest is that it occurred because he refused to cross a field of beans when an angry mob was chasing him. This behavior was eccentric even by the standards of ancient Greece and only makes sense in light of the Pythagorean beliefs taught at the school.

Pythagorean Beliefs

Pythagoreans believed that numbers were a fundamental property of the universe and that the cosmos operated in harmonies that could be represented as ratios of whole numbers. The purpose of life was to achieve harmony with the universe through a process of purification to counter the corrupting influence of the body. One of the features of this purification was that Pythagoreans were vegetarian—a strong political statement. At this time, one of the duties of a citizen was to participate in civic religious events. Avoiding such events or refusing to perform them properly could draw the ire of the gods. Almost all festivals required the sacrifice of an animal, usually an ox or a goat. The fat and bones would be offered to the gods on the altar and meat would be part of a communal meal. A vegetarian was therefore separating himself from the community.

As for the material that made the cosmos, Pythagoras thought it was governed by numbers. He is said to have come to this conclusion after discovering that musical harmonies can be represented as ratios of whole numbers. The connection between two such different practices such as music and mathematics led Pythagoras to believe that there must be something cosmically significant about numbers. These ratios are embedded in the tetractys symbol—a triangle of 10 dots in four rows, one dot at the top, then two dots, then three, and finally four. The ratios of the motions of the planets were also assumed to be harmonious, and it is said that Pythagoras claimed to be able to hear "the music of the spheres," the harmonies generated by these motions. Numbers that could not be represented by ratios of whole numbers were therefore a serious problem in Pythagorean cosmology.

The Pythagorean Legacy

It is hard to be sure that the theorem that bears his name was actually a Pythagorean concept. While 3-4-5 triangles were used before Pythagoras's time, he may have been the first to prove the Pythagorean theorem, or this might be a later proof attributed to the inspiration of the school. However, there is reason to consider the interest in irrational numbers to be a Pythagorean innovation. Quite how this was discovered is uncertain. The Pythagorean theorem can be used to prove that $\sqrt{2}$ is irrational, but irrationality can also be found in the "pentalpha," a five-pointed star more commonly known as "pentagram," adopted as a symbol by the Pythagoreans. The discovery of irrational numbers is sometimes credited to Hippasus of Metapontum. Usually in these tales, Hippasus meets a grisly end at the hands of Pythagoras who resents the existence of irrational numbers. While this might be a fantastical tale, it is believed that the Pythagoreans were sworn to secrecy concerning the existence of irrational numbers because it was a significant threat to their belief system.

A celebrated legacy of the Pythagorean school is that its approach to applying mathematics to the natural world led to the establishment of the quadrivium: arithmetic, astronomy, geometry, and music that, with grammar, logic, and rhetoric, formed the "liberal arts" that were the foundation of medieval university courses. While the philosophy of liberal arts has changed in modern times, mathematics remains an important feature, as it can be found in many areas in higher education.

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Alun Salt

See Also: Greek Mathematics; Harmonics; Numbers, Rational and Irrational; Pythagorean and Fibonacci Tuning; Pythagorean Theorem.

Pythagorean Theorem

Category: History and Development of Curricular Concepts.
Fields of Study: Communication; Connections; Geometry.
Summary: The Pythagorean theorem is a fundamental theorem of mathematics and has numerous applications in number theory and geometry.

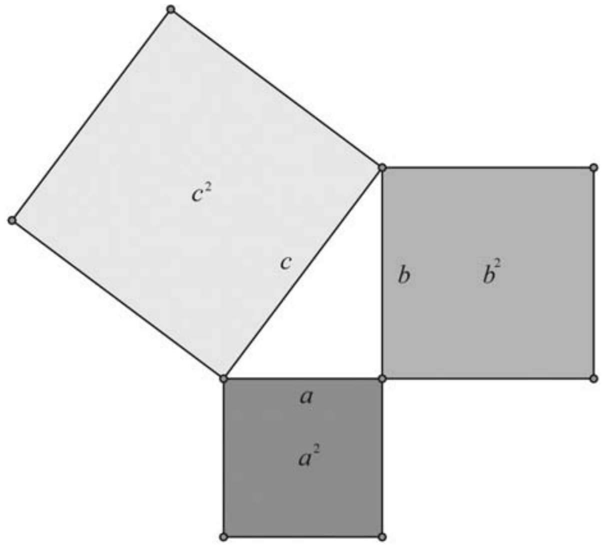
The Pythagorean theorem stands as one of the great theorems of mathematics. Ancient peoples appear to have used the Pythagorean theorem to calculate the duration of lunar eclipses or to create right angles in their pyramids or buildings. Archeological evidence suggests that the truth of the result was known in Babylon more than 1000 years before Pythagoras, approximately 1900–1600 B.C.E. Mathematicians and historians continue to debate the early history of the theorem and whether it was discovered independently in such places as Mesopotamia, India, China, and Greece. For instance, some theorize that Pythagoras may have learned the theorem during a visit to India, which in turn may have been influenced by Mesopotamia. The theorem is the culminating proposition of the first book of Euclid's *Elements*. While Euclid (c. 350 B.C.E.) did not mention Pythagoras, later writers such as Cicero and Plutarch referred to it as his discovery. As phrased in the twenty-first century, the theorem states the following:

In any right triangle, the square of the hypotenuse c is equal to the sum of the squares of the legs a and b. That is, $a^2 + b^2 = c^2$.

The theorem has inspired countless generations, and it is useful in a wide variety of contexts and applications, such as in chemistry cell-packing and music.

In Pythagoras's day, humankind had not yet invented algebra. As such, this theorem was not viewed with algebraic perspective but rather in a distinctly geometric way. Visually, as shown in Figure 1, on the right triangle with legs a and b and hypotenuse c, the sum of the areas of the darker gray squares is equal to the area of the lightest gray square.

Figure 1.



Proofs

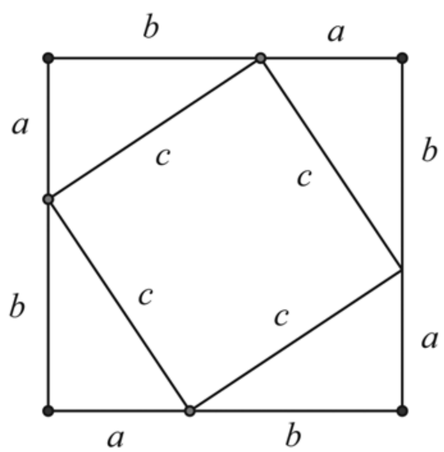
Among the many remarkable features of the Pythagorean theorem, one of the most prominent is that the result admits so many different proofs, including one by former U.S. President James Garfield in 1876. Some of the shortest representations of the Pythagorean theorem are geometric figures called "dissections." For example, Indian mathematician Bhaskara's dissection figure was accompanied by the word "Behold." The Chinese also presented dissection figures that are now called "Pythagorean," and some theorize that these may have led to the development of tangram puzzles. Complete Pythagorean proofs based on dissection figures often combine algebra and geometry.

Given a right triangle with legs of length a and b, construct a square of side length $a+b$. Then, along each side, mark a point that lies a units along the side. If consecutive pairs of these points are connected with line segments, four identical (congruent) copies of the original triangle have been constructed inside the large square (see Figure 2).

In addition, these four line segments have generated a quadrilateral (a four-sided polygon) in the interior of the large square. This quadrilateral's sides each have length c, which is the hypotenuse of the given right triangle. Further, a straightforward argument involving angle measurements in the triangles shows that each of the four angles in the interior quadrilateral measures 90 degrees. Hence, the inside quadrilateral is in fact a square.

Consider the area of Figure 2 in two different ways. First, the area A of the entire outside square, which has sides of length $a+b$, must therefore be $A=(a+b)^2$.

Figure 2.



At the same time, one can view the area of the outside square as having been subdivided into five parts. Four of those pieces are congruent right triangles whose area is each $ab/2$. The fifth part is the interior square, whose area is c^2. Thus, the area A of the outer square also satisfies the relationship that

$$A=\frac{4ab}{2}+c^2.$$

Equating the two different expressions for A, one finds

$$(a+b)^2=\frac{4ab}{2}+c^2.$$

Expanding the left side and simplifying the right, it follows that $a^2+2ab+b^2=2ab+c^2$.

Finally, subtracting $2ab$ from both sides, the conclusion of the Pythagorean Theorem follows: $a^2+b^2=c^2$.

Applications

Furthermore, the Pythagorean theorem is rightly viewed as one of the most central results in Euclidean geometry. Its statement is equivalent to Euclid's parallel postulate, and therefore is directly tied to the truth of a large number of other key results.

In addition to the geometric ideas the Pythagorean theorem evokes, it generates key new ideas and questions about numbers. For instance, if one takes the legs of a right triangle to each have length 1, then it follows that the hypotenuse c is a number such that $c^2=2$. There is no rational number (that is, no ratio of whole numbers) whose square is 2. This situation forced Greek mathematicians to reconsider their original conviction that all numbers were "commensurable": that any possible number must be able to be expressed as the ratio of whole numbers. Remarkably, it took mathematicians another 2000 years to put the so-called real numbers, the set of numbers on which calculus is based, on solid footing.

Another Pythagorean idea that has generated a remarkable amount of mathematics is the notion of a "Pythagorean Triple," which is an ordered triple of whole numbers like (3, 4, 5) that represents a solution to the Pythagorean theorem, since $3^2+4^2=5^2$. A Babylonian clay tablet, named the "Plimpton 322 Tablet," contains many Pythagorean triples. Some suggest that these were a set of teaching exercises, though historians and mathematicians continue to debate their role. Euclid is credited with the development of a formula that will generate a Pythagorean triple, given any two natural numbers. Indeed, there are even infinitely many "primitive" Pythagorean triples, triples in which *a*, *b*, and *c* share no common divisor. Algebraic extensions include investigating solutions to Pythagorean-like equations with other powers, such as $a^3 + b^3 = c^3$. Remarkably, no three positive numbers satisfy such equations; Pierre de Fermat, a French lawyer in the seventeenth century, wrote this note (as translated by historians) in the margins of Diophantus of Alexandria's *Arithmetica*:

> I have discovered a truly marvelous proof that it is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second into two like powers. This margin is too narrow to contain it.

No one ever discovered Fermat's proof, yet Fermat's Last Theorem stimulated the development of algebraic number theory in the nineteenth century, and many results in mathematics were shown to be true if Fermat's Last Theorem was true. Andrew Wiles finally proved it to be true near the end of the twentieth century.

There are many other extensions of the Pythagorean theorem. Pappus of Alexandria generalized the theorem to parallelograms. In the 1939 film *The Wizard of Oz*, the Scarecrow recites a version using square roots instead of squares. The Scarecrow's theorem is false in planar geometry, but it can hold in spherical geometry. However, the Pythagorean theorem does not hold on a perfectly round planet. In this case, $a^2 + b^2 > c^2$. Writers for the animated television show *Futurama* named this the Greenwaldian theorem, after mathematician Sarah Greenwald. In the twenty-first century, physicists and mathematicians investigate whether the Pythagorean theorem holds in our universe.

The Pythagorean theorem is also a fundamental idea in several other areas of mathematics and applications. Essentially all of plane trigonometry rests on the Pythagorean Theorem as its starting point, and the modern notion of "orthogonality" in linear algebra is an extension and generalization of the work of Pythagoras. Both trigonometry and orthogonality lead to a wide range of interesting and important applications, including the theory of wavelets and Fourier analysis, mathematics that enables prominent image compression algorithms to help the Internet function.

Its own inherent beauty, the multitude of possible proofs, the rich mathematical ideas it spawns, and the applications that follow all contribute to making the Pythagorean theorem one of the genuine masterpieces in all of mathematics.

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Matt Boelkins

See Also: Geometry and Geometry Education; Geometry of the Universe; Mathematics, Elegant; Parallel Postulate; Pythagorean and Fibonacci Tuning; Pythagorean School; Wiles, Andrew.

Q

Quality Control

Category: Business, Economics, and Marketing.
Fields of Study: Data Analysis and Probability; Measurement.
Summary: Industrial productions and processes can be mathematically studied to help ensure quality.

Statistical quality control, or more broadly, quality assurance, seeks to improve and stabilize the production and delivery of goods and services. A central concern of quality control is the testing and reporting of measurements of quality—typically as part of a monitoring process—to ensure that the quality of the item being studied meets certain standards.

Quality standards are determined by those who produce the goods or services. Some standards are specification limits imposed by engineering or design concerns that define conformance to a standard. For example, in making airplane engines, a certain part may need to have a diameter between 12 and 14 millimeters or it will not fit into a housing. However, for many processes, there are no specification limits and quality standards may be defined internally from data on past behavior of a process that is judged to be "in control" or "stable." For example, in examining the safety of a large production line, it may be that in each week of the last five years, the average number of person hours lost to accidents has been 1.3. There is no specification limit for this quantity, but control limits can be based on this historical average.

In order to analyze a process for statistical quality control effectively, a process must first be declared to be "in control." To be in statistical control, the vast majority of the products or services must be of sufficient quality for the producers to be satisfied. Moreover, the process must be stable (the mean and variance of the quality measurements must be roughly constant). If a process is in control, then statistical analysis can provide meaningful control limits to the process for monitoring. Graphical methods play a significant role in statistical quality control.

History

Some measure of quality control was in evidence during the building of the Great Pyramids of Egypt. Archeologists have long been impressed not only with the complexity of the construction process, but also by its precision. In the Middle Ages, medieval guilds were formed, in part, to ensure some level of quality of goods and services. The use of statistical methods in quality control—also called "statistical process control" or (SPC)—is more recent, with most of the development in the twentieth century. Graphical methods for quality control were introduced in a series of memos and papers in the 1920s by Walter E. Shewhart of Bell

Telephone Laboratories. The charts he developed and promoted are known today as "Shewhart control charts." H. F. Dodge and H. G. Romig, also of Bell Laboratories, applied statistical theory to sampling inspection, defining rules for the acceptance of many products. Joseph M. Juran, whose focus was more on quality management, rather than SPC, was another early quality pioneer at Bell Laboratories and later Western Electric.

W. Edwards Deming applied SPC to manufacturing during World War II and was instrumental in introducing these methods to Japanese industry after the war ended. He and Juran are generally credited with helping Japanese manufacturing shed the negative image that "made in Japan" had in the 1950s and transforming the country into a source of high quality goods consumed all over the world. In the early twenty-first century, quality control issues continue to appear in the media as concerns proliferate over the quality of goods produced in China.

Common-Cause and Special-Cause Variation

Shewhart and Deming defined two types of variation that occur in all manufacturing and service processes in their 1939 book *Statistical Methods from the Viewpoint of Quality Control*. A certain amount of variation is a part of all processes and can be tolerated even when the goal is to produce goods and services of high quality. This variation is called "common-cause variation," and it comprises all the natural variation in the process. The second variation, called "special-cause variation," is unusual and is not part of the natural variation. Special-cause variation needs to be detected as soon as possible. Quality control charts are designed to detect special-cause variation and distinguish it from common-cause variation.

Quality Control Charts

A quality control chart plots a summary of the quality measurements from each item (or a sample) in sequence against the sample number (or time). A center line is drawn at the mean, or at the desired center of this statistic. Upper and lower control limits are drawn indicating thresholds above or below which will signal an "out of control" measurement. Sometimes, various warning lines are drawn as well, and a variety of rules for deciding if the measurement is really out of control are available. The simplest chart, called an "individual" (or "runs") "chart," plots a single measurement for each item. The control limits are based on the Normal probability model, which implies that for a process in control, only 0.27% of the observations will lie more than three standard deviations (σ) from the center. Therefore, if the process stays in control, a false alarm will occur only once in about 1/0.0027 or once every 370.4 observations. The central idea of a control chart is that a special cause will cause the mean to shift (or the standard deviation to increase), and so the measurement will fall outside the 3σ limits with higher probability. If the shift is great enough, the time to detection will be very short. However, if the special cause results in a subtle shift, it may take many observations before such a signal is detected. Various other types of charts are available that have generally better performance in terms of both false alarm rates and failure to detect shifts.

Total Quality Management and Philosophy

The ideas of Deming, Juran, Shewhart, and others have inspired numerous other people and quality movements. One such movement is total quality management (TQM) also known as "total quality" and "continuous quality improvement." As the name implies, this approach to quality involves more than the monitoring of manufacturing or service processes. It includes all parts of the organization and, specifically, the role of management to help ensure that in providing goods or services, that "all things are done right the first time." Implementing these ideas throughout a large organization gave rise to an abundance of books, experts, and quality "gurus" in the latter part of the twentieth century. One approach to total quality focuses on reducing variation (decreasing σ). If the common-cause variation can be reduced enough, while the process is in control, essentially no measurements will fall outside the 3σ limits. This notion is the essential idea behind the 6σ approach, first popularized by the Motorola company and later the General Electric Company in the 1980s. By the late 1990s, a majority of the Fortune 500 companies were using some form of the 6σ approach.

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See Also: Deming, W. Edwards; Normal Distribution; Scheduling; Water Quality.

Quilting

Category: Arts, Music, and Entertainment.
Fields of Study: Geometry; Measurement; Representations.
Summary: Quilting can incorporate and help teach mathematical concepts, such as symmetry and tessellations.

Quilting is a needlework technique in which two layers of fabric are sewn together, usually with an inner layer of padding (called "batting") between them. Often, one or both outer layers are formed by sewing together (or "piecing") smaller pieces of fabric. Sometimes, designs are appliquéd (sewn onto a larger piece of fabric) or embroidered on the quilt. The quilting itself (the stitches holding the layers together) is often also decorative. Many traditional quilt designs display mathematical concepts, such as symmetry and tessellations, that generalize into the abstract mathematics of group theory and tiling theory. In diverse parts of the world, people create quilts not only to warm the body at night, but also to use as clothing, furnishings, or to share family or cultural history. A carving of an ancient Egyptian Pharaoh figure containing what may be a quilt and a quilted carpet found in the mountains of Mongolia dates to approximately the first century. Directions can be found to quilt coded designs that may have been used on the Underground Railroad.

Quilt Designs

Some traditional quilts are "crazy quilts" in which scraps of fabric are sewn together in no particular pattern. Others are formed of similar or identical square "blocks," each of which may be pieced together. Often, quilt patterns involve careful measurement (using common fractions) in the cutting and sewing of the pieces.

Quilt designs are often symmetrical—the entire design can be folded in half along a line such that one half falls directly onto the other half. Each half is a reflection of the other along that line, which is called a "line of symmetry." These lines may be vertical, horizontal, or diagonal. Some quilt blocks, such as the traditional Amish Star, are symmetric along many lines. Quilts and quilt blocks may also have rotational symmetry—the design can be rotated around a point through less than a full rotation in a way that leaves the overall design unchanged. Quilts in the Hawaiian Islands are known for their distinctive radial symmetry.

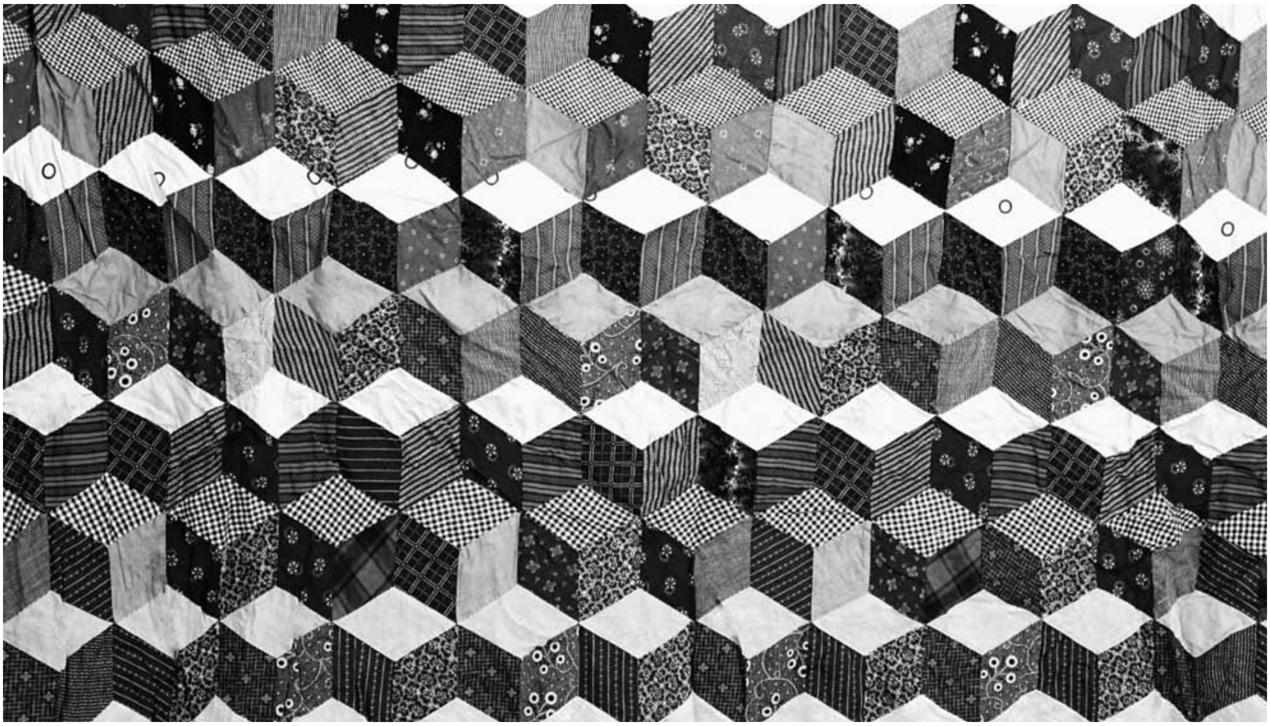
Mathematics generalizes this everyday concept of symmetry. A mathematical object (not necessarily a geometric shape) is symmetric with respect to a particular mathematical operation if the operation, applied to the object, preserves some property of the object. A mathematical group consists of a set of operations that preserve a given property of a given object. Group theory is central to abstract algebra and has many applications.

Fabric quilts, construction paper versions, or computerized models of quilt designs have been used to introduce students as early as elementary school to geometric concepts, such as symmetry and transformations. They help children develop, at a basic level, fundamental algebraic properties, such as inverse, identity, and equivalence. Students also make quilts to explore many other concepts, such as the Pythagorean theorem, polar coordinates, group theory, the Fibonacci sequence, and Pascal's triangle, named after mathematician Blaise Pascal.

Tessellations

A tessellation (or tiling) is an infinitely repeating pattern composed of polygons covering a plane without any openings or overlaps. Many quilt designs are formed from tessellations. A regular tessellation uses one polygon with equal sides and equal angles, such as equilateral triangles, squares, or regular hexagons. For example, the traditional Grandmother's Flower Garden and Honeycomb quilt designs use tessellations of regular hexagons. Many modern watercolor quilts use tessellations of one-inch squares.

A semi-regular tessellation uses a combination of squares, triangles, and hexagons that are arranged



Many traditional quilt designs display mathematical concepts, such as symmetry and tessellations, that generalize into the abstract mathematics of group theory and tiling theory.

identically around each vertex. Demi-regular tessellations, with two vertices in each repetition, form more complicated quilt patterns. Many quilt blocks, such as Log Cabin variations, consist of non-regular tessellations.

Mathematicians have generalized tiling theory to higher dimensional Euclidean spaces and to non-Euclidean geometries. These generalizations reveal links to group theory and to classical problems in number theory. Much of the art of M.C. Escher is based on non-Euclidean tessellations.

Other Designs

Contemporary quilters like mathematician Irena Swanson have also incorporated other mathematical concepts in their designs, such as infinite geometric series and fractals, as well as portraits of mathematicians. Mathematician Gwen Fischer created quaternionic quilts to visually showcase the algebraic structure of the group. For example, the lack of reflection symmetry across the main diagonal highlights the lack of commutativity of the group elements.

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See Also: Escher, M.C.; Symmetry; Textiles; Transformations.

R

Racquet Games

Category: Games, Sport, and Recreation.
Fields of Study: Algebra; Data Analysis and Probability; Geometry.
Summary: The equipment, game play, and scoring of racquet sports can be analyzed using mathematical concepts, such as vector operations and probability.

Racquet games include sports such as tennis, badminton, squash, and table tennis, as well as other less popular games like real tennis, racquets, and racquetball. Mathematics has many roles to play in these games—from equipment testing and court marking to training and analysis of play.

For example, the scoring system in tennis is not a simple counting or linear progression. Mathematicians model a ball's spin in multiple axes, along with trajectories and deflections, as functions of other variables. Markov chains and vector operations can be used to analyze the progression of games and both probability and statistical methods are used to describe performance, seed players for competition, and predict outcomes of matches.

Racquets

Racquet weight distribution, shape, and string material are important factors in the resultant power, accuracy, and comfort of a racquet. Increasing power, for example, can lead to a decrease in accuracy and it is important to balance these properties. Computer-aided design is the natural choice for this process because of its fast and powerful recalculation abilities.

Projectiles

Racquet sport projectiles such as balls and shuttlecocks are subject to strict regulations and must adhere to these for as long as possible at the highest levels of play. For example, the World Squash Federation allows balls that are 40 millimeters in diameter and each must be tested at 23 degrees Celsius (73 degrees Fahrenheit) and 45 degrees Celsius (113 degrees Fahrenheit), room temperature and play temperature, respectively. There are several dot grades according to level of rebound but an average squash ball rebounds at around 30% (dropped from a height of 3.2 feet, it should reach 12 inches on the bounce). A tennis ball rebounds at around 50%, although changes in ambient air pressure (because of altitude) can affect this figure. Table tennis balls rebound at 85%.

A popular way to gauge the overall performance of these projectiles is to measure their maximum speed. Tennis balls seem to hold the record for the being the fastest, and indeed Andy Roddick can propel a tennis ball very fast (152 miles per hour). However, the fastest badminton stroke left the racquet at over 186 miles per hour. This figure seems counterintuitive because a shuttlecock slows down much more quickly than a tennis ball.

Training

One of the most important roles for mathematics in racquet sports is in training. Sports science researchers study muscle and joint strain and develop nutritional guidelines that allow the player to remain comfortable and energetic during play. Of the racquet sports, squash is regarded as the most intense—players burn roughly 50 percent more calories per hour than badminton or tennis. However, tennis games can run several hours, whereas badminton and squash games are typically decided in under an hour. The total number of calories burned is the product of the calories per hour and the number of hours.

Scoring

In all of the major racquet sports (and many others), a feature of the scoring system may mean that the player who wins more individual points or rallies can still lose the match. Consider the scores of the 1972 British Open final decided by the best of five games, each played to nine points: 0–9, 9–7, 10–8, 6–9, and 9–7. The loser (Geoff Hunt) scored 40 points and won two games; the winner (Jonah Barrington) scored 34 points, won three games and the title.

The same quirk appears in any scoring system where victory is decided by the most wins over a specific number of games. In tennis, this feature exists on two levels. It is possible to win more points and more games but still lose the match. For example, if a match ends 6–4, 0–6, 6–4, 0–6, 6–4, the winner wins 18 games, the loser wins 24 games. The maximum difference in points or rallies in this case is 60 (72–132) in favor of the loser.

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See Also: Hitting a Home Run; Hockey; Probability; Rankings; Tournaments.

Radar

See *Doppler Radar*

Radiation

Category: Weather, Nature, and Environment.
Fields of Study: Algebra; Data Analysis and Probability; Measurement; Number and Operations.
Summary: Radiation research has a heavy mathematical component, especially in modeling distribution of or shielding from radiation.

Radiation is the transmission of energy via waves or particles, such as energetic electrons, photons, or nuclear particles. These waves or particles, called "quanta," travel radially in all directions from the source, leading to the name "radiation." Radiation exists everywhere, from both natural sources, like the sun, and many manmade sources, like radio stations and particle accelerators. The various types of radiation that exist may be harmful or beneficial to people, depending on source and application. Ionizing radiation contains enough energy per quantum to detach electrons from atoms, like X-rays or the radiation emitted by particle accelerators. High energy particles are created constantly by all luminous objects in the universe. Most of these particles never reach the surface of Earth. They may be deflected by magnetic fields or interact with atmospheric particles. Common types of nonionizing radiation include visible light, radio waves, and microwaves.

Many mathematicians have contributed to radiation research, like Wilhelm Wien, who derived a distribution law of radiation and won a Nobel Prize for his work on heat radiation. Physicist Max Planck used some of Wein's mathematics as the basis for quantum theory. Paul Ehrenfest contributed to quantum statistics, in part by applying Plank's quantum theory to rotating bodies. Subrahmanyan Chandrasekhar won the Royal Society Copley Medal for his work in mathematical astronomy, including the theory of radiation. Victor Twersky was widely regarded as an expert on radiation scattering. His work has been used in diverse applications, such as studying the effect of atmospheric dust on light propagation. Mathematicians continue to work on radiation problems, including applications such as detecting radiation or shielding satellites from the harmful effects of cosmic radiation, as well as creating mathematical methods for formulating and investigating radiation problems, such as Monte Carlo simulations.

Electromagnetic Radiation

Electromagnetic radiation (EMR) includes both ionizing and nonionizing forms of radiation. EMR waves result from the coupling of an electric field and a magnetic field. The fields are perpendicular to one other and to the direction of energy propagation. Electromagnetic radiation behaves like both a wave—with properties including reflection, refraction, diffraction, and interference—and a particle, because its energy occurs in discrete packets or quanta. Maxwell's equations, named for physicist and mathematician James Maxwell, are cited as the most elegant way to express the fundamentals of electromagnetism. The set of four equations, which have integral and differential forms include: Gauss's laws for electricity and magnetism, named for mathematician Carl Freidrich Gauss; Faraday's law of induction, named for physicist and chemist Michael Faraday; and Ampere's law with Maxwell correction, named for physicist and mathematician Andre-Marie Ampere. Many have derived theories and applications from these building blocks, such as mathematician Josef Stefan, who showed that total radiation from a blackbody is proportional to the fourth power of its absolute temperature.

Properties

Properties of radiation waves can be used to determine their potential effects on people and objects or their usefulness for applications. Wavelength is the length of one cycle of the wave, or the distance from one peak to the next. Frequency is the number of cycles of the wave that travel past a fixed point along its path per unit time. All electromagnetic waves travel in a vacuum at a speed of about 3×10^8 meters per second. A fundamental relationship between wavelength and frequency is that wave speed is the product of wavelength and frequency, which means that greater wavelengths correspond to lower frequencies. The energy of electromagnetic photons is the product of wave frequency and Planck's constant, so higher frequencies produce greater photon energies. Among the common types of EMR radiation, radio waves have the longest wavelengths, resulting in low frequencies and low energies. Higher frequency ultraviolet radiation has the most energy and is the most harmful component of the cosmic radiation that penetrates Earth's atmosphere. X-rays, discovered by physicist Wilhelm Röntgen, occur naturally when solar wind is trapped by Earth's magnetic field in the Van Allen belts, named for physicist James Van Allen.

Black holes are also sources of X-rays in the universe. While photons have no mass, some forms of radiation are particles with positive mass produced in the atomic decay of radioactive materials. For example, beta radiation is composed of high-energy electrons, which are dangerous because they can penetrate skin to the layer where new cells are produced. Mathematician Jesse Wilkins's work on mathematical models to compute the penetration and absorption of electromagnetic gamma rays has been used in the design of nuclear radiation shields.

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See Also: EEG/EKG; Elementary Particles; Energy; Light; Medical Imaging; Microwave Ovens.

Radio

Category: Communication and Computers.
Fields of Study: Algebra; Measurement; Representations.
Summary: Radio waves have numerous applications and are described, analyzed, encoded, and "jammed" using mathematics.

Radio is a means of sending information by transmitting signals using radio waves, which are a type of electromagnetic radiation with frequencies in the spectrum of approximately 3 kilohertz (kHz) or 1000 cycles per second, to 300 gigahertz (GHz), or 1 billion cycles per second. These units are named for German experimental physicist Heinrich Hertz. Radio waves are used not only to carry radio and television signals but are also used in many other common technologies including wireless computer networks, wildlife tracking systems, cordless and cellular phones, baby monitors, and garage door openers. One interesting way that mathematics connects to radio is through mathematically based radio shows, like *Math Medley*, which was hosted by Patricia Kenschaft. Mathematicians have also spoken on programs like National Public Radio's *Science Friday*.

Radio waves are sinusoidal, meaning that they are characterized by a smooth, repetitive oscillation whose function at time t can be described algebraically as

$$y(t) = (A)\sin(\omega t + \phi)$$

where A is the wave's amplitude (peak deviation), ω is the wave's angular frequency (described in radians per second), and ϕ is the wave's phase (where the wave cycle is at time $t = 0$).

Brief History and Unique Properties

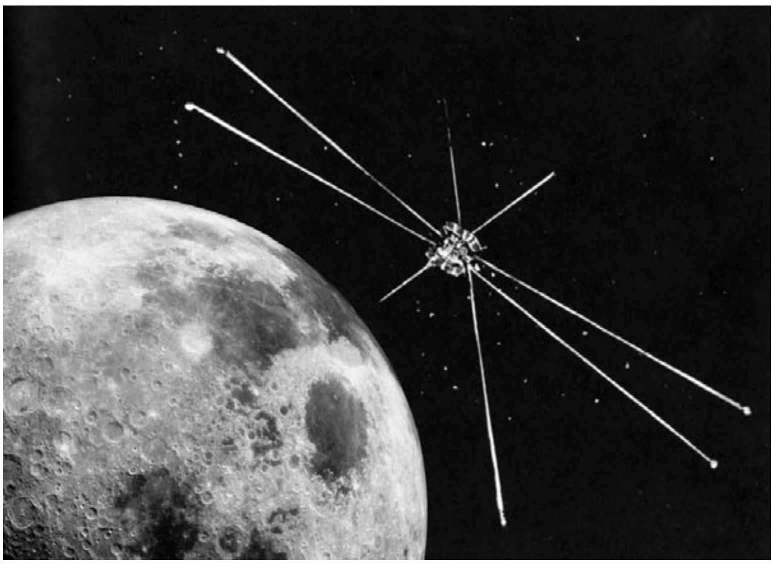
In 1864, the British physicist James Clerk Maxwell predicted the existence of radio waves as part of his theory of electromagnetism. Hertz confirmed Maxwell's theory between 1886 and 1888 and is generally credited with being the first person to send and receive radio waves. Several individuals played an important role in developing a practical system of radio transmission including the Serbian-American engineer Nikola Tesla, who demonstrated wireless radio communication in 1893; the British physicist Oliver Lodge, who demonstrated the transmission of Morse Code using radio waves in 1894; and the Italian physicist Guglielmo Marconi, who in 1896 was granted the first patent for a radio. Radio communications between ships and coastal stations were in use by 1897, and the first radio time signal (used to synchronize clocks) was transmitted from a U.S. Naval Observatory clock in 1904.

Radio waves may be broadcast over long distances because of the Heaviside Layer (also called the "Kennelly–Heaviside layer"), a conducting layer in the ionosophere predicted independently in 1902 by the British mathematician and physicist Oliver Heaviside and the British physicist Arthur Edwin Kennelly. The existence of the Heaviside Layer was established in 1924 by the British physicist Edward Appleton, who also determined that the height of this reflective layer was about 100 kilometers (62 miles) above the Earth's surface. The Heaviside Layer allows radio signals to follow the curvature of the Earth (rather than disappearing into space) because they are reflected by the Heaviside layer and thus "bounce back" to Earth.

Applications

Radio astronomy, which led to the discovery of objects such as pulsars and quasars, dates from the 1931 discovery by American physicist Karl Guthe Jansky of radio waves emitted from the Milky Way galaxy. American astronomer Grote Reber created the first radio frequency sky map in 1941, and in the 1950s, the British astronomers Martin Ryle and Antony Hewish produced two notable catalogues of celestial radio sources.

Historically, most radio broadcasts used one of two techniques for sending their signals: amplitude modulation (AM) or frequency modulation (FM). AM is the older technology (the first AM broadcast took place in 1906) and it was the dominant radio technology for most of the twentieth century. AM encodes information by modifying the amplitude of the transmitted signal. The technology for FM broadcasting, which encodes information by varying the frequency of the transmitted signal, was developed in the 1930s and became common by the late 1970s. The information in these analog signals is inherently part of the signal itself—the information influences the wave's shape, and thus information loss can occur with any disruption of the signal. One example is the audible static that occurs when a radio receiver begins to travel beyond the range of a radio transmitter. In the twenty-first century, digital modulation has been increasingly used to minimize this problem. Digital modulation transfers digitized information using a broad spectrum of radio frequencies—far more than the AM or FM systems. Further, each signal is sent many times, reducing the chance of interference and signal loss because separate bits from many streams may be pieced together. Further, since the radio waveforms are not altered by the information, multiple signals may be carried at the same time in the form of one composite signal that is decoded by the receiver, a technique called "multiplexing." Satellite radio systems take advantage of multiplexing and the wider angle of coverage to offer many hundreds of specialized channels across broad geographic areas. Television is also transitioning from analog to digital signals.



The Radio Astronomy Explorer was a radio telecope placed in a moon orbit in 1973 to obtain radio measurements of the planets.

Radio transmissions are used for communication during wartime, but because a radio signal may be picked up by anyone with a receiver, various coding methods have been developed. One famous example is the code talkers used by the American Army during World War I and World War II. This program capitalized on the fact that Native-American languages such as Navajo and Choctaw were almost unknown outside those tribes and also developed a simple code for terms like "tank" and "submarine," which allowed them to code and encode messages rapidly and with little risk of comprehension by the enemy. Also in World War II, the German Army used mechanical circuits to encrypt information. Although supposedly unbreakable because of the large number of combinations possible, the British mathematician William Tutte was able to deduce the pattern of the encoding machines after British intelligence intercepted two long coded messages, each of which was transmitted twice (the second time with corrected punctuation).

Interference

Radio waves can be blocked by weather formations, geographic features, and many other natural phenomena. Further, if several stations are broadcasting on a similar frequency, they may interfere with each other. Use of an antenna tuned to a particular frequency (so it will pick up the signal at the frequency more strongly than signals at other frequencies) and aimed at the source of the signal can improve reception. Radio signals can be deliberately jammed by broadcasting noise on the same frequency as the signal. For example, the Soviet Union regularly jammed broadcasts by Radio Free Europe and Voice of America.

To minimize unintentional interference, different parts of the radio spectrum are reserved for different uses and broadcast stations are assigned specific frequencies for their use. In the United States, AM radio uses frequencies from 535 to 1700 kHz, and FM radio uses frequencies between 88 megahertz (mHz) and 108 mHz. A radio station that identifies itself as "90.7 FM" is broadcasting at the frequency of 90.7

mHz, or 90,700,000 cycles per second (technically, 90.7 mHz is the station's mean frequency). Other parts of the spectrum are reserved for other uses. For instance, 30–30.56 mHz is allocated for military air-to-ground and air-to-air communications systems for tactical and training operations and for land mobile radio communication in support of wildlife telemetry and natural resource management.

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Sarah Boslaugh

See Also: Satellites; Television, Mathematics in; Televisions; Tides and Waves; Wireless Communication.

Raghavan, Prabhakar

Category: Friendship, Romance, and Religion.
Fields of Study: Communications; Connections.
Summary: Prabhakar Raghavan has made important contributions to Internet and Web analysis, as well as online social networking, through his work at Yahoo! Research Labs.

Prabhakar Raghavan is the head of Yahoo! Research Labs, where he pursues research in text and Web mining and algorithm design, in addition to overseeing the lab's work. He has received honors like being elected as a member of the National Academy of Engineering and as a fellow of both the Association for Computing Machinery and the Institute of Electrical and Electronic Engineers. He is listed as a Consulting Professor of Computer Science at Stanford University and a member of the editorial board of *Internet Mathematics*, a journal on the mathematics of managing huge databases like the Internet. Beginning in 2007, Raghavan served as a member of the board of trustees for the Mathematical Sciences Research Institute. Raghavan attended the Indian Institute of Technology in Madras, where he earned his Bachelor's of Technology in Electrical Engineering in 1981, before coming to the United States to complete his education with a Master's of Science in Electrical and Computer Engineering from the University of California at Santa Barbara and a Ph.D. in Computer Science from University of California at Berkeley. While at Berkeley, Raghavan won the 1986 Machtey Award, given by the annual IEEE Symposium on Foundations of Computer Science, for his paper "Probabilistic Construction of Deterministic Algorithms: Approximating Packing Integer Programs."

Career

After graduate school, he worked for IBM's T. J. Watson Research Center and Almaden Research Center before becoming vice president and chief technology officer at Verity, Inc., an intellectual capital management software developer. Verity had first made its name with a text retrieval system called Topic that allowed users to search for the information they were looking for based on conceptual keywords, rather than being limited to searching for words actually in the text—much like Yahoo!'s later hierarchical organization of Web sites by topic. In 2005, Raghavan was hired to head the newly established Yahoo! Research Labs, the same year that Verity was bought out by rival Autonomy Corporation.

As head of Yahoo!'s labs, Raghavan has spoken of the need to determine the science and mathematics underlying online communities and social networks, saying: "Is it better to pay a celebrity $10,000 to tweet about your product, or find 10,000 non-celebrities to tout you? The nascent research suggests your money is better spent on the crowd—but the key is finding the people who are slightly more influential than most." Mathematicians, computer scientists, and social scientists work to understand the motivations and responses of online users. "We have this huge mountain of data, and it raises fascinating questions about how we can use that to better the experience for our users," says Raghavan, who refers to researchers in this area as "Internet social scientists," who combine mathematical analysis of large databases and algorithmic understanding with techniques from the social sciences and economics, including sociology and psychology. He notes that while his computer science education was heavily grounded in mathematics and engineering, he also believes that these other disciplines should become a fundamental component of a computer science education. The science of optimizing monetization of Internet services is better understood; although still in development, it is a significant interest of Raghavan's as he seeks to monetize social networks. An eclectic group of computer scientists and social scientists came together and figured out how to take computing from a "glass house" to where a billion people could use it. In the twenty-first century, the ways people interact with computers are becoming mundane. How people will interact with each other to create rich social experiences is the crux of this new and ever-expanding science.

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Bill Kte'pi

See Also: Internet; Predicting Preferences; Search Engines; Social Networks.

Randomness

Category: History and Development of Curricular Concepts.
Fields of Study: Communication; Connections; Data Analysis and Communication.
Summary: While a seemingly simple idea, the concept of randomness has been studied by mathematicians for thousands of years and has many modern applications.

The philosophical concept of determinism supposes that all events that occur in the world can be traced back to a specific precipitating cause and denies the possibility that chance may influence predestined causal paths. Mathematical determinism similarly states that, given initial conditions and a mathematical function or system, there is only one possible outcome no matter how many times the calculation is performed.

Historical Studies of Randomness and Certainty

Many ancient cultures embraced the idea of fate. For example, the Greek pantheon included goddesses known as Fates. At the same time, the existence of ancient gambling games and deities like the Roman goddess Fortuna suggest that these people understood the notion of randomness or chance on some level. Around 300 B.C.E., Aristotle proposed dividing events into three different categories: certain events, which were deterministic; probable events, which were because of chance; and unknowable events.

In the 1600s, the work of mathematicians such as Blaise Pascal and Pierre de Fermat laid some foundations for modern probability theory, which quantifies chance. Abraham de Moivre published *The Doctrine of Chances* in 1718. Around the same time, Daniel Bernoulli investigated randomness in his *Exposition of a New Theory on the Measurement of Risk*. Nonetheless, determinism continued to maintain a prominent place in mathematics and science. Researchers often assumed that seemingly observed randomness in their data was because of measuring error or a lack of complete understanding of the phenomena being observed.

The emergence of fields like statistics and quantum mechanics in the nineteenth century helped drive new work on randomness. Mathematician Émile Borel wrote more than 50 papers on the calculus of probability between 1905 and 1950, emphasizing the diverse ways in which randomness could be applied in the natural and social sciences as well as in mathematics. Applied probabilistic modeling grew very quickly after World War II.

Randomness in Society

In twenty-first-century colloquial speech, the word "random" is often used to mean events that cannot be predicted, similar to Aristotle's unknowable classification. However, probability theory can model the long-term behavior of random or stochastic systems using probability distribution functions, which are

Casinos monitor roulette wheel performance and rebalance and realign them to keep results random.

essentially sets of possible outcomes having mathematically definable probabilities of occurring. They describe the overall relative frequencies of events or ranges of events, though the specific sequence of individual events cannot be completely determined. Stochastic behavior is observed in many natural systems, such as atmospheric radiation, consumer behavior, the variation of characteristics in biological systems, and the stock market. It is also connected to mathematical concepts like logarithms and the digits of π. Elementary school children discuss some of the basics of randomness when studying data collection methods, like surveys and experiments. Formal mathematical explorations typically begin in high school and continue through college.

Society depends on the use of randomness or the assumption that randomness is involved in a given process. Examples include operating gambling games and lotteries; encrypting coded satellite transmissions; securing credit card data for e-transactions; allocating drugs in experimental trials; sampling people in surveys; establishing insurance rates; creating key patterns for locks; and modeling complex natural phenomena such as weather and the motion of subatomic particles.

Generating Randomness

Generating random numbers, however, is very different from observing random behavior. For example, in 1995, graduate students Ian Goldberg with David Wagner discovered a serious flaw in the system used to generate temporary random security keys in the Netscape Navigator Web browser. Almost every civilization in recorded history has used mechanical systems, such as dice, for generating random numbers and randomness has close ties with gaming and game theory. Physical methods are not generally practical for quickly generating the large sequences of random numbers needed for Monte Carlo simulation and other computational techniques. Flaws in shuffling and physical characteristics, like a worn-down corner on a die, or deliberate human intervention, can also introduce bias. In fact, some people have proven their ability to flip a coin in a predetermined pattern. Motivated by the mathematical unreliability of these physical systems, mathematicians and scientists sought other reliable sources of randomness. Leonard Tippet used census data, believed to be random, to create a table of 40,000 random digits in 1927. Ronald Fisher used the digits of logarithms to generate additional random tables in 1938. In 1955, RAND Corporation published *A Million Random Digits with 100,000 Normal Deviates*, which were generated by an electronic roulette wheel. Random digit tables are still routinely used by researchers who need to perform limited tasks like randomizing subjects to treatment groups in experimental designs as well as in many statistics classes.

The development of computers in the middle of the twentieth century allowed mathematicians, such as John von Neumann, and computer scientists to generate "pseudorandom" numbers. The name comes from the fact that the digits are produced by some type of deterministic mathematical algorithm that will eventually repeat in a cycle, though relatively shorter runs will display characteristics similar to truly random numbers. Using very large numbers, or trigonometric or logarithmic functions, tends to create longer nonrepeating sequences. Linear feedback shift registers are frequently used for applications such as signal broadcast and stream cyphers. Linear congruential generators produce numbers that are more likely to be serially correlated, but they are useful in applications like video games, where true randomness is not as critical and many random streams are needed at the same time. Hardware random number generators, built as an alternative to algorithm-driven software generators, are based on input from naturally occurring phenomena like radioactive decay or atmospheric white noise and produce what their creators believe to be truly random numbers.

Randomness Tests

Mathematicians and computer scientists are perpetually working on methods to improve pseudorandom number algorithms and to determine whether observed data are truly random. Randomness can be counterintuitive. For example, the sequences 6, 6, 6, 6, 6, 6 and 2, 6, 1, 5, 5, 4 produced by fair rolls of a six-sided die are equally likely to occur, but most people would say that the first sequence does not "look" random. Irregularity and the absence of obvious patterns are useful ideas, but they are difficult to measure. Distinctions between local and global regularity must also be made, which include the ideas of finite sets and infinite sets. Irenée-Jules Bienaymé proposed a simple test for randomness of observations on a continuously varying quantity in the nineteenth century. Florence Nightingale David published a power function for randomness tests shortly after World War II. Another technique from information theory measures randomness for a given sequence by calculating the shortest Turing machine program that could produce the sequence. The National Institute of Standards and Technology recommends many such tests, including binary matrix rank, discrete Fourier transform, linear complexity, and cumulative or overlapping sums. As of 2010, the digits of π had passed all commonly used randomness tests.

Classical probability theory is not the only way to think about randomness. Claude Shannon's development of information theory in the 1940s resulted in the entropy view of randomness, which is now widely used in many scientific fields. By the latter half of the twentieth century, fuzzy logic and chaos theory also emerged. Fuzzy logic was initially derived from Lotfali Zadeh's work on fuzzy sets and non-binary truth values, while chaos theory dates back to Henri Poincaré's explorations of the three body problem. Bayesian statistics, based on the eighteenth-century work of Thomas Bayes, challenges the frequentist approach by allowing randomness to be conceptualized and quantified as a partial belief, which shares characteristics with fuzzy logic. Spam filtering is one application that relies on Bayesian notions of randomness.

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See Also: Coding and Encryption; Probability; Sample Surveys.

Rankings

Category: Games, Sport, and Recreation.
Fields of Study: Number and Operations; Measurement.
Summary: Ranking is a widely used to create ordered lists of people or objects, and there are many ways to assign and analyze ranks.

Throughout human history, people have been ordering objects into hierarchies based on criteria such as measurements or qualitative properties. In the twenty-first century, people rank many objects, such as quarterbacks, political candidates, and restaurants. Every spring, high school seniors eagerly wait to see who will be the valedictorian, or top-ranked student, of their high school class. However, there is not usually a single unique ranking for a set of objects, since ranks depend on the criteria selected and the specific method in which they are combined. *US News and World Report* aggregates multiple quantitative and qualitative indicators in its annual ranking of colleges. Mathematicians use a variety of techniques to study ranking, such as algebra, geometry, graph theory, game theory, operations research, and numerical methods. An entire subset of statistical techniques based on ranks, called *nonparametric* or *distribution-free tests*, are used to transform and analyze data that do not conform to the assumptions or parametric tests.

These techniques are often used in the social sciences. There are also debates about whether ranks are true numbers, given that the spacing between ranks need not be equal in the manner of most common measurement scales. For example, the difference between one inch and two inches is the same as between two inches and three inches. The difference between first and second place, however, is not necessarily quantitatively or qualitatively the same as the difference between second and third place.

Tiebreakers

Some ranking strategies result in ties between one or more individuals. Sometimes there is a tiebreaker, and other times there is not. The ranking of items occurring after the tie can vary depending on the type of ranking used. The most common is called *standard competition ranking*, where a gap is left in the numbering after the tie takes place corresponding to the number of elements in the tie. For example, if there were six items and a three-way tie for second occurred, the ranking would be given as "1, 2, 2, 2, 5, 6" with third and fourth place omitted. Some methods, especially those used in statistical analysis, assign an average rank. In a three-way tie for second place out of six objects, the assigned rankings would be "1, 3, 3, 3, 5, 6," since the average of 2, 3, and 4 is 3.

Sports

Athletic competitions are one very visible use of rankings. During the ancient Olympic Games, athletes would compete in events, such as running, boxing, and the pentathlon, to determine which athletes were better than others. Ultimately, they would be ranked by their performance in these events. Even during the modern Olympics, though the events are more numerous and athletes generally compete in only a few events, the result is a ranking of the best athletes, with prizes being awarded to the top three finishers. There are rankings for other sports as well. For example, the Associated Press ranks the top 25 NCAA football teams by polling sportswriters across the nation. Each writer creates a personal, subjective list of the top 25 teams from all eligible teams (more than 25). The individual rankings are then combined to produce the national ranking by giving a team 25 points for a first place vote, 24 points for a second place vote, and so on down to one point for a 25th place vote. Teams are also regularly ranked by their number of wins or other game-related metrics, as are individual players.

Tests

Rankings also occur on standardized tests. Rather than give each individual a unique rank, tests such as the SAT separate the scores into percentages and then rank test takers according to the percentage they fall into. Percentile ranks can also be seen in other places, such as height and weight charts for children. Whereas many rankings place an emphasis on small numbers (it is better to be ranked first or second than twenty-fifth), percentiles are considered in the opposite manner—a larger value percentile ranking is a better rank. Percentiles indicate what percentage of the test-taking group performed the same or worse than a test-taker in that percentile. For example, being in the 57th percentile would indicate that 57 percent of the test takers scored the same or worse. When considering rakings, it is important to determine how the ranking is arranged to properly interpret the data.

Other Mathematical Connections

The word "rank" carries many specific definitions in various fields of mathematics. For example, the rank of a matrix is the number of linearly independent rows or columns. In graph theory, the rank of a graph is the number of vertices minus the number of connected components. Other definitions of rank can be found in set theory and Lie algebra (named for mathematician Sophus Lie). In chess, a game studied by many mathematicians, a rank is a row on the chessboard.

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Chad T. Lower

See Also: Growth Charts; Infinity; Measurement in Society; Nielsen Ratings; Statistics Education; Voting Methods.

Rational Numbers

See *Numbers, Rational and Irrational*

Reasoning and Proof in Society

Category: School and Society.
Fields of Study: Connections; Reasoning and Proof.
Summary: Many aspects of society have inherited from mathematics the desire for a method of proof that is demonstrable and irrefutable.

Reasoning and proof are fundamental components of human existence. Children begin applying reasoning as soon as they can make connections between actions and consequences. They then go on to explore more formal methods of reasoning and proof throughout their educational careers, not just in mathematics. Although people often associate mathematics solely with deductive proofs, many other types of reasoning are important to mathematics, including inductive logic, evidence-based reasoning, and computer-assisted arguments. Furthermore, the concept of truth being produced by reasoning and proof also pervades other fields, including philosophy, the natural and social sciences, and political and legal discourse.

Origins of Mathematical Proof

What proves a statement? Generally, it is believed that statements are proved by deducing the statement as a logical consequence of something already believed to be true. One might think that proofs are necessary only when what is being proved is not apparent. The Greeks, however, did not limit proving to non-obvious statements; they gave a logical structure to all of geometry, assuming as its basis the smallest possible number of "already believed" statements. They also employed a method called "proof by contradiction" in which a truth is not demonstrated directly, but rather by showing that its opposite cannot be maintained.

Why did Greek culture give geometry this kind of logical structure, and why did the Greeks think that doing so was significant? The question is important because the causes that produced mathematical proof still exist in the twenty-first century, where they continue to operate and promote the use of proof.

First, proofs give a way to reconcile discordant opinions. Greek mathematics was heir to two earlier traditions, Egyptian and Babylonian mathematics, whose results did not always agree. For instance, in studying circles, the Babylonians approximated π first as 3, and later as 3.125. Egyptian computations give a value for π of about 3.16. The Greeks wanted to know π's true value. One way to avoid having multiple answers to the same question is to make no assumptions other than those with which nobody could disagree, like "all right angles are equal," and then deduce other facts solely from those un-doubtable assumptions. What is amazing is how many results this approach produced.

Second, proofs are a natural outcome of the search for basic principles. The pioneering Greek philosophers of nature of the fifth and sixth centuries b.c.e. sought simple explanatory principles that could make sense out of the entire universe. Thales, for instance, said that "everything is water," and Anaximenes claimed that "everything is air." The Pythagoreans asserted that "all is number," while Democritus said that "everything is made of atoms." As in nature, so in mathematics, the Greeks wanted to develop explanations based on simple first principles, on the so-called elements.

Third, the logic of proofs can arise from the process of discovery. One effective way to solve a problem is to reduce it to a simpler problem whose solution is already known. For instance, Hippocrates of Chios in the fifth century B.C.E. reduced finding the area of some lunes (areas bounded by two circular arcs) to finding the area of triangles. In reducing complicated problems to simpler problems, and then reducing these to yet simpler problems, the Greek mathematicians were creating sets of logically linked ideas. If such a set of linked ideas is run in reverse order, a proof structure emerges—simple statements on which rest more complex statements on which rest yet more complex statements. The simplest statements at the beginning are called the "elements"; the intermediate ones are the fruitful results that are now called "lemmas"; and these in turn demonstrate the final and most important results.

Fourth, logical reasoning played essential roles in classical Greek society. In the sixth and fifth centuries B.C.E., Greece was largely made up of small city-states run by their citizens. Discourse between disputing parties, from the law courts to the public assemblies, required and helped advance logical skills. A good way, then and now, to persuade people is to understand their premises, and then construct one's own argument by reasoning from their premises. A good way to disprove someone's views is to find some logical consequence of those views that appears absurd. These techniques are beautifully illustrated in Greek legal proceedings and political discourse, as well as in the dialogues of Plato.

Finally, Greek mathematics developed hand in hand with philosophy. Greek philosophers began by trying to logically refute their predecessors. Zeno, for instance, presented his paradoxical arguments not to prove that motion is impossible but to challenge others' intuition and common-sense assumptions. That Plato wrote in dialogue form both illustrates and demonstrates that Greek philosophy was as much about the method of logical argument as it was about conclusions. Aristotle wanted every science to start, like geometry, with explicitly stated elementary first principles, and then to logically deduce the key truths of the subject. Greek philosophy issued marching orders to mathematicians, and men like Euclid followed these orders.

Philosophy returned the favor. Plato made mathematics the center of the education of the rulers of his ideal Republic and mathematics has remained at the heart of Western education. Plato championed mathematics because it exemplified how, by reasoning alone, one could transcend individual experience. Such transcendence is most striking in the case of proof by contradiction. The argument form, "If you accept *A*, then you must also accept *B*, but *B* contradicts *C*," was part and parcel of the educated Greek's weapons of refutation. But proof by contradiction is not merely destructive, it also allows people to rigorously test conjectures that cannot be tested directly and, if they are true, to demonstrate them.

For example, Euclid defined parallel lines as lines in the same plane that never meet. But it can never be shown directly that two lines can never meet. However, it can be assumed that the two lines do, in fact, meet and then prove that this assumption leads to a contradiction. This process made Euclid's theory of parallels possible.

As another example, consider the Greek proof that $\sqrt{2}$ cannot be rational (it cannot be the ratio of two whole numbers). Because the Pythagorean theorem holds for isosceles right triangles, $\sqrt{2}$ must exist.

But no picture of an isosceles right triangle can allow one to distinguish a side of rational length from one of irrational length.

Nor can one hope to prove the irrationality of $\sqrt{2}$ by squaring every single one of the infinitely many rational numbers to see if its square equals 2. However, if one assumes that there is a rational number whose square is two, logic then leads to a contradiction, so it is proved that $\sqrt{2}$ cannot be rational.

By such means the Greeks proved not only that $\sqrt{2}$ was irrational but also that a whole new set of mathematical objects existed: "irrational numbers."

Proof in general, and proof by contradiction in particular, transformed the nature of mathematics. Logic lets people reason about concepts that are beyond experience and intuition—about ideas that cannot be observed. Mathematics had become the study of objects transcending material reality, objects visible only to the eye of the intellect. There could be truths about such objects and such truths could be proved. These developments had profound consequences far beyond mathematics.

Beyond Mathematics

The ideal of logical proof in mathematics took on a life of its own. Since mathematicians apparently had achieved truth by means of proof, practitioners of other areas of Western thought wanted to do the same. So in theology, politics, philosophy, and science people tried to imitate the mathematicians' method.

In 1637, Rene Descartes wrote in his *Discourse on Method*, "Those long chains of reasoning . . . which enabled geometers to reach the most difficult demonstrations, made me wonder whether all things knowable to men might not fall into a similar logical sequence." If so, he continued, there cannot be any propositions that cannot be eventually discovered and proven.

Building on Descartes's ideas, Baruch Spinoza in 1675 wrote a book called *Ethics Demonstrated in Geometrical Order*. Like Euclid, Spinoza first explicitly defined his terms, including "God" and "eternity." He then stated axioms about existence and causality. On the basis of his list of definitions and axioms, Spinoza logically demonstrated his philosophical conclusions, including the existence of God.

Isaac Newton wrote his great *Principia* in 1687. This work includes Newton's laws of motion and theory of gravity. He did not structure the *Principia* like a modern physics book; he gave it the same definition-axiom-theorem structure that Euclid had given the *Elements*. Newton expressly called his famous three laws "Axioms, or Laws of Motion." From these axioms, Newton logically deduced the laws of the universe, including universal gravitation, just as Euclid had deduced his own theorems.

The American Declaration of Independence of 1776 also pays homage to the ideal of Euclidean proof. The principal author, Thomas Jefferson, was well versed in the mathematics of his time. Jefferson began with axioms, saying, "We hold these truths to be self-evident," including the axioms "that all men are created equal" and that, if a government does not preserve human rights, "it is the right of the people to alter or abolish it, and set up new government." The declaration then says that it will "prove" that King George III's government had not protected human rights. Once Jefferson proved this, the Declaration of Independence concludes: "We therefore . . . publish and declare that these United Colonies are and of right ought to be free and independent states." Indeed, Jefferson could have ended his argument, as had Spinoza and Newton, with the geometer's "QED."

Jefferson's argument exemplifies the characteristic program of Enlightenment philosophy—using reason to reach conclusions on which everyone will agree. This program is epitomized in the words of Voltaire in his *Philosophical Dictionary*: "There is but one morality, as there is but one geometry."

Abstraction, Symbolism, and their Power

Logical proof in mathematics and the use of mathematical models of reasoning in the larger intellectual world were not limited to geometry. In mathematics in the seventeenth and eighteenth centuries, proof methods moved beyond the geometric to include the algebraic. This shift began when François Viète, in 1591, first introduced general symbolic notation in algebra, an idea with incredible power.

School children learn that for every pair of distinct numbers, not only does $9 + 7 = 16$, so does $7 + 9$. Viète's general symbolic notation allows one to write down the infinite number of such facts all at once: $B + C = C + B$.

A century later, Isaac Newton summed up the power and generality of Viète's idea by calling algebra "universal arithmetic." Newton meant that one could prove algebraic truths from the universal validity of the symbolic manipulations that obey the laws of ordinary arithmetic. For instance, consider the quadratic equation $2x^2 - 11x + 15 = 0$. Simply stating, "3 and 2 1/2 are the solutions" gives no information about how those answers were obtained. But every quadratic equation has the general form of $ax^2 + bx + c = 0$. Solving that general equation by the algebraic technique of completing the square gives the well-known quadratic formula for the general solution:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This general solution contains the record of every operation performed in getting it. The original example had $a = 2$, $b = -11$, $c = 15$. As such, it is known exactly how the answers, 3 and 2 1/2, are obtained from the coefficients in the equation. More important, this process proves that these and only these must be the answers.

In the seventeenth century, Gottfried Wilhelm Leibniz was so inspired by the power of algebraic notation to simultaneously make and prove mathematical discoveries that he invented an analogous notation for his new differential calculus. Furthermore, he envisioned an even more general symbolic language that would, once perfected, find the indisputable truth in all areas of human thought. Once such a language existed, Leibniz said, if two people were to disagree, one could say to the other, "let us calculate, sir!" and the disagreement would be resolved. This idea made Leibniz the prophet of modern symbolic logic.

By the eighteenth century, many mathematicians thought discovery and proof should be based on abstract symbolic reasoning. Imitating mathematics, scientists introduced analogous notations in other fields. For instance, Antoine Lavoisier and Claude-Louis Berthollet developed a new chemical notation that they called "chemical algebra," which is used when balancing a chemical equation.

These ideas, both within and beyond mathematics, led the Marquis de Condorcet to write in 1793 that algebra contains within it the principles of a universal instrument, applicable to all combinations of ideas. Such an instrument, he said, would eventually make the progress of every subject embraced by human intelligence as sure as the progress of mathematics.

In the nineteenth century, George Boole produced the first modern system of symbolic logic and used it to analyze a wide variety of complicated arguments. His system, developed further, underlies the logic used by digital computers in the twenty-first century, including applications embodying Condorcet's dream, from automated theorem-proving to translators, grammar checkers, and search engines.

Non-Euclidean Geometry: The Triumph of Euclidean Logic

Unthinkable as it may have been to Enlightenment philosophers like Voltaire, there are alternatives to Euclid's geometry. But non-Euclidean geometry was not invented by imaginative artists or by critics of mathematics speculating about alternative realities. Like irrational numbers, non-Euclidean geometry was discovered by mathematicians. Its discovery provides another example of human reason and logic trumping intuition and experience and it—like Euclid's geometry—has had a profound effect on other areas of thought.

Non-Euclidean geometry grew out of attempts to prove Euclid's parallel postulate:

If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, then the two straight lines, if produced indefinitely, meet on that side where the angles are less than two right angles.

Such attempts were made because the postulate seemed considerably less self-evident than his other postulates. From antiquity onward, mathematicians felt that it ought to be a theorem rather than an assumption, and many eminent mathematicians tried to prove it from the other postulates. Some attempted to prove it indirectly; assuming it to be false, they deduced what appeared to be absurd consequences from that assumption. For instance, that parallel lines are not everywhere equidistant, and that there is more than one line parallel to a given line through a point in the same plane. These results contradict our deep intuitive sense of symmetry.

But in the nineteenth century, three mathematicians independently realized that these conclusions were not absurd at all, but were perfectly valid theorems in an alternative geometry. Nicolai Ivanovich Lobachevsky, by analogy with imaginary numbers, called his new subject "imaginary geometry." Janos Bolyai more theologically called it "a new world created out of nothing." But Carl Friedrich Gauss, acknowledging the logical move that made it possible, called the new subject "non-Euclidean geometry."

The historical commitment of mathematicians to the autonomy of logic and to logical proof enabled them to overcome their scientific, psychological, and philosophical commitments to Euclidean symmetry to create this new subject. Logical argument once again let mathematicians find and demonstrate the properties of something neither visual nor tangible—something counter-intuitive. Non-Euclidean geometry is the ultimate triumph of the Euclidean method of proof. But there are wider implications.

From this discovery, nineteenth-century philosophers concluded that the essence of mathematics (as opposed to the natural sciences) is its freedom to choose any consistent set of axioms that meets the mathematician's sense of what is important, beautiful, and fruitful—just as long as the logic is right. There could even be real-world applications of systems that contradict all past mathematical orthodoxies. In physics, for instance, the type of non-Euclidean geometry studied by Bernhard Riemann in the 1850s turned out to be exactly what Albert Einstein needed for his general theory of relativity; the new mathematics can

Proof and the Law

Though the goal of legal arguments is persuasion as well as proof, legal arguments require evidence, and thus require discerning what follows logically from such evidence. Some legal thinkers have carried this view quite far.

For instance, Christopher Langdell, pioneer of the case method in legal education and Dean of Harvard Law School in the 1870s, saw law as a science. By analogy with geometry, law, according to Langdell, is governed by a consistent set of general principles. The correct legal rules should be logically deduced from those general principles and then applied to logically produce the correct legal ruling in line with the facts of a particular case.

Most Anglo-American legal theorists do not follow Langdell's "classical orthodoxy," agreeing instead with Oliver Wendell Holmes that the life of the law has not been logic but experience. Yet Holmes, too, employed logical argument within every case he discussed. For instance, he used a proof by contradiction to argue that freedom of speech is not absolute when he famously said that the most stringent protection of free speech would not protect a man in falsely shouting "fire!" in a theater and causing a panic.

Finally, the adversary system of Anglo-American law not only allows but also requires, that in order for a case to prevail in court, the winning argument must not only support that case but also explicitly answer the arguments on the other side, with these counter-arguments presented as strongly as possible. Thus, logical proof pervades all legal argument.

explain gravitation, describe the curvature of space, and account for black holes.

Knowing that alternative systems of mathematical thought are logically possible has also had philosophical and social implications. José Ortega y Gasset, for instance, contrasted the view of the old geometry (interpreted as saying that nations may perish but principles will be kept) to the new perspective, which he interpreted as saying that people must look for such principles as will preserve nations, because that is what principles are for.

Proof and the Citizen

Citizens of democracies need to be able to evaluate arguments presented to them, whether by friends, adversaries, politicians, or advertisers. In the words of Jacques Barzun, "The ability to feel the force of an argument apart from the substance it deals with is the strongest possible weapon against prejudice."

Citizens also need to be free to work out the logical implications of the principles they treasure. In the words of Winston Smith, a character in George Orwell's novel *1984*, "Freedom is the freedom to say that two plus two make four. If that is granted, all else follows." This kind of "proving" has driven the progress of the idea of universal human rights. For instance, building on the Declaration of Independence, Elizabeth Cady Stanton, a pioneer in fighting for women's rights in America, wrote in the Seneca Falls Declaration of 1848, "We hold these truths to be self-evident; that all men *and women* are created equal." Similarly, Martin Luther King, Jr., in his "I Have a Dream" speech, spoke of "the promise that *all* men, yes, black men as well as white men, would be guaranteed the unalienable rights of life, liberty, and the pursuit of happiness."

Now, just as in ancient Greece, the ability to reason and prove and the liberty of expressing and acting upon the results of proofs are essential to a free and democratic society. The historical function of proof in mathematics has not been just to prove theorems but also to exemplify and teach logical argument in areas such as philosophy, law, politics, religion, and every area of modern life.

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See Also: Geometry and Geometry Education; Mathematical Certainty; Parallel Postulate; Proof; Strategy and Tactics.

Recycling

Category: Weather, Nature, and Environment.
Fields of Study: Algebra; Data Analysis and Probability; Number and Operations.
Summary: Efficient recycling requires the use of sophisticated mathematical models to maximize product use and reuse and minimize energy consumption.

Recycling is the extraction of usable materials out of used objects. Materials that are often recycled at the start of the twenty-first century include metal, paper, glass, and plastic. One important mathematical problem of recycling is the comparison of environmental and monetary costs of recycling and virgin production. Mathematicians are also involved in developing new methods for recycling and modeling both economic and environmental impacts. The notion of "algorithm recycling" applies to resources used in some mathematical investigations. For example, statistical bootstrap recycling reuses samples to minimize demands on computational resources. Some mathematicians, scientists, educators, and others use recycling for education, recreation, and art. Mario Marin has designed polyhedral outdoor play spaces and kinetic sculptures from recycled and remaindered materials and has published many creative ways to recycle household objects, like plastic bottles, into interesting polyhedral structures. With regard to learning, some have even suggested a concept called "neuronal recycling," which refers to adaption of neuronal circuits for new uses.

Proportion-Based Regulations and Labeling

To motivate recycling, companies and governments set rules that demand the recycling of a certain proportion of materials and the use of a certain proportion of recycled material in production. Because recycling is the third desirable option in the waste management hierarchy, after reduction of waste and reusing of objects and materials, setting high recycling quotas is never a goal in its own right. However, recycling is often preferable to disposal. Governments sometimes directly mandate minimum recycled content in certain classes of manufactured goods. Labeling laws, which require companies to display the percent of recycled content in goods and packages, may also promote recycling if consumers support it, or hinder recycling if consumers do not find recycled goods in this particular industry appealing. Companies advertise their recycling efforts—typically by disclosing the percent of recycled material—to present ecofriendly images to their customers.

A common scheme to promote the recycling of packaging is to include a refundable fee in the price of the product. Once the customer returns the packaging to the store, the fee is refunded.

Measuring Efficiency

Because recycling is a complex process, there are ecological and economical costs involved in it. For recycling to make sense, the benefits have to outweigh the costs. Computing costs and benefits is a complex problem. Costs are incurred at all stages of recycling: collecting materials, sorting them, and re-making them. Benefits include the reduction of landfill costs, reduction of pollution, and revenues from the use of recycled materials. In the cases of nonrenewable natural resources, recycling is the only option to keep using these resources in the future.

Metal Recycling

Because of the relative difficulty and high cost of mining and smelting of metals, and the ease of collecting and recycling, metals are the most recycled materials in the world. For example, recycling aluminum takes only 5% of the energy that it would take to make it from the raw materials. About three-quarters of steel and a third of aluminum is recycled in the United States as of 2010. Some applications of science and mathematics metal recycling involves the separation of impurities, such as paint.

Paper Recycling

One category of paper recycling, post-consumer paper, is familiar to most people because paper is ubiquitous in modern society. "Mill broke" is scraps that pulp mills accumulate from making paper, which they can also recycle. Preconsumer paper is scraps collected and recycled in paper mills. Unlike metal recycling, where the cost-benefit ratio is low, paper recycling is more complicated and controversial. For example, burning paper for energy may be more environmentally sound than recycling it and harvesting and replanting forests may be cheaper than recycling.

Estimates for energy saving are 40% to 65% for recycled paper, compared to creating new paper. However, pulp mills frequently produce energy by burning roots, bark, and other byproducts, whereas recycling plants have to be close enough to collection (usually urban) areas to minimize transport cost and frequently

Atlas Recycled *by Tom Tsuchiya is a sculpture made of used atlases and maps that also serves as a recycling receptacle for bottles and cans—one of the works from the EcoSculpt 2010 exhibition in Cinncinnati, Ohio.*

depend on fossil fuels for energy. Thus, the environmental costs of conserving the same amount of energy is different, as one process uses renewable resources and the other uses nonrenewable resources. Water and air pollution benefits of paper recycling are more pronounced than energy benefits because of highly toxic bleaching used in making new paper.

Plastic Recycling

Recycling of plastics involves a scientific challenge not found in recycling of other materials. Because of the ways polymer chains are formed in plastics, different plastics do not blend well. Removing dyes, glue, paper stickers, and other impurities is also difficult. Plastics are coded with the Resin Identification Codes, numbers 1–7, inside the triangular recycle symbol.

There are several processes for recycling plastic. The most straightforward is melting similar plastics together, with some steps to remove impurities. Heat compression mixes all types of plastics in high-heat, high-pressure drums. Thermal depolymerization is currently an experimental procedure that "reverses" the process of making plastic and turns it into a substance similar to crude oil. Another experimental procedure, called "monomer recycling," reverses plastic-making halfway, turning polymers into the mix of monomer chemicals that formed them.

The short-term cost-benefit analysis may not support plastic recycling because of the high energy and labor requirements of the known processes. However, crude oil (the raw material of plastic) is a nonrenewable resource, which makes plastic recycling attractive in the long term.

Glass Recycling

The main benefits of glass recycling are saving landfill space and saving energy on producing new glass. However, because glass is sturdy and easy to clean, glass container reuse is vastly preferable to recycling. Through changing their infrastructures, along with using clear bottle standards and monetary incentives, some countries can reuse more than 95% of their glass bottles.

Crushed glass can be added to concrete. This process can be considered reuse rather than recycling because the glass is serving a different purpose. Measurements of glass-infused concrete include its insulation properties and strength properties, both of which are improved by the addition of glass. Also, concrete with glass is more aesthetically pleasing and can be used for countertops and other highly visible places.

Mathematical Modeling

Mathematical models are widely used in logistics—controlling the efficient flow and storage of goods, services, and information from the point of origin to the point of consumption. Reverse logistics is the extension of this principle that addresses concepts such as returns, source reduction, recycling, and reuse. Mathematicians have researched models for logistics that address these reversals of flows. For example, Italian researchers created a staged mathematical model of the options for recycling a broad range of appliances, electronic equipment, and other household items commonly thrown away. The model suggested that recycling can offer what is known as *economies of scale* to businesses, which are increasingly being held liable for end-of-life product disposal.

Others have used techniques such as dynamic quantitative models to simulate recycling systems and flows to better understand the driving variables and relationships among the activities and participants. These models can aid planners in making decisions about recycling policies and procedures. Nutrient recycling for trees, which has implications for issues such as global warming, has been modeled using linear and quadratic functions, along with data-based numerical simulations. However, some scientists argue that mathematical models must be contextually evaluated and used with caution for decision making and legislation. Models based on limited data may generate what appear to be useful results, but extrapolation or subsequent modeling can create bias and propagation of errors.

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See Also: Advertising; Carbon Footprint; Deforestation; Fuel Consumption; Green Design; Green Mathematics; Synchrony and Spontaneous Order.

Relativity

Category: Space, Time, and Distance.
Fields of Study: Algebra; Geometry; Measurement; Representations.
Summary: Albert Einstein's theory of relativity is one of the most well-known theories in physics and helps describe the nature of the universe.

Albert Einstein's theory of relativity forms one of the two pillars of modern physics, the other being quantum mechanics. It consists of two parts: the special theory of relativity from 1905, and the general theory of relativity from 1915, which both rely on significant mathematics.

The special theory of relativity describes how space and time are perceived by observers in different inertial systems. Einstein derived this theory from a single physical principle of relativity. It was discovered in 1632 by Galileo Galilei that the laws of mechanics are the same in all inertial systems—a discovery, known as "Galileo's principle of relativity," that constituted a radical break with the prevailing Aristotelian physics. Einstein's principle of relativity generalized this concept to all laws of nature, including Maxwell's laws of electromagnetism, which govern the propagation of light. It thus follows from Einstein's principle of relativity that the speed of light is the same in all inertial systems, a central result in the theory of relativity. Prior to Einstein, it was believed that light propagates through a luminiferous aether in the same way as sound propagated through air, but all attempts to measure the speed of the Earth relative to this aether, such as the Michelson–Morley experiment in 1887, failed. Special relativity explained the negative results of these experiments and made the aether hypothesis superfluous.

The general theory of relativity unifies special relativity with Isaac Newton's law of universal gravity. Its basis is Einstein's equivalence principle, according to which an accelerated system of reference (such as a so-called Einstein elevator) is indistinguishable from a system at rest in a gravitational field. Mathematically, Einstein's field equations describe how the presence of mass, energy, and momentum gives rise to a curvature of space and time. Although this idea has little significance in weak gravitational fields, such as that of the Earth, general relativity is essential in the study of the universe as a whole. For example, Karl Schwarzschild in 1915 found an exact solution to Einstein's equations that explains the existence of black holes.

The many surprising consequences of the theory of relativity have been described in numerous popularizations, most notably by George Gamow. Einstein's theory must not be confused with the various relativist positions in philosophy, such as aesthetic, moral, cultural, or cognitive relativism.

Special Relativity

The Lorentz transformation forms the basis of the special theory of relativity. It is a set of equations describing how to translate suitably chosen coordinates of space and time between two inertial systems (S) and (S') moving with the speed (v) relative to one another:

$$x' = \gamma(x - vt) \text{ and } t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

where c denotes the speed of light of 299,792,458 meters per second, and the dimensionless number

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

is the so-called Lorentz factor. In 1908, Hermann Minkowski gave a mathematical description of the Lorentz transformation as a rotation of the coordinate axes in four-dimensional space-time.

When v is much smaller than c, the Lorentz factor is close to 1, and the Lorentz transformation reduces to the classical Galilean transformation. When v approaches c, however, the Lorentz transformation has a number of consequences that radically contradict classical physics as well as common sense. For example, clocks in motion are slowed down (called "relativistic time dilation"), objects in motion are contracted in the direction of movement (called "relativistic length contraction"), and clocks in motion that are seen as synchronized by an observer moving with the clocks are seen as nonsynchronized by an observer at rest (called "relativity of simultaneity").

It is another consequence of special relativity that no material objects—or signals of any kind—can travel faster than light. This "speed limit" exists because anything traveling faster than light relative to one observer would appear to be traveling backwards in time relative to another observer, thus leading to paradoxes regarding cause and effect. There is a quantum-mechanical phenomenon, the so-called Einstein–Podolsky–Rosen paradox, that seems to contradict this principle. According to quantum mechanics, the wave function of two entangled particles is affected by a measurement of the state of one of the particles, causing an instantaneous change to the state of the other, even if the two particles are located in different galaxies. But this phenomenon, which has since been verified experimentally, does not really contradict relativity since it cannot be used to transmit information from one galaxy to the other.

Special relativity dictates that mass and energy are connected by the equation $E = mc^2$, undoubtedly the most famous formula in all of physics. Any particle with mass m has a rest energy given by this equation. If the same particle is accelerated to the speed v, its energy is multiplied by the Lorentz factor γ, and its kinetic energy is found as the difference between total energy and rest energy, expressed algebraically as

$$E_{kin} = \gamma mc^2 - mc^2 \approx \frac{1}{2}mv^2.$$

The approximation, valid for v much smaller than c, equals the expression for kinetic energy in classical mechanics. This formula shows that it would require an infinite amount of energy to accelerate a particle with positive mass to the speed of light.

General Relativity

Einstein noted that special relativity implies that space appears to be curved, or "non-Euclidean," to observers in accelerated systems (for example, on a rotating disc) and inferred from the equivalence principle that the same must be true in gravitational fields. However, after realizing this fundamental principle in 1907, it took him eight years to find the field equations that describe the exact curvature of space-time. The idea that physical space might be curved was not new. Already in 1823, Carl Friedrich Gauss investigated this question empirically by measuring the sum of angles of a triangle formed by three mountaintops but found no curvature. Bernhard Riemann further developed the mathematics of curved space in 1854 and this work would become an essential part of Einstein's theory.

General relativity predicts that a body falling freely in a gravitational field, such as the Earth in its orbit around the sun, follows a "geodesic" in curved space-time. This geodesic is called the body's "world-line." In a curved space, geodesics are the least curved lines, in the same way as the equator is a least curved line on the surface of Earth. Although the predictions of general relativity are nearly the same as those of classical mechanics for bodies in weak gravitational fields, the interpretation of gravity is radically different: whereas classical mechanics explains the elliptical orbit of the Earth as a consequence of a gravitational force emanating from the sun, general relativity postulates that the mass of the sun gives rise to a curvature of space-time, and that the world-line of Earth is in fact a geodesic.

It is a consequence of general relativity that clocks in gravitational fields are slowed down. This effect is called "gravitational time dilation." For a clock at rest in the gravitational field of Earth, the dilation factor is

$$\sqrt{1 - \frac{2GM}{rc^2}} \approx 1 - \frac{GM}{rc^2}$$

where G is Newton's gravitational constant, M is the mass of Earth, and r is the distance between the clock and the center of Earth.

Proofs and Applications of Relativity

Einstein showed in 1915 that general relativity explains the perihelion precession of the planet Mercury. This phenomenon, which had mystified astronomers since its discovery in 1859, is that the elliptical orbit of Mercury rotates around the sun with 43 arc seconds per century.

Also in 1915, Einstein predicted that light emitted from distant stars is deflected when passing through the gravitational field of the sun. Although this effect had previously been derived from Newtonian gravity alone, Einstein showed that the angle of deflection following from general relativity is twice the angle following from classical physics. Einstein's prediction was confirmed dramatically by Arthur Eddington during the total solar eclipse of May 29, 1919.

Contrary to quantum mechanics, the technological implementations of which are ubiquitous, relativity has few practical applications. One notable exception is the global positioning system (GPS). GPS satellites revolve around the Earth twice per sidereal day at a height of about 20,000 kilometers (12,400 miles) and with a speed of about 4 kilometers (2.5 miles) per second. Because of the speed and altitude, the atomic clocks aboard the satellites are subject both to relativistic time dilation and to a reduced gravitational time dilation.

The first effect amounts to a loss of 7 microseconds per day, the second to a gain of 45 microseconds per day. In total, therefore, the atomic satellite clocks gain 38 microseconds per day relative to clocks on the ground. Failure to take these relativistic effects into account would render GPS useless since the resulting positional error would accumulate to 11 kilometers (6.8 miles) per day.

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See Also: Black Holes; Einstein, Albert; GPS; Geometry of the Universe; Gravity.

Religion, Mathematics and

See *Mathematics and Religion*

Religious Mathematicians

See *Mathematicians, Religious*

Religious Symbolism

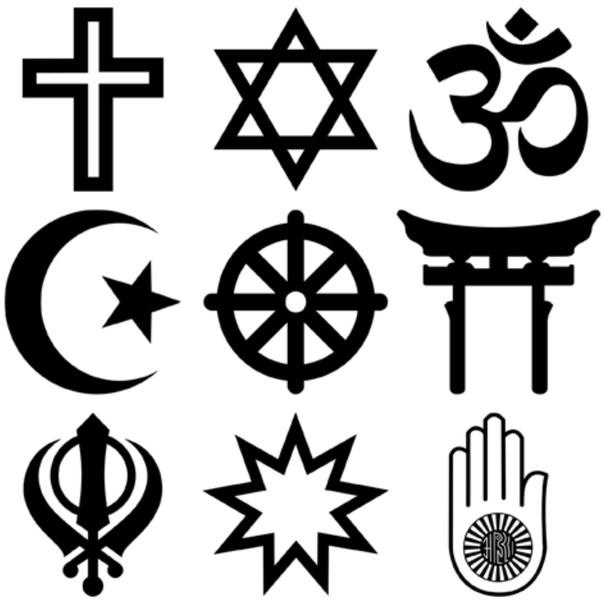
Category: Friendship, Romance, and Religion.
Fields of Study: Communication; Geometry; Number and Operations; Representations.
Summary: Many religious symbols are mathematical in nature.

Archaeological research suggests that religion predates people's ability to read and write but that symbols were often used to express religious ideas and to convey meaning. In this context, such symbols might be pictures, geometric objects, or numbers that hold a particular meaning within a given faith. Long after the introduction of the written word, symbols still hold a powerful place in most religions. There are many highly recognizable symbol forms that are used in various ways by different faiths around the world, though they often share similar underlying structures, themes, or meanings. Symmetry is common in religious symbolism, as are certain numbers or concepts that some believe to have special significance beyond mathematical interpretations. For example, some have proposed a stylized version of the empty set symbol to represent atheism.

Stars

Stars have been used for millennia in a variety of religions. The most common is a five-pointed star, also known as a "pentagram" (*penta* means "five"). At times, the five points have represented the five senses (vision, hearing, touch, smell, and taste). Wiccans use the points to represent five elements (spirit, fire, air, water, and earth) as do Taoists (fire, earth, metal, water, and wood). Other times, it has represented the human body with the "points" of the body relating to the head, arms, and legs outstretched, as seen in the Baha'i Faith where the pentagram is its official symbol. Christians have used the pentagram to denote the wounds (five stigmata) received by Jesus Christ when he was crucified—hands, feet, and side. Judaism has used the pentagram to represent the Pentateuch (Genesis, Exodus, Leviticus, Numbers, and Deuteronomy) and later, Solomon's Seal. Muslims use a crescent moon and a pentagram to denote the religion of Islam.

Some religions have used a point-down pentagram as part of their symbolism. For example, Anton LaVey's Satanists (who have nothing to do with Satan—they do not believe Satan exists) use the upside down pentagram for their symbol and often impose a goat's head in the symbol with the upper points representing horns, the side points being the ears, and the lower point as the chin and beard area. Mormons (belonging to the Church of Jesus Christ of Latter-Day Saints) have used



From top left: Christian cross, Jewish Star of David, Hindu Aumkar, Islamic star and crescent, Buddhist wheel of Dharma, Shinto torii, Sikh Khanda, Bahá'í star, and Jain ahimsa symbol.

the inverted symbol in some temple architecture as representing the "morning star" (Venus' path in the sky).

Another star variation is the six-sided star, sometimes referred to as a "hexagram" (*hexa* means "six") or the Star of David as a symbol of Judaism. Most often, this star is drawn as two equilateral triangles drawn on top of each other with one pointing up, the other down, and slightly offset. Hindus have a variation of the hexagram called the *Shatkona*, which show the triangles weaved together denoting the interlocking of fire and water, or male and female. The hexagram is also a symbol for Rastafarians and is usually solid black. The Raelism Movement uses a different variation of the hexagram as their official symbol; it contains a right-facing swastika embedded in the center of the star.

A seven-pointed star (called a "heptagram") is sometimes used by Jews and Christians to denote a seven-day creation. Faery Wiccans and Blue Star Wiccans also use the seven-pointed star, but the Blue Star Wiccans refer to it as a "septagram" instead.

There are a few variations of eight-pointed stars. Islam has a star referred to as *rub el hizb*, which appears as two squares superimposed with one slightly offset the other. It is used to help facilitate the recitation of the Qur'an. The same shape (without the center circle) is referred to as the Star of Lakshmi by Hindus, where it represents the eight forms or kinds of wealth. This shape is referred to as an 8/2 "octagram" (*oct* means "eight"). The "8/2" indicates that there are eight sides on the star and every second point (or vertex) is connected with a line. An 8/3 octagram would have every third vertex connected to each other. This symbol has been used by Christians to represent baptism and resurrection. Ancient Mesopotamia calls their eight-pointed star the Seal of Shamash. The center was a circle representing the sun (Shamash) with eight points emanating from the center. Most likely, the vertical and horizontal points represent the four directions of the compass while the diagonal points represent the equinoxes and solstices.

Although the Baha'i uses a pentagram for their official symbol, a nine-pointed star is more commonly associated with the religion. The star is often drawn similar to the hexagram, but with three equilateral triangles slightly offset and a single point at the top of the star, but without the inner lines. The Baha'i Faith also uses another version of the nine-pointed star with symbols of the "nine world religions" at each point.

Crosses

The cross is sometimes thought of as a universal symbol for Christianity since, in the Christian faith, Jesus is believed to have been crucified on a Roman cross. However, there are many types of "Christian crosses" and many religious crosses that are not Christian at all. The original Christian cross probably resembled an "X" for the first Greek letter in the word "Christ." It is not related to the crucifixion and came much later than Jesus's death, as many early Christians opposed its use. When placed so that its arms pointed vertically and horizontally, the meaning was the four directions of the compass—where the gospel should be spread. Eventually, the Greek cross made way for the Latin cross, which resembles a lower case "t." Orthodox Christians add a small horizontal line above the arms of the cross denoting the sign hung by Pilate, and a small diagonal line below the arms of the cross denoting a footrest. Other denominations, like Methodists, show a flame behind the cross indicating the Holy Spirit. Sometimes, the cross is displayed upside down, known as a reversed cross or the cross of Saint Peter. Although the original meaning for this cross probably originated from Peter's request to be crucified upside down (so was Christian in origin), many have associated it with the occult and Satanism. Because satanists inverted the Christian pentagram, people believe they inverted the cross as well.

The ankh has a cross for a base, but an oval in place of the head of the cross. Sometimes, the ankh is referred to as an *ansata*, or handle, cross. This symbol was primarily used in Egypt as a symbol of life and fertility. Since its context was often in regards to resurrection, this symbol was used by Gnostic sects of early Christians to symbolize the resurrection of Christ. The ankh was actually used by Christians before the Latin cross. Wiccans currently use this symbol today to mean immortality and completion.

Another misunderstood religious symbol is the swastika. The swastika is a cross with its arms bent at right angles, most commonly so that the top arm is bent to the right and each remaining arm is bent in a similar clockwise direction (from the center) to give the impression of movement. When the arms are bent in the other direction, it can be called a "swastika" or it is sometimes referred to as a "sauwastika." The name is Sanskrit in origin and can be loosely translated as "good luck charm." Historical records show that the swastika is an ancient symbol (older than the ankh). Hindus use both forms of the swastika; the right facing means the evolution of the universe, whereas the left facing indicates the involution of the universe. Together, both versions are thought of as a balance of opposites. Buddhists primarily used the right facing swastika, although recently they have changed to using the left facing version, as the right facing version has become known as an anti-Semitic hate symbol since World War II. The swastika used by the Nazis was right facing but also rotated 45 degrees and appears different from the religious symbols. In Jainism, the swastika is the symbol for their seventh saint (or Jina). Jainists draw swastikas using rice to begin and end ceremonies around altars and idols. The swastika has also been used by Native Americans to represent the sun, the four directions, and the four seasons. Raelians use the swastika in a hexagram to denote that time is infinite.

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See Also: Houses of Worship; Incan and Mayan Mathematics; Mathematicians, Religious Mathematics and Religion; Numbers and God.

Religious Writings

Category: Friendship, Romance, and Religion.
Fields of Study: Connections; Communication; Number and Operations; Representations.
Summary: Mathematics and religious thought have been driven by the same motive: the need to better understand the nature of life and the universe.

In addition to its computational and problem-solving power, mathematics has long been joined to religious faith to form systems of mutual support. Evidence of the most productive relationships can be found in a variety of texts that call attention to mathematical concepts and knowledge as part of a religious or theological treatise. In other cases, the purported significance of mathematics to religion is a cause for antagonism and tension. Among the most persistent relationships evoked in writings that combine mathematics and religion is one that is understood to exist between their particular ways of knowing. Whether by way of analogy or more direct linkages, predominate characteristics of mathematical knowledge—its clarity, certainty, and timelessness—have often been called upon to serve theological contemplation.

Plato

Several Platonic dialogues feature extended discussions of mathematical knowledge in relation to philosophical and cosmological considerations, most notably the *Meno*, the *Timaeus*, and the *Republic*. Coaxing a geometric argument from an unsuspecting slave boy in the *Meno* serves as an epistemological lesson in humankind's ability to access certain and timeless knowledge. In the *Timaeus*, the power of mathematics as a system that provides a way of comprehending the physical world legitimizes adopting a cosmological perspective organized around identifiable characteristics, such as intelligence and goodness. The significance of mathematics to training philosopher-rulers as presented in the *Republic* is predicated on their need to reason effectively about ideal forms such as morality and justice. Although it would be incorrect to refer to them as "religious" in a strict sense, these Platonic dialogues establish a crucial link between mathematical and metaphysical contemplation frequently reflected in later theological writing.

Gregory of Rimini

Gregory of Rimini (c. 1300–1342) followed an Aristotelian mode of thinking, according to which abstract mathematical concepts exist only in the mind of mathematicians. Unlike their characterization within the Platonic tradition, mathematical entities have no existence independent of the objects that possessed them in terms of size, quantity, or other qualitative features. Even so, Gregory of Rimini's compiled *Lectures* undertake discussions of the continuum that ultimately challenge Aristotle's opinion on the impossibility of infinity as an actual or completed notion. This work intertwines discussions of divine omniscience, the temporal and spatial characteristics of angels, and the divisibility of the continuum, placing it squarely in a scholastic tradition that incorporates mathematical considerations within commentaries that focus primarily on religious subject matter.

Nicholas Cusanus

Although the author of several texts dedicated to Classical problems, such as squaring the circle, the philosopher and theologian Nicholas Cusanus (1401–1464) explicitly elaborated on the connection between mathematics and religion in *Learned Ignorance* (c. 1440). The significance of mathematical reason to theological contemplation discussed in this text is founded upon its ability to provide reliable and infallible knowledge about objects that transcend direct human experience. For Casanus, relations that exist between all things meant that one is able to develop an appreciation of unknowable objects based on other, better-understood objects. Polygonal approximations to a circle underscore this relationship. At the same time, Cusanus was aware that obtaining knowledge in this way depended on using various symbols and symbolic relationships in consistent and correct ways. The study of mathematics employed immutable symbols that avoided interpretive ambiguity and, thus, appealed to Cusanus as an appropriate framework for working with them.

Michael Stifel

In his 1532 *Book of Arithmetic About the Antichrist, A Revelation in the Revelation*, Michael Stifel (1468–1567) used computation skills and numerological inclinations to predict the end of the world. By doing so, he contributed to the fervor of the Reformation by associating the pope with the antichrist of the Book of Revelations. Indicative of his talents as a mathematician who pursued a lifelong fascination with numbers and their meaning, Stifel's 1544 book, *Arithmetica Integra* is considered his major achievement. In it, he explores and extends Pythagorean number theory, the construction of magic squares, the theory of irrationals, and the algebra of quadratic equations.

Galileo Galilei

Galileo Galilei (1564–1642) articulated a connection between mathematics and the divine that many found problematic. Like others before him, much of his writing asserted the superiority of mathematical reasoning, acknowledging it as the most certain way to both read and describe truths pertaining to the natural world. However, his praise of mathematics went considerably further in some texts, including the 1632 book, *Dialogues Concerning the Two Chief World Systems*. Specifically, Galileo maintained that human knowledge was indistinguishable from divine knowledge regarding those areas of mathematics to which it turned its attention. Consequently, mathematical reasoning provided unmitigated and unparalleled access to God's designs. As a threat to longstanding theological hierarchies, Galileo's pronouncements on mathematics were part of the indictments brought against him by church inquisitors.

René Descartes

A mathematical approach to reasoning is evident in the prescriptions set down by René Descartes (1596–1650) in his book, *Discourse on the Method*. Compelled by both skepticism and consistent criteria, he promoted a reductive framework for investigating problems that requires breaking up the analysis into pieces. Examining and understanding the simplest of the pieces would then lead to a solution. The first principle of this analytic approach allows one to establish a simple truth by virtue of its evident nature. Using mathematics as an exemplar for all reasoning therefore demanded an assurance of certainty. Descartes addresses this requirement in his 1641 book, *Mediations*. In particular, this work contains proofs of the existence of a benevolent and non-deceiving God, by virtue of which humans are able to recognize eternal truths for themselves. Although not above philosophical criticism, Descartes's work embraces mathematical and religious concerns of the time.

George Berkeley

George Berkeley (1685–1753) adopted a significantly antagonistic perspective on mathematics and theology. Some of his early writing evidences his affinity and appreciation for mathematics. However, later commentaries published while he served as the Bishop of Cloyne criticized mathematicians. Most notable among these are the 1732 book *Alciphron, or the Minute Philosopher* and the 1734 book, *The Analyst, or a Discourse Addressed to an Infidel Mathematician*. Berkeley asserted that mathematicians made unjust claims to exactness. His belief that the persuasive power of its problematic reasoning undermined the precepts of revealed religion only exacerbated this concern. Associating it with dogmatism and obscurantism, Berkeley was particularly hostile to the use of fluxions and infinitesimals, respectively, in the Newtonian and Leibnizian developments of calculus. One of his overarching objections pertained to the unacceptable admission of infinity in mathematics. Consequently, he attempted to establish the rule for computing the derivative of x^n in the *Analyst* by avoiding the use of either fluxions or infinitesimals.

Charles Babbage

Exemplary of natural theology in the nineteenth century, the *Bridgewater Treatises* were intended to provide commentary on modern scientific discoveries in relation to the Creation. In all, eight manuscripts were commissioned that discussed topics such as chemistry, geology, meteorology, and physiology. Mathematics was not one of the subjects included in the original commission, and Charles Babbage (1791–1871) took its omission as an opportunity to write his *Ninth Bridgewater Treatise*. Considered the father of modern mechanical computing, Babbage dedicated much of his life to designing the difference and analytic engines. His treatise highlights this work by arguing that events appearing miraculous can be accounted for as part of a grand design. As consummate a promoter as he was a mathematician, Babbage publicly illustrated this point several times with a model of the difference engine. These demonstrations involved programming the machine to break an identifiable recursive pattern at a moment that defied explanation by his audience.

Edwin Abbott

The enduringly popular 1884 book, *Flatland: A Romance of Many Dimensions*, introduced the concept of higher dimensional space to a wide readership. As its author, Edwin Abbott (1838–1926), drew upon his strengths as an educator, an expositor, and a theologian to convey multiple messages that relate to the mathematical imagination. Among these, scholarship has focused attention on progressive theological imperatives that he developed elsewhere and subtly incorporated into *Flatland*. Specifically, Abbott was keen to promote a form of theology that would be able to respond positively to new scientific attitudes and investigations. Mathematical research provided an ideal vehicle for Abbott, as discussions of non-Euclidean geometries suggested a loss of certainty within the discipline concomitant with a loss of religious certainty. Though perhaps the best known, Abbott joined and influenced other writers who used new developments in geometry as the impetus for renewed spiritual reflection that continued into the twentieth century, including Charles Hinton, Arthur Schofield, Peter Ouspensky, and Claude Bragdon.

Other Connections

There are other ways in which religion and mathematics are connected in writing. For example, mathematician Blaise Pascal produced many specifically religious writings, including *Provincial Letters* and the *Pensées*. Literary and religious scholars continue to study not only these works but also his mathematical and scientific writings to gain greater insight into his religious beliefs. A systematic study of the contributions of people from other cultures and religions to mathematics, such as Muslims or Hindus, or the geometric discussions in rabbinical writings also interest historians and mathematicians. Finally, while there are countless historical examples of mathematicians whose religious beliefs and mathematical work are philosophically intertwined, philosopher and mathematician Bertrand Russell's 1927 lecture, and later essay, *Why I Am Not a Christian,* has been called "devastating in its use of cold logic" in critiquing religious beliefs. A book containing this and related essays was included in the New York Public Library's list of the most influential books of the twentieth century.

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K. G. Valente

See Also: Greek Mathematics; Infinity; Mathematical Certainty; Mathematicians, Religious; Mathematics and Religion; Numbers and God; Proof.

Renaissance

Category: Government, Politics, and History.
Fields of Study: Geometry; Representations.
Summary: The Renaissance's resurgence in humanism also benefited mathematics and engineering.

The Renaissance or Rinascimento (both words mean "rebirth") was a flourishing of philosophy, art, architecture, science, and high culture more generally beginning in fourteenth-century Europe. Renaissance thinkers thought of themselves as restoring the civilization of Greece and Rome after what they called "the Middle Ages." The Renaissance saw the rise of humanism, hermeticism, Neoplatonism, and realist art involving optical perspective; the decline of feudalism; increased circulation of ideas due to printing; the Protestant Reformation; a strong interest in classical literature and history; a strengthened interest in science and mathematics and their applications; and world exploration.

Early Renaissance (c. 1300–1450)

The Renaissance can be traced back to the thirteenth-century writings of Dante Alighieri, Francesco Petrarca, and Brunetto Latini and the paintings of Giotto di Bodone. Such work was sponsored by bankers, merchants, and industrialists who rose to great wealth and influence, displacing the Church and landed nobility as primary sponsors of high culture.

Starting in the mid-fourteenth century, humanist scholars searched libraries to recover the lost texts of classical Rome. Many edited texts went to print, increasing their accessibility at (relatively) low cost. After approximately 50 years, attention turned to recovering the Greek heritage, which—though mostly lost in the West—had continued on in Byzantium. Many Greek scholars migrated west at this time, bringing their expertise and manuscripts to Venice, in particular. The recovery and translation of Plato's works, along with several tracts in neoplatonism and hermeticism,

The mid-Renaissance Basilica di San Lorenzo has a geometric regularity and an open lightness.

fueled an interest in applying simple numerical ratios and geometric regularity in fields as diverse as art and architecture, cosmology, alchemy, and musical tuning. The intentions included occult efforts to replicate cosmic structures, invoking astral influences at the human scale. More visceral results were achieved by composers, such as Josquin des Prez, who brought polyphonic techniques to Italy from the Low Countries, laying foundations for important Italian composers (such as Giovanni Pierluigi di Palestrina) toward the end of the sixteenth century.

Renaissance (c. 1450–1500)

The Renaissance spread north from Tuscany and across the Alps during the second half of the fifteenth century. Political philosophy, exemplified by Niccolò Machiavelli's *Prince* and *Discourses on Livy,* attempted a rational analysis of political structures contextualized by cultural difference and the practicalities of everyday life. Vernacular languages came to be used for scholarly writing, making texts more widely readable as did printing, which advanced rapidly with the establishment of fine publishing houses in the Veneto. Examples include the Aldine Press, where italic typefaces were invented and Erhard Ratdolt's press, which pioneered the printing of mathematical diagrams when producing the first edition of Euclid's *Elements* in 1482.

The mid-Renaissance was centered on the Republic of Florence, largely sponsored by a powerful banking family, the Medici. The ideals of this period are expressed in Florentine architecture, such as Filippo Brunelleschi's Church of San Lorenzo, which has a legible geometric regularity, bright and even light, openness, and a delicately balanced stillness. Ideals in painting included realism based on optical theory. Artists could occupy the leading edge of mathematical research; Piero della Francesca, for example, produced treatises on perspective theory in addition to painting with perspective techniques. Sculpture also developed a scholarly foundation through both historical study of the classical texts that had survived and hands-on dissection of fresh cadavers.

High Renaissance (c. 1500)

The High Renaissance lasted only briefly before transforming into Mannerism. It was focused on Rome, owing to the patronage of Pope Julius II. Art gained a level of dynamism best known through the works of Rafaello Sanzio (Raphael) and Michelangelo Buonarotti in Rome, and Tiziano Vecelli (Titian) and Giorgione in Venice. Leonardo da Vinci's *Last Supper,* Raphael's *School of Athens*, and Michelangelo's ceiling in the Sistine Chapel were painted during the High Renaissance.

Further north, the Renaissance adapted to local cultures and circumstances. In Germany, for example, goldsmiths crafted clocks, automata, and mathematical and astronomical instruments for their patrons. Reformation printers published a wide range of medieval texts alongside Lutheran tracts, largely shedding the refined typography of Venice in favor of speed and quantity. Gothic elements remained strong in the art and architecture of England, the Netherlands, and

Scandinavia and Renaissance influences reached those countries only after they had become Mannerist. Because of Protestantism, secular authorities replaced the Catholic Church as the primary sponsor of cultural works.

Renaissance Science and Mathematics

Renaissance scholars initially reacted against Scholastic natural philosophy by turning to Neoplatonism, taking an often mystical and magical approach to nature, often with practical goals. This shift can be seen in the intertwining of alchemy and astrology, for example, and in the wide range of applications described in Giambattista della Porta's 1558 book *Natural Magic*. The title reflects a distinction drawn between natural magic, which invoked empirical knowledge of nature to achieve results; in contrast to spiritual magic, which regulated astral influence using amulets and talismans; and demonic magic, which invoked supernatural beings.

The Church's need for calendrical reform led Nicolaus Copernicus to develop heliocentric astronomy as an improvement upon the Hellenistic methods maintained and developed throughout the Middle Ages. Astronomy was favored also in Protestant territories owing to the educational reformer Philip Melanchthon arguing that it was an ideal way to learn about divine creation.

Artillery motivated studies in ballistics, leading to stellated polygonal designs for fortresses, such as Naarden in the Netherlands and the Kronborg in Denmark. Aristotelianism, however, still provided qualitative theory for ballistics and other practical endeavors, such as hydraulic engineering.

The development of machines and engineering techniques inspired efforts to classify and theorize about them, as shown by the published "theaters of machines" by Jacques Besson and Agostino Ramelli.

The influences of exploration can be dated at least as far back as 1488, when Bartholomeo Dias found a connection between the Atlantic and the Indian Ocean that led to trade routes established beginning in 1498 with Vasco da Gama's arrival in Calicut, six years after Christopher Columbus found the West Indies. Such journeys motivated developments in navigation and shipbuilding as well as an outward-looking attitude. Trade expanded, especially in Spain, Portugal, and—as the new knowledge spread north—the Netherlands. Descriptions and specimens brought back from foreign regions caused disputes and reforms in biological taxonomy that were eventually settled in the eighteenth century by Charles Linnaeus.

Progressive rational problem-solving, combined with the growth of theoretical method and a growing preference for naturalistic rather than occult explanations, provided many elements needed for the eventual emergence of modern empirical science.

Mathematics was boosted early by the ascendance of merchants and bankers who needed computational methods to manage money and later to solve problems in navigation and cartography. Some advanced material was assimilated from Arabic sources, such as geometric methods and high-precision trigonometric tables. Solving polynomial equations became a display of virtuosity; the quadratic had been solved in antiquity, now Girolamo Cardano and other mathematicians developed solutions for cubics and higher order problems. As algebra developed, many algebraic symbols were invented and evolved into the forms used today. Hindu-Arabic numerals replaced Roman numerals but the calculation of the products, ratios, and square roots of large numbers in astronomy and navigation was still onerous and error-prone. These operations were facilitated by conversion into addition and subtraction problems using prosthaphaeresis (based on trigonometric transforms), and later through the invention of logarithms.

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Alistair Kwan

See Also: Algebra in Society; Castles; Exponentials and Logarithms; Marine Navigation; Middle Ages; Multiplication and Division.

Representations in Society

Category: School and Society.
Fields of Study: Connections; Representations.
Summary: Symbols, equations, and images are all used to teach mathematical concepts and to convey mathematical information in society.

Representations are at the forefront of the focus standards of the National Council of Teachers of Mathematics to improve mathematics teaching and learning. Representations allow students to see and experience mathematics from different perspectives. The role of multiple representations in promoting students' conceptual understanding of mathematics has long been emphasized by researchers. Thus, representations are among the essential parts of mathematics lessons. Further, in the twenty-first century, even people who had very little exposure to mathematics in school will encounter various mathematical representations in their daily lives. Familiarity with mathematical representations or representational literacy has become an essential skill. Many mathematical concepts are defined in terms of representations. A function may be represented by a Taylor series of infinite terms, which is named after Brook Taylor. There is also an entire branch of mathematics called "representation theory" that expresses algebraic structures using linear transformations.

Representations

Mathematics has its own native beauty and inspirational aesthetic to represent the physical world and the world of intellect. One of the strengths of mathematics is its resources to seek for new solutions and explore frameworks to answer problems related to the real world. To achieve this goal, mathematical representations in society should be explored and important ideas of modern mathematics should be communicated properly. Representations in mathematics can be described as constructs that symbolize or correspond to real-world mathematical entities, features, or connections. Gerald Goldin broadly defined representations as any configuration of characters, images, or concrete objects that can symbolize or represent something else. Representations take various forms, such as informal representations used in preschool settings or more formal representations used in mathematics classrooms or by mathematicians. For example, children represent groups of five with their hand or, even further, they develop proportional thinking as they relate five fingers to one hand and 10 fingers to two hands. More formally, mathematics students or mathematicians use mathematical equations, for example, to represent curves or relationships among financial variables.

Internal and External Representations

Representations can be both internal and external in nature and can be created by forming individual representations, such as letters, numbers, words, real-life objects, images, or mental configurations. Internal representations are mental images or cognitive constructs of individuals that relate to external representations or to experiences in the external world. James Kaput referred to internal representations as *mental structures* and defined them as instruments that are used to organize and manage the flow of an individual's experience. Internal representation systems exist within the mind of an individual and consist of constructs to assist in describing the processes of human learning and problem solving in mathematics. Internal representations of mathematical concepts can take various forms, such as individual visualization of mathematics concepts, idiosyncratic notation systems, or attitudes toward mathematics.

External representations, on the other hand, include all external entities or symbols. External representations provide a medium to communicate mathematical ideas, concepts, or constructs. Richard Lesh defined external representations as the embodiment of internal systems of thought. Lesh also referred to external representations as mathematical representations that are simplifications of external systems. Learners use external representations, such as marks on paper, sounds, or graphics on a computer screen, to organize the creation and elaboration of their own mental structures. Unlike internal representation systems, external representation systems can be easily shared with and seen by others.

Multiple Representations in Mathematics Education

In mathematics education, there has been a shift from classic to nontraditional teaching and learning practices with multiple representations, where educators use various representations to effectively present information. Multiple representations refer to different kinds of representations that present the same mathematical ideas from different perspectives or representations that present different aspects of the same mathematical concept. For example, teaching fractions concepts using multiple representations may involve presenting fractions in real-life contexts such as partitioning a pizza or a pie, allowing students to explore equivalent fractions using kinesthetic or virtual manipulatives, or providing students with pictorial representations of fraction operations in addition to formal mathematical representations. Teaching and learning with various kinds of representations provide students with hands-on and minds-on experiences and support a better understanding of mathematical concepts. Also, using multiple representations in mathematics education can help to alter the focus from a computational or procedural understanding to a more comprehensive understanding of mathematics using logical reasoning, generalization, abstraction, and formal proof. A substantial amount of research has demonstrated the effectiveness of multiple representations in enhancing students' conceptual understanding of mathematical concepts.

The notion of multiple representations in mathematics education commonly refers to external representations. However, one of the essential goals of mathematics education is to develop internal representation systems that interact well with external representation systems. James Kaput identified five interacting types of internal and external representations: (1) mental representations—internal representation—that learners construct by reflecting on their experiences; (2) computer representations that model mental representations through computer programs, which allow for arrangement and manipulation of information; (3) explanatory representations consisting of models or analogies that create the interaction between mental and computer representations; (4) mathematical representations, where one mathematical structure is represented by another mathematical structure; and (5) symbolic representations, such as formal mathematical notations.

To understand James Kaput's taxonomy of internal and external representations, consider the different types of representations related to the concept of "slope." When learning about positive slopes, a student might internally imagine a hill, which constitutes an internal (or mental) representation. This mental representation can be replicated on a computer screen. The student can create a unique model that incorporates the mental representation through a computer representation. If the model is viable, then it can be an explanatory representation for the concept of "slope." The student, then, can sketch a similar mathematical graph of the hill and can name the steepness of the hill with the mathematical notation, "slope." This graphical representation of slope can, then, provide support to represent the slope in a symbolic form as a rate of change ($y = mx + b$, where slope is represented with m and indicates the ratio of change on the y-axis to the change on the x-axis). As portrayed in this example, internal and external representations are not separate. Rather, they

Children learn groups of five with their hands or may develop proportional thinking as they relate five fingers to one hand and 10 fingers to two hands.

are intrinsically connected, and they interact continuously. Furthermore, a concept like slope is itself a type of alternative representation. In calculus, a curve is represented by the changing nature of its tangent vector, where the solution to the first derivative at a particular point is the slope of the tangent vector.

Translational Skills Among Different Modes of Representations

In addition to the importance of the effective interactions between internal and external representations in the acquisition and use of mathematical knowledge, it is essential that students develop fluency among different external representations. Richard Lesh enumerated multiple modes through which representations can be constructed: manipulatives, pictures, real-life context, verbal symbols, and written symbols. To demonstrate deep understanding of mathematics, students need to represent their mathematical ideas with different modes of representations and smoothly translate within and between those modes. For example, in algebra, students should be able to make the connection between graphical and algebraic or symbolic representations of equations. Similarly, students need to link what they learn using concrete or virtual manipulatives to both pictorial representations and abstract symbols. For instance, students who initially learn fraction operations using concrete or virtual manipulatives should be able to relate this knowledge when they later on learn fraction operations using symbolic and more abstract mathematical representations. Connecting different modes of representation simultaneously has been demonstrated to improve conceptual understanding as well as positive attitudes toward mathematics.

In mathematics education research, there is strong evidence that students can grasp the meaning of mathematical concepts by experiencing different mathematical representations and making connections and translations between these modes of representations. Using translational skills among different representational modes encourages students not to merely memorize theorems and facts but also to think analytically to reproduce and use them in real life problems or even in pure mathematical problems.

To deepen students' understandings, teachers should provide students with multiple representations of a single mathematical concept and focus on students' transition ability from one representation to another. Teachers need to be able to present one concept in multiple modes without relying on a single mode and provide students with appropriate transitions among these representations. Teachers should provide also students with ample opportunities to represent mathematical concepts in multiple ways and to connect these representations, thereby developing representational fluency. For example, asking a student to restate a problem in unique words, to draw diagrams to illustrate the concept, or to act out the problem are some ways to provide students with opportunities to translate among representations. If teachers fail to implement the transitioning among different representations, students will be less likely to see how different representations are related and will be more likely to develop misconceptions.

Multiple modes of representation can be used by teachers and students to enhance understanding of mathematics. Most research has shown that providing students with accurate representations improves student learning. However, different representational modes might have different impacts on student understanding. One mode might be more relevant or effective than another for teaching a specific concept. Or, some representational modes can be more appropriate at different developmental stages of the same concept. For example, research on teaching and learning of fractions has shown that students should be given the opportunity to develop mental representations of fractions using manipulatives before they are presented with symbolic representations. Thus, in addition to using multiple representations, choosing effective and appropriate presentations of information is crucial in teaching and learning. Representations that allow students to actively interact with the subject matter are more effective in student learning than representations that do not support students' active involvement.

Despite the research support for development of higher order thinking skills afforded by different representational forms, little is understood about how students interact with multiple representations in various learning environments. Even though each representation provides similar information, the strain that each representation puts on students' cognitive resources may differ. Not only do individual representations have different impacts on students' conceptual understanding but integrating multiple representations may also result in interaction effects among different modes presented. Therefore, integration of multiple representations becomes an important consideration in the design of instructions. Educators should employ caution as they integrate different modes into instruction, because delivering redundant information with different modes might interfere with learning.

Mathematical Thinking and Representations in the Twenty-First Century

An increasing number of daily activities in the twenty-first century require familiarity with mathematical representations and mathematical thinking. Mathematical thinking, which is a crucial tool for every member of society, includes skills such as pattern recognition, generalization, abstraction, problem solving, proof, and analytical thinking. Most companies prefer employees who are equipped with mathematical literacy or general mathematical skills. However, many students either do not necessarily understand these qualifications or do not value them enough. It is important to emphasize that all humans use mathematical thinking tools in their every day lives and workplaces, with or without noticing they are doing so.

It is not very hard to realize the extent to which mathematical representations are integrated into mundane objects and activities. Consider the number of newspaper columns that provide their readers with different kinds of mathematical representations to explain current issues. Topics in such columns include sports, economics, advertisements, and weather reports. For example, the growth of players, the statistics and ranking of teams, and teams' transfer budgets are represented in several representational modes, such as tabular data, textual information, visual representations, or graphical interpretation. Not only do sports fans need to understand the mathematical information provided readily to them but they also may need to use the mathematical information in problem solving situations, such as estimating the chances of their team's victory. More surprisingly, when a rivalry game is present, the provided data get even more complicated to analyze the chances of each team.

Even though the use of mathematical representations and information in economic and weather columns in various modes is apparent, the ones used within advertisements or political columns may be overlooked. Understanding the mathematical information included in advertisements and deciding which product to buy requires effective use of mathematical thinking tools. In most advertisements, companies present several payment options with different price ranges instead of giving just one price for a product. In particular, mortgage plans to buy houses and installment plans to buy cars require serious analyses of options to choose the best for a given budget. In political columns, on the other hand, one would not be surprised to see percentages representing the proportion of the population that supports various political parties in a country or the votes of a poll. Such information is not only presented as tabular data, visual charts, or graphs, but also as textual information, which is another mode of mathematical representation.

Representations in Problem Solving

Problem solving is one of the essential tools for mathematical thinking. A person equipped with problem solving skills does not necessarily need to have the knowledge base for the solution to each problem encountered but needs to know how to approach problems, locate and access information from different resources, and process information to solve the problem. For example, when one faces a novel problem, an approach to solving that problem can be forming an analogy between the new problem and another, previously solved problem. In other words, known information from an earlier problem can be mapped onto the novel problem. Brainstorming may be another valuable approach to gather different ideas on solution paths to unfamiliar problems. If a problem is too complex, problem solvers can try to break it down into more manageable parts (more solvable problems). One approach to problem solving is solving the problem step-by-step and taking an action at each step to get closer to the goal. Another solving approach can be conducting extensive research to analyze existing ideas and then adjusting possible solutions to the problem in hand. Finally, trial-and-error may be an approach to find a solution to an existing problem. It is emphasized in problem solving that there are many solution paths to a problem and a willingness to try multiple approaches is encouraged. Multiple approaches and strategies may be available and some of these approaches may be more efficient than the others.

Problem solving in mathematics, and in other fields as well, requires both knowledge of different representational systems and representational fluency that enables flexible use of various representational systems. For example, when solving a mathematical problem that asks how many quarters there are in 2 1/2, various strategies that involve different representations exist to approach the problem. A student may choose to translate this problem, which is represented in words, into a real-life context, such as how many quarter slices of pizza there are in 2 1/2 pizzas. Another student may opt to draw a picture that represents the given problem and solve the problem using the pictorial representation. Or, some students may represent the problem using symbolic representations and solve the problem accordingly. There may be other approaches where students start with a real-life context and then translate it to a pictorial representation, or where students come up with various relevant representations and choose the most efficient one for them. In more complex problems, different parts of the problems may require different representations. Thus, representational fluency is an essential part of problem solving.

Problem solving is such an important skill that is not only required to help students solve mathematical problems but also provides them with necessary tools to approach and solve problems in the real world. Because the real word does not have recipes to solve a problem, and problem solving requires structured, thoughtful, and careful analysis of problems (especially ill-defined problems) in various situations, people equipped with problem-solving skills are highly valued by employers.

Representational Skills

The National Council of Teachers of Mathematics presents representation as an important skill needed for students and teachers in teaching and learning mathematics in *Principles and Standards for School Mathematics*. Students should lucidly and coherently be able to express mathematical ideas through various representational modes, especially in writing and speaking. Through representational skills, abstract concepts can be manipulated into concrete concepts. Developing appropriate representation manipulation skills is necessary to improve conceptual understanding. Further, using various modes of representations, such as graphics, tabular data, mental images, physical objects, mathematical symbols and notations, drawings, and textual information, provides students with organizational skills to systematize their thinking and approach a concept from multiple views, leading to a more coherent understanding. With this ability, students can represent phenomena in a way that is meaningful to them. More importantly, the capability of representing a concept in numerous modes eliminates possible communication problems.

Mathematics as a Language

Mathematics is, to some extent, a language that is universal and can be understood in any part of the world without much difficulty. The mathematics language, which consists of both symbolic and verbal languages, has evolved as the most efficient medium to communicate mathematical ideas and information. Mathematics language also includes graphical images to effectively communicate mathematical concepts and ideas. Thus, different representational modes are used in communicating mathematical ideas and concepts. For example, when a mathematics teacher writes an equation and explains the equation in spoken language to a class, both verbal and written representational forms are in play. Communication in mathematics often involves a constant representational translation between symbolic and verbal representations. Symbolic and verbal languages of mathematics help to express ideas in a meaningful and efficient way. The evolution of mathematics language has been in progress for thousands of years. The goal of this progress is to improve the efficiency of communication, which is central to learning and using mathematics.

Before the emergence of mathematical notations and symbols, mathematicians found it difficult to share their knowledge with the community, even with other mathematicians. Even if a mathematician were able to prove a theorem, for example, geometrically without using mathematical notations and symbols, the mathematician might not have easily written down the proof to share it with others. Difficulties in representing mathematical ideas (writing in a concise and meaningful way using various mathematical notations and symbols) forced mathematicians to seek alternative (especially short and easy) forms to present their knowledge. The need for an effective and efficient mode of communication to convey mathematics ideas resulted in the development of the symbolic mathematical language.

Although the symbolic mathematical language is universal, the verbal mathematical language differs across societies or cultures. For example, although the American and the Japanese use the same symbolic notations to convey mathematical ideas, the verbal language each of these nations uses to communicate about mathematics is different. Differences in verbal languages to communicate mathematics have implications for teaching and learning mathematics. Verbal languages that are clearer about mathematical terms or that relate better to mathematical entities or ideas can support mathematical understanding. For example, counting in the verbal Chinese language is based on the concept of base-10 system. In Chinese, the number 11 is not an arbitrary word in the verbal language. Rather, in Chinese, 11 is "ten-one," 12 is "ten-two," 21 is "two-ten-one," 22 is "two-ten-two," and so on. In other words, the Chinese verbal language clearly conveys that there is one 10 and one 1 in 11 or there are two 10s and one 1 in 21. Such a clear relation between mathematical ideas and verbal language can be an important cognitive tool that supports mathematical understanding.

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See Also: Communication in Society; Connections in Society; Geometry in Society; Mathematical Modeling.

Revolutionary War, U.S.

Category: Government, Politics, and History.
Fields of Study: All.
Summary: The American Revolutionary War saw advances in mathematics cryptography and education.

The American Revolutionary War was a political and armed conflict between Great Britain and the British colonies on the North American continent between 1775 and 1783. Colonists who sought to end British rule and declare their political and economic independence supported the establishment of 13 colonial governments, each of which in turn sent representatives to Philadelphia to set up the Second Continental Congress.

This congress debated the state of political and economic ties to Britain, plied for support from other European powers, and discussed the possibilities and potential of a collective effort to make the separation official. Shortly after its inception, the Second Continental Congress formed a Continental Army and issued the Declaration of Independence. These actions announced the birth of a new nation: the United States of America. The "War of American Independence," as the American Revolutionary War is also called, saw fierce fighting in a wide variety of locations throughout the new nation and on the soil of virtually every new state. Some key battles were fought in Lexington, Concord, and Boston, Massachusetts; Saratoga and Ticonderoga, New York; Trenton, New Jersey; King's Mountain and Cowpens, South Carolina; and Yorktown, Virginia; among many other places.

The war lasted almost a decade and ended with the Treaty of Paris, which was signed at the Palace of Versailles in 1783 and recognized the sovereignty of the United States of America. There are many statistics available that relate to aspects of the war, including casualties and cost. For instance, some report that the British spent about £80 million while incurring a national debt of 250 million pounds, while the United States spent approximately $135 million, of which $37 million became the national debt. Mathematics was used in a wide variety ways, including in the design and implementation of artillery and in planning strategy and tactics. Mathematicians fought in the war, conducted surveys, and created and decoded ciphers. The mathematics educational system also changed significantly as a result of the war.

Louis-Antoine de Bougainville

Many historians agree that the Americans would have been unable to win the war without the political and military support of France and other allies. Louis-Antoine de Bougainville was a French mathematician who became the first Frenchman to sail around the world. In 1752, he wrote a calculus book, *Traité du calcul–intégral*, which brought him recognition within the mathematical community for his clear exposition and updates to differential and integral calculus. After a second edition and election to the Royal Society of

Cryptography

Early U.S. military intelligence began during the Revolutionary War. Paul Revere, William Dawes, and others used light signals to warn of invading forces before the battles at Lexington and Concord, which are generally considered to be the first military engagements of the war. James Lovell, who has been called the "father of American cryptanalysis," broke the British ciphers, which were rearrangements of letters. He used a method known as *frequency analysis*, which involves determining letters based on the frequency of symbols in the coded message.

Lovell discovered that the British often changed ciphers by shifting them instead of creating a new rearrangement and this made them easier to decode. Lovell also created his own cipher forms but these were deemed too confusing for those wanting to send and receive messages.

Paul Revere's ride used light signals to warn the public.

This belief was even true for Benjamin Franklin, who was well versed in mathematics and enjoyed magic squares recreationally. Franklin commented, "If you can find the key & decypher it, I shall be glad, having myself try'd in vain." American diplomats began to rely increasingly on replacements of words and other techniques instead of alphabet substitutions, and spies for both sides conveyed information about supplies and troop movements using codes. For instance, U.S. spy Benedict Arnold used book ciphering, in which a word is represented by a number that corresponds to a location in a book, in his communication with British intelligence officer John Andre.

London in 1756, he turned to a career in which he participated in numerous wars, including the Revolutionary War.

His astronomical observations became important to later explorers. He stated, "geography is a science of facts: one cannot speculate from an armchair without the risk of making mistakes which are often corrected only at the expense of the sailors." During the Revolutionary War, he was a commodore who supported the U.S. side.

Simeon DeWitt

U.S. Army geographer Simeon DeWitt subscribed to *The Mathematical Correspondent*, generally regarded as the first U.S. special-interest scientific publication. DeWitt was a student at Rutgers University when British troops burned the college buildings. He continued his study of mathematics and surveying on his own and was appointed the geographer of the army by General George Washington. After the war, he became surveyor-general of New York State.

Education

Mathematics education changed dramatically in the United States during and after the war. Before the war, students usually learned mathematics from British works, although Americans like Isaac Greenwood had written arithmetic texts. Advanced mathematics included algebra, geometry, trigonometry, calculus, and surveying techniques. Many colleges were shut down during the war because students and professors served as soldiers, and buildings were used for other purposes. However, some members of the army were trained in mathematics during the war. After the war, new primary schools and colleges were established. Between 1776 and 1815, numerous mathematics texts were published in the United States. Some of these were reprints of English works, and others were compilations or new works by American writers. In 1788, American Nicholas Pike published his text, *The New and Complete System of Arithmetick: Composed for the Use of the Citizens of the United States*, which contained both arithmetic and geometry. It was popularized by patriotic recommendations. There was also a change in the education of women. Prior to the war, it was thought that mathematics beyond simple arithmetic was unnecessary for women. After the war, mathematics educational opportunities began slowly to increase, as women were educated in mathematics to help in family businesses.

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Calli A. Holaway
Michael G. Lovorn

See Also: Artillery; Coding and Encryption; Strategy and Tactics.

Ride, Sally

Category: Space, Time, and Distance.
Fields of Study: Communication; Connections.
Summary: The first American woman in space, Sally Ride was a Mission Specialist and has become a science and mathematics education advocate.

Sally Kristen Ride, the first American woman in space, was born May 26, 1951, in Los Angeles, California. She attended Stanford University, and in 1973, earned Bachelor's degrees in physics and English. By 1978, Sally had earned Master's and Doctorate degrees in physics. After answering a newspaper advertisement for space program applicants, she was selected to complete the National Aeronautics and Space Administration's (NASA) rigorous astronaut training program. Upon completion, she served as capsule communicator on early space shuttle missions.

Time in Space

On June 18, 1983, Ride became the first American woman in space, serving as a mission specialist aboard the space shuttle *Challenger* for STS-7, commanded by Captain Robert L. Crippen and piloted by Captain Frederick H. Hauck. Soon after this historic 146-hour mission, Ride was selected as a mission specialist for STS 41-G. On October 5, 1984, again aboard the space shuttle *Challenger*, she began a mission that logged an additional 197 hours in space. Ride was training for her third space flight when the space shuttle challenger accident occurred in January 1986. As a result, her mission was cancelled but she was appointed to the Presidential Commission investigating the accident. After the investigation, Ride was assigned to NASA Headquarters in Washington, D.C., where she helped found NASA's Office of Exploration. Later, she worked at the Stanford University Center for International Security and Arms Control.

Post-Astronaut Career

In 1989, Dr. Ride accepted a faculty position at the University of California, San Diego, as a professor of physics, and she was appointed director of the California Space Institute. More than a decade later, she founded Sally Ride Science, an innovative science education company dedicated to supporting girls' and boys' interests in the sciences, mathematics, and technology. The company designs science education projects for elementary and middle school students. Ride has also authored several science books for elementary and middle school students, including *To Space and Back* (1989), *Voyager* (2005), *The Third Planet* (2004), *The Mystery of Mars* (1999), and *Exploring Our Solar System* (2003).

In 2003, Ride was assigned to the Space Shuttle Columbia Accident Investigation Board, and has since been named to several national committees, including the President's Committee of Advisors on Science and Technology, the National Research Council's Space Studies Board, and the Review of United States Human Space Flight Plans Committee. She has also served on the boards of the Congressional Office of Technology Assessment, the Carnegie Institution of Washington, the NCAA Foundation, the Aerospace Corporation, and the California Institute of Technology.

The Sally Ride Science Academy, which was created in 2009, focuses on training teachers to increase their students' interest in science and mathematics by changing the image of scientists. As Ride told *USA Today*, the perception that a scientist "is some geeky-looking guy who looks like Einstein, wears a lab coat and pocket protector . . . [is] not an image that an 11-year-old girl or a 10-year-old boy aspires to." In particular, Ride asserts that girls have difficulty seeing themselves as scientists: "A girl doesn't look at that stereotype and say, 'That's what I want to be when I grow up.'" The Academy trains teachers on how to utilize readings that show scientists and mathematicians in real-world roles, which helps students to visualize themselves as being able to take on those roles. Ride believes that society's view that girls are not good at mathematics and science is persistent and needs to be rectified. In order for girls to become interested in mathematical and scientific careers, society needs to portray those careers as "normal" for girls to pursue. Ride views herself as a role model, particularly for girls, and describes herself as "a pretty normal 10-year-old girl who grew up to be an astronaut."

In addition to having been inducted into the National Women's Hall of Fame and

Astronaut Sally Ride monitors control panels from the pilot's chair. Floating in front of her is a flight procedures notebook.

the Astronaut Hall of Fame, Ride has been the recipient of numerous honors and awards. She has received the NASA Space Flight Medal, the Jefferson Award for Public Service, the von Braun Award, the Lindbergh Eagle, and the NCAA's Theodore Roosevelt Award.

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Calli A. Holaway
Michael G. Lovorn

See Also: Spaceships; Weightless Flight; Women.

Risk Management

Category: Business, Economics, and Marketing.
Fields of Study: Algebra; Problem Solving.
Summary: Effectively assessing and mitigating risk can involve sophisticated mathematical analysis and modeling.

A feeling of security is essential for the welfare of all people, ancient or modern. There are many threats in the twenty-first century that can reduce the feeling of security, including financial problems, diseases, and crime. Threats feature different causes, which may be grouped into two main categories: natural (random), and intentional (malicious).

Natural causes are independent from human will (for example, natural disasters), while intentional causes relate to the action of some adversary (for example, a terrorist). Some origins of threats, such as illness or accidents, are not completely random; though an actual intentionality is missing, correlations can be found between human behavior and the unwilled events. It is clear that the intention of any intelligent being, humans in particular, is to maximize one's own benefit throughout an entire lifetime on the base of trade-offs between expenses and medium or long term returns. This goal justifies, among other risk management strategies, the common use of insurance policies and alarm systems.

Risk Assessment

In order to predict human behavior with respect to issues of risk, as well as to support the choice of protection strategies of any nature, risk assessment is employed. In order to assess the risk, a mathematical model is required. The most common and simple mathematical model for risk assessment consists of the following formula: $R = P \cdot V \cdot D$.

Risk (R) with respect to a specific threat (T) is a combination of three different factors:

- P, the expected probability of the occurrence of T (how probable is the threat?)
- V, the expected vulnerability with respect to T (how probable is it that T will cause the expected consequences?)
- D, the expected damage caused by T (if the consequences caused by the threat are endured, how damaging are the consequences?)

Note that the combination operator "·" is not necessarily a multiplier. Depending on the criteria used for the analysis and on the type of scale (linear or logarithmic), it can play different roles (even as a sum).

Risk can be evaluated both using qualitative and quantitative approaches. Qualitative indices use reduced scales of values of intuitive meaning; for instance: low, medium, and high. The advantage is that estimations can be more straightforward (though rougher) and computations can be easier. The disadvantage is that results are usually less rigorous, and the combination of qualitative indices is questionable. Quantitative approaches, on the other hand, use and produce values of parameters using well-specified metrics. The disadvantage is the difficulty of getting input data, which—being produced by expert judgments, statistical analyses, and stochastic modeling—are always affected by more or less relevant uncertainty errors. The advantage is that quantitative approaches enable possible automatic optimizations using appropriate algorithms.

In some approaches the $P \cdot V$ factor is compacted into a single factor, which will be defined as the frequency (F) of "successful" threats, expressed algebraically as $F = P \cdot V$.

An example of qualitative risk evaluation using associative matrices is reported in Table 1 using the estimated values of F and D to obtain R.

In quantitative approaches, risk is evaluated using a more formal approach, defining rigorous metrics for the three factors P, V, and D of the risk formula; for instance as follows:

- P is measured in number of threat events per year.
- $V = P(T \text{ success} \mid T \text{ happens})$, which is the conditional probability that a threat will succeed given that it happens.
- D is measured in monetary damages.

Therefore, in this case, the "·" operator is actually a multiplier, and the risk can be measured; for example, in dollars per year, which is a measurement of an expected periodic monetary loss. The input values of the risk formula can be obtained in several ways, including statistical approaches and stochastic process modeling.

Risk Mitigation

In order to reduce the risk, several mechanisms can be adopted. The (possibly iterative) process of assessment and mitigation is sometimes referred to as "risk management." The objective of this process is to find an optimal trade-off between the expense in protection mechanisms and the expected risk reduction.

Countermeasures can be very different, depending on the type of risk being faced. They include organizational modifications, periodic diagnostic checks, norms, insurance policies, patrols of agents and first responders, sensors and alarm systems, preventive maintenance, early warning, mechanisms for delaying the threat, emergency preparedness, and disaster management.

With reference to the risk formula, a countermeasure should be able to significantly reduce P, V, or D, or all of them at once. For example, in the case of a viral epidemic, a behavioral change (such as staying at home, using cars instead of public transportation, and frequently washing hands) can reduce P, a vaccine or a strengthening cure can reduce V, while warmth, rest, and medicines can reduce D.

Cost-Benefit Optimization

Countermeasures employed to reduce the risk feature their own cost. While the objective of organizations (such as companies, enterprises, or countries) is to maximize the so-called return on investment, the objective of human beings is to maximize their average welfare throughout their lives. Therefore, countermeasures are adopted whose cost and effectiveness is judged to be "adequate." A more formal approach consists in analytically predicting the benefits resulting from the selected countermeasures, which needs appropriate mathematical models. In quantitative approaches, the periodic Expected Benefit (EB) is defined as $EB = RR - CC$, where RR is the expected risk reduction in a specified time slot, and CC is the countermeasures cost in a specified time slot.

The RR parameter is evaluated using standard risk assessment methodologies. Depending on the countermeasures, the CC can depend on the length of the time slot. For instance, a vaccine can last a whole lifetime with no additional costs, while insurance has periodic costs; alarm systems have an initial expense for the buying and installation of devices and additional costs because of maintenance and power consumption. Furthermore, a reliable payback analysis requires considering not only the initial investment but also the financial

Table 1. Qualitative risk evaluation using associative matrices.

F ↓ D →	Low	Medium	High
Low	Low	Low	Medium
Medium	Low	Medium	High
High	Medium	High	High

concepts of cash flow, opportunity cost, and final value of the capital invested.

Once a suitable mathematical model for computing the *EB* has been defined, it is possible to perform a set of analyses, including parameter sensitivity and automatic optimizations.

The parametric sensitivity analysis aims to evaluate the impact of data uncertainty on the computed results. To be performed, it requires that input data are modified (increased or decreased by a certain percentage) and that corresponding results are evaluated. Depending on the results of the sensitivity analysis, models can be assessed as more or less robust to certain input parameters: the more the results are affected by variations in input parameters, the less the model is suitable to be evaluated using uncertain data.

Automatic optimizations can be performed using appropriate algorithms with the aim of maximizing the *EB* with possible external constraints, like a limited budget. For linear problems, operations research provides a set of algorithms, which can be suitable for multi-variable and multi-objective optimization of a specific function. For large non-linear problems, genetic algorithms, which mimic the evolution of live beings, can be adopted. Genetic algorithms, in particular, are based on the concepts of populations of solutions, selection, crossover, and mutations. Genetic algorithms have proven useful in solving a large number of optimization problems, including the ones regarding risk minimization, which are difficult or impossible to manage using traditional approaches.

In conclusion, when security relates to personal benefit maximization, mathematical techniques are involved, which can be very complex since they fall in the area of multi-objective optimization with external constraints and contrasting requirements. Operations research has investigated similar problems, which have even attracted interest from the communities of researchers in statistics and probabilistic modeling. In particular, Bayesian networks are among the formalisms suitable for the stochastic cause–consequences modeling using a graph-based approach, which can also be extended with decision and cost nodes (in such a case, they are named "influence diagrams"). Bayesian networks are direct acyclic graphs (DAGs) in which nodes represent random variables, and arcs represent stochastic dependencies quantified by conditional probability tables (CPTs). It can be formally demonstrated that a well-formed Bayesian network represents the joint probability density function of the problem described by the network. Several user-friendly graphical tools are available for the solution of Bayesian networks. However, solving algorithms belong to the NP-hard class, therefore, their efficiency tends to significantly worsen as the size and complexity of the network increases.

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Francesco Flammini

See Also: Earthquakes; Floods; Insurance; Life Expectancy; Mathematics Research, Interdisciplinary.

Robots

Category: Architecture and Engineering.
Fields of Study: Algebra; Data Analysis and Probability; Geometry; Number and Operations.
Summary: Robots, their motion driven by mathematical algorithms and coordinate or polar geometries, have long been incorporated into society and popular culture.

Robots and robotic systems are increasingly commonplace in many areas of daily life, such as manufacturing, medicine, exploration, security, personal assistance, and entertainment. In general, a robot is a mechanical device that can perform independent tasks guided by some sort of programming. Sometimes, robots are intended to replace humans in tedious or hazardous tasks. In others tasks, such as some surgeries, robots may actually exceed human capabilities. For many, the word "robot" brings to mind both futuristic androids, which are robots that are designed to look human and cyborgs, which contain both mechanical and biological components. Robots used in many industrial applications, such as in medicine, bomb disposal, and repetitive jobs, rarely resemble humans. However, several humanoid robots and robots that realistically mimic the look and behavior of animals have been produced. In 2008, a Japanese play was written and produced for both robots and human actors, and robot animals have sometimes been marketed as replacements for biological pets. The word "robot" can also refer to software-like Web crawlers that run automated tasks over the Internet to gather data, though "bot" is a more common name. The field of robotics generates many interesting problems in both theoretical and applied mathematics and benefits from the contributions of mathematicians. For some, the ultimate quest in the twenty-first century and beyond is to develop materials, technology, and algorithms to create robots that meet or perhaps exceed human levels of perception, behavior, and intelligence. Nano-robots, which are ultra-small robots about the size of a nanometer, might one day be developed for tasks like hunting and destroying cancer cells.

Brief History

Playwright Karel Capek is typically credited with introducing the word "robot" from the Czech word for "laborer," in his 1920 play *R.U.R.* (Rossum's Universal Robots). Another writer who popularized robots was Isaac Asimov, who introduced the term "robotics" in his 1941 short story *Runaround*. However, robotic devices can be found much farther back in history. One early robotic device was a water clock produced by the Babylonians, which used the mathematics of volumes and rates of water flow to calculate time. Greek mathematician Hero of Alexandra described the use of weights and ropes to construct a mobile cart that could be programmed to move along a path. In the thirteenth century, Muslim mathematician and scientist Abu Al-'Iz Ibn Isma'il ibn Al-Razaz Al-Jazari created a set of programmable musicians. The drummer was operated by a rotating shaft that manipulated levers to produce rhythms. Around 1495, Italian painter and mathematician Leonardo da Vinci used his knowledge of the mathematics of anatomy and bodily movement to sketch designs for a warrior robot outfitted in medieval armor.

Interest in robotics accelerated in the nineteeth century as early computer technology with punch cards began to be incorporated into systems such as that used for the Jacquard loom, named for Joseph Jacquard. Others, such as Pafnuty Chebyshev, studied the theoretical mathematics of linkages, inventing the Chebyshev linkage that converts rotating motion to approximate straight-line motion. Charles Babbage's mathematical engines were some of the first mechanical computers. These engines used finite differences to calculate the values of polynomials. Such inventions were forerunners of computer-controlled robot technology that quickly progressed in the mid-twentieth century to transistors and integrated circuits. Mathematician Norbert Weiner is often known as the "father of cybernetics," which is the science of self-regulating feedback systems, for his work and 1948 book *Cybernetics*: *Or Control and Communication in the Animal and Machine*.

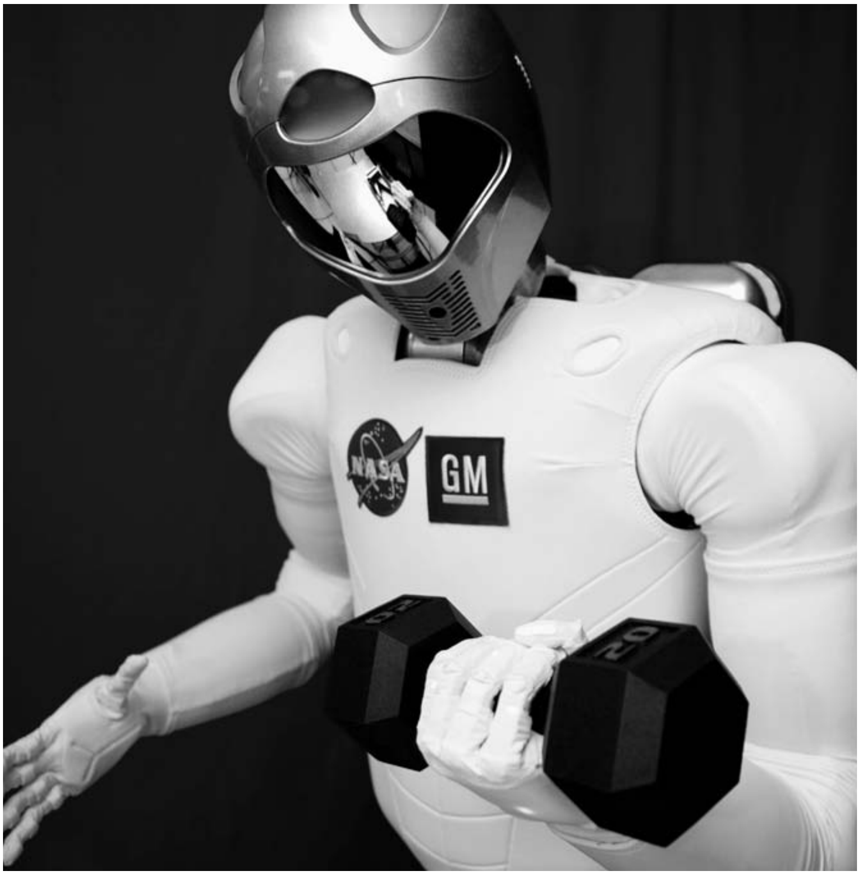
Cybernetics is not synonymous with artificial intelligence or robotics, but this mathematical discipline is essential for environmentally responsive or adaptive robots. Some other areas of mathematics that have contributed to the development and implementation of robots included algebraic and differential geometry, which is used to help solve problems, such as orientation and movement in three dimensions; partial differential equations, which are used to model many aspects of behavior; optimization algorithms to help sequence tasks; combinatorics, which is used to investigate modular components and systems; and Bayesian statistical methods, named for Thomas Bayes, which can be employed in dynamic perception and machine learning.

Robotic Motion

In the twentieth and twenty-first centuries, many robots are complex, electromechanical devices that move and interact with physical objects, often replacing or augmenting human actions by carrying out certain tasks. Some mobile robots use articulated legs or wheels. Somewhat more common are stationary robotic arms with joints that allow for motion similar to the way joints allow human limbs to move. Having more joints increases the possible angles for movement and degrees of freedom, and hence increases fluid motion and accuracy. Articulated robots, used widely in various industries to perform tasks such as welding components or spray-painting parts, look much like human arms and have at least three joints. If the joints are slide-only, called "prismatic joints," then the robot arm can reach any position in a rectangular workspace by means of translations. If one joint is hinged, which is called a "revolute joint," then all points within a cylindrical workspace can be reached by a combination of rotation and translation. If two of the joints are hinged, a robot arm with a polar geometry is achieved. Inventor George Devol and engineer Joseph Engelberger developed one of the first modern-day programmable robots, Unimate, which began operation in 1961 at a General Motors plant. In 1969, Stanford University student Victor Scheinman created the predecessor for all robotic arms, the Stanford arm.

Mathematical programming and calibration for proper movement of robots depends on kinematics, which is the study of motion; and dynamics, which is the study of how force affects motion. With articulated or jointed robots, for example, the mathematics of kinematics is at the heart of positioning, collision avoidance, and redundancy. Direct kinematics makes use of given joint values to determine the end position that a robot arm may achieve. The mathematics of inverse kinematics is used to determine the required values for the joints when the end position of the robotic arm motion is known. Getting the robot arm to the right position is only half of the mathematical problem. The other half involves calculating forces using dynamics. For example, a robot designed to fight fires would need motors to move the robot and its arms. Calculations incorporated in determining which motors to use would involve dynamics. Inverse dynamics would help determine the required values of forces to generate the desired acceleration of the robot or its components. The movement involved in robotics most often occurs in three-dimensional space, so geometry plays a role in the positioning and movement of robots. Matrices can be used to represent the points through which robots navigate. These algebraic representations are then reviewed and coordinated using sophisticated applications of basic calculus principles, like differentiation, to ensure maximum efficiency when designing and operating robots.

Movement and action in robots are driven by algorithms. Some robots respond to direct human input from keyboard commands or from haptic devices that respond to tactile or body motion. Others autonomously perform programmed tasks. Some robots are "smart" or "intelligent," meaning that they are able to sense and adapt to their surroundings while completing their tasks. Even then, these robots are able to



In November 2010, Robonaut 2 was brought to the International Space Station where it will remain as the first humanoid robot to work in space.

accomplish tasks only because they have been programmed to do so. For example, "smart" mobile robots make use of a variety of sensors with terrain-identification and obstacle-detection programs using input data and probabilistic models to guide trajectory and avoid collisions. Probabilistic robotics is increasingly of interest, with the goal of developing algorithms that facilitate accurate autonomous decision making in the face of real-work complexity and uncertainty, which would increase the reliability of automated behavior and more closely replicate the type of processing that occurs in the human brain.

Robots: Fiction and Fact

Robots are widely used in entertainment, especially science fiction. Mary Shelley's 1818 novel *Frankenstein* is cited by some as showing that scientific creations able to perform human tasks long preceded television and movies. Some well-known examples include C-3PO from the *Star Wars* series and Wall·E from the 2008 Pixar movie of the same name. Data, from the 1987–1994 television series *Star Trek: The Next Generation*, is an example of a fictional android. The Borg species from the *Star Trek* series and the Terminator robot from *The Terminator* movie series are examples of cyborg characters, usually hybrid humans whose biological capabilities are sustained or enhanced through robotic elements—though the Terminator may be thought of by some as a robot enhanced by biology. Enhancing human capabilities through robotic elements, like pacemakers and prosthetic devices, is common in the twenty-first century. However, the medical applications of robotics have not focused on humans achieving superhuman powers (as is done in fiction) but rather on helping those with medical conditions and disabilities.

Robots in Education

Robots are often used in schools to motivate learning of mathematics concepts, such as two- and three-dimensional coordinate geometry. The roBlocks construction system was developed by computational design scientists Mark Gross and Eric Schweikardt. Users can build robots using modular sensor, logic, and actuator blocks to study concepts like kinematics, feedback, and control. They can also create their own control programs to further explore robot mathematics and dynamics. The Lego Group produces a robotic construction and programming system called Mindstorms NXT that has been marketed for both education and entertainment.

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See Also: Coordinate Geometry; Interplanetary Travel; Matrices; Nanotechnology; Neural Networks; Science Fiction; Surgery.

Roller Coasters

Category: Games, Sport, and Recreation.
Fields of Study: Algebra; Calculus; Geometry; Measurement.
Summary: Roller coasters are mathematically designed to provide safe and thrilling rides.

Roller coasters are entertainment rides designed to put the rider through loops, turns, and falls, inducing sudden gravitational forces. The rapid ascents and descents coupled with sharp turns create momentary sensations of weightlessness. One known precursor of roller coasters are seventeenth-century Russian ice slides, which sent riders down a tall, ice-covered incline of roughly 50 degrees. Modern roller coasters can be traced to the late 1800s. As of 2010, Ohio's Cedar Point held the record for most roller coasters (17) in a single amusement park.

Conservation of Energy

The law of conservation of energy states that energy can neither be created nor destroyed, but can only be converted from one form to another. Roller coasters exploit this law by converting the potential energy gained by the car as it ascends to the top of a hill into kinetic energy as it descends and goes through the turns and loops. The potential energy of the car at the top of the loop is given by

$$E = m \times g \times h$$

where E is the total potential energy (joules), m is the total mass of the car (kg), g is the acceleration due to gravity (9.8 m/s^2), and h is the height (m).

For example, consider a roller coaster car weighing 2200 pounds perched at the top of Cedar Point's Top Thrill Dragster, which is about 426 feet high. The car, at this point, has accumulated $1000 \times 9.8 \times 130 = 1{,}274{,}000$ joules or 1.2 megajoules of energy—the same amount of energy released by the explosion of a quarter kilogram of TNT. This potential energy is converted into kinetic energy as the car hurtles down the loops.

As the car expends potential energy, it is converted into kinetic energy, propelling it forward. In an ideal situation where there is no friction or air drag, the car would travel forever. However, because of friction and other resistive forces, the car decelerates and finally stops when it has expended all its potential energy.

Centripetal Force

Centripetal force is responsible for keeping the rider glued to the seat as the car executes turns and loops and even puts the rider upside down. Centripetal and centrifugal forces act on a body that is traveling on a curved path. Whereas centrifugal force is directed outwards, toward the center of curvature, centripetal force acts inward on the body.

G-Force and Loop Design

G-forces are non-gravitational forces, and can be measured using an accelerometer. Humans have the ability to sustain a few *g*'s (a few times the force of gravity), but deleterious effects are a function of duration, amount, and location of the g-force. Many roller coasters accelerate briefly up to six *g*'s, depending on the shapes, angles, and inclines of loops, turns, and hills. Early roller coaster loops were circles. To overcome gravity, the cars entered the circle hard and fast, which pushed riders' heads continually into their chests as the coaster changed direction. In the 1970s, coaster engineer Werner Stengel worked with National Aeronautics and Space Administration (NASA) scientists to determine how much force riders could safely tolerate. As a result of this and other mathematical investigations, he began to use somewhat smoother clothoid loops, which are based on Euler spirals, named for Leonhard Euler. In 2010, using the same equations that describe how planets orbit the sun, mathematician Hanno Essén drew a new and unique series of potential rollercoaster loops. Riders would get the thrilling visual experience of a loop without any of the typical jolting and shaking, because the force that riders would feel pushing them into their seats would stay exactly the same all the way around the loop.

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Ashwin Mudigonda

See Also: Energy; Gravity; Weightless Flight.

Roman Mathematics

Category: Government, Politics, and History.
Fields of Study: Connections; Number and Operations; Representations.
Summary: The ancient Romans, who are often remembered for their applied mathematics, made important contributions to surveying, time-keeping, and astronomy.

The Roman period for mathematics could be said to have started when a Roman soldier was sent to seize Archimedes during the capture of Syracuse. Told by Archimedes to wait as he finished his diagrams, the soldier lost patience with the old man and slew him. The popular stereotype of the Romans is that they did little to advance Greek discoveries in mathematics, instead merely applying Greek methods to practical problems. This conception is not entirely fair. The Roman Empire was not one homogenous zone, but was rather a collection of culturally diverse provinces. For this reason, many works produced during the time of Roman rule, like the books of Ptolemy, writing in Alexandria, Egypt, are written in ancient Greek rather than Latin. Therefore, these books could be considered Greek, Roman, or Greco-Roman depending on the context. However, despite this diversity, the Roman period led to the dominance of some mathematical practices that still have an influence in the twenty-first century.

Roman Numerals

One of the most distinctive remnants of Roman mathematics is the use of Roman numerals, which are letters that stand for specific values and usually work as additive values. The numerals are

I = 1	V = 5	X = 10
L = 50	C = 100	D = 500
M = 1000.		

So: LXXVII = $50 + 2(10) + 5 + 2(1) = 77$.

The numerals are written with the largest values at the left, proceeding to the smaller values. They can also have subtractive constructions. I preceding subtracts one from a 10 to make nine. X before an L or C produces 40 or 90, and C before D or M produces 400 or 900. So

MCMXLVIII =

$$1000 + (1000 - 100) + (50 - 10) + 5 + 3(1) = 1948.$$

The origins of the system are unknown. It has been proposed that they were based on tally marks, with I being a notch, V being a double notch to mark five, and 10 as crossed-notches (though it could also be that X was formed from two V symbols). The number IV to represent 4 is a later addition based on medieval Latin and does not seem to have been used by the Romans, who instead used IIII.

This system is not very helpful for arithmetic, and so it is little surprise to find that the Romans developed the portable abacus to ease mathematical operations. This device was a tray with a number of columns etched into it that could hold pebbles. A pebble (in Latin, the word "calculus") had a value depending on the column that held it. Moving a pebble a column to the left increased its value by a factor of 10. Such an abacus could be used by merchants in the city or by surveyors working for the military.

Survey

Roman surveyors employed geometry to divide the landscape and lay out cities with effects that can still be seen in the twenty-first century. The key to Roman survey was a tool called a *groma*, which was a tall staff with a beam, known as a *rostro*, at right-angles to the staff at the top. The rostro supported a wooden cross, and at each end of the cross-beams was hung a plumb line. Sighting across these lines allowed Roman surveyors to lay out grids of perpendicular lines in the landscape. Surveyors could then divide land for agricultural purposes, and some field systems in Europe are based on these ancient surveys. The *groma* also left an impression on modern cities. The Romans frequently built new cities in conquered territories, for either native inhabitants or new settlements of veteran soldiers. At the heart of a Roman settlement lay the forum, the central civic space, which usually lay at the intersection of the Cardo maximus (the main north-south street) and the Decumanus maximus (the main east-west street). This system created new cities with grid-plans in which the main intersection was laid out by a *groma*. These perpendicular grids were the origins of many European settlements and was adopted in the planning of many U.S. cities in the nineteenth century.

The Roman Calendar

The Roman calendar instituted by Julius Caesar made a radical change to time-reckoning in Europe. Before this development, European calendars outside Rome were usually luni-solar calendars. As such, each month was related to the lunar cycle, which is not commensurate with the solar year, and so periodically whole months, known as "inter-calary months" would be inserted into the year to keep the months in step with the seasons. Insertions would usually have to be done every two or three years. Even ancient authors recognized that this system was inefficient, including Herodotus, who wrote in the late fifth century B.C.E. that the Egyptians had a much more accurate solar calendar. In 45 B.C.E., Julius Caesar adapted the Egyptian method of time-keeping for Roman use.

Each month was counted as a period of days, usually 30 or 31 but with 28 or 29 in February. In addition, Julius Caesar laid down rules for when an inter-calary day would be added to February. The Egyptians corrected the calendar by adding a day every fourth year. Unfortunately, the Romans counted inclusively, meaning that the leap year was in the fourth year, rather than after the fourth year. For example, 2020 is a leap year. For the ancient Romans, the second year in the cycle is 2021 and the third is 2022. Therefore, 2023 is the fourth and the Romans of Julius Caesar's time would have made this a leap year, rather than 2024. Augustus Caesar corrected this error in the early years of the first century C.E.

This method of keeping the years remained until the reforms of Pope Gregory XIII in 1582, though Britain and the American colonies did not implement the Gregorian calendar until 1752. The difference between the two calendars is that years divisible by 100 are not leap years, unless the year is divisible by 400. Otherwise, years are marked by the same cycle of months as the ancient Romans did.

Mathematics and the Cosmos

Even though ancient mathematicians had a relatively small set of tools based in geometry and arithmetic, these could be used to create incredibly intricate models. Ptolemy proposed a model of the universe that contained circles rotating upon circles to reproduce the movement of the planets. The connections between mathematics and cosmology made mathematics attractive to philosophers of the Roman period. The assertion that mathematics could reveal truth became increasingly contentious in late antiquity. Pagan philosophers came into conflict with a new religious sect, Christianity, which was increasingly powerful. One notorious incident was the killing of Hypatia, a female mathematician philosopher, in the city of Alexandria by a Christian mob. For some ancient historians, her death marks the end of the period known as classical antiquity.

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Alun Salt

See Also: Arabic/Islamic Mathematics; Archimedes; Calendars; Greek Mathematics; Sacred Geometry.

Ross, Mary G.

Category: Mathematics Culture and Identity.
Fields of Study: Algebra; Communication; Connections; Data Analysis and Probability.
Summary: Mary Ross was a prominent Native-American mathematician and engineer.

Mary G. Ross (1908–2008), a Native American of Cherokee heritage, had a distinguished career as a mathematician, space scientist, and engineer. She was the first female engineer to work at the Lockheed corporation and also the first female Native-American engineer. Ross was born in the Oklahoma territory and as a child lived with her grandparents in the Cherokee Nation of Tahlequah in order to pursue her education. She often credited a strong family and tribal focus on equal education for boys and girls as being crucial to her career. At age 16, she enrolled in Northeastern State Teachers College (Oklahoma), receiving her bachelor's degree in mathematics in 1928. Ross taught high school mathematics and science in Oklahoma for nine years before moving to Washington, D.C., to work as a statistical clerk in the U.S. Bureau of Indian Affairs. Her talent and education were quickly recognized and she was reassigned to work as an advisor (similar to a dean) for a coeducational Indian boarding school in Santa Fe, New Mexico (later to become the Institute of American Indian Art). At the same time, she pursued graduate studies in mathematics and astronomy, receiving her master's degree from Colorado State Teachers College in 1942. Ross received numerous awards during her lifetime.

Aeronautical Engineering

In 1942, Ross began working as a mathematician at the Lockheed Aircraft Corporation. She was given the opportunity to study aeronautical and mechanical engineering, taking evening classes at UCLA as well as an emergency war training course offered at Lockheed and, in 1949, received professional engineering classification as a mechanical engineer (there was no classification for aeronautical engineering at the time). As a research engineer at Lockheed, Ross worked on a number of projects related to transport and fighter aircraft and, in 1953, was chosen to be one of 40 engineers who became the nucleus of Lockheed Missiles and Space Company, now known as Lockheed Martin. In this group, she worked on a number of missile systems, including the Polaris ballistic missile, which required her to work in the new field of hydrodynamics because the Polaris missile was designed to be launched underwater from a submarine.

Ross continued to advance at Lockheed, becoming a research specialist in 1958, an advanced systems engineer in 1960, and a senior advanced systems engineer in 1961. She worked on the Agena series of rockets and the Polaris reentry vehicle. She also helped develop criteria for missions to Mars and Venus, designing orbital space systems and interplanetary expeditionary systems and writing a volume of the *NASA Planetary Flight Handbook*. About her career, she said, "I have always considered my work a joint effort. I was fortunate to have worked on great ideas and with very intelligent people. I may have developed a few equations no one had thought of before but that was nothing unusual—everybody did that . . . it has been an adventure all the way."

Other Accomplishments

Ross became an advocate of women's and Native-American education following her retirement from Lockheed in 1973. Her great-great-grandfather was principal chief of the Cherokee for 40 years, and she expressed the idea that, "there is a lot of ancient wisdom from Indian culture that would help solve the problems of today." She co-founded the Los Angeles section of the Society of Women Engineers and also worked to expand educational opportunities within the American Indian Science and Engineering Society and the Council of Energy Resource Tribes.

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Sarah Boslaugh

See Also: Airplanes/Flight; Interplanetary Travel; Minorities; Women.

Ruler and Compass Constructions

Category: History and Development of Curricular Concepts.
Fields of Study: Communication; Connections; Geometry; Measurement.
Summary: Ruler and compass constructions form the basis of geometry and have challenged mathematicians for thousands of years.

Ruler and compass constructions have long been important in mathematics. In geometry, a ruler and compass construction refers to a geometric construction that uses only an unmarked ruler and a compass. The ancient construction problems of squaring the circle, duplicating the cube, and trisecting the angle were unsolved until they were proved impossible by algebraic techniques. Early tile makers and architects were also interested in these constructions. Aside from historical considerations, limiting constructions to these two tools is important because the restrictions generate a variety of rich problems. In the twenty-first century, dynamic geometry software programs allow students, teachers, and researchers to explore, save, and share constructions.

Euclid

The most significant early compendium of ruler and compass constructions is Euclid's *Elements* written c. 300 b.c.e. In fact, Euclid's book organizes everything around these constructions in an attempt to build as much geometry as possible starting with the most basic tools. Drawing a line using a ruler and a circle using a compass are seen as elementary in Euclid's tradition—hence, the title *Elements*—and it is preferred to reduce as much of geometry as possible to these elementary tools. *Elements* begins with five common notions and five "self evident" postulates. The first three postulates specify the rules for geometric constructions:

- A straight line segment can be drawn joining any two points.
- Any straight line segment can be extended indefinitely in a straight line.
- Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.

The final two postulates of Euclid are

- All right angles are congruent.
- If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

The last one is the famous fifth postulate and is equivalent to the more common parallel postulate: from a given point not on a given line, one can draw exactly one line parallel to the given line. Euclid based the whole edifice of rigorous geometry on these axioms, hence ruler and compass constructions are at the center of Euclidean geometry.

The Three Classical Problems

Three ancient construction problems captured the imagination of mathematicians for many centuries: doubling a cube, trisecting an angle, and squaring a circle.

- *Doubling a cube*: Given the side of a cube, can one construct, using an unmarked ruler and a compass, the side of another cube whose volume is twice the first one?
- *Trisecting an angle*: Given an arbitrary angle, can one draw a line, using an unmarked ruler and a compass, that trisects the angle?
- *Squaring a circle*: Given a line segment that is the radius of a circle, can one construct, using an unmarked ruler and a compass, the side of a square that has the same area as the original circle?

None of these constructions are possible, but surprisingly, despite more than 2000 years of effort, a satisfactory answer to these three questions was given only in the nineteenth century.

Each of these classical problems has a long history. For example, the problem of doubling a cube was known to the Egyptians, Greeks, and Indians. In one version of the Greek legend, the citizens of Athens consulted the oracle of Apollo at Delos to put a stop to a plague in Athens. The oracle prescribed that the Athenians double the size of their altar. Efforts to find a way of doubling the volume of the cube failed, and it is claimed that Plato (427–347 B.C.E.) had remarked that the oracle really meant to "shame the Greeks for their neglect of mathematics and for their contempt of geometry." The original legend did not specify the tools to be used, and, in fact, solutions using a number of tools were found. However, a construction using the elementary tools of an unmarked ruler and a compass remained elusive.

Tool Variations

Variations on the tools are possible. For example, if one were allowed to make two marks on the ruler, then with the use of this marked ruler and a compass, one can trisect an arbitrary angle.

An interesting variation arose in the work of Abu'l Wafa Buzjani (940–997 C.E.). Abu'l Wafa in a work aimed at artisans (such as tile makers, designers of intricate patterns, and architects) limited the geometric tools to an unmarked ruler and a "rusty" compass. In other words, he wanted to only use a compass that had a fixed opening and could not be adjusted to draw different sized circles. He believed that working with such a fixed compass would be more accurate, less error-prone, and more useful for artisans. Abu'l Wafa constructs, among other polygons, regular pentagons, octagons, and decagons using a rusty compass. Since the opening of the compass used in Euclid's *Elements* could vary, Abu'l Wafa could not rely on the constructions in *Elements*. Hence, he constructed anew, using the rusty compass, all the needed basic results.

In Europe, the Danish mathematician Georg Mohr (1640–1697) showed, rather surprisingly, that all ruler and compass constructions can be done with a compass alone. In such constructions, one cannot draw a line segment, and a line segment is considered constructed as long as its two endpoints are found. This result is now known as the Mohr–Mascheroni theorem. The Italian Lorenzo Mascheroni (1750–1800) had independently found the same result. Georg Mohr also proved that all ruler and compass constructions can be done with a ruler and a rusty compass. Finally, the German mathematician Jacob Steiner (1796–1863) and the French mathematician Jean-Victor Poncelet (1788–1867) proved that all constructions using a ruler and a compass can be made with a ruler and only one use of the compass.

Proofs

Going back to the classical problems, the first rigorous proof of the impossibility of doubling the cube and trisecting an arbitrary angle using a ruler and a compass was given by the French mathematician Pierre Laurent Wantzel (1814–1848). In 1882, the German mathematician Ferdinand Lindemann (1852–1939) proved that π is transcendental. From this, it followed that one cannot square a circle using a ruler and a compass. In general, using only these tools, it is possible to construct line segments of any rational length as well as line segments whose length is the square root of the length of any already constructed segment. However, one can prove that it is impossible to construct other lengths using the theory of fields that was developed with the help of Niels Henrik Abel and Évariste Galois on the solvability of equations. The proof essentially boils down to the fact that, using a ruler and a compass, one can draw only straight lines and circles, and the only new points are the intersections of these lines and circles. Since lines have linear equations and circles have quadratic equations, finding the points of intersection of these shapes is the same as equating their equations and finding the solutions. These all can be achieved using the quadratic formula, which involves only square roots.

Polygons

Constructing regular polygons with a straightedge and compass is also an interesting ruler and compass construction problem. An n-gon is a regular polygon with n sides. Ancient Greeks could construct regular n-gons for $n = 3, 4, 5$, and 15 (triangles, squares, regular pentagons, and regular pentadecagons). They also knew that if one can construct a regular n-gon with a straightedge and compass, then one can also construct a regular $2n$-gon. Carl Friedrich Gauss (1777–1855) added to this knowledge, by constructing, when he was 19 years old, a regular heptadegon (a 17-gon).

A Fermat prime is a prime number of the form $2^{2^k} + 1$, where k is a non-negative integer. The only Fermat primes known are 3, 5, 17, 257, and 65537. It is not known whether there are any other Fermat primes or not. In any case, Gauss stated, and Wantzel gave a proof, that a regular n-gon is constructible with ruler and compass if and only if n is an integer greater than two such that the greatest odd factor of n is either one or a product of distinct Fermat primes.

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Shahriar Shahriari

See Also: Arabic/Islamic Mathematics; Greek Mathematics; Measurement, Systems of; Measurements, Area; Measurements, Length; Parallel Postulate; Pi; Squares and Square Roots.

S

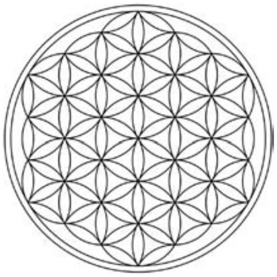
Sacred Geometry

Category: Friendship, Romance, and Religion.
Fields of Study: Connections; Geometry; Number and Operations; Representations.
Summary: Cultures have long imbued various spaces, shapes, forms, ratios, and geometric concepts with special significance and ritual power.

Humanity has long attributed sacred meaning to certain geometric forms and concepts. The term "sacred geometry" was popularized during the twentieth century to represent the religious, philosophical, and spiritual beliefs surrounding geometry. The core of its teachings may be found in very ancient cultures, with varying metaphysical systems and worldviews. Some attribute the modern renaissance of the movement to artist Jay Hambridge. The image of a nautilus shell with overlaid golden rectangles is common in the twenty-first century, but when Hambridge investigated mathematical proportion and symmetry in Greek art and architectural design in the beginning of the twentieth century, his work on dynamic symmetry led to debate about definitions of dynamic versus static symmetry.

The development of sacred geometry led to more debate as some asserted that it showed the continuity and universality of mathematical concepts or forms, such as the golden proportion, the logarithmic spiral, or the flower of life, across cultures, millennia, and the universe. In its most common conception, sacred geometry is then a metaphor for universal order—a metaphor found in the artistic expression of many cultures, especially in religious architecture. In its most ambitious conception, it is itself a practice for enlightenment or self-development, similar to meditation, prayer, or artistic techniques. The knowledge and exercise of geometrical skills can be taken to form a practice that awakens the practitioner to underlying order or truth. The movement has inspired its followers, who look for these forms in art, architecture, nature, and science. People like Drunvalo Melchizedek, who originally planned to major in physics and minor in mathematics but graduated with a fine arts degree, have organized spiritual workshops related to sacred geometry. Some attribute sacred geometry to people's needs to seek out connections. Astrophysicist Mario Livio found some of the analyses "rather contrived . . . with lines drawn conveniently at points that are not obvious terminals at all. Furthermore, some of the ratios obtained are too convoluted . . . to be credible."

Sacred diagrams and figures are omnipresent across ages and cultures. For example, the square has religious significance in Hindu architecture and design. The diagram known as the circular "mandala," for instance, symbolizes to some the cosmos through its symmetry and sectors, which represent elements,



Flower of Life mandala

seasons, divinities, and various categories of religious and metaphysical interest. Practitioners believe that meditating on The Flower of Life icon, one example of a mandala, will reveal the mysteries of the universe. The Egyptians used regular geometric polygons and pyramids in important architectural structures and in representations of the gods. Geometric figures, such as the platonic solids, were assigned additional significance in ancient Greece.

For instance, Earth was associated with the cube, air with the octahedron, water with the icosahedron, fire with the tetrahedron, and the dodecahedron was a model for the universe. In his work *The Timeas*, Plato noted: "So their combinations with themselves and with each other give rise to endless complexities, which anyone who is to give a likely account of reality must survey." In the twentieth century, sacred geometry has become the universal language of nature, mastering shapes and patterns equally found in stars, snowflakes, and DNA, which ultimately represent a sort of blueprint of creation.

Golden Ratio

A common element in sacred geometry is the golden ratio. Many of the sacred geometry principles of the human body are found and subsumed into the famous "Vitruvian Man" drawing by Leonardo Da Vinci. "Vitruvian Man" was inspired by the work of Marcus Vitruvius Pollio, a first century Roman architect who wrote *De architectura*, or *The Ten Books on Architecture*. Vitruvius detailed systems of ratios he believed were found in the human body and that could be used to construct buildings, including temples, to achieve his three necessary criteria for structural perfection: beauty, durability, and utility. Da Vinci also lived and studied with the fifteenth-century mathematician Fra Luca Pacioli and drew the illustrations of the book *De Divina Proportione* (About Divine Proportion). In it, Pacioli explains and illustrates mathematical proportion in its direct relation of artistic patterns and forms and explores architecture and the vital proportion of the golden ratio, the ultimate divine proportion extensively.

Devotees of twentieth-century sacred geometry note the high occurrence of the golden ratio, such as its recursive occurrence in the Parthenon; the Notre Dame Cathedral; the great pyramid of Giza; the relations between platonic solids; the ratio of segments in a five-pointed star (called a *pentagram*); the ratio of adjacent terms of the famous Fibonacci Series, named after Leonardo Fibonacci; the symmetrical pattern of aperiodic tilings, thanks to which Roger Penrose discovered new aspects of quasicrystals; in movements of the stock market; and even in Erik Satie's compositions.

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See Also: Houses Of Worship; Numbers and God; Religious Symbolism; Symmetry.

Sales Tax and Shipping Fees

Category: Business, Economics, and Marketing.
Field of Study: Number and Operations; Measurement.
Summary: Different types of sales taxes and shipping fees affect the final price of a purchase.

Benjamin Franklin famously noted, "Our Constitution is in actual operation; everything appears to promise that it will last; but in this world nothing is certain but death and taxes." When someone makes a purchase, often times there are extra charges added to the customer's bill. These costs may include a tax, shipping charges, or fees. These extra amounts, however, have a special purpose and they are each computed differently. For example, a sales tax is based on a percentage of the total amount of the sale and that percent is regulated by local and state governments. On the other hand, shipping is charged to cover the delivery of merchandise from the retailer to the customer's location. These fees are based on the policies of the company selling the goods as well as how quickly the customer would like their purchase delivered. Lastly, fees can be special charges; for example, insurance might be added to a purchase to cover the cost of the merchandise in the event it is lost or damaged during delivery. Albert Einstein commented that preparing a tax return "is too difficult for a mathematician. It takes a philosopher." The calculations to determine sales tax and shipping fees utilize percentages, multiplication, and addition, but Einstein may have been referring to the ever-changing instructions.

Both mathematicians and philosophers have long been involved in issues related to taxation. The *Jiuzhang suanshu* (*Nine Chapters on the Mathematical Art*) contains related problems. In the tenth century, astronomer and mathematician Abu'l-Wafa wrote a text on mathematics for scribes and businessmen, with part four of the book containing seven chapters devoted to various kinds of taxes and related calculations. In the seventeenth century, lawyer and amateur mathematician Étienne Pascal worked as a tax assessor and was appointed as the chief tax officer. In order to help his father in his tax work, mathematician and philosopher Blaise Pascal invented the Pascaline, which is reported to be the first digital calculator. In the twenty-first century, financial planners, mathematicians, and actuaries create mathematical models and investigate a variety of mathematical concepts related to taxes and fees, including the impact of flat rate, progressive, symmetric, or asymmetric taxation; and game theory applied to the interaction between taxpayers and tax collectors. They also investigate equilibrium states and how increasing or decreasing sales taxes or shipping and handling fees or using a nonlinear structure impacts consumer decisions about purchases and business sales.

Sales Tax

Many states, counties, and municipalities levy a sales tax as a way to increase revenues for their government or to balance their budget; however, not every state or local government charges a sales tax. The rate of the tax varies depending on the laws of the governmental unit. In other words, a purchaser will encounter different sales tax rates throughout the United States. The charges in 2010 varied from 0% in states like Alaska or Delaware to a high of 8.25% in California. This means that a person in Alaska who pays $100 for an mp3 player would not be required to pay any tax on the sale. However, a person buying that same mp3 player in California would be required to pay this tax. In other words, that $100.00 purchase would have an 8.25% tax added to the cost, meaning the new purchase price would be the original cost ($100.00) plus the sales tax ($8.25) for a total of $108.25.

Many localities exempt certain classifications of goods from their sales tax. Some common exceptions include groceries and prescriptions. On the other hand, special items such as gasoline, cigarettes, and alcohol have a significantly higher sales tax, as they have the potential to add sizeable revenue to a state's budget. A federal law called the Internet Tax Freedom Act (ITFA) specifically addresses sales over the Internet. The law provides that no governmental unit is allowed to add any special or additional tax on Internet purchases. This means that a sales tax may be charged on Internet purchases at the same rate as items purchased in person or by phone but no extra tax charge can be added.

Shipping and Handling Fees

Shipping and handling fees vary dramatically by seller as well as by the type of shipping the buyer requests. Common factors used to compute delivery costs include (1) how many items are being purchased, (2) how much the order weighs, and (3) how quickly the customer would like to receive their merchandise. However, common shipping types include free shipping, overnight delivery, two day or expedited delivery, and standard shipping, which may vary from three to seven days. In addition, the cost may change based on the number of items purchased or the weight of the merchandise. The following three examples illustrate different types of shipping options:

- *Flat fee*: The seller charges a flat shipping fee for all purchases regardless of price, weight, or number of items.
- *Progressive*: The seller charges a progressively larger shipping charge based on the cost of

the purchase. Shipping for a $50 purchase might cost $5, while shipping for a $100 purchase might cost $10.

- *Flat fee and item charge*: The seller charges a flat shipping rate plus an item charge (shipping + charge × number of items). Assume that the base shipping is $3.99, and there is a charge of $.99 for each item. A one item purchase would have a charge of $3.99 + $0.99 = $4.98. However, suppose the purchaser buys three items. In that case, the charge would be $3.99 + 3($0.99) = $6.96.

Shipping and fees are often grouped together as one charge; however, some vendors are known to charge each of these as separate and distinct charges. Vendors often add an additional charge to deliver a purchase. One example would be a package that requires special handling based on size or weight, such as a piece of furniture. Higher cost items such as jewelry might have an insurance charge added to the customer's total.

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See Also: Income Tax; Money; Shipping.

Sample Surveys

Category: History and Development of Curricular Concepts.

Fields of Study: Communication; Connections; Data Analysis and Probability.

Summary: Mathematicians and statisticians help design sampling methods and techniques to better represent populations and account for biases and missing data.

A survey is a statistical process by which data are collected from a representative sample of some population of interest in order to determine the attitudes, opinions, or other facts about that population. A census is the special case where everyone in the population is surveyed.

For example, the Babylonians are known to have taken a population census around 3800 B.C.E. In one of the first modern surveys, the *Harrisburg Pennsylvanian* newspaper polled city residents about the 1824 presidential election. Polling continued to be largely a local phenomenon until a 1916 national survey by *Literary Digest* magazine, which predicted the winners of several presidential elections despite using highly unscientific survey methods. Their famously incorrect assertion that Alf Landon would beat Franklin Roosevelt in the 1936 election is cited as contributing to the magazine's failure. Journalist and market researcher George Gallup, who correctly predicted Roosevelt's 1936 victory, was a pioneer in statistical sampling in the early twentieth century, though at the time, many considered his ideas quite radical. A post–World War II boom in manufacturing led companies to survey consumers to tailor products to preferences and increase sales. In the twenty-first century, public opinion polls on all aspects of society are pervasive and surveys frequently shape society's opinions and actions in addition to simply measuring them.

Students begin learning how to collect survey data in the primary grades. Researchers in many disciplines also routinely rely on data gathered via surveys. Mathematicians and statisticians work on mathematically valid methods for selecting samples that are random and representative as well as methods to reduce bias in surveys, effectively analyze data, present results that adjust for random error, and account for the effects of missing data. Many of these individuals belong to the Survey Research Methods Section of the American Statistical Association. Leslie Kish, a recipient of the association's prestigious Samuel S. Wilks Award, was especially cited for his worldwide influence on sample survey practice and for being "a humanitarian and true citizen of the world . . . [whose] concern for those living in less fortunate circumstances and his use of the statistical profession to help is an inspiration for all statisticians."

History of Surveys

In practice, surveys are collections of questions administered to individuals. Organizations like Gallup (founded as the American Institute of Public Opinion in 1935) specialize in conducting scientifically valid surveys. In the early part of the twentieth century, surveys were mostly conducted door-to-door by trained surveyors, a procedure used by both Gallup and the U.S. Census. Frequently, surveyors used the mail, like in the case of *Literary Digest*. Telephone surveys increased notably in the 1960s, which was attributed in large part to the fact that the costs of in-person research were escalating and trends in non-response suggested that people were growing less willing to answer face-to-face surveys, which diminished their prior advantage over phone surveys. Around 1970, statisticians Warren Mitofsky and Joseph Waksberg developed an efficient method of random digit dialing that revolutionized telephone survey research. However, some major organizations, like Gallup, continued door-to-door surveys into the mid-1980s, at which point they determined that a statistically sufficient proportion of U.S. homes had at least one telephone.

In 2008, Gallup notably expanded its methodology to include cell phones, since an increasing proportion of people no longer use landlines. In the twenty-first century, surveys are increasingly conducted via the Internet, though the U.S. Census still uses a combination of mail and house-to-house surveys. Harris Interactive, which went public in 1999, is a company that specializes in interactive online polls like the Harris Interactive College Football Poll, which ranks the top 25 Bowl Conference Series football teams each week.

Bias

Each survey method has different implications for both response bias and nonresponse bias. It is unclear when mathematicians and pollsters first began to recognize the negative influences of these biases, though adjustments were made in the latter half of the twentieth century. Systematic investigations can perhaps be traced to the mid-twentieth century, coincident with similar concerns in experimental design, like the placebo effect and psychologist Henry Landsberger's naming of the Hawthorne effect. Overall, these biases are problematic because they are non-random and cannot be accounted for by most traditional statistical methods. As a result, they may produce misleading results. Methods to combat these biases are the subject of a great deal of ongoing research and are typically addressed via incentives and proactive planning rather than adjustments after the fact.

Sampling

Randomness is a critical component of survey methodology. Statistical techniques commonly assume that the sample is a random subset of the population. When this is true, the results are more likely to be representative and informative of the population. Though random sampling is the standard in modern scientific polling, early pollsters like Gallup tended to use convenience or quote sampling—taking a sample of whomever was accessible or convenient, sometimes grouped according to other influential variables like political party, gender, or neighborhood. In some cases, this was simply an issue of practicality in terms of time and financial resources. Mathematical statistician Jerzy Neyman is credited with presenting the first developed notion regarding making inferences from random samples drawn from finite populations, what is now called "probability sampling," at a professional conference in 1934. He also contrasted probability sampling with non-random methods. The U.S. Department of Agriculture, in partnership with the statistical laboratory at Iowa State University, began researching probability sampling methods in the late 1930s, as did the U.S. Census Bureau. One of these influential survey researchers was William Cochran, who also helped build many academic statistics programs, including at Harvard. Through the 1940s and beyond, the formal methods of probability sampling and analysis sampling were developed, implemented, and refined in a wide variety of situations.

In the late 1970s and beyond, some researchers' attention turned to more advanced concepts like model-dependent sampling. In probability sampling, the characteristics of the population are wholly inferred from the sample. Model-dependent sampling, in contrast, assumes some probability model for the population beforehand and designs both a sampling and an analysis plan around this model. This method allows the researchers conducting the survey to optimally match the statistical properties of chosen estimators to the population. Statisticians Morris Hansen, William Madow, and Benjamin Tepping discussed many of the principal advantages and limitations of this method in a 1978 presentation and 1983 publication. Morris Hansen was an internationally known expert on survey research, an associate director for research and development at the Census Bureau, and later chairman of the board for polling company Westat, Inc. He also served as president of the American Statistical Association and Institute for Mathematical Statistics.

U.S. Census

Though the U.S. Constitution calls for a count of the population in the decennial census, the U.S. Census Bureau conducts other types of surveys and has been using sampling since 1937. In 1940, the bureau began asking a random sample of people counted in the decennial census extra questions to allow better characterization of population demographics as well as to estimate coverage errors. The ongoing American Community Survey helps determine how billions of federal and state dollars are distributed each year. In the late twentieth century, in large part because of substantial difficulties during the 1990 census, many statisticians proposed completely substituting sampling methods for the decennial counting process or at least substantially increasing the role of sampling. They felt that issues like undercoverage of certain subpopulations could be better addressed with increasingly sophisticated statistical methods. Cost was also considered. They had the support of many cities, states, civil rights groups, and members of Congress. The proposal was opposed by many other politicians and segments of the general population for both political reasons and because of skepticism regarding the sampling process. It ultimately required a ruling by the U.S. Supreme Court, which allowed supplemental sampling for some purposes but required a count to determine congressional apportionment.

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See Also: Census; Data Mining; Elections; Internet; Measurement in Society.

Satellites

Category: Communication and Computers.
Fields of Study: Algebra; Geometry; Measurement.
Summary: Mathematics is fundamental to the design, function, and launch of satellites.

Astronomy and mathematics have long developed together. Many early mathematicians studied the motion of celestial objects. The term “satellite” comes from the Latin *satelles* (meaning “companion”), which was used by mathematician and astronomer Johannes Kepler to describe the moons of Jupiter in the seventeenth century. Mathematician Giovanni Cassini correctly inferred that Saturn’s rings were composed of many small satellites in the seventeenth century. Mathematicians Jean Delambre and Cassini Jacques both published books of astronomical tables, including planetary satellites, in the eighteenth century. When artificial satellites were developed, the term “satellite” largely came to refer to those in common speech, while “moon” was applied to natural bodies orbiting planets. Mathematicians like Michael Lighthill and engineers like John Pierce helped develop satellites in the 1960s.

By the first decade of the twenty-first century, there were several hundred operational satellites orbiting the Earth to facilitate communication, weather observation, research, and observation. The advantage of satellites for communication are that signals are not blocked by land features in the same manner as a lower-altitude signal would be, making long-distance communication possible without multiple ground-based relays. Early communication satellites simply reflected signals back to Earth to broaden reception. Modern satellites use many different kinds of orbits to facilitate complex functioning, including low Earth orbit; medium Earth orbit; geosynchronous orbit; highly elliptical orbit; and Lagrangian point orbit, named for mathematician Joseph Lagrange. Mathematics is involved in the creation and function of such satellites, as well as for solving problems related to launching satellites, guiding movable satellites, powering satellite systems, and protecting satellites from radiation in the Van Allen belt, named for physicist James Van Allen. For example, graph theory is useful in comparing satellite communication networks. Techniques of origami map folding, researched by mathematicians like Koryo Miura, have been used in satellite design. Chaos theory has been used to design highly fuel-efficient orbits, derived in part from mathematician Henri Poincaré's work in stable and unstable manifolds. Government agencies like the U.S. National Aeronautics and Space Administration (NASA) and private companies like GeoEye employ mathematicians for research and applications. The Union of Concerned Scientists (UCS) maintains a database of operational satellites.

Orbits

The orbit of a satellite about the Earth determines when it will pass over various points on the Earth's surface and how high it is above the Earth. In general, orbits are characterized by altitude, inclination, eccentricity, and synchronicity. As defined by NASA, low Earth orbits have altitudes of 80–2000 kilometers. This orbit includes the majority of satellites, the International Space Station, and the Hubble Space Telescope. Statistical estimates at the start of the twenty-first century suggest that the number of functional satellites and nonfunctional debris in low orbit ranges from a few thousand (tracked by the U.S. Joint Space Operations Center) to millions (including very small objects). Objects in low orbit must travel at speeds of several thousand kilometers per hour, so even a small object can cause damage in a collision. Medium Earth orbit extends to about 35,000 kilometers (21,000 miles), the altitude determined by Kepler's laws of planetary motion for geosynchronous orbits. Inclination is an angular measure with respect to the equator, while eccentricity refers to how elliptical an orbit is. Geosynchronous satellites rotate at the same rate as the Earth spins, so they appear stationary relative to Earth. They usually have inclination and eccentricity of zero; they circle the equator to balance gravitational forces. The Global Positioning System (GPS) is one example of satellites at this orbital level. Sun synchronous orbits are retrograde patterns that allow a satellite to pass over a section of the Earth at the same time every day. They have an inclination of 20–90 degrees and must shift by approximately one degree per day. These orbits are often used for satellites that require constant sunlight or darkness. The maximal inclination of 90 degrees denotes a polar orbit. A halo or Lagrangian orbit is a periodic, three-dimensional orbit near one of the Lagrange points in the three-body problem of orbital mechanics, which was used for the International Sun/Earth Explorer 3 (ISEE-3) satellite.

Signals

Antennas and satellite dishes are used to receive satellite signals on Earth. Most satellite dishes have a parabolic shape. A signal striking a planar surface reflects directly back to the source. If the surface is curved,

Antennas and satellite dishes generally have a parabolic shape and are used to receive satellite signals on Earth.

the reflection is in the plane tangent to the surface. A parabola is the locus of points equidistant from a fixed point and a plane, so a parabolic dish focuses all incoming signals to the same point at the same time, increasing the quality of the signal. Mathematics is used to compress, filter, interpret, and model vast amounts of data produced by satellites. Reed–Solomon codes, derived by mathematicians Irving Reed and Gustave Solomon, are widely used in digital storage and communication for satellites. Much of the data from satellites is images, which utilize mathematical algorithms for rendering and restoration. One notable case that necessitated mathematical correction is the Hubble Space Telescope. An incorrectly ground mirror was found to have a spherical aberration, which resulted in improperly focused images. Mathematical image analysis allowed scientists to deduce the degree of correction needed. Some of the mathematical concepts involved in these corrections include the Nyquist frequency, which is a function of the sampling frequency of a discrete signal system named for physicist Harry Nyquist, and the Strehl ratio, named for mathematician Karl Strehl, which quantifies optical quality as a fraction of a system's theoretical peak intensity.

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See Also: Digital Storage; GPS; Interplanetary Travel; Planetary Orbits; Wireless Communication.

Scales

Category: Arts, Music, and Entertainment.
Fields of Study: Algebra; Measurement; Number and Operations; Representations.
Summary: Musical scales have distinct mathematical properties and patterns.

Western music is based on a system of 12 pitches within each octave. The interval between adjacent pitches in this 12-tone system is called a "half step" or "semitone." Pitches separated by two successive semitones are said to be at the interval of a "whole step," or a "tone." Based on a variety of theoretical underpinnings, the concept and sound of tones and semitones have evolved throughout the history of Western music. In modern music practice, a uniform division of the octave into 12 equally spaced pitches, known as "equal temperament," holds sway. Scales are arrangements of half and whole step intervals in the octave. Denoting a half step as *h* and a whole step as *w*, the familiar diatonic major scale is defined by the sequence *wwhwwwh.* The diatonic natural minor scale is *whwwhww*. Beginning these patterns from each of the 12 pitches results in 24 distinct diatonic scales. This suggests a set-theoretic description by which each major scale can be represented as a transposition (in algebra this would be called a "translation") of the set of pitches C, D, E, F, G, A, B, and C. In the twentieth century, such mathematical formalisms have led to the conceptualization of non-diatonic scales with special transposition properties.

Octave Equivalence

The concept of octave (the musical interval between notes with frequencies that differ by a factor of two) is fundamental to understanding musical scales. In Western music notation, pitches separated by an octave are given the same note name. The piano keyboard provides a visual representation of this phenomenon. Counting up the white keys from middle C as "1," the eighth key in the sequence is again called C. This eight-note distance explains the etymology of the word "octave." The perception and conceptualization of such pairs of pitches as higher or lower versions of the same essential pitch is called "octave equivalence." Octave equivalence is thought to be common to all systematic musical cultures. Evidence of octave equivalence is found in ancient Greek and Chinese music. Recent psycho-acoustic research suggests a neurological basis for octave equivalence in auditory perception.

The mathematical explanation of octave equivalence comes from the fact that the sound of a musical pitch is a combination of periodic waveforms that can be modeled as sinusoidal functions of time. In the two periodic functions, $f(t) = \sin(t)$ and $g(t) = \sin(2t)$, with frequencies 2π and π, every peak of the lower

Table 1. The diatonic scale in three intonation schemes, Pythagorean, just, and equal temperament.

	C	D	E	F	G	A	B	C
Pythagorean	1:1	9:8	81:64	4: 3	3:2	27:16	243:128	2:1
interval		9:8	9:8	256:243	9:8	9:8	9:8	256:243
Just	1:1	9:8	5:4	4:3	3:2	5:3	15:8	2:1
interval		9:8	10:9	16:15	9:8	9:8	9:8	16:15
Equal	1:1	1.1225:1	1.2600:1	1.335:1	1.4983:1	1.6818:1	1.8878:1	2:1
interval		$2^{1/6}$	$2^{1/6}$	$2^{1/12}$	$2^{1/6}$	$2^{1/6}$	$2^{1/6}$	$2^{1/6}$

frequency function coincides with a peak of the high-frequency function. In sonic terms, this is the highest degree of consonance possible for two pitches of different frequencies.

History of Scales

As Western music developed from the Middle Ages through the twentieth century, the central construct was the diatonic scale. This arrangement spans an octave with seven distinct pitches arranged in a combination of five whole steps and two half steps. Interestingly, the pattern of intervals (and not the absolute pitch of the starting note) was the only distinguishing feature of scales until the rise of tonal harmony in the seventeenth century. Pitch-specific examples help illustrate the interval patterns.

The diatonic scale traces its origins to the ancient Greek *genus* of the same name, referring to a particular tuning of the four-stringed lyre (tetrachord) consisting of two whole steps and one half step in descending succession. An example of this tuning can be constructed with the pitches A, G, F, and E. Concatenization of two diatonic tetrachords [A-G-F-{E]-D-C-B} produces the pitches of the diatonic scale (the piano white keys). In medieval European musical practice, the distinct Church Modes (such as Lydian or Phrygian) developed from the diatonic scale by the assignment of a tonal anchor or final tone. For example, the Dorian mode is characterized by the sequence of ascending half and whole steps in the diatonic scale *whwwwhw*; for example D-E-F-G-A-B-C-D, while the Phrygian mode is *hwwwhww*: E-F-G-A-B-C-D-E. The diatonic major scale *wwhwwwh* (C-D-E-F-G-A-B-C) came into widespread use in the seventeenth century. The diatonic natural minor scale is *whwwhww* (A-B-C-D-E-F-G-A).

Intervals, Ratios, and Equal Temperament

The simplest musical interval is the octave. The frequency ratio between pitches separated by an octave is 2:1. The interval of a perfect fifth has frequency ratio 3:2. Using these two ratios, pitches and corresponding intervals for the diatonic scale can be assigned according to Pythagorean tuning. Simpler diatonic scales based on ratios of small integers are known as "just tunings." Western music in the modern era uses a symmetric assignment of intervals known as "equal temperament." In equal temperament, the 12 half steps that comprise the frequency doubling octave each have frequency ratio $2^{1/12} \approx 1.0595$. For these three tuning schemes, frequency ratios relative to the starting pitch and intervals between adjacent scale notes are illustrated and compared in Table 1. For each intonation, the first row gives the frequency ratio from the tonic C to the given note. The second row in each case gives the frequency ratio between adjacent diatonic pitches.

Modern Scales

In contrast to the idiosyncratic pattern of intervals that comprise the diatonic scales, the chromatic scale *hhhhhhhhhhhh* is perfectly symmetric. In particular, the set of pitches that form the chromatic scale is unchanged by transposition—there is only one set of pitches with this intervallic pattern. This set of pitches is referred to as having order one. The elements of the pitch set forming a diatonic scale, which generates 12 diatonic scales by transposition, has order 12. This point of view suggests other scales of interest with respect to transposition. The set of six pitches in a whole-tone scale *wwwwww* (for example, C-D-E-F♯-G♯-A♯-C) are unchanged by transposition by an even number of half steps. A transposition by an odd number of half steps results in the whole tone scale containing the remaining six pitches (for example, C♯-D♯-F-G-A-B-C♯). Thus, the set of pitches in the whole-tone scale has order two. Whole-tone scales are a characteristic feature in much of the music of Claude Debussy.

The twentieth-century composer and music theorist Olivier Messiaen codified a number of eight-tone "scales of limited transposition." Among these are the order three scales *hwhwhwhw* and *whwhwhwh*, which are called "octatonic scales" in the music of Stravinsky and sometimes referred to as "diminished scales" in jazz performance. It can be seen that transposition by one and two half steps produce new diminished scales, but transposition by three half steps leaves the original set of pitches unchanged.

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Eric Barth

See Also: Composing; Geometry of Music; Harmonics; Pythagorean and Fibonacci Tuning.

Scatterplots

Category: History and Development of Curricular Concepts.
Fields of Study: Communication; Connections; Data Analysis and Probability.
Summary: Scatterplots are useful tools for mathematicians and statisticians to graph and present data.

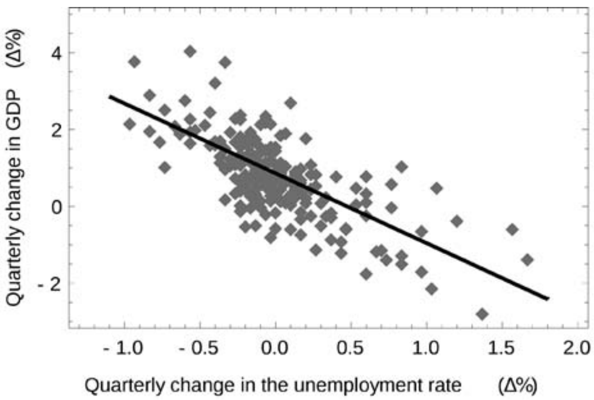
Human beings are constantly exploring the world around them to discover relationships that can be used to explain past and current events or phenomena and perhaps to predict future occurrences.

The colloquial expression "a picture is worth a thousand words" is traced back to many possible historical sources, including French leader and noted student of mathematics Napoleon Bonaparte, who purportedly said, "A good sketch is better than a long speech." In the twenty-first century, graphing is a fundamental first step in any exploratory data analysis, and graphical representations are common in the media. Scatterplots, which most often represent values of paired variables in a Cartesian plane, help data investigators identify relationships, describe patterns and correlation, fit linear and nonlinear functions using techniques like regression analysis, and locate points known as "outliers" that deviate from the predominant pattern. In the primary grades, students often use line graphs, which some consider to be a special case of scatterplots, while scatterplots for data may be explored beginning in the middle grades in both mathematics and science classes.

Early History

Mathematicians and others have long sought alternative methods of representation for researching, presenting, and connecting the mathematical concepts they studied. The Cartesian plane, named for René Descartes, facilitated graphing of algebraic equations and data beginning in the seventeenth century. Historians have traced scatterplots to 1686, though the term "scatter diagram" is attributed to early twentieth-century researchers such as statistician Karl Pearson, and "scatterplot" seems to have first appeared in a 1939 dictionary.

Examples of early pioneers of data graphing include "political arithmetician" Augustus Crome, who studied the relationships between nations' population sizes, land areas, and wealth; mathematician and sociologist Adolphe Quetelet, who conducted studies of body measurements that helped contribute to the measure now known as the Body Mass Index, which relates height and weight; and engineer and political scientist William Playfair, who called himself the "inventor of linear arithmetic," a term he used for graphs. He said: ". . . it gives a simple, accurate, and permanent idea, by giving form and shape to a number of separate



A scatterplot chart showing the relationship between gross domestic product growth and unemployment.

ideas, which are otherwise abstract and unconnected." Playfair's eighteenth-century graphical summaries of British trade across various years are perhaps the earliest example of what would now be referred to as "time series plots" (or in some cases "line graphs"), which may be considered a special case of scatterplots.

While Playfair plotted many economic variables as functions of time, the most extensive early use of scatterplots to relate two observed variables is probably the anthropometric and genetic research of Francis Galton, a cousin of scientist Charles Darwin. After studying medicine and mathematics in college, he became interested in the investigation and characterization of variability and deviations in many natural phenomena. He established a laboratory for the measurement and study of human mental and physical traits, focusing on empirical and statistical studies of heredity in the latter half of the nineteenth century. Many of Galton's scatterplots involved graphing parental characteristics on one axis, usually the *X*, and offspring characteristics on the other. Like scientist Gregor Mendel, some of his initial genetic experiments were conducted on peas; later, he investigated measurements of people. Scatterplots of height appeared in his 1886 publication *Regression Towards Mediocrity in Hereditary Stature*, which is the origination of the name for the statistical technique of regression analysis. The word "mediocrity" in this context was a reference to the mean or average height (not a qualitative judgment) and was used to describe a pattern observed in the data: very short parents tend to have taller children, and very tall parents tend to have shorter children, in both cases closer to the mean.

Recent Developments

Prior to the development of computers and data analytic software, data had to be graphed by hand. In the twenty-first century, computers facilitate many types of scatterplots. In addition to the standard plots of two variables in the Cartesian plane, there are three-dimensional scatterplots that display point clouds to explore the ways in which three variables relate and interact. Symbols used to represent points on a two- or three-dimensional scatterplot may also be coded using different colors or shapes to indicate additional variables and uncover patterns. Matrix plots are square grids of scatterplots for a set of variables that plot all possible pairwise sets, usually arranged such that all of the plots in the same row share the same *Y* variable and all plots in the same column share the same *X* variable. Mathematicians, statisticians, computer scientists, and other types of researchers have explored the theoretical and methodological links between scatterplots and map surfaces for use in applications such as data mining and spatial analysis of geospatial information system (GIS) data.

While they are useful tools for exploration and representation, scatterplots are often subject to misinterpretations. For example, sometimes relationships or correlations shown in scatterplots are mistakenly taken as evidence of cause and effect, which must be inferred from the way in which the data were collected rather than from the strength of the association.

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See Also: Coordinate Geometry; Forecasting; Graphs; Visualization.

Scheduling

Category: Business, Economics, and Marketing.
Fields of Study: Data Analysis and Probability; Number and Operations.
Summary: Scheduling can be a complex mathematical exercise and is necessary to keep businesses and supply chains running efficiently.

Intense competitiveness forces companies to optimize performance in terms of cost, time, and resources. Scheduling is the process of developing and implementing optimal operational plans. Formal concepts of scheduling date to the Industrial Revolution and innovations like Henry Ford's assembly line, although the basic ideas probably existed from antiquity in any society where people manufactured goods.

In manufacturing, multiple tasks are carried out in sequence to produce a final output from raw materials. Further, steps in a manufacturing process may be performed on different machines that require variable time to deliver outputs and it is possible that materials will be transported between facilities. A mathematically determined schedule that takes into account all relevant variables in the process serves to optimally allocate resources with respect to demand of the tasks, including shortening time intervals to reduce unproductive time and minimizing costs from wasted time and materials. Operations research is a field of applied mathematics and science that uses mathematical tools, such as simulation and modeling, linear programming, numerical analysis, graph theory, and statistical analysis, to arrive at optimal or near-optimal solutions to complex problems like scheduling. It may also tackle problems in which the resources are not materials but people. The scheduling of airplane crews is a highly constrained and difficult problem because of legal limits on work and rest times as well as the need for crews to return to a home base. Allocation of police, fire, and ambulance services is also a widely used and very important application of scheduling theory.

Production Management

As a part of production management, scheduling interferes with many different aspects of business such as the supply chain, inventory maintenance, and accounting. For example, consider a paint company that makes provisions of sales for the next month by analyzing previous data. In light of these provisions, schedulers determine the expected arrival time and amount of different types of chemicals, which have different delivery times.

The supply chain should be able to deliver the correct amounts of chemicals in time. In a similar way, accounting of the cost of supply and inventory should be accessible for the schedulers. Because of the number of operational parts of business that scheduling is related with, it is apparent that scheduling is a very complex process. It gets more complex with larger variation in types of products and larger numbers of machines varying in processing times. Thus, schedulers demand thorough knowledge of factors such as the processing time of each machine, delivery time, the amount of resources to allocate among machines, and the size and flow of operations for each product.

Manufacturing

In many manufacturing processes, different machines might share the same input, or inputs of a machine might consist of outputs from multiple machines. Scheduling operations in these type of cases requires extensive mathematical modeling. Two basic types of modeling for production scheduling are distinguished by the presence of randomness within. Deterministic models do not include the probability of faults in processes or critical changes in capacity or resource availability. They are based on previous averages of production figures and output rates, so they do not easily adapt to changes in demand or capacity constraints. In these cases, rescheduling is needed, which causes time and resource loss if repeated too many times. They are best suited to manufacturing productions that involve less risk of defects. Stochastic models, on the other hand, involve the probability of unexpected malfunctions or critical changes by distributing probability analytically to individual steps of the schedule. Usually, they are appropriate for processes consisting of many individual operations. For example, these models examine machine failure rates and aim to provide options for when a breakdown occurs. Also, these models maintain an inventory of materials, which may prove critical in maintaining production. Simulations of models provide schedulers an environment to test possibilities that can obstruct the flow of production.

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See Also: Data Mining; Mathematical Modeling; Parallel Processing.

Schools

Category: Architecture and Engineering.
Fields of Study: Data Analysis and Probability; Geometry; Number and Operations.
Summary: Principles of geometry affect school design and mathematical models of risk may help identify safety issues.

When people think of mathematics in schools, most probably envision the teaching and learning of mathematics that occurs inside classrooms. However, there are many aspects of twenty-first century schools that depend on mathematics. For example, the transition of school design from one-room schoolhouses that were common in the nineteenth century, through the often rectangular and symmetric classroom buildings of the latter nineteenth and early twentieth centuries, to the open-plan schools initiated in the 1950s, to twenty-first century schools that consider contemporary concerns about renewable energy, technology, and safety. Changes in teaching philosophies over time, such as loop education and emphasis on science, technology, engineering, and mathematics (STEM) education principles in the lower grades, led to some of these changes, as did studies on tragedies like the shootings at Columbine High School and Virginia Polytechnic Institute and State University, popularly known as Virginia Tech (VT). Mathematics principles can be used to map the flow of students to and from classes, optimize locker placement and access, build accommodations and accessibility for students with disabilities, and plan for athletic facilities and other non-classroom spaces. These applications are increasingly important as schools seek to educate students to live and work within the rapidly changing economies, technologies, and environments of the twenty-first-century global society. Other studies may determine whether to retrofit old buildings or construct new facilities using mathematical methods like cost-benefit analysis. There are many organizations and publications devoted to discussing the mathematics, engineering, and technical aspects of school design and construction.

Optimizing School Design

The notion of what constitutes "optimal school design" has markedly changed over time. There are some who consider the classic one-room schoolhouse to be the original open-plan design, since the teacher accommodated all students in all grades in a single space, dividing class time among the various grades. Famed Boston architect Gridley J. F. Bryant, who also studied engineering, is credited with revolutionizing the design of many public buildings. His Quincy School, which opened in 1847, was among the first multi-classroom schools. The school was three stories tall, with four identical and symmetrically arranged classrooms on each floor. This model was used for schools throughout the late nineteenth and early twentieth centuries and would be further evolved with movable desks and tables to allow for some flexibility within the "box in a box" construction, as it was called by some. This design led to other considerations such as optimal selection and placement of furniture such as desks, tables, chairs, and later computers, as well as features such as lockers and storage spaces, all of which must be fit into a limited amount of space yet be accessible and functional for a varying student body. Proper placements rely on mathematical concepts such as volume and are related to mathematical packing problems. Detractors often likened Bryant's school configuration to prisons, which he also designed. The evolution of open-plan schools of the latter twentieth century was motivated by cost and changes in teaching philosophies, derived in part from research in mathematics education. There was and continues to be controversy regarding the efficacy and desirability of open plan schools. Mathematicians, architects, facilities planners, and others continue to research effective strategies for design and construction. For example, architect Prakash Nair is internationally recognized as a leader in school design, and has been cited for using educational research as a basis for designs that optimize teaching and learning. He helped develop a "pattern language" that draws on geometric ideas and uses a modular set of design patterns, sub-patterns, and groupings to match school designs to goals and needs. It can be used to develop new schools and assess existing structures. Other education professionals like C. Kenneth Tanner, whose background includes work in design, mathematics, statistics, and operations research, have also used a combination of data-based research with mathematical techniques and tools to address a broad spectrum of school planning issues, such as technology integration. Organizations like the School Design and Planning Laboratory at the University of Georgia use data-driven methods and models for assessing school design and forecasting student populations and demographics, which may impact design, use, and sustainability.

Safety

The safety of children in U.S. schools has become a growing concern for parents, teachers, and society in general. The 1999 shootings at Columbine High School focused national attention on issues of school security, safety, and patterns of police response to such incidents. Even more debate occurred after the 2007 shootings at VT. The Secret Service, the Department of Education, and the Federal Bureau of Investigation (FBI) conducted broad studies into the causes and prevention of school violence. For example, the Secret Service and the Department of Education studied all 37 shootings in U.S. schools between 1974 and 2000. Data analysis revealed no identifiable statistical patterns; school shooters came from a variety of ethnic, economic, and social classes, and most had no history of violent behavior that would reliably predict later actions. Using statistical methods, profilers from the FBI also concluded that the "oddball" students that society commonly perceives to be potential troublemakers were not in fact more likely to commit violence, though such studies are limited by the relatively small number of incidents and data available for modeling. Probabilistic, predictive profiling is quite controversial, but many educators and others still advocate its use in risk assessment.

Another mathematically based strategy schools may employ for risk assessment is actuarial methods. Actuarial models for school risk statistically combine empirically chosen threat factors to produce probabilities for particular outcomes or behaviors, and sometimes they may be standardized for specific student populations. In some cases where there are sufficient data and the models can be validated, they have often performed better at identifying in-school threats than subjective human judgments. However, other model-based assessments of risk that are based on sparse data or with a short window for prediction have not been shown to be as reliable. Some researchers have tried to develop expert systems for school threat assessment and decision making, which are automated or semi-automated tools that use artificial intelligence and algorithms developed from data, achieving mixed success. Both actuarial models and expert systems for schools may be revised to incorporate new data as it is identified, making them flexible mathematical modeling tools.

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See Also: Engineering Design; Forecasting; Learning Models and Trajectories; Packing Problems; Risk Management.

Science Fiction

Category: Arts, Music, and Entertainment.
Fields of Study: Communication; Connections; Representations.
Summary: Mathematics plays many roles in science fiction, sometimes as content or characters, other times bringing elements to life on the screen.

Like mathematics, writing science fiction is a craft grounded in deduction and extrapolation. The writer begins with certain axioms: the world as we know it and the world as we believe it could be. He introduces certain new variables: a thinking robot, an alien invasion, human clones. Explicitly or implicitly, the story explores the consequences, the corollaries of these new things according to the implications of those initial axioms. Stories that do not do this are considered fantasy or sometimes science-fantasy or soft science fiction, if they otherwise contain the set-dressing of the science fiction genre.

The setting is often the future, an alternative history, or some alternate reality, and may include use of time travel. Alien characters frequently interact with humans in science fiction. Many books, movies, comics, graphic novels, computer games, and Internet applications use science fiction themes, sometimes as a context in which to explore deeper philosophical questions. Mathematics and science play a variety of roles within the science fiction genre. Sometimes, mathematics and science are written into the story to give validity and believability to the futuristic setting or to the technology. Mathematics is also used to bring fantastic science fiction elements to life on screen, such as in the groundbreaking *Star Wars* franchise or the 2010 film *Avatar*. At other times, characters in science fiction works are mathematicians or scientists who act as the primary heroes or villains, or who explain scientific elements to the audience. The inclusion of mathematically talented characters in science fiction is sometimes done to exploit commonly held stereotypes about mathematicians for narrative purposes, such as genius or aloofness. In other works, mathematics becomes the explicit subject of the story, and the mathematics of science fiction in both written and visual media has been explored in college courses and mathematics research. Mathematicians or individuals that have mathematical training often create science fiction, and science fiction may inform mathematical research. The widely noted "Big Three" authors of twentieth-century science fiction—Arthur C. Clark, Robert A. Heinlein, and Isaac Asimov—all had mathematical training or mathematically based science backgrounds and made nonfiction contributions to areas such as satellites, rocketry, robotics, and ethics.

Early History of Science Fiction

Because of the varying definitions of science fiction, it is difficult to determine exactly what might be the first science fiction story. The Mesopotamian epic poem *The Epic of Gilgamesh*, which is among the oldest surviving works of literature, is cited by some scholars as

Science fiction is often set in an alternate reality and may involve time travel or alien characters.

containing elements of science fiction. Some researchers note that the Bible, when examined as a work of literature, has stories that could be classified as science fiction, such as the ascension of the prophet Elijah to heaven in a fiery chariot. In the second century, the Greek satirist Lucien of Samosata wrote about interplanetary travel and alien life forms in his *True Histories (*or *True Tales).* English lawyer and philosopher Thomas More's 1516 work *Utopia* described a perfect society, which became a common theme among later science-fiction writers. Some scholars argue that such early works cannot be claimed as the first science fiction because neither the audience nor the authors likely knew enough about the underlying science. Correspondingly, they might claim that the origin of science fiction coincided with the post-medieval scientific revolution and discoveries in science and mathematics made by people such as Isaac Newton and Galileo Galilei. Mathematician and astronomer Johannes Kepler wrote a story in 1634 called *Somnium*, which imagined that a student of astronomer Tycho Brahe had been transported to the moon and described how Earth might look when viewed from that location. It contained mathematical computations and is considered by some to be a scientific treatise, while others cite it as the first science fiction, including both Asimov and astronomer and author Carl Sagan. Author Brian Aldiss asserts that science fiction derives many of its structure and conventions from the Gothic horror genre, which suggests that Mary Shelley's 1818 novel *Frankenstein* is "the first seminal work to which the label SF can be logically attached." This labeling is perhaps because of its introduction of science fiction themes like a mad scientist, the potential misuse of technology, and the presence of an non-human being as a main character.

The Foundations of Twentieth-Century Science Fiction

Jules Verne and Herbert George (H. G.) Wells are often jointly known as the "fathers of science fiction" for their creative influence on the development of twentieth-century science fiction. Jules Verne consistently incorporated the newest technological discoveries and experiments of his lifetime into his work. Many of his most popular novels, like *A Journey to the Center of the Earth* (1864), *From the Earth to the Moon* (1865), *Twenty Thousand Leagues Under the Sea* (1869), and *Around the World in Eighty Days* (1873), have been widely translated into other languages and adapted into plays, movies, television shows, and cartoons. Some scholars have called Verne's books visionary and even prophetic for describing mathematical and scientific phenomena such as weightlessness and heavier-than-air flight before they were well-known or understood. His attention to realistic scientific principles and detailed descriptions of problems and solutions would later challenge many real-life mathematicians, scientists, and engineers. Physicist and engineer Hermann Oberth and scientist Konstantin Tsiolkovsky, who are known as the "fathers of rocketry and astronautics" along with physicist Robert H. Goddard, reported being inspired by Verne's books.

Like Verne's work, the novels of H. G. Wells have been widely adapted into various other media, and the 1895 novel *The Time Machine*, in particular, is cited as inspiring many other works of fiction. The invention of the now commonly used term "time machine" is attributed to Wells, as is the notion of time being the fourth dimension. In the 1897 novel *The Invisible Man,* a scientist named Griffin makes himself invisible by changing the refractive index of his body so that it neither absorbs or reflects light. Some of Wells's books were considered to be exceptionally bold and compelling. His 1898 novel *The War of the Worlds* is well-grounded in mathematical and scientific theories from the time it was written, like mathematician Pierre-Simon Laplace's formulation of the nebular hypothesis. It shared a vision of space travel common to late nineteenth-century novels, including Verne's *From the Earth to the Moon*. Large cylinders were fired from cannons on the Mars surface to transport the aliens to Earth. Later mathematical models and calculations necessary to send people into Earth's orbit and to the moon, as well as to guide probes to Mars and the far reaches of the solar system, demonstrated that the parabolic trajectories were often quite complex and that the forces required to propel a cylinder from Mars to the Earth would likely be lethal to passengers. Wells was also a science teacher and political activist who recognized and asserted the importance of quantitative knowledge, noting: "Statistical thinking will one day be as necessary for efficient citizenship as the ability to read and write."

Mathematical Science Fiction

While mathematics is widely used to help build or validate the setting or technology of a science fiction story, such as in the works of Verne and Wells, in some cases it is a central component of the plot. There are many mathematical science fiction novels that have been written about a variety of themes. The 1946 short story *No-Sided Professor*, written by mathematician and author Martin Gardner, disusses the Möbius strip, a one-sided figure named for mathematician August Möbius. It addresses the possibility of a zero-sided figure and other concepts in topology. *Occam's Razor*, by author David Duncan in 1956, posits the notion of discontinuous time, which can be bridged by minimal surfaces in certain topologies. Asimov's 1957 novel *The Feeling of Power* addresses scientific computing in a futuristic society in which people have lost the ability to perform basic arithmetic calculations. The rediscovery of hand-multiplication therefore becomes a new "secret weapon" for the society's military. Author Stanislaw Lem discusses countably infinite sets in his 1968 novel *The Extraordinary Hotel*, while author and mathematician Greg Egan's 1991 work *The Infinite Assassin* includes a discussion of the Cantor set, named for mathematician Georg Cantor, an important concept in topology and some other mathematical fields. Other mathematical science fiction urges appreciation of mathematics as if it is a form of poetry. Examples include author Kathryn Cramer's 1987 work *Forbidden Knowledge*, author Normal Kagan's 1964 work *The Mathenauts*, and multiple stories by author Eliot Fintushel. Mathematician and author Vernor Vinge often addressed the mathematical themes of superhuman artificial intelligence and a predicted technological singularity: a point in time where the exponential growth of technology results in essentially instantaneous change. These themes are also found Clark's *2001: A Space Odyssey* and its sequels. The term "technological singularity" is credited to mathematician Irving Good and is also linked to Moore's law, named for Intel co-founder Gordon Moore, which mathematically models the trend in the evolution of computer processor speeds.

Several science fiction novels challenge the foundations of mathematics itself or the commonly proposed notion of mathematics as a universal language. Author Ted Chiang's *Division by Zero*, a 1991 short story, discusses the discovery of a proof that mathematics is inconsistent, which may be possible according to Gödel's Incompleteness Theorems, named for mathematician Kurt Gödel. Chiang's later 1998 work *Story of Your Life* involves humans trying to communicate with aliens whose mathematics is based on variational formulations rather than algebra. In the same year, mathematician and author David Reulle's *Conversations on Mathematics With a Visitor from Outer Space*, which was published in a collection of nonfiction mathematical essays, argued that mathematics on Earth is essentially human in nature, so humans should not expect aliens to share human's unique mathematical language. Sagan's 1985 novel *Contact* alternatively suggested that humans and aliens may communicate via mathematics, but rather than the typical mode of

Mathematical Characters and Stereotypes

Science fiction authors often include mathematicians or mathematically talented individuals as characters in order to explain scientific elements to the audience or to exploit commonly held associations and stereotypes, which can be a shortcut for characterization, including intelligence, logic, emotional coldness, eccentricity, arrogance, or general strangeness or differences between mathematicians and supposedly "normal" people. For example, in Michael Crichton's 1990 novel *Jurassic Park,* mathematician and chaos theorist Ian Malcolm, sometimes cited as having been modeled in part on Ian Stewart, expounds with some arrogance on the mathematics that shape the increasingly dangerous situations in which the characters find themselves. However, he otherwise defies many of the stereotypes associated with mathematicians, such as social ineptness. The notion of logic and mathematical reasoning as male modes of thinking and understanding, versus understanding via female emotion and intuition, is also pervasive in older science fiction. Some point to lessening trends in this theme in the latter twentieth century, and a few works like Chiang's *Division by Zero* contain female characters who are coldly logical and distant rather than emotional. It is an issue of debate whether this should be seen as a positive or negative shift.

receiving radio waves containing messages from space, communications are instead embedded within the very framework of mathematics itself.

Since Wells introduced the notion of the fourth dimension in *The Time Machine*, dimensionality in many forms has been a widely used theme in science fiction, including mathematical science fiction. In the 1940 novel *And He Built a Crooked House* by Heinlein, a mathematical architect designs a house that is constructed as an "inverted double cross" representation of an unfolded tesseract net in three-dimensional space. Following an earthquake, the structure spontaneously shifts and folds itself into an actual tesseract, whose four-dimensional properties are explored and described by characters. The satirical Edwin Abbot novel *Flatland: A Romance of Many Dimensions,* which was written largely as a social commentary on Victorian norms and mores, may also be considered science fiction because it depicts an alternate two-dimensional world inhabited by polygonal creatures, which is visited by three-dimensional creatures in a manner that resembles twentieth- and twenty-first-century depictions of human-alien interactions. More than a century after its initial publication, *Flatland* remains popular in the mathematical community because of its entertaining and enlightening discussions of what some people consider to be an abstract mathematical concept, and it was once described by Asimov as, "The best introduction one can find into the manner of perceiving dimensions."

Other authors have used the novel as inspiration. Mathematician and author Ian Stewart's 2001 work *Flatterland: Like Flatland, Only More So*, explores several mathematical topics such as Feynman diagrams, named for physicist Richard Feynman, superstring theory, quantum mechanics, fractal geometry, and the recurring science fiction theme of time travel. He includes mathematical jokes and puns such as a one-sided cow named Moobius to make concepts relatable to a broader audience. Stewart also co-authored the semi-fictional *Science of Discworld* series, which compares mathematically and scientifically the natural laws of sperical planets or "roundworlds" like Earth to the created or imagined physical laws of the flat, disc-shaped setting of author Terry Pratchet's *Discworld* novels. Some other works that are commonly cited as extensions of ideas found in *Flatland* include mathematician and author Dionys Burger's 1953 novel *Sphereland: A Fantasy About Curved Spaces and an Expanding Universe* and two works by mathematician and author Rudy Rucker: the 1983 short story *Message Found in a Copy of Flatland* and the 2002 novel *Spaceland*.

Mathematics is a living discipline that is constantly evolving, and mathematical science fiction sometimes underscores this point. Gardner's 1952 story *The Island of Five Colors* is the sequel to the *No-Sided Professor.* The characters in the story attempt to solve the Four Color theorem, which was unproven at the time. It illustrates the inherent time dependence of some elements of science fiction, since imagined creations and the mathematics on which they are based frequently become reality later. Gardner stated: "the true four-color theorem, unproved when I wrote my story, has since been established by computer programs, though not very elegantly. As science fiction, the tale is now as dated as a story about Martians or about the twilight zone of Mercury." At the same time, others argue that the themes of such novels are still useful and relevant when considering the nature of mathematics and that these stories do not automatically lose value as entertainment or inspiration simply because the mathematical or scienctific frameworks become somewhat out of date.

Mathematicians as Science Fiction Authors

Mathematicians and mathematically trained individuals such as Martin Gardner, Isaac Asimov, Greg Egan, Ian Stewart, and Vernor Vinge often contribute to both science fiction writing and mathematical or scientific research, and several mathematicians have won the Hugo Award, the premier prize in science fiction and fantasy literature. Perhaps one of the most well-known of these is Rudy Rucker, who is considered among the founding fathers of the science fiction subgenre of cyberpunk, a style that draws inspiration from Gothic horror like *Frankenstein*, film noir, punk, computer science, and cybernetics, a discipline whose twentieth-century development is attributed to mathematician Norbert Weiner. Rucker credits his mathematical background for influencing not only the content of what he writes but also the way in which he writes: "I think of the writing process itself as a fractal. I have the big arc of the plot, the short-story-like chapters, the scenes within the chapter, the actions that make up the scenes, and nicely formed sentences to describe the actions, the carefully chosen words in the sentence. And hidden beneath each word is another fractal, the entire language with all my ramifying mental associations."

He also notes that both mathematics and science fiction writing can be thought of as ways of exploring the consequences of imposed constraints or assumptions, and that science fiction writing provides a structure for carrying out interesting "thought experiments" about concepts such as alternative mathematical structures to explain the nature of reality.

Visual Media

Science fiction has long been translated to other forms of media, and mathematics plays a dual role as a subject and a technique for bringing both realistic and fantastic images to life. Further, mathematicians are often involved as writers or consultants. Stanley Kubrick's *2001: A Space Odyssey* is one of the most well-known science fiction films. Mathematician Irving Good consulted on the film, as did novelist Clark, and it is praised for its scientific realism and pioneering special effects. The *Star Wars* franchise, launched in 1977, now includes books, comics, movies, video games, and Web media. It was groundbreaking in its use of mathematically based special effects techniques, including extensive stop motion animation and then later computer animation for backgrounds, props, costumes, and even entire characters. Effects that were once limited to big-budget films have now made their way onto television. The *Star Trek* franchise is notable not just for its visual imagery but also for references to real-world mathematical concepts including π and Fermat's last theorem, named for mathematician Pierre de Fermat. Other examples of shows that contain frequent real-world mathematical include SyFy's *Eureka* and the animated series *Futurama*. Producer and writer Ken Keeler has a mathematics Ph.D. from Harvard. Along with other mathematically trained writers, he co-creates many of the mathematical references found on *Futurama*, and once notably constructed a new mathematical proof to validate an episode's plot twist.

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See Also: Geometry of the Universe; Interplanetary Travel; Literature; Mathematics, Elegant; Movies, Mathematics in; Universal Language; Writers, Producers, and Actors.

Sculpture

Category: Arts, Music, and Entertainment.
Fields of Study: Geometry; Representations.
Summary: Mathematics may be necessary to assure the stability of a sculpture and sculptures can represent mathematical concepts in three dimensions.

The word "sculpture" comes from Latin *sculpere*, meaning "to carve." Sculptures can be made from variety of materials, including wood, metal, glass, clay, textiles, or plastic that is carved, cast, welded, cut, or otherwise formed into shapes. Topiary and bonsai are living sculptures. Modern sculptors even experiment with light and sound. Additionally, sculptures may be free-standing objects or appear as reliefs on surfaces like walls.

The Taj Mahal, one of the most recognizable structures on Earth, includes many geometric reliefs. Sculptures can be static or kinetic, like Rube Goldberg contraptions, and projection sculptures change appearance when viewed from differenent sides. The outdoor Penrose tribar sculpture in East Perth, Australia, appears to be the illusory figure developed by Roger Penrose when viewed from the correct angle. While mathematical forms have long been used to create sculpture, mathematicians have come to embrace this incredibly flexible art form to investigate many mathematical concepts that might otherwise be difficult to visualize. Many mathematical sculptures are quite aesthetically pleasing in addition to being highly functional in clarifying and representing mathematical ideas. Displays of mathematical sculptures are now a regular part of many art exhibitions and mathematics conferences.

Mathematical Sculptures

Researchers who explore higher degrees of dimensionality often find it challenging to represent these concepts to people whose everyday perception is three-dimensional. Mathematician Adrian Ocneanu's work includes modeling regular solids mathematically and physically. His "Octatube" sculpture, on display in Pennsylvania State University's mathematics building, maps a four-dimensional space into three dimensions using triangular pieces bent into spherical shapes. "Octatube" is conformal; the angles between faces and the way the faces meet are uniform. It was sponsored by Jill Grashof Anderson, whose husband was killed on September 11, 2001. Both graduated with mathematics degrees in 1965. Mathematician Nigel Higson said, "For professionals the sculpture is very rich in meaning, but it also has an aesthetic appeal that anyone can appreciate. In addition, it helps to start conversations about abstract mathematical concepts—something that is generally hard to do with anyone other than another expert."

Other concepts explored by mathematical sculptures include minimum variation surfaces, such as spheres, toruses, and cones, which humans tend to judge to be aesthetically pleasing because of their constant curvature; zonohedra, a class of convex polyhedra with faces that are point-symmetrical polygons, such as parallelograms; and Möbius loops, Klein bottles, and Boy's surfaces, named for mathematicians August Möbius, Felix Klein, and Werner Boy. Sculptures on exhibit at the Fermi National Laboratory, like "Monkey-Saddle Hexagon," focus in part on saddle-shaped minimal surfaces.

Mathematicians Who Sculpt

Art and mathematics have been intertwined for centuries and many historical sculptors such as Leonardo DaVinci were also mathematicians. Cubist sculptors explored many new perspectives on dimension and geometry. Spouses Helaman and Claire Ferguson have created and written extensively about mathematical sculpture. Helaman developed the PLSQ algorithm for finding integer relationships, considered by many to be among the most important algorithms of the twentieth century. He creates his award-winning sculptures to represent mathematical discoveries, and the pair's worldwide presentations have been praised for their accessibility and for initiating dialogue among multiple disciplines.

George Hart, another mathematician-sculptor, has worked in fields like dimensional analysis. He regularly hosts "sculptural barn raisings," where people are invited to help assemble large mathematical sculptures and discuss their properties. This includes a traveling sculpture for use at schools and conferences. Hart also uses rapid prototyping technology for mathematics and sculpture work. In 2010, he left Stony Brook University to be chief of content at the interactive Museum of Mathematics, with an opening date of 2012.

An "impossible" Penrose triangle sculpture built in 2008 in Gotschuchen, Austria, by a physics association.

Computer-Generated Sculpture

Self-taught artist and mathematician Brent Collins and computer scientist Carlo Séquin created their Fermi mathematical sculpture exhibit as part of a prolific ongoing collaboration. Séquin started researching geometric modeling in the early 1980s and Collins created saddle-form sculptures during the same period, though he only later learned their mathematical names. The Séquin-Collins Sculpture Generator combines the aesthetics of sculpture, mathematical theory, and computer visualization to allow sculptors to rapidly prototype and refine ideas electronically before beginning to work in their chosen medium. A designer can move around and through the model as well as slice and transform it. Some consider the computer images themselves to be "virtual sculpture." In contrast, some sculptors see computer modeling as too restrictive on the symbiotic processes of design and implementation. Some directions of mathematical sculpture include knots, three-dimensional tessellations, surfaces defined by parametric equations, fractal structures, and models of complex natural entities such as organic molecules.

Other Representations and Projects

The Hyperbolic Crochet Coral Reef project combined mathematics and marine biology to call attention to global warming and other environmental issues using three-dimensional crocheted sculptures of reef lifeforms. Artists create reef components using iterative patterns, which can be permuted to produce a broad variety of lifelike designs. The project is an extension of the hyperbolic crochet work pioneered by mathematician Daina Taimina, who demonstrated that hyperbolic surfaces can be modeled physically.

Some mathematically themed sculptures represent the connections between mathematics and other aspects of society rather than trying to model explicit mathematical concepts. Oakland University's Department of Mathematics and Statistics has a sculpted ceramic mural called *Equation*, which was created to explain the development of mathematics and its relationship to the universe and humanity. Though not a mathematician, artist Richard Ulrish stated that he has fond memories of the mathematics courses he took at Oakland.

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See Also: Crochet and Knitting; Escher, M.C.; Painting; Surfaces; Visualization.

Search Engines

Category: Communication and Computers.
Fields of Study: Algebra; Data Analysis and Probability; Number and Operations.
Summary: Using complex and sometimes proprietary algorithms, search engines locate and rank requested information, usually on the Internet or in a database.

Search engines are used for finding information from digitally stored data. Based on a search criterion like a word or phrase, search engines find information from the Internet and personal computers and present search results appropriately. A search engine is a very efficient tool for effortless finding information from millions of Web sites and their Webpages. For example, information on movies or weather forecast from the Internet can be easily found using search engines. To sort through vast amount of data, search engines use statistics, probability, mathematics, and data analysis.

Types of Search Engines

Different types of search engines are developed for different purposes. The simplest one is a desktop search engine, which is used for finding information stored within a computer. An enterprise search engine searches for digitally stored information within only one organization. A Web search engine looks for information on the World Wide Web (WWW). Sometimes, federated search engines are used for searching online databases or related items. Though there are different types, the term "search engine" generally refers to Web search engines.

Search Mechanisms

Searching for a word or phrase in a document in a computer is very simple and sophisticated search engines are not needed for this. A program simply reads the whole or selected part of the document, looks for where the intended word or phrase is located, and highlights the locations in the document.

Desktop search engines perform more complicated searches. These engines read all files and folders kept in the computer to collect information and index them. Indexing is a method of storing information about files and folders considering several factors like file names, contents, types, authors, and locations of files. It uses mathematical manipulations involving numbers, operations, and data mining. Once indexing is finished, the engine follows that index for searching. For example, if the word *algebra* is searched in a computer, the engine reads the index and tries to find out where the word *algebra* is located (if anywhere), and it shows the resulting files or folders.

The most complicated and interesting search engines are Web search engines. The Web contains billions of Web pages, and each page contains information. These search engines search for information from almost all of them. These engines generally work in three major steps: (1) collecting information from the Web, (2) indexing, and (3) presenting search results.

For reading Webpages and collecting information, almost all Web search engines have their own computer program, often called a "crawler." A Web search engine may have one or more crawlers. The information collected by crawlers contains subject matters, hyperlinks, images, and other information. Next, the search engines index the collected data and store them for future retrieval. The index is like a giant catalogue and involves huge mathematical applications to prepare. When a search criterion is given for searching, search engines follow this index; they find which Webpages contain the information and present results as lists of links to those pages.

A challenging task for Web search engines is to present the search results properly and quickly. While showing the results, it is expected that the more relevant pages corresponding to the search criterion should appear earlier than less relevant pages. Different search engines have different algorithms for arranging pages based on relevance. For example, the Google search engine uses an algorithm called PageRank for this purpose. It uses probability, data analysis, matrix algebra, and related fields.

Examples of Search Engines

Web search engines began to be developed in the 1990s and are constantly improving to handle the increasing size and content of the Web. Many of the individuals who develop and refine search engines have degrees in mathematics. Popular search engines like AltaVista (launched in 1995), Google (1998), Yahoo Search (2004), and Bing (2010) are only a few examples. Google Desktop, GNOME Storage, Windows Search, and Easyfind are among the most popular desktop search engines, while OpenSearchServer and DataparkSearch are good examples of enterprise search engines.

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See Also: Data Analysis and Probability in Society; Data Mining; Internet; Matrices; Number and Operations; Probability; Randomness; Statistics Education.

Segway

Category: Travel and Transportation.
Fields of Study: Algebra; Calculus; Geometry; Measurement; Number and Operations.
Summary: The Segway is a personal transporter built on the principle of dynamic equilibrium.

The Segway is an electric, two-wheeled personal transportation device that utilizes principles of balance and equilibrium both to create and control its motion. The Segway transporter was developed in part to combat the congestion and pollution caused by automobiles. In many cities, Segway tours are now alternatives to walking or bus tours. The Segway is often cited as an

Riders lean forward to make the wheels move forward and lean back to stop the Segway.

application of a classical dynamical systems problem: the inverted pendulum problem. It can also be referenced in illustrating the phenomenon of dynamic stability that also occurs in human walking.

Inverted Pendulum

In a traditional pendulum problem, the pendulum is composed of a mass attached to a string that is itself attached to a pivot point. In this case, the mass hangs below the pivot point. The position in which the mass hangs below the pivot point is stable—the pendulum eventually returns to that position even if pushed away from that position. In fact, it is relatively easy for the pendulum to rest in this equilibrium position. In an inverted pendulum problem, the situation in which the mass is above the pivot point is considered. Frequently, one can visualize this scenario as a "cart and pole." With the cart at rest, if the pole is perfectly positioned, it will stand upright on top of the cart. However, this condition is unstable; if the pole is moved away from this resting position, it falls.

An interesting property about the inverted pendulum (or cart-and-pole) problem is that as long as the base, or cart, is resting, the upright position is unstable. However, if the base or cart is in motion, oscillating at the right frequency, the upright position becomes stable. Imagine that the cart is moving forward and backward ever so slightly and very rapidly; in this case, the pole can remain upright. Now, the pole is in a dynamically stable position. This type of motion-induced stability is similar to what happens as humans walk. If an individual leans forward with his or her feet firmly planted on the ground, the individual will fall. However, if the feet are allowed to move, the individual will not fall but instead will move forward (or backward, depending on the direction of the lean). Allowing the feet to move has made the leaning position dynamically stable. With the feet moving, it is much harder for the individual to fall.

Dynamic Equilibrium

The Segway transporter operates on this principle of dynamic equilibrium. Riders lean forward to cause the wheels to move forward and lean back to cause the Segway to stop or reverse. The wheels and base are dynamically moving to keep the rider in an upright position instead of falling to the ground. Balance sensors in the base of the Segway regulate and control the motion by incorporating the pitch angle (or tilt) of the rider, the change in pitch angle, the wheel speed, and the wheel position. Mathematicians, physicists, and engineers relate all these variables through differential equations describing motion; these equations have long been studied in each of these fields. The Segway transporter is one example of a project resulting from the interplay of all three fields.

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Angela Gallegos

See Also: Mathematical Modeling; Mathematics, Applied; Trigonometry.

Sequences and Series

Category: History and Development of Curricular Concepts.
Fields of Study: Algebra; Communication; Connections; Number and Operations.
Summary: Sequences and series are important mathematical representations with numerous, interesting applications.

A sequence is a list of objects, called "terms," arranged in a fixed pattern such as 1, 3, 5, 7, 9, . . . or Monday, Tuesday, Wednesday, Thursday, Friday, ... In a series, the terms of a sequence are typically added together. Series have a long history of being used to approximate functions or represent geometric quantities. For example, in the seventeenth century, James Gregory showed how the areas of a circle and hyperbola could be obtained using series. In the early days of calculus, series represented geometric quantities and were manipulated using methods extended from finite procedures. Mathematicians like Niels Abel critiqued the rigor of series and expressed concerns with the foundations of calculus. The theory of series was later made rigorous within the field of analysis. Series are important to many areas in science and engineering. Sequences are explored in the primary and middle grades, while series are introduced in high school.

Famous Sequences

One very famous sequence emerges when considering the reproductive habits of rabbits. Consider two rabbits that are too young to reproduce after their first month of life but can and do reproduce after their second month of life. That pair of rabbits produces another pair after its second month and for each month thereafter. If one assumes that none of the rabbits die and that each pair reproduces in the same manner as the first, the number of pairs of rabbits at the end of each month corresponds to the elements of the sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, ... This sequence is known as the "Fibonacci sequence." It is named after Leonardo de Pisa who was called Fibonnaci, a nickname meaning "son (*filius*) of Bonaccio." He wrote about it in his 1202 book *Liber Abaci*. With the exception of the first two terms, each successive term is found by adding the two terms prior to it. This sequence appears in nature in other situations, including the arrangement of leaves on the stems of certain plants, the fruitlets of a pineapple and the spirals of shells. Some mathematical historians suggest that a Fibonacci-like sequence of integers is also represented in stone balance weights excavated in the 1960s that originated in the eastern Mediterranean during the Late Bronze Age.

Other specific types of sequences have been explored. In 1940, Pavel Aleksandrov introduced a concept called "exact sequences," which found relevance in a wide variety of mathematical fields. In 1954, Jean-Pierre Serre was awarded a Fields Medal, the most prestigious award in mathematics, in part because of his work on spectral sequences.

Series

A series is often the sum of the terms of a sequence. Series originate as early as the Indian mathematician and astronomer Brahmagupta who gave rules for summing series in his 628 c.e. work *Brahmasphutasiddanta* (*The Opening of the Universe*). The sum of the terms of an arithmetic sequence is called an *arithmetic series*. The arithmetic series $1 + 2 + 3 + 4 + \cdots + 97 + 98 + 99 + 100$ is also a well known one, as it is related to mathematician Carl Friedrich Gauss (1777–1855). At a very young age (around 6 years old), Gauss found the sum of the natural numbers (1, 2, 3, 4, …) from 1 to 100. That is, the sum given by the series $1 + 2 + 3 + 4 + \cdots + 97 + 98 + 99 + 100$. He was given this task by his teacher to keep him busy while the teacher worked with the other students in the class who were not as mathematically gifted as Gauss. After a relatively short time, Gauss returned to the teacher with the sum 5050. Gauss's method was to pair up the terms of the series. Taking the sum of the first and last term $(1 + 100)$ yields 101. This is the same as the sum of the second and second to last $(2 + 99 = 101)$, the third and third to last (3+ 98 + 101), and so forth. In all, there are 50 such pairs, each of which sums to 101. Thus,

$$1 + 2 + 3 + 4 + \cdots + 97 + 98 + 99 + 100$$
$$= (50)(101) = 5050.$$

Many mathematicians advanced the theory of important series such as power series, trigonometric series, Fourier series, and time series. For example, Nicholas Mercator represented the function $\log(1+x)$ as a series in 1651. Taylor series, named after Brook Taylor, is a representation of a function as an infinite sum of terms calculated from the values of its derivatives at a single point. From the early history of analysis, these power series were important in the study of transcendental functions. Data given as a sequence of data points over time led Wilhelm Lexis to develop time series in 1879.

Applications of Series

Some other series arose in the context of questions related to physics and sparked controversy. The mathematics and physics of a vibrating string and solutions of the wave equation led to trigonometric series. Daniell Bernoulli, Jean Le Rond d'Alembert, Leonhard Euler, and Joseph-Louis Lagrange debated the nature of trigonometric series in the eighteenth century. Joseph Fourier developed Fourier series for the heat equation in the nineteenth century, which was criticized at the time because it contradicted a theorem by Augustin-Louis Cauchy but was explored more rigorously by Johann Dirichlet. An overshoot or ringing in Fourier series was first observed by H. Wilbraham and later explored by Josiah Gibbs. The Gibbs phenomenon has implications in signal processing. The three-body problem, which investigates the behavior and stability of three mutually attracting orbiting bodies in the solar system, was solved by Delaunay in 1860 via representing the longitude, latitude, and parallax of the moon as an infinite series.

However, in 1892, Jules Henri Poincaré showed that these and similar solutions were not in general uniformly convergent, and this criticism created doubt about proofs of the stability of the solar system and eventually led to the formation of the field of deterministic chaos. A prize was offered by King Oscar II of Sweden for a solution to the extension of the three-body problems to n bodies. It has since been proven that no general solution is possible, but the n-body problem was also connected to series in Quidong Wang's 1991 work.

Series were also important as mathematicians searched for efficient ways to represent π and find its digits. Keralese mathematician Madhava of Sangamagramam may have been the first when he used 21 terms of a series and stated π correctly to 11 places. In the 1800s, William Shanks used a series to calculate digits of π in the morning and check them in the evening. He calculated 707 digits of π using this method. However, there was a suspicious lack of the number "7" in the last digits, and it was later found that only the first 527 digits were correct. Johann Lambert used the same series to show in 1761 that π must be irrational—it cannot be expressed as a ratio of whole numbers and has an infinite, non-repeating decimal expansion. Srinivasa Ramanujan found series that converged more rapidly than others, and these efficient series were used as the foundations of computer algorithms.

Binary Series

A very famous series is the binary series that consists of powers of 2: $2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + \cdots$. It is theorized that the King of Persia, finding himself very bored, asked that a game be invented for his amusement. The inventor of the game the king found most enjoyable would be given a reward. A servant of the king created the game of chess that was most pleasing to the king. When asked what prize he would like, the servant replied that he wanted grains of rice. The chessboard consists of 64 small squares. As a reward the servant asked for 1 grain of rice for the first square, 2 for the second square, 4 for the third square, 8 for the fourth square and so forth, until all 64 squares had been accounted for. The number of grains of rice requested is the sum $2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + \cdots 2^{63}$, and it amounts to 274,877,906,944 tons of rice, which is more rice than has been cultivated on Earth since recorded time. The story goes that the king grew furious at the servant once he knew what was requested. The servant was taking the rice as he received it and distributing it among the poor. At some point, the king indicated that he did not have the rice to pay the servant. The servant indicated that he was content with the amount that he had already received and that it was the king who offered a reward not he who made the initial request. Both parties were pleased.

Applications in Economics

Series appear in many other contexts as well. For example, the future value of an ordinary annuity can be found using a series. An ordinary annuity is an account where an individual makes identical deposits on a regular schedule. The money in the account earns interest that is compounded with the same frequency as the deposits. Suppose an individual deposits $100 every year into an account that earns 6% interest annually. Three years later, the first year's deposit has earned interest over two years, the second account over one year, and the last deposit not at all. The money in the account after three years is given by: $100(1.06)^0 + 100(1.06)^1 + 100(1.06)^2$. The general series can be expressed as a single number

$$A = P\left(\frac{(1+i)^n - 1}{i}\right)$$

where A is the future value of the annuity, P is the payment made at the end of each period, i is the interest rate per period, and n is the number of periods.

Limits

Though infinite sequences consist of infinitely many terms, it may be the case that the sum of the terms of such sequences converges on a given value. Such is the case of the geometric series $.9 + .09 + .009 + .0009 + \cdots$. In this series, the first term is .9 and the common ratio is

$$\frac{1}{10}$$

Applying the formula for the sum of the first n terms of the series yield

$$S_n = \frac{.9\left(1 - \frac{1}{10}^n\right)}{1 - \frac{1}{10}} = \left(1 - \frac{1}{10}^n\right).$$

As the number of terms approaches infinity (as $n \to \infty$), the fraction

$$\left(\frac{1}{10}\right)^n$$

becomes so small that one may consider it zero. Therefore,

$$S_n = \left(1 - \frac{1}{10}^n\right) = 1 - 0 = 1$$

as n grows infinitely large. Since

$$S_n = .9 + .09 + .009 + .0009 + \cdots = .\overline{9}$$

one arrives at the very famous result that $.\overline{9} = 1$.

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See Also: Archimedes; Functions; Limits and Continuity; Numbers, Complex.

Servers

Category: Communication and Computers.
Fields of Study: Algebra; Communication; Number and Operations.
Summary: Servers help users connect to networks, including the Internet.

ARPANET, the first network of time-sharing computers, was connected in 1969. In subsequent decades, technology developments and the increasing benefits of distributed, shared access spurred network growth, ultimately resulting in the Internet and World Wide

Web. Most local, national, and global networks rely on servers, which manage network resources for client computers that are connected to it. A server may be a physical computer, a program, or a combination of hardware and software. In some cases, a system is a dedicated server. In other cases, software servers operate on multipurpose systems. A distributed server is a scalable grouping in which several computers act as one entity and share the work. In general, a network server manages overall network traffic, while specialty servers handle other tasks. CERN httpd (or W3C httpd), which debuted in 1990, is considered to be the first Web server. It was developed by scientists Tim Berners-Lee, Ari Luotonen, and Henrik Frystyk Nielsen at the European Organization for Nuclear Research (CERN). Servers and clients use communication protocols to exchange information to carry out tasks. There are server-to-server and client-server variations. Mathematicians, computer scientists, and others work to create technology and algorithms that make servers possible and increase their efficiency. They also study the properties of networks and servers, which facilitates advances in both mathematics and computers. For example, in a system with multiple parallel servers, jobs may be assigned to any server. Often, jobs are modeled with an exponentially distributed processing time or some other probabilistic distribution with some resource cost per unit of time. Mathematical methods may be used to find the optimal strategy for allocating jobs to servers to minimize costs.

Function

The term "server" does not describe a specific type of computer in the same sense that "desktop" or "Windows machine" does. When used in reference to hardware, a server is any computer running a server program, which can—and in practice does—include all configurations and operating systems. Since the 1990s and the increased demand for Internet services, there have been more and more computers that have been designed specifically to be used as Internet servers. Because they need to run for long periods of time without interruption, they must be durable, reliable, and have uninterruptible power supplies. Typically, hardware redundancy is incorporated, so that if a hard drive fails, another one is automatically put on line—a feature rarely found in personal computers. There is also a great deal of server-specific hardware, such as water cooling systems, which help reduce heat, and Error-Correcting Code (ECC) memory, which corrects memory errors as they happen, preventing data corruption. Many components are designed to be hot-swappable, meaning that they can be replaced while the server runs—without needing to power it down. Furthermore, ordinary server operations including turning the power on or off can often be conducted remotely; for example, from a home computer. Some system operators maintain watch over multiple servers in multiple locations and physically visit the site only when necessary because of a crisis.

Communication

Sockets are the primary means by which network computers in a network communicate. They are the endpoints of the flow of interprocess communication (IPC) and provide application services. They are also the place where many security breaches take place. Mathematicians and computer scientists study the different socket types and their states to understand how they work and to improve function and security. Servers create sockets on start-up that are in listening state, waiting for contact to be made by client programs. For instance, a Web browser, like Firefox, is a client program used to access content from Web servers. Most servers connected to the Internet use a protocol known as Transmission Control Protocol (TCP), developed by computer scientists Vinton Cerf and Robert Kahn for ARPANET. An Internet socket is referred to by its socket number, a unique integer that includes Internet Protocol (IP) address and socket number. Listening sockets using TCP are usually assigned the remote address 0.0.0.0 and the remote port number 0. TCP servers can serve multiple concurrent clients by creating what is called a "child process" associated with each client and establishing TCP connections between child processes and clients. Each connection uses a unique dedicated socket. Two communicating sockets—the local socket created by the server and the remote socket of the client—are called a "socket pair," and their activity is referred to as a "TCP session."

A common feature of Web servers is server-side scripting, which allows Web pages to be created in response to client activity. For instance, a search for a book on Amazon.com results in a unique search results page. Without this capacity, every possible search would need to be conducted in anticipation of client needs.

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Bill Kte'pi

See Also: Cerf, Vinton; Parallel Processing; Personal Computers; Wireless Communication.

Shipping

Category: Business, Economics, and Marketing.
Fields of Study: Geometry; Measurement; Number and Operations.
Summary: A variety of mathematical concepts, including packing, routing, and tracking, are necessary to make the process of shipping goods more efficient.

The shipping and delivery industry is a vast global business that is responsible for delivering packages, postal mail, and commercial cargo all over the world. In 2009 alone, express delivery companies made $130 billion in revenue worldwide, the U.S. Postal Service delivered 177 billion pieces of mail, and ocean liners transported more than $4.6 trillion worth of goods between nations. With so many items being delivered to so many different places, there is a need for mathematics to help manage the complex delivery network and ensure that deliveries are made correctly, safely, cheaply, and quickly. Mathematics has had a significant impact in three key areas of the shipping industry: container packing, vehicle routing, and package tracking.

Container Packing

To minimize transportation costs and maximize profit, a shipper would naturally prefer to pack cargo into as few shipping containers as possible. Determining the optimal way to arrange items in a container is a deceptively difficult problem. Given a set of differently sized objects, the Bin-Packing Problem is to find the order in which to place the objects so that they fill the minimum number of bins. Testing every permutation of packing the objects would be too time-consuming, so an efficient and simple algorithm is required.

A common packing procedure is the First-fit algorithm, where the objects are ordered from largest to smallest, and each object is placed in the first available bin that will hold it. It can be proven mathematically that this algorithm is not guaranteed to produce the optimal packing. In the worst case, the result can be far from optimal and require the use of more bins than a more sophisticated packing. The First-fit algorithm is an example of an approximation algorithm, which means it produces a good approximate answer but not necessarily the optimal arrangement of objects. Other more sophisticated bin-packing algorithms have been developed, but as of 2010, no efficient algorithm was known that always produced the optimal packing.

In practice, there are more considerations to packing shipping containers. Some packages will be irregularly shaped and do not stack well. Some cargo is fragile

NP-Complete Problems

The NP-complete set is a list of mathematical problems for which there is no known fast algorithm for solving the problem exactly. The Traveling Salesman Problem is an example of a NP-complete problem. While there are fast algorithms for finding a good answer, the only known algorithm for finding the single shortest route is extremely slow. However, just because there is no known fast algorithm for solving these problems, it does not mean that such an algorithm does not exist. In 2000, the Clay Mathematics Institute offered a $1 million prize to anyone who could devise an algorithm that would solve an NP-complete problem quickly or prove that no such algorithm exists. While not technically an NP-complete problem, the Bin-Packing Problem is in a related category of problems known as "NP-hard."

and must be secured separately. Sometimes, a delivery vehicle will make several stops, so the packages that are delivered first should be packed into a container last to make them easily accessible.

Through World War II, most cargo was shipped in wooden crates of various sizes. A big step forward came in 1956, when trucker Malcolm McLean patented the modern shipping container made of corrugated steel. This sturdy container was easier to move between truck, rail, and ocean liner. More importantly, having a standard-size container meant that packing procedures could be standardized. Prior to 1956, it was estimated that loose cargo cost $5.86 per ton to load. After the standardized container was introduced, it was estimated the loading cost dropped to 16 cents per ton, a 3600% improvement.

Tjalling Koopmans (1910–1985)

Tjalling Koopmans was a Dutch economist who helped develop the mathematical field of operations research. Working for the British Merchant Shipping Mission in the 1940s, Koopmans derived a mathematical model for finding the most cost-effective shipping routes. Later, he became a professor of economics at University of Chicago and then at Yale University. In 1975, Koopmans received the Nobel Prize for Economics for developing mathematical tools for the analysis of corporate management and efficiency.

Vehicle Routing

Cargo travels by a variety of transportation modes, including truck, rail, air freight, and ocean liners. The goal of routing is to determine a vehicle for each piece of cargo to be delivered and then find the shortest delivery route for each of the vehicles. The Traveling Salesman Problem is a simple mathematical example of a routing problem. In practice, the value of a route is not determined by just the distance. The problem is complicated by considerations such as personnel, fuel costs, traffic, tolls, and tariffs.

Mathematical analysis of delivery routes can lead to huge improvements in shipping efficiency. As the first Postmaster General of the United States, Benjamin Franklin ordered careful surveying of delivery routes, refined the post office accounting practices, and increased public access to mail. Under this new system, the U.S. Postal Service became profitable for the first time, and it is estimated that the mail delivery time between major cities was cut in half.

The routing problem is an example of a problem studied in operations research, the branch of mathematics that studies the cost-effectiveness of decisions made by corporate management such as scheduling and personnel assignments. The field of operations research has its origins in World War II, when the Allied Forces were interested in coordinating the manufacturing and organization needed to mobilize the military. One of the early researchers in operations research was Tjalling Koopmans, who proposed a mathematical model for the routing problem for shippers.

Package Tracking

It is important for a shipper to carefully track a package until it reaches its destination. A common system for identifying a package is the barcode. By encoding the destination as a sequence of black and white bars, the packages can be sorted quickly by automated sorting machines equipped with laser scanners. The U.S. Postal Service has developed a special barcode that encodes the address as a sequence of short and tall black bars. The mail is first read by an Optical Character Recognition (OCR) program, which translates the handwritten address into a barcode. The barcode is stamped onto the package and then automatically sorted to be sent to the next distribution center.

Radio-frequency identification (RFID) is a tracking technology that could potentially have a large impact on the shipping industry. A small electronic tag that emits a radio signal would be placed on each item to be shipped. Generally, this tag is a microchip just a few millimeters on a side. Potentially, this microchip would allow a shipper to determine the entire contents of a shipping container without ever opening the container. However, the technology still needs to be refined to make RFID a cheaper alternative to the barcode. Furthermore, since an item could theoretically still be tracked after the delivery is made, RFID technology is somewhat controversial because of privacy concerns.

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Todd Wittman

See Also: Bar Codes; Scheduling; Traveling Salesman Problem.

Similarity

Category: History and Development of Curricular Concepts.
Fields of Study: Communication; Connections; Geometry; Measurement.
Summary: The concept of mathematical similarity has been studied since antiquity.

The concept of "similarity" is universal, playing a particularly large role in the field of geometry. In general, objects may be called "similar" if they share features that look alike, such as shape, color, or value. However, it is a much stronger statement to say that two objects are "mathematically similar." Similarity can be a powerful simplifying assumption in modeling situations. Scaling an object appears in many applications, such as in architecture. Scaling notions can also explain the speed of a hummingbird's heartbeat as compared to a human heart, and why certain insects would collapse under their own weight if they were scaled to a large size. Julian Huxley asserted that the evolutionary struggle to maintain similar surface-to-volume relationships is important in anatomy. Recognizing a similar object is also important. Logician and philosopher Willard Van Orman Quine felt that learning, knowledge, and thought all require similarity so that humans can order objects into categories with similar meaning. Similarity is often connected to triangles in mathematics, starting in grades three through five, but there are many other mathematical situations where it is also useful, such as in the definition of trigonometric functions, in axiomatic arguments, in matrices, in analysis of differential equations, and in fractals.

Early History

Distance calculations contributed to the development of similarity. Thales of Miletus is said to have measured the height of a pyramid using its shadow, but historians are unsure of the method that he used. A method that makes use of similar triangles is attributed to Thales by Plutarch of Chaeronea. In classrooms in the twentieth and twenty-first century, similar experiments are conducted. By measuring the length of the shadow of a tall object, like a pyramid, tree, or building, at the same time as measuring the length of a shadow of a known meter or other stick, a proportion with similar right triangles can be formed. The method assumes that light rays are parallel. In ancient China, instruments such as the *L*-shaped set-square or gnomon also needed similar triangles. In chapter nine of the *Nine Chapters on the Mathematical Art*, problems were posed and solved using similarity concepts. One of the problems has been translated as

> There is a square town of unknown dimensions. There is a gate in the middle of each side. Twenty paces outside the North Gate is a tree. If one leaves the town by the South Gate, walks 14 paces due South, then walks due West for 1775 paces, the tree will just come into view. What are the dimensions of the town?

Many other mathematicians have worked on a variety of similarity concepts and applications. In Euclid of Alexandria's *Elements*, the various definitions of similarity depend on the figure being examined. Apollonius of Perga explored the similarity of conic sections. During the seventeenth and eighteenth centuries in China, the proportionality of corresponding sides of similar triangles in the plane was quite useful in solving problems in spherical trigonometry. In some twenty-first-century college classrooms, students explore the reason why spherical triangles with shortest distance paths and the same angles must be congruent—there is no concept of similarity on a sphere. Mathematics educators also study the conceptual difficulties in teaching and learning similarity.

Other concepts of similarity arose from mechanics concerns. In his work on the equilibrium of the plane, Archimedes of Alexandria postulated that plane figures that are similar must have similarly placed centers of gravity. Galileo Galilei tried to generalize the notion of geometric similarity to mechanics. Isaac Newton, Hermann von Helmholtz, Joseph Fourier, James Froude, Osborne Reynolds, Lord Rayleigh (John Strutt), and others also worked on variations of similarity in physical situations. Building on their work, and motivated by the lack of a theoretical foundation for flight research, Edgar Buckingham articulated a formal basis for mechanical similarity in 1914. Aside from physical applications, in computer graphics, transformations that preserve similarity can be used to scale mechanical and dynamical behavior in addition to static images.

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See Also: Archimedes; Digital Images; Matrices; Transformations.

Six Degrees of Kevin Bacon

Category: Friendship, Romance, and Religion.
Fields of Study: Algebra; Geometry; Number and Operations.

Summary: Concepts from graph theory help explain the idea that people, including actor Kevin Bacon, are surprisingly closely connected with each other.

Six degrees of Kevin Bacon is an example of a network showing a high level of interconnection, known as the “small world” phenomenon. In the language of graph theory applied to films, nodes are film actors, and two nodes are connected by an edge if the corresponding actors have appeared together in a film. It is also a game that tests cinematic knowledge. The task is to find the shortest connection between a given actor and Kevin Bacon. For example, John Wayne is two connections from Kevin Bacon. They were never in a film together, so the distance is greater than one. John Wayne starred with Eli Wallach in *How the West Was Won*, and Eli Wallach starred with Kevin Bacon in *Mystic River*, establishing a shortest distance of at most length two.

The idea of quantifying distance by interpersonal connections dates at least to a 1929 short story called *Chain-Links* by the Hungarian writer Frigyes Karinthy, wherein the narrator determines a five-step connection between a riveter at the Ford Motor Company and himself. Almost 40 years later, the social psychologist Stanley Milgram, best known for his experiments on obedience to authority, devised an experiment to quantify interpersonal connections empirically. Letters were given to some 300 participants, each charged with forwarding the letter to an acquaintance who should move the letter toward the intended recipient. Writing in 1969 with Jeffrey Travers, Milgram stated, “The mean number of intermediaries observed in this study was somewhat greater than five; additional research (by Korte and Milgram) indicates that this value is quite stable.” Rounding up, this value became the popular notion “six degrees of separation”—that any two people on the planet are connected by six links. It served as the title of John Guare’s 1990 play and 1993 movie about the confidence man David Hampton. In the play, a character speaks to the audience, “Six degrees of separation. Between us and everybody else on this planet. The President of the United States. A gondolier in Venice. Fill in the names. I find that A) tremendously comforting that we’re so close and B) like Chinese water torture that we’re so close.” Exactly how close people are is something sociologists continue to debate, since the nodes and edges of this network are not precisely known.

Mathematics Networks

There are large networks where the nodes and connections are exactly known, allowing for precise analysis. In a collaboration network, nodes are researchers, and two nodes are connected by an edge if the corresponding researchers worked together on a published paper. As early as 1957, mathematicians determined their "Erdös numbers," the collaboration distance from Paul Erdös, the most prolific mathematician of recent years, with some 1500 published research papers and more than 500 collaborators. For instance, the author never wrote a paper with Erdös, but Robin Wilson wrote a paper with Erdös in 1977, and the author wrote a paper with Robin Wilson in 2004, so the author's Erdös number is two. The American Mathematical Society's MathSciNet electronic publication computes the "collaboration distance" between any two authors in its database of some 500,000 authors and 2.5 million publications.

Kevin Bacon

Film Networks

Of more interest to the general public than mathematicians and their papers, the Internet Movie Database (IMDb, found at imdb.com) includes over 1 million actors around the world and some 250,000 films from the 1890s to titles in production. The Web site OracleOfBacon.org accesses the IMDb and determines the shortest link between any two actors. The network is very tightly connected; it is surprisingly difficult to name any pair of actors even four apart. Consider Kevin Bacon, who has been in over 60 films with over 2200 total co-stars. That is a very small percentage of the total number of actors in the database, but there are over 225,000 actors who, like John Wayne, are co-stars of co-stars of Kevin Bacon. Actors within four links of Kevin Bacon comprise approximately 98% of the database. About 99% of the actors in the IMDb all connect to one another. Finding actors within the last 1% who are five or more from Kevin Bacon is another entertaining part of the game. As of 2010, there are 17 actors with a distance of eight from Kevin Bacon, so that "six degrees" is a misnomer.

Another variant of the game is to determine the actor who is best connected on average. The average every actor's Kevin Bacon number is 2.980. This number means, roughly, that a randomly chosen actor is within three links of Kevin Bacon. It is interesting to consider which sorts of actors have the lowest averages. John Wayne, with significantly more movies and co-stars than Kevin Bacon, has an average of 3.026 links to the rest of the connected actors. The best-connected actor, as of 2010, is Dennis Hopper, with an average distance of 2.772. The IMDb is regularly updated with new actors and films, and the connection data change accordingly.

Why is it six degrees of Kevin Bacon, and not some other actor? The game was created by students at Albright College in January 1994; they had watched *Footloose* earlier in the day, then saw a commercial for another Kevin Bacon film, *The Air Up There*, and a pop culture phenomenon was born. There are similar games based on other large databases, such as baseball players connected by teams, and "six degrees of" remains a very common phrase in society. Kevin Bacon himself used the notion to build a Web-based charity fundraiser, SixDegrees.org. The notion of "small world" networks is being used by scientists in applications as diverse as neural networks of worms, the interconnection of power grids, analysis of the World Wide Web, and genealogical connections.

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Brian Hopkins

See Also: Mathematics Genealogy Project; Movies, Mathematics in; Social Networks.

Skating, Figure

Category: Games, Sport, and Recreation.
Fields of Study: Algebra; Geometry.
Summary: The elements, equipment, and scoring system of figure skating all involve a mathematical framework.

Figure skating is a winter Olympic competitive sport, which involves artistically gliding on ice using metal blades. Ice skating rinks are generally shaped in the form of rectangles with rounded corners. The patterns skaters form on the ice can be explained in geometric terms. Physical principles are observed when watching figure skating. The scoring system used to judge figure skating involves algebraic computations.

Patterns

The bottoms of ice skating blades are not flat, but rather slightly curved, like arcs taken from the edge of a circle about seven to nine feet in radius. This enables the skater to angle and tilt to form patterns on the ice. These patterns can be represented geometrically. For instance, the most famous geometric pattern on ice is a figure eight, which can be formed by two circles of equal size tangential to each other. A skater could start the first circle of the figure eight on the right forward outside edge and skate the second circle on the left forward outside edge. The possible edge combinations include using the left or right foot, traveling forward or backwards, and using the inside or outside edges.

Mathematical Principles of Spinning and Jumping

In addition to basic compulsory figures, modern skating requires participants to execute increasingly difficult jumps and spins. In a jump, the skater's center of gravity follows a parabolic arc with respect to the ice, and a jump is frequently measured in terms of its vertical displacement (the height off the ice) as well as horizontal displacement (the distance). Both are a function of many variables, such as the takeoff angle and velocity immediately prior to the jump.

Spinning, whether in the air as part of a jump or on the ice, is also a complex function of many variables. Factors include the skater's body mass and speed when entering the spin, as well as the extension of the arms or legs from the body. For example, a spinning skater rotates more slowly with extended arms than when the arms are tucked in because as the radius between the body and the arms decreases, the angular velocity increases.

Judging

Four disciplines of figure skating are competitive at the Olympic level: singles (ladies' and mens'), pairs, and ice dance. In each of these disciplines, a choreographed program is skated to music in competition and is judged according to the International Skating Union's International Judging System. The International Judging System awards points for technical difficulty and artistry.

There are many types of skating elements. Jumps vary from their takeoff edges as well as numbers of rotations between one and four. Throw jumps are also performed by the pair teams. A variety of spins are possible, but there are three basic spin positions: upright, camel, and sit. Some spins involve a change of foot, change of position, flying entrance, or difficult variation. Footwork is an element in every program and requires steps and turns that fully cover the ice surface in a circular, straight line, or serpentine pattern. For pairs and ice dance skaters, combination spins, lifts, and other elements requiring two skaters are also scored.

Each of the skating elements performed in a program is assigned a numerical base value, which varies according to difficulty. For example, in the 2010–2011 skating season, the base value of a triple toe loop was 4.1 points, and the base value of the single toe loop was 0.4 points, indicating that the triple toe loop was a much harder jump. Judges add to or subtract from the base value of each element depending upon its execution. For instance, a poorly performed toe loop would receive fewer than 0.4 points. The sum of the values given for each element is called the "technical score."

In addition to a technical score for performance on the individual elements, overall scores for artistic aspects of the program, such as choreography, interpretations, transitions, and skating skills, are awarded as the program components score, which is added to the technical score for a total overall score. The skaters with the highest scores earn the highest rankings.

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Diana Cheng

See Also: Arenas, Sports; Ballet; Connections in Society; Hockey.

Skydiving

Category: Games, Sport, and Recreation.
Fields of Study: Algebra; Calculus.
Summary: Principles of calculus can be used to model a sky dive and to calculate the effect of the parachute on velocity.

Skydiving is the act of leaping out of an airplane at a sufficient altitude and placing your life in the hands of a piece of cloth—although a fairly large piece of cloth. Leonardo da Vinci left drawings of parachutists in his *Codex Atlanticus* circa 1485. The modern parachute was invented by Louis-Sébastien Lenormand in France, making the first public jump in 1783. In 1797, André Garnerin was the first to use a silk parachute, earlier versions being made of linen. The first parachute jump from an airplane was in Venice Beach, California, in 1911. The parachute was held in the arms and thrown out as the jumper left the plane. The soft-pack parachute was developed in 1924. There are two types of parachutes used for skydiving: round, and ram-air (square). The U.S. Army uses the round 35-foot diameter parachute to train its paratroopers because they are reliable and give the jumper a terminal velocity of about 15 feet per second. Most skydivers in the United States started using a 28-foot round canopy. They produced a terminal velocity of about 17–18 feet per second—a somewhat hard landing. The switch to ram-air types came in the 1970s; these give more comfortable landings and maneuverability. The rates of descent vary from canopy to canopy, but terminal velocities usually run from eight feet per second (5.5 mph) to 14 feet per second (9.5 mph).

A canopy's performance is determined by its wing-load, which helps determine the terminal velocity and speed at landing. Most canopies are flown with a wing-load between 0.8 and 2.8 pounds per square foot. To compute the right size of canopy, take the total weight (W) of the jumper and equipment divided by the assigned wing-load factor (WLF):

$$Area_{\text{canopy}} = \frac{W_{\text{jumper}} + W_{\text{equipment}}}{WLF}.$$

To model the parachute jump itself is much more complicated. It involves a first order differential equation to find the speed. The forces on a skydiver are the gravitational force, F_g, and the drag force, F_d, of air resistance and buoyancy. There are two factors to the drag: the time before and the time after the canopy deploys. If x is the distance above the Earth's surface, then $a = dv/dt$ is acceleration and $v = dx/dt$ is velocity. For most jumps, the gravitational force stays essentially constant.

In a first approximation to the problem, take the drag force to be proportional to the velocity. The coefficient of drag has one value when the skydiver is falling and a second value when the parachute is fully deployed. During the fall, the velocity satisfies the initial value problem:

$$m\frac{dv}{dt} = -mg - k_1 v \qquad v(0) = 0.$$

This is a separable ordinary differential equation. Its solution can be found by most students in a calculus class. The jumper's position then is found by integrating the velocity with initial condition that at time $t = 0$ the jumper is at the jump altitude. After the chute deploys, the velocity and position can be found exactly as above, except that the drag coefficient and initial conditions change.

A second approach is to assume that the drag force is proportional to the square of the speed. Then, a falling object reaches a terminal velocity:

$$V_T = \sqrt{\frac{2mg}{\rho A C_d}}$$

where V_T is the terminal velocity, m is the mass of the falling object, g is the acceleration due to gravity, C_d is the drag coefficient, ρ is the density of the fluid through which the object is falling, and A is the projected area of the object.

Based on air resistance, the terminal velocity of a skydiver in a belly-to-Earth free-fall position is about 122 miles per hour (179 feet per second). A jumper reaches 50% of terminal velocity after about three seconds and reaches 99% in about 15 seconds. Skydivers reach higher speeds by pulling in limbs and flying head down, reaching speeds close to 200 miles per hour. The parachute reduces the terminal velocity to the five to 10 miles per hour range. This is achieved by increasing the cross-sectional area and the drag coefficient, lowering the terminal speed.

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David Royster

See Also: Airplanes/Flight; Calculus and Calculus Education; Calculus in Society.

Skyscrapers

Category: Architecture and Engineering.
Fields of Study: Algebra; Geometry; Measurement.
Summary: Mathematicians and engineers work together to design and build skyscrapers.

A skyscraper is a building noteworthy for its great height. As the name suggests, the building appears to touch the sky. There is no agreed-upon minimum height that classifies a building as a "skyscraper"; the term is used for any building that commands attention because of its height. Many people are fascinated by building, visiting, and measuring skyscrapers. The Eiffel Tower, designed by engineer Gustave Eiffel, revolutionized civil engineering and architectural design. In the design of a skyscraper, architects and engineers must consider load distribution and the impact of the wind and earthquakes. Scientists and mathematicians also investigate how to improve features such as seismic dampers. Many skyscrapers resemble rectangles or pyramids, but they may have other geometries, like the plan for the Helicoidal Skyscraper in New York or the sail-shaped skyscraper in Dubai—the Burj al-Arab Hotel. In Tokyo, St. Mary's Cathedral incorporates eight hyperbolic parabolas, and the HSB Turning Torso in Sweden uses five-story cubes that twist as they rise, with the top cube 90 degrees from the bottom cube. Buckminster Fuller proposed a city consisting of huge floating spheres, which he called Cloud Nine. The Wing Tower in Scotland was designed to rotate at the base in order to respond to changes in the direction of the wind. Proposed dynamic skyscrapers allow each floor to rotate independently, creating changing shapes, and using turbines to harness the power of the wind. There are various ways of ranking skyscrapers by height, and these buildings have other characteristics that can be quantified as well. Mathematician Shizuo Kakutani invented a mathematical skyscraper in ergodic theory called a "Kakutani skyscraper," so named because the mathematical process resembles the floors of a skyscraper. Students in some mathematics classrooms play a multiplication skyscraper game.

History

Throughout history, there have been buildings that were considered unusually tall, including pyramids, towers, and religious structures. The 10-story Home Insurance Building in Chicago, designed by William Le Baron Jenney and completed in 1885, is considered by many to be the world's first skyscraper. A variety of technological developments made the first skyscrapers possible. These included the mass production of steel, the invention of the elevator, the ability to achieve water pressure at altitude, the fireproofing of flooring and walls, and the development of reinforced concrete. The 792-foot Woolworth Building in New York City, completed in 1913, was typical of how skyscrapers would be constructed for the rest of the twentieth century. It had a steel skeleton and a foundation of concrete. Modern skyscrapers typically have a frame that supports the building's weight, with walls suspended from the frame. This feature distinguishes them from smaller buildings in which the walls are usually weight-bearing.

The Empire State Building in New York City reigned for 41 years as the world's tallest skyscraper and entered the public consciousness when the 1933 film *King Kong* depicted a giant ape that climbed the building. The movie had innovative special effects, including the use of scale modeling. In the twenty-first century, numerous television and FM radio stations transmit their signals from atop the Empire State Building and from skyscrapers in other cities.

Measurement

There are many different ways to measure the height of a skyscraper. It can be measured by the number of floors, highest occupied floor, spire height, or total height including such things as an antenna. Consequently, different figures can be found for the height of a single skyscraper. When lists of the world's tallest skyscrapers are published, a single skyscraper often ranks in different places on lists that use different rules of measurement. For example the Willis Tower in Chicago, formerly known as the Sears Tower, is the world's second tallest building when ranked by number of floors or when antennae are included, but it places seventh worldwide when spires are counted, but antennae are not.

Since 1998, a number of skyscrapers in Asia have surpassed the tallest American buildings in height. The Burj Khalifa, which opened in 2010 in Dubai, United Arab Emirates, is the world's tallest skyscraper as of 2010, whether ranked by its 163 floors, its 2,093-foot highest floor, or its spire height of 2,717 feet. The progression of record skyscraper heights over time can be graphed and modeled by a regression equation.

Other Aspects

Skyscrapers are noteworthy for other quantities besides their heights. When known geometric solids are used to model a skyscraper's shape, the building's surface area can be estimated. Because of differences in elevation, a skyscraper often experiences measurably different weather conditions at its top and bottom. In addition to its noteworthy height measurements, the Burj Khalifa contains over 20 acres of glass, has over 5 million square feet of floor space, and has elevators that travel over 26 miles per hour. Tall buildings are known to sway slightly in windy conditions. A rule of thumb for estimating a building's sway is to divide its height by 500 to arrive at the amount of horizontal sway near the top of the building. In many skyscrapers, steel tubes, or bundles of tubes, give the building strength against this swaying. The distance one can see from the top of a skyscraper can be computed. When the curvature of Earth is considered, the sight line is tangent to Earth's surface. On a clear day it is possible to see over 100 miles from atop the world's highest skyscrapers.

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David I. Kennedy

See Also: City Planning; Clouds; Elevation; Elevators; Engineering Design; Measurement in Society; Movies, Making of; Radio.

SMART Board

Category: Communication and Computers.
Fields of Study: Geometry; Representations.
Summary: Interactive whiteboards use a touch-sensitive display to mimic the functionality of a whiteboard while enhancing the user's options.

The SMART Board is a brand of interactive whiteboard. Unlike traditional whiteboards and chalkboards, the SMART board does not require markers, chalk, or erasers. Instead, the SMART Board utilizes a projector and a touch sensitive display. The projector displays computer images onto the screen. The screen itself allows the user to interact directly with applications similar to a large touch screen. For instance, touching the screen is equivalent to left-clicking with a mouse. Typically, SMART Boards come with four digital pens and a digital eraser. These digital devices allow the user to write on the screen using digital ink. SMART Board interfaces are available for Windows and for the Mac operating system.

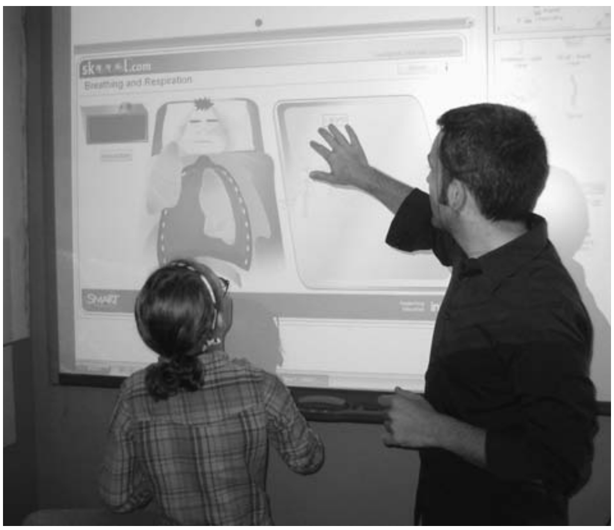
Development

David Martin and Nancy Knowlton, inventors of the SMART Board, initially devised the idea in 1986 and began promoting it in 1991. Knowlton previously taught accounting and computer science, while Martin has a bachelor's degree in applied mathematics and began his career working on computer simulations. The SMART Board was the first interactive whiteboard that gave users touch control of computer applications. In 2003, their company developed and later patented Digital Vision Touch (DViT) technology, which relies on concepts of three-dimensional geometry, such as projection, reflections, and parallel lines to effectively display information and allow the user to interact with the board. It uses digital cameras and sophisticated recognition algorithms to determine the position of the user's fingertip and to make a distinction between single clicking, double clicking, and drag and drop. These recognition algorithms differentiate it from other touch technologies, like tablet personal computers. As of 2010, SMART Board was the most popular interactive whiteboard on the market in the United States.

Advantages

There are several advantages to SMART Board technology in the mathematics classroom. First, lectures done using the SMART Board can be saved, which allows instructors to access information written minutes, weeks, or even years earlier. By exporting these files as a pdf or a similar universal format, the instructor can post classroom notes on their course Web page, allowing students to review notes from previous classes, either to prepare for a test or to catch up on material that was covered when they were absent. In addition, the digital images saved by the SMART Board can more easily be read and transcribed for students with disabilities. Further, images on the SMART Board can be individually selected and copied to additional pages, which allows complex mathematical formulas and diagrams to be reproduced accurately and quickly. SMART board systems are typically connected to computers, meaning any application that is accessible on the computer is available on the SMART Board. Instructors may access spreadsheets, word processors, and the Internet. For these reasons, SMART Boards can greatly enhance the educational experience for both the instructor and the student. SMART Board–type lectures can also be accomplished using a tablet computer installed with the appropriate software and a projector system.

Since its introduction in 1991, SMART Boards have been incorporated into classrooms of all levels from kindergarten to college. In addition, many corporate boardrooms feature SMART Boards allowing for interactive presentations. As of 2010, over 1 million SMART Board systems have been installed across the world. It is likely that SMART Boards and similar systems will continue to replace or supplement the more traditional whiteboards and chalkboards found in current classrooms.



A projector displays video output on the SMART board, which acts as a large touch screen.

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See Also: Calculus and Calculus Education; Curricula, International; Curriculum, College; Curriculum, K-12; Digital Images; File Downloading and Sharing; Geometry and Geometry Education; Internet; Personal Computers; Schools; Statistics Education.

Smart Cars

Category: Travel and Transportation.
Fields of Study: Algebra; Geometry; Measurement.
Summary: A smart car is able to respond to the conditions it detects, such as sounding an alarm if it detects that a driver is becoming drowsy.

A smart car is also sometimes referred to as a "biometric car." The overall design and technology of such vehicles should incorporate many functions: protection of the driver and passengers, reliable and easy navigation, and better mechanical and fuel efficiency. Mathematicians, engineers, and many others are involved in the development of improved vehicle technology, including aerodynamics and computerized systems that use mathematical techniques from geometry; mathematical and computer modeling; and statistical analyses of data regarding safety, ergonomics, and consumer preferences. Methods from artificial intelligence, such as cellular automata, are also very useful. According to mathematician John von Neumann, cellular automata can be thought of as "cells" or agents that behave according to relatively simple sets of mathematical rules or algorithms. These rules include responses to neighboring cells' behaviors, making them useful in modeling many biological processes, like flocking birds or traffic.

Ideal Functions

In many peoples' minds, the primary purpose of a smart car should be to help a driver in ways that prevent accidents and encourage safe driving. For example, many car accidents occur because drivers do not realize that they are drowsy, so they consequently fall asleep at the wheel. A biometric smart car could alert drivers to such conditions by measuring eye movements relative to typical alert driver behavior to detect inattention and lack of scanning of the instruments and the road. Drivers that deviated too far from established safety norms would then be alerted. Other systems may involve a steering detector that responds to angular movements of the steering wheel that exceed a specified degree or a system that measures the angles of a driver's head and sound an alert if the head nods too far forward. In 2010, a Japanese company launched a system designed for commercial truck drivers that analyzes a driver's unique patterns and variability taking into account variables such as time. It then uses mathematical algorithms to proactively recommend rest breaks and measures to increase alertness and safety.

But What Actually Makes a Car Smart?

In addition to reactive systems like driver alertness warnings, some feel that a truly smart car should anticipate conditions to be avoided. Speeding when road conditions are poor or attempting to pass another car in low visibility could be predicted and avoided. Smart car systems would not only anticipate but also correct any anomaly so that a driver has time to recover. Further, they might suggest actions to a driver in advance of adverse conditions by monitoring the road and weather. Aspects of these features are present in many models of cars at the start of the twenty-first century facilitated by the introduction of real-time technology, such as interactive maps and global positioning systems (GPS), which depend on external communication with the environment to provide data beyond the drivers' senses. For example, many agencies provide data on road grade and surface, work zones, hazards, or speed restrictions. A smart car also monitors its internal state, taking measures of aspects like tire pressure and fluid levels using electronic sensors—functions that used to have to be performed by hand.

Advanced instrumentation, once found mostly in luxury cars, is becoming commonplace in vehicles. These systems may include smart starting that relies on electronics embedded in the car's keys; biometric features, like fingerprint scans; or keyless entry that may also require a computer chip, code, or fingerprint to activate. Many hybrid gas–electric vehicles balance energy usage to obtain maximum performance in mileage. Future smart cars may automatically sense variables like weight distribution and suggest load adjustments for better balance and braking. There are even notions that future smart cars will be able to dynamically reshape their surfaces for maximum aerodynamic efficiency. There is work being done on systems such as neural networks that may monitor and analyze all driver decisions in order to better provide feedback for safety and performance for particular geographic regions. Networks within smart cars may also interact with other cars and "smart roads," which could use computer technologies and mathematical modeling or algorithms, coupled with control and communications features, to improve issues like road safety and traffic capacity by directing traffic and helping drivers make better and safer decisions.

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Julian Palmore

See Also: GPS; Highways; HOV Lane Management; Neural Networks; Street Maintenance; Traffic.

Soccer

Category: Games, Sport, and Recreation.
Fields of Study: Data Analysis and Probability; Geometry; Measurement.
Summary: Mathematical modeling and statistical analysis can help inform individual techniques and team tactics in soccer.

Soccer is a sport that has been enjoyed worldwide for more than a century by both players and spectators. In the early part of the twentieth century, mathematician Harald Bohr, founder of the field of almost periodic functions and brother of famed physicist Niels Bohr, was a skilled player and a silver medalist on the 1908 Danish Olympic soccer team. He was reported to be so popular that his doctoral dissertation defense was attended by more soccer fans than mathematicians. In general, the sport is often cited for its equal emphasis on individual skills and team tactics. As in other sports, statistics are frequently cited by sports commentators. In addition, technically demanding individual actions, as well as masterfully executed plays, can all be described and analyzed using statistics and mathematics, which is done worldwide by numerous sports scientists. One could even say that the players, perhaps unconsciously, use or display "mathematics in motion."

Individual Technique

The effectiveness of any of the various moves a player uses (kicking, heading, or dribbling) depends on a combination of physical qualities and technical skills. This idea can be demonstrated using the instep kick as an example; the instep kick, with the aim to kick the ball as hard as possible, is by far the most studied soccer movement by sport scientists. In order to maximize the forward swinging speed of the shank, physical qualities (such as strength and speed of contraction) of the knee extensor muscles and the hip flexor muscles are important. However, research has shown that technical skills are equally important. The specific technical skill required for optimal kicking is coordination—how the shank moves relative to the thigh.

Coordination is one of the topics studied in the scientific field of biomechanics, which relies heavily on

The skill required for optimal kicking is coordination—how the shank moves in relation to the thigh.

mathematics. Biomechanics researchers use high-speed cameras in their laboratories to record kicking performance from top level players. From the video footage, the researchers can obtain the three-dimensional position in space of selected points on the kicking leg. Using mathematical concepts from vector algebra and trigonometry, joint and segment angles can subsequently be calculated. These data, in turn, allow calculations of a number of kinematic parameters of the foot, shank, and thigh, comprising linear velocity and acceleration and angular velocity and acceleration.

In mathematics, the most common way to calculate velocities and accelerations from position data is to use calculus. This method, however, requires the position data to be specified as a mathematical function. This is not the case with position data obtained from video footage, which are discrete in nature—they consist of thousands of numbers, specifying the three-dimensional position of numerous points on each video frame. From the cameras' frame rate, the elapsed time between frames can be calculated, which instead allows numerical differentiation of the position data using a computer. Finally, by combining the kinematic data with data for each segment's mass and moment of inertia (a measure of a segment's inertia when rotating) and using the principles from Newtonian mechanics, the researchers can calculate how the movement of the thigh affects the movement of the shank and vice versa. The forward swing of the thigh generates a force at the knee that causes the shank to swing faster forwards. The force is larger, the faster the thigh moves, while the effect of the shank is larger, the closer the knee angle is to 90 degrees. Top players instinctively coordinate thigh and shank movements in order to take maximum advantage of these intersegmental forces, although science so far has failed to determine precisely what optimal coordination is.

Team Tactics

When a midfielder executes a beautiful play that a forward picks up between defending opponents and scores, a lot of "hidden" mathematics is occurring. The midfielder's team members and opponents are all moving simultaneously in different directions with different speeds, yet the midfielder still manages to precisely calculate the required ball speed and direction to execute his play, so the ball and forward meet at the intended spot out of reach of defending opponents. Situations like this are analyzed by sport scientists and coaches using the methods of notational analysis. With video footage and specialized software, the various actions (sprinting, moving sideways, tackling, or heading) of each player from both teams can be registered. Statistical calculations can reveal which situations are most likely to lead to a certain outcome, such as scoring a goal, and which general tactics lead to most of these situations. Digital representations have also been used to assist with tactics and analysis. Researchers from the University of Sheffield digitized a soccer ball (including even the stitching) and computed airflow around the ball. They found that the specific shape and surface of the ball, and its initial orientation, are significant in determining the ball's trajectory through the air. Measurements on actual balls in a wind tunnel at the University of Tsukuba verified these mathematical simulations.

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Henrik Sørensen

See Also: Arenas, Sports; Connections in Society; Hockey; Kicking a Field Goal.

Social Networks

Category: Friendship, Romance, and Religion.
Fields of Study: Geometry; Number and Operation; Representations.
Summary: Social networks can be described and analyzed using graph theory.

A social network is a set of actors and the relationships that connect them. The actors are usually people, but may be other individual or collective actors, such as organizations, gangs, clubs, municipalities, nations, or social animals. Social network analysis is a cross-disciplinary method for analyzing social networks that integrates techniques from science, social science, mathematics, computer science, communication, and business. In keeping with its diverse origins, various types of social relationships have been studied using social network analysis, such as friendship, sexual relationships, kinship and genealogy, competitions, collaboration, and disease spread.

Sociogram, Sociomatrix, Graph, and Network

Modern social network analysis can be traced to Austro-American psychiatrist Jacob Levy Moreno, though many of the methods he employed in his work had been used before in a more piecemeal fashion. For example, French probabilist Irénée-Jules Bienaymé modeled the disappearance of closed families (for example, aristocrats) and family names in the nineteenth century. In his 1934 book *Who Shall Survive*, Moreno used diagrams he called "sociograms" to analyze friendships among girls in a training school in New York State. The girls were represented by points, and pairs of girls who were friends were connected by a line. In sociograms of relationships such as liking, which are not necessarily reciprocated, an arrowhead indicates the direction.

While very simple social networks can be analyzed by visual inspection, the power of social network analysis arises from the conceptualization of the sociogram as a mathematical graph, which can be analyzed using the concepts and methods of graph theory. Moreover, a graph can be represented by a square adjacency matrix in which each row and column represent a point, and the cell entries represent the presence or absence of lines between points. A graph can be generalized in several ways. Lines can have numerical values representing, for example, the strength, intensity, or frequency of a relationship. There can be multiple types of lines between pairs of actors, each representing one type of relationship. Actors can have various attributes with numerical values or qualitative labels. In social network analysis, real-life social networks are modeled by mathematical networks, then the properties of the networks are analyzed mathematically in order to draw conclusions about the structure of the social relationships.

Social Cohesion

Social cohesion is a fundamental issue in the social sciences; it is the "glue" or bond that holds a social group together. According to social network analysis, it is the network of social ties among members of the group. Therefore, to measure the level of social cohesion in a social group or subgroup, one must measure the extent of ties among the members. The density of ties among members is the simplest measure of connectedness. It is defined as the ratio of the number of actual ties to the number of possible ties and ranges from 0 to 1. In a network with one symmetric (undirected) type of tie, and k members, the total possible number of ties is

$$\frac{k(k-1)}{2}.$$

A network in which every actor is connected is called a "complete" graph, or a "clique."

It is easy to imagine four people all being friends with one another but less realistic to postulate a clique with a large number of members. For example, in a clique with 30 members, each would have to maintain ties with the 29 other members—an onerous task. Limits on human beings' time, energy, and memory constrain the number of people with whom they can maintain social ties. Therefore, social networks tend to become more sparse (the ties become less dense) as they become larger. Residents of a small village may know all the other residents, but this is impossible for city-dwellers. Thus, the village will tend to be more socially cohesive than the city. Density of ties has also been used to study social cohesion in such areas of social life as marriage, the family, small groups in laboratories, community elites, intercorporate relationships such as share ownership and interlocking directorates, scientific communities, and the spread of ideas and diseases.

The overall density of ties is a rather crude measure of connectivity and cohesion in a social network, because it is insensitive to local variations. Real-life social networks tend to contain islands of actors tied relatively densely to one another but disconnected or only loosely connected by sparse ties to other such islands. In the friendship network of a high school, there are likely to be a number of small cliques, perhaps loosely connected into larger subgroups that are in turn perhaps totally disconnected from one another. Detection of relatively cohesive subgroups in a network and delineation of their articulation into larger, less cohesive groups are a major theme in social network analysis.

Centrality

The centrality of an actor in a network is an important attribute, because centrality is associated with power, prestige, prominence, and popularity. In a network of ties representing flows or potential flows of valued social goods, such as information, a central actor is in a privileged position for both reception and transmission. The centrality of an actor may be intuitively evident from visual inspection of the drawing of a graph, especially if the graph is small or highly centralized. In larger graphs, a precise definition and formula are needed. The four main definitions of centrality are degree, closeness, betweenness, and power (or "eigenvector") centrality.

Degree centrality is the proportion of the other actors to which an actor is directly connected. The closeness centrality of an actor is based on how close the actor is to each of the other actors in the network and is the inverse of distance. The betweenness centrality of an actor is the extent to which the actor is "between" other actors; in other words, how often the shortest paths between pairs of other actors pass through the actor. Power centrality is defined recursively taking into account the power centrality of the actors to which an actor is adjacent.

Applications of Social Networks

The popular party game Six Degrees of Kevin Bacon tries to connect any movie actor to actor Kevin Bacon via costars in movies using the shortest number of steps. That value is an actor's Bacon Number. The Web site "The Oracle of Bacon," originally implemented in 1996, can be used to find the shortest path for any actor that can be linked to Kevin Bacon. The average path length as of September, 2010, was about three. It also allows a user to find a measure of centrality for the Hollywood network based around any actor in the database in terms of the average path length.

On a more personal level, the social network Web site Facebook includes an application called Friend Wheel that lets users visualize the interconnections among their friends as nodes and ties. Further, it selectively arranges the friends' names around the circumference of the wheel so that closely-knit groups or cliques are placed together and color-coded. Thomas Fletcher, a computer science and mathematics student at Bath University, developed the application and made it available in 2007.

Harkening back to Moreno's study, in 1995 a team of sociologists was the first to map the romantic and sexual relationships of an entire high school. Unlike similar adult networks, which tend to have several highly interconnected cores with loose interconnections (like airline hubs), the students were connected via long chains, more like a rural phone network. One chain linked 288 of the 573 romantically active students, though there were also many unconnected dyads or triads. Researchers attributed this finding in part to the often-elaborate teenage social rules about who may date. The surprising finding had important implications for educational practices like sex education programs.

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See Also: Connections in Society; Graphs; Matrices; Six Degrees of Kevin Bacon; Visualization.

Social Security

See *Pensions, IRAs, and Social Security*

Software, Mathematics

Category: Communication and Computers.
Fields of Study: Algebra; Geometry.

Summary: Mathematics software has long been used as a teaching aid and has become an important tool in applied mathematics.

Mathematics software refers to a wide variety of computer programs designed to manipulate, graph, or calculate numeric, symbolic, or geometrical data. Along with the development of computer technology and wider access to personal computers, these types of programs gained popularity at end of the twentieth century. Within the mathematics community, it has influenced instruction, applications, and research. Instruction has changed so that mathematics is more accessible to larger numbers of students; it is more engaging, more visual, and more focused on conceptual understanding rather than on computational facility.

Mathematics software has changed research and the nature of mathematical proof so that computers are now tools for exploration, for applications, and for performing repetitive tasks. There are numerous journals devoted to the development, use, or implementation of software in research and teaching, such as the *Transactions on Mathematical Software* journal. Computer software has influenced what mathematics is being taught, how it is being taught, the nature of its applications, and the way mathematics is explored. Computer software provides society a different modality for learning, understanding and applying mathematics.

What is Computer Software?

"Computer software" is a general term reserved for a collection of computer programs that provide step-by-step instructions for a computer to perform specific tasks. There are four major types of software:

- *Operating systems*: System software, often called the computer "platform" (for example, Microsoft Windows, Mac OS, and Linux);
- *Computer languages*: The code and syntax used for developing software (for example, Java, C/C++, Visual BASIC, Pascal, and Fortran)
- *General applications software*: Software designed for general purposes (for example, word processors, database systems, spreadsheets, and communications software)
- *Specific applications software*: Software designed for performing content-based tasks (for example, MATLAB, Mathematica, Geometer's Sketchpad, SPSS, and MINITAB)

Software for the subject of mathematics falls in the category of Specific Applications Software.

Mathematics Software

The term "mathematics software" refers to computer programs designed to manipulate, graph, or calculate numeric, symbolic, or geometrical data. The journal *Transactions on Mathematical Software* (TOMS), produced by the Association of Computing Machinery (ACM), provides current information on available mathematics software. Through TOMS, the reader can gain access to large indexed mathematics software repositories. The majority of the software is written in Fortran or C++ for solving mathematics problems occurring in the sciences and engineering. Research scientists are invited to use these modules in developing their own software.

Software for Applied Mathematics

According to the National Research Council (NRC), computer software has had a major impact on applied mathematics and has illuminated new areas for mathematical research. The use of computer software in research in applied mathematics is prevalent, especially when repetitive computations are necessary. The most prominent computer software packages used in college-level instruction in the early twenty-first century are MATLAB, Mathematica, and Maple. These are computer algebra systems (CAS) that perform both symbolic and numeric computations. Software is used for statistics applications by professionals in mathematics, sciences, education, and social sciences, such as SPSS, SAS, BMDP and SPlus. These allow users to easily explore and visualize data and automate the computational aspects of many commonly used statistical procedures, which can be significantly difficult for larger data sets. It also facilitates more complex modeling and computer-intensive methods like exact tests, resampling techniques like bootstrapping, and many types of Bayesian statistical procedures, which are named for mathematician Thomas Bayes.

Software for Mathematics Research

Mathematics software is also gaining prominence in the fields of pure mathematics such as number theory, abstract algebra, and topology. An outstanding example of the impact of software on topological research occurred in 1976 when a computer program was used to check all of the possible cases in the Four-Color Map conjecture.

To understand the Four-Color Map conjecture, consider a map of the United States. Suppose the task is to color the individual states so that no two contiguous states are the same color. How many different colors are necessary to complete the task? Such a question arose in 1852. The Four-Color Map conjecture states that, at most, four colors are needed to color the map. In 1976, Kenneth Appel and Wolfgang Haken finally proved this conjecture (thus establishing it as a theorem) by using a computer program, representing the first time computer software was used in the proof of a mathematics theorem.

This computer-based proof led to considerable controversy within the mathematics community. The controversy centered on the nontraditional nature of the proof, which required a computer program for testing all of the possible cases, namely 1936 maps. Some mathematicians argued that this procedure did not constitute a formal mathematical proof, which is typically based on deductive logic and mathematical principles (such as definitions, axioms, and theorems). Instead, it was an exhaustive test of all possible cases, made possible by a computer program. Thus, neither deductive logic nor mathematical structure was required. Regardless of the controversies surrounding the proof of the Four-Color Map theorem, the result was to alter the attitudes of mathematicians toward the role of computer software in formal mathematical proof. Consequently, since the 1970s, computer software has become a major research tool for both pure and applied mathematicians.

A further consequence of the use of computer software in mathematical research is a trend for mathematicians to use open-source software, rather than proprietary software. Many commercial or proprietary software programs were originally developed and sometimes freely distributed as part of grant-funded projects or by individual mathematicians, computer scientists, and others to meet specific research or teaching needs. Some of these programs were also developed in conjunction with educators and students. With proprietary software, the user is denied access to the algorithms used in solving problems and thus cannot have complete confidence in the fidelity of the mathematical results obtained by the programs. On the other hand, open-source software provides the source code to its users so they can modify and apply it with confidence to their research endeavors. Sage is an important example of open-source software that contains one of the world's largest collections of computational algorithms. For this reason, it is gaining in popularity among contemporary research mathematicians.

Software for Mathematics Education

Since the 1980s, computer software has been utilized regularly in the research of both pure and applied mathematics and it has made its way into mathematics classrooms. However, the adoption of mathematics software in teaching has not been without controversy. For instance, in 1993, students at the University

Algebra, Trigonometry, and Calculus Software

Mathematics instruction at all levels has changed considerably because of the profusion of graphing calculators in schools. These so-called calculators are actually handheld computers that have numerous built-in mathematical functions and programming capability. As a consequence, the mathematics curriculum is now more focused on conceptual development rather than building computational facility. Additionally, classroom computers and the use of interactive whiteboards (large interactive computer panels) have served to make mathematics instruction far more interactive and engaging for students.

Popular commercial software packages for college instruction are MATLAB, Mathematica, and Maple. A powerful piece of free-ware for algebra instruction at the high school level is Winplot (for Windows platforms only), which is available in 14 languages. It is a virtual graphing utility that can plot and animate functions, relations, and three-dimensional surfaces in a variety of formats.

of Pennsylvania complained about frustrations with Maple in calculus classes, citing a lack of support and faculty expertise. Some students even wore shirts printed with vulgarities about Maple, which attracted national attention. The use and implementation of software in classes has continued to generate debate regarding the balance between students exploring concepts and solving problems using traditional methods and computers. There are also questions regarding how much teaching time should be focused on instructing students in software use versus addressing concepts.

More recently, mathematics instruction in grades K–12 has benefited from computer software. This trend is due in part to the recommendations of major professional educational organizations and from federal programs and legislation. In 2000, the National Council of Teachers of Mathematics predicted that technology would enhance the learning of mathematics, support mathematics teaching, and influence the content that is taught. Educators have also praised the advantages of interactive software on student motivation and for providing a different modality for instruction—a modality that is visual, concrete, and interactive. Thus, anticipated impacts of computer technology on student achievement are encouraging. In 2002, the No Child Left Behind Act provided $15 million for research on the effects of computer technology on K–12 instruction.

Geometry Software

Computer software for teaching geometry is prevalent in American schools. The software of choice is dynamic software, which allows students to construct geometric shapes and actively explore their properties on the computer screen by (1) dragging vertices, (2) measuring component parts, (3) transforming them in the coordinate plan, (4) animating them, and (5) tracing points, and so on. Examples of dynamic geometry software are Cabri II Plus, The Geometer's Sketchpad (GSP), and GeoGebra.

When using dynamic geometry software, high school students have been able to make new discoveries in Euclidean geometry. For example, in 1994, Ryan Morgan, a sophomore at Patapsco High School in Baltimore, used GSP to discover a generalization to Marion Walter's theorem.

First, consider Marion Walter's theorem: If the trisection points of the sides of any triangle are connected to the opposite vertices, the resulting hexagon has area one-tenth the area of the original triangle.

Based on the prior theorem, Morgan discovered the following: If the sides of the triangle are instead partitioned into n equal segments (for n = an odd integer) and each division point is connected to the opposite vertex, a central hexagon is still formed.

Morgan's theorem states that this hexagon has an area

$$A = \frac{8}{(3n+2)(3n-1)}$$

relative to the original triangle.

Discoveries by high school students, such as Morgan's theorem, lend credence to using dynamic software for geometry instruction in the nation's high schools.

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See Also: Calculators in Society; Curriculum, College; Educational Manipulatives; Geometry and Geometry Education; Personal Computers; Statistics Education; Telephones.

Solar Panels

Category: Architecture and Engineering.
Fields of Study: Algebra; Geometry; Measurement.
Summary: The angle of inclination of a solar panel array is key to its efficiency, among other factors.

Solar panels are interconnected assemblies of photovoltaic cells that collect solar energy as part of a solar power system, either on Earth or in space. Typically, several solar panels will be used together in a photovoltaic array along with an inverter and batteries to store collected energy. Photovoltaic cells convert the energy of sunlight into electricity via the photovoltaic effect (the creation of electric current in a material when it is exposed to electromagnetic radiation), which was observed by French physicist Alexandre-Edmond Becquerel in 1839. Prior to that time, many scientists and mathematicians built and researched parabolic burning mirrors, which are another way to focus solar energy. Diocles of Carystus showed that a parabola will focus the rays of the sun most efficiently. Archimedes of Syracuse may have built burning mirrors that set ships on fire. George LeClerc, Comte de Buffon, apparently tested the feasibility of such a mirror by using 168 adjustable mirrors in order to vary the focal length to ignite objects that were 150 feet away. It was also investigated experimentally in the early twenty-first century on the television program *Mythbusters*. Mathematics teacher Augustin Mouchot investigated solar energy in the nineteenth century and designed a steam engine that ran on sun rays. Some consider this invention to be the start of solar energy history. The first working solar cells were built by the American inventor Charles Fritts, in 1883, using selenium with a very thin layer of gold. The energy loss of Fritts's cells was enormous—less than 1% of the energy was successfully converted to electricity—but they demonstrated the viability of light as an energy source. Engineer Russell Ohl's semiconductor research led to a patent for what are considered the first modern solar cells, and Daryl Chapin, Calvin Fuller, and Gerald Pearson, working at Bell Labs in the 1950s, developed the silicon-based Bell solar battery. There were fewer than a single watt of solar cells worldwide capable of running electrical equipment at that time. Roughly 50 years later, solar panels generated a billion watts of electricity to power technology on Earth, satellites, and space probes headed to the far reaches of the galaxy. Scientists and mathematicians continue to collaborate to improve solar panel technology. One such focus is creating scalable systems that are increasingly efficient and economically competitive with various other energy technologies.

Physics and Mathematics of Solar Panels

In 1905, Albert Einstein published both a paper on the photoelectric effect and a paper on his theory of relativity. His mathematical description of photons (or "light quanta") and the way in which they produce the photoelectric effect earned him the Nobel Prize in Physics in 1921. In general, the photons or light particles in sunlight that are absorbed by semiconducting materials in the solar panel transfer energy to electrons—though some is lost in other forms, such as heat. Added energy causes the electrons to break free of atoms and move through the semiconductor. Solar cells are constructed so that the electrons can move in only one direction, producing electrical flow. A solar panel or array of connected solar panels

The solar field at Nellis Air Base in Nevada has more than 72,000 panels and supplies the base with more than 30 million kilowatt-hours of power.

produces direct current, like chemical batteries, which can be stored. An inverter can convert the direct current to alternating current for household use.

Mathematics is involved in many aspects of solar panel design, operation, and installation. For example, the perimeter of an array of multiple solar panels may change with rearrangement of the panels, but the area stays the same. Since area is one critical variable for power collection, this suggests different optimal arrangements for surfaces where solar panels might be arranged, like walls and roofs. Satellites often use folding arrays of solar panels that deploy after launch, and folding portable solar panel arrays have been designed for applications like camping and remote or automated research and monitoring stations. Space scientist Koryo Miura developed the Muria–Ori map folding technique, which involves mathematical ideas of flexible polygonal structures and tessellations. It has been incorporated into satellite solar panels that can be unfolded into a rectangular shape by pulling on only one corner.

Arrays

A solar panel array may be fixed, adjustable, or tracking. Each method has trade-offs in installation cost versus efficiency and energy over the lifetime of the installation, which can be analyzed mathematically in order to optimize an individual setup. Fixed arrays are solar panels that stay in one position. Optimal positioning of such arrays usually involves facing the equator (true south, not magnetic south, when in the northern hemisphere), with an angle of inclination roughly equal to their latitude. Using an angle of inclination slightly higher than the latitude has been shown in some studies to improve energy collection in the winter, which can help balance shorter days or increased heating energy needs. Setting the inclination slightly less than the latitude optimizes collection for the summer. Adjustable panels can have their tilt manually adjusted throughout the year. Tracking panels follow the path of the sun during the day, on either one or two axes: a single-axis tracker tracks the sun east to west only, while a double-axis tracker also adjusts for the seasonal declination movement of the sun. Tracking panels may lead to a gain in power, but for some users, the cost trade-off might suggest adding additional fixed panels for some applications instead. Solar power companies and other entities provide maps showing the yearly average daily sunshine in kilowatt hours per square meter of solar panel. Combined with the expected energy consumption of a building, this data helps determine how many solar panels and batteries will be needed for an installation. Science and mathematics teachers often have students build solar panels and collect data to facilitate mathematical understanding and critical thinking, as well as make mathematics, science, and technology connections.

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See Also: Electricity; Light; Origami; Satellites.

South America

Category: Mathematics Around the World.
Fields of Study: All.
Summary: Long before European settlement, mathematics flourished in South America.

South America includes Argentina, Bolivia, Brazil, Chile, Colombia, Ecuador, French Guiana, Guyana, Paraguay, Peru, Suriname, Uruguay, and Venezuela. The history of South American mathematics begins with pre-Columbian developments like the Nazca lines and *quipus* ("KEE-poos") and continues through the astronomy boom of the colonial period to work by modern mathematicians and ethnomathematics studies in Brazil.

Quipus

The Incan empire, with its capital in Cuzco, Peru, dominated pre-Columbian South America. The Incan civilization emerged from the highlands in the early thirteenth century and extended over an area from what is now the northern border of Ecuador, Peru, western and south central Bolivia, northwest Argentina, northern and central Chile, and southern Colombia. The Incas reached a high level of sophistication with remarkable systems of agriculture, textile design, pottery, and administration. Since the Incas had no written records, the *quipu* (or *khipu*) played a pivotal role in keeping numerical information about the population, lands, produce, animals, and weapons.

Quipus were knotted tally cords that consisted of a main cord from which hung a variable number of pendant cords containing clusters of knots. These knots and their clusters conveyed numerical information in base-10 representation. For example, if the number 365 was to be recorded on the string, then five touching knots were placed near the free end of the string followed by a space, then six touching knots for the 10s, another space, and finally three touching knots for the 100s. Specific information was conveyed via the number and type of knots, cluster spacing, color of cord, and pendant array. Inca administrators and accountants employed this complex system for numerical storage and communication. Quipus were mathematically efficient and portable. Unfortunately, the Spanish destroyed many quipus, potentially hiding clues to understanding Incan architectural processes, irrigation, and road systems.

Nazca Lines

The Nazca lines are a set of figures that appear engraved in the surface of the Nazca desert in southern Peru. The lines include hundreds of geometric shapes and renderings of animals and plants, including birds, a spider, a monkey, flowers, geometric figures, and lines—some of them miles long. The Nazca lines, best appreciated from an airplane, are one of the world's enduring mysteries. It is hard to explain how the ancient people of Nazca (900 B.C.E.–600 C.E.) achieved such geometrical precision in an area over 300 square miles. German-born mathematician and archaeologist Maria Reiche spent five decades studying and preserving these lines. She, like many other scientists, believed that the Nazca lines represented an astronomical calendar and observatory, while other theories suggest that they map areas of fertile land.

Mathematics in the Colonial Era

The accidental arrival of navigator Christopher Columbus in the Americas in 1492 marked the beginning of a 300-year period of Spanish and Portuguese colonial rule in South America that ended in the early nineteenth century. Under the Treaty of Tordesillas (1494), Portugal claimed what is now Brazil, and Spanish claims were established throughout the rest of the continent with the exception of Guyana, Suriname, and French Guiana. Roman Catholicism and an Iberian culture were imposed throughout the region, and mathematical systems and practices of ancient cultures were replaced by the Hindu-Arabic decimal system used by the Spanish.

Mathematical activity in Spain between the sixteenth and nineteenth century decisively influenced mathematical thinking and practices in South America. In sixteenth-century Spain, two lines of mathematical thought existed: the arithmeticians (calculators, interested in the uses of mathematics) and the algebraists (abstract or pure mathematicians). Because the European countries used the colonies to enhance their trade and economic resources, the emphasis in South America was on applied mathematics.

Later, the Spanish and the Portuguese established schools—mostly run by Catholic religious orders—which concentrated mathematics teaching on economic applications related to trade. There was also an interest on mathematics related to astronomical observations. The first nonreligious book published in the Americas was an arithmetic book related to gold and silver mining printed in 1556.

Astronomy was a major area of interest in South America in the seventeenth century. In Brazil, research on comets was of major importance, as exemplified by the work of Valentin Stancel (1621–1705), a Jesuit mathematician from Prague who lived in Brazil from 1663 until his death (his astronomical measurements are mentioned in Newton's *Principia*). As in many cultures, most astronomical interpretations attempted to explain divine messages to humankind. Other developments in Brazil included the first aircraft known to fly: the Passarola, invented by Bartolomeu de Gusmão, a Brazilian priest and scientist from Sao Paulo. De Gusmão, also known as the "Flying Priest," studied mathematics and physics at the Universidade de Coimbra in Portugal. The Passarola was an aerostat heated with hot air and flew in Lisbon, Portugal, in 1709.

Mathematics in the Era of Independence

In the first quarter of the nineteenth century, many successful revolutions resulted in the creation of independent countries in South America. Mathematical activity increased throughout Latin America in the twentieth century. For instance, Argentinian mathematician Alberto P. Calderon (1920–1998) developed new theories and techniques in classical and functional analysis. Professor Calderon worked at the University of Chicago for many years. He was awarded the National Medal of Science in the United States.

Research by Professor Ubiritan D'Ambrosio and his students in the slums and indigenous communities in Brazil focused on ethnomathematics—a sub-field of mathematics history and mathematics education. The goal of ethnomathematics is to understand connections between culture and the development of mathematical processes and ideas. Other researchers have explored specific mathematical habits and methods in South American cultures. In the 1980s, Terezinha Nunes and her collaborators studied differences between street mathematics and school mathematics in Brazil by comparing how street vendors (including children) and farmers solve problems compared to those who encounter similar problems in formal school situations.

For example, in their study of young street vendors in Recife, the interviewers acted as customers and asked questions that required the use of arithmetic skills (such as making change). The children did much better in this "real" situation than on a formal test given a week later that used similar numbers and operations. One possible explanation is that the children were better able to keep the meaning of the problem in mind in the "real" situation. Many others, such as Geoffrey Saxe, have found similar results. An implication of these studies is that the essence of school mathematics, which the Recife children were not as successful at, is highly symbolic and possibly devoid of meaning. These studies have been important in advancing the goal of mathematics education that students must initially construct appropriate meanings for the various concepts and methods they encounter.

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See Also: Astronomy; Calendars; Incan and Mayan Mathematics; Knots.

Space Travel

See *Interplanetary Travel*

Spaceships

Category: Travel and Transportation.
Fields of Study: Algebra; Geometry; Measurement.
Summary: Every task involving spaceships, from their design to their launch to effective collision avoidance and communication, is mathematically intensive.

Spaceships, also called "spacecraft," are manned or automatic vehicles for flying beyond planet atmospheres. Different types of spaceships serve different purposes, including scientific or applied observations and data collection, exploration of celestial bodies, communication, and recreation.

According to the routes they take, spaceships can be classified as suborbital, orbital, interplanetary, and interstellar. According to the type of propulsion used, spacecraft engines can be designated as reaction engines, including rockets; electromagnetic, such as ion thrusters; and engines using fields, such as solar sails or gravitational slingshots. Mathematics is fundamental for spaceship design, operation, and evaluation. For example, mathematics is used to plan efficient trajectories, avoid collisions, communicate with satellites, transmit data over vast interplanetary distances, and solve complex problems like those that occurred in the famous Apollo 13 mission.

Mathematics in Spaceship Systems

Propulsion of a spaceship poses scientific and engineering problems that involve balancing forces and computing sufficient fuel, energy, work, and fluid mechanics. For any type of engine, the impulse it gives to the craft has to be calculated and compared to the craft's tasks such as leaving the gravity well of a planet or maintaining an orbit. For example, calculations for rocket engines involve variables including the changing mass of the craft as its fuel is spent, the efficiency of the engine, and the velocity of the rocket's exhaust. Solar sail theories involve such variables as radiation pressure of the light, the area of the sail, and the weight of the craft.

Mechanics and material sciences problems involved in the structure of spacecraft include withstanding the forces, temperatures, and electromagnetic fields involved in moving through space. For example, moving through a planetary atmosphere at speeds necessary to leave the planet's gravity well involves high temperatures from friction.

The guidance and navigation systems of a spaceship collect data and then compute position, speed, and the necessary velocity and acceleration to reach the destination. These systems also determine the relative position of the spaceship to nearby celestial bodies, which influence the craft's motion by their gravitational and electromagnetic fields. For example, mathematical description of a craft orbiting a planet includes the six Keplerian elements (for example, inclination and eccentricity) defining the shape, the size, and the orientation of the orbit, named for Johannes Kepler.

Most twenty-first-century spacecraft do not carry living organisms, but when they do, life support systems are necessary. Life support systems protect people, animals, or plants in the spaceship from harmful environments and provide air, water, and food. The design of life support systems involves biology, physiology, medical sciences, plant sciences, ecology, and bioengineering. Mathematical models for life support typically include calculations of safety margins, such as maximum allowable radiation doses. All organisms need some inputs (such as food, water, or oxygen) and produce some outputs depending on a variety of variables, such as activity levels. Spaceship ecosystem designers strive to produce waste-free, closed systems where water is reclaimed and plants are used to purify the air. Because of the complexity of the closed ecosystem problem, most current flights employ simpler, machine-driven life support systems.

Escaping a Planet's Gravity

The problem of escaping the gravitational field of a large celestial body, such as Earth, is different from the problem of flight in space far from large bodies. For example, a certain velocity, called *escape velocity*, is required to leave any given planet. At the sea level of Earth, the escape velocity is about 11 kilometers per second (km/s) or 7 miles per second (mi/s). However, spaceships usually fly slower at first. The escape velocity is inversely proportional to the square root of the distance from the planet's center of gravity. Spaceships leaving the Earth reach these lower escape velocity levels at some distance from the surface. For comparison, the escape velocity from the Sun is about 600 km/s (373 mi/s) and the speed record as of 2010 for a spacecraft leaving the Earth is about 16 km/s (10 mi/s). This means that flights near the Sun are not technologically possible in the early twenty-first century. The escape velocity of a black hole is greater than the speed of light (over 300,000 km/s or 186,000 mi/s), which is the highest theoretical speed possible.

Atmospheric Flight

Flight within an atmosphere presents very different problems compared to flight in a vacuum. The problems solved by applied mathematicians who study atmospheric flight include friction, turbulence, wing lift, aerodynamic shapes, and control of temperature. Spaceships launching or landing on planets have to be equipped for atmospheric flight. Because of differences in the vacuum and atmosphere flight requirements, many spaceships are designed to change their configuration when they cross atmospheric boundaries. For example, mathematical theories originally developed for origami are used to fold and unfold solar batteries, which can be used only in a vacuum because of their large area.

Science Fiction and Computer Game Mathematics

Space travel frequently appears in science fiction, where plots deal with various existing engineering or physics limitations. Hard science fiction is the more scientifically oriented subgenre, and it frequently includes extensions, discussions, and speculations dealing with the current scientific research. This tradition of blending science and literature started in the late nineteenth century with the works of Jules Verne; many of his then-fantastic devices and ideas (for example, televisions and submarines) were implemented relatively soon after.

As an example of experiments with scientific limits in literature, science-fiction spaceships may travel at superluminal (faster than light) speeds, often through non-physical spaces such as "hyperspace," "subspace," or "another dimension." These are terms from existing mathematical theories, which hard science fiction sometimes discusses.

Sci-fi spaceships may also be living organisms, completely or partially. This idea is a reflection of the current interest in bioengineering and has connections with exciting research in ecology, genetics, cybernetics, and artificial intelligence, as well as social sciences such as philosophy and bioethics.

Computer games and movies about space flight created a demand for applied mathematicians who can model fantastic situations with passable realism. The physics and mathematics of three-dimensional modeling is a fast-growing area, with new courses and programs opening in universities and an expanding job market. What started in the nineteenth century as an exotic occupation for very few writers has become a profession for many programmers and applied mathematicians.

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See Also: Airplanes/Flight; Elevation; Energy; Fuel Consumption; Interplanetary Travel; Origami; Radiation; Satellites; Solar Panels; Vectors; Weightless Flight.

Spam Filters

Category: Communication and Computers.
Fields of Study: Number and Operations; Data Analysis and Probability; Problem Solving.
Summary: Spam filters use probability and Bayesian filtering to sort spam from legitimate e-mails.

Most people with an e-mail address receive unsolicited commercial e-mail, also known as spam, on a regular basis. Spam is an electronic version of junk mail, and has been around since the introduction of the Internet. The senders of spam (called spammers) are usually attempting to sell products or services. Sometimes, their intent is more sinister—they may be trying to defraud their message recipients. Since the cost of sending spam is negligible to spammers, it has been bombarding e-mail servers at a tremendous rate. Some estimate that as much as 40% to 50% of all e-mails are spam. The cost to the message recipients and businesses can be considerable in terms of decreased productivity and unwelcome exposure to inappropriate content and scams. As frustrating and potentially damaging as spam e-mail is, fortunately, much of it does not reach recipients thanks to spam filters. Spam filters are computer programs that screen e-mail messages as they are received. Any e-mail suspected to be spam will be redirected to a junk mail folder so that it does not clutter up a user's inbox. How does the filter decide which messages are suspect? Spam filters are implementations of statistical models that predict the probability that a message is spam given its characteristics. The filter classifies messages with large predicted probabilities of being spam, as spam.

Filters

Primitive filters simply classified a message as spam if it contained a word or phrase that frequently appeared in spam messages. However, spammers only need to adjust their messages slightly to outsmart the filter, and all legitimate messages containing these words would automatically be classified as spam. Modern spam filters are designed using a branch of statistics known as "classification." Bayesian filtering is a particularly effective probability modeling approach in the war on spam. Bayesian methods are named for eighteenth-century mathematician and minister Thomas Bayes. He formulated Bayes' theorem, which relates the conditional probability of two events, A and B, such that one can find both the probability of A given that one already knows B (for example, the probability that a specific word occurs in the text of an e-mail given that the e-mail is known to be spam); the reverse, the probability of B given that one knows A (for example, the probability that an e-mail is spam given that a specific word is known to appear in the text of the e-mail).

The underlying logic for this type of filter is that if a combination of message features occur more or less often in spam than in legitimate messages, then it would be reasonable to suspect a message with these features as being or not being spam. An extensive collection of e-mail messages is used to build a prediction model via data analysis. The data consist of a comprehensive collection of message characteristics, some of which may include the number of capital letters in the subject line, the number of special characters (for example, "$", "*", "!") in the message, the number of occurrences of the word "free," the length of the message, the presence of html in the body of the message, and the specific words in the subject line and body of the message. Each of these messages will also have the true spam classification recorded. These e-mail messages are split into a large training set and a test set. The filter will first be developed using the training set, and then its performance will be assessed using the test set. A list of characteristics is refined based on the messages in the training set so that each of the characteristics provides information about the chance the message is spam.

However, no spam filter is perfect. Even the best filter will likely misclassify spam from time to time. False positives are legitimate e-mails that are mistakenly classified as spam, and false negatives are spam that appear to be legitimate e-mails so they slip through the filter unnoticed. An effective spam filter will correctly classify spam and legitimate e-mail messages most of the time. In other words, the misclassification rates will be small. The spam filter developer will set tolerance levels on these rates based on the relative seriousness of missing legitimate messages and allowing spam in user inboxes.

Spam filters need to be customized for different organizations because some spam features may vary from organization to organization. For instance, the word "mortgage" in an e-mail subject line would be quite typical for e-mails circulating within a banking institution, but may be somewhat unusual for other businesses or personal e-mails. Filters should also be updated frequently. Spammers are becoming more sophisticated and are figuring out creative ways to design messages that will filter though unnoticed. Spam filters must constantly adapt to meet this challenge.

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See Also: Internet; Predicting Preferences; Search Engines; Social Networks; Software, Mathematics.

Sports Arenas

See *Arenas, Sports*

Sport Handicapping

Category: Games, Sport, and Recreation.
Fields of Study: Algebra; Data Analysis and Probability; Number and Operations.
Summary: Various calculations are used to set fair, competitive handicaps in sports.

Sport handicapping is an important methodology that affects millions of people worldwide and potentially impacts billions of dollars worth of bets. In many sports, handicaps are calculated for individuals or teams and are used as a way of "equalizing" performance by giving a scoring advantage or other in-game compensation to some players. This process allows lower skilled players to compete with higher skilled players while preserving perceived fairness. The term "handicap" refers to both the adjusted scores and the process of determining them, and may also be used for whole tournaments that rely extensively on the method. Handicap in this context derives from a seventeenth century lottery game called "hand-in-cap," where players put their bets in a literal cap. A point spread, frequently used in sports betting, is a related idea for computing or estimating relative advantage to equalize teams in competitive sports. Examples of sports using handicapping at various levels include bowling, golf, horse racing, and track and field.

Handicapping

In sport, a handicap is usually imposed to enable a more equal competition to take place. The handicap is calculated according to specific criteria set down for each of the sports that use the technique, meaning that some are much more complex than others. To understand why a handicap may be used, consider one of the most well-known sports that employ a handicap system, golf.

If a recreational golf player were to compete against the best golfer in the world in a round of golf, then the outcome would almost certainly be a win for the better golfer. A win by a large margin would also have been very likely. If a handicap were applied that was based on each player's average scores, then the outcome would be much less certain. There would have been a distinct possibility that, if the recreational player had played well, they would have had the opportunity to beat the better golfer—or at least not loose by many shots—after the handicap was applied.

In most sports when professionals compete against each another, the events are usually free from handicapping. A professional golf tournament will usually engage those who play with a scratch (or zero) handicap.

One of the primary reasons for using a handicap is to make an event more competitive. In many respects, this makes the given sport more enjoyable and can help to make it more appealing and increase the number of those wishing to participate.

Tenpin bowling is a sport that has more participants worldwide than most other sports. The overwhelming majority of players are recreational, although many take part in annual league competitions. Most leagues are not scratch based (on actual total pin fall) but are handicapped. In tenpin bowling handicap leagues, the scores that are used to determine who has won are a combination of the total pins actually knocked down and the handicap value. This method allows players (and teams) with lower averages to compete against players (and teams) that have much higher averages.

The handicap in Tenpin Bowling is usually of the form: Handicap value (per game) = 80% of the difference between the player's average and 200 pins.

If a bowler averages 100 pins, then the bowler would, using the handicapping system, gain a handicap value of 80 pins: $(200 - 100) \times 0.80$. The total pinfall for a game would be 80 plus whatever number of pins the bowler actually knocked down.

This handicap system is versatile in that the two values used (the 80% and the 200 pins, in the example above) can be manipulated to suit the particular league. For instance, if there are a number of players who average over 200, for example 210 or 220, then the handicap may be 80% of the difference between each bowler's average and 220 pins. Alternatively, if the players are grouped quite closely together, then the handicap may be 66% of the difference between each bowler's average and 200 pins.

Athletics

Athletics, or track and field, is another mass participation sport, but one in which, at the highest level, age is intrinsically linked to performance—few athletes compete internationally in their late 30s and beyond. There is still huge participation in the sport by people older than 30, and there are obvious health benefits to doing so.

There is a scoring system that takes age into account by comparing race time to that of the world record holder in each age group. It is often known as a World Association of Veteran Athletes (WAVA) Rating and is expressed as a percentage between zero and 100. If one gets a WAVA rating of 50%, it means that the competitor is half the pace of the world record holder. WAVA rating is a useful way to make comparisons between runners of all ages and can form the basis of a handicap league.

Horse Racing

A further important application of handicapping is that seen in horse racing, a sport on which billions of dollars worth of bets are made each year. In a handicapped race, the horse must carry a certain additional weight, which when added to the weight of the jokey gives it an assigned impost (or total weight). These weights are held in saddle pads with pockets.

The calculation for the weight a horse is required to carry is based on a number of factors. A great deal of work is done with past data to create and then ensure that the handicaps are as fair as possible. These handicaps allow for horses of differing abilities to race against each other over a given distance.

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See Also: Algebra in Society; Betting and Fairness; Data Analysis and Probability in Society.

Squares and Square Roots

Category: History and Development of Curricular Concepts.

Fields of Study: Algebra; Communication; Connections; Geometry.

Summary: Squares and square roots have long challenged mathematicians and have led to various expansions of the number system and developments in number theory.

The square of a number x, denoted x^2, is the number $x \times x$. The inverse operation is called the *square root*: the number x is a square root of y if $y = x^2$, the notation used being $x = \sqrt{y}$. Historically, these operations have been a major source of new problems, ideas, and systems of numbers in the early and modern development of mathematics. Square roots have also appeared in many applications, such as computing the standard deviation of a data set, and have often presented a challenge to scientists and mathematicians in the days before readily available calculating technology. Middle-grade students in the twenty-first century continue to use squares and square roots to simplify computations and solve problems, as do carpenters and engineers.

Definition

Geometrically, the "square" of a number x measures the area of a square whose side has length x. This idea explains the name and is likely the way that ancient civilizations were first confronted with the operation. The Pythagorean theorem is an equality between sums of areas of squares constructed on the sides of a right triangles, namely $a^2 + b^2 = c^2$ if a and b are the two legs and c is the hypotenuse. Applied to the triangle obtained by halving a square of side length one along one of its diagonals, it shows that such a diagonal has length equal to $\sqrt{2}$.

A member of the Pythagorean School sometimes identified as Hippasus of Metapontum (c. fifth century B.C.E.) discovered that this number cannot be expressed as the ratio of two integers—it is irrational. The discovery was a sensation amid the Pythagorean School where it was preached that all numbers were rational and called for an extension of the number system.

In the centuries that followed, extensions of the number system would include all numbers expressible with an infinite number of decimal digits, so that each positive number has a square root (for example, $\sqrt{2} = 1.4142136\ldots$) and negative numbers, which can be multiplied according to the usual associative rules, and the following additional ones governing signs: $-1 \times x = -x$; $(-1) \times (-1) = 1$, which implies that $(-1)^2 = 1$ so that -1 should also be counted as a square root of 1.

More generally, both extensions can be combined to yield the system of real numbers, which are the numbers with sign and infinite decimal expansions. In this system, each square of a number is a positive number (or zero), and each positive number has exactly two square roots, which differ by a sign. For example, 2 has as square roots the numbers $1.4142136\ldots$ and $-1.4142136\ldots$, a fact denoted by the expression $\sqrt{2} = \pm 1.4142136\ldots$.

Computation

Square roots can be computed by hand, by calculator, or by computer (up to the desired numerical approximation) by several methods, including those using sequences, exponentials, logarithms, or continued fractions. Mathematicians in ancient Egypt and Babylonia are some of the first who are thought to have extracted square roots. Early Chinese, Indian, and Greek mathematicians also contributed to this area. According to some historians, the first method to be introduced in Europe was that of Aryabhata the Elder, a Hindu mathematician and astronomer. One of the oldest ones, still at the basis of many currently used algorithms, is the so called Babylonian method (which is also an instance of the modern Newton–Raphson method for solving general equations in one variable). Given a positive number S and choosing an initial "guess" x_0, the method produces a sequence of numbers x_n converging to the square root of S by the rule

$$x_{n+1} = \frac{1}{2}\left(x_n - \frac{S}{x_n} \right).$$

For example, the first approximations to $\sqrt{2}$ starting from $x_0 = 1$ are $x_1 = 15$, $x_2 = 1.416\ldots$, $x_3 = 1.414215\ldots$, $x_4 = 1.4142135623746\ldots$, the last one already having 11 correct decimal digits.

Solving the Quadratic Equation

Square roots are used to solve the general quadratic equation $ax^2 + bx + c = 0$, where a, b, and c are parameters, and a is not zero. The formula, at least partially known to the ancient Greek, Babylonian, Chinese, and Indian mathematicians, is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

provided that the so-called discriminant of the equation, the number $b^2 - 4ac$, is not negative.

Imaginary Numbers

The Italian mathematician Rafael Bombelli, in his book *L'Algebra* written in 1569, proposed the introduction of a new number i, which should denote the square root of -1. Multiplying the number i by real numbers would yield square roots of negative real numbers. The new numbers so obtained are called "imaginary numbers," a name introduced by René Descartes (who meant it to bear a derogatory connotation). A new number system is obtained with the numbers formed by adding a real and an imaginary number; such numbers are called "complex numbers." Complex numbers can be added, multiplied, and divided, and the preceding quadratic formula shows that any quadratic equation has two solutions that are complex numbers; this remains true even if the parameters a, b, c are allowed to be complex number themselves. Actually, a stronger result holds true: Carl Friedrich Gauss (1777–1855) discovered that any equation of the form $a_0 x^d + a_1 x^{d-1} + \cdots + a_{d-1} x + a_d$ has d solutions in complex numbers, an important theorem known as the fundamental theorem of algebra. Partly thanks to this property, complex numbers are of fundamental importance in modern mathematics and in many fields of science and engineering, such as telecommunications.

Implications in Number Theory

Questions regarding squares, square roots, and quadratic forms have played a particularly important role in number theory, often giving rise to the simplest instances of rich theories. Numbers that are squares of integers are called "perfect squares," the first examples being 1, 4, 9, 16, 25, ... Galileo Galilei examined perfect squares in the attempt to understand infinity. Leonardo Fibonacci wrote a number theory book called *Liber Qudratorum*, the book of squares.

The problem of representing integers as sums of perfect squares has also received much attention. Pierre de Fermat (c. 1607–1665) proved that the odd prime numbers that are sums of two perfect squares are exactly those that have remainder 1 when divided by 4, an example being $13 = 2^2 + 3^2$ (whereas, for example, the prime number 7 has no such representation). Joseph Louis Lagrange (1736–1813) proved that every positive integer can be written as the sum of at most four perfect squares (for example, $15 = 9 + 4 + 1 + 1$); three squares suffice only for those numbers which are not of the form $4^k(8m + 7)$, as was later proved by Adrien-Marie Legendre.

In his 1801 masterpiece *Disquisitiones Arithmeticae*, written at the age of 21, Gauss investigated two problems whose generalizations are still major topics of current research. The first one is related to the question of representing integers as the sum of squares and asks for a classification of binary quadratic forms, which are functions of two variables x and y of the shape $f(x, y) = ax^2 + 2bxy + cy^2$, where a, b, and c are integer parameters, in terms of the set of integers they represent—the set of possible values of $f(x, y)$ as x and y range among the integers. The second problem considered by Gauss is the following: given two odd prime numbers p and q, is it possible to write p as the difference of a perfect square and a multiple of q (in symbols $p = n^2 - mq$)? Conversely, is it possible to write q as the difference of a perfect square and a multiple of p? Gauss proved that if at least one of p, q leaves remainder 1 when divided by four, then the two questions have the same answer; and that if p and q both leave remainder 3 when divided by 4, then the answer to the second question is "no" whenever the answer to the first question is "yes" and vice versa.

As a consequence of this result (known as the "quadratic reciprocity law") he was able to give an efficient method for answering the question. In fact, Gauss found not one but eight different proofs of this fact, which is so central in modern number theory that about 200 more proofs were later found.

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See Also: Babylonian Mathematics; Carpentry; Chinese Mathematics; Numbers, Complex; Numbers, Rational and Irrational; Pythagorean School.

Stalactites and Stalagmites

Category: Weather, Nature, and Environment.
Fields of Study: Algebra; Geometry.

Summary: The growth, age, and shape of stalactites and stalagmites can be mathematically calculated, depending on a variety of variables.

Stalactites and stalagmites are secondary minerals, also called "speleothems," formed as calcium carbonate, calcium oxide, and other minerals first dissolved in water and are then precipitated as water drips. Stalactites hang from cavern ceilings and concrete structures, and stalagmites rise from floors, sometimes meeting to create columns. Mathematicians, statisticians, geologists, and other scientists involved in studying stalactites and stalagmites develop complex, interdisciplinary theories and models as well extensions and applications. This work draws from many areas of mathematics, chemistry, and physics, especially fluid dynamics.

Growth and Dating

Stalactites and stalagmites form from chemical reactions involving ground water and minerals in the earth and the open areas of caves. The reactions typically consist of dissolving, precipitating, and—sometimes—evaporation. Chemical reactions of minerals first dissolving in water and then precipitating out of water are directly opposite to one another. The mathematical analogy of this relationship is an inverse function, and in either case, these processes may be quantified mathematically using standard chemical notation and formulas. Some stalactites and stalagmites are slow-forming, such as those made of calcium carbonate. Concrete or gypsum stalactites, which are made from more water-soluble materials, form much faster. For example, calcium hydroxide, which originates concrete stalactites, is about 100 times more soluble than calcium carbonate. Gypsum stalactites are formed by simple evaporation.

Dating of stalactites and stalagmites is complex because fluctuations in temperature or humidity can affect the pace of growth in such ways that length is not directly proportional to age. In some caves, because of minerals dissolving in water seasonally, stalactites and stalagmites may have annual bands, much as trees have rings, visible by the naked eye or under ultraviolet light. Dating with such direct methods, when available, can then be used to mathematically estimate and reconstruct temperature and humidity variation patterns in ancient times. However, the process is currently not reliable for anything less than very drastic climate changes.

Another method of dating involves collecting data on stalactite and stalagmite growth over several years. Then, data are used to determine the relationship between the size and the age, with approximations such as the method of least squares.

Dating with radioactive isotopes measures the ratio between a radioactive element, usually uranium, and the product of its radioactive decay. Electron spin resonance (ESR) dating is based on measuring radiation damage on calcium that forms stalactites and stalagmites.

These three methods of dating consistently produce average growth rates of about 0.1 millimeters (0.004 inches) per year in lime cave stalactites, with several times slower rates for stalagmites. Gypsum and concrete stalactites, formed by different reactions, grow several hundred times faster.

Unique, Optimal Shape

Plato supported the notion that there are true or ideal forms in nature, many of which may be expressed geometrically. While stalactites vary widely in size, they all tend to have a distinct, uniform shape that varies only by scale or magnification. Physicist Raymond Goldstein, part of an interdisciplinary team that investigated the mathematics of stalactite shape, said, "Although any particular stalactite may have some bumps and ridges that deform it, one might say that within all stalactites is an idealized form trying to get out." Using equations from fluid dynamics and other information about stalactite growth, the team developed a simulation and grew virtual stalactites under a variety of conditions, which they compared to real stalactites. The broad range of initial conditions for the mathematical model as well as for situations in real caves produced the same shape, though in caves, shapes can be distorted by impurities or breaks. The findings relate to other natural growth situations, including thermal vents and mollusk shells. To measure stalactites' shapes exactly without destroying them, the researchers use high-resolution digital cameras and scaled photography. This work also facilitates the mathematical study of stalactites' rippled patterns.

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See Also: Carbon Dating; Caves and Caverns; Probability; Transformations.

State Legislation

See *Government and State Legislation*

Statistics Education

Category: History and Development of Curricular Concepts.
Fields of Study: Communication; Connections; Data Analysis and Probability.
Summary: Statistics education has grown and adapted since the nineteenth century.

At the start of the twentieth century, science fiction author H. G. Wells asserted, "Statistical thinking will one day be as necessary for efficient citizenship as the ability to read and write." While use of statistical methods dates to earlier times, the first college statistics departments were founded in the early twentieth century, and many textbooks were written on statistical subjects like the design of experiments. A century after Wells's prediction, the notion of statistical thinking permeates all levels of education from kindergarten through college. In the early twenty-first century, there are increasing calls for statistical literacy in the United States and abroad in order to help people manage an increasingly complex and data-driven world.

Etymology

The word "statistics" derives from the term "state arithmetic," which refers to the various counting and calculating operations necessary for governments to operate effectively. The ancient Babylonians, Egyptians, Greeks, Romans, Chinese, and others appear to have used various kinds of mathematics for activities like partitioning land and determining army sizes. The eleventh-century *Domesday Book*, a survey of England ordered by William the Conquerer, is another example of such state arithmetic. Statistician Maurice Kendall cites the first possible occurrence of the term "statistics" in the sixteenth-century work of Italian historian Girolamo Ghilini, who wrote about "*civile*, *politica*, [and] *statistica e militare scienza*." However, he also traces the conceptual beginnings of the field to the "political arithmetic" of the seventeenth century and the work of researchers like pioneer demographers John Graunt and William Petty, who examined population growth and commerce in London versus Rome and Paris; and mathematician Edmond Halley, who some consider to be the founder of actuarial science for his work on life expectancy tables and insurance calculations. German historian and economist Gottfried Achenwall is frequently credited with inventing the German form of the word "statistics" in the eighteenth century and the related term *Staatswissenschaft* for political science. Their shared root *staats* means "state." Scottish politician John Sinclair appears to have been the first to use the term "statistics" in English in his *Statistical Accounts of Scotland*, a late eighteenth-century work addressing people, geography, and economics. He said: "I thought a new word might attract more public attention, I resolved to use it."

Historical Applications

In the nineteenth century, the ideas of statistical counting and calculating began to spread into a wider variety of political, social, scientific, and financial applications. For example, British physician William Farr received statistical training in France and applied statistics to medicine and models of epidemic diseases, calling his methods "hygiology" after the word "hygiene." He is credited as the founder of the field of epidemiology. Another pioneering epidemiologist was physician John Snow, who famously used statistical methods to trace the source of an 1854 cholera outbreak in London. His conclusions were politically controversial. In approximately the same period in the United States, self-taught statistician Lemuel Shattuck was appointed to plan a census of Boston in 1845 and later helped plan national census activities. He ultimately helped implement many local and state public health measures. Governments, businesses, and academic institutions increasingly used data and statistical methods to inform decisions. During this period, countless mathematicians, statisticians, economists, scientists, and others contributed to the development of statistical methods and the mathematical foundations of statistics, as well as the related field of probability. Many of them addressed both the theory and application of statistics.

Historical Education

Universities had existed in Europe since the Middle Ages. In other parts of the world, there were centers of learning at which scholars gathered to exchange ideas and teach. However, education in many academic subjects was often accomplished through mentorships or private tutoring. For example, nineteenth-century statistician and nurse Florence Nightingale was tutored in arithmetic, algebra, and geometry. She, in turn, tutored others before becoming involved in nursing. One of her tutors was the well-known mathematician of the period, James Sylvester. She was also influenced by the work of Farr and corresponded with mathematician Adolphe Quetelet, who was a pioneer in the use of statistics for anthropometry and criminology. She called him "the founder of the most important science in the world."

Other statisticians formed relationships with universities for research. For example, Karl Pearson, Francis Galton, and Walter Weldon worked at University College London. Pearson gave statistics lectures starting in 1894, and the trio founded the journal *Biometrika* in 1901 "as a means not only of collecting or publishing under one title biological data of a kind not systematically collected or published elsewhere in any other periodical, but also of spreading a knowledge of such statistical theory as may be requisite for their scientific treatment." Upon his death in 1911, Galton bequeathed the university a large endowment. Pearson became the first Galton Professor of Eugenics, sometimes called Galton Professor of Applied Statistics, perhaps because of the controversial nature of eugenics. That same year, Pearson was instrumental in creating the university's Applied Statistics department, now the Department of Statistical Science, which was recognized as the world's first college statistics department. It merged biometrics and eugenics (genetics) laboratories that had been founded by Pearson and Galton—though the Galton Laboratory later moved to the Department of Biology. Some other statisticians who worked or studied at University College London in the early nineteenth century include William "Student" Gossett, who is credited with the development of the Student's *t* distribution; Karl Pearson's son, Egon Pearson, who became the head of the Applied Statistics department when it split with the Department of Eugenics; Ronald Fisher, who was the first head of the Department of Eugenics and is referred to by some as the "father of modern statistics;" and Jerzy Neyman, who co-developed what is often called "Fisher–Neyman–Pearson inferential methods" or "classical" methods of statistical inference. These techniques typically use what is known as the "frequentist approach" to statistical analysis, which is based on defining probabilities of events as the limits of their relative frequencies over a large number of trials or experiments. It is perceived by many as being wholly objective and therefore "scientific." This approach is in contrast to Bayesian methods, named for mathematician Thomas Bayes. Bayesian statistical methods allow for subjective or belief-driven probabilities that may or may not be derived from observation or experimentation. The Applied Statistics department at University College London temporarily relocated during World War II; the war was to have a broad impact on mathematics and statistics in Europe and the United States.

Education in the United States

The Unites States was also developing its own college-level education programs at the beginning of the twentieth century. Similar to the department at University College London, many programs and other efforts started with individuals offering courses and partnerships between researchers and universities. One often-cited example is Iowa State University. George Snedecor, a professor in the Department of Mathematics, taught courses that included statistics content starting in 1914. He often focused on agriculture problems, a significant research area at the university. In 1924, he co-wrote a worldwide publication about computational statistical methods with Henry Wallace, who would later become Secretary of Agriculture and vice president of the United States. Iowa State created a statistical consulting and computing service in 1927, which was available to researchers in many disciplines. This service led to Iowa State's creation of the first recognized statistical laboratory in the United States, in 1933, and its Department of Statistics, in 1947. However, statistics degrees were offered before that time, beginning with Gertrude Cox's master's degree in 1931. Cox went on to help found the Department of Statistics at North Carolina State University, one of the oldest statistics departments in the United States. She was the first female full professor and first female department head at the school and went on to start other college programs as well. An anecdote about her hiring at North Carolina State reports that, when Snedecor was asked to recommended five men for the job, he added to his letter: ". . . if you would consider a woman for this position I would recommend Gertrude Cox."

European statisticians also proved influential on U.S. statistics education and, in some cases, on government policy. Fisher visited Iowa State in the 1930s, and his agricultural work at the Rothamsted Experimental Station made a great impact on Snedecor. William Cochran, who was born in Scotland, also worked at the Rothamsted Experimental Station and taught at Iowa State. He went on to help create many statistics departments, including the one at Harvard, and he served on the committee that produced the 1960 *Surgeon General's Report on Smoking and Health*. Statistics proliferated, and similar efforts took place elsewhere, such as at the University of California, Berkeley. Neyman, who was born in Poland and also studied in England, France, and Russia, started working at Berkeley in 1938. Like many mathematicians and statisticians of the time, he was fleeing the growing Nazi influence in Europe. Prior to World War II, colleges sometimes offered a few undergraduate and graduate statistics courses but entire departments were still fairly rare. Thanks largely to Neyman's efforts, Berkeley had a department by 1955. He would also contribute significantly to experimental design, including some methods used by the United States Food and Drug Administration to test new medicines. Berkeley would become a center for mathematical statistics and was chaired for a time by statistician and mathematician David Blackwell, the first tenured African-American professor at Berkeley.

Post–World War II Statistics Education

Statistics and statistics education exploded after World War II, influenced by developments that occurred during the war and the subsequent Cold War. Statisticians had contributed significantly to the war effort in both the United States and Europe. For example, Hungarian mathematician and statistician Abraham Wald, who had suffered persecution for being Jewish, helped solve the problem of where to armor British bombers against antiaircraft fire. Others, like French-German Wolfgang Doeblin, would die as a result of the war. Later studies of Doeblin's works showed that he was an early pioneer of Markov chains, named for Andrei Markov. John Tukey was one of the most influential statisticians working in the mid- and later twentieth century. According to statistician Frederick Mosteller, the first chair of Harvard's statistics department and an influential force in statistics education: "He probably made more original contributions to statistics than anyone else since World War II." Tukey worked at the government's Fire Control Research Office during World War II, among his many roles. At the same time, he was often praised for his teaching. Mathematician Robert Gunning called him a "very lively presence on campus" and "a good and energetic teacher," who also helped schedule class and exam times in his head. As a member of Princeton's mathematics department, Tukey helped found the school's Department of Statistics in 1966, following earlier work by statistician Samuel Wilks, who had worked for the Office of Naval Research and profoundly influenced the application of statistics to military planning. The American Statistical Association's Samuel S. Wilks Award was named in his honor. Later, the department became the Committee for Statistical Studies, which encourages cross-disciplinary study of statistics and coordinates courses in many departments and programs.

The post-war extension of statistics into areas like clinical trials (pioneered by statistician Austin Bradford Hill), business, manufacturing (influenced by statisticians like W. Edwards Deming), and financial economics (for which economists Harry Markowitz, Merton Miller, and William Sharpe won a Nobel Prize), as well as the revival of Bayesian methods, meant that statistics was reaching a broader audience. It also meant that, more often, statistics courses were taught outside traditional mathematics and statistics departments. The debate over who should teach statistics was not new. Given that the discipline had been developed within so many fields—agriculture, psychology, biology, sociology, business, just to name a few—it was only natural that teaching would occur within these fields. Statistician John Wishart, who had worked with Pearson at University College London and with Fisher at Rothamsted, asserted that non-statisticians were not equipped to teach statistics or supervise statistical research. Fisher took a different approach, citing statistics' basis in research and applications and arguing for focused statistics offerings in departments in which statistics were often used, like psychology and biology. Around 1940, Harold Hotelling, who taught at Stanford University, Columbia University, and the University of North Carolina Chapel Hill, presented the idea that being a strong mathematician is not sufficient for teaching statistics, so mathematicians and statisticians were not always superior instructors versus individuals in other disciplines. He asserted that a statistics teacher must meld quantitative skills with "a really intimate acquaintance with the problems of one or more empirical subjects in which statistical methods are taught." Hotelling recognized that in typical academic structures, there might be some reluctance among faculty to teach courses that lay outside their specialty areas and that keeping current with statistics might be a daunting task for nonspecialists. These issues remain matters of debate at the start of the twenty-first century. A study published in 2000, funded by the National Science Foundation, suggested that students were more likely to receive statistics education from instructors outside mathematics or statistics departments.

Employment

Through the 1970s, universities in the United States and elsewhere produced many statisticians or statistically trained practitioners in other disciplines, many of them to meet growing industry demands. However, employers were showing increasing concern that their new employees did not know how to practice statistics on the job, even if they had been instructed in current applied methods and practices in their academic programs. The American Statistical Association (ASA), which was founded in 1839, created a committee in the late 1940s to consider matters related to the training of statisticians. In 1980, the ASA Committee on the Training of Statisticians for Industry presented guidelines for programs that train industrial statisticians. One conclusion that spurred further debate stated: ". . . it is generally agreed that the MS degree is a minimum requirement for the professional statistician . . . it is recommended that someone interested in statistics as a profession obtain solid foundations in science or engineering and mathematics." Some discussion centered on balancing theory, applications, and employer-desired skills such as communication and teamwork. In Great Britain, the 1986 report *Supply of and Demand* *for Statisticians* cited both teaching factors and unrealistic expectations on the part of employers. Overall, in the 1980s, there were many general calls from statisticians to increase both the number and quality of programs, with mixed success. In the 1990s, there were also calls to increase the quality of undergraduate education and provide more interdisciplinary opportunities to graduate-level statisticians to "modernize" statistics for the twenty-first century. This call hearkened back to statistics' inherently interdisciplinary roots in previous centuries.

New Emphasis

The hallmarks of statistics education in the latter twentieth century and into the twenty-first century would be an increased focus on concepts over computation, statistical literacy, statistical thinking, use of real data, use of technology for both data analysis and conceptual understanding, and assessment to gauge student learning and understanding. Reports by several professional mathematical and statistical organizations contributed to this shifting educational emphasis. For example, the 1991 Focus Group on Statistics Education, part of the Curriculum Action Project of the Mathematical Association of America, produced *Heeding the Call for Change*. Later, the ASA Undergraduate Statistics Education Initiative (1999) focused on many aspects of education. One concern they noted was that many students were having a negative first experience in introductory statistics. In 2005, the Guidelines for Assessment and Instruction in Statistics Education (GAISE) committee, sponsored by ASA, produced

Impact of Computers

The advent of computers contributed to changes in statistical practice and new debates related to statistics teaching. Until then, many statistics courses had, of necessity, focused on teaching computational formulas, and statistical practice relied on techniques that were computationally tractable for researchers analyzing data by hand. Larger and larger data sets were becoming more common, requiring computer assistance for analyses. In the late 1960s, social scientist Norman Nie and computer scientists C. Hadlai Hull and Dale Bent developed the Statistical Package for the Social Sciences (SPSS) for mainframe computers. Academic use of the program soared when McGraw-Hill published a user's manual in 1970. By 1984, SPSS was the first statistical package offered for disk operating system (DOS) personal computers. Also in the 1970s, Numerical Algorithms Group (NAG) introduced its Algol 60 and Fortran algorithm libraries for mainframe systems.

The Statistical Analysis System (SAS) Institute emerged in 1976 from roughly a decade of work, starting with the University Statisticians Southern Experiment Stations, a consortium of universities funded by the U.S. Department of Agriculture and National Institutes of Health. SPSS, NAG, and SAS continue in the twenty-first century to offer a breadth of statistical software. Many other software packages and algorithms to graph and analyze data also emerged, some for general purposes and others for specific applications. One example is LISREL (an abbreviation of "linear structural relations"), which is used for structural equation modeling. It was developed by statisticians Karl Jöreskog and Dag Sörbom in the 1970s. Computers also revitalized interest in computationally intensive exact tests, iterative methods such as bootstrapping, and Bayesian analyses. Instructors debated the role of computers in the classroom. Many argued for statistical programming in languages such as Fortran or C, rather than point-and-click packaged routines, believing that statisticians should understand what the computer was doing. On the other hand, some classroom instructors advocated for the pedagogical utility of programs that computed statistics in a quick and easy manner, leaving the students free to focus on interpretation of results and "statistical thinking." The debate is ongoing. In the twenty-first century, many statistical programs contain both programming and menu-driven options, such as S-PLUS and its freeware version R.

K–12 and undergraduate reports focusing on instructional practice and assessment. There have also been recurring meetings, such as the International Conference on Teaching Statistics (ICOTS), which allow statistics instructors to address and debate issues, including the place of "classical" statistical methods versus Bayesian or computationally intensive exact methods in introductory classrooms; how best to meet the needs of non-majors taking statistics courses in mathematics and statistics departments; the "best" structure for introductory statistics textbooks; or the role of online tools and distance education.

The 2000 edition of the *National Council of Teachers of Mathematics Principles and Standards for School Mathematics* outlined standards for mathematics education that included statistics threaded from kindergarten through the last year of high school. Previously, statistics had been offered in various forms in high schools, though it presented some difficulty because many did not think it fit neatly into the traditional algebra, geometry, trigonometry, calculus sequencing used by many schools. The Advanced Placement (AP) Statistics exam was first offered in 1997. More than 7000 students took the exam, the most for a first offering of any AP exam as of 2010, and between 1996 and 2010 the rate of enrollment increased more quickly than any other course offered by AP.

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See Also: Blackwell, David; Data Analysis and Probability in Society; Expected Values; Measures of Center; Normal Distribution; Permutations and Combinations; Probability; Randomness; Scatterplots.

Step and Tap Dancing

Category: Arts, Music, and Entertainment.
Fields of Study: Communication; Geometry; Representations.
Summary: Step and tap dancing each involve rhythms and combinations that can be analyzed mathematically.

Step dance is the type of dance focusing on feet movements. It de-emphasizes the other two spatial dance aspects—hand and body movement—and repositions dancers relative to the ground to form movement patterns. There are forms of step dancing in several cultural traditions, such as Malambo from Argentina, Irish stepdance, African-American stepping, and traditional Cherokee dancing. Related forms include clog and tap dancing.

The movements of these styles of percussive dance may be performed by a single dancer or choreographed among several dancers. Tony Award–winning choreographer and dancer Danny Daniels noted that, while an individual dancer may improvise, groups must be coordinated. The rhythms and counts for the dances he designed or performed on Broadway could be organized and detailed using mathematically based musical notation. Dance theorist Rudolf Laban used ideas from various fields, including crystallography, when he modeled dance dynamics. Scientists and dancers continue to develop notation and models to express human movement in tap and other dances. Dance algorithms may help create natural robotic movement. Dancer Gregory Hines said: "My style is part choreography, part improvisation. That gives me a chance to show people the possibilities of tap dancing, which, at its heart, is mathematics with endless possibilities."

Ratio and Proportion

There are several ratios related to music and choreography that determine movement in step dancing. Music time signature is written as a fraction with the denominator signifying the size of the notes used, and the numerator signifying the total length—in such notes—of a bar, which is the unit of music. For example, traditional music for Irish slip-jig has 9/8 time signature in the note pattern: quarter, eighth, quarter, eighth, dotted quarter (three-eighth). The five notes in the time signature correspond to two-and-a-half dance steps per bar, with long graceful slides between the steps.

The formula for a dance includes the number of bars in each repeating cycle (sometimes performed symmetrically) first for one starting foot and then the other. For example, a song that has 40 bars may be choreographed to include five step cycles, each spanning eight bars. Another ratio important for step dancing is the tempo of music, measured in beats per minute (bpm). Dancing competitions specify the tempo range for each type of dance. For example, single jig must be 112–120 beats per minute. Tap dancers of the past used their signature "time steps" (particular combinations of taps) to communicate the tempo to the accompanying band.

Patterns and Improvisation

In step dances, themes are expressed using sequences of the basic elements or steps. For example, common elements in tap dancing include shuffles, flaps, pullbacks, wings, and stomps. These sequences may be strictly choreographed from beginning to end, sometimes with repeating patterns or permutations of shorter elements, which can be repeated by any dancer who has learned the sequence. Improvisation allows the dancer to take basic elements and rearrange them in ways that may appear to be random to the casual observer.

Some step dance music has built-in departures from the standard bar structures. For example, Irish step-dance "crooked tunes" may have seven-and-one-half bar parts in addition to eight bar parts. Step dance patterns have multiple levels: steps within a bar, combinations of steps spanning multiple bars, and patterns of these step combinations. Order and perceived randomness can be manifested at all levels.

Dance-Dance Revolution

Dance-Dance Revolution (DDR) is a step dancing video game. The goal of the game is to match the pattern of steps on the screen and their rhythm on the special gaming pad with four or eight foot positions. The combination of visual, audio, and kinesthetic representations of the same rhythm have kept versions of the game popular around the world since its release in 1998.

Later versions of DDR use a mathematical visualization of multi-dimensional data, called radar diagrams, to rate the difficulty of individual dances. The variables describe different characteristics of the dance, such as steam (the density of steps) and chaos (the amount of steps that do not occur on beat).

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See Also: Ballroom Dancing; Contra and Square Dancing; Permutations and Combinations; Video Games.

Stethoscopes

Category: Medicine and Health.
Fields of Study: Algebra; Geometry.
Summary: Some modern stethoscope designs digitize sound waves, which can be modeled and analyzed.

The stethoscope is perhaps one of the most iconic pieces of medical equipment and is used by doctors in nearly every area of clinical practice around the world. From its beginnings as a simple tube to amplify sound, in the twenty-first century the stethoscope is evolving into a highly mathematical and computerized tool. It can record, analyze, and display diagnostic information using software and algorithms developed from clinical data using a variety of concepts and techniques from statistics, signal processing, spectral analysis, and related sciences. Further, mathematical models and simulations are increasingly used to support and validate clinical results.

History and Development

French physician René Laennec is credited with the invention of the "stethoscope" in 1816. The name comes from the Greek words meaning "chest" and "to examine." Knowing that solid bodies conduct and amplify sound, Laennec used tightly rolled and glued sheets of paper to hear patients' heartbeats. Experimenting with cylinders of various materials, he observed that an aperture maximized magnification of internal body sounds. His ultimate design was a straight, eight-inch wooden tube with a conical chest piece and a funnel-shaped stopper. Later physicians developed stethoscopes from materials like rosewood, papier-mâché, and even glass. The binaural form was popularized in the United States in the early 1900s by William Osler.

In the twenty-first century, the binaural acoustic stethoscope consists of a chest piece with a plastic disc (called a "diaphragm") on one side and a hollow cup (called a "bell") on the other. The bell transmits low frequency sounds and the diaphragm transmits high frequency sounds. A majority of clinicopathological correlations and diagnostic techniques used today result from patient data acquired by physicians listening with stethoscopes or a bare ear. Refinements in design and the increasingly widespread use of stethoscopes—coupled with training—improved observations. With respect to the heart, these included better precision in timing cardiovascular sounds, focusing on segments of the cardiac cycle in turn, and devising quantitative symbols to describe sounds. On the other hand, stethoscopes have also been investigated as a vector of disease transmission in busy clinical settings like emergency rooms.

Mathematical Modeling

Electronic systems of collecting and analyzing data have begun to supplement or even supplant the use of the stethoscope. Some predict that before 2020, manual stethoscopes will become obsolete. Electronic stethoscopes convert acoustic sound waves into electrical signals, which can be amplified and enhanced, producing both visual and audio output. Software can then represent cardiopulmonary sounds graphically and interpret them using mathematical algorithms. Signals may also be recorded or transmitted, facilitating remote diagnosis and teaching. Some research suggests that mathematical methods improve accuracy in diagnosing conditions, such as heart murmurs, but some methods have not yet shown clinical usefulness. Mathematicians and physicians continue to investigate and model cardiac sounds from murmurs and prosthetic valves, as well as other types of hemodynamic data, using techniques from spectral waveform analysis and physics concepts like damped oscillations of viscoelastic systems. They have also sought to quantify pulmonary sounds, like wheezing and crackles, and address signal processing issues, such as noise reduction, amplification, and filtration.

Measuring Blood Pressure

Blood pressure is the amount of pressure exerted by the blood upon the arterial walls. A clinician uses a device known as a "sphygmometer"—a device that pumps air into a cuff wrapped around a patient's arm—and listens for pulse sounds with a stethoscope, observing the height in millimeters of a column of mercury supported by the blood pressure. The sounds are known as "Korotkoff sounds," named for Russian physician Nikolai Korotkoff. A contraction of the heart that causes a pulse beat that supports a column of mercury 120 millimeters high is called a "systolic reading of 120." The reading in the period between contractions of the heart or pulses is called the "diastolic blood pressure." If the diastolic reading is 80 millimeters, the blood pressure is recorded as 120/80 and is read as "120 over 80." These numbers represent a ratio rather than a true fraction. The U.S. National Heart, Lung and Blood Institute defines normal blood pressure to be <120 for systolic *and* <80 for diastolic pressure and defines hypertension to be >140 *or* >90 for systolic and diastolic, respectively. These values are derived in part from statistical studies of typical human variation in blood pressure and associations with medical conditions like stroke and heart disease. Early diagnosis and appropriate treatment of hypertension is recognized as one of the most significant advances of modern medicine in reducing morbidity and mortality.

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See Also: Diagnostic Testing; EEG/EKG; Mathematical Modeling.

Stock Market Indices

Category: Business, Economics, and Marketing.
Fields of Study: Algebra; Data Analysis and Probability; Measurement.
Summary: Stock market indices use sophisticated mathematical formulas to track the performance of the stock market and to help inform investors.

Mathematical stock market indices are used for a variety of purposes: as indicators of overall market health and activity, as measures of specific corporate profitability and activity, as performance metrics against which institutional investors (such as mutual fund managers) are measured, and for individual portfolio optimization and risk assessment. Some mathematicians and economists were developing price-based indices as early as the nineteenth century as well as analyzing pricing trends for explanation and prediction of market behavior. The Dow Jones Industrial Average (DJIA), named for journalist Charles Dow and statistician Edward Jones, appeared in 1896. Initially, it was a simple sum or average of the stock prices from 12 large companies. Since then, stock market indices have increased in their variety and mathematical complexity. For example, technical analysts use Fibonacci retracement levels, named after mathematician Leonardo Pisano Fibonacci, in order to model support and resistance levels in the currency market. Mathematicians and statisticians are instrumental in producing these indices. They also conduct theoretical and applied studies of market performance using these indices as data. In 1999, French-American mathematician Benoit Mandelbrot showed that market volatility can be modeled by fractal geometry, which contradicted some aspects of modern portfolio theory. Author and mathematician John Allen Paulos addressed many mathematical stock market issues in his popular book *A Mathematician Plays the Stock Market.*

Definition and Examples

When describing the performance of the stock market as a whole (or a segment of the market, such as selected large-company stocks, or all small-company stocks, or stocks of all companies belonging to a particular industry), one is usually referring to a stock market index. Such an index is a representation of a hypothetical portfolio that contains a certain quantity of each of the stocks in the market (or market segment). The quantity of each stock in the fictitious portfolio depends upon the "weighting" technique employed.

Some of the more commonly encountered stock indexes include the following:

- S&P 500, comprised of 500 large-company U.S. stocks that cover about 75% of U.S. equities
- DJIA, comprised of 30 large-company U.S. stocks
- Wilshire 5000, comprised of the most common stocks in the United States (although not necessarily exactly 5000 of them)
- Nikkei 225, an index of Japanese equities
- FTSE, a collection of indices of British stocks

Building a Stock Market Index

The wide variety of stock market indices fall into several weighting categories, each involving a different mathematical approach to combining stocks within a hypothetical portfolio. One can imagine a potentially unlimited number of ways of creating a portfolio that includes numerous company stocks: for example, a portfolio comprised of one share of each stock, a portfolio comprised of the same dollar amount of each stock, and so on. The most common methods of weighting stocks within an index are price-weighting and market-value-weighting. (To simplify, stock performance is treated as only a function of changes in the stock price over time—as capital gains and losses. In reality, dividends, stock splits, and a variety of other issues must be taken into account, which makes the specific mathematical applications more complex than represented in this entry.)

Price-Weighted Indices

A price-weighted stock index represents a theoretical portfolio that includes one share of each stock comprising the index. The price or value of the index is then equal to the average of individual stock prices. Therefore, the relative impact of a given company stock on the index is a function of the company's stock price per share: larger prices per share imply greater influence on the index.

Suppose that $S_i(t)$ represents the per-share price of stock i at time t, and let $S_I(t)$ be the value of the index at time t. Then, the price of a price-weighted index could be defined as simply the arithmetic average of the stock prices in the index:

$$S_I(t) = \frac{\sum_{i=1}^{n} S_i(t)}{n}(t)$$

where n is the number of stocks comprising the index.

While the value of a price-weighted index is simple to calculate, typically the measure of most interest to an investor is not the actual price of the index, but rather the percentage change (the rate of return) in the index over a period of time. Let $r_i(t,t+1)$ be the rate of return on stock i during the period from time t to time $t+1$, and let $r_I(t,t+1)$ be the return on the index between times t and $t+1$ (assume an annual return period for purposes of this discussion, but returns can also be calculated daily, monthly, quarterly, or over any other period of time).

Then, the return on a price-weighted index is

$$r_I(t,t+1) = \frac{S_I(t+1)}{S_I(t)} - 1 = \frac{\sum_{i=1}^{n}\left[r_i(t,t+1) \times S_i(t)\right]}{\sum_{i=1}^{n} S_i(t)}.$$

Multiplying this value by 100 yields the return expressed as a percentage change. The DJIA and other Dow Jones averages are examples of price-weighted indices.

Market-Value-Weighted Indices

A market-value-weighted (also called "value-weighted") stock index is one that weights the individual per-share stock prices according to the relative market values, or market capitalizations (called "market cap" for short), of the component stocks. A company's market cap is simply the totally value of its outstanding equity and is calculated as the per share stock price multiplied by the number of stock shares outstanding. Thus, an individual company's influence on a value-weighted index is a function of the overall equity value, or size, of the company—larger companies have greater influence on the movement of the index.

Using the notation introduced above, and letting N_i represent the number of shares of stock i outstanding, the rate of return on a value-weighted stock index would be

$$r_I(t,t+1) = \frac{\sum_{i=1}^{n}\left[r_i(t,t+1) \times S_i(t) \times N_i\right]}{\sum_{i=1}^{n}\left[S_i(t) \times N_i\right]}.$$

The S&P 500 and other Standard & Poor's indices are examples of market-value-weighted indices.

Other Types of Index Weightings

While price-weighted and value-weighted indices are common, there are other weighting techniques that can be used. For example, it is possible to create an index that gives equal weight to the return of each stock comprising the index. In such a case, the return on the index would be calculated as

$$r_I(t,t+1) = \frac{\sum_{i=1}^{n} r_i(t,t+1)}{n}.$$

With such an index, the performance of each stock has the same impact on the overall index return as every other stock.

Another possibility in creating an index would be to use geometric, as opposed to arithmetic, averaging. A geometric average is calculated by multiplying n numbers together and taking the n-th root of the product (as opposed to summing the numbers and dividing by n, as with an arithmetic average).

The key in interpreting the various types of stock market indices is to know their underlying construction and to understand and interpret them appropriately. Price-weighting and equal-weighting, for example, can result in very different index performance indications than value-weighting, even relative to the same underlying stock return data. The appropriate index to use in a given situation depends upon the specific purpose in mind. If one wants a measure of market performance that is more influenced by the price movements in the stocks of larger companies, for example, a value-weighted index may be most appropriate. If the sizes of companies are not relevant for analytical purposes, or if the companies that comprise an index are very similar in size and other attributes, a price-weighted or value-weighted index may be appropriate.

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See Also: Money; Mutual Funds; Pensions, IRAs, and Social Security; Probability.

Strategy and Tactics

Category: Government, Politics, and History.
Fields of Study: Geometry; Measurement; Problem Solving; Representations.
Summary: Mathematical concepts and processes can be used to analyze optimal strategies in a variety of situations.

In a competitive situation, such as businesses selling similar products, armies engaged in battle, opponents playing games, oil companies deciding where to drill, and employees bargaining for better salaries, successful outcomes depend on choosing the best plan of action from among a set of strategies to achieve a specific outcome. In many cases, mathematics can be used to analyze the situation and help to choose the best strategy. Mathematical techniques have been—and will continue to be—developed to address a wide range of problems in areas such as military logistics, intelligence, and counterintelligence.

The first step in the process is to determine the objective. That goal may be to maximize profit, beat the opposing army, or win the game. Next, the possible strategies to choose from and the limitations or constraints that may affect the choice of strategy need to be identified.

In competitive situations, the opponent's choice of strategy must be taken into consideration as well. While there are many examples of systematically analyzing and selecting the "best" strategies throughout history, the twentieth century—especially the World War II era—saw the emergence of operations research as the discipline that explores and develops systematic techniques for making decisions that are the "best" in some sense, usually maximizing profits/benefits or minimizing costs/liabilities.

Decision making can be approached mathematically in a number of ways depending upon the situation involved and the information available.

Linear Programming: Choosing the Best Option When Resources are Limited

Many decision problems arose out of troop supply needs during World War II. With a war on several fronts, deciding how to ship the limited troops and supplies to maximize their effectiveness was daunting. Many of the situations had the following characteristics:

- There were resources needed in specific combinations by a number of end users, and the amount of each resource was limited.
- The resources were used proportionally for each combination (in other words, to assemble whole units from raw materials, the number of raw materials needed was the same for each unit produced).
- The goal was to maximize the benefit or minimize the cost, and the cost or benefit was proportionally related to the number of units produced (in other words, the more produced, the higher the benefit or cost).

These characteristics yield a mathematical structure that is linear. Each resource corresponds to an equation or inequality that is a linear combination of unknown quantities representing the units to be combined or produced The objective function is also a linear combination of the number of units. See Example 1 for a very simple, classic example that involves deciding how to prepare a "balanced" meal.

Example 1. A linear programming problem.

A dietician wishes to prepare a salad meal that has a minimum amount of calories but still satisfies nutritional requirements. In particular, it must have at least 30 grams of protein and at most 9 grams of fat. The foods available are an ounce of lettuce with 4 calories, no fat, and 1 gram of protein; and slices of roast beef with 90 calories, 3 grams of fat, and 16 grams of protein. What amounts of lettuce and beef should the dietician serve with a diet salad dressing? Minimize calories $= 4L + 90B$, where $1L + 16B \geq 30$ and $3B \leq 9$.

These problems are easy to solve when they are small, like the problem in Example 1. The problems that arise in practice—such as those under consideration during World War II—are usually much larger and can involve hundreds of unknowns. During World War II, British and U.S. mathematicians looked for an approach that could make use of computers, which were being developed at that time and offered the possibility of performing many simple calculations quickly. In 1947, too late for the war effort, U.S. mathematician George Danzig (1914–2005) developed the simplex algorithm for solving linear programming problems. The simplex algorithm is an efficient recipe for solving linear programming problems of any size and is very easy to program on a computer. In the decades since the development of the simplex algorithm, many industries have used this procedure to solve problems in fields as diverse as banking, natural resources, manufacturing, and farming.

Linear programming problems are usually used to model static situations in that the final solution is essentially the result of one decision made under a clear set of assumptions. Many decision problems are more complicated, with a number of intermediate decisions to be made. These more dynamic problems often involve a probabilistic component as well, with uncertainty playing a complicating role in each decision.

Game Theory

Often, people are faced with a decision in which the resulting payoff will depend on external forces that are hard to predict (like natural forces). One option may always be best, but it is more likely that the best choice will simply "depend" on other factors. For example, when deciding which crop to plant, a farmer can list seed costs and profits based upon yield, but the yield will depend on the weather. A table can be made for each crop choice based upon several different weather scenarios, with past experience used to assign a probability to each possible weather scenario. Example 2 provides a standard format, usually called the "payoff matrix."

Example 2. A payoff matrix.

	List the possible states of the external forces
List the possible actions to choose from in making the decision	List the gain (profit, benefit, etc.) for each combination of actions and states.

Many decisions can be similarly structured, including determining what stocks to buy, what products to market, and what wars to wage. Different people will make different decisions depending upon their comfort level with risk.

Strategies for systematic decision making can be placed in four categories:

1. *Optimist strategy*: "MaxiMax" (Maximize the maximum gain). Find the best gain for each possible action and choose the largest of these maximums. Of course, that action may have the most risk associated with it, since the maximum gain may also coincide with the least likely state for the external force. In this case, the farmer may plant something that would have huge profits but only in the most unlikely weather conditions.
2. *Pessimist strategy*: "MaxiMin" (Maximize the minimum gain). Find the smallest gain for each action, and choose the largest of these minimums. This is a safe choice because it yields the minimum guaranteed gain regardless of external forces. In this case, the farmer may choose a "safe" crop to plant. If weather is really good, another crop would have been a better choice.
3. *Balanced strategy*: "MiniMax Regret." Calculate the "regret" for each possible action by determining the cost of choosing that

action compared to benefits of the best state of the external forces. Find the worst (largest) regret for each action and pick the action with the smallest worst-case regret.

4. *Averaging strategy*: "Expected Value." Use the probabilities governing the external forces to determine the expected gain for each action and choose the highest one. Expected gain or payoff is calculated as a weighted average of the gain for each state of the external force where the weight for each state is the probability of that state occurring. This strategy can be thought of as determining the action that, when chosen repeatedly, provides the best average benefit over the long term. For the farmer, this may not seem reasonable, since the decision under consideration is what to plant in a single, given year.

When the external force is an opponent with choices to make rather than a natural phenomenon with a random component, these decision situations can be examined as mathematical games. Two-person games can be represented with a payoff matrix as in Figure 2. The "row" player lists strategies on the left and the "column" player lists strategies across the top. The entries of the matrix are pairs of numbers, the row player's payoff, and the column player's payoff, respectively. In situations where the row player's winnings are equal to the column player's losses, and vice versa, the payoff matrix entries can be completely defined with one number, conventionally the row player's payoff. These games are called "zero-sum games" because for a particular pair of strategies, the row player's payoff and the column player's payoff, being negatives of each other, sum to zero.

Example 3. The prisoner's dilemma.

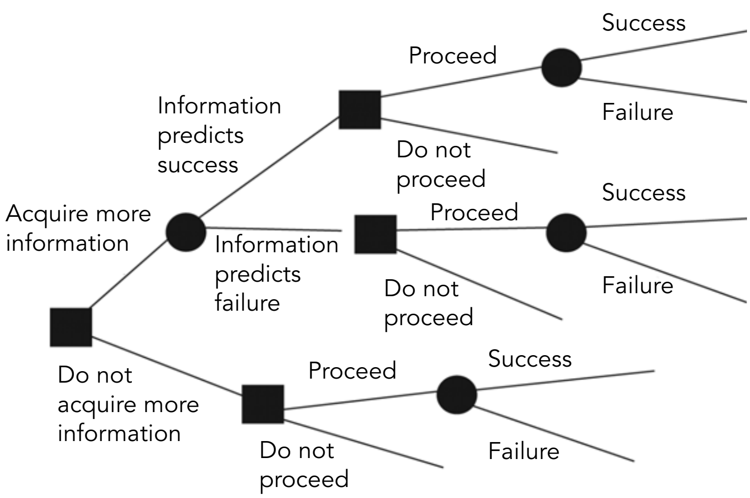
Two suspects are arrested by the police. They are each offered the same deal: Confess and receive a reduced sentence. If one confesses and the other does not, the confesser goes free and the other gets a 10-year sentence. If both confess, each gets a five-year sentence. If neither confesses, both get a one-year sentence on reduced charges. Neither prisoner knows what the other will say. What should they do?

	Confess	Refuse
Confess	$(-5,-5)$	$(0,-10)$
Refuse	$(-10,0)$	$(-1,-1)$

While mathematicians have been studying decision making and games of strategy systematically for several centuries, game theory emerged as a recognized mathematical approach to analyzing these decision processes in the 1930s and 1940s through research published by John von Neumann (1903–1957). The "prisoner's dilemma" (Example 3) was investigated in the 1950s and led to additional interest in the field.

The prisoner's dilemma captures many interesting features of competitive situations. Analysis shows that the intelligent prisoner should always confess, since the "best" outcome will occur no matter what the other prisoner decides to do: −5 is better than −10 if the other prisoner confesses; 0 is better than −1 if the other prisoner refuses to talk. However, this individual "best choice" results in each prisoner confessing and getting

Figure 1. A "decision tree."



a five-year sentence, whereas if neither confesses, they only get one-year sentences. This feature of competitive behavior and strategies can be thought of as the friction between basing strategic decisions on individual goals or on the common good.

With appropriate choices for the values in the table, these games could model a number of competitive situations, such as two companies trying to determine what price to set for competing products or two armies determining how to wage war.

Decision Trees

In situations where the ultimate decision depends on an intermediate choice, a decision tree can help to organize the information and facilitate a systematic analysis. A company may be ready to bring a product to market and needs to decide whether or not to invest funds up front in a test market exercise. The test market may bring in better information about how to market the product on a larger scale, thus increasing profit, but the cost of the test market exercise would also take away from the profit. An oil drilling company could choose to invest funding in test wells before determining the final drilling location. A university may be trying to hire a senior administrator and could choose to invest funds in a head-hunter search firm.

In all of these situations, the outcomes can be organized into a tree diagram like the one in Figure 1. Each "decision fork" is represented by a square, and each event fork—governed by external, possibly random forces—is represented by a circle. The branches leading from the event forks have probabilities assigned based upon the likelihood that an outcome will occur. Typically, acquiring additional information will result in an increased probability of success (or failure), and so the probabilities of success and failure will be different for different event forks.

Each terminal branch represents a final outcome. If current assets, the cost of the information acquisition, and the gains or losses under success and failure are known, then each terminal branch can be labeled with the net gain (or loss) for that option. Once those values are determined, the tree can be "folded back" through calculating the expected outcomes from the probabilities to determine which decisions to make to maximize the gain.

The decision points and events may include more than two options or outcomes, and there may be more than two decisions to be made before the final outcome, so the tree may have more forks and branches than the one in Figure 1 but the analysis process is the same.

From these trees, the value of the additional information acquired can be calculated. This calculation can assist companies in determining how much they should be willing to pay for that information. Also, the amount of risk a company is willing to assume can be incorporated into the process, allowing companies that are willing to shoulder a larger risk for the (slimmer) chance of a larger gain to include that information into the analysis.

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Holly Hirst

See Also: Coding and Encryption; Intelligence and Counterintelligence; Predicting Attacks; Risk Management; Scheduling.

Street Maintenance

Category: Travel and Transportation.
Fields of Study: Data Analysis and Probability; Geometry; Measurement.
Summary: Street maintenance requires planning, preparedness, and risk assessment, all of which involve mathematics.

Stone paved roads date back thousands of years and mathematicians and architects have long investigated ways to lay paving stones. Another connection between mathematics and streets dates to when Hermann Minkowski proposed numerous metric spaces, one of which is referred to in the twenty-first century as "taxicab geometry." Some streets are laid out on a grid system, leading to mathematical investigations in taxicab geometry. The surface curvature of roads is also mathematically interesting and important in drainage and safety issues. Street maintenance is a combination of services that includes resurfacing of streets and curbs, pothole patching, sweeping, snow removal, and maintenance of drains. Mathematical problems that arise within street maintenance have to do with engineering, applied physics and chemistry, logistics, budgets, and communication.

Types of Streets

Different types of roads call for different maintenance. Civil engineers can use tools such as falling weight deflectometers to measure properties of street coverings—in this case, deformation under dropped weight. A heavily loaded truck can damage the street surface approximately 10,000 times more than a small passenger car. This fact explains why streets with industrial traffic require more frequent maintenance, owners of trucks pay more taxes, and trucks are not allowed on most streets.

There are many materials used to cover streets, and the choice of material provides interesting mathematical optimization problems. For example, rubberized asphalt contains recycled tires, which is an environmental bonus and can reduce the noise of the road by about 10 decibels, which is valuable for nearby homes. However, it can only be laid in certain temperatures. Concrete is more durable than asphalt but is more expensive and harder to repair. Brick and cobblestone coverings do not form potholes and can hold heavy loads. However, they are noisy, they require manual installation and maintenance, and they can damage cars.

Potholes and Fatigue

Most potholes happen because of what is known in the materials science as "fatigue" of the surface. Fatigue occurs when materials are subject to periodic forces, such as heavy cars passing through. Small cracks start to appear, which then aggregate into networks of cracks, which then give way to a pothole. Calculus, differential equations, and statistics models are used to test road surface materials for resistance to fatigue and to predict fatigue's time through the statistically derived fatigue curves (*S-N* curves). Cycles of heating and cooling can quickly extend existing cracks and make potholes larger as well as freezing water that has seeped into cracks.

Cleaning

The mathematics of cleaning schedules involves balance among many random variables, such as traffic or seasonal leaves removal. In a typical city, urban streets with heavy pedestrian traffic are swept daily, and other streets are swept every week or two. Statistical data on street use determines where to place garbage cans and how often to empty them, when to send heavy sweep machines for cleaning, and how to avoid disrupting regular street use and events with cleaning activities.

Some street maintenance measures prevent street dirt. Highly visible trash cans can drastically reduce littering. In many communities, residents are invited to participate in street cleaning and maintenance to some degree, from sites where they can report potholes to street cleaning celebrations on holidays or weekends. Birds can be attracted to appropriate places, and dog owners are guided to special parks and runs. Mathematical models behind such measures come from studies of human and animal behavior.

Accidents and disasters—from dust storms to spilled poisons—may require special cleaning activities. Because such events are rare but require special knowledge and equipment, it usually makes sense to maintain tools and specialists for these special events only in large cities and to send teams to smaller places that need help.

Snow Maintenance

Streets under snow require special maintenance, including mechanical removal of the snow by snowplows, snow-blowers, or shovels; inert surface treatment for traction with sand or sawdust; and chemical surface treatment. The mathematics of dealing with snow includes economical and environmental factors. When snow immobilizes traffic, productivity and sales are lost. However, snow-removal measures cost money and take time. In cities where it snows infrequently, it is usually cheaper to wait for the snow to melt rather than to maintain a fleet of removal machines.

Most of the chemical treatment of snow is done with sodium chloride (table salt). Salt makes snow melt at about 10 degrees Fahrenheit less than usual (freezing-point depression). Switzerland uses more than a pound of salt a year for every square yard of its roads. Chemical treatments can damage plants and animals throughout the watershed. Safe amounts of chemicals can be determined based on ecological models. Chemicals also cause vehicle damage and faster road deterioration. These costs are part of the decision of which type of snow maintenance is more economically sound.

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Maria Droujkova

See Also: Bicycles; Green Design; Traffic.

String Instruments

Category: Arts, Music, and Entertainment.
Field of Study: Geometry; Number and Operations; Representations.
Summary: The harmonics and timbre of wind instruments are described and computed using mathematics.

All stringed instruments exhibit a fundamental property of physics in that when impacted, they vibrate at numerous frequencies. The vibration of the string displaces the air around it, which—when impacted on the human eardrums—creates the sensation of sound. Some of the common instruments in the string family are violin, guitar, harp, mandolin, cello, and banjo. A modern violin has about 70 parts, and the overall design of such complex string instruments is inherently mathematical. Features such as string tension, area, and shape of the top plate, and spacing of frets all have mathematical properties that influence sound. For any string, at a given tension, only one note will be produced. To generate multiple notes from the instrument, many strings may be used to span the desired frequency spectrum (for example, harps) or the string may be forced to vibrate at different lengths, thereby changing the frequency (for example, guitars). On an equally tempered instrument like a guitar, the spacings of the frets, which help a player adjust string length, have to be scaled by the ratio $2^{1/12}$. This problem is mathematically equivalent to duplicating a cube, which is one of the classic problems of antiquity. Mathematician Jim Woodhouse has studied violin acoustics using linear systems theory and mathematically modeled "virtual violins," as well as related vibration problems like vehicle brake squeal.

Harmonic Series and Fundamental Frequency

When a string is plucked, struck, or bowed, it resonates at numerous frequencies simultaneously. The waves travel up and down the string. These waves reinforce and annul each other, which results in standing waves. The one-dimensional wave equation is used to model string instruments. A harmonic series is composed of frequencies that are an integer multiple of the lowest frequency. Fundamental frequency is the lowest frequency in a harmonic series. The musical pitch of a note is usually perceived as the fundamental frequency. The fundamental frequency (f) of a string can be computed as

$$f = \frac{\sqrt{\frac{T}{\frac{m}{L}}}}{2L}$$

where T is the string tension in newtons, m is the string mass in kilograms, and L is the string length in meters. The fundamental frequency is also known as the "first harmonic."

Timbre

Timbre is the quality of a musical note and is what defines the character of a musical instrument. When two different instruments play the same note, the note could have the same frequency. The human ear distinguishes the source of the note because of timbre. Hermann Helmholtz was the first to describe timbre as a property of sound. When an instrument plays a certain note, the outputted sound consists of the fundamental frequency and its harmonics. These harmonics differ from instrument to instrument—what is known as "timbre."

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Ashwin Mudigonda

See Also: Harmonics; Percussion Instruments; Pythagorean and Fibonacci Tuning; Wind Instruments.

Stylometry

Category: Arts, Music, and Entertainment.
Fields of Study: Data Analysis and Probability; Measurement.
Summary: Stylometry is a descriptive science that uses statistical techniques to identify authorship of written materials.

Stylometry is a descriptive science that uses statistical techniques to identify authorship or written materials. In addition to comparing simple frequency patterns of words, stylometry focuses on the groupings of words and the position of these words in sentences. Using stylometry, scholars have tried to determine if Homer wrote the last book of the *Odyssey*, if the Apostle Paul wrote the *Letter to the Ephesians*, and if Shakespeare wrote the first act of the play *The Booke of Sir Thomas Moore*. Because of the successful use of stylometry, its techniques have been expanded to help identify composers from their musical compositions and analyze artists from their paintings.

Beginnings of Stylometry

In 1851, August de Morgan, an English mathematician, initiated the field of stylometry when he suggested that authors could be identified by the average number of letters in their written words. Because de Morgan's suggestion was simplistic and often misleading, stylometry did not gain validity until 1944, when Udny Yule published his pioneering work that suggested that an author's vocabulary usage did not depend on sample size. Analyzing Paul's Epistles and the words of the physician Hippocrates in 1957, W. C. Wake was the first to produce an acceptable test of authorship using distributions, sampling methods, and periodic effects within distributions. In 1961, A. Q. Morton and others used computer technology to both extend and verify Wake's approach.

Uses of Stylometry

Stylometry has other constructive uses, such as the use of statistical techniques to examine concordances (13 million words) of the works of Saint Thomas Aquinas (illustrated above). As a result, scholars not only identified spurious additions introduced by editors of Aquinas's works, but also successfully reconstructed lost passages. Scholars used a similar approach to examine stylistics differences among the three Greek tragedians—Euripides, Aeschylus, and Sophocles—trying to also establish chronological progressions across a single author's works in terms of vocabulary, themes, and use of iambic trimester.

A specific example of scholars' use of stylometry involves *The Booke of Sir Thomas Moore*, a play about a martyred Englishman in 1535. Scholars first concluded that the play was a composite effort of five authors, with handwriting analyses accepted as proof that William Shakespeare was the sole author of two of the play's sections. Then, computer analyst Thomas Merriam created computer databases of the play in question and three other Shakespearean plays—*Julius Caesar*, *Pericles*, and *Titus Andronicus*.

The concordances generated for all four plays revealed significant similar frequencies of "word habits" or repeated combinations of words and phrases. Though Merriam concluded that Shakespeare was the sole author of *The Booke of Sir Thomas Moore*, his stylometric data did not convince all scholars. Skeptics such as these claim that Merriam's techniques are at best informative, being suspect because the three comparison plays are not the best representatives of Shakespeare's style.

Modern Applications

Stylometry has been used in court cases to identify "fraudulent" wills and "false" criminal confessions. In the late 1970s, defense attorneys for kidnap victim and accused bank robber Patty Hearst tried to introduce stylometric evidence that "proved" the tape-recorded "communiqués" read by Hearst were not her own words.

Their evidence was based on concordances built from previous essays by Hearst, oral conversations, her confession, and materials produced by the Symbionese Liberation Army. The attorneys carefully analyzed these concordances using statistical discrimination, cluster analysis, and *t*-test comparisons to examine factors such as average sentence lengths, parsing patterns involving conjunctions, and linguistic habits. Despite the defense's protests, the trial judge and the appeals court both ruled that the stylometric evidence was not admissible and thus was never used.

Donald Foster, a Vasser College English Professor, used stylometry to identify with 99% confidence the "anonymous" author of the political text, *Primary Colors*. Though *Newsweek* columnist Joe Klein originally denied being the suspected author, he eventually admitted to the deed. Since that time, Foster has helped confirm Ted Kaczynski's authorship of the *Unabomb Manifesto* and identify Eric Rudolph as a suspect in the 1996 Atlanta Olympics bombing.

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See Also: Diagnostic Testing; Literature; Probability.

Submarines

See *Deep Submergence Vehicles*

Succeeding in Mathematics

Category: School and Society.
Fields of Study: Communication; Connections; Problem Solving.
Summary: Poor mathematics performance can be attributed to a variety of factors and numerous organizations and strategies are believed to help students achieve mathematics success.

Many educational initiatives are designed to motivate U.S. students to excel in science and mathematics, with the goal of building the strong science, technology, engineering, and mathematics (STEM) workforce needed to meet twenty-first century challenges. Three overarching goals of the 2009 federal Educate to Innovate program are increasing STEM literacy for everyone; improving teaching so that American students meet or exceed those in other nations; and expanding STEM education and career opportunities for underrepresented groups. However, success in mathematics, or even literacy, can be difficult to define. Some see it as some minimum skill set, number of courses, or type of courses taken. Others conceptualize it by what sorts of problems students are able to solve or by their ability to manage real-world mathematical problems, such as budgets or loans.

Measuring Success

There are many barriers to achieving success. Broad application of standardized testing in mathematics education sometimes reduces the measure of success to a single score or change in scores over time. Modern educational approaches and programs at all levels increasingly emphasize problem solving, which the National Council of Teachers of Mathematics asserts is not well measured by standardized tests, since problem solving reaches beyond simply remembering some encapsulated set of concepts, formulas, and skills to include broader applications, novel situations, and mathematical thinking and reasoning. There are calls for innovative assessment alongside changes in educational practice in order to attempt to capture what it means to know, do, and be successful in mathematics at home, school, and work. Among professional mathematicians, measures of success vary as well, with ongoing debate about various aspects of teaching and scholarship, including how and whether to assess measures like the number of publications, the number of citations to an author, the quality of a journal, or letters from peers, and the role of other measures like student evaluations. Educational researcher Christopher Jett, who examined mathematics success among African-American men, uniquely defined mathematics success as, “being able to use mathematics as an analytical tool to educate, stimulate, and liberate the (my) people.”

Failure and Anxiety

Albert Einstein was once quoted as saying, “Do not worry about your difficulties in mathematics. I can assure you mine still are greater.” This assertion seems contrary to many people’s belief that success in mathematics is binary: people are successful or not, with no middle ground. Popular culture portrayals of mathematicians as geniuses often inadvertently support the mistaken belief that one must be gifted to succeed in mathematics. Further, while mathematicians are portrayed as wizards, mathematics itself is often shown as a sort of mysticism—a secret and arcane knowledge accessible only by a select few. In reality, mathematics encompasses a diversity of fields and professional mathematicians have varying sets of competencies, personalities, and working styles. Likewise, students at any level may have command of a wide variety of skills and concepts, and while those concepts are related and may build on one another, competence is not uniform. A student who struggles all through algebra may still be successful in geometry. A student who labors over constructing a proof may have a flair for data analysis and statistics.

Mathematics is inherently cumulative in nature. The feeling of failure at mathematics, especially given the common binary view, can seemingly be caused by a small problem that actually immediately impacts only one small area. This partial temporary failure can result in a long-lasting loss of confidence. Mathematics anxiety is an increasingly recognized phenomenon that interferes with students’ ability to learn mathematics and perform at the best of their abilities, regardless of their actual skill. Many people who are perfectly capable of learning and using mathematics feel anxiety about it and will avoid using mathematics whenever they have the option. Over time, this can lead to degradation of their abilities as they fall out of practice, which cyclically reinforces the anxiety. In addition to avoidance, mathematics anxiety can sometimes negatively affect working memory. As the anxiety grows, the student has more trouble keeping track of tasks, leading to poor performance and yet again reinforcing the anxiety. Some believe that mathematics anxiety is caused in part by poor performance on mathematics achievement tests and in part by early difficulty in mathematical skill development. The anxiety remains even after actual performance has improved and may be related to a belief that the earlier difficulties reflect some inherent character trait rather than a situational difficulty. People who later in life describe themselves as “terrible” at mathematics but who display education-appropriate competence in mathematics may not remember the original event that inculcated in them this belief that they are poor performers. Beyond performance, some cite teacher or classroom practices as contributing to mathematics anxiety. Like other mathematicians, mathematics teachers do not possess equal skills and levels of comfort in all areas of mathematics. This may be especially true of elementary school teachers who must teach a wide array of subjects on a daily basis. Classroom practices such as emphasizing the "right" answer, which also frequently occurs on standardized tests, can increase anxiety because some students attach great significance to being "wrong." Other students may feel anxiety over being asked to "show their work," because they are less confident in their mathematical thinking than in their ability to produce a correct answer.

The Mathematics Anxiety Rating Scale (MARS) was developed by psychologists in the early 1970s and exists in several versions, including foreign language adaptations. Researchers using this scale and other measures have identified other situations more likely to trigger anxiety. For example, tests where problems become progressively more difficult appear to trigger mathematics anxiety more often than tests in which the distribution of problems by difficulty is more random, which is true even when all the problems on the test are well within the skill level of the test-taker. Timed tests and the possibility for public embarrassment, such as working at a board in the front of the class, are also factors that can induce anxiety. Many studies suggest an association between mathematical anxiety and gender; female students are more often anxious about mathematics, perhaps because they have embraced the belief that women are not as good at mathematics as men and thus have difficulty building self-confidence in their abilities. These stereotype effects can also extend to other underrepresented groups.

Stereotype Threat

One widely studied phenomenon regarding success on standardized tests is known as the "stereotype threat" in which the stereotyping of groups in society affects an individual. Researchers found that proficient white males performed more poorly on a difficult mathematics test when researchers induced the threat of superior performance by Asians as compared to a control group. The impact of stereotype threat for many groups of students has been researched under a wide variety of conditions. For instance, Asian women performed more poorly on mathematical tests in which they were cued as women, while they performed better when cued as Asians as compared to control groups. Some researchers theorize that students must contend with a subconscious whisper of inferiority when their abilities

Neuroscience

Research including verbal descriptions of problem solving, observation, and biological data from tools such as magnetic resonance imaging has confirmed that spatial ordering, temporal-sequential ordering, higher order cognition, memory, language, and attention are among the functions of the mind that are at work when children think with numbers—a wide array of functions, not a single "mathematics center" of the brain. Sequential ordering may be used to solve multi-step problems. Spatial ordering lets a child recognize symbols and geometric forms, among others. Higher order cognition lets a child "think about thinking," considering different problem-solving strategies, being aware of what they are doing as they do it, and generalizing and applying old skills to new problems. Language skills affect a student's ability to understand instructions given to them and articulate their thinking. They are also critical because of the discipline-specific vocabulary at work in mathematics. Some terms, like "exponent" have no common, everyday usage. Other words like "times" and "multiply" are generally used interchangeably. Words like "normal" and "random" overlap with everyday non-mathematical vocabulary and may have broader colloquial meanings. Given the number of interacting components involved in the process of thinking about, working with, and learning mathematics, and the fact that a temporary "stall" in one area can seemingly magnify a negative, reflexive "I just don't get it" feeling in students, the roots of mathematics success or failure can be difficult to identify and address.

are most taxed. Whether they consciously or unconsciously accept the stereotype or not, they may still work harder in order to avoid confirming it, because failure has a more devastating meaning. The extra burden may be enough to impact performance. The stereotype cues may be subtle, like self-identification of gender, race, or culture before an exam. Researchers have also found that removing the cues can positively impact test performance. For example, in a 2009 meta-analysis of 18,976 students from five countries who were matched by researchers using past performance, stereotyped students performed better under conditions that reduced the threat.

Research and Strategies

In the 1970s, mathematics education researchers Elizabeth Fennema and Julia Sherman developed the Fennema–Sherman mathematics attitude scales to examine eight components considered critical for success in mathematics: attitude toward success in mathematics, mathematics as a male versus female domain, parent support, teacher support, confidence in learning mathematics, mathematics anxiety, motivation for challenge in mathematics, and mathematics usefulness. Their work has been cited among the most quoted social science and educational research studies of the latter twentieth century, and many versions of their scales exist.

The cumulative body of research suggests several strategies that may help students succeed in mathematics, though one point of general agreement seems to be that the key to mathematics success is active participation, active study, and engaging the material. For example, younger students can be encouraged to ask questions in and out of class and can be given mathematical exercises that are interactive rather than requiring them to only passively listen to explanations of the material. Hands-on activities with even simple objects like buttons, dried beans, or animal counters can help children develop number, counting, and arithmetic skills. More sophisticated tools, like tangrams and algebra tiles, develop geometric concepts and thinking about functions. This method has come to include computer-based virtual manipulatives. Asking questions and engaging in hands-on learning is also valuable in the later grades and college. With regard to attitudes, educators frequently encourage students to recognize that the act of learning and doing mathematics is likely to be different than other school subjects, particularly with regard to its cumulative nature and fact that working with a variety of mathematics problems is usually the only way to learn mathematics. Some instructors include explicit problem-solving and test-taking strategies in their instruction in addition to concepts. Students may also benefit from instruction in methods of note-taking, reviewing, and reading mathematics textbooks that encourage them to think reflectively about mathematics content rather than simply summarizing. At the same time, teachers may use a variety of presentation and engagement methods, including using real-world problems, considering different learning styles, being aware of anxiety and stereotypes that may affect students, and engaging parents in an ongoing dialogue about mathematics education to gain support and make them partners in their children's success.

Organizations

Beyond the classroom, clubs, professional organizations, and scholarship programs have been shown to contribute to success and some are particularly targeted toward groups that may be more at-risk, such as women and minorities. The Meyerhoff Scholars Program is a notable example of such a program. It was initially created in 1988 to target African-American men, though admission is no longer restricted by gender or ethnicity. A 2010 statistic noted that program participants were 5.3 times more likely to be attending or have graduated from a STEM Ph.D. or M.D./Ph.D. program than others who were invited to join but declined and attended another school. The program's success is attributed in large part to its emphasis on mentorship, particularly since women and minorities interested in mathematics may never have met a woman or a minority mathematician or have not been exposed to research and challenges to excel rather than to imply succeed. The Hypatia Scholarship program for women at the University of South Australia, founded in 1997 and named for woman mathematician Hypatia of Alexandria, awards not only financial support but also provides women with shared office space and computer resources in close proximity to faculty to encourage interaction and build confidence. It funds summer employment to encourage the participants to use their mathematical training in industry or academia. Feedback from students indicated that the women valued the social network more than the financial support, saying it motivated them and helped reduce anxiety. Other organizations offer financial scholarships and some opportunities for networking, such as the American Statistical Association's Gertrude Cox Scholarship. Mathematics clubs and honor societies, like Pi Mu Epsilon, provide social and academic opportunities for students with mathematics interests, and researchers have found some evidence that participation is associated with increases in retention, positive attitudes about mathematics, and higher grade point averages.

The successful Upward Bound program targets disabled, low-income, homeless, and foster care youth, as well as those who would be first-generation college students to encourage comprehensive success in secondary and higher education. It provides instruction in mathematics, laboratory sciences, composition, literature, and foreign languages, along with support like counseling, academic tutoring, and assistance with college admission and financial aid.

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Bill Kte'pi

See Also: Curriculum, K–12; Learning Models and Trajectories; Mathematicians, Amateur; Minorities; Women.

Sudoku

Category: Games, Sport, and Recreation.
Fields of Study: Algebra; Number and Operations.
Summary: The game of Sudoku is explained by and informs graph theory and randomness.

Sudoku is a number puzzle based upon the mathematical concept of a "Latin square." Latin squares are arrays of numbers in which each number is listed only once in any row or column. Leonhard Euler originated the term, calling them "Latin squares" because he used Latin letters rather than numbers in his investigations. Completed Sudoku grids are Latin squares that are further subdivided into subgrids in which the numbers also appear only once in the subgrid. The graphic below shows a completed 9-by-9 Sudoku puzzle.

7	3	1	4	9	5	8	6	2
9	8	4	6	2	1	3	5	7
5	2	6	3	7	8	9	4	1
4	1	9	5	3	2	7	8	6
8	7	5	1	6	9	4	2	3
3	6	2	7	8	4	1	9	5
1	9	8	2	5	7	6	3	4
6	5	7	8	4	3	2	1	9
2	4	3	9	1	6	5	7	8

Originally published in the 1970s in Europe and the United States, Sudoku surged to popularity in Japan in the late 1980s and reappeared in Europe and America in the mid-1990s, becoming popular among puzzlers. The popularity of the puzzle has continued to grow in the twenty-first century, leading the puzzle to become the subject of mathematical scrutiny.

Sudoku is usually based on a Latin square with nine rows and columns; puzzles of other sizes are possible such as 4-by-4, 16-by-16, and 25-by-25. Some of the numbers are filled in to start. The goal is to quickly and accurately complete the puzzle, with the digits 1–9 placed once in each row, column, and subgrid. Below is an unsolved Sudoku puzzle:

	1	3	5		9	2		
2				8				
9	8		6		7		1	
3	5						8	
		6	8		1	7		
	2						9	1
	6		3		4		2	5
				5				9
		2	9		6	8	4	

Graph Theory

There are a number of interesting mathematical questions associated with Sudoku puzzles, in particular the conditions under which they have one solution. In 2007, Agnes Herzberg and M. Ram Murty showed that Sudoku puzzles can be recast as graph coloring problems allowing the broad, well-developed theory of graphs to be applied to the solution question. In particular, they showed that a standard Sudoku puzzle can be thought of as a graph where each cell in the puzzle is represented by a vertex with 20 edges, each edge connecting the cell to another cell in the row, column, or sub-grid. Graphs for which all vertices have the same number of edges are called "regular," so Sudoku graphs made in this way are "20 regular" graphs.

Since each digit can appear only once in any row, column, or subgrid, putting the nine digits into the cells is equivalent to coloring the vertices of the graph with nine colors such that no vertices connected by an edge are the same color—or in graph theory terminology, finding a "proper 9-coloring" of the graph. The number of ways to color a regular graph with *n* colors is a well-known formula that is a function of the number of colors and the number of vertices. Using this and other ideas about coloring graphs, Herzberg and Murty proved that 9–by-9 puzzles must have at least eight different digits shown in the starting configuration to have a unique solution.

There are still many unanswered questions about when Sudoku puzzles have one solution. Assuming that eight different digits are used in the starting configuration, how many numbers total must be shown in the starting configuration to ensure a unique solution? The answer is not known; a small number of distinct puzzles with 17 entries in the starting configuration are known to have a unique solution. There are no known puzzles with 16 or fewer entries that have unique solutions. Does one exist? Would the answer be different if all nine digits are used in the starting configuration? Mathematicians and puzzlers are investigating these and other interesting questions.

For example, in 2010, mathematicians Paul Newton and Stephen DeSalvo demonstrated that the arrangement of numbers in Sudoku puzzles is more random (by some definitions of randomness) than 9-by-9 matrices produced by random generators, since Sudoku rules excludes some of the possible arrangements that have innate symmetry.

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Holly Hirst

See Also: Acrostics, Word Squares, and Crosswords; Mathematical Puzzles; Matrices; Puzzles.

Sunspots

Category: Weather, Nature, and Environment.
Fields of Study: Algebra; Data Analysis and Probability.
Summary: Sunspots have long been observed and mathematicians and scientists continue to try to understand them and their effects.

Sunspots are a not yet fully explained phenomenon tied to solar activity. The sun is Earth's richest source of heat and light. Furious eruptions of energy take place on the surface of the sun. In the core, nuclear reactions occur because of the immense temperature and pressure. Through a process known as "convection," millions of tons of hydrogen are converted into helium every second and are then expelled at the surface of the sun as light and heat. Sunspots have a magnetic field strength that is thousands of times stronger than Earth's magnetic field. These magnetic fields inhibit convection to create relatively cooler areas, which appear as dark spots on the surface of the sun. Scientists and mathematicians have long attempted to understand their behavior and oscillations and have used mathematical tools like differential equations, hexagonal planforms, and time series analyses. They also count the number of sunspots and examine possible relationships between this number and factors on Earth, like radio disruptions, land temperature, and weather phenomena.

History

Direct observation of the sun is very dangerous, which historically made sunspots hard to study and quantify.

In ancient times, Chinese astronomers recorded solar activity. Mathematician and astronomer Thomas Harriot, noted for his work on algebra, is also credited as the discoverer of sunspots. Increased understanding of the nature of sunspots, including the observation that they often occurred in groups and that they moved relative to one another as the sun rotated, is tied to the development of the telescope in the seventeenth century. One of Galileo Galilei's works on sunspots offered evidence for the heliocentric system of Nicolaus Copernicus, and this led to debate about sunspots, as evidenced in astronomer, mathematician, and Jesuit Christoph Scheiner's views and works.

In the eighteenth century, Alexander Wilson used a geometric argument to show that sunspots were depressions. In the nineteenth century, pharmacist and amateur astronomer Heinrich Schwabe collected data on the periodicity of sunspots. Systematic observations, such as the approximately 11-year cycle, were made by Rudolph Wolf starting in 1848, who also measured the number of sunspots present on the surface of the sun. Wolf was primarily an astronomer but he also taught mathematics and physics. His observations were disputed by other astronomers, but his methods, which were based on statistical analyses, were eventually accepted as correct. Wolf's formula continues to be used in the twenty-first century as one of the sunspot indices. The International Sunspot Number is compiled worldwide by the Solar Influences Data Analysis Center in Belgium and by the U.S. National Oceanic and Atmospheric Administration. In the twenty-first century, sunspots are observed with solar telescopes, which use various filters, and specialized tools such as spectroscopes and spectroheliscopes. Amateurs generally observe sunspots using projected images.

Waxing and Waning

Scientists know that the sun had a period of relative inactivity in the seventeenth century, which corresponds to a climatic period called the "Little Ice Age." Evidence suggests that similar periods existed in the distant past, which means there might be a connection between solar activity and terrestrial climate. The magnetic activity that accompanies the sunspots can change the ultraviolet and soft X-ray emission levels, affecting Earth's upper atmosphere. Some researchers have proposed that sunspots and solar activity are the main cause of global warming rather than carbon dioxide greenhouse gas emissions.

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See Also: Telescopes; Temperature; Weather Forecasting.

Subtraction

See *Addition and Subtraction*

Surfaces

Category: History and Development of Curricular Concepts.
Field of Study: Communication; Connections; Geometry.
Summary: Surfaces are two-dimensional manifolds, some of which have been studied for their special properties.

Living beings interact with much of the world through surfaces. Humans walk on surfaces and eat and sleep on them. Surfaces like the one-sided Klein bottle, named for Felix Klein, stretch the imagination and are the subject of mathematical investigations. They are often represented using physical models as well as computer models, including sculptures and computer animations. In twenty-first-century classrooms, students investigate a variety of surfaces and their properties, including area and volume. The National Council of Teachers of Mathematics recommends an understanding of the area and volume of rectangular solids for primary school students; of prisms, pyramids, and cylinders for middle grades students; and of cones, spheres, and cylinders for high school students. The parametrization and volume of surfaces is further explored in a multivariable calculus course.

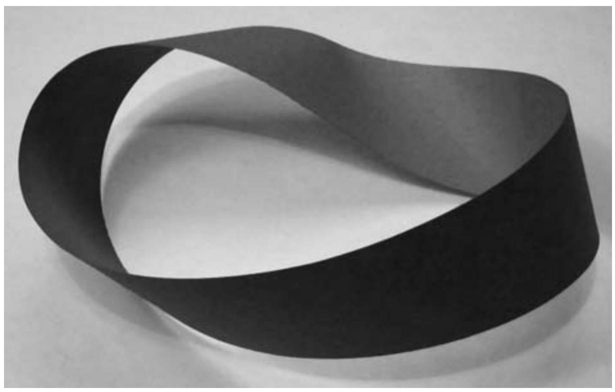
History of Study

Mathematicians have long developed the theory of surfaces, and they continue to investigate their properties. In addition to the plane, polyhedra, such as the surface of a cube or an icosahedron, are among the first surfaces studied by the ancient Greeks in geometry. Their view of surfaces was entirely different from the functional description used in investigations of surfaces in the twenty-first century. The Greeks also had a good knowledge of surfaces of revolution and pyramids. With the introduction of analytic geometry in the seventeenth century, the study of surfaces developed into one of the most studied branches of mathematics. Mathematicians like Carl Friedrich Gauss, Pierre Bonnet, Barnhard Riemann, Gaspard Monge, and their followers firmly established surfaces on a rigorous basis during the eighteenth and nineteenth centuries.

One of the greatest achievements of the theory of surfaces is the Gauss–Bonnet theorem. Versions of the theorem were explored by Gauss in the 1820s and Bonnet and Jacques Binet in the 1840s. The form of the theorem that is standard in undergraduate differential geometry courses is attributed to Walther von Dyck in 1888. Smooth surfaces are defined as those surfaces in which each point has a neighborhood diffeomorphic to some open set in the plane. This added structure allows the use of analytic tools. The parametric functions for a smooth surface define two quadratic differential forms: the first and second fundamental forms, which are local invariants defined as functions of the arc length. The Gaussian curvature of the surface is an isometric invariant; hence, an intrinsic property of a surface, which is known as the Theorema Egregium of Gauss. The Gaussian curvature measures the deviance of the surface from being flat at each point. The parametric equations of a surface determine all six coefficients of its first and second fundamental forms; conversely, the fundamental theorem of surfaces states that given six functions satisfying certain compatibility conditions, then there exists a unique surface (up to its location in space). A geodesic curve on a smooth surface is characterized as a locally minimizing path. In a sense, geodesics are the straight lines of surfaces, which are essential for defining the notions of distance, area, and angle on a surface. Bonnet investigated the geodesic curvature, which measures the deviance of a curve on the surface from being a geodesic. The Gauss–Bonnet theorem states that for an orientable compact surface, the total Gaussian curvature is 2π times the Euler characteristic of the surface, named for Leonhard Euler. A consequence of this theorem is that the sum of the interior angles of a geodesic triangle is greater than, less than, or equal to π, depending on if the Gaussian curvature of the surface is positive, like on a sphere; negative, like on a hyperboloid; or zero, like in the plane.

Types of Surfaces

The classification of surfaces is another topic that is explored in undergraduate geometry or topology classes. In 1890, Felix Klein asked what surfaces locally look like the plane. In Klein's Erlangen Program, a space was understood by its transformations. Heinz Hopf published a rigorous solution in 1925 that arose from groups of isometries acting on the plane without fixed points. The surfaces are the plane; the cylinder; the infinite Möbius band, named for August Möbius; the flat Clifford torus or donut, named for William Clifford; and the flat Klein bottle. Intuitively, surfaces seem to always have two sides; however, the Möbius band and the Klein bottle have only one side. Other surfaces like the projective plane resemble the sphere, and many-holed donuts



A paper Möbius band is a developable surface (it has zero Gaussian curvature).

resemble hyberbolic space. The Euler characteristic is used to classify the topology of a surface.

Some important types of surfaces that are studied intensively in geometry and analysis are minimal surfaces with zero mean curvature, such as catenoids and helicoids; developable surfaces with zero Gaussian curvature, like the plane, the cylinder, the cone, or a tangent surface; and ruled surfaces that can be generated by the motion of a straight line, like the cylinder and the hyperboloid of one sheet. While some of these surfaces date to antiquity, others are more recent. In the eighteenth century, Euler described the catenoid and Jean Meusnier the helicoid. Discoveries in 1835 by Heinrich Scherk and in 1864 by Alfred Enneper included minimal surfaces that are now named for each of them. In the 1840s, Joseph Plateau's experiments indicated that dipping a wire ring into soapy water will create a minimal surface. Jesse Douglas won a Field's Medal in 1936 for his solution to Plateau's problem in minimal surfaces. A minimal surface that originated at the end of the twentieth century is because of Celso Coasta in 1982.

Representations and Investigations

Algebraic geometers investigate algebraic surfaces, such as cubic or quartic surfaces that can be represented by polynomials. These led to rich mathematical investigations in the nineteenth and twentieth centuries. For instance, in 1849, Arthur Cayley and George Salmon showed that there were 27 lines on a smooth cubic surface. Quartic surfaces were of interest in optics, and mathematicians such as Ernst Kummer studied them.

While mathematicians had long built physical models of surfaces, which were typically housed in universities and museums, in the late twentieth and early twenty-first centuries, computer-generated surfaces revolutionized the visualization and construction of surfaces and led to many interesting mathematical questions. For instance, numerous mathematicians and computer scientists have explored the method of subdivision of surfaces, including Tony DeRose, and Jos Stam, who won a Technical Achievement Award in 2005 from the Academy of Motion Picture Arts and Sciences. This method often takes advantage of the similarity between the local structure of a surface and a small piece of a plane. For instance, surfaces may be represented using small flat triangle or quadrilateral mesh representations. These representations are easier to manipulate, but they can still appear smooth to the eye.

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See Also: Animation and CGI; Crochet and Knitting; Geometry in Society; Measurements, Volume; Polyhedra; Sculpture.

Surgery

Category: Medicine and Health.
Fields of Study: Data Analysis and Probability; Number and Operations; Representations.
Summary: Mathematical models can be used for various aspects of surgical operations in order to predict effects and improve recovery.

Surgery is the branch of science that typically involves medical treatment through an operation. There are a variety of reasons for why a surgery is performed. Diseases such as cancer and various forms of heart disease may be treated through surgical procedures.

When cancer is found, surgery may be performed to remove a tumor in order to reduce the likelihood of the cancer spreading or to alleviate pressure caused by a tumor pressing against another organ. Heart surgeries include a heart transplant, a coronary artery bypass, or a heart-valve repair or replacement. Injury is usually treated with surgery when the body is unable to repair itself. Torn ligaments or tendons can be surgically treated through reattachment or replacement. Burn victims can be treated with a skin graft as a permanent replacement for the damaged skin. Deformities can also be treated with surgery. Spinal fusion surgery can be used to treat spinal deformities like scoliosis. A cleft lip and palate is a fetal deformation that can be corrected with surgery soon after birth. The surgery usually involves an incision with medical instruments. Surgical incisions can vary in size from large incisions, such as in some open-heart and brain surgeries, to tiny incisions, such as in the case of laparoscopic procedures. The advantage to smaller incisions is that the wound heals faster, leaves a smaller scar, and reduces the likelihood of infection.

Mathematically Modeling Surgery

Improving surgical techniques and devising new surgical procedures is an active area of research. Multidisciplinary approaches are required for developing successful techniques and procedures. The National Institutes of Health (NIH) has stated that these approaches should include computational and mathematical simulations to facilitate this biomedical research. Simulations can be accomplished, in part, through mathematical modeling.

Mathematical modeling is the process of using a mathematical language in order to describe, in this case, a biological phenomenon treated with a biomedical procedure. While any mathematical model is a simplification of reality, computational solutions of the mathematical model may provide useful insights for researchers and clinicians when the model has been formulated under biologically and physically sound principles with realistic treatment strategies.

In order to develop a mathematical model for surgical treatment, it is common to have a team of researchers work closely on a given problem because of the different areas of expertise needed to address what is likely a complex biomedical problem. The first step is for the researchers or clinicians to define, as clearly as possible, the problem or question that they want the modeler to analyze and identify the benefits of the modeling project. From there, they should work to determine the appropriate scales with which to study the problem. Is the most appropriate scale at the molecular, cellular, or tissue level or is it more of a systemic problem? Is the time scale (if there is assumed to be temporal variation) on the order of minutes, days, or years? Answers to these questions will help determine if the model is best described in terms of discrete units or continuous variables, whether temporal or spatial variation should be included, and if a deterministic or statistical approach is more appropriate. This will also help determine what computational platform and method might best be used to analyze the question at hand. Developing a model diagram can help visualize the model formulation and process. From there, the research team needs to decide how best to access the quality of the model. From the model formulation, what are the assumptions and model limitations? Are the assumptions biologically reasonable? Are there data that can be used to quantify some of the model parameters? Are there data that can be used to quantitatively or qualitatively compare to the initial model simulations? If the research team is at the beginning stages of a new surgical treatment, can the model be developed to put forth hypotheses tested for animal experiments or clinical trials? The mathematical model works best as an iterative process when the modeler and experimentalist or clinician exchange ideas. The first set of suggested guidelines for biomedical research teams with mathematical modelers was proposed for the mathematical modeling of acute illness.

Applications

In connection with surgical procedures, mathematical models can be used in a variety of ways. Mathematical models may be used to predict the likelihood of a surgical procedure's success. For example, in reconstructive microsurgery (where skin tissue is moved from one location to another), a mathematical model was developed to predict successful tissue transfer based on oxygen delivery, tissue volume, and blood-vessel diameter. Mathematical models can be used to explore ways of making existing surgical procedures more successful. Stents are tubes inserted into a blood vessel (or another tubular body part) to keep open the vessel but are associated with a higher risk of a heart attack or a blood clot. A mathematical model was developed to analyze drug delivery to stent locations where two or more arteries meet.

Mathematical models can be developed to analyze controversial questions. For example, a model was developed to analyze whether a more liberal or constrictive allowance of fluid level allows for a more successful recovery from abdominal surgery. Mathematical models can be used to predict changes following surgery. A model was developed to predict changes in the knee joint following a wedge osteotomy, which is the removal of a wedge region of bone around the knee. The model was validated by predicting the results of 30 patients undergoing the surgical procedure, then the results were compared to actual measurements 14 months after surgery. In spite of these efforts, mathematical modeling of surgical procedures (and questions related to the surgical procedures) is still a relatively new concept.

Ideally, mathematical models can be used on individual patients to predict a likely and optimal outcome when considering surgical treatment. With advances and improvements in imaging techniques and computer software, this may be possible for treatment of some diseases and injuries. For example, when dealing with more complex arterial geometries near stent locations, researchers may be able to predict appropriate drug treatment strategies. However, in the absence of patient-specific models, mathematical models may be used to help clinicians make decisions based on patient variability. Patient variability implies that although there are differences in individual patients, there may be common characteristics in subpopulations of patients with a similar disease or injury. These common characteristics might be measured in common biomarkers from urine or blood analysis or similarities in imaging analysis. Mathematical models can be used to investigate surgical treatment strategies for patients with similar characteristics.

Mathematical models can also help with the development of new treatment surgical protocols or the analysis of existing treatment strategies. When exploring these questions, it is common for researchers to conduct experimental trials on animal models. However, animal experiments can be time consuming and costly. Mathematical models used to analyze a given question can provide a significant cost savings. For example, computer simulations of the model can be used to initially screen different experimental trials in order to decide which ones are worth pursuing and which are not. Furthermore, successful experimental results on animals do not guarantee the same level of success in clinical trials on humans. Mathematical models can not only give an idea as to how experimental trials on animals translate into surgical treatment on humans but also provide necessary insights when animal experiments are not possible or clinical trials on humans are unethical.

To help address the many questions that arise from current surgical procedures and the development of new surgical methods, an interdisciplinary team of researchers is required to formulate and analyze the problem at hand. It has been suggested that the team include mathematical modelers. Mathematical modeling can potentially provide a way to investigate novel treatment strategies and predict possible problems that may arise for a given surgical procedure. Furthermore, there exists the possibility of significant cost savings, in part, by reducing the number of animal experiments or clinical trials performed. In these ways, the mathematics underlying the description of the biology can be beneficial to the surgeon or biomedical researcher.

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See Also: Mathematical Modeling; Medical Imaging; Transplantation.

Swimming

Category: Games, Sport, and Recreation.
Fields of Study: Data Analysis and Probability; Measurement; Number and Operations.
Summary: Swimming performance can be modeled and improved mathematically.

Mathematical modeling and statistical analysis have been applied to swimming in a variety of ways. Modeling the properties of fluids in motion is the subject of fluid dynamics, a sub-branch of mechanics. Placing objects in the fluid complicates the physics enormously. The interaction of the fluid and object at the point where the object meets the fluid (called the "boundary") is of particular interest. Problems studied in this way include why flags "flap" in a breeze and how fish swim. Statistical analysis has been applied to a number of questions about swimming performance, including the prediction of future world record times, the modeling of deterioration in swimming performance as a function of age, and the evaluation of whether triathlons are fair to swimmers.

Improving Human Performance in Swimming

Modeling human swimming presents serious challenges for researchers. The use of arms and legs to propel the swimmer through the water adds complexity to the fluid dynamics models. Because the human swimmer is not completely immersed in the water but keeps part of the body above the surface, the interaction between the swimmer and the surface is particularly difficult to model. Researchers have applied smoothed particle hydrodynamics to the study of human swimming performance which, unlike traditional fluid dynamics, treats fluid flow as the motion of individual particles. This method enables researchers to more accurately model and simulate the interactions of the swimmer at the surface. The goal of this research is to help individual swimmers improve their performance in competition.

Predicting World Record Swim Times

Statistical analysis of human swimming performance encompasses a number of different approaches and methods. An analysis of world records in swimming from 1960 to 2010 shows a nearly steady decrease in times, resulting in between 15% to slightly more than 25% improvement, depending on the event. The question remains how long times can continue to decrease, how much is because of increased participation in swimming (especially women's swimming), and how much is because of advances in technique and conditioning.

Predicting the Swimming Performance of Aging Swimmers

On a different tack, Ray Fair modeled the performance of elite swimmers of different age groups and modeled the performance at various swimming distances by age. For example, he predicted that a 60-year-old will swim a time about 10% slower than the swimmer had done at age 35, while a 70-year-old will be 25% slower.

Are Triathlons Fair to Swimmers?

Richard De Veaux and H. Wainer investigated the relative disadvantage of swimmers to runners and cyclists in a triathlon. Because the times taken for the three events are so different, they argued that the standard triathlon proportions (including the Iron Man and Olympic triathlons) are grossly unfair to swimmers. The best marathon runners in the world take about two hours, seven minutes to run the 26.2-mile marathon (with variation due to course and weather). An elite cyclist can cover about 60 miles in the same time, and an elite swimmer can travel 7.5 miles. Thus, to be fair in terms of average time taken, a triathlon based on a marathon should also contain a 60-mile bike leg and a 7.5 mile swim. In reality, the Iron Man is a 26.2-mile run, a 112-mile bike leg, and only a 2.4-mile swim, and thus disadvantages swimmers enormously.

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Richard De Veaux

See Also: Data Analysis and Probability in Society; Mathematical Modeling; Tides and Waves.

Symmetry

Category: Architecture and Engineering.
Fields of Study: Geometry; Measurement.
Summary: An ancient mathematical concept, there are various forms of symmetry.

"Symmetry," which comes from the Greek word roots meaning "same" and "measure," describes a picture, shape, or other object that looks the same when viewed from another perspective or that can be transformed in some way without changing its important properties. The word "symmetry" can refer to this property, to the transformation itself, or more holistically to an aesthetically pleasing sense of balance. Eighteenth-century mathematician Adrien-Marie Legendre revolutionized the concept of symmetry when he connected it to transformations. There are a wide variety of uses of the word "symmetry" in different domains, including art, architecture, and science, and many of these have existed from antiquity. The concept of symmetry is inherent to modern science and architecture, and its evolution reflects in many ways the dynamic nature of these fields.

Visual Symmetry

In the context of geometric figures drawn in the plane, there are three fundamental types of symmetry:

1. A figure has "reflection" symmetry if it coincides with its own mirror image across some line. The capital letters M and W have a single reflection symmetry, while the letter H has two symmetries, horizontal and vertical.
2. A figure has "rotational symmetry" if it can be rotated around a fixed point, leaving the figure unchanged. For example, the capital letters N, Z, and S are unchanged when rotated 180 degrees. The pattern of black squares in traditional crossword puzzles also has this half-turn symmetry.
3. A figure has "translational symmetry" if it can be slid or moved without changing. A typical example is a repeating pattern on wallpaper.

Construed in the broadest terms, symmetry plays a role in almost all art and is related to balance and harmony. One of the many ways in which the narrower geometric notion of symmetry applies to art is tessellations. A tessellation is a covering of the plane by copies of a limited set of tiles. Such figures are often highly symmetric. Tilings by squares, hexagons, and triangles are common enough, both in art and on kitchen floors, and more fanciful tessellations involving animal and plant shapes are also possible. Tessellations, dynamic symmetry, and mathematical sophistication are especially evident, for example, in the art of M.C. Escher (1898–1972).

Abstract Symmetries

Symmetry is not just a geometric concept. Any structure or object can have symmetry. Abstractly, a symmetry is any transformation of an object resulting in an object that is "the same" in the sense of having all the same properties that are important in context. Often,

Many plants are radially symmetric with almost identical petals or leaves growing at regular intervals.

the object is a geometric figure, and the relevant properties are length, angle, and area, but it need not be so.

Consider the game rock-paper-scissors. Renaming the scissors gesture to "paper," renaming paper to "rock," and renaming rock to "scissors" would leave the rules of the game unaltered. This is an abstract symmetry of the game. Then, there are enough symmetries to identify any move with any other, so all three options are intrinsically "equally good." In this example, there is symmetry but no geometry whatsoever.

Symmetry and Groups

In higher mathematics, notions of symmetry are expressed in the language of group theory. A "group" is a set (G) of objects that can be composed together (in other words, if x and y are elements in a group, $x \times y$ is also an object in the group), subject to three conditions: associativity, identity, and inverse criteria. The salient feature of this definition is that the set of all the symmetries of any object satisfies these conditions. The associativity property is automatic from function composition; but what about the other two? These are restatements of the convention that the transformation that does nothing is a symmetry and the idea that symmetries are "undo-able." Symmetries leave an object "structurally the same as it was," so there will always be another symmetry to undo any given symmetry.

The symmetries of any object that preserve any desired features form a group, called the "symmetry group" of the object. Often, one can understand a complicated object much better by studying the size and structure of its symmetry group.

Klein and the Erlangen Program

Felix Klein (1849–1925) greatly strengthened the connection between geometry and group theory. His insight was that, if one really wants to understand a geometric structure, then one should study the group of symmetries that preserve the structure. This philosophy has proved very fruitful and is now known as the Erlangen program.

For example, in ordinary Euclidean plane geometry, the focus is on lengths and angles. The group of symmetries that preserve lengths and angles consists of translations, rotations, reflections, and combinations of these. Given any two points, each with an arrow pointing away from it in a given direction, one can always translate and rotate the plane so that the image of the first point lies on the second point, and the arrows are pointed in the same direction. This is the sophisticated way to understand the notion that every point and direction in the plane are functionally the same as every other point and direction.

The Erlangen program has played a fundamental role in the development of nineteenth- and twentieth-century geometric thinking, clarifying the relationships and distinctions between geometry and topology; projective and affine geometry; and Euclidean, hyperbolic, and spherical geometry.

Symmetry and the Universe

Those who study the shape of space are greatly concerned with symmetry. Consider the question of whether the universe is "homogeneous." That is, do the laws of physics treat every place the same as every other place? Is every direction physically like every other direction? What answers to those questions are believed to be correct determines what shapes, structures, and geometries are viable candidates to model the universe.

Time symmetry is another issue of importance in physics research; one wants to know to what extent the physical laws of the universe treat the past and future symmetrically. On a small enough scale, particle interactions have time symmetry. If one watched a "movie" of particle interactions on a small enough time-scale, it would be impossible to tell whether the movie was playing forward or backward. On the other hand, the large-scale events observed in everyday life do not possess such past-future symmetry; for example, eggshells break but do not spontaneously assemble, people age but do not become more youthful. This discrepancy between small-scale symmetry and large-scale asymmetry is rather mysterious, and one can hope that reconciling the two will lead to greater understanding of physics.

Symmetry and Architecture

Symmetry has long been connected with architecture. In Greek and Latin, symmetry was used to indicate a common measure or a notion of something well-proportioned, rather than as a reflection. However, reflection symmetries can be found in many buildings from different cultures, where the left side is a mirror image of the right side. Architects have also used symmetry in external views, layout, stability, or building details, such as stairs or windows. Some authors claim that the first recorded instance of the use of symmetry as a mirror reflection was in 1665, when Gian Lorenzo Bernini was asked to design an altar for the church of Val-de-Grace, while others assert that it was first found in Claude Perrault's 1673 treatise on columns. Perrault is best known as the architect of the east wing of the Louvre.

Concepts such as the symmetry groups of the plane also originate in architecture. Beginning with mathematician Edith Muller's 1944 analysis, experts continue to debate how many of the 17 groups can be found in the mosaics of the Alhambra at Granada, a fourteenth-century Moorish palace. Some assert that all 17 can be found there and in many other examples in Islamic art and architecture. A formal mathematical proof that there are no additional symmetry groups was proven independently by Evgraf Fedorov in 1891 and George Pólya in 1924. Partly because of a prohibition against using anthropomorphic forms, symmetry appears in many instances of Islamic-influenced architecture, such as the Taj Mahal.

The connections between symmetry and architecture continue into the twenty-first century. In numerous texts in the twentieth and twenty-first centuries, mathematicians such as Hermann Weyl illustrate concepts using architectural references. Architects and engineers also frequently use symmetry, though architects working in the modernist aesthetic reject symmetry in their designs.

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Michael "Cap" Khoury

See Also: Escher, M.C.; Geometry and Geometry Education; Geometry of the Universe; Puzzles; Sudoku.

Synchrony and Spontaneous Order

Category: Weather, Nature, and Environment.
Fields of Study: Algebra; Connections; Data Analysis and Probability.
Summary: The world is filled with examples of spontaneously emerging order.

Humans are familiar with order: people order homes by placing belongings in one place; people also watch football games with players who follow orders given by a quarterback who directs the play. There are many examples of order in nature. Birds and fish order themselves by flying in flocks and swimming in schools. How is order created in a complex system with many parts? Experience indicates that order emerges from the actions or directions of a leader, just as the quarterback is the leader of a football team. It is possible, however, for a system to be ordered without the help of a single leader—an attribute that occurs in a spontaneously ordered system. Systems have a global (group) level and a local (individual) level. A school of fish is made up of thousands of individual fish, and a laser is a collection of particles of light (photons) that are emitted from trillions of atoms. When a system is spontaneously ordered, the order occurs because of local level interactions without global level direction. Imagine a spontaneously ordered football team. The quarterback on this team does not need to direct or call a play. This team is able to organize and execute plays simply by communicating with each other (individually) as each play unfolds. There will likely never be a team like this, but spontaneously ordered phenomena are all around if one knows where to look.

When multiple events are ordered in time, the result is "synchrony." Without synchrony, life would be very different. People would not enjoy watching a football game with unsynchronized players who run in different directions after—or before—the ball is snapped. Many of the technological devices that people use, including GPS, cell phones, and lasers, rely on synchrony to work properly. Scientists have even published evidence of synchrony in cloud patterns. When spontaneous order occurs, the result is often synchrony. Mathematicians and statisticians are involved in the collection of data that help define important variables related to synchrony and fuel the development of theories and models, as well as the formulation of mathematical models to describe and explain synchrony. This work draws from many areas of mathematics, including logic, probability, decision theory, geometry, and statistics, as well as related scientific fields.

Examples of Synchrony and Spontaneous Order

In some regions of Southeast Asia, large numbers of male fireflies flash on and off at the same time, creating a spectacular array of synchronized lights. It is believed that the males are flashing in unison to attract females. Physiologically, these fireflies have an internal firing mechanism that can generate a rhythmic flashing sequence. Experiments with individual fireflies demonstrate that the timing of their flashes can be altered to mimic that of an external stimulus, which is flashing rhythmically. This suggests that synchronized firefly flashing is the result of a spontaneously ordered process. To test this hypothesis, mathematicians Renato Mirollo and Steven Strogatz created a simple mathematical model by using an equation to describe an individual firefly as a biological oscillator (just as a plucked guitar string is a mechanical oscillator). They coupled multiple, identical oscillators together to form a system. Their mathematical model is a system of coupled differential equations. Mirollo and Strogatz analyzed the system and proved that in almost all cases, no matter how many oscillators there are or how the oscillations are started, synchrony is the result.

Fish often travel in schools. One advantage of this behavior is to allow fish to better avoid predators by performing highly synchronized, evasive maneuvers. Experimental data suggests that schooling fish have a preferred distance, elevation, and orientation relative to their nearest neighbor. Scientists Andreas Huth and Christian Wissel have modeled fish schooling as a spontaneously ordered system. They assume that schooling originates not because of a particular fish directing the group's movements but because of simple behavioral rules for individual fish. Their assumptions include that each fish desires to be close (but not too close) to another fish, each fish moves according to its perception of the position and orientation of neighboring fish, and individual fish movement is random. Huth and Wissel tested different movement rules for their model since there are no data that supports specific movement rules for schooling fish. They used the data generated from computer simulations of their model to determine the average direction of movement as a group and the average angular deviation by individual fish from the group's direction, which is defined as the "polarization" of the school. The polarization is a way to quantify the synchrony of the school because the larger the polarization, the more disoriented the school is. Since polarization depends on the movement rules, they used polarization to find movement rules for which their model best simulated synchronized schooling.

A fluorescent light bulb consists of a long tube filled with an inert gas. The light that we observe originates from the atoms in the gas. Each atom has multiple electrons that exist at specific energy levels. Electricity forces electrons through the tube and these electrons collide with the atoms in the gas. The collision raises the energy level of the atom's electrons, which then spontaneously revert back to a state of lower energy. This loss of energy causes a light particle (photon) to be emitted and the light that we see is from the emission of millions upon millions of photons. The light from a fluorescent light bulb consists of many different wavelengths and is scattered in many directions. Alternatively, the light from a laser, which stands for "light amplification by stimulated emission of radiation," is highly synchronized with a single frequency, direction, and phase. The first laser was constructed in 1960, but in 1917, Albert Einstein developed the quantum physics that predicted how a laser is able to synchronize the photons. When lasers were invented no one knew what to use them for.

Today, laser light is used for everything from grocery store checkout scanners to eye surgery. Just as with fluorescent light, raising and lowering the energy levels of individual electrons generates the light from a laser. An external energy source (such as electricity) continually stimulates electrons and raises them from lower energy states to higher energy states. Initially, when the laser is turned on and some electrons spontaneously fall back to their lower energy states, the emitted photons move in random directions. But a laser has mirrors at both ends and the photons are trapped between the mirrors for a long period of time before they can escape. Furthermore, a laser is constructed so that the photons will perfectly synchronize and amplify a light wave with a specific frequency and direction while filtering out the other light waves. One of the mirrors allows some of the light to escape in the form of a laser beam, an example of synchrony that we encounter each day.

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John G. Alford

See Also: Clouds; Light; Mathematical Modeling; Mathematics Research, Interdisciplinary.

T

Tao, Terence

Category: Mathematics Culture and Identity.
Fields of Study: Number and Operations; Representations.
Summary: One of the most accomplished contemporary mathematicians, Terence Tao is a groundbreaking number theorist as well as a popular blogger.

Terence Chi-Shen "Terry" Tao (1975–) is a South Australian–born mathematician. Some rank Tao among the greatest living mathematicians in the early part of the twenty-first century. A child prodigy, he was taking college-level mathematics classes as early as 9 years old and was awarded a Ph.D. from Princeton University at the age of 20. He is, as of 2010, a professor at the University of California, Los Angeles. As a mathematician and writer, he is extremely productive and has contributed elegant solutions to difficult problems in diverse areas in mathematics. His primary research interests are analytic number theory, harmonic analysis, combinatorics, and partial differential equations.

Honors and Contributions

Dr. Tao's contributions to mathematics, and his awards for them, are numerous. In 2006, Terence Tao was awarded the Fields Medal. The Fields Medal is sometimes called the "Nobel Prize of mathematics" and is generally regarded as the most prestigious award in mathematics. It is awarded once a year for superlative achievement by a mathematician up to the age of 40. At the age of 13, Tao won a gold medal at the International Mathematics Olympiad, an annual competition intended to challenge the world's brightest students of high-school age; as of 2010 he remained the youngest person to ever win such a gold medal. His accolades also include the Salem Prize, the Clay Research Award, the SASTRA Ramanujan Award, the Australian Mathematical Society Medal, and the King Faisal International Prize. Tao has been pleased with his success, but he would like to continue focusing on mathematics research rather than reflect on his achievements.

One particularly significant contribution by Terence Tao to number theory came in 2004. In joint work with Ben Green, he proved a remarkable result about arithmetic progressions of prime numbers. An arithmetic progression is a sequence of numbers with a constant difference between them. For example, $5, 11, 17, 23, 29$ is an arithmetic progression with length 5 and constant difference 6. The five numbers in the sequence are prime. Green and Tao proved that it is possible to find arithmetic progressions of five primes, or 50 primes, or 50,000 primes. Indeed, they showed that arithmetic progressions of primes exist that are as long as desired. Understanding the distribution of the prime numbers is of paramount importance in number theory, and results of this type are often notoriously difficult to establish.

Communication and Strategy

In addition to all his papers and books, Terence Tao is a very well-respected and prolific blogger. On the "What's New" blog, Terence Tao frequently posts remarks on his ongoing projects, links to and commentary on current articles, and other mathematical topics. There are numerous active mathematical blogs at all levels of sophistication, but many consider "What's New" to be the "grandfather" of mathematical blogging. "What's New" is considered by many active mathematicians to be an important and influential source of information. As of 2010, the American Mathematics Society had published two books of excerpts from his blog.

While he has been described as the "Mozart of Math" because of his creativity and the mathematics that seems to flow out of him, Tao attributes his success to strategies that enable him to break up difficult problems into easier ones. Often, he focuses on one question at a time and tries a variety of techniques. He stated: "When I was a kid, I had a romanticized notion of mathematics—that hard problems were solved in Eureka moments of inspiration. With me, it's always, 'let's try this that gets me part of the way. Or, that doesn't work, so now let's try this. Oh, there's a little shortcut here.'"

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Michael "Cap" Khoury

See Also: Number Theory; Mathematics, Elegant; Mathematics, Theoretical.

Tax

See *Income Tax; Sales Tax and Shipping Fees*

Telephones

Category: Communication and Computers.
Fields of Study: Geometry; Measurement; Number and Operations; Representations.
Summary: Mathematicians have played key roles in efficiently managing telecommunication networks and developing newer and more powerful phones and wireless networks.

Inventors Elisha Gray and Alexander Graham Bell independently designed devices to electrically transmit speech in the 1870s; however, Bell patented his device first. The American Bell Telephone Company created the first telephone exchange in 1877. A subsidiary company, American Telephone and Telegraph (AT&T), was incorporated in 1885 to develop and implement long-distance telephone service, and Bell Laboratories was founded in 1925 for research and development. Later, the labs would be managed by both AT&T and Lucent Technologies. The telecommunications industry has long relied on the contributions of mathematicians for its success and hundreds of companies continue to employ mathematicians to address the increasingly complex problems of twenty-first-century communication. They develop technology and algorithms for wired and wireless communication, which facilitate speed and efficiency for a variety of applications. They also research ways to increase security and prevent unauthorized listening or wiretaps. Some create business models or study issues such as customer satisfaction.

Finite Phone Numbers in an Expanding Communication Network

One notable mathematical problem of the early twenty-first century is assignment of phone numbers. In 2007, the Federal Communications Commission stated that 582 million of 1.3 billion available phone numbers in the United States were already assigned, increasingly to cell phones. Some mathematical models have suggested that exponential growth would exhaust the supply. Similar concerns have been raised about social security numbers and Internet addresses, since the number of digit permutations for any given string length is finite.

Contributions of Mathematicians

Many mathematicians have contributed to telephone systems. For example, George Boole, whose Boolean algebra was used in switching systems; Oliver Heaviside, who adapted complex numbers to study electrical circuits and worked on long-distance systems; and Agner Erlang, who modeled phone call waiting time using probability theory, in collaboration with the Copenhagen Telephone Company. Mathematical modeling has been used to design and study telecommunications systems since the beginning of the twentieth century. Many mathematical and scientific advances, both theoretical and applied, were developed by the multidisciplinary working groups at Bell Labs, AT&T, and Lucent, which included notable mathematicians and at least one Nobel Prize winner. Examples of significant advances with applications both within and beyond telephones include transistors, solar cells, lasers, satellites, the Unix operating system, the C programming language, and digital signal processing chips. Mathematician Claude Shannon is often referred to as the "father of information theory," which he developed while at Bell Labs. William Massey created performance models for telecommunication systems using queuing, stochastic methods, and special functions, and he has cited Bell Labs as especially supportive to minority mathematicians and scientists.

Cell Phones and Smartphones

Mathematical methods were also important in the development of cell phones and smartphones. In 1947, Bell Labs engineer W. Rae Young suggested a hexagonal tower arrangement for cellular mobile telephone phone systems, which was expanded upon by engineer Douglas Ring—though the technology did not exist to implement the idea until the 1970s. The Motorola DynaTAC 8000x, released in 1983, was the first truly portable cell phone. Cellular technology proliferated rapidly, and society has widely embraced smartphone technology. Described as a new generation of telephone, smart phones are, essentially, computers small enough to fit in a palm or pocket. The IBM Simon Personal Communicator, created by IBM and BellSouth and sold beginning in 1994, is cited as the first smartphone, while at the start of the twenty-first century, Apple's iPhone and the Motorola Droid are very popular. The Android open-source operating system, which forms the basis for the Droid smartphones, was invented by computer scientist Andrew Rubin. It has been compared to Lego system building blocks because of the structure of its software solution stack, which many consider to be more compatible than the discretely packaged and isolated programs of some other operating systems.

Mathematics has played an increasingly large role in cell phone and smartphone service. Smartphones not only serve as cell phones for verbal or textual communication, they also play media, provide access to the Internet, serve as GPS and navigation devices, and run various other software. Electronic signals from smartphones carry digitized speech and data, requiring mathematical algorithms to construct and compress information, as well as to correct errors. Mathematicians and information theorists, such as David Huffman and Jorma Rissanen, developed compression techniques using concepts from probability theory and entropy. Mathematical methods from signal processing and graph theory prevent interference between multiple callers and help to establish networks that provide uninterrupted coverage. The International

Mobile Telecommunications-2000 or 3G (third generation) is a global standard for mobile telecommunications introduced in 2000. It addresses critical issues such as data rates, bandwidth, frequencies, broadband compatibility, and issues of authentication, confidentiality, and privacy. As of 2010, scientists and mathematicians were developing further standards for mobile networks and devices, including a next generation 4G network. The Open Handset Alliance is a group of companies that develops and advocates for open standards for mobile devices.

Apps

As smartphone popularity booms, so do the tools developed for smartphones by computer scientists and others. Downloadable applications (commonly called "apps") are readily accessible for free or for purchase.

Many of them are aimed at education or academic subject areas, including mathematics. One set of apps offers the opportunity to practice with mathematics concepts and skills, like Math Flash Cards and Advanced Mental Math. Gamer-style apps like Math Ninja require players to answer challenge questions to advance. Other apps are electronic versions of mathematically based board games, like Mancala and Dominoes, while the popular game Tetris involves performing geometric transformations quickly to stack variously shaped objects, which is related to classical packing problems.

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Norma Boakes

See Also: Cell Phone Networks; Fax Machines; MP3 Players; Software, Mathematics; Solar Panels; Wireless Communication.

Telescopes

Category: Space, Time, and Distance.
Fields of Study: Algebra; Geometry; Number and Operations.
Summary: Image clarity in telescopes is achieved through extremely precise measurements and mathematics.

In 1608, the Dutch lensmaker Hans Lippershey applied for a patent on what was soon named a "telescope." It is not clear if Lippershey was the true inventor; at least two other Dutch lensmakers also claimed credit. The news of this new invention quickly spread. In 1609, Galileo Galilei in Italy started using telescopes to observe heavenly objects. Among other findings, he discovered the rotation of the sun, the phases of Venus, and the first four satellites of Jupiter. A mathematician as well as a physicist and astronomer, Galileo also used geometry to measure the heights of lunar mountains by determining how long they remained illuminated after the lunar sunset.

Other mathematicians and physicists helped develop the modern telescope. Isaac Newton determined that lenses acted like prisms in spreading out the spectrum of visible light (a phenomenon known as *chromatic aberration*). Newton and the mathematician James Gregory independently invented the reflecting telescope, which does not have this problem. Leonhard Euler made a mathematical analysis of chromatic aberration, and in England so-called achromatic lenses (a combination of two lenses that together bring light of different colors to a focus) were invented in the early eighteenth century.

Optics

A telescope is an optical device for seeing objects that are either far away, or very dim, or both. Consider a typical magnifying glass, as shown in Figure 1, which is a piece of glass or other transparent substance shaped so that both sides are sections of spheres. Light rays from an object (such as a candle) come to a focus on Screen 1. In other words, light rays from any given point on the candle converge onto a single point of Screen 1, forming an image. Screen 2 is at the wrong distance, meaning the light rays do not converge properly on Screen 2. Screen 1 is said to be "in focus," and Screen 2 is "out of focus."

Figure 1.

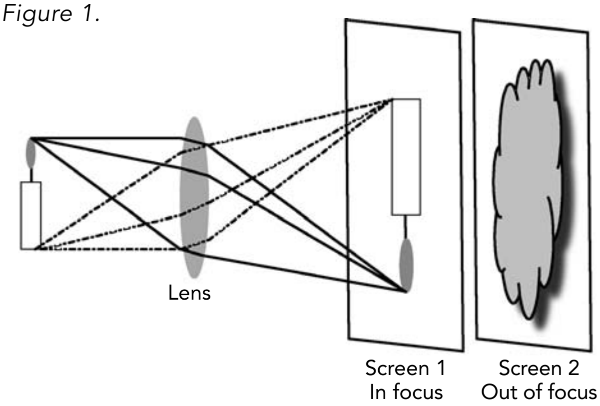


Figure 1a.

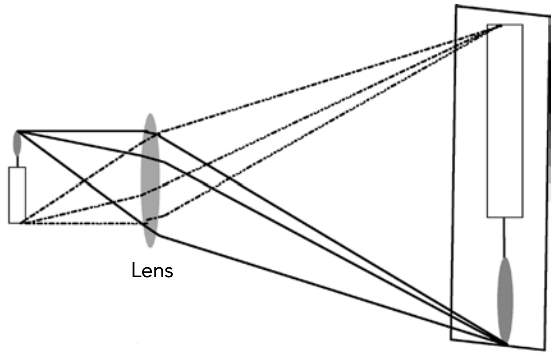


Figure 1 repeated, but with a lens with twice the focal length.

The first lens the light goes through is the called the "objective lens." The plane (Screen 1) where the image is in focus is called the "focal plane." The "focal length" is the distance to the focal plane for a source at infinity (incoming parallel rays).

Magnification is measured in diameters. If the image is twice as tall and also twice as wide as the original, then there is a magnification of two diameters. Some optical devices, however, are measured in power, which is the square of the magnification in diameters; for example, a microscope advertised as "100 power" actually magnifies 10 diameters.

Figure 1a shows the same configuration as Figure 1, but the focal length is twice as long. The image on the focal plane is thus twice as high and twice as wide—it is magnified twice as many diameters. Since the image is spread out over four times the area, it is only one-fourth as bright. Conversely, for a given focal length, doubling the size of the objective lens lets in four times as much light, hence the image is four times as bright.

In Figure 1, if one were to put a light-tight box around screen 1, set up a shutter to control when light enters the box, and replace screen 1 with photographic film, the result is a camera. Replace the photographic film with an electronic light-sensitive screen, and the result is a digital camera. If the camera is used to take pictures of far-away or dim objects, then it qualifies as a telescope.

Astronomical Telescopes

Since astronomers are interested in dim celestial objects, a big objective is necessary for astronomical telescopes. Amateur astronomers frequently use a 6-inch (15-cm)

Figure 2. Eye plus magnifying glass.

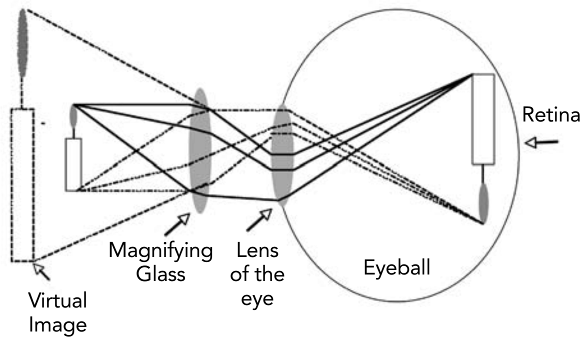
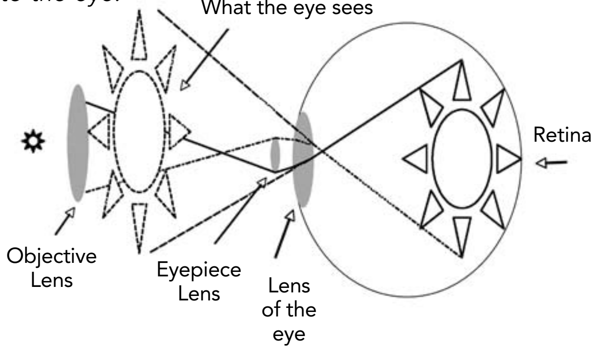


Figure 3. How a telescope delivers an enlarged image to the eye.



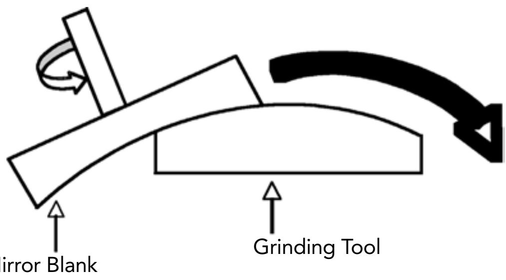


Figure 4. Grinding a mirror by hand (curvature greatly exaggerated).

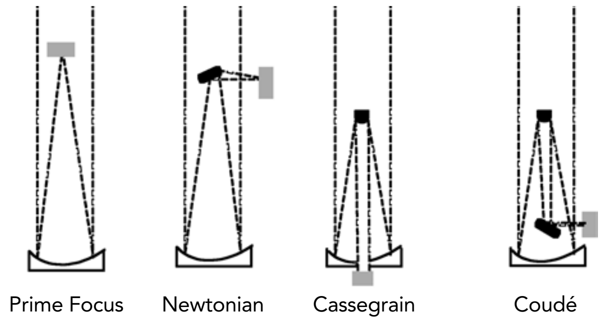


Figure 5. Four designs of reflecting telescopes (Gray bar shows focal plane).

objective as a good compromise between light-gathering power and cost. Professional astronomers rarely use objectives less than about half a meter (1.5 feet) in diameter. The largest objective lens in the world as of 2010 is 40 inches (1.106 meters) at Yerkes Observatory in Wisconsin.

The eye has its own lens, and the telescope has two lenses (or sets of lenses): the objective and the eyepiece. Figure 3 shows how the two-lens telescope delivers a greatly magnified image to the eye. The magnification in diameters is equal to the focal length of the objective divided by the focal length of the eyepiece. For example, the 40-inch telescope at Yerkes has a focal length of 744 inches. With a one-inch eyepiece, this telescope magnifies 744 diameters.

A microscope operates in the same way, except that the object being viewed, instead of distant and dim, is well lit and close to the objective lens.

Diffraction and Refraction

The useful magnification of a telescope is limited by diffraction. Light rays at the edge of the objective lens are diffracted—they are bent around the edge of the lens. These diffracted light rays cause a pattern of light and dark circles around bright images, which will blur adjacent images together. An empirical formula traditionally used to specify the limit of useful magnification is the Dawes Limit (also called the Rayleigh Limit): the resolution in arc-seconds is $4.56/D$, where D is the diameter of the objective in inches; or $11.6/D$, where

Non-Optical Telescopes

Gravitational fields bend light, as predicted by Albert Einstein's general theory of relativity. Hence, large gravitational fields act as lenses. The first test of general relativity was during a solar eclipse in 1919, when the effect of the sun's gravity was to make stars very near the sun's edge appear to be at a small—but measurable—angle further away from the sun than when they are viewed when the sun is not almost in front of them. In effect, the sun acted as a lens and magnified the image of the area around the sun. There are no lenses for radio waves, but radio telescopes that observe radio waves from astronomical objects, such as quasars, do exist. Most radio telescopes use a metal parabolic mirror to reflect the astronomical radio waves to a receiver at the focus of the parabola.

There also exist what might be called *sound telescopes*. One variety, for picking up sounds from a distance, uses a parabolic dish to reflect sound waves to a microphone at the focus of the parabola. Ultrasound machines, used for monitoring pregnant women, use the woman's own bladder to focus the ultrasound waves onto the receiver.

D is in centimeters. For example, the diameter of the pupil of the human eye when dark-adapted is approximately 8 mm. By the Dawes Limit, the eye can resolve 11.6/.8 (14.5 arc-seconds), or about 1/125 of the diameter of the full moon. The Yerkes telescope can resolve about 0.1 arc-seconds.

A telescope using a lens as its objective is called a *refracting telescope*, since light is "refracted" (bent) by the lens. As of 2010, the 40-inch Yerkes instrument is the largest refracting telescope. A lens that size has to be thick to stand up to gravity, and thick lenses absorb so much light that beyond the size of Yerkes, absorption begins to outweigh the increased light gathered by a wider lens. Hence all current telescopes with objectives greater than 40 inches are "reflecting telescopes" in which the objective is a mirror rather than a lens.

Observer Placement

Unlike a lens, an objective mirror has a parabolic rather than a spherical surface. There is also the mechanical problem of where to place the observer or camera. There are several possibilities, some of which are shown in Figure 5.

One method, called "prime focus," places the photographic film (or other astronomical instrument) inside the path of the incoming light. A few very large reflecting telescopes, such as the 200-inch Hale Telescope at Mount Palomar, actually allow for a human observer to ride in a cage at the prime focus.

A more common arrangement, invented by Isaac Newton and called the "Newtonian," consists of a small flat mirror at an angle, which moves the focal plane to the side of the telescope. Two other common arrangements have a convex mirror at the prime focus, reflecting the light back down the length of the incoming light and also increasing the focal length. In the Cassegrain arrangement, a hole is cut in the middle of the mirror for the light to pass through. In the *coudé* arrangement, the light is reflected one more time into the mounting of the telescope, allowing the use of stationary instruments too heavy to be loaded onto the tube of the telescope.

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See also: Conic Sections; Digital Cameras; Planetary Orbits; Relativity.

Television, Mathematics in

Category: Arts, Music, and Entertainment.
Fields of Study: Communication; Problem Solving.
Summary: Television shows routinely help shape the public's view of mathematics and mathematicians.

Like many other academic disciplines, mathematics has found its way to the small screen in the form of children's educational programming, various puzzle challenges on reality television and other game shows, and mathematically talented characters on a variety of scripted shows. These categories of programming and their attendant themes help shape and reflect the public's image of mathematics and mathematicians at different times. It is important to note that television viewership is determined though the statistically based Nielsen ratings, which networks use to calculate advertising revenue. As a result, the fate of a show is often tied to its Nielsen ratings.

Some of these programs promote mathematics as an exciting learning area (often in children's educational programming) or as a technical skill, which can give characters power and control. Problematic stereotypes persist, especially the still-common portrayal of mathematicians predominantly as white men. The stereotype of the mathematically talented character as a "nerd" is also prevalent and suggests that popular television representations of mathematics reflect both respect for the technical knowledge and fear about an expertise sometimes portrayed as mystifying or as the exclusive domain of obsessive "geeks."

Children's Educational Programming

The focus in children's educational programming that addresses mathematics is often on encouraging children to be excited about the subject area, along with helping them master skills and gain understanding. Most notably, the Children's Television Workshop (CTW), founded in 1967, ultimately created or inspired much of children's educational programming. Funded by federal and private sources, CTW designed *Sesame Street* to teach letter and number skills, as well as foundations of critical thinking, to preschoolers. The program revolutionized children's programming when it premiered in 1969 and has been broadcast continually ever since. Its core focus is on educational content that is presented using attention-getting and retaining tactics, such as fast movement, humor, puppets, and animation. The Count, for example, is a flamboyant Dracula-like character who loves to count. A popular animated segment, "Pinball Countdown," taught children to count using an elaborate pinball machine. Mathematics is also contextualized in segments involving real-life skills, like going to the grocery store.

Studies suggest that *Sesame Street* is viewed by almost half of all U.S. preschoolers on a weekly basis, and there are at least 10 foreign-language versions that have been broadcast in more than 40 countries. Not only is mathematics presented in the show, but it has also been used in shaping decisions about content and presentation. A multidisciplinary team, including Edward L. Palmer, who held a Ph.D. in educational measurement and research design, systematically studied early episodes of the show using data collection and statistical methodology to address both appeal and content comprehension. Other researchers in the early 1970s, including the Educational Testing Services, found both gains in learning and improvements in attitudes toward school in children who watched *Sesame Street*, but at the time it did not help close the gap between some groups of children as had been originally hoped. A longitudinal study found that exposure in the preschool years was significantly associated with better grades in English, mathematics, and science in secondary school, though one cannot infer direct causality from such a study.

In 1973, ABC premiered *Schoolhouse Rock!* as short, musical cartoons aired in between full-length shows on Saturday morning. The show was reportedly inspired by David McCall, the chairman of a public relations firm, whose son had difficulty with multiplication tables but could easily recall song lyrics. In the "Multiplication Rock" series, multiplication of numbers was set to music. Though there is no song about 10, "My Hero, Zero" discusses powers of 10 and the importance of zero. "Little Twelvetoes" examines the base-ten numeral system by imagining a world in which humankind is born with 12 fingers and toes instead of 10. The series "Money Rock" and "Computer Rock" also included applied mathematical concepts. Teachers often show "Multiplication Rock" in their classrooms, and the series is available as both audio and video recordings.

Other shows featured mathematical content as well. In the late 1980s, Children's Television Workshop created the mathematics show *Square One*, which featured guest stars and explored mathematical concepts through segments that parodied aspects in popular culture. In 2002, PBS premiered *Cyberchase*, in which three children and their bird use mathematics to prevail against evil schemes to destroy Cyberspace. Other examples of mathematics in educational programming for various levels of students are *The Metric Marvels*, *Math Can Take You Places*, *Bill Nye the Science Guy*, and *Blue's Clues*. One notable addition to the family of education mathematics programs in the twenty-first century is Nickelodeon's *Team Umizoomi*. This series, which premiered in early 2010, mixes 2-D and 3-D animation with live action to create a virtual world in which a team of characters helps children solve problems. Like many modern television programs, *Team Umizoomi* has an accompanying Web site. According to Nickelodeon executive Brown Johnson, "Math surrounds us everywhere we go, which is why we wanted to create a fun, adventure-filled, interactive series that engages preschoolers and encourages them to practice and refine their mathematical thinking skills."

Reality Television and Game Shows

This spirit of mathematics as an adventurous challenge also appears in other programming, especially on reality television and game shows, which include it as a key test of skill. Often, players must solve a puzzle that falls under the umbrella of some classical problem from the fields of game theory or probability. Instances of mathematics in *Survivor*, *The Mole*, and *The Real World/Road Rules Challenge* have been examined and catalogued. *The Price is Right* has been used to study probability in the classroom, and *Friend or Foe* has been used to analyze and study the Prisoner's Dilemma in the classroom.

Another way in which mathematics is applied to reality television is through its application of voting theory. Many reality shows use formulas to calculate voting results. The fall 2010 season of *Dancing with the Stars* was marked by a controversy in which contestant Bristol Palin, daughter of 2008 Republican vice presidential candidate and former Alaska governor Sarah Palin, consistently received low scores from the judges and yet escaped elimination week after week. The controversy prompted the ABC network to, for the first time, specify its voting scheme on the show and explain it on its Website. Under the system, the judges' scores for each couple are recalculated as a percentage of the judges' total scores for that night. Then, the votes each couple receives from home viewers are calculated as a percentage of the total number of votes received for that week. These two percentages are added together, and the couple with the lowest combined total is eliminated. Palin's high percentage of the popular vote meant that her combined share was rarely the lowest. While reality programs typically refuse to reveal the exact number of votes contestants receive, as in highly popular shows such as *American Idol*, in the case of *Dancing with the Stars*, viewer curiosity and voting controversy prompted an unusually detailed discussion of the mathematics involved.

Nerd-Genius

Moving beyond such simple tests of skill, scripted series sometimes treat mathematics on a deeper thematic level. One common theme is the "nerd-genius." Since the late 1990s, mathematicians and scientists have more frequently been appearing as the unlikely heroes of shows ranging from police procedurals (*NUMB3RS*) to sitcoms (*The Big Bang Theory*), from animated shows (*Futurama*) to reality gamedocs (*Beauty and the Geek*). The increasingly positive portrayal of "nerd-genius" may reflect a greater acceptance of the Information Age and of technical expertise and knowledge as positive attributes.

The popularity of *The Big Bang Theory*, which premiered in 2007 on CBS, speaks to this larger fascination with the nerd-genius. The sitcom follows four young scientists, two of whom are physicists at the California Institute of Technology (one in experimental physics, the other in theoretical physics), a third who is a Caltech astrophysicist, and the fourth, who is an aerospace engineer at a NASA field center. By the show's third season, it was drawing over 14 million viewers per week and ranked in the top 15 shows. In 2010, it won a People's Choice Award for Favorite TV Comedy, and star Jim Parsons won an Emmy Award for Outstanding Lead Actor in a Comedy Series.

The characters are all teased for being socially awkward and obsessive about mathematics and science, as is typical for the stereotype. However, they are also lauded for their intellect and the program presents their thought processes as both humorous and fascinating. The program takes the scientific content seriously, retaining a UCLA physics and astronomy professor, David Saltzberg, to review scripts for accuracy and provide mathematical equations and diagrams. The show has addressed such topics as string theory, loop quantum gravity, and dark matter.

Mathematics is featured in educational shows, reality and game shows, and with mathematically talented characters.

Women and Minority Mathematicians

Whenever mathematicians are depicted on screen, some audience members may form (sometimes prejudicial) opinions about what mathematicians look like or how they act. Mathematicians are often presented as nerdy white men. There are possible downfalls of such limited portrayals. For example, Ron Eglash describes how the dearth of African-American geek characters in popular culture reflects and somewhat reinforces the stereotype that white male nerds are the gatekeepers to full participation in science and technology. But, to their credit, some television shows have made an effort to broaden the demographic range of their mathematical characters, including women and African Americans among their number.

There have been a few female characters with mathematical ability on television. Early examples include three characters from the *Star Trek: Voyager* (1995–2001) series: captain Kathryn Janeway, chief engineer B'Elanna Torres, and Seven of Nine, who was rescued from the Borg (and thus joining the series) in season four. Often, these characters discuss intricacies of twenty-fourth-century physics, including warp speed travel and altering the time line. The show situates these three women (and the Vulcan Tuvok) as leaders among their shipmates in terms of knowledge of and ability in physics, engineering, and mathematics.

Another woman character with mathematical talent is Winifred "Fred" Burkle on the show *Angel* (1999–2004), created by Joss Whedon as a spinoff of his popular *Buffy the Vampire Slayer* series. Though a physicist by training, Fred displays her talents in mathematics, engineering, and invention on the show. Moreover, her character is supported by most of the other characters on the show—she is seen as a key player on the team. The Fred character has also been used as a case study of how Hollywood representations impact girls' mathematical education.

The show *NUMB3RS* (2005–2010) contains another mathematically talented female character: Amita Ramanujan, a Southern Californian of Indian origin. Throughout the series, Amita was a Ph.D. student, then colleague and fiancée, of mathematician Charlie Eppes. Charlie's brother, Don, works for the FBI and uses Charlie's mathematical skills to help solve crimes. Amita and physicist Larry Fleinhardt form Charlie's problem-solving team and inner social circle. As with Fred from *Angel*, Amita is supported by the other series characters who value her mathematical talents. However, the role of Amita has also been controversial because of her romantic relationship with her thesis adviser, Charlie.

Lisa Simpson, from the long-running animated show *The Simpsons* (1989–), also displays mathematical ability (among other nerdish qualities) at various times throughout the series. For instance, in the episode "Girls Just Want to Have Sums," which originally aired April 30, 2006, on FOX, Principal Skinner makes disparaging remarks about girls' mathematical abilities. As a result, the school is split into two single-sex schools. Upset by the lack of rigor in her mathematics class, Lisa is forced to dress as a boy, Jake, in order to attend the boys' mathematics class and learn "real" mathematics. When Jake wins an award for mathematical achievement, Lisa reveals her true identity, to which her brother Bart claims that she did so well in mathematics only because she learned to think like a boy. In the 2010 episode "MoneyBart," Lisa used the statistical methodology of Sabermetrics to manage Bart's baseball team.

Though African-American characters possessing mathematical talent are admittedly not common, two notable exceptions aired on television shows in the late 1980s. *A Different World* (1987–1993), a spinoff of the popular *Cosby Show* (1984–1992), featured Dwayne Wayne as a lead character. At different points throughout the series, Dwayne was a mathematics major and a calculus teacher. Known for his flip-up glasses, Dwayne was involved in romantic relationships with several of the female characters on the show. By contrast, Steve Urkel, on *Family Matters* (1989–1997) was the stereotypical geeky character, depicted in thick glasses, suspenders, and with a high-pitched voice. Whereas Dwayne was portrayed as popular with the opposite sex, Urkel was portrayed as an annoying neighbor of the Winslows who was grimly tolerated from week to week, though even he ultimately gained the audience's sympathy and became engaged to the Winslows's daughter Laura near the end of the show's run.

Other black characters with mathematical talent include Geordi LaForge of *Star Trek: The Next Generation* (1987–1994) and Turkov, of *Star Trek: Voyager*. Geordi eventually became the chief engineer on the Enterprise and a close friend of the android character Data. Often the two of them would discuss various details of twenty-fourth-century physics. Tuvok, though chief security officer of Voyager, also displayed a deep knowledge of science and mathematics. Both Geordi and Tuvok were valued members of their respective crews and were portrayed as scientific experts.

Such depictions of mathematicians and diversity offer great promise for the future, as television shows continue to reflect how society views mathematics and also impact those views themselves. In the twenty-first century, some have noted an increase in the portrayals of mathematics and mathematically talented individuals on television. Examples include mathematical discussions by the main characters on Bones (2005–); an intern on *House* (2004–) named Martha Masters has a Ph.D. in applied mathematics, who joined the cast in 2010; and forensic pathologist Dr. Maura Isles on *Rizzoli and Isles* (2010–), who often discusses mathematical concepts.

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See Also: Movies, Mathematics in; Nielsen Ratings; Plays; Science Fiction; Televisions; Writers, Producers, and Actors.

Televisions

Category: Architecture and Engineering.
Fields of Study: Data Analysis and Probability; Geometry; Measurement; Number and Operations.
Summary: Innovations in television technology rely upon a sophisticated use of mathematics, physics, and engineering.

Humans process reality by initially recording light and sound waves through the eyes and ears and then transmitting these data to the brain where they are transformed and synthesized into intelligible matter. In a similar manner, the engineering challenge of television from its conception has been to record data, transmit them (via electricity), and then reconstitute them at a physical distance from its origin. Television is a relatively recent invention. The first appearance of the word (a combination of Greek and Latin words, meaning "far-seeing") occurred in 1900 at the International Electricity Congress at the Paris Exhibition. It was not one single person who invented television, but a number of scientists, engineers, and visionaries working independently in different countries who devised the necessary technology and mathematics. Television has changed dramatically from its first appearance as an electromechanical system to electronic systems, including cathode ray tube (CRT), liquid crystal display (LCD), plasma, and three-dimensional (3D) television.

Image Scanning and Aspect Ratio

Scanning the image required it to be disassembled into discrete pieces of picture that could then be transmitted separately and reassembled as a sequence of images on a screen, with each image recomposed from those smaller pieces of the picture. If the sequence of images could be displayed on the receiver's end rapidly enough, they would appear to the human eye as a continuous whole of moving images. This approach makes use of the fact that the human eye can distinguish two parallel lines only if they are about one-thirtieth of a degree apart and will blend 12 images per second into a moving whole. In the 1920s, the transmission of images went from an unacceptably choppy five per second to 12.5 and more.

The earliest scanning mechanism is known as the "Nipkow disk," named for the German physicist Paul Nipkow, and versions and refinements of this were used as late as the 1930s. It consisted of a disk with a spiral of small holes in it and a photosensitive cell made of selenium on the other side of the plate from the image. One revolution of the disk corresponded to one complete image, with the holes as they rotated capturing the image in a series of lines. The number of such lines depended on the number of holes, which thus determined the degree of resolution of the image. A second disk was then rotated at the receiving end, playing back the captured image. One drawback of the Nipkow disk was that the scanned lines were not linear, which changed the geometry.

Historians debate why Thomas Edison chose to represent the geometry of television using the rectangular 4:3 aspect ratio, which indicates the ratio of the width to the height of the image. Some hypothesize that Edison chose this because the ratio approximates the golden mean while others assert that his motivation was to save money by cutting 70 mm film stock in half. The Society of Motion Picture Engineers adopted this ratio in 1917 and it was standard for many years. The international standard for high-definition television was devised mathematically in 1980 by electrical engineer Kerns H. Powers. Powers analyzed the common aspect ratios in use at the time and normalized them to a constant area to fit them in a rectangle. When overlapped via their centers, they shared a common inner rectangle. He computed the geometric mean to obtain the 16:9 aspect ratio that continues to be the standard for televisions in the twenty-first century.

A uniform aspect ratio for television created another problem of how to capture the ratio on 35 mm film. Mathematical principles were used to develop lenses that were "anamorphic," which stemmed from the Greek words meaning "formed again." Ultra Panavision used counter-rotated prisms, Technirama used curved mirrors and reflection principles, and CinemaScope used a cylindrical lens. However, the lenses created distortion problems as compared to spherical lenses. In the twenty-first century, mathematics continues to play a role in anamorphic widescreen processes.

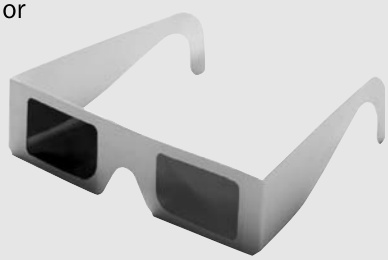
CRT Television

While electromechanical televisions such as the Nipkow disk were being developed, an electronic alternative that used a CRT rather than mechanical parts was also being explored. Philo Farnsworth and Vladimir Zworykin, among others, worked independently on this technology in the United States in the late 1920s. The diameter of the round picture tube, which was also the diagonal of the rectangular cover, was the critical parameter. Televisions are still measured on the diagonal in the twenty-first century.

The innovation involved harnessing electrical properties of matter. At the receiving end is a CRT—a glass vacuum tube, which receives the incoming transmitted

3D Television

Stereoscopic effects produced by special televisions add a perceived depth of dimension to standard television that has previously been represented by only height and width, though this technology was still in its infancy at the start of the twenty-first century. Mathematics plays an important role in the evolution of home 3D technology. For example, it is used in determining the proper viewing distance and angle, which depend on the geometry of the display and the location of the viewer in a (often) small space. However, there is a great deal of variability in the process. At the start of the twenty-first century, some people complain of headaches caused by improper parallax, interocular distance in the images or display, or difficulty in interpreting the motion.



signal that represents the picture, known as the "video signal" (audio and visual components are transmitted separately). At one end of the CRT is a cathode, which is heated so that it will radiate electrons (negatively charged particles) that are then attracted along the circuit to the other end of the tube (called the "anode end"), which is at positive electric potential for this purpose. This beam of electrons is focused electrically by charged plates and can be delicately manipulated by interactions with a magnetic field produced by electric current passing through coils.

At this end of the tube is a photosensitive phosphor-coated screen, which has the property of responding to the beam of electrons by emitting light that is proportional in intensity, point for point, to the beam that is moved across it. The video signal is synchronized with the electron beam so that the variations in the beam relay image information. The beam moves line-by-line, lighting the phosphor that illuminates the screen on which the image is viewed. Color images necessitate a more complicated technology than black-and-white images: three signals, one for each of the primary colors (red, green, and blue) and three electron beams are exploited to produce color images.

LCD and Plasma

CRT television was standard through the 1980s but the line-by-line sweeping of the electron beam across the screen takes time and faster technology is available on high-definition television (HDTV), which depends on either an LCD or a plasma screen. The image received via these newer technologies is still comprised of small units, called "pixels" (an abbreviation of "picture elements"), but these operate differently. In an LCD system, each pixel is deployed by an electrically stimulated liquid crystal, which undergoes internal molecular rearrangement in such a way as to polarize (filter) light that is shone from the back. Intensity of light is adjusted by a blocking procedure similar to sunglasses. In a plasma screen, however, each pixel functions like a miniature fluorescent light, since it contains a mixture of gases and mercury that respond to electric charge by radiating energy that in turn causes phosphor on a screen to emit light.

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See Also: Digital Images; Digital Storage; Electricity; Energy; Television, Mathematics in.

Temperature

Category: Space, Time, and Distance.
Fields of Study: Algebra; Measurement.
Summary: Scientists and mathematicians have developed and investigated a variety of principles and scales associated with the measurement and definition of temperature.

Quantification of temperature is necessary for many reasons, including scientific experiments, weather prediction, and many manufacturing processes. Temperature, by its formal definition, measures the movement of molecules in an object. Greater movement results in higher temperatures; conversely, less movement results in lower temperatures. The byproduct is heat, so temperature is often thought to measure the heat of an object. Mathematicians, many of whom are also physicists, have made significant contributions in quantifying heat and developing the temperature scales widely used in the twenty-first century.

History

Joseph Fourier began heat investigations in the early nineteenth century. His work *On the Propagation of Heat in Solid Bodies* was controversial at the time of its publication in 1807. Joseph Lagrange and Pierre-Simon Laplace argued against Fourier's trigonometric series expansions; however, Fourier series are widely used in a variety of theories and applications in the twenty-first century. Jean-Baptiste Biot, Simone Poisson, and

Laplace objected at various times to Fourier's derivation of his heat transfer equations. In 1831, Franz Neumann formulated the notion that molecular heat is the sum of the atomic heats of the components. Studying mixtures of hot and cold water, which did not produce water that was the average of the two temperatures, he concluded that water's specific gravity increases with temperature. This relationship was later shown by other researchers to be true only for a certain range of temperatures. In the late nineteenth century, James Maxwell and Ludwig Boltzmann independently developed what is now known as the "Maxwell–Boltzmann kinetic theory of gases," showing that heat is a function of only molecular movement. Their equations have many applications, including estimating the heat of the sun.

Around the same time, Josef Stefan proposed that the total energy emitted by a hot body was proportional to the fourth power of the temperature, based on empirical observations. In the twentieth and twenty-first centuries, scientists continued to study heat and have developed mathematical and statistical models to estimate heat. These models are used in areas like astronomy, weather prediction, and the global warming debate.

Measuring Tools and Temperature Scales

Heat can be difficult to quantify. Scientists and mathematicians developed many methods and instruments to measure and describe perceived temperature. Some of the earliest were called *thermoscopes*, often attributed to Galileo Galilei. In the early 1700s, Gabriel Fahrenheit created mercury thermometers and marked them with units that became known as "degrees Fahrenheit." He empirically calibrated his thermometer using three values. Icy salt water was assigned temperature zero. Pure ice water was labeled 30. A healthy man would show a reading of 96 degrees Fahrenheit. Later, Fahrenheit would measure the temperature of pure boiling water as 212 degrees Fahrenheit, adjusting the freezing point of water to be 32 degrees Fahrenheit so there was 180 degrees between the freezing and boiling point of water.

Anders Celsius created a different temperature scale in the mid-1700s. The Celsius temperature scale was numerically inverted with respect to Fahrenheit. He used 100 to indicate the freezing point of water and 0 for the boiling point of water. Because there were 100 steps in his temperature scale, he referred to it as a "centigrade" (*centi* means "a hundred" and *grade* means "step"). A few years later, Carolus Linnæus allegedly reversed the scale to make zero the freezing point and 100 the boiling point.

About a century after Celsius created his scale, William Thomson, Lord Kelvin, is given credit for the idea of an absolute zero, a temperature so cold that molecules do not move. The Kelvin scale was precisely defined much later after scientists and mathematicians better understood the concept of conservation of energy. Near-absolute zero conditions produce many interesting problems in mathematics and science. For example, clumping of atoms as they approach an unmoving state can be studied as a classic packing problem, which has extensions in areas like materials science and digital compression. The Kelvin temperature scale uses the same scale as centigrade, with absolute zero about 273 degrees below the freezing point of pure water. Converting from degrees centigrade to Kelvin is as simple as shifting the scale by adding 273.

In the mid-twentieth century, the centigrade scale was replaced with the Celsius scale. The changes were relatively minor, so one estimates the freezing and boiling points of water to be 0 degrees Celsius and 100 degrees Celsius. In actuality, 100 degrees Celsius (the boiling point of water) is now 99.975 degrees Celsius. Converting from degrees Celsius to degrees Fahrenheit, or degrees Fahrenheit to degrees Celsius, involves multiplicative rescaling, not just translation, since 1 degree Celsius is 1.8 times larger than 1 degree Fahrenheit.

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See Also: Climate Change; Clouds; Cooking; Geothermal Energy; Measuring Tools; Thermostat; Weather Forecasting.

Textiles

Category: Arts, Music, and Entertainment.
Fields of Study: Geometry; Representations.
Summary: Mathematics is integral to creating both traditional and modern weave patterns in textiles.

Textiles are flexible sheets made out of fibers. Natural textiles are made from plants, animals, or minerals; artificial textiles use human-made fibers, like plastic or synthetic proteins. Woven textiles combine longer fiber threads either by hand or by using looms or knitting machines. In nonwoven textiles, like felt, short or microscopic fibers are bonded by chemical or physical treatments. Nonwovens are often meant to be highly durable or disposable and have many applications in health, construction, and filtration technologies. Mathematical methods are used to design, produce, and analyze textiles. In 1804, Joseph Jacquard invented a weaving system using cards with patterns of holes to control loom threads. These cards were later modified by Charles Babbage into computer punch cards. Weaver and mathematics teacher Ada Dietz wrote *Algebraic Expressions in Handwoven Textiles* in 1949. She outlined a method for using expansions of multivariate polynomials to generate weaving patterns.

Weave Formulas

On a loom, "warp" threads are held parallel and "weft" threads are passed over and under them. A pattern formed in one pass of weft can be either repeated exactly, transposed, or otherwise changed in the next passes. Let *A* stand for warp threads on top and *E* stand for the weft thread on top. In plain fabric, a pattern *AEAE* . . . indicates that the weave is transposed by one thread in the next row. Basket weave uses *AAEEAAEE* . . . , so the pattern is repeated for two rows and then transposed by two (or some other whole number) for the next two rows. Satin is *AAAAEAAAAE* . . . , giving four repeats followed by one transposition. A satin weave results in the majority of the threads being parallel, so light is minimally scattered, producing the characteristic sheen. In contrast, twill has a distinct, textured diagonal pattern formed by using an *EEAEEA* . . . weaving scheme. Patterns may be added to plain weaves by printing or dying the fabric. The U.S. group Complex Weavers provides a forum for sharing advanced weaving methods and patterns, such as manifold twill.

Patterns and other factors like the thread intersections per area also dictate other properties. For example, plain weave fabrics tear the easiest, because force is applied to the single thread immediately next to the tear. Crimp is how easily the fabric morphs under tension. Plain weaves generally morph the easiest. Wrinkle resistance is the opposite; the more freedom of movement threads have, the easier it is for them to return to smoothness. Satin is an example of a wrinkle-resistant weave. On the other hand, satin silks shrink the most because their weave pattern is loose. Twill has a relatively high resistance to tearing, which makes fabrics such as jean popular for working clothing.

Cultural Textiles

Textiles are a significant cultural art form for many people in Africa. The three most well-known forms are *kente*, *adire*, and *adinkra*. Kente cloth is woven in long narrow strips, traditionally by Asante and Ewe men, and then sewn together into larger pieces of fabric that may be used for clothing or household goods. The cloth was often a sign of wealth and kept for special occasions. There are more than 300 known kente patterns, many of which represent people or historical events. Widely found adira cloth has patterns made by resistance dying. The cloth is tied, stitched, or stenciled, often with geometric patterns, to prevent the dye from adhering to some portions of the cloth. Adinkra cloth is printed, usually by drawing a square grid and stamping symbols into each square. This highly developed symbol language expresses concrete and abstract concepts, such as transformation or unity. Like kente cloth, adinkra often tells stories or proverbs. Tessellations and other repeating patterns are also common. In Ghana, the cloth was originally worn for mourning and some is still reserved for that purpose.

In Scotland, tartans represent families, clans, or regions. A "sett" is a specific plaid pattern, specified by sequences and widths of colored stripes. The pattern is formed by interweaving bands of stripes at right angles. Most are symmetrical, which means the sett is reflected 90 degrees around a pivot or center stripe. Asymmetrical setts have no pivot point. Symmetry has implications in kilt making. A kilt "pleated to the sett" has pleats folded to visually reproduce the tartan pattern across the back of the kilt, often not possible with an asymmetric pattern. Tartan patterns have been investigated with mathematical methods, such as group theory, and they are used in classrooms as examples of symmetry. Artist Andrew Hennessey has proposed “stella tartan” in which tartan setts would be woven radially and overlap in irregular polygon patterns.

High-Technology Textiles

The Industrial Revolution made rapid mass production of textiles feasible and the textile industry has since used many mathematical and computational techniques to continue its evolution. These techniques include differential equations, numerical methods, image processing, pattern recognition, and statistics. Computer-aided design (CAD) and computer-aided looms (CAL) are widespread. Application areas include supply chain management, quality control, and product development. The latter may involve structural modeling and simulation, as well as thermal or biomechanical bioengineering, particularly for specialty textiles. Some competitive swimwear has tiny triangular projections that mimick shark skin to reduce drag. An absorbent, nonwoven textile called *air-laid paper* is used in diapers. Integrating tiny light-emitting diodes into fabric allows clothes to change color or display text or animation. Thermal self-regulation may be achieved with phase-changing microcapsules that become fluid for cooling or solid to release heat, as needed. Weak link theory and bundle theory, as well as research in twisted continuous filaments, helical modeling of yarns, two-dimensional elasticity theory, aerodynamics, and many other investigations have also revolutionized the individual threads that compose fabric, often changing its properties even when using traditional weaves.

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See Also: Algebra in Society; Crochet and Knitting; Engineering Design; Matrices.

Thermostat

Category: Architecture and Engineering.
Fields of Study: Algebra; Measurement.
Summary: Thermostats are mathematically calibrated according to physical principles to regulate temperature in a variety of settings.

Thermostats and thermometers are related instruments that perform different tasks. A thermometer measures (“meter”) heat (“thermo”) to determine and display a current temperature. On the other hand, a thermostat is designed to keep the heat (“thermo”) stationary (“stat”) to help maintain a desired temperature. Inventor and college professor Warren S. Johnson produced the first electronic room thermostats in 1883. He installed them in classrooms to keep students more comfortable in cold weather and to minimize outside interruptions. In the twenty-first century, thermostats are most commonly found inside vehicle engines and as a part of residential, commercial, or industrial heating systems—though they can also be found in appliances, like gas stoves.

Automobiles

In an automobile engine, the thermostat helps regulate temperature so that the engine operates properly and efficiently. The thermostat acts as a control valve for the coolant fluid, which flows within an engine and to a separate radiator that helps to cool the hot coolant. When an engine is first started, the thermostat is closed, and the coolant flowing within the engine cycles through only the engine until it warms up to an ideal temperature. The thermostat measures the temperature change using a special type of wax. Initially, the wax is solid but as the temperature of the surrounding coolant increases, the wax melts and expands to allow hot fluid to flow from the engine to the radiator and cooler fluid to flow from the radiator back in to the hot engine. If the engine gets too hot, the thermostat will open more to allow coolant from the radiator to permeate through the engine. On the other hand, if the engine begins to get too cold, the thermostat will begin to close, allowing less coolant into the radiator and more coolant to cycle through the engine to heat it back up. The thermostat is mathematically calibrated to the engine type and will automatically make the needed corrections as the vehicle is in use.

Buildings

A thermostat used to control temperature in a building similarly does not directly heat (or cool) the rooms. In this situation, it controls a heating (or cooling) unit, which is used to help regulate the temperature. In many systems, a bimetallic strip is used to measure the temperature of a room. Metals expand and contract as they heat and cool. Bimetallic strips work because different metals expand and contract at different rates. A strip of steel and a strip of copper (or brass) will be placed together and the ends secured to each other. If the temperature does not change, the strip remains flat. When the temperature changes, the different rate of expansion or contraction will cause the flat strip to develop a curve toward the metal that has changed less. The amount of curvature can be matched mathematically to a specific degree or range of change in temperature, triggering the system to adjust accordingly.

To increase the sensitivity of the thermostat, most bimetallic strips are long and coiled inside the thermostat. The coil loosens or winds more tightly with a change in room temperature. At a certain point, the bimetallic strip's movements will trigger the heating unit to turn either on or off. Once turned on, the thermostat uses weights or magnets to keep the heating unit from turning off too quickly. Without these devices, the thermostat would create short cycles (turning on and off quickly), which are generally inefficient and could cause a premature failure of the heating unit. Since the bimetallic strip's movement depends directly on the temperature of the immediately surrounding air, the thermostat should not be placed in a location that would cause an inaccurate reading. One common mistake is placing the thermostat by a heat register, where hot air flowing out will trigger the thermostat to turn the heating unit off before the rest of the room has acclimated.

Electronic Variations

More advanced thermostats frequently use electronic rather than electromechanical sensors and may have more than a simple on-off setting. Setpoint staging uses one type of heating process, or *stage*, when the room temperature is within two degrees of the thermostat setting and another when the difference is greater than two degrees from the thermostat setting. Time-based staging activates a secondary stage or unit after the first stage runs for a predetermined amount of time, indicating that the room is colder or hotter than some preset value. Multistage thermostats analyze variables such as the current room temperature, the desired temperature, and the amount of time it takes for a space to warm or cool one degree to determine mathematically when to use a second heating stage.

Other Thermostat Applications

The term "thermostat" is also used in statistical thermodynamics, which applies probability theory to systems made up of a large number of particles. This field of study helps relate the large-scale properties of materials observed by people in everyday life to the microscopic properties of the atoms and molecules from which they are made. Here, a thermostat mathematically maintains a constant temperature in computer simulations of molecular dynamics by realistically exchanging the energy of endothermic and exothermic processes that happen during the simulation. For example, the Gaussian thermostat, named for mathematician Carl Friedrich Gauss, maintains system temperature by rescaling the velocities of the simulated atoms at each individual step of the simulation.

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See Also: Auto Racing; Budgeting; Measuring Tools; Temperature.

Tic-Tac-Toe

Category: Games, Sport, and Recreation.
Fields of Study: Geometry; Number and Operations; Problem Solving.
Summary: Traditional Tic-Tac-Toe has a limited number of possible games, which can lead players to quickly discover an unbeatable strategy as long as they move first.

Tic-tac-toe is a famous game often played by children. It requires a playing board of a 3-by-3 arrangement of square cells, usually quickly drawn by making two vertical lines cross two horizontal lines and imagining an outer border. Two players alternate marking cells with either an *X* (usually the first player) or an *O* (the second player). Each attempts to put three of their marks in a straight line, while trying to block the attempts of the other. The winner is the player who first makes the three-in-a-row line. Unfortunately for the challenge of the game, the first player can always win by putting an *X* in the center cell and playing carefully. Children often learn this strategy and the game can become mundane if this strategy is always employed.

Play Possibilities

However, tic-tac-toe is simple enough that it can serve as a fairly easy example of game analysis, where all possible positions and plays are determined. Most other games are so complex that such analyses are overwhelmingly complex.

Ignoring symmetric patterns, there are three possible first plays—a corner, a side, or the center. The second play patterns are based on these three openings. Again, ignoring symmetries, the corner opening leads to five possible second moves, the side opening also allows five possible second moves, but following a center opening there are only two possible second plays. Hence, there are a total of 12 noncongruent, nonsymmetrical second plays. Similar exploration of the possibilities shows a total of 66 possible third moves, though 26 are duplications, so there are only 40 noncongruent arrangements after the third play. Then, it becomes much more complicated because of overlaps of first- and third-move *X*s and second- and fourth-move *O*s. This fact demonstrates that even in such a simple game as tic-tac-toe, the full analysis becomes quite complex.

Variations

The 3-by-3 magic square (with numbers 1–9 arranged in the cells so that each row, column, and diagonal sums to 15) looks like a tic-tac-toe board with numbers. A game can be played where players take turns choosing numbers 1–9 (without repeats), trying to reach a sum of 15 with three numbers. Playing this game and placing the numbers onto the 3-by-3 magic square turns out to follow the same general games strategies as tic-tac-toe.

Tic-tac-toe can become a much more interesting—and challenging—game by expanding the board to three dimensions. If the game is played on a stack of three 3-by-3 boards (a cube of 27 cells), any row of three is a win. Some have suggested that a 4-by-4-by-4 cube, with a line of four to win, is a smoother game. Winning lines can lie entirely on a horizontal level, drop vertically from top to bottom, slant along a vertical plane, or go from one corner to the opposite corner along the body diagonal. New players often have difficulty even noticing winning lines! For even more complexity, the game can be played in four dimensions, usually displayed as a two-dimensional array of two-dimensional boards, assuming the boards can be stacked in any of the horizontal, vertical, or diagonal ways, with winning lines in any of the stacks according to the three-dimensional patterns, a variation that can be either 3-by-3-by-3-by-3 or 4-by-4-by-4-by-4.

Alternatively, the traditional board can be imaged to extend infinitely, allowing more possibilities for winning lines. One version keeps the traditional board but assumes the left column wraps to be next to the right column, so a line of three can be the upper center, the right center, and the left bottom corner. Similarly, the top and bottom rows can be considered as wrapping around to be next to each other.

Nine-Men's Morris

Many games from around the world pick up on the ideas of tic-tac-toe, especially the goal of making three (or more) counters in a row. Probably the most famous is called Nine-Men's Morris in English (also called "mill" or, in French, *merelles* or *morelles*); some suggest early versions were even played in ancient Egypt. The board is three concentric squares connected in the middles of the sides, with each junction and corner marked with a dot. Two players each have nine counters, marked to distinguish those of each player. They take turns playing their counters onto the dots of the board, trying to get three in a row, which is called a "mill." After players use up the nine counters each, play continues by sliding already-played counters along the lines on the board. Anytime a row of three is made by one player, the player is allowed to remove one of the other player's counters (but they cannot take a counter that is already in a mill). Eventually, one player either has no counters left or cannot move any remaining counters, and the other player wins.

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See Also: Acrostics, Word Squares, and Anagrams; Board Games; Dice Games; Sudoku.

Tides and Waves

Category: Weather, Nature, and Environment.
Fields of Study: Geometry; Number and Operations.
Summary: Mathematicians study and model the forces that cause tides and waves.

Approximately 70% of the Earth's surface is covered with water, most of which is in a constant state of motion. The causes of this motion include the gravitational pull of celestial bodies in space, like the sun and moon; the rotation and shape of the Earth; and the influence of natural phenomena, like wind and earthquakes. Mathematicians have long studied tides and waves, following in the path of ancient scholars and others who sought to understand these phenomena for many spiritual and practical reasons, such as sailing. In the twenty-first century, people still travel both above and below the surface of the oceans for research, commerce, and pleasure, and there are many problems old and new to be explored. Some interesting mathematical investigations related to tides and waves at the start of the twenty-first century include three-dimensional modeling of extreme waves (also called "rogue waves"), such as those observed during the 2004 Indian Ocean tsunami and the Hurricane Katrina storm surges in 2005. Mathematicians, scientists, and engineers have also explored methods and developed technology to harness tide and wave power as an alternative energy source, including methods that actually create waves in addition to using naturally-occurring ones. Some colleges and universities teach courses on tides and waves that involve substantial mathematics. The theme of Mathematics Awareness Month in 2001 was "Mathematics and the Ocean," underscoring the importance and relationship of ocean phenomena and mathematics, as well as the depth and breadth of the topics studied.

Tides

Water in Earth's oceans moves in a variety of ways, including many scales of currents, tides, and waves. Mathematicians and scholars from ancient times up through the Renaissance observed, identified, and quantified tidal patterns. The term "tides" generally refers to the overall cyclic rising and falling of ocean levels with respect to land—though tides have been observed in large lakes, the atmosphere, and Earth's crust, resulting largely from the same forces that produce ocean tides.

The daily tide cycles are caused by the moon's gravity, which makes the oceans bulge in the direction of the moon. A corresponding rise occurs on the opposite side of the Earth at the same time, because the moon is also pulling on the Earth itself. Most regions on Earth have two high tides and two low tides every day, known as "semidiurnal tides," which result from the daily rotation of the Earth relative to the moon. Since the angle of the moon's orbital plane also affects gravitational pull on Earth's curved surface, some regions have only one cycle of high and low, known as "diurnal tides." The height of tides varies according to many variables, including coastline shape; water depth ("bathymetry");

Officers of the National Oceanic & Atmospheric Administration Corps photographed the devastation caused in New Orleans by the 2005 Hurricane Katrina's storm surges.

latitude; and the position of the sun, which also exerts gravitational force. "Spring" tides, not named for the season, are extremely high and low tides that occur during full and new moons when the sun and moon are in a straight line with the Earth, and their gravitational effects are additive. A proxigean spring tide occurs roughly once every 1.5 years when the moon is at its proxigee (closest distance to Earth) and positioned between the sun and the Earth. Neap tides minimize the difference between high and low tides. They occur during the moon's quarter phases when the sun's gravitational pull is acting at right angles to the moon's pull with respect to the Earth.

A few of the many contributors to the theory and mathematical description of tides include Galileo Galilei, René Descartes, Johannes Kepler, Daniel Bernoulli, Leonhard Euler, Pierre Laplace, George Darwin, and Horace Lamb. Some mathematicians, like Colin Maclaurin and George Airy, won scientific prizes for their research. Work by mathematician William Thomson (Lord Kelvin) on harmonic analysis of tides led to the construction of tide-predicting machines.

Waves

There are many mathematical approaches to the study of waves in the twenty-first century, and some mathematicians center their research around this topic. In contrast to tides, a wave is a more localized disturbance of water in the form of a propagating ridge or swell that occurs on the surface of a body of water. Despite the fact that surface waves appear to be moving when observed, they do not move water particles horizontally along the entire path of the wave. Rather, they combine limited longitudinal or horizontal motions with transverse or vertical motions. Water particles in a wave oscillate in localized, circular patterns as the energy propagates through the liquid, with a radius that decreases as the water depth or distance from the crest of the wave increases. Wind is a primary cause of surface waves, because of frictional drag between air and water particles. Larger waves, like tsunamis, result from underwater Earth movements, such as earthquakes and landslides.

The Navier–Stokes equations, named for Claude-Louis Navier and George Stokes, are partial differential equations that describe fluid motion and are widely used in the study of tides and waves. Solutions to these equations are often found and verified using numerical methods. The Coriolis–Stokes force, named for George Stokes and Gustave Coriolis, mathematically describes force in a rotating fluid, such as the small rotations in surface waves. A few examples of individuals with diverse approaches who have won prizes in this area include Joseph Keller, who has researched many forms and properties of waves, including geometrical diffraction and propagation; Michael Lighthill and Thomas Benjamin, who jointly posed the Benjamin–Lighthill conjecture regarding nonlinear steady water waves, which continues to spur research in both theoretical and applied mathematics; and Sijue Wu, who has researched the well-posedness of the fully two- and three-dimensional nonlinear wave problem in various function spaces, using techniques like harmonic analysis. In other theoretical and applied areas, some techniques from dynamical systems theory, statistical analysis, and data assimilation, which combines data and partial differential equations, have been useful for formulating and solving wave problems.

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See Also: Coral Reefs; Gravity; Mapping Coastlines; Marine Navigation; Moon; Radiation; Swimming.

Time, Measuring

See *Measuring Time*

Time Signatures

Category: Arts, Music, and Entertainment.
Fields of Study: Measurement; Number and Operations.
Summary: Musical time signatures are mathematically defined and are cyclical in nature.

A time signature is a musical notation that defines the meter of a particular composition or a portion of a composition. It establishes a hierarchical, cyclic relationship among beats and among the subdivisions of those beats, which are inherently mathematical in nature. The history of time signatures is somewhat unclear. Some suggest that time signatures first made their appearance around 1000 C.E., though they may not have looked like the ones used in the twenty-first century. Others date the development of the fractional-form time signature closer to the fifteenth century. Nearly all modern Western music uses time signatures or some type of grouped pulses. Along with tempo (rate of beats), musicians use time signatures to gain an understanding of the relation of the elements of a piece of music to one another in time, particularly with regard to a contextual temporal metric.

A time signature normally consists of two integers,

$$\begin{matrix} n \\ b \end{matrix}$$

written with one directly above the other. Although it is often notated in prose as a fraction (for example, n/b), it is not a fraction and does not contain a dividing bar or solidus. A time signature appears in the first measure of a composition (in the staff following the clef and key signature), where it defines the default meter for the composition as a whole or until any subsequent time signature occurs that establishes a new default.

Meters and Beat

Time signatures may define various types of meters: simple, compound, complex, additive, or open. In simple meters (those in which the beats have a binary division), the upper integer indicates the number of beats in any one measure. The lower integer is conventionally expressed as a power of two, $b = 2^m$, and specifies what rhythmic value receives the beat. For instance, the time signature

$$\begin{matrix} 2 \\ 4 \end{matrix}$$

indicates a simple meter in which every measure contains two beats and the quarter-note value is the relative duration of each beat. In compound meters (where beats divide into triples), the upper integer n, which is larger than three and divisible by it, designates that each measure contains $n/3$ beats. The lower integer, $b = 2^m$, indicates that the dotted $1/2^{m-1}$-th note receives the beat (the total relative duration of a $1/2^{m-1}$-th note and a $1/2^m$-th note). For example,

$$\begin{matrix} 6 \\ 8 \end{matrix}$$

is the time signature for the compound meter in which each measure has two beats, and the dotted quarter-note duration (a quarter-note value plus an eighth-note value, or equivalently three eighth-notes) represents the beat.

Meters: Complex and Open

Complex meters incorporate beats that normally divide into a mixture of twos and threes. For example, the time signature

$$\begin{matrix} 5 \\ 8 \end{matrix}$$

(each measure has the duration of five eighth-notes) might divide into two unequal beats: one with two subdivisions and one with three. The time signature for a complex meter might also be notated as an additive meter, wherein the upper value is actually an arithmetic expression that agrees with this pattern. For instance, the complex meter

$$\begin{matrix} 5 \\ 8 \end{matrix}$$

could be indicated by the time signature

$$\begin{matrix} 2+3 \\ 8 \end{matrix}$$

An open meter is notated by the symbol **0** in place of a more traditional time signature. It indicates that the duration of each measure is defined merely by the rhythmic values or graphic spacing of the notes it contains and does not incorporate a recurring or otherwise specified pattern of beats.

Cyclic Groups

Because of its cyclic nature, meter suggests a modular temporal space, similar to clock time. Algebraically, one might use cyclic groups to model different types of meters. The time signature is useful in determining the order of such a cyclic group, n from above, and what relative duration represents a generating unit, b from above. Then, the first beat of a measure, beginning at time-point zero, would associate with the identity element of the cyclic group, and so on through the nth beat of the measure. Any subsequent measures would represent additional cycles through these sequential group elements.

Interesting Time Signatures

Some time signatures are frequently used, like the lilting rhythm of the following:

$$\text{the waltz } \begin{matrix} 3 \\ 4 \end{matrix} \text{ or the quick Sousa march } \begin{matrix} 6 \\ 8 \end{matrix}.$$

A mathematician might argue that the number of time signatures is limited because the number of beats per measure quickly becomes divisible by a smaller number, making it a multiple of another time signature. However, in music theory, time signatures have a broader meaning in terms of tempo and musical phrasing, not just counts of beats. Interesting compositions have been constructed by considering the mathematical properties of time signatures. Robert Schneider of indie rock band The Apples in Stereo composed a score for a play written by mathematician Andrew Granville and his sister Jennifer Granville in which all the time signatures had only prime numbers of beats per measure. It also included Greek mathematics related to primes in musical form. An entire subgenre of music called *math rock*, which emerged in the 1980s, is typified by uncommon time signatures such as

$$\begin{matrix} 13 \\ 8 \end{matrix} \text{ or } \begin{matrix} 7 \\ 8 \end{matrix}.$$

These complex rhythms can also be found in some mainstream music, such as the song "Anthem" by Rush, which is partially written in

$$\frac{7}{8} \text{ time.}$$

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See Also: Ballet; Ballroom Dancing; Composing; Popular Music; Step and Tap Dancing.

Toilets

Category: Architecture and Engineering.
Fields of Study: Algebra; Geometry; Measurement.
Summary: The geometry of modern toilets has been analyzed by engineers using a variety of mathematical and statistical methods.

In all human societies, the disposal of bodily waste has been a primary health concern. It has been estimated that the average human being produces one to two liters of urine and one-quarter to one-half kilogram of feces each day. Fecal matter, in particular, can contribute to the spread of a wide range of diseases, as bacteria and other pathogens can enter food and water when waste is not treated properly. Such problems are especially prevalent in areas of high population density and limited water resources. Over time, a range of toilets and treatment systems have been developed to deal with sewage. Because of the lack of resources and infrastructure, many places in the world in the twenty-first century still contend with waterborne diseases that originate in human waste.

History

Given that many mammals, including most primates, choose to defecate in selected areas in their habitat, it is likely that humans have had specific defecation sites throughout history. Dry toilets, such as pit latrines and outhouses, are ways communities formalized the locations in which humans defecate and are still used in many parts of the world in the twenty-first century. In these systems, waste is concentrated in one place, ideally where it will not infect drinking water. The earliest sitting toilets that used running water to carry waste away date to at least 2500 B.C.E. in the civilizations of the Indus Valley, in what is now India and Pakistan. In 1596, Queen Elizabeth I's godson, Sir John Harrington, invented the first indoor flushing toilet. In 1775, Alexander Cummings, a Scottish watchmaker who studied mathematics, filed a patent for a flush toilet. However, it was not until the late 1700s in Europe and 1800s in America that further modifications and inventions ushered in an age of modern plumbing.

Design and Operation

The geometry of modern toilets is essential to their efficiency and is extensively analyzed by design engineers using a variety of mathematical and statistical methods. The modern home tank toilet consists of a storage tank, a bowl, and an *s*-shaped siphon. Water is stored in the tank. When the toilet is flushed, this water is released into the bowl through rim jets on the underside of the toilet's rim and through a tube called the "siphon jet" that allows most of the water to flow directly into the bowl. The bowl is attached to an "s"-shaped tube, and the influx of water from the tank into the bowl pushes the waste and water over the lip of the "s" and down to an attached waste system. The bowl clears because of the siphon-action created. When the toilet finishes flushing, air enters the siphon tube and stops the siphon. Meanwhile, a flapper valve in the toilet tank closes the connection between the tank and the bowl and allows the tank to refill.

New Developments

The flush toilet takes a large volume of water to operate. In an era of increasingly limited resources, there has been a movement to create low-flush and no-

Coriolis Effect

There is a frequently recurring question of whether the swirl of the water in toilets in the southern hemisphere is opposite that in the northern hemisphere. This notion has been perpetuated in many ways, including popular television shows and scientific programming or textbooks. It is true that large oceanic and atmospheric phenomena, such as hurricanes, will spin in opposite directions in the two hemispheres because of the Coriolis effect. In a small-scale system like a toilet, the geometry of the apparatus, along with water turbulence or temperature, is a much more important factor—a fact that has been verified through systematic experimentation.

flush toilets. For example, toilets manufactured in the United States prior to 1994 used 13 liters of water per flush. The Energy Policy Act of 1992 required that toilets use six liters or less per flush, and as of 2011, high-efficiency toilets used 4.8 liters per flush. In Europe, dual flush toilets are common, providing the user with a choice of how much water to use depending on whether urine or feces is being flushed. Other technologies, including composting toilets that require no water and allow waste to biodegrade for use as fertilizer, have been developed for use by ecologically conscious consumers and people in areas of the world where water or sewage treatment facilities are limited. In addition, a number of toilets have been developed that include warmed seats, water and air jets for cleaning and drying the user, and built-in stool and urine analysis for health assessments.

Modeling Toilet Use

Many modern homes now have multiple toilets and ensuring adequate toilet facilities in public places requires planning and calculation. Two statistical studies of public-restroom use in the late 1980s are still referenced into the twenty-first century. They focused on the amount of time men and women spent in the restroom and they provided some of the first quantitative evidence that women take longer and thus require more toilets. This equity principle is known as "potty parity" and has been enacted into law in many places.

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Jeff Goodman

See Also: Energy; Green Design; Water Quality.

Tools, Measuring

See *Measuring Tools*

Tornadoes

See *Hurricanes and Tornadoes*

Tournaments

Category: Games, Sport, and Recreation.
Fields of Study: Algebra; Data Analysis and Probability; Number and Operations; Representations.
Summary: Mathematical methods can be used to seed the bracket for a tournament.

A tournament is any of a variety of competitions in which a relatively large number of players or teams compete at a sport, game, or other competitive activity. While formats differ widely, tournaments generally involve teams or individuals playing a large number of games in a relatively brief period of time. Typically, the ostensible purpose is to determine a single overall winner when the total number of players is (much) larger than the number of players who can participate in a single match. Tournaments of various kinds are held for most competitive activities.

Considerable mathematics goes into the design of tournaments and the choice of format for a particular tournament, often drawing from disciplines such as combinatorics and graph theory. Different choices about the rules of the tournament affect the appeal of the tournament for participants and spectators and, more importantly, can affect which players will be more likely to win. The situation is somewhat analogous to voting systems in which the outcome of a decision can change based on the form of ballot, even when the voters' preferences are unchanged.

Common Types of Tournaments

In a single-elimination knockout tournament, the players compete in pairs. The loser of each game is eliminated from the tournament; the winners go on to the next round. This process continues until only one player is left, who is declared the winner. If the number of competitors is not a power of two, then some competitors sit out one or more initial rounds, automatically advancing to the next round. Which players sit out can be determined randomly or based on some prior rankings. The schedule for which players meet in the first round, the winners of which of these games will meet in the second round, and so on, is called the *bracket* for the tournament. In situations where the competitors are ranked in advance (for example, seeds in a tennis tournament), care must be taken in designing the bracket. It would be undesirable for a player to gain an advantage in a tournament by deliberately underperforming in order to obtain an artificially low prior ranking. The most commonly used brackets involve the highest ranked player meeting the lowest-ranked player in the first round and are used because they are optimized to prevent such manipulation. Double-elimination and triple-elimination tournaments (participants are not eliminated until suffering a second or third loss also exist, though the latter are rather rare. These formats are tolerant of one (or two) lost matches by the player or team that will go on to be champion but the problem of arranging the brackets and scheduling the matches can be more complicated.

In a round-robin tournament, each participant competes against every other participant. Typically, each pairing competes in a single match but variants exist in which more games are played. Such a format gives more information about the relative strength of the players at the expense of requiring more games. Another drawback is that it is generally difficult to identify a canonical choice for first-place champion after a round-robin tournament.

Of course, much more complicated systems exist. Consider, for example, the FIFA World Cup. In the 2010 format, the 32 competing teams are first randomly divided into eight groups. The teams within each group all play against one another. Based on the results of these round-robin matches, a winner and a runner-up emerge from each group. These 16 teams then compete in a single-elimination knockout; the first round of knockout matches involve the group winners each competing against the runner-up from another group.

Graph-Theoretic Tournaments

The term "tournament" is also used with a specialized meaning in the subject of graph theory. A tournament in this sense is a collection of any number of vertices and arrows, where each pair of vertices is connected by a single arrow. Such a picture can represent a round-robin tournament in which each participant competes against every other participant exactly once, and there are no ties. The vertices are the players, and the direction of the arrow indicates who won each game (the arrow points from the winner of the game to the loser). Such configurations were originally studied by H. G. Landau to study the dominance relationships among populations of chickens. Tournaments have gone on to find important applications to social voting theory and public choice.

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See Also: Competitions and Contests; Rankings; Sport Handicapping.

Traffic

Category: Travel and Transportation.
Fields of Study: Algebra; Geometry; Problem Solving.
Summary: Mathematical models and statistical analysis of traffic flow suggest solutions.

Traffic flow is studied using mathematical and statistical techniques and computer simulations in order to better understand the movement of vehicles on roads and highways. Americans drive their vehicles almost 3 trillion miles per year on approximately 4 million miles of public roads. Mathematical models have shown that the behavior of even a single driver can have a broad impact on overall traffic flow in this dynamic system. As every driver knows, traffic patterns can often be unpredictable and frustrating, leading to driver stress, accidents, pollution, wasted fuel, and wasted time. Mathematical analysis of traffic congestion can provide transportation engineers with insights leading to improvements in efficiency and safety in the transportation of goods and people. A mathematical understanding of traffic flow patterns can also provide guidance for the design of roadways and provide more accurate calculations of trip itineraries and real-time driving times. These can be disseminated to the public and used in intelligent transportation systems.

The use of mathematics to describe traffic flow patterns slowly originated in the 1930s in order to study road capacity and also to begin to address traffic-related questions, such as how does traffic move through intersections. The mathematical investigations of vehicular traffic increased rapidly in the 1950s, mainly because of the expansion of the highway system after World War II. In the twenty-first century, theoretical models of traffic are utilized by high-performance computers, which can simulate the motions of vehicles on virtual road networks of entire cities and regions.

Traffic engineers distinguish between uninterrupted traffic flow situations (for example, traffic streams on highways and other limited-access roads) and interrupted flow circumstances (for example, where two or more traffic streams meet at a road intersection). The methods suited to analyze a particular traffic scenario depend on whether the flow is interrupted or uninterrupted. When formulating a mathematical description or model of traffic, one must attempt to account for the interplay between the vehicles and the drivers, the layout of the road system, traffic lights, road signs, and other factors.

Queuing theory, which is essentially the mathematical theory of waiting lines, is a probabilistic framework used for analyzing various traffic flow problems, such as optimizing vehicle passage through an intersection or traffic circle, calculating vehicle waiting times at tollbooths, and other similar waiting problems. On the other hand, car-following models and hydrodynamic modeling are deterministic approaches for analyzing traffic flow on long stretches of road.

Car-Following Traffic Models

Car-following models, also known as *microscopic models*, are considered from the point of view of tracking the movements of a line of $n = 1, \ldots, N$ individual cars driving in the same direction down a road in order to try to predict their exact positions $x_n(t)$, velocities $v_n(t)$, and accelerations $a_n(t)$. The starting point for car-following problems is to model how the driver of a car reacts when the vehicle directly in front of it changes speed (it is assumed for simplicity that there no passing is allowed). As a first crude estimation, one could assume a driver adjusts instantaneously according to the relative speed of the driver's car and the vehicle in front:

$$a_n(t) = C\left[v_{n-1}(t) - v_n(t)\right]$$

where C is a constant of proportionality, called the *sensitivity parameter*, which can be measured experimentally. A more realistic assumption would be that a driver adjusts with a lag response time of about one or two seconds, to a maneuver by the vehicle in front of it:

$$a_n(t) = C\left[v_{n-1}(t-T) - v_n(t-T)\right]$$

where T is the time lapse because of the driver's delayed reaction. Equations with delays such as these are then solved to keep track of each vehicle as the traffic moves. Numerous additional assumptions and effects have been incorporated into more sophisticated theories of car-following, such as considering the impact of spacing between cars, the effect of aggressive or cautious driving, and the effect of drivers looking ahead in the road and reacting to the motions of multiple vehicles in front of it.

Hydrodynamic Traffic Models

Hydrodynamic modeling, also called "continuum modeling," considers the flow of a traffic stream to be analogous to the flow of a compressible fluid in a pipe. Continuum traffic models do not keep track of the positions of individual vehicles, like car-following models, but track averaged, macroscopic quantities. For a long stretch of crowded road, such as an interstate highway, three important quantities of interest are flow rate (Q in vehicles per hour), vehicle speed (V in miles per hour), and vehicle density (ρ in number of vehicles per mile). These variables, of course, can vary along the stretch of road in both space and time, and their relationship is described algebraically as $Q = \rho V$. Furthermore, based on observations of traffic patterns over the years, it has been posited that for a given stretch of road, there exists a direct relationship between the flow rate and density. What has essentially been observed is that, on a road having some maximum flow rate, there is a critical vehicle density below which speed is not severely impacted but above which speed reduces. As the density continues to increase, then eventually flow rate reduces, and traffic becomes completely congested. For a concrete example, Greenshield's model postulates a simple linear relation between vehicle speed and density,

$$V = V_{free}\left(1 - \frac{\rho}{\rho_{jam}}\right)$$

where the parameter V_{free} is the free flow speed of a vehicle that is unencumbered, and ρ_{jam} is the density corresponding to bumper-to-bumper traffic. Then, the flow-density relation would be given by

$$Q = V_{free}\,\rho\left(1 - \frac{\rho}{\rho_{jam}}\right).$$

This parabolic function begins to capture some of the flow-density behavior that is observed on some real roads, although it is certainly an oversimplification. If the traffic density is zero ($\rho = 0$), then the flow rate must also be zero ($Q = 0$). Additionally, in bumper-to-bumper traffic ($\rho = \rho_{jam}$), the flow rate is zero, or very nearly zero in reality.

In the Lighthill–Whitham–Richards (LWR) theory of traffic, a long stretch of road is considered that has no entries or exits. On such a stretch of road, the number of vehicles must be conserved, and this fact combined with a flow-density relation gives rise to an equation, called a "conservation law," that predicts how vehicle density varies along the stretch of road. When a traffic jam occurs, it manifests as a sudden disturbance, or shock-wave, in the vehicle density along the road. LWR theory and other much more sophisticated continuum models of traffic can predict conditions under which traffic jams will form, propagate, and dissipate. Common reasons for traffic jams are accidents, construction, lane merges, and other changes in road capacity. However (as all drivers have experienced) sometimes "phantom jams" occur on highways for no apparent reason. These phantom jams can also be explained by continuum traffic models.

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See Also: Auto Racing; Highways; Smart Cars; Travel Planning.

Trains

Category: Travel and Transportation.
Fields of Study: Algebra; Geometry; Measurement; Number and Operations.
Summary: Trains and railways present interesting mathematical problems related to force and load, scheduling, and geometry.

Railroads influenced nearly every aspect of nineteenth and early twentieth century U.S. society. Companies building infrastructure for railroads (and railroads themselves) dominated the U.S. economy as more goods and people were transported via rail. Investors clamored to profit from the railway boom, inspiring engineers and mathematicians to improve the technology used in the railway system. As more people traveled by train, punctuality and reliability needed to improve. Time zones in the United States were established primarily because competing rail companies used different standard times for their schedules. In addition, Christophorus Buys-Ballot and others conducted experiments using trains to explore the Doppler effect, named for mathematician and physicist Christian Doppler. At the start of the twenty-first century, wooden and electric railway sets remain popular toys with children of all ages, while railroad enthusiasts design elaborate model train layouts in various scales reflecting the days when towns were centered around train stations.

Locomotives

Locomotives are classified using the Whyte system, named for mechanical engineer Frederick Whyte, which utilizes numbers to describe the wheel arrangement of the engine. For example, a 4-8-4 type locomotive has four wheels in the front, 8 driving wheels in the middle, and 4 wheels in the rear. The capacity of a locomotive depends on the amount of friction the driving wheels have with the track and the weight of the engine over the driving wheels. These quantities are related by the equation $F = MW$, where F represents the maximum pulling force of the train, M represents the coefficient of friction between the wheels and the track, and W is the portion of the weight of the locomotive over the driving wheels. While this relationship indicates that heavier trains can pull larger loads, more power is needed to move the train, leading to higher fuel costs. Increasing the coefficient of friction gives the train better traction and thus more pulling force, so most locomotives have a sandbox on the front from which sand is sprayed onto the track when the rails are slippery. Though friction is needed to get the train started, reducing M increases efficiency once the train is in motion, lowering operating costs.

Modern diesel-electric locomotives use high-tech designs to achieve more horsepower while reducing engine weight significantly. Equipped with a sophisticated array of sensors, onboard computers, and control systems, twenty-first-century trains maintain their hauling capacity while reducing fuel consumption and emissions. The future may see more magnetic levitation (Maglev) trains, which use magnetic fields to suspend the train above the track. The first commercial Maglev train opened in 1984 in Birmingham, United Kingdom, but ceased operations in 1995 in part because of design problems. A Maglev train in Japan recorded a maximum speed of 581 kilometers per hour (361 miles per hour) in 2003, the highest ever speed for a Maglev transport.

Passengers and Timetables

Commercial trains, whether passenger trains or freight trains, follow carefully written schedules. Composing these intricate timetables is a daunting task. Railways must ensure that trains do not collide on the tracks, and that goods and people are transported in a timely and efficient manner. In 2006, the Netherlands introduced a new railway timetable for all trains and mathematical modeling played a key role in developing the timetable. To determine how a set of trains should be routed through a station, researchers listed all feasible routes through the station for every train. Each combination of a train and a feasible route is represented by a node on a graph. Nodes on this large graph are connected if they belong to the same train or if there is a routing conflict between the train/route combinations. Presenting the scheduling problem in graph form enables sophisticated computer programs to generate a usable timetable. Additional modifications improve the efficiency of the timetable in the case of unexpected delays.

Railway passengers expect trains to be on time and to have sufficient space for a comfortable ride. Timetables can be fine-tuned to meet these customer demands using another type of mathematical modeling called "peak load management." Consultants work with railways to determine when trains are the most crowded and when passenger demand is highest. Mathematicians quantify the notion of "attractiveness," a measure of how satisfied a rider on a given train will be as a function of the journey time on the train, the time the passenger would like the train to arrive at its destination, and the actual arrival time. Another constant is added to the equation to determine how much attractiveness is reduced for each minute the actual arrival time differs from the customer's ideal arrival time. More terms can be added to measure the crowding on the train—overcrowding having a significant impact on attractiveness. Using this model, railways can develop timetables that increase the probability that a customer will ride on an "attractive" train. Further refinements to the model attempt to minimize the chance that a passenger will need to stand while riding.

Trains as Teaching Tools

Creative elementary school teachers have devised ways to use the appeal of toy trains to teach addition and subtraction. A colorful cardboard train is taped to a bulletin board and children count the number of cars on the train. Train cars are easily removed or added and the students see addition and subtraction in action by counting the number of cars on the new train. Wooden railway systems with magnetic couplings between cars also allow for easy joining and separating, making these toys excellent mathematical manipulatives when working with small groups of children. Older students may encounter the Two Trains puzzle. Two trains are on the same track traveling toward one other at a constant speed. A fly starts on the front of one train and flies toward the other train at a constant speed, faster than either train. Once the fly reaches the other train, the fly immediately turns around and continues buzzing back toward the first train. How far does the fly travel before being smashed when the two trains collide?

Track Geometry

Freight yards use combinations of switches, sidings, and turnaround loops to sort railway cars, assembling them into trains bound for various destinations. The fact that trains cannot pass each other on a single track leads to many challenges. The optimal arrangement of freight cars in the most efficient manner is another problem for mathematical modeling, but these fascinating switching systems have inspired mathematicians to investigate interesting questions involving train track layouts and railway switching puzzles.

A switch (also known as a “turnout,” or “point”) is a *Y*-shaped structure used to split tracks into two lines or to combine two lines into one. The directional nature of a switch makes the dynamics interesting: trains entering at the “top” of the *Y* will always exit through the bottom branch, but trains entering through the bottom have the option of traveling on the left branch or the right branch. Switches are used to sort cars in freight yards, enable locomotives to move onto a siding to allow a train traveling the opposite direction on the track to pass, and make it possible via a turnaround loop for a train traveling one direction to reverse direction.

How can two trains traveling in opposite directions, say eastbound and westbound, pass one another? If there is a siding long enough to contain one of the trains, the problem is easy. But what if only one car can occupy the siding at a time? Variations on this train-passing puzzle have been around for over a century. The trains can still pass each other through clever use of the siding. The eastbound train leaves its cars behind, moves onto the siding, and waits for the westbound train to pass through. After the eastbound engine emerges from the siding, the westbound train backs through the siding, bringing along one of the eastbound train’s cars and leaving that car on the siding. After the westbound train has pulled forward past the siding, the eastbound train can pick up its car, and the process repeats until the entire eastbound train is through.

Imagine a child playing with a toy railroad. Given a set of switches and plenty of track, how many different layouts can the child make? To determine whether two track layouts are different, the structure is transformed into a graph, with nodes representing lengths of track. Nodes are connected if there is a switch allowing a train to travel from one length of track to another. Layouts are said to be different if their graphs are the same. A child with two switches can make five distinct layouts. Using more switches and combinations of other types of switches, like the three-way pitchfork-shaped variety, even more layouts can be made and counted using mathematics.

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See Also: Bus Scheduling; Graphs; Mathematical Modeling.

Trajectories

See *Learning Models and Trajectories*

Transformations

Category: History and Development of Curricular Concepts.
Fields of Study: Communication; Connections; Geometry.
Summary: Numerous mathematicians since antiquity have studied and worked on the concept of transformations.

In mathematics, transformations have a rich history that connects various disciplines, including geometry, algebra, linear algebra, and analysis with applications in statistics, physics, computer science, architecture, art, astronomy, and optics. In general, a transformation changes some aspect while at the same time preserving some type of structure. For example, a dilation of an object will shrink or enlarge it but will preserve the basic shape; while a reflection of the plane will produce a mirror image, which flips figures while preserving distances between points. Mathematicians and geometers often transform an object, equation, or data to something that is easier to investigate, such as transforming coordinates to simplify algebraic expressions. The theory of transformations has important implications as well. There are many types of transformations including geometric transformations, conformal transformations, *z*-score transformations, linear transformations, and Möbius transformations, named for August Möbius. Geometric transformations have long been implicitly used in aesthetically pleasing design patterns in pottery, quilts, architecture, and art, such as tessellations in the Moorish Alhambra Palace. Historians and anthropologists compare and contrast these patterns to track the spread of groups of people. Mathematical transformations can be represented in a variety of ways, such as matrix representations of linear transformations, which are useful in algorithms and computer graphics. In school, young children study geometric transformations and this study continues through high school, where students represent various geometric and algebraic transformations using coordinates, vectors, function notation, and matrices. Students also investigate transformations using computers and calculators.

Early History

The early development of geometric transformations is tied to motions that were useful in modeling the Earth and the stars and in creating artistic works, architectural buildings, and geometric objects. The Pythagoreans thought that points traced lines and lines traced surfaces. Aristotle objected to the use of physical concepts like movement in these abstract mathematical objects. Euclid of Alexandria mostly avoided the concept of motion in his work. However, he used the notion of “superposition,” where one object is placed on top of another, in triangle congruence theorems, such as in his proof of side-angle-side congruence. In modern proofs, mathematicians would likely use transformations in order to place these triangles on top of each other. Euclid also defined a sphere as the rotation of a semicircle, and he defined a cylinder as the rotation of a rectangle. Archimedes of Syracuse investigated axial affinity motions in his work on ellipses, and Apollonius of Perga explored inversion. Marcus Vitruvius described the projections that were important in architecture, and he also investigated the concept of stereographic projection, which was useful in astronomy and map making.

Mathematicians around the world generalized these motions and applied them to a variety of fields. Most mathematicians in later times relied on transformations in geometry, although Omar Khayyám criticized Ibn al-Haytham's extensive use of motion by questioning how a line could be defined by a moving point when "it precedes a point by its essence and by its existence." Both Thabit ibn Qurra and his grandson Ibrahim ibn Sinan investigated "affine" transformations of the plane that preserved straight lines, like dilations. Alexis Clairaut and Leonhard Euler defined and explored general affine transformations. Sir Isaac Newton investigated various coordinate systems and the transformations between them, such as what are referred to as "rectangular and polar coordinates." Girard Desargues systematically investigated projective transformations, although many earlier mathematicians had investigated perspective drawing and projection in mathematics, art, and optics. Edward Waring and Gaspard Monge also studied projective transformations. Mobius represented affine and projective transformations analytically in terms of homogeneous coordinates.

Carl Friedrich Gauss linked transformations with linear algebra when he represented linear transformations of quadratic forms as rectangular arrays of numbers. A linear transformation of the plane is a map that preserves addition and scalar multiplication of vectors. Linear transformations of the plane are combinations of rotations, reflections, dilations, shears, and projections, and they are important in modeling movement in computer graphics. In general, a linear transformation is a map between vector spaces that preserves addition and scalar multiplication. Linear transformations of coordinates were important in the development of analytic geometry and some multivariate statistical methods and linear transformations were also linked to projective geometry and Möbius transformations, which are also called "fractional linear transformations." Henri Poincaré connected these transformations to hyperbolic geometry. Gotthold Eisenstein and Charles Hermite tried to extend Gauss's work on forms and in this context they defined the addition and multiplication of linear transformations. Arthur Cayley defined a general notion of matrices and recognized that the composition of linear transformations could be represented using them. James Sylvester explored properties of matrices that were preserved under transformations and defined the nullity of a matrix. Matrices continued to be connected to linear transformations and the theory of linear transformations extended to infinitely many dimensions.

Modern Developments

At the beginning of the twentieth century, Felix Klein revolutionized mathematics and physics with the idea of a transformation group. In his Erlanger Program, the properties of a space were now understood by the transformations that preserved them. Thus the classification, algebraic structure, and invariants of these transformations provided information about the corresponding geometries. His ideas unified Euclidean and non-Euclidean geometry and became the basis for geometry in the twentieth century. Klein's collaboration with Sophus Lie impacted the development of the Erlanger program. Lie also developed the notion of continuous transformation groups and associated these with a differential equation. Physicists and mathematicians continue to study the local structure of a so-called Lie group by the infinitesimal transformations in the Lie algebra. Earlier mathematicians and physicists had already used invariants in a several ways. For instance, Cremona transformations are named for Luigi Cremona, who studied birational transformations. These transformations were important in the study of algebraic functions and integrals. Max Noether investigated the invariant properties of algebraic varieties using birational transformations. In physics, Hermann Minkowski explored Maxwell's equations for electromagnetism, named after James Maxwell. These equations were invariant under Lorentz transformations, named for Hendrik Lorentz, and led to a geometry of space-time and the beginning of relativity theory.

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See Also: Animation and CGI; Composing; Coordinate Geometry; Equations, Polar; Quilting; Symmetry.

Transplantation

Category: Medicine and Health.
Fields of Study: Algebra; Data Analysis and Probability; Number and Operations.
Summary: Locating and allocating available compatible organs is an important task of surgery, as is determining the likelihood of success and survival.

Organ transplantation involves replacing a damaged organ or body part with an organ taken from another body, a location on the patient's own body, or sometimes another source. Relatively common organ transplants include hearts, lungs, livers, corneas, bone marrow, and skin. In the twenty-first century, there are increasing instances of transplantations involving parts that have proven more difficult in the past, including a human face in 2010. Transplantation is one of few medical fields where practice is driven by statistical analysis of large-scale national datasets. Collecting comprehensive data about transplantation in the United States is mandatory, and researchers use statistics to inform clinical practice and national policy. Still, there are too few living and deceased organ donors to meet the need. Optimization tools make the best use of scarce resources, like donated organs. With kidney paired donation, optimization can even increase the supply of available organs. An artificial pancreas employing control theory was under development in 2010.

Statistics

Statistical analyses inform transplant policy and individual decisions. The transplant community seeks equity in allocating organs, so the allocation system is frequently analyzed for gender and racial disparities. Understanding outcomes with and without transplantation helps patients decide if they will benefit from a particular transplant.

Survival analysis is the branch of statistics concerned with the distribution of time to an event. Survival analysis is commonly used in medicine to study time-to-death but can also be used to study time to any event, such as time from joining a transplant waitlist until receiving a transplant. The survival function $S(t) = \Pr(T > t)$ indicates the probability that the random time of an event T is later than a given time t.

Complications

Survival analysis is complicated by censoring; not all patients in a study have reached the event of interest. In a time-to-death analysis, some patients are likely still alive. The first technique for estimating a survival function with censoring was the product-limit estimator of statisticians Edward Kaplan and Paul Meier.

Confounding is another challenge. One could perform a survival analysis of the association between gender and time-to-transplantation to see whether men and women receive transplants at the same rate. However, not all patients are expected to wait the same amount of time. Other factors (such as age and blood type) confound studies of the effect of the factor of interest (gender) on time-to-transplantation. Cox proportional hazards analysis methods, named for statistician David Cox, can account for confounding, using a regression model based on the hazard function $\lambda(t)dt = \Pr(t \le T < t + dt \mid T \ge t)$, which indicates the

Doctors from Walter Reed Army Medical Center Organ Transplant Service perform Guyana's first kidney transplant operation in 2008.

instantaneous probability of an event at some time (t) conditional on having survived to at least that time.

Optimization

Donated organs are scarce and each organ must be allocated to one of many potential recipients. Optimization techniques allocate scarce resources by maximizing an objective function. A person's Lung Allocation Score is largest when the transplant has the largest lifespan benefit, and available lungs are offered to the nearby person with the largest score.

Kidney paired donation in which two living donors who are incompatible with their intended recipients exchange kidneys for compatible transplants requires more complex optimization techniques. More people can obtain better transplants when the paired donations are arranged using either a maximum weight matching in a graph or a maximum weight cycle decomposition (if more than two donors and recipients are involved in each exchange). By optimizing an individual's outcome rather than the overall good, a Markov decision process model, named for mathematician Andrei Markov, can determine whether it is better for a patient to accept a certain organ offered or wait until a possibly better organ is offered later. Another Markov decision process model can establish the best time for a patient to receive a liver transplant from a living donor.

Control

Control theory studies systems where adjustments over time maintain some desired set point, like a thermostat heating or cooling a room to maintain a comfortable temperature. In transplantation, control theory is used in an experimental artificial pancreas. A healthy person's pancreas maintains blood glucose levels over time by regulating insulin in response to eating a meal or exercising. An artificial pancreas uses a blood glucose monitor and a mathematical control system to drive an insulin pump. The control algorithms are tested on mathematical models of blood glucose levels before being tested in human subjects.

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See Also: Cochlear Implants; Disease Survival Rates; Life Expectancy; Surgery.

Travel Planning

Category: Travel and Transportation.
Fields of Study: Geometry; Measurement; Problem Solving; Representations.
Summary: Mathematical models are used to plan and evaluate short- and long-term transportation infrastructure decisions.

Travel planning is a broad field that covers everything from individual journey planning (for example, deciding which form of transportation to use when commuting) to regional transportation planning (for example, deciding on the layout for a new train line or arterial road). Professional transportation planners use equations and computer programs to directly compare different transportation modes and routes. These equations can be very simple with just a handful of terms, or incredibly complex models with hundreds of interacting variables. Regardless of the complexity of the analysis, the ultimate goal is to satisfy the objectives of the planning project and often the most important objective is to minimize journey costs.

Governments aim to maintain effective road networks and public transportation systems. Within certain cities, however, there can be a distinct bias toward either private or public transportation. This bias reflects the fact that governments prioritize certain planning decisions over others. These decisions fall into two broad categories: long-term and short-term planning.

Professional transport planners use equations and computer programs to compare different modes and routes when trying to maintain effective road networks and public transportation systems.

Long-Term Planning

Long-term planning includes decisions such as the use of land and placement of new freeways or bypasses. The objectives for these projects are often manifold: reduce costs, reduce pollution, reduce noise, maintain traffic flow, and maintain priority for public transport and carpool vehicles. It is the challenge of the transport planner to balance these objectives and ensure that the final decision satisfies these criteria.

Long-term planning also incorporates projections of future effects. For example, a wider freeway leads to better accessibility in certain urban areas, which eventually leads to more construction in those areas. Construction, in turn, leads to more traffic on the freeway and a renewed need to widen the road, thus creating a cycle. Long-term projects typically take several years to implement and even longer to monitor their impact.

Short-Term Planning

Short-term planning includes the introduction of bus priority lanes, changing the timing of traffic light signals, using trains with a greater numbers of cars during peak travel times, changing the price of parking in a particular area, changing taxi regulations, introducing new public transport fare systems, and so on. These are changes that can be implemented and evaluated within weeks as opposed to years.

Comparing Alternatives

To make any long-term or short-term planning decisions, it is necessary to compare a range of alternatives, side by side, using as few indices as possible. As a simple example, a set of four time and money measurements can be reduced to a single measurement of cost (C) using the equation

$$C = a(P) + b(t_T) + c(t_W) + d(t_J)$$

where P is the fare price, t_T is the transit time, t_W is the wait time, t_J is the journey time, and the coefficients a, b, c, and d are used to weight the components relative to one another (for example, for a given individual, wait time may be perceived to be twice as costly as journey time). This equation is particularly useful for comparing different forms of public transport. The single cost values will paint a very clear picture of which mode and route has the optimum mix of short times and low costs. Cost is often expressed in minutes, as opposed to dollars, as this measure will remain stable even as prices increase.

Some other measures used by transport planners include traffic density (number of vehicles on a given stretch of road), traffic flow (number of vehicles passing through a given stretch of road every minute), and performance index (an aggregate measure of the delays experienced in a given transport network). Each of these measures must be interpreted in context because acceptable ranges for the values will vary depending on road type, city size, and network connectivity.

Another common method used to estimate the amount of traffic passing between two zones (for example, a neighborhood and a commercial center) is called the "gravity model." It was given this name because the form of the equation is similar to Isaac Newton's equation of gravity. The traffic passing between two zones, *A* and *B*, is proportional to the product of the traffic originating in zone *A* and the traffic arriving in zone *B* but inversely proportional to a function of the distance between the two zones.

Governmental transport planners use these measures to test for weaknesses in the transport network—places where demand exceeds supply—and to gauge the effects of previous planning decisions. The act of planning is therefore firmly rooted in the interpretation of numerical output from mathematical analyses.

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See Also: Bus Scheduling; City Planning; Traffic; Traveling Salesman Problem.

Traveling Salesman Problem

Category: Travel and Transportation.
Fields of Study: Geometry; Problem Solving.
Summary: The traveling salesman problem is a notable applied mathematics problem that is simply constructed and may be unsolvable.

Imagine a salesperson that needs to travel to 30 cities. The salesperson wants to begin in his or her hometown, visit every city exactly once, and return to the hometown. In what sequence should the salesperson visit the cities in order to minimize the total amount of traveling time on the road between cities? The significance of the traveling salesman problem (TSP) lies in the fact that many other problems can be translated into a traveling salesman formulation and that a brute force check-all-the-possibilities approach will take prohibitively long—even for moderately sized problems (like the example) and with the use of fast computers.

Many problems can be translated to the TSP. The travel time between cities can be replaced by distance, cost, or other measures. Hence, in essence, this problem captures many sequencing problems where a number of tasks have to be sequenced and the costs can be modeled appropriately. Problems as diverse as optimizing the routes of garbage trucks, planning the sequence of motions performed by a robot, and ordering genetic markers on a chromosome have been modeled by the TSP.

Solving the TSP

Why is solving the TSP hard? If one decides to solve the problem by checking all the possibilities and then choosing the best one, then the sheer number of possibilities will make the problem impossible to solve. For example, with 30 cities and starting at a hometown, initially there are 29 cities to choose as a first destination. Regardless of the first choice, there are 28 cities to choose from next and so on. The total number of possible ways to start from a hometown, traverse each of the 30 cities exactly once and return to the hometown is

$$\begin{aligned} 29! &= 29 \times 28 \times \ldots \times 3 \times 2 \times 1 \\ &= 8{,}841{,}761{,}993{,}739{,}701{,}954{,}543{,}616{,}000{,}000 \\ &\approx 8.8 \times 10^{30} \end{aligned}$$

possibilities.

Even if a computer checked a million possibilities per second, checking all the possibilities would take more than 200,000,000,000,000,000 years—much longer than the age of the universe. Making the computer twice or 10 times faster still will not be enough to make the problem worth attempting.

Solution Through Algorithms

Could there be clever algorithms that solve the TSP faster? The TSP is among the problems that computer scientists call *NP-hard*. Given any algorithm for solving the TSP, certainly the number of steps needed by the algorithm grows as the size of the problem—namely the number of the cities—grows. If the number of steps in an algorithm as a function of the size of the problem is a polynomial, then it is generally believed that the problem is tractable. In other words, if there is one such polynomial time algorithm, then one can hope to find other more efficient ones and be able to solve even large-sized problems efficiently. At the start of the twenty-first century, it is not known whether the TSP has such a polynomial time algorithm. But it is known that if there is such an algorithm, then there is also efficient algorithms for a host of other problems of interest to computer scientists. For many years, researchers have looked for such algorithms and have not been able to find one, and the strong prevailing opinion is that no such algorithm exists (this is the famous $P \neq NP$ problem).

Even though the TSP is a difficult problem to solve in general, progress has been made in developing algorithms that do much better than the brute force method. In fact, very large instances—for example, one with 85,900 cities—of the TSP have been solved exactly. On another front, many approximation algorithms have been devised. These algorithms do not aim to find the absolute best solution but rather find a solution that is close to the best one. A simple approximation algorithm using minimum spanning trees, for example, can find a solution that is guaranteed to be no worse than twice the optimal solution. More sophisticated algorithms can find a solution within a few percentages of the optimal solution for a problem with the number of cities in the millions.

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See Also: Birthday Problem; Bus Scheduling; Cocktail Party Problem; Scheduling; Tournaments.

Trigonometry

Category: History and Development of Curricular Concepts.
Fields of Study: Communication; Connections; Geometry; Measurement.
Summary: Trigonometry is one of the most essential branches of mathematics in engineering and science.

The principal value of trigonometry rests in its numerous and practical applications throughout the world. It has always been viewed as one of the most applied areas of mathematics. When combined with the latest technologies, trigonometry impacts society in immeasurable ways. It is essential in engineering and in all of the sciences. Some of the applications of trigonometry include astronomy, aviation, architecture, engineering, geography, physics, seismology, surveying, oceanography, cartography (mapmaking), navigational systems, space sciences, medical imaging, music, and video games. Clearly, the applications and influences of trigonometry are fully embedded within contemporary society. To appreciate modern applications of trigonometry and the contributions of societies throughout the world, it is important to consider its historical evolution.

Origins of Trigonometry

The beginnings of trigonometry date back to prehistoric cultures and mirror the evolution of civilization itself. The word "trigonometry" comes from the Greek word *trigonon* (meaning "triangle") and *metron* (meaning "measurement"). However, trigonometry did not originate as the study of triangles. It was initially viewed as a combination of geometry and astronomy used in studying the movements and locations of celestial bodies in the sky and in time keeping. The foundations of trigonometry originated in prehistoric cultures with their vigilant observations of the night sky. Some of their discoveries are inherent in the designs of Stonehenge and ancient Egyptian monuments. The early civilizations of Egypt, Babylonia, and India (c. 2000 B.C.E.) contributed significantly to the origins of trigonometry. The Egyptians applied the properties of geometry and astronomy in constructing the pyramids, and the Babylonians (c. 1900 B.C.E.) used angles and ratios to keep track of the motions of celestial bodies in the sky. They followed the paths of planets, the lunar and solar eclipses, and gave coordinates to the stars, all of which required familiarity with angular distances measured on the "celestial sphere." While these early civilizations used trigonometric principles for astronomical measurements, in designing monuments, and in time keeping, they did not fully develop the mathematical system that is now known as "trigonometry."

The invention of trigonometry emerged as a defined body of knowledge from work conducted by Greek astronomers around 350 B.C.E. in the city of Alexandria, Egypt, the intellectual center of the ancient world. The Greek astronomer, Hipparchus (190–120 B.C.E.), is frequently called "the Father of Trigonometry. " He used ratios to determine the distances of the Earth from the sun and the moon and was responsible for tabulating the measures of arcs and their corresponding chord lengths for angles within a circle. The trigonometry of ancient Alexandria would now be called "spherical trigonometry" since it was confined primarily to the study of the properties of great circles on spherical bodies.

The earliest recorded work on spherical trigonometry appears in the book, *Sphaerica*, written by Menelaus, a Greek mathematician working in Alexandria (70–140 C.E.). It describes Menelaus' theorem, which associates the ratios of the lengths of intersecting arcs of great circles on a sphere.

Development of Trigonometric Functions

A fundamental difference between ancient Alexandrian trigonometry and modern trigonometry is that the former used arcs and chords in its trigonometric tables instead of the trigonometric functions that are used in the twenty-first century: sine, cosine, and tangent. The oldest surviving table of arcs and chords was created by Ptolemy of Alexandria (90–168 C.E.). However, when the chords of the circle are rotated to a vertical position and radii are inserted to connect the endpoints and midpoint of the chord to a common center, it is possible to translate half-chord lengths to a sine function. Thus, Ptolemy's famous table of arcs and chords was equivalent to a modern table of sines. His most famous mathematical work was the *Almagest*. It included a table of chords for angles for one-half degree to 180 degrees, in increments of half degrees. His chords were accurate in length to five significant digits. In addition, Ptolemy proved (using chords) the formulas for the sum and difference of two angles that are equivalent to the current sine functions of two combined angles. These discoveries were applied to astronomy, time keeping, and in locating the direction of Mecca for the daily five prayers required by followers of Islam.

In the first century B.C.E., trigonometric principles were used primarily in navigation and map-making. Fortunately, Ptolemy recorded all of the geographical knowledge collected by the ancient world in his eight volumes, titled *Geographia*. It included the latitudes and longitudes of 8000 places on Earth and was the world's first atlas, similar to those used in the twenty-first century. It took astronomers nearly 400 years to shift from using tables of angles and chords to a reliance on tables of sines. Indian astronomers of the fifth century gave trigonometry its current interpretation of the sine function, which quickly spread to Arab and Islamic astronomers.

It is important to recognize that the six trigonometric functions that are used in the twenty-first century were developed at the end of the tenth century by Arab and Islamic astronomers. The West learned about Arab and Islamic trigonometry at the beginning of the twelfth century through translations of Arabic and Islamic astronomy handbooks. Indeed, the maps created by the Alexandrian Greeks and the trigonometric functions developed by the Arab and Islamic astronomers were employed during the world explorations of the fifteenth and sixteenth centuries. Christopher Columbus (1451–1506) utilized these materials to guide him to discover the "new world" in 1492. There is no denying the meritorious impacts of trigonometry on exploration and navigation during the fifteenth and sixteenth centuries.

Trigonometry of the Renaissance

Calculations using trigonometric functions to determine the sides of right triangles did not become prevalent until the sixteenth century. German mathematician Bartholomew Pitiscus (1561–1613) is recognized for creating the word "trigonometry," meaning triangle measurement. He chose *Trigometria* for the title of his book, which described the applications of trigonometry in surveying.

During the seventeenth century, numerical calculations in trigonometry were simplified by the invention of logarithms by the Scottish mathematician John Napier (1550–1617). Fifty years following Napier's invention of logarithms, English mathematician Isaac Newton (1642–1727) invented the calculus in which he represented trigonometric functions as infinite series in powers of x. Specifically, Newton discovered that

$$\sin(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9 - \cdots$$

and

$$\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8 - \cdots.$$

These representations of the sine and cosine functions continue to play important roles in applied and pure mathematics in the twenty-first century.

For most of the history of trigonometry, angles were measured in degrees, which were defined as fractions of the circumference of a circle. This practice was not efficient nor consistent because the radius of the circle was not fixed. The creators of tables of sines often chose a radius convenient for their calculations. Ptolemy used a radius of 60 since his fractions were expressed in 60ths. The Austrian mathematician Georg Rheticus (1514–1574) used a radius of 10^{15}, which permitted him to tabulate the six trigonometric functions with 15-digit accuracy without the use of decimals or fraction manipulation.

Further innovations and interpretations in trigonometry were developed in the eighteenth century by the Swiss mathematician Leonhard Euler (1707–1783). He explained that computations of trigonometric functions would be more efficient if the line lengths were measured in the same unit. Consequently, he chose "1" to be the radius of a circle centered at the origin. Thus, the circle's circumference would be 2π; the arc for 45 degrees would be $\pi/4$; the arc for 30 degrees would be $\pi/6$; and so on. This system led to the later development of radian measures for angles and arc lengths.

More significantly, Euler discovered that trigonometric functions could be defined in terms of complex numbers (the union of real and imaginary numbers). Specifically, Euler's famous equation states that for any real number x,

$$e^{ix} = \cos(x) + i\sin(x).$$

The relevance of this equation was that trigonometry could then be viewed as only one of the numerous applications of complex numbers. This formula served as a unifying concept for the study of mathematics as a whole.

Contemporary Applications of Trigonometry

Although trigonometry originated in ancient civilizations from studying the movements of astronomical bodies and triangle relationships, its current applications encompass far more. Trigonometry now spans the diverse fields of architecture, engineering, science, music, navigation, medicine, digital imaging, and games of entertainment.

Trigonometry is a perfect partner for architecture and engineering. Contemporary buildings with curved surfaces in glass and steel would be impossible without trigonometry. Although the surfaces are perceived as curved, they are frequently composed of numerous triangles. Furthermore, since the triangle is an ideal shape for evenly distributing the weight of a structure, an understanding of relationships among the parts of triangles is essential in the design and construction of buildings, bridges, and monuments. Specifically, if an engineer knows the lengths of the beams that will be attached to a structure, the angles at which they must be attached can be calculated using trigonometry. Additionally, in the architectural design of an amphitheatre, the engineer's task is to design the structure so that all sounds from the stage are funneled into the audience's ears. Engineers and architects use trigonometry to identify the perfect shape to balance this sound as it reflects off the walls and ceilings.

Since trigonometry facilitates the understanding of space, it has numerous applications in the physical sciences. In optics and statics, trigonometric functions are vital in understanding the behavior of light and sound. It is particularly useful for modeling the periodic processes found in music because of the cyclical and periodic nature of trigonometric functions (sine, cosine, and tangent). Harmonics are determined by the form of their sine waves with respect to their periods and frequencies. The period of a sine wave is the length of the interval of repetition of the sine wave. The frequency of a sine wave is the number of cycles a sine wave goes through in a standard distance or time interval. In music, the frequency is often expressed in units of hertz, (Hz), where 1Hz means one period per second. For example, the pitch of every note in music is determined by the length of its sine wave (period) and by its frequency. Musical notes with wide sine waves are lower in pitch because they have fewer cycles per second, while notes that have narrow sine waves are higher in pitch and have more cycles per second.

Trigonometry plays a vital role in modern technologies through the process of triangulation. It is used by global positioning systems (GPS), in computer graphics, and in gaming. Specifically, computer generation of complex images is made possible by coloring numerous, microscopic squares (called "pixels") that define the precise location and points on the image. The technique of triangulation is used to make the image highly detailed and clearly focused. In GPS, triangulation is used for object location. Similar imaging technologies have also revolutionized the medical fields through the development of Computed Axial Tomography (CAT) scans, ultrasounds, and magnetic resonance imaging (MRI).

Trigonometry Instruction

It has been shown that trigonometry emerged approximately 4000 years ago through careful observations of the movements of celestial objects in the sky. These observations gave rise to the study of the celestial sphere and relationships among arcs, chord lengths, and central angles of great circles. In the twenty-first century, this form of trigonometry is called "spherical trigonometry." It was not until the end of the tenth century that Arab and Islamic astronomers defined the six trigonometric ratios. It took another 500 years before right triangle trigonometry became prominent in surveying, navigation, and architecture throughout the Western world. Therefore, it is interesting that this evolutionary sequence of the development of trigonometry is not reflected in the order in which it is typically introduced to students.

The current order of instruction is to first introduce the six trigonometric functions as ratios between the sides of right triangles. These concepts are then applied in finding the missing parts of right triangles. Finally, the unit circle and the periodic nature of trigonometric functions are introduced. This instructional order is often supported by the fact that right triangle trigonometry is a natural extension of the study of the Pythagorean theorem and ratios between parts of triangles, both of which are covered in the middle grades. Furthermore, trigonometric ratios are not as difficult to comprehend as periodic functions.

Some educators may argue that instruction should follow the evolutionary sequence of a topic. In support of that argument, students of today's technological world (with microwaves, wifi's, electrocardiograms, and electronic music) gain early familiarity with the periodicity of the sine and cosine functions. They are also more likely to model scientific and social phenomena with trigonometric functions than to apply right triangle trigonometry in finding the missing parts of triangles. Thus, for research purposes in identifying "best practices" for instruction, some educators have considered assessing the effects of teaching trigonometry in the same order that it historically evolved.

In conclusion, trigonometry is more valuable to society today than ever before in recorded history. The foundations of trigonometry emerged about 4000 years ago in the Egyptian and Babylonian civilizations. It developed into a well-defined mathematical discipline in the city of Alexandria, Egypt, circa 350 B.C.E. In the centuries that followed, trigonometry continued to evolve through the contributions and insights of diverse cultures and societies throughout the world. Trigonometry, with its advanced measurement facilities and associated technologies, remains one of the most applicable and practical fields of mathematics, vital to the advancement of the sciences, engineering, and technologies.

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See Also: Animation and CGI; Functions; Geometry and Geometry Education; Geometry in Society; GPS; Harmonics; Measurements, Length.

Tunnels

Category: Architecture and Engineering.
Fields of Study: Algebra; Geometry; Measurement; Number and Operations.
Summary: Tunnels have long presented interesting mathematical and engineering problems.

A tunnel is a connecting passageway through materials like rock, earth, or water. Tunnel engineers must take into consideration issues like seepage and weight. Scientists and mathematicians create mathematical models of tunnels to investigate aspects like aquifers and safety issues. Analytic and closed form solutions are useful in engineering. Mathematical fields like graph theory, differential equations, geometry, probability, and trigonometry are important for modeling and measuring tunnels.

Mathematically Challenging Tunnels

Five centuries after it was completed, Hero of Alexandria gave a theoretical explanation that may explain how the Tunnel of Samos was constructed. Mathematical physicist Renfrey Potts had an undergraduate degree in mathematics. He worked as a consultant for General Motors and created car-following models. This work led to experiments on a testing track with just two cars that successfully predicted the optimum speeds for congested traffic in the Holland Tunnel in New York, named for engineer Clifford Holland. The Channel Tunnel between England and France represented a significant engineering and mathematical challenge. At the time of its building and into the twenty-first century, it had the longest undersea length of any tunnel in the world. It presented significant challenges including problems related to the topology and geology of the rock through which it was bored; significant water

If the height and width of a parabolic tunnel are known, one can determine the tunnel's height at different distances from the base center by modeling using coordinate geometry and equations.

pressure; ventilation; communication; and the fact that construction was started at the same time from both ends, requiring exceptional precision to meet in the middle. This tunnel serves as a model for other underwater tunnel projects and many teachers use it to present mathematics concepts. Scientists and mathematicians also experiment with digital and physical wind tunnels as well as quantum tunnels.

Ancient Tunneling

The problem of delivering fresh water to large populations has been an ongoing human endeavor since ancient times. In the sixth century B.C.E., a one-kilometer tunnel was dug through a large hill of solid limestone to bring water from the mountains to the main city on the island of Samos. The Eupalinian aqueduct on Samos was designed by the ancient Greek engineer Eupalinos of Megara. The tunnelers worked from both ends and met in the middle, with an error less than 0.06% of the height. To achieve this remarkable result, Hero of Alexandria theorized that the tunnelers used a method based on similar triangles in order to determine the correct direction for tunneling. Mathematicians and scientists continue to debate the pros and cons of various theories of how this engineering marvel was constructed.

Modeling Tunnels

Tunnels can be modeled using coordinate geometry and equations. For example, knowing the height and width of a parabolic tunnel, one can determine the tunnel's height at different distances from the base center. To solve this problem, one needs to find the equation for the parabola choosing convenient x-y axes.

Frictionless Tunnels

The possibility of mathematical modeling allows for innovative and challenging ideas. What if a frictionless tunnel would be bored through Earth's center? Paul Cooper, a mathematician fond of Jules Verne's books, tried to answer this question in an issue of the *American Journal of Physics*. He set up and solved by computer a set of differential equations for tunnels that would provide minimum gravity-powered travel time between any two cities on Earth.

According to Cooper's differential equations, by freefalling in airless, frictionless, straight-line tunnels, passenger vehicles powered only by the pull of gravity could theoretically travel between any two points on the Earth's surface in a total time of only 42.2 minutes. Accelerated by the force of gravity on the first half of the trip, the vehicle would gain just enough kinetic energy to coast up to the other side of the Earth. However, significant obstacles make such a project impossible in the twenty-first century. Subterranean temperatures reach extremes, even for relatively shallow tunnels of only a few miles deep, requiring huge cooling systems for vehicles. Also, it is almost certainly impossible to create a completely frictionless path without a rail or track of some type, leaving the vehicle with insufficient kinetic energy to complete its trip without a source of additional power. Consequently, such a tunnel is still science fiction more than science.

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See Also: Caves and Caverns; Coordinate Geometry; Energy; Traffic; Wind and Wind Power.

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Ultrasound

Category: Medicine and Health.
Fields of Study: Algebra; Geometry; Representations.
Summary: Ultrasound uses mathematical principles to create images of the human body.

Although ultrasound cannot be heard by humans, it has been produced and used for a vast number of applications in many different fields. In industry, ultrasound has been used as a technique to assess the structural integrity of materials. The interaction between ultrasound and live systems has been studied since the 1920s. During the 1960s, it was used in medicine, initially as a therapeutic option and then later as a diagnostic resource. In the twenty-first century, ultrasound is a major medical imaging technology widely used in clinical facilities around the world because it causes no harm to the human body and results can be achieved in real time, besides the fact it is considerably cheap and easy to use. The available technologies using ultrasound are in constant development. Every new application depends on the advance of computer sciences that work with many concepts of physics and the solution of mathematical problems in this field seems inexhaustible.

Sound is a form of energy consisting of the vibration of molecules of an environment that can be air, water, solid, or biological tissues (such as bones and muscles). This kind of energy propagates across the medium in the form of waves. Sound is a mechanical wave whose fundamental characteristics are amplitude, which is the distance between the highest and lowest point of the wave and frequency, which is the number of cycles that occur in a second, measured in hertz (Hz). Humans are able to detect sounds with a frequency of 20–20,000 Hz–the normal limits of the human hearing. The term "infrasound" refers to sound waves that have a frequency lower as 20 Hz, and sounds with a frequency higher than 20,000 Hz are called "ultrasound." Unlike humans, some animals, such as bats, dolphins, whales, dogs, cats, and mice can hear ultrasound.

Imaging the Human Body

While traversing a material, the properties of ultrasound change in intensity and speed of propagation, which means that ultrasound waves travel at different speeds depending on the material. Consider two samples of human bone, one from a 30-year-old person and the other from an 80-year-old person. If ultrasound waves cross these two bony samples, the speed at which the sound propagates in the bones can be represented algebraically by the following equation:

$$v = \sqrt{\frac{E}{\rho}}$$

where ν is the speed of ultrasound in the bone sample, E is the modulus of elasticity of the bone sample, and ρ is the density of the bone sample.

The speed of sound (ν) can be calculated by measuring the time required for the wave to propagate through the bone and then dividing by the width of the bone. Knowing the density of the bones (ρ), this equation could be used to determine the values of the modulus of elasticity (E) that indicates the elastic properties of the bone. In a 30-year-old person, the speed of the sound through the bone is approximately 4000 m/s. In an 80-year-old person, this rate drops to 3800 m/s. This fact means that the higher the speed of the sound through the bone, the better is the quality of bone. A low speed could reveal a bony fragility and a fracture probability. This principle is used in ultrasonometry, a technique used to estimate the bony fracture or osteoporosis risk in patients. Ultrasound medical imaging is one of the most powerful diagnostic tools in modern medicine. Along with other imaging methods, it is based on advanced mathematical techniques and numerical algorithms that are necessary to analyze the data and produce readable pictures or three-dimensional images of inner body structures without surgery or use of radiation. It has been widely used to identify the sex or to detect malformations in fetuses during gestation.

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See Also: Diagnostic Testing; Digital Images; Harmonics; Medical Imaging.

Unemployment, Estimating

Category: Government, Politics, and History.
Fields of study: Algebra; Data Analysis and Probability.

Summary: Unemployment rates are calculated using intricate statistical models and sampling methods.

An unemployed person is generally defined as an individual who is available for work but who currently does not have a job. Overall unemployment is typically quantified using the unemployment rate, which represents the number unemployed people as a percent of the labor force. The Bureau of Labor Statistics is an independent statistical agency of the U.S. federal government primarily responsible for measuring labor market activity. Many mathematicians and statisticians are involved in data collection, modeling, and estimation of employment activity, including the highest levels of direction and management. For example, Janet Norwood was the first woman commissioner of the U.S. Bureau of Labor Statistics and frequently spoke to the Joint Economic Committee and other congressional Committees. She was also president of the American Statistical Association and chair of the Advisory Council on Unemployment Compensation. Regarding her work, she noted, "These data figure very prominently in most of the political debates, so it is extremely important that they be accurate and of high quality, and that they be released in a manner that is totally objective."

Economist John Maynard Keynes's revolutionary work, *The General Theory of Employment, Interest and Money*, was published in 1935–1936. The Industrial Revolution and shift away from an agrarian economy had significantly changed the way in which researchers in many fields looked at economic measures, including employment, and the Great Depression brought even greater attention and emphasis to these concepts. Because of labor-market volatility in the late 1920s, the 1930 U.S. census attempted the first comprehensive federal measure of unemployment, but data from the decennial census were not timely enough to be useful in assessing the effectiveness of Depression legislation to aid unemployed workers. Statisticians used newly emerging polling methods to develop better measures and mathematical models. Better methods also changed, at times, the definition of unemployment. Overall, it is commonly accepted that unemployment induces negative effects on the financial and economic status of societies and individuals with respect to many variables. As workers become unemployed, the goods and services that they could have produced are lost along with the purchasing power of these workers, thus leading to the unemployment of more workers. In addition, a large unemployment rate can induce significant social changes and has been the foundation of civil unrest and revolutions. Mathematicians and statisticians continue to create explanatory and forecasting models that are used to guide policies and decisions intended to stabilize economies and aid unemployed workers at local, state, and national levels. These models draw from mathematical ideas and techniques in a wide range of areas, including time series analyses, equilibrium modeling, structural component modeling, neural networks, and simulation.

Sample Design and Collection of Unemployment Data

In most countries, the task of collecting and analyzing unemployment-related information is assigned to certain governmental agencies. In the United States, the Current Population Survey (CPS), conducted by the Census Bureau for the Bureau of Labor Statistics since the mid-twentieth century, provides most of the necessary data. Counting every unemployed person each month is impractical in terms of both cost and time, so the Census Bureau conducts a monthly survey of the population using a sample of households that is designed to represent the civilian population of the United States. At the start of the twenty-first century, the (CPS) surveyed about 50,000 households per month. The selection is generally a multistage stratified sample selected from many different sample areas. The sample provides estimates for the nation and serves as part of model-based estimates for individual states and other geographic areas.

In the first stage of sampling, the United States is divided into primary sampling units (PSUs) that usually consist of a metropolitan area, a large county, or a group of smaller counties. PSUs are then grouped into strata based on some factor that divides the population into mutually exclusive homogeneous groups. The homogeneity of the stratum ensures that the within-strata variability is very small compared to the variability between strata. One PSU is then randomly selected from each stratum with a probability of selection proportional to the PSU's population size. The second stage of sampling consists of randomly selecting small groups of housing units from the sample PSUs. Elements from this sample of housing units are called "secondary sampling units" (SSUs). These households are usually selected from the lists of addresses obtained from the last decennial census of the population. Housing units from blocks with similar demographic composition and geographic proximity are grouped together in the list. The final sample is usually described as a two-stage sample but occasionally, a third stage of sampling is necessary when actual SSU size is extremely large. In this situation, a third stage, called "field subsampling," is needed in order to keep the surveyor's workload manageable. This involves selecting a systematic subsample of the SSU to reduce the number of sample housing units to a more convenient number. Once a survey is designed and the sample is drawn, field representatives and computer-assisted telephone interviewers contact and interview a responsible person living in each of the sample units selected to complete the interview.

Seasonal Adjustment of Unemployment Data

The collected data by the CPS are subjected to a series of transformations and adjustments before the analytical tools are applied to fit adequate models to the unemployment rate and explain its behavior in terms of relevant factors. Because some types of employment are seasonal or cyclical over time, such as December holiday retail sales or fall farm harvesting, adjustments must often be made to account for such cycles. In fact, throughout a one-year period, the level of unemployment experiences continuous variations because of such seasonal events as changes in weather, major holidays, agricultural harvesting, and school openings and closings. Since seasonal events follow an almost regular periodic pattern each year, their influence on the overall pattern can be easily estimated and eliminated. There are two popular methods for removing seasonality. The first estimates the seasonal component using a regression model with time series errors. The explanatory variables in the regression equation are 12-period harmonic terms. Once the regression coefficients are estimated, the fitted values are evaluated for each month subtracted from the corresponding actual values leading to seasonally adjusted series. The second method consists of simply taking seasonal differences of the unemployment series. The removal of the anticipated seasonal component makes it easier for data analysts to observe fundamental variations in the unemployment level, such as trends, gains, nonseasonal intrinsic cycles, and effects of external events, especially those related to economic factors.

Rate Estimation and Prediction

Since the unemployment survey is conducted in the same manner on a monthly basis, the type of data collected is called "time series data." Dependence or autocorrelation among the observations in such data is common, which means that most classical mean-variance types of statistical models are not applicable for estimation and prediction with most unemployment data. Mathematical and statistical models that take into account the particularity of time-dependent data are called "time series models." Among the most popular and useful are autoregressive integrated moving average (ARIMA) models and their seasonal extension (SARIMA). Such models can be used to describe the relationship between a current unemployment rate and past ones using differencing operations and linear equations. As a consequence, the model can also be used to predict future realizations of the unemployment rate. The ARIMA models are very flexible in the sense that they allow for the inclusion of external factors, which can help explain the movement of the unemployment rate and lead to estimators and predictors with smaller variability errors.

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See Also: Census; Forecasting; Gross Domestic Product (GDP).

Units of Area

Category: Space, Time, and Distance.
Fields of Study: Algebra; Geometry; Measurement; Number and Operations; Representations.

Summary: Numerous units of area have been used throughout history for measuring land.

Specific measurements of land area date back to ancient times to define land ownership (for the purposes of taxation, among other reasons). Some of these measurements are still used in the twenty-first century.

Ancient Units of Measurement

In Mesopotamia, land area was divided into a *bur* (an estate), which covered about 64,800 square meters. The bur, in turn, was divided into *iku* (fields), each of which covered about 3600 square meters. Further measurements and sub-divisions are recorded on surviving land documents. The Egyptians also had their own system based on the *kha-ta* (100,000 square cubits), which in turn was divided into 10 *setat*, which consisted of 10 *kha* (1000 square cubits, or 275.65 square meters).

The Romans had a very specific system of measuring land with the basic measure being an *actus quadratus* (acre), which covered about 1260 square meters. Smaller measurements were described as being a *pes quadratus* (square foot) or *scripulum* (or square perch). These measurements were based on the *pes* (foot) being the basic unit of measurement throughout the Roman Empire, a length that was fixed throughout the Empire.

By contrast, the Greeks used a different system of land measurement by which land was divided into a "plethron"—a variable area of land that consisted of the amount of land a yoke of oxen were able to plough in a single day. As a result, the exact measurment varied from some parts of Greece to other parts (and indeed for different parts of a city), although it was thought to approximate to about four English acres. In rocky and hilly areas, the land area was larger than in other parts of the city state. This method of measuring land area—based on what could be done with it—is quite different to the Roman system and largely emerged from a method of equitable taxation by which those with poorer land could be taxed fairly alongside those with more fertile land.

This Greek concept of land measurement was later followed by the Anglo-Saxons in England with their use of the "hide" as a measure of land. This measurement was used in the *Domesday Book* in 1086 and continued until the end of the twelfth century. Traditionally, it was thought that a hide consisted of the land needed to support 10 families, because it is used instead of the term *terra x familiarum* (land of 10 families) in the Anglo-Saxon version of Bede's ecclesiastical history. In Scotland during the same period, the term "groatland" was used to describe the land that could be rented for a particular coin—in this case, a groat. It would represent a larger area for poorer agricultural land than for richer land.

Medieval Era

By medieval times, in Europe and especially England, the terms of measuring land were standardized, and these tended to follow the Roman measurements of a "perch," a "rood," and an "acre." In spite of these measures (although the "hide" was being phased out), there were other measures including the "carucate," which covered the land that an eight-ox team could plough in a year (approximately 120 acres); a "virgate," which covered land that could be ploughed by two oxen in a year (about 30 acres); and a "bovate," which covered the land that could be ploughed by a single ox in a year. There was also an area known as a "knight's fee," which was the land expected to be able to produce a single armed soldier in times of war. Although early in medieval England, an acre was supposed to be the land that could be ploughed in single day, by late medieval times, it had been formalized as 4840 square yards.

Other Systems of Measuring Area

Elsewhere in the world, many other places had their own system of measuring area. The Chinese had a system based on the *li* (7.9 square yards), the *fen* (10 *li*), the *mu* (10 *fen*), the *shi* (10 *mu*), and the *qing* (10 *shi*). The Japanese also had a system of measurement by *tsubo*, which covered the land that was the same size as two tatami mats (about 3.306 square meters). In Korea, there is a similar measure called the *pyeong*, which covers 3.3058 square meters. These measures are generally used to measure the size of rooms and buildings rather than large areas of land. The *tsubo* and the *pyeong* are both still used in the twenty-first century to help describe the size of houses or apartments for sale, in the same way as the term "square" is used by Australian estate agents (approximating to 100 square feet, or 9.29 square meters).

Metric System

The metric system was devised during the 1790s following the French Revolution in an attempt to standardize measurements and it was adopted by the French after Napoleon Bonaparte came to power in 1799. It focuses on the meter as the main measurement of length, and the square meter as the measurement of area. This is used throughout most of the world in the twenty-first century.

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See Also: Measurements, Area; Roman Mathematics; Units of Length; Units of Mass; Units of Volume.

Units of Length

Category: Space, Time, and Distance.
Fields of Study: Algebra; Geometry; Measurement; Number and Operations; Representations.
Summary: Numerous units of length exist and are used according to the distance measured.

Measuring length or distance has been necessary as far back as the oldest hunter-gatherer peoples in order to perform necessary tasks, such as traveling and finding or hunting food. Many of the first units of length were derived from bodily measurements. Modern units of length can broadly be divided into two categories: the U.S. customary system and the international system. The U.S. customary system is more commonly known as the "American system." The International system is more commonly known as the "metric system." The basic unit of length in the American system is the foot, while the basic unit of length in the metric system is the meter. The American system is used more often in the United States, while the metric system is more common in other parts of the world. Scientific journals almost always report measurements in metric units. The exact values of length measurements depend on the units chosen but certain constants (like π) that are fundamental to related measurements (like circumference) are unitless.

American System

The American system of lengths is similar to the British imperial system from which the American system takes its historical roots. The basic unit of measurement is the foot (ft), which originally was set to be the length of an adult man's foot. Each foot is approximately 0.3048 meters. Smaller distances in the American system are typically measured in inches (in) or less commonly in mils. There are 12 inches in a foot and 1000 mils in an inch. Rather than mils, it is much more common to use fractions of an inch to obtain additional accuracy in the American system. Longer distances in the American system are usually measured in yards (yd) or miles (mi). There are three feet in a yard and 1760 yards in a mile (5280 feet in a mile).

Metric System

The meter was originally established in France as one ten-millionth of the distance from the Earth's equator to the North Pole along the meridian passing through Paris. However, in 1983 it was defined as 1/299,792,458 of the distance traveled by light in a second in a vacuum. Smaller units of length in the metric system are often measured in centimeters (cm), millimeters (mm), micrometers (μm, also known as the *micron*), and nanometers (nm). There are 100 centimeters in a meter and 1000 millimeters in a meter. Similarly, there are 1 million micrometers in a meter and 1 billion nanometers in a meter. Longer distances are usually measured in kilometers (km). There are 1000 meters in a kilometer, which are sometimes referred to as "klicks" in the military. The fermi and the angstrom are also units of length in the metric system, though they are not officially part of the international system. There are 10^{15} fermis in a meter and 10 trillion angstroms in a meter. Because of their small length, the fermi and the angstrom are best suited for very small distances. Less common units of length in the metric system include the decimeter (one-tenth of a meter), picometer (10^{-12} meters), decameter (10 meters), megameter (1 million meters), gigameter (1 billion meters), and petameter (10^{15} meters).

Atomic and Astronomic Measurements

Atomic measurements are also given in terms of either Planck length or the Bohr radius. The Planck length is defined in terms of Planck's constant, the gravitational constant, and the speed of light in a vacuum. The result is that the Planck length is based entirely on universal constants rather than human constructs, such as the second. A Planck length is approximately 1.61625×10^{-35} meters. The Bohr radius is defined as the expected distance between the nucleus of a hydrogen atom and its electron in the Bohr model of the atom. The Bohr radius is approximately 5.29177×10^{-11} meters.

Astronomical distances are typically given in terms of light-years, astronomical units, or parsecs. The light-year is defined as the distance light travels in a vacuum in a Julian year (365.25 days). The light year is approximately 9,460,730,472,581 kilometers or 5,878,630,000,000 miles. Distances such as the light-second, the light-minute, and the light-month are defined analogously to the light-year. The astronomical unit is defined as the average distance between the Earth and the sun, approximately 149,597,871 kilometers or 92,955,807 miles. The parsec is defined in terms of the astronomical unit and an angle with measure one arc second. The imaginary right triangle that defines the parsec has one angle with measure one arc second. The opposite side of the triangle from this angle has length equal to one astronomical unit. The length of the adjacent side to this angle is defined as a parsec and can be derived using basic trigonometry. There are approximately 3.26 light-years in a parsec.

Other Measurements

There are a number of units of length that are based on the American system and still in use in certain professions in the twenty-first century. A furlong is often used in horse racing and is defined as one-eighth of a mile (220 yds). The hand is a unit of length used to describe the height of a horse and is equivalent to four inches. Rods (5.5 yds) and chains (66 ft) are often used in surveying. A fathom is often used to measure the depth of water and is equal to six feet. A nautical mile is approximately equal to one minute of latitude. Thus, there are

1872 meters (approximately 6076 feet) in a nautical mile. Fathoms and nautical miles are often used by mariners.

There are also a number of archaic units of length that may be familiar to the reader, most significantly the cubit (1.5 ft) and the league. The dimensions of Noah's Ark as well as other Biblical artifacts are given in cubits. The league has several different values, however the most common is the distance that a person can walk in an hour (approximately three miles). The league was featured in the title of Jules Verne's *Twenty Thousand Leagues Under the Sea*.

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See Also: Measurements, Length; Units of Area; Units of Volume.

Units of Mass

Category: Space, Time, and Distance.
Fields of Study: Algebra; Geometry; Measurement; Number and Operations; Representations.
Summary: A variety of measurement systems have been used throughout history to measure weight and mass.

Throughout history, there have been many ways of measuring mass. Until modern times, these methods were those used to measure what was known as "weight." A number of ways of assessing weight existed in prehistoric times. The Sumerians used a system similar to that later used throughout the ancient Middle East, with 180 grains making a shekel (or *gin*), and 60 of these forming a pound (or *ma-na*), and 600 of these making a load (or *gun*). A wall painting from ancient Egypt, dating from 1285 B.C.E., shows the god Anubis weighing the heart of Hunefer using scales, indicating that the Egyptians had a system of using weights and measures. There were, however, slight differences between the Middle Kingdom and the New Kingdom in Egypt.

Greeks and Romans

The Greeks, with the extensive use of coinage, used a scale that was based on the barley corn but it was actually more fixed on the weight of individual coins. The Romans adapted the Greek system for their own use, with the basic measure of an *uncia* (or ounce). Twelve of these made up one *as*, with different names were given to parts of an *as*: *quadrans* were a quarter of an *as* and *semis* were half an *as*.

Middle Ages

During the Middle Ages in Europe, there were a number of measures that were used for a variety of purposes. For apothecaries, jewelers, and the making of coins, there were "grains," "scruples," and "drams." Two systems were heavily used in Western Europe. The Troy weights, named after the French city of Troyes, were based on the troy ounce (the name "ounce" coming from the Roman "uncia"). By contrast in England, until 1526, there was the Tower ounce, which was slightly lighter than its continental measure (18.75 dwt/pennyweight, rather than the Troy ounce which was 20 dwt). For both measures, 12 ounces made up a pound. In England, eight pounds equaled a "butcher's stone," and 12 pounds a "mercantile stone." The larger measurements were in tons, which consisted of 2240 pounds—now known as a "long ton." The United States later adopted a measure in which 2000 pounds equals a "short ton."

Throughout Europe, there were regional varieties and customary names. Scotland was divided between using the "Troy" measures, and the "Tron" measures, the latter being used in Edinburgh—the system was standardized in 1661. The Portuguese used a system maintained at a national level and was based on the *onca* (ounce), with 16 of these making an *arratel* (pound), 128 *arrateis* making a *quintal*, and 1728 making a *tonelada*. These Portuguese measures, also used in Brazil, were abandoned when both countries adopted the metric system: Portugal and its colonies (or overseas provinces) in 1852, and Brazil 10 years later. The

Standardization

An attempt to standardize the measurement of mass started in France, which, on December 10, 1799, passed a new law establishing a kilogram that consisted of 18,827.15 grains, although the kilogram had already beem used for the previous four years. It was defined as being a 1 cubic decimeter of distilled water at 4 degrees centigrade, its maximum density. This standardization led to the metric system, and in turn it led to the introduction of what became known as the International System of Units (SI units).

The SI units define mass in kilograms as the base unit. It is almost exactly the same as one liter of water, although the exact measure is the same as a piece of platinum-iridium alloy, which is called the International Prototype Kilogram and is stored in a vault in France. An anomaly meant that it was the only base unit with an SI prefix "kilo-" (meaning thousand). There are a number of multiples and submultiples, but only some are commonly used. A one-thousandth part of a gram is called a "milligram," a millionth part called a "microgram," and 10^{-9} g called a "nanogram." Going the other way, although there are terms such as a "zettagram" (10^{21} g) and a "yottagram" (10^{24} g), these are rarely used. One curiosity is that instead of using the term "megagram," for 1000 kg, the term "tonne" is used; its spelling denoting its difference from the pre-decimal "ton." It has been relatively easy to convert from the old "imperial" system of pounds to the metric SI system, with one kilogram essentially being 2.2 pounds. Most of the world uses kilograms in the twenty-first century, the United States being a prominent exception.

Russians also had their own system, which had emerged from that used by the Mongols—although Peter the Great (r. 1682–1725) overhauled the system and used one based on the English system.

Asia

Elsewhere in the world, there were many other systems of measuring mass. The Chinese used a system with 1000 *cash* making a *tael*, and ten *taels* equaling a *catty*, and 100 of those making up a *picul*. The Japanese system relied on the *momme* (about 3.75 g), with 100 of these forming a *hyakume*, 160 of them making one *kin*, and 1000 of them equaling one *kan*. The *momme* is still used as a measure of mass in the pearling industry, which is still dominated by Japan.

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See Also: Roman Mathematics; Units of Area; Units of Length; Units of Volume.

Units of Volume

Category: Space, Time, and Distance.
Fields of Study: Algebra; Geometry; Measurement; Number and Operations; Representations.
Summary: A variety of units are used to measure volume throughout the world.

Measuring of volume intrigued many scientists in the ancient world. For the most part, crops, stones, and other items were measured by weight rather than volume because of the relative ease of doing so—especially given the irregular shapes of many items. For solid items with irregular shapes, it seemed far too complicated to work out their volume, even if this could be done with any degree of accuracy. This notion changed dramatically with the ideas that have been attributed to the famous Greek mathematician and inventor Archimedes of Syracuse (c. 287–212 B.C.E.). The tale of Archimedes is that he was given the task of determining the purity of the gold used to create the crown of King Hiero II—the king was worried that silver or base metals might have been used in its manufacture and were cleverly disguised. Pondering the problem while getting into a bathtub, Archimedes, according to the story, noticed that the water rose and the amount it rose was equal to the size of the parts of his body that were submerged. This led Archimedes to deduce that water could be used to measure the volume of a particular item, such as the king's crown. It could then be weighed against a block of pure gold of the same volume. It is said that when he realized that this could be done, Archimedes shouted "Eureka!" ("I have found it!") and ran through the streets to tell everybody of his discovery, forgetting that he had not put his clothes on.

Whether or not the story of Archimedes is actually true—and some historians doubt its veracity, although Galileo stated that he believed that it might well be true—the story does illustrate the use of fluid displacement, which can be used to easily measure the volume of irregularly shaped objects. This method does not seem to have been known before the Greeks. Certainly, the ancient Egyptians had major problems working out volume and there are complicated equations and formulae on the Rhind Mathematical Papyrus, which dates to about 1700 B.C.E., illustrating that the Egyptians were already grappling with the subject.

The Romans had two systems for recording the measurement of volume. The first and most often used was for liquid measures and was based on a sextarius (from *sester*), which is roughly 0.54 liters. Six *sesters* made up one *congius*; four of these made up one *urn*; and two *urns* make one *amphora*. For dry measures, although a *sester* was still used and was of equivalent size, eight *sesters* equaled a gallon; two gallons made up one modius (also called "peck"); and three of these one quadrantal (also called "bushel").

English

In use from the Middle Ages, the English ended up with an extremely complicated system of measuring volume, which was formalized as Imperial Measurements. The smallest measure was a "mouthful," with two of those making a "pony," two ponies making a "jack," two jacks making one "gill," two gills making one "cup," and two of those making a "pint." The system continued with two pints making a "quart" (from "quarter gallon"), with two quarts equaling one "pottle," and two pottles making a "gallon." The next levels of measurements were "pecks," "kennings," "bushels," "strikes," "coombs," "hogsheads," and "butts" (also called "pipes"). A slightly different scale was used to measure wine and beer. Even when British adopted the metric system in 1965, some of the old terminology (and measures) were still used, especially pints (for milk) and gallons (for gasoline). A bushel is also the standard measurement for wheat and some other items in agriculture.

Metric System

Although the English had numerous terms, medieval and early modern France had a vast range of measures of volume, which varied from one part of France to another, most arising for customary reasons. After the French Revolution, the new government sought to standardize all systems of measurement, including volume. This process saw the introduction, under Napoleon Bonaparte, of metrication, and in turn it led to the International System of Units (SI units).

The original metric system had liters (or litres) as the measure of volume, and from 1901 until 1964, it was defined as being the volume of one kilogram of pure water heated to 4 degrees Centigrade and measured under a pressure of 760 millimeters of mercury. When it came to devising the SI units, the liter was dropped as a measure, and the official measurement was in cubic meters. The difference is not significant in all but scientific terms, although liters continued to be used by many people throughout the world.

Gas Volume

While it was possible to measure the volumes of liquids easily (and also of solid objects by measuring the displacement of a water of a similar quantity), the measuring of the volume of gas has long posed a problem. The problem was solved by the British civil engineer Samuel Clegg (1781–1861), who had worked on natural gas flues and was able to design a dry meter and then a water meter, which were able to measure the amount of gas used by consumers. This invertion helped the gas industry in Britain—and later in other countries—measure gas and thereby charge customers based on usage.

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See Also: Archimedes; Cubes and Cube Roots; Measurements, Volume; Roman Mathematics; Units of Length; Units of Mass.

Universal Constants

Category: Space, Time, and Distance.
Fields of Study: Number and Operations; Measurement.
Summary: Universal constants help describe the universe and are believed to be fixed for all times and places in the universe.

A universal constant is a physical quantity whose value remains fixed throughout the universe for all time. However, most constants are known only approximately; humans started measuring them relatively recently and it is an assumption that they are—and have always been—fixed. There may be other assumptions that scientists and mathematicians have implicitly made that turn out to be false and undermine the universality of these constants. For example, the ratio of the circumference of a circle to its diameter in Euclidean space is π, but with Albert Einstein's conceptualization that the universe could have non-Euclidean geometry, this circumference-to-diameter ratio in the real world may be some value not equal π.

The international Committee on Data for Science and Technology defines and modifies physical constants and quantifies their levels of certainty. Three constants in particular are fundamental to the current understanding of the physical world. Together, they underlie the mathematics of gravity, relativity, and quantum physics. They are G (the gravitational constant), c_0 (the velocity of electromagnetic radiation in a vacuum (in other words, the speed of light), and h (Planck's constant).

Universal Constant: "G"

G first appeared in Isaac Newton's famous equation $F = Gm_1 m_2 / r^2$, which quantifies the force (F) of gravitation between two masses (m_1 and m_2), where r is the distance between their centers of mass. G is approximately $6.67 \times 10^{-11} m^3 kg^{-1} s^{-2}$ (meters-cubed per kilogram per second-squared), which is a very small number. Gravity is thus a very weak force. Although every mass is attracted to every other mass, the effects of gravity are obvious only when the masses involved are very large (such as with planets).

Using another of Newton's equations, $F = ma$, it follows that the acceleration due to gravity on Earth is the same for all masses. This acceleration is known as g and its value is around 9.81 ms^{-2} at sea level. This value varies with distance from the Earth's center of mass (r in the equation above), so acceleration due to gravity decreases to around 9.78 ms^{-2} at the top of Mount Everest. Knowing g to be about 9.81 ms^{-2} and the radius of the Earth to be roughly 6,378,000 meters, one can use G to show that the mass of the Earth is about 5.98×10^{24} kg. One can also estimate the mass of the Sun and other celestial bodies, such is the applicability of G.

Universal Constant: "c_0"

The velocity of light in a vacuum, c_0, is probably the most widely known universal constant. Since the length of a meter is defined by it, c_0 is fixed at exactly 299,792,458 ms^{-1}. The constancy (or invariance) of c_0 is a principle that was made famous by Albert Einstein in his theory of special relativity. Einstein's principle states that no matter how fast you or the light source are travelling, you will always measure c_0 to be 299,792,458 ms^{-1}. This principle is counterintuitive, but both the constancy of c_0 and related predictions of relativity theory have been verified empirically. From relativity theory, it is known that as velocity increases, measurements of time and space change because duration and displacement are relative—they depend on how fast one is moving. The amounts by which they change are determined by c_0.

What is actually traveling at c_0 in electromagnetic radiation are massless particles called "photons."

As carriers of the electromagnetic force, all light, electricity, and magnetism are the result of photon motion. The relationship between the photon energies and the frequency of their electromagnetic radiation is the basis of quantum physics and the third constant, *h*.

Universal Constant: "*h*"

Named after Max Planck, *h* has an approximate value of 6.63×10^{-34} kgm^2s^{-1}. The units of *h* can be understood as joule-seconds, also known as "action." This unit is distinct from *power*, which is joules *per* second; for example, 10 joules expended every second for 10 seconds is 100 joule-seconds.

The first appearance of *h* was in the Planck's relation $E = hv$. Planck discovered that photons only had certain discrete energy values, the $E = hv$ equation relates the energy (E) of the photon to the frequency (v) of its electromagnetic radiation. The fact that *h* exists implies that energy comes in discrete lumps, not in a continuous stream. The unit of *h* appears in a number of important and fundamental relations, such as Werner Heisenberg's uncertainty principle and Niels Bohr's model of the atom.

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See Also: Einstein, Albert; Elementary Particles; Gravity; Pi; Relativity.

Universal Language

Category: Space, Time, and Distance.
Fields of Study: Communication; Connections Representations.
Summary: Mathematics has been proposed as a universal language; attempts have been made at a mathematics notation that would be recognizable on any planet.

From the beginnings of humanity, people needed to establish connections. Along with speaking, counting developed from the early stages of human evolution. Numbers and counting were necessary in the first civilizations to describe ownership, for trade, or for calculating taxes. Shapes and measures were needed to make furniture, buildings, and ritual places, as well as in landscaping, time-keeping, sky-charts, and calendars. Mathematics is present everywhere in the real world: in science, art, entertainment, business, and leisure. People use mathematics to describe the universe, and mathematics is commonly referred to as the "language of science or the universe." Albert Einstein questioned:

> At this point an enigma presents itself, which in all ages has agitated inquiring minds. How can it be that mathematics, being after all a product of human thought which is independent of experience, is so admirably appropriate to the objects of reality? Is human reason, then, without experience, merely by taking thought, able to fathom the properties of real things?

Some take this idea a step further and view mathematics as a universal or interstellar language or explore the creation of a universal language.

Debate

Those who consider that mathematics is a universal language reason that because mathematics arises naturally and humans possess the ability to be literate in the shared language of mathematics then it must be universal. Others criticize this viewpoint and note that learning mathematics is challenging for many people. Some scientists and mathematicians point to the fact that despite differences between cultures and natural languages, the discoveries in mathematics are the same all over the world because mathematics is so well-suited to describe reality. Discoveries that were simultaneous, like the formulations of calculus by physicist Sir Isaac Newton and mathematician and philosopher Gottfried Leibniz, appear to give even more credence to this viewpoint. However, Newton and Leibniz were able to share ideas and build upon the contributions of the same earlier mathematicians and they developed different mathematical approaches and terminology. In some examples of simultaneous discoveries, like for mathematicians in the Soviet Union and the United States, the researchers were quite separated. Other philosophers and mathematicians assert that humanity invents mathematics and distorts reality in accepting its postulates.

Physicist Werner Heisenberg's uncertainty principles seem to give rise to questions about whether anyone can objectively measure or quantify reality. Attempts to model the universe on a quantum and grand scale have led to both calls for and rejection of a theory of everything.

Creating a Universal Language

Scientists, mathematicians, philosophers, and linguists have long contemplated a language that is universal. Linguists explore languages for commonalities, and Search for Extraterrestrial Intelligence (SETI) researchers analyze signals for mathematical patterns. Some visual or graphical representations are also viewed as universal. In *De Arte Combinatoria*, Leibniz imagined

> . . . a general method in which all truths of the reason would be reduced to a kind of calculation. At the same time this would be a sort of universal language or script . . . for the symbols and even the words in it would direct the reason . . . It would be very difficult to form or invent this language or characteristic, but very easy to understand it without any dictionaries.

Leibniz cited earlier attempts at universal languages, such as correspondances that converted words into numbers by physician Johann Becher or scholar Athanasius Kircher. George Dalgarno had invented a system for translating numbers into words. In 1678, Leibniz also developed this type of system: 81,374 would be written and pronounced as *mubodilefa*. For Leibniz, the digits 0–9 became the first nine consonants of the alphabet and powers of 10 were represented using vowels. Leibniz also planned to explore the logical foundations of geometry via a universal language but he did not continue this work.

Philosopher Sundar Sarukkai noted that: "The search for 'universal' language or 'pure' language is part of human history in all civilizations. In part, this reflects an enormous distrust of ambiguity in meaning." However, he also asserts that, "it is semantic ambiguity that allows individuals and societies to develop and flourish."

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See Also: Calculus and Calculus Education; Mathematics: Discovery or Invention; Mathematics, Utility of; Universal Constants; Visualization.

V

Vectors

Category: History and Development of Curricular Concepts.
Fields of Study: Communication; Connections; Measurement; Number and Operations.
Summary: Vectors express magnitude and direction, and have applications in physics and many other areas.

There are some quantities, like time and work, that have only a magnitude (also called "scalars"). If one says the time is 6 A.M., it is adequate. When discussing velocity or force, however, then magnitude is not enough. If a particle has a velocity of five meters per second, this is not sufficient information because the direction of movement is unknown. Quantities that require both a magnitude and a sense of direction for their complete specifying are called "vectors." Pilots use vectors to compensate for wind to navigate airplanes, sport analysts use vectors to model dynamics, and physicists use vectors to model the world.

History and Development of Vectors

The term "vector" originates from *vectus*, a Latin word meaning "to carry." However, astronomy and physical applications motivated the concept of a vector as a magnitude and direction. Aristotle recognized force as a vector. Some historians question whether the parallel law for the vector addition of forces was also known to Aristotle, although they agree that Galileo Galilei stated it explicitly and it appears in the 1687 work *Principia Mathematica* by Isaac Newton. Aside from the physical applications, vectors were useful in planar and spherical trigonometry and geometry. Vector properties and sums continue to be taught in high schools in the twenty-first century.

The rigorous development of vectors into the field of vector calculus in the nineteenth century resulted in a debate over methods and approaches. The algebra of vectors was created by Hermann Grassmann and William Hamilton. Grassmann expanded the concept of a vector to an arbitrary number of dimensions in his book *The Calculus of Extension*, while Hamilton applied vector methods to problems in mechanics and geometry using the concept of a "quaternion." Hamilton spent the rest of his life advocating for quaternions. James Maxwell published his *Treatise on Electricity and Magnetism* in which he emphasized the importance of quaternions as mathematical methods of thinking, while at the same time critiquing them and discouraging scientists from using them. Extending Grassman's ideas, Josiah Gibbs laid the foundations of vector analysis and created a system that was more easily applied to physics than Hamilton's quaternions. Oliver Heaviside independently created a vector analysis and advocated for vector methods and vector calculus. Mathematicians such as Peter Tait, who preferred quaternions, rejected the methods of Gibbs and Heaviside. However, their methods were eventually accepted and they are taught as part of the field of linear algebra. The quaternionic method of Hamilton remains extremely useful in the twenty-first century. Vector calculus is fundamental in understanding fluid dynamics, solid mechanics, electromagnetism, and in many other applications.

During the nineteenth century, mathematicians and physicists also developed the three fundamental theorems of vector calculus, often referred to in the twenty-first century as the "divergence theorem," "Green's theorem," and "Stokes' theorem." Mathematicians with diverse motivations all contributed to the development of the divergence theorem. Michael Ostrogradsky studied the theory of heat, Simeon Poisson studied elastic bodies, Frederic Sarrus studied floating bodies, George Green studied electricity and magnetism, and Carl Friedrich Gauss studied magnetic attraction. The theorem is sometimes referred to as "Gauss's theorem." George Green, Augustin Cauchy, and Bernhard Riemann all contributed to Green's theorem, and Peter Tait and James Maxwell created vector versions of Stokes theorem, which was originally explored by George Stokes, Lord Kelvin, and Hermann Hankel. Undergraduate college students often explore these theorems in a multivariable calculus class.

The concept of a space consisting of a collection of vectors, called a "vector space," became important in the twentieth century. The notion was axiomatized earlier by Jean-Gaston Darboux and defined by Giuseppe Peano, but their work was not appreciated at the time. However, the concept was rediscovered and became important in functional analysis because of the work by Stefan Banach, Hans Hahn, and Norbert Wiener, as well as in ring theory because of the work of Emmy Noether. Vector spaces and their algebraic properties are regularly taught as a part of undergraduate linear algebra.

Mathematics

A vector is defined as a quantity with magnitude and direction. It is represented as a directed line segment with the length proportional to the magnitude and the direction being that of the vector. If represented as an array, it is often represented as a row or column matrix. Vectors are usually represented as boldface capital letters, like ***A*** or with an arrow overhead: $\vec{A}$.

*Figure 1. **B** and **C** add up to **A**.*

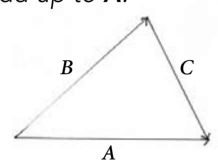
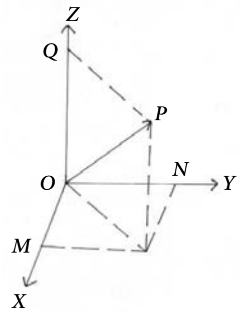


*Figure 2. **OP** can be split into mutually perpendicular components **OM**, **ON**, and **OQ**.*



The Triangle Law states that while adding, "if two vectors can be represented as the two sides of a triangle taken in order then the resultant is represented as the closing side of the triangle taken in the opposite order" (see Figure 1).

Any vector can be split up into components, meaning to divide it into parts having directions along the coordinate axes. When added, these components return the original vector. This process is called "resolution into components" (see Figure 2). Clearly, this resolution cannot be unique as it depends on the choice of coordinate axes. However, for a given vector and specified coordinate axes, the resolution is unique. When two vectors are added or subtracted, these components along a specific axis simply "add up" (like $2 + 2 = 4$ or $7 - 2 = 5$) but the original vectors do not, which follow the rule of vector addition that can be obtained by the Parallelogram Law of Vector Addition. Vector addition is commutative and associative in nature.

Multiplication for vectors can be of a few types:

1. For scalar multiplication (multiplication by a quantity that is not a vector), each component is multiplied by that scalar. Vector multiplication by a scalar is commutative, associative, and distributive in nature.
2. For the multiplication of two vectors, one can obtain both a scalar (dot product) or a vector (cross product). For a cross product the resultant lies in a plane perpendicular to the plane containing the two original vectors. Dot product is both commutative and distributive. But cross product is neither commutative nor associative in nature because the result is a vector and depends on the direction.

Applications

Theoretical sciences have a wide spread of applications of vectors in nearly all fields:

- *Obtaining components*: Occasionally, one needs a part (or component) of a vector for a given purpose. For example, suppose a rower intends to cross over to a point on the other side of a river that has a great current. The rower would be interested to know if any part of that current could help in any way to move in the desired direction. To find the component of the current's vector along any specified direction, take the dot product of that vector with a unit vector (vector of unit magnitude) along the specified direction. This method is of particular importance in studying of particle dynamics and force equilibria.
- *Evaluating volume, surface, and line integrals*: In many problems of physics, it is often necessary to shift from either closed surface integral (over a closed surface that surrounds a volume) to volume integral (over the whole enclosed volume), or from closed line integral (over a loop) to surface integrals (over a surface). To accomplish these shifts, it is often very useful to apply two fundamental theorems of vector calculus, namely Gauss's divergence theorem and Stokes's theorem, respectively.
- *Particle mechanics*: In the study of particle mechanics, vectors are used extensively. Velocity, acceleration, force, momentum, and torque all being vectors, a proper study of mechanics invariably involves extensive applications of vectors.
- *Vector fields*: A field is a region over which the effect or influence of a force or system is felt. In physics, it is very common to study electric and magnetic fields, which apply vectors and vectorial techniques in their description.

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See Also: Function Rate of Change; Gravity; Matrices; Numbers, Complex.

Vedic Mathematics

Category: Government, Politics, and History.
Fields of Study: Number and Operations; Problem Solving.
Summary: Vedic mathematics involves challenging mental calculations and was transmitted orally.

Vedic mathematics is a system of mathematics associated with India's Upper Indus Valley prior to 1000 B.C.E. Originally transmitted orally, the Vedic mathematics known in the twenty-first century was abstracted from ancient Sanskrit texts, known as "Vedas." Sri Bharati Krsna Tirthaji rediscovered the Vedas in the early 1900s, but his scholarly results were not published until 1965.

The Vedas covered all areas of knowledge, with the mathematics created to support this knowledge. Since recording mechanisms were not available, Vedic mathematics involves creative mental calculations, often at very challenging levels. Through Arab and Islamic writers in the 770s C.E., some Vedic mathematics was transmitted and became part of European mathematics, including elements such as the Arabic numerals, the multiplication sign, and a symbol for zero. However, the mental aspects of Vedic mathematics were not known until 1965, and these "secrets" have provided scholars, mathematicians, and students interesting explorations into multiple areas, including basic arithmetic computations, factoring, exponents, algebra in the form of linear through cubic equations, elementary number theory, analytic geometry involving the conic sections, the Pythagorean theorem, and differential calculus.

The Sutras

Sixteen formulas (or *Sutras,* which means "thread") form the foundation of Vedic mathematics, along with fourteen "sub-Sutra" corollaries. Expressed as word phrases, each formula acts as a "thread" woven throughout the Vedic mathematics system, assuming the role of a unifying element.

For example, Sutra #2 states: "All from 9 and the Last from 10." Sutra #3 states: "Vertically and Crosswise." The combined importance of both Sutras is best explained within the context of mental multiplication, such as finding the "sum" 88×98. Both numbers are close to the "base" 100, involving "deficiencies of 12 and 2," respectively. The desired product is obtained using these deficiencies (Sutra #2), then represented either mentally or symbolically (by Sutra #3):

$$\begin{array}{c} 88 \text{ -- } 12 \\ 98 \text{ -- } 2 \\ 86/24 \end{array}$$

In these operations, the deficiencies 12 and 2 are placed to the right of the original numbers, 88 and 98. The 86 is found by subtracting a deficiency from the other number in the product ($98 - 12 = 86 = 88 - 2$), while the 24 is the product of the deficiencies. Finally, the desired result is found: $88 \times 98 = 8624$, as the 86 actually represented 8600. Though this process involves a sense of magic, it is much easier than the modern computational algorithm commonly used in the twenty-first century.

It is not only important to investigate why this Sutra-based technique works, but also determine possible constraints or exceptions. For example, applying the Sutra to the product 25×57, the process becomes:

$$\begin{array}{c} 25 \text{ -- } 75 \\ 57 \text{ -- } 43 \\ -18/3225 \end{array}$$

Because the desired product can be obtained via $-1800 + 3225 = 1425$, the power and the limitations of the Sutra become more evident, especially the emphasis on the numbers 9 and 10. The technique is not useful in this example because the large internal products and need for a negative quantity become obtrusive. However, the method does work, and it can be proven true algebraically. Suppose the desired product is $a \times b$, where a and b are whole numbers less than 100. Using the respective deficiencies ($100 - a$ and $100 - b$), the Sutra's process leads to the algebraic identity $ab = 100[b - (100 - a)] + (100 - a)(100 - b)$. Thus, the numbers a and b could be any numerical values—positive, negative, fractions, irrational, or even complex numbers.

Finding Decimals

As another example, Sutra #1 states: "By One More than the One Before." This Sutra is used in the construction of the number system, as each whole number is one greater than its predecessor (akin to the Peano postulates formulated in the nineteenth century). However, the Sutra's power is its application in other situations as well. Suppose the problem was to find the repeating decimal equivalent to the "vulgar" fraction 1/19, usually obtained by laboriously dividing 19 into 1. The Sutra suggests a focus on "one more than the number before" the 9, or the number 2, which is one more than the 1 which appears before the 9. The 2 (called *Ekadhika* for "one more") becomes the new divisor in lieu of the troublesome 19. The "strange" decimal resulting from this division of 2 into 1 is

$$0._{1}05_{1}263_{1}1_{1}5_{1}7_{1}89_{1}47_{1}3_{1}68421 \ldots.$$

To explain this strange expression, start with a 0 and a decimal point. Then, 1 divided by 2 is 0 remainder 1, represented by placing a 0 in the decimal expression, preceded by a subscripted 1 as the remainder. The process is repeated, where 10 (or the visual of the subscripted 1 and adjacent 0) is divided by 2, resulting in 5 with remainder 0. Thus, the 5 in the decimal expression now is not preceded by a subscripted number. Next, 5 divided by 2 results in 2 remainder 1, which are represented as before with the remainder becoming the preceding subscript. And, in subsequent divisions, 12 divided by 2 is 6 remainder 0, 6 divided by 2 is 3 remainder 0, 3 divided by 2 is 1 remainder 1, and so on. Finally, to get the final value of the decimal expression for 1/19, the subscripted values are removed: $1/19 = 0.052631578947368421$, as they are needed only as "mental" reminders of the division process by 2. The mathematical explanation underlying this process is quite complex, but can be found in Chapter 26 of Tirthaji's *Vedic Mathematics*.

These two examples illustrate the enjoyment of investigating Vedic mathematics. On one level, the 16 Sutra and their corollaries provide efficient mental algorithms that become very powerful and efficient in special instances. On a second level, the careful examination of the Sutra and its application provides a rich opportunity to understand the role of generalization and algebraic identities.

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Jerry Johnson

See Also: Arabic/Islamic Mathematics; Asia, Southern; Multiplication and Division.

Vending Machines

Category: Architecture and Engineering.
Fields of Study: Algebra; Geometry.
Summary: Ubiquitous vending machines use algebra and Boolean logic to function.

Vending machines are finite state machines, also known as "automata," that transition between states based on customer input data, such as product selection. Vending machine designers use mathematical models and Boolean algebra to determine the states the machine should transition into based on input data variables, with the outcome often expressed as a table. The control unit reads the data as either "true," meaning the machine recognizes the input language, or "false," meaning that it does not.

The first documented vending machine, invented by the Egyptian mathematician Hero of Alexandria, appeared c. 215 b.c.e. By the twentieth century, vending had developed into a billion dollar industry, and vending machines dispensed a variety of products. Older vending machines relied on the mechanical activity of knobs or levers activated by the customer to dispense the desired product. Vending machine operators utilize mathematics to determine potential and actual expenses and profits, as well as to process sales and stock data. For example, net income can be determined through the simple formula: Net Income = Income – Expenses.

Modern vending machines, however, utilize basic computing system processors to analyze customer input data, such as a letter and number, that corresponds to the desired product, which is then electronically dispensed. Modern advances in vending machine technology include card validators for debit and credit

A woman buying a beverage on a Tokyo street. Vending machines are extremely popular in Japan and there are machines that sell ramen noodles, alchoholic beverages, fruit and vegetables, batteries, and even clothing.

cards; voice activation; electronc message displays for insufficient funds, lack of change, or sold out products; and remote wireless diagnostics and data collecting to alert venders of the need for restocking or repair.

Vending machine control units are part of a class of abstract machines known as "finite state machines" or "automata"; in particular, they are deterministic or discrete finite state automata (DFA). Finite state machines are always in a position known as a "state," transitioning between these states based on input data. Designers use mathematical models in the design of finite state machines, such as vending machines. The machines are designed to recognize a regular language, converting computation into language recognition. Each state is labeled either "true" (accept the data) or "false" (reject the data) based on whether the machine recognizes the language of the input data.

Vending machine design utilizes Boolean logic or algebra, or algebra based on two logical values, in this case the values of "true" and "false." The general Boolean function is expressed through the formula

$$y = \sum (x, \ldots)$$

where $(x, \ldots)$ is equal to a set of Boolean variables with the values "true" or "false." Diagrams of the various states of the vending machine and the possible transitions between them can be converted into Boolean operations.

The control unit reads each string of input data, generally input from the vending machine customer, such as the diameter, thickness, or number of ridges of coins followed by product selection codes. Transition functions tell the machine which state it should enter based on input data. Transition functions are often represented in tabular form. The control unit changes its state with each data string entered until the final input, after which it outputs either "true" or "false" based on its final state. Vending machines also use the algebraic relationship between range and domain, where the range is the machine's output and domain is the customer's input. For example, a customer must input an equal or greater amount of money than the cost of the desired product.

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See Also: Algebra in Society; Closed-Box Collecting; Functions.

Video Games

Category: Games, Sport, and Recreation.
Fields of Study: Algebra; Data Analysis and Probability; Geometry.
Summary: Video games use the mathematical concepts of algorithms, matrices, and random numbers as part of their programming.

Video games are pervasive in modern society, from computers to television-based systems to applications that can be downloaded easily onto cell phones. There is an ongoing debate over what should be called the first video game. The narrower definition is a game generated by a computer and displayed on a video device. Others consider it to be any electronically based game displayed with video output. The most likely candidate is a 1940s invention by physicists Thomas Goldsmith and Estle Ray Mann. Their "Cathode-Ray Tube Amusement Device" was inspired by World War II radar displays and allowed the player to shoot virtual missiles at targets. Though patented, at the time it was too costly to produce commercially and only a few prototypes were ever made. Much of the mathematics used to design and operate computers also applies to video games and the various fields and professions are closely connected. Video game design programs offered by many colleges emphasize physics and mathematics education along with computer programming, as these skills are necessary to represent the real world in increasingly realistic ways. The new generation of body-sensing game controllers uses optics to detect a player's motion in three axes and translate it to corresponding movements within the game environment. While most people think of video games as entertainment, they are increasingly being incorporated into the classroom and other learning applications. In 2009, U.S. President Barack Obama initiated a campaign called Educate to Innovate, which seeks to use interactive games, among its other strategies, to improve the mathematical and scientific abilities of American students.

Simple Modeling Using Polygons

Any video game that has graphics needs to have a way of drawing a picture on the screen. A very basic program can take a turtle (or curser) on a screen and move it forward and rotate its direction clockwise. Many geometrical shapes are easy to draw using a turtle. For example, to tell the turtle to draw a rectangle, a simple program might tell the turtle to move 100 steps (which could be measured by pixels on the screen), turn 90 degrees, move 50 steps, turn 90 degrees, move 100 steps, turn 90 degrees, and move 50 steps. At this point, a 100×50 rectangle has been drawn, and the turtle is perpendicular to the position where it started.

A circle (or any object with a curve) would be much more difficult to draw using these commands because of the thickness of a pixel and the fact that the turtle cannot move half degrees. A user could try to tell the turtle to move one step then turn one degree. After repeating those commands 360 times, the turtle will be back where it began, and will have drawn a circle that is slightly less than 115 steps across. Technically, it did not draw a circle, but rather a polygon with 360 sides. A slight modification may be to tell the turtle to move two steps then turn one degree. After 360 repetitions, the turtle will appear where it started and the shape appears to be a circle that is twice as wide as the first shape drawn, about 229 steps wide.

There is a big gap between 115 steps and 229 steps wide. If a programmer needs a circle between those dimensions (or beyond those dimensions), the programmer can use mathematics to adjust the step length to get a circle of the desired size. The length across a circle is called the "diameter" and the distance around a circle is called the "circumference." The relationship between these two measurements is $C = \pi d$, where C is the circumference, and d is the diameter.

Since the turtle will be tracing the outside of the circle, it will travel the length of the circumference. The turtle will also be making 360 turns during its travel. Since each step should be the same length, one can find the length of each step by taking the circumference and dividing by 360. Since π is approximately 3.14, one can estimate the length of the step by multiplying 3.14 and the desired diameter and then dividing by 360.

Depending on the video game being created, a programmer will probably desire to draw more than circles and polygons. Using the above steps for a circle but only repeating the steps 180 times will yield a half circle, which could approximate the shape of a setting sun, the top of a silo, or the ice cream in a cone. More complex shapes, like drawing a long-haired cat, could be made by the turtle but the programmer now has a time concern. The programmer creating the directions to draw the cat and the fur on the cat would require a long time to type in the programming for the cat—and even more if the cat is supposed to move—since the repeat step would be used sparingly, if at all. On the users end, a large program with a lot of steps would take a long time to draw, depending on the speed of the computer or gaming system on which it is to be played.

Although video games are displayed on a two-dimensional screen, programmers now commonly create elements of the game in three dimensions. To mimic the body of an object, programmers create the outer shell of the object using a mesh of triangles or quadrilaterals. Depending on the detail desired, more meshes could be created. Once the object is created, it needs to be displayed on the screen. This process involves using a "point-of-view camera," which will change how the object is drawn based on where the camera is and how far away it is from the object. The triangle mesh of the object is adjusted accordingly. For example, as the object approaches the camera (gets closer to the screen), the triangles will elongate and become larger. A programmer that wants the object to get closer to the camera and rotate will use vectors and matrices (linear algebra) to adjust the size and the dimensions of each triangle in the meshes. Once the computer does the mathematical calculations to modify the triangle mesh, the point-of-view camera creates a two-dimensional image of the three-dimensional mesh in the orientation it has been set to. This two-dimensional image then gets projected to the viewing screen.

Interesting geometry is also found in the movement of objects through the game. In some cases, like the games Portal and the older PacMan, players can exit

A game programmer working with multiple monitors. The screen on the right shows a portion of the large amount of source code used.

the playing field on one side of the screen and return from another side or in a different orientation. This property involves concepts like a torus and higher-dimensional analogs.

Color

When programming colors (assuming the screen is not monochrome), a programmer needs to remember that the primary colors for light are different than the primary colors for pigment. When drawing on paper, the three colors magenta (red), cyan (blue), and yellow can be combined in such a way as to create almost any other color. For example, many color printers only use three colors to print. Since most screens work based on a projection of light (whether a computer monitor or a television screen), the primary colors of light must be used. For light, the colors red, blue, and green are the primary colors; with these colors, any other color can be created. All three together make white, and no light at all makes black.

When coding colors, each of the three primary light colors is given an intensity value 0 – 255. This value is then converted to a two-digit hexadecimal number, where 00 is the decimal number zero and FF is the hexadecimal number 255. The hexadecimal number 12 would be an intensity level of 18. The hexadecimal number A0 would be an intensity level of 160. The programmer then takes these three intensity numbers and combines them to make a six-digit color number by placing the intensities in order for red, then green, and finally blue. For example, pure red would be FF0000 (intensity 255 for red and intensities 0 for both green and blue). Similarly, 00FF00 would be pure yellow and 0000FF would be pure blue. The color white would be represented FFFFFF (a combination of all three colors), whereas black would be 000000 (no light whatsoever).

Random Number Algorithm

Many video games that have been created offer a storyline or, at least, a progression to get from one stage or level to the next. Moving to the next level often requires a certain level of skill or collecting certain objects. On the other hand, there are video games that are created, like video poker or Tetris, where skill alone is not enough to do well. There is a certain random element that will determine the outcome. However, computers are not capable of creating random numbers. Instead, the video game console is pre-programmed with a list of pseudo random numbers. For example, every TI-84 calculator that has its memory reset will create the number 0.943597402492 13 as its first "random" number. Obviously, if everyone obtains the same result, it cannot be random.

Using the TI example, every random number it produces will be a decimal between zero and one. If the game requires a number higher than one, the programmer merely multiplies the random number times the highest number they desire. For example, in Tetris, there are seven tetrominoes that could be selected for the next drop. A programmer may want a random number generated to determine the shape of the next piece. As a result, the programmer would create a random number, and then multiply it by seven to get a number between zero and seven; however, this number is still a decimal. The computer programmer can then tell the console to truncate the number, which would ignore everything beyond the decimal point giving an integer between zero and six. A final (optional) step would be to add one to this truncated integer resulting in a number between one and seven. Each block would get assigned a number, and the pseudo-random number that resulted would select the next block.

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See Also: Animation and CGI; Polygons.

Vietnam War

Category: Government, Politics, and History.
Fields of Study: All.
Summary: Because of the importance of cryptography in World War II and the emergence of game theory in the 1950s, mathematics was heavily involved in the Vietnam War.

The Vietnam War, a conflict transpiring in Vietnam, Cambodia, and Laos from 1955 to 1975, involved the Communist forces of North Vietnam, the Viet Cong, the Khmer Rouge, the Pathet Lao, the People's Republic of China, the Soviet Union, and North Korea, against the anti-Communist forces of South Vietnam, the United States, South Korea, Australia, the Philippines, New Zealand, Thailand, the Khmer Republic, Laos, and the Republic of China. Most American involvement was concentrated from 1963 to 1973, with the last U.S. troops leaving with the fall of Saigon in 1975. It eventually resulted in a Communist victory, with U.S. forces and their allies withdrawing, Communist parties taking control of Laos and Cambodia, and South Vietnam unified with the North under Communist rule.

Mathematicians and the War

Mathematicians fell on both sides of the disagreement regarding the Vietnam War. Some served in the war effort, such as William Corson, an economist with an undergraduate degree in mathematics who later wrote the book *The Betrayal*. Grace Murray Hopper returned to active duty in 1967 because of an increased demand for naval computer systems. Others engaged in war-related research. Warren Henry helped develop the hovercraft for nighttime fighting during the 1960s while working at Lockheed Space and Missile Company and this was used in the war.

In 1966 and 1970, mathematicians at the International Congress of Mathematicians appealed to their colleagues to avoid war-related work. Mathematicians around the world organized or participated in protests, including Alexander Grothendieck in France and Steven Smale in the United States. Mathematicians in Japan at the University of Kyushu in South Japan organized "demonstrations of the 10" against the war on the 10th, 20th, and 30th of the month. Funding originally designated for teacher development during the New Math movement was instead directed to the war. Some have asserted that this diversion of funds was one of the main reasons that the educational movement failed. Mathematics played a role in the war in a number of ways, including war strategy, precision weapons, airplane computers, cryptography, and a statistically flawed 1969 draft drawing. Statisticians and others have used statistical techniques to study the long-term effects of Agent Orange on soldiers. Decision theory has been used to model the war. Systems analysis and game theory may have contributed to U.S. involvement and defeat, such as in the decisions of Secretary of Defense Robert McNamara.

Game Theory

One of the key political leaders of the American forces during the Vietnam War was Robert McNamara, a student of game theory, who served as the secretary of defense from 1961 to 1968—the period corresponding with the nation's first serious engagement with the war and its major expansions and escalations. McNamara was also responsible for the policy of Mutually Assured Destruction (MAD), a nuclear policy grounded in game theory. It said that the best deterrent to full-scale use of nuclear weapons was for opposing sides to each possess sufficient firepower to completely destroy the other so that neither side dares attack, knowing it cannot survive the counterattack. A chilling take on foreign policy, history may be on McNamara's side with the Cold War. The escalating war in Vietnam is another story. From a game theory perspective, those escalations make perfect sense. Consider that fact that North Vietnam had the options to escalate or to negotiate a peace. The United States also had those options as well as the option to pull out. The only way for the United States to gain a military advantage—and potential victory—was to escalate, with the worst possible outcome of such escalation being a stalemate. Despite increased desertion and plummeting morale, as well as growing anti-war sentiment at home, McNamara continued to escalate the engagement because it was the most promising option he was trained to see.

This was later used as an example of "escalation of commitment," a phenomenon identified in Barry Straw's 1976 paper "Knee Deep in the Big Muddy: A Study of Escalating Commitment to a Chosen Course of Action," wherein cumulative prior investment becomes the motive to continue to escalate one's investment even when rational thought says it is the wrong choice. That initial error of judgment becomes the motive to continue, to stay committed to the course of action, in order to justify it. The more one continues, the greater error one must admit to if one disengages, which is why psychologists sometimes refer to this phenomenon as the "commitment bias," a natural tendency to want to believe that one has been making the right choices and to ignore evidence to the contrary.

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See Also: Asia, Southeastern; Cold War; Game Theory; Infantry; Military Draft; Predicting Attacks; Vietnam War.

Viruses

Category: Medicine and Health.
Fields of Study: Algebra; Geometry.
Summary: The spread of viruses in a population—and the internal structure of viruses themselves—can be analyzed mathematically to help epidemiologists study viral infections.

A virus is a parasite. It cannot reproduce on its own. Instead, it must invade a cell of another organism and use the host cell's machinery to make copies of itself. The newly replicated viruses then leave the host cell and infect other cells. In the process, the virus often damages the host. For example, different viruses cause measles, polio, and influenza in people; hoof-and-mouth disease in cattle; and leaf curl in many vegetables. Mathematics provides a language to describe viral structures. Furthermore, mathematical models of the spread of a virus in a population are powerful tools in public health policy.

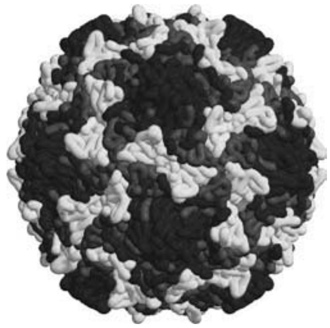
Capsid Geometry

A virus consists of genetic material (either DNA or RNA) surrounded by a protein coat called a "capsid." Viruses have much less genetic material and are much smaller than single-celled organisms like bacteria. With limited genetic material, a virus can encode only a few proteins of its own, and so must use them efficiently. Often, the entire capsid is assembled from many copies of a single protein, which means the capsid should be highly symmetric.

One of the first virus structures to be determined was that of the Tobacco Mosaic Virus (TMV). Copies of the TMV capsid protein are arranged in a helix around the viral RNA. Many other viruses have helical capsids as well. In contrast, poliovirus, the Hepatitis B virus, tomato bushy stunt virus, and other viruses have icosahedral capsids. Figure 1 shows a computer-generated image of the poliovirus capsid with protein subunits colored to highlight the icosahedral symmetry. Other, more complicated capsid shapes are possible.

While the capsids do not have flat triangular faces, they have axes of five-fold rotational symmetry, like those through the vertices of the icosahedron; axes of three-fold rotational symmetry, like those through the centers of the triangular faces of the icosahedron; and axes of two-fold rotational symmetry, like those through the centers of the edges of the icosahedron.

Figure 1: Poliovirus capsid.



Modeling the Spread of Viruses

Models of virus transmission in a population help researchers understand which interventions might slow the spread of a virus. The SIR model, first proposed by W. O. Kermack and A. G. McKendrick in 1927, is one of the simplest and is suitable for viruses such as measles and influenza. Each person in a population is in one of three categories: (1) susceptible to the virus, (2) infected and infectious, or (3) recovered and immune.

Let S, I, and R be the proportion of the population that is susceptible, infected, and recovered, respectively. The SIR model is given by the following system of differential equations:

$$\frac{dS}{dt} = -\beta SI, \ \frac{dI}{dt} = \beta SI - \gamma I, \text{ and } \frac{dR}{dt} = \gamma I$$

where the constant β depends on the probability that an infected person transmits the virus to a susceptible person, and the constant γ depends on how long it takes an infected person to recover. This model does not lead to simple expressions for S, I, and R as functions of time but it can be explored computationally. One simple way to do so is to treat time discretely and approximate

$$\frac{dS}{dt}$$

by $\left(S_{t+1} - S_t\right)$, where S_t is the value of S at time step t. This method yields the difference equations

$$S_{t+1} = S_t - \beta S_t I_t$$

$$I_{t+1} = I_t + \beta S_t I_t - \gamma I_t$$

$$\text{and } R_{t+1} = R_t + \gamma I_t.$$

The basic SIR model can be modified to fit other scenarios. For example, immunity might wear off over time, or some part of the population might be at higher risk of infection, or a vaccination campaign might begin.

The SIR model assumes that all possible contacts between infected people and susceptible people are equally likely (hence the factor of SI in dS/dt). Modifying the model to reflect the social structure of the population allows researchers to ask crucial questions. If the supply of influenza vaccine is limited, is it more effective to vaccinate school children, who spread the disease, or the elderly, who may suffer more complications from infection? Will closing airports slow an epidemic enough to justify the costs to travelers? In such situations, mathematical models allow public health officials to test the effects of different interventions before choosing a course of action.

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Catherine Stenson

See Also: Diseases, Tracking Infectious; HIV/AIDS; Mathematical Modeling; Polyhedra; Symmetry.

Vision Correction

Category: Medicine and Health.
Fields of Study: Geometry; Measurement; Problem Solving; Representations.
Summary: Modern optometry depends on precise measurements to construct corrective lenses.

Human vision is subject to a variety of ailments and disorders. Some are congenital; others are age-related. Faulty vision results in blurriness, coupled with headaches and ocular tiredness. However, for many years, humans have been perfecting the art of using external implements to aid vision. Technologies exist in the twenty-first century that can restore perfect vision to people suffering from common vision-related problems, such as myopia or astigmatism. The methods used to diagnose vision issues and to construct corrective lenses rely on precise mathematical measurements and understanding of the geometric principles behind light refraction. Vision may also be modeled in various ways, including using a concept called "orthonormal polynomials," such as the Fourier series and optic wavefronts. This has many applications, including laser vision correction. In stereoscopic vision, two-dimensional projections of the world onto the retina of each eye are combined and compared to form a three-dimensional image. It was once thought of as virtually impossible to cure stereoblindness, but in the early twenty-first century, vision therapists use a variety of techniques to help patients perceive stereoscopic depth in three spatial dimensions.

Lens Power

The optical power of a lens, also known as "dioptic power," "refractive power," or "focusing power," is a measure of the curvature of the lens and the degree to which a lens converges or diverges light. It is equal to the reciprocal of the focal length of the lens in meters. Its unit is "diopter." Prescriptions for eyeglasses specify the optical power of the lenses. The human eye has a refractive power of 60 diopters. Stacking lenses helps to combine their optical power.

Eyeglasses and Bifocals

A simple pair of eyeglasses contains nothing more than two pieces of glass shaped in such a way that they act like a pair of lenses. Lenses exploit the physical property of light called "refraction." Refraction occurs when light travels between mediums of different densities, such as air and glass. The change in the medium causes light to bend in a certain calculable way. This property of lenses is suitable to refocus the image back onto the retina in people suffering from long-sightedness and short-sightedness.

The focal length of a lens in air can be calculated using the lensmaker's equation, given by

$$\frac{1}{f} = (n-1)\left[\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n-1)d}{nR_1 R_2}\right]$$

where f is the focal length of the lens, n is the refractive index of the material, R_1 is the radius of curvature of the lens surface closest to the light source, R_2 is the radius of curvature of the lens surface farthest from the light source, and d is the thickness of the lens.

To address people suffering from vision problems such as myopia, hyperopia, and astigmatism, bifocal lenses were invented. These lenses have a section of magnification at the lower portion of the frames to allow the wearer to read small print. Benjamin Franklin is generally associated with the invention of the first pair of bifocals.

Contact Lenses

Contact lenses are corrective or cosmetic lenses placed on the cornea of the eye. Their performance is similar to that of eyeglasses but they can be shaped somewhat differently. Spherical lenses are the typical shape of contact lenses on both the inside and the outside surfaces, whereas toric contact lenses, often used for people with astigmatism, are created with curvatures at different angles and cannot move on the eye. Contact lenses are extremely lightweight and are virtually invisible when compared to eyeglasses. However, they are also not held in place by a rigid framework like glasses. Mathematical models are useful for understanding the various movements of lenses within the eye, especially hard contact lenses.

In the twenty-first century, technology has advanced to a level where it is possible to imprint electronics onto the contact lenses themselves, resulting in the ability to project a virtual display onto the eye directly. While this technology by itself does not directly correct any vision problems, it could be used to assist people in their everyday activities, such as locating objects, or reading street signs by magnifying letters.

LASIK

Laser-assisted in situ keratomileusis (LASIK) is becoming an increasingly popular alternative to contact lenses and eyeglasses. LASIK is a type of refractive surgery performed using a laser. A "laser" (Light Amplification by Stimulated Emission of Radiation) is a highly concentrated beam of light capable of focusing high energy in a small area.

The technology was invented by a Colombia-based Spanish ophthalmologist Jose Barraquer. His technique involved cutting thin flaps in the cornea and altering

LASIK VISX surgery being performed by a U.S. Navy surgeon. A close-up of the eye is seen on the monitor.

its shape. After the laser was invented, Dr. Bhaumik, in 1973, announced the breakthrough in using lasers to treat vision problems.

LASIK involves creating a flap of corneal tissue, remodeling the cornea underneath the flap with the help of a laser, and then repositioning the flap. Mathematical computations are used to determine the depth of the cuts used in the surgery, and these are often a function of the average cornea thickness of 550 micrometers. One alternative is to leave some fixed tissue depth.

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Ashwin Mudigonda

See Also: Light; Surgery.

Visualization

Category: History and Development of Curricular Concepts.

Fields of Study: Algebra; Communication; Connections; Data Analysis and Probability; Geometry; Representations.

Summary: Visualization is a useful practice when doing or learning mathematics and computers can help create visualizations of difficult concepts.

The ability to form a mental image is a fundamental process and has been incorporated in many theories about knowledge acquisition. The advent of the printing press and perspective drawings allowed for an unprecedented sharing of realistic pictures, graphs, and inventions, and this led in part to the Industrial Revolution.

The development of coordinate geometry gave rise to graphical representations of data and algebraic concepts. With the popularity of computers and computer graphics, mathematicians, artists, and programmers have created visualizations of mathematical objects and huge amounts of data. Mathematicians also found new ways to visualize and share abstract ideas such as the fourth dimension. Dynamic image manipulation features, such as rotation or zooming, further increased the accessibility of visualized objects by facilitating new perspectives and comprehension of hard-to-see surfaces. Mathematical visuals have been fundamental in both research and entertainment contexts like for computer-generated imagery (CGI) used in modeling, computational geometry, or movies. Various types of visualization, including spatial visualization and visuals of data and graphs, are important components of all levels of mathematics and statistics classrooms in the twenty-first century. Visualization is an interdisciplinary topic and researchers from a diverse range of fields contribute, including mathematicians, computer scientists, psychologists, engineers, and neuroscientists. Educators and researchers create visualizations, study visualization ability, and design new ways to help students visualize.

Early History

Visualization has been as important in mathematics and statistics research as in education and mathematicians in many fields throughout history created visual representations. Representations of maps are as ancient as the earliest societies from which there exists evidence of stone tablets and animal skins. Another important historical research area related to visualization and mathematics was the field of optics. For example, ancient people created lenses. Euclid of Alexandria investigated geometry and perspective in his book on optics. Many mathematicians and scientists worked to understand vision, including mathematician Abu Ali al-Hasan ibn al-Haytham, who wrote a seven volume work on optics and visual perception, which is noted by some as the first work to correctly demonstrate understanding that light is reflected from an object to the eye. Self-taught mathematician and scientist Tobias Mayer was one of many to formulate a theory for color perception and he also modeled the limits of vision, noting, "there is a certain visual angle below which an object presented to the eye appears either not distinct enough or not even distinct at all, but only confused and as though it had vanished from sight. . . . We shall call this angle the limit of vision, and we shall investigate its angle by experiment."

In the seventeenth century, René Descartes made significant progress in coordinate geometry. The Cartesian plane that is named for him allowed for new representations of data and algebraic equations. Mathematicians, statisticians, social scientists, and others began to investigate ways to visually present graphs and data to facilitate analysis, interpretation, and understanding. Social issues motivated many researchers in the nineteenth century. For example, William Playfair created color-coded graphical representations of the English national debt and the trade balances between England and other countries. Adolphe Quetelet graphed the distributions of anthropometric data to show both the center and variability, leading in part to the measure now known as Body Mass Index. Florence Nightingale developed the polar area chart as part of her campaign for improved sanitation in medical facilities. John Snow used graphical mapping techniques to trace the source of a London cholera outbreak. Graphs of mortality statistics and many other naturally occurring phenomena also proliferated. Philosopher and logician John Venn developed Venn diagrams in 1881, which are also used in many mathematics classrooms.

Recent Developments

The rise of computers in the twentieth century led to mind-bending visualizations and new fields of research in mathematics as well as beautiful artistic forms. Mathematician Benoit Mandelbrot popularized the field of fractals. The computer visualization of some objects helped clarify their mathematical properties. One example is Enneper's surface, which had been introduced by Alfred Enneper in the nineteenth century. In the mid-twentieth century, Steven Smale proved that it was possible to turn a sphere inside out in three dimensions without creating any creases. This idea stretched the imagination and mathematicians tried to visualize it. For instance, mathematicians at the Geometry Center for the Computation and Visualization of Geometric Structures produced a video called *Outside In*, which visualized William Thurston's sphere eversion method. Geometer Thomas Banchoff pioneered visualizations of four-dimensional objects. Mathematicians in the twenty-first century attempted to visually model the Internet using hyperbolic geometry in order to reduce the load on routers. Researchers from interdisciplinary fields have participated in conferences on topics like visualization algorithms or data visualization. Mathematicians have designed visualization software and techniques for many areas in mathematics, including linear algebra, group theory, and complex analysis. Some of these visualizations are used in classrooms, while others are the focus of research investigations or artistic exhibitions.

Visualization Ability

The connections between visualization ability and mathematical success also have a long and varied history. In the nineteenth century, scientist and mathematician Sir Francis Galton conducted studies to examine the relationship between visual imagery and abstract thought. Some have noted that nineteenth-century mathematician Henri Poincaré had poor eyesight as a student and scored a zero on an entrance exam for the École Polytechnique; however, he had a great memory because he was able to mentally translate concepts he heard aurally into visual representations of the same concepts. Poincaré later wrote about the ability to form retina images and what he referred to as "pure visual space." The Poincaré disc model of hyperbolic geometry is named for him, and twenty-first-century students explore this in interactive computer models that are designed to help visualize and explore mathematical topics, including the variation in the sum of the angles for differently sized triangles.

Other visual challenges, like "stereoblindness" and "subitizing" difficulties, have also been tied to mathematics. Stereoblindness, the inability to properly combine images in the mind to see in three dimensions, was once thought of as impossible to cure. Subitizing is the ability to rapidly perceive and differentiate the number of distinct items in a small group of objects, like dots on a cube. Some researchers in the first part of the twentieth century investigated the importance of subitizing to the understanding of numbers, counting, and abstract thinking and educational psychologists in the second half of the twentieth century continued this work and developed a variety of theories. While the specific mechanisms are still the topic of debate, in the twenty-first century, vision and subitizing therapies have been successfully implemented in the optometry profession and are thought to help mathematics students. Some proponents of left-brain versus right-brain dominance theories assert that visualization is focused in the right brain, while other mathematical skills, like logic and analysis, are focused in the left side of the brain. Psychobiologist Roger Sperry was awarded the Nobel Prize in 1981 in part for his split-brain experiments. However, medical imaging scans of people performing mathematical tasks has shown regions from both sides of the brain highlighted and researchers continue to investigate this issue.

Gender

In the latter half of the twentieth century, researchers investigated gender differences in spatial visualization ability. In 1978, geneticists Steven Vandenburg and Allan Kuse developed a mental rotation test that has been used in part to quantify spatial visualization ability. In 1980, Camilla Benbow and Julian Stanley, referred to as psychologists and educators, asserted that gender differences in mathematics might result from "greater male ability in spatial tasks." Their statements were widely publicized in the media. Later researchers found that visual training by video games or certain changes in testing conditions, like removing "I don't know" as an answer or eliminating time constraints, could reduce these observed gender differences. Research on stereotype vulnerability, where the effort to counter societal perceptions about a whisper of inferiority can negatively impact performance, has further complicated visualization research efforts.

Education

Various educational learning models and theories stress the importance of visualization. In Piaget's theory, named for epistemologist Jean Piaget, spatial skills develop at various age levels or stages and according to experience. For instance, he proposed that young children could understand two-dimensional space, while the mental manipulation of three-dimensional objects in space comes later on. Mathematician Walter Whiteley has proposed research questions related to visualization and suggested a variety of ways in which teachers might intentionally train students to "see like a mathematician." He noted:

> Curriculum suggests that 2-D is easier than 3-D, although it is cognitively less natural for many modes of reasoning, and 3-D skills are the needed goal for later work. The domination of analytic over synthetic reasoning encourages the pattern that 2-D is the starting point, and the disconnection between early childhood reasoning, and latter problem solving both of which engage 3-D reasoning.

The van Hiele model of geometric thought, developed by educators Dina van Hiele-Geldof and Pierre van Hiele, listed visualization as its first level. Additional learning models presented by mathematicians and educators have also stressed the importance of interweaving visualization training with other skills.

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See Also: Animation and CGI; Coordinate Geometry; Graphs; Maps; Optical Illusions; Painting; Sculpture; Telescopes.

Volcanoes

Category: Weather, Nature, and Environment.
Fields of Study: Data Analysis and Probability; Geometry.
Summary: Mathematical models and data analysis can help geologists better understand the activity of volcanoes and the fluid dynamics of their eruptions.

Volcanoes are openings of channels connecting the molten interior of a planet with its surface. Active volcanoes emit magma, ash, and gasses, and inactive volcanoes are reminders of past eruptions, consisting of solidified lava and ash. The science of studying volcanoes is known as "volcanology." Many scientists and philosophers throughout history, including mathematicians Johannes Kepler and René Descartes, theorized about their nature and formation. Mathematics continues to play a role in modern volcanology through both the coursework and degrees that are required and in the mathematical research prevalent in the exploration of various volcanic phenomena. Computer-based numerical simulations and digital imagery, often from satellite observation, combined with mathematical and statistical methods, such as neural networks and data mining, are increasingly used to model, describe, and visualize the complex mathematical representations of volcanic processes. Predicting eruptions is also a challenge, which is necessary not only for safety and response at the time of the eruption but also for larger issues such as global climate change. Benjamin Santer of Lawrence Livermore National Laboratory, who specializes in mathematical and statistical analyses of climate data, has used volcanoes as one variable in explaining climate change. Scientists at the Yellowstone Volcano Observatory also collect data to monitor and mathematically study the enormous Yellowstone caldera, sometimes known as the Yellowstone supervolcano.

Measuring Volcanoes

The most destructive volcanic effect comes from pyroclastic flow, which is a mixture of solid to semi-solid fragments of rock, ash, and hot gases that flows down the sides of the volcano. It is a type of gravity current, similar to an avalanche, that can be modeled with theories and equations from fluid dynamics. A useful metric for comparing eruptions is the volume of volcanic ejecta. For example, the 1980 eruption of Mount St. Helens produced about 1.3 cubic kilometers of ash, but the ancient eruption of the Toba volcano on Sumatra around 75,000 years ago produced more than two thousand times more ash. It is possible to measure the fragmentation of the airborne volcanic matter, called "tephra," even for ancient eruptions. Fragmentation is associated with the strength of the volcanic explosion. The dispersion of tephra over an area has been found to be related to the height of the eruption column. Finding and analyzing dispersion allows estimation of heights for ancient eruptions and an additional way to measure heights for modern eruptions. Volcanologists have created the Volcanic Explosivity Index (VEI), which takes into account the volume of ash and the height and duration of the eruption. There are nine types of volcanoes according to VEI, scaled 0–8. For example, the low-strength, low-height Type 0 is called "Hawaiian," and the high-strength, low-fragmentation Type 6 through Type 8 are called "Plinian eruptions," named for Roman historian Pliny the Younger, who described in detail the first century eruption of Mount Vesuvius that destroyed Pompeii. Plinian eruptions can have global environmental effects. Similar to the Richter scale, VEI is logarithmic: each level type is about 10 times greater in magnitude than the previous level.

Geometry of Volcanoes

Shapes of volcanoes depend on their explositivy, viscosity of magma, the composition of the surrounding crust, and other geological factors. The familiar, iconic cone shape such as Mount Fuji defines a "stratovolcano," so named because of its many layers (or "strata") of ash and hardened lava. Eruptions of these volcanoes have high explosivity and low-viscosity lava, making lava and tephra deposit near the opening in layers of diminishing thickness, thus forming the cone.

In contrast, broad, very fluid lava fields produce shield volcanoes that resemble a rather flat warrior shield. Lava domes, as the name suggests, are proportionally higher than shield volcanoes and more rounded than cone volcanoes, resembling semispheres. Lava domes are formed by high viscosity lava combined with low explosivity, where lava either accumulates under the crust and pushes it up, or flows over the crust and solidifies in the dome shape.

Eruption Forecast

Because volcanic eruptions depend on many variables, eruption forecasting relates to such areas of science and mathematics as chaos theory and systems science. Overall, prediction means collecting multi-variate data in volcano observatories and matching variable patterns to those that occurred before eruptions of similar types of volcanoes in the past. For example, the pattern of earthquakes becoming stronger and shallower with time, called "earthquake swarm," can be used to forecast the eruption time. Mathematical models of volcanoes are

After the 1980 eruption of Mount St. Helens, 24 square miles were filled by a debris avalanche.

based on equations from thermodynamics, fluid dynamics, and solid mechanics. The systems science principles of prediction describe qualitative trends in variables. For example, the principle of coinciding change says that unrelated, co-evolving trends in several parameters are more significant than changes in any one parameter.

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Maria Droujkova

See Also: Earthquakes; Geothermal Energy; Measurement, Systems of; Measuring Tools; Plate Tectonics; Prehistory; Probability.

Volleyball

Category: Games, Sport, and Recreation.
Fields of Study: Data Analysis and Probability; Geometry; Measurement.
Summary: Mathematics is fundamental to player motion, strategy, and scoring in volleyball.

Volleyball, which began in the late nineteenth century as a non-contact recreational sport, quickly developed into a globally popular competitive sport. Two teams, typically with two to six players, face one another on opposite sides of a rectangular court divided by a net. Beach volleyball is played on sand courts rather than a hard surface. Game strategy uses mathematical concepts such as angles, rotation, and parabolic motion in an effort to impart optimal trajectories, speeds, and spins on the ball to prevent the other team from successfully returning it. The receiving team must understand three-dimensional motion and vectors in order to intercept the ball and change its direction, often using a sequence of hits coordinated among several players. The strategies of beach volleyballers often differ from those of hard court volleyballers because of differences in the ability to jump or dive for an incoming ball. Mathematics is also used to analyze and model body kinetics, such as the motions of a player's shoulders and arms while serving. Statistics are used to analyze and describe both team and individual proficiencies and success. These include measures like number of attacks, kills, and assists; hitting percentages; and kill average and efficiency as a function of total attempts.

General Game Play and Scoring

Volleyball teams work together to hit the ball over the net in such a way as to prevent the other team from returning it. A match consists of three or five games. The third game of a three-game match or the fifth game of a five-game match is the deciding game. A single sequence of back and forth hitting is known as a "rally," which begins with one side serving the ball and ends when one team or the other fails to legally return it. Each side gets three attempts and the same player may not touch the ball twice in a row. At the end of the rally, the winning side may earn a point, the right to serve the ball, or both.

There are two different scoring systems used in volleyball. In side-out scoring, only the serving team may earn a point. In rally scoring, either side earns a point. Winners always get the serve. Deciding games are played to 15 points; nondeciding games are played to 25. However, the winning team must be ahead by at least two points or play continues. Sometimes, a scoring cap is used, which nullifies this requirement. Statistical analyses show that rally point scoring makes matches shorter and match lengths more predictable versus side-out scoring. However, there appears to be no significant effect on scoring margins between teams; on average, after an even number of serve changes, points awarded to non-serving teams balance. In addition to statistics, Markov chains are useful for analyzing volleyball games in terms of the proportion of points won and the probabilities of winning a point, game, and match.

Player Roles and Strategy

Hard court teams typically consist of six players with specialized roles, with the left, center, and right forwards in a row along the frontcourt and the left, center, and right backs in a row along the backcourt. However, players usually rotate through positions during play, requiring analysis of permutations and the timing of substitutions. Beach volleyball teams typically consist of two players each, generally front and back. Players seek to control the ball through the angle, force, and timing with which the ball is struck and by choosing whether or not to impart spin on the ball. The volleyball typically travels along a parabolic path, modified by its spin and additionally influenced by player efforts and external factors, such as air resistance. The basic skills used in volleyball include the serve, pass, set, spike, block, and dig. A variety of serves can be used as the server hits the ball into the opponent's court. Different types of serves affect the ball's direction, speed, and acceleration with the goal of increasing the difficulty of handling the ball for the opposing team. Serves that have flatter parabolic paths tend to preserve more of the initial force and velocity and are usually more difficult to return.

The opposing team's first reception of the ball is known as the "pass," the second contact is known as the "set," and the third contact is known as the "attack" (also called "spike"), though a team may not opt to use all three contacts in every play. A block is a team's attempt to prevent the opposite team from spiking the ball into their court, and a dig is an attempt to prevent a ball from hitting the court. Shots include the hard angle, deep angle, seam shot, line shot, angled line shot, swiping shot, high and hard, and the save. Achieving different shots relies on affecting the ball's speed, spin, and angle of trajectory through shoulder and hip positions, aiming at gaps between opposing players, and the amount of force applied. Spin tends to make the ball more difficult to return successfully, since the appropriate counterforce to control the ball and change its directional vector is more difficult to determine and apply quickly.

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Marcella Bush Trevino

See Also: Curves; Kicking a Field Goal; Mathematical Modeling.

Voting

See *Elections*

Voting Methods

Category: Government, Politics, and History.
Fields of Study: Algebra; Number and Operations; Problem Solving.
Summary: Social choice theory concerns itself with the mechanics of group decisions such as elections and the impact methodology can have.

Voting theory (also known as "social choice theory") is concerned with how group decisions are made when there are a number of alternatives from which to choose (for example, finding the winner of an election). When there are only two options, voting is straightforward—the winning alternative (also called the "social choice") should be the one that receives the most votes. However, when the choice is among three or more alternatives, determining the social choice is significantly more complex. There are many reasonable methods for selecting a winner and the methods can produce different winners even when given the same sets of votes. All voting methods have inherent flaws and, regardless of the method used, strange and paradoxical situations can occur. For example, in the 2000 U.S. presidential election, George W. Bush and Al Gore were major party candidates, while Ralph Nader, representing the Green Party, had much less support. Although Bush won the election, exit polls at the time indicate that had Nader not been on the ballot in some states, Gore almost surely would have won the election. In other words, in the U.S. electoral system, the presence (or lack thereof) of an "also-ran" candidate can have a profound outcome on the winner. This disturbing property is one of many that interests mathematicians, economists, and political scientists who study voting theory.

Preference Ballots

Preference ballots, where voters rank the alternatives in order of preference, are among the most useful ways of gathering information from voters. A voting method aggregates these preferences in some way and determines a social choice (or choices, in the case of ties). In this way, a voting method can be thought of as a function whose typical input is a set of individual ballots and whose output is the winning alternative, or—in the case of a social welfare function—a ranking of the alternatives, perhaps with ties. Many such functions are possible:

- *Plurality method:* A procedure that returns as the social choice the alternative that is the top preference on the most ballots (the candidate with the most first place votes).
- *Weighted voting method:* Also called the "positional method," this process assigns points to an alternative based on its position on a ballot, with higher placings on a ballot earning more points. The winning alternative is the one having the most points.
- *Borda count:* A special positional method whereupon a voter's lowest-ranked alternative earns zero points, the voter's second lowest-

ranked alternative earns one point, and so on, with the voter's top choice earning $n-1$ points, assuming n candidates.

- *Hare system:* Also called "instant runoff voting" or "plurality with elimination," this method arrives at the social choice by successively eliminating less desirable outcomes. In this procedure, ballot-counting proceeds in rounds, with the candidate having the fewest first-place votes eliminated at the end of each round. A ballot on which an eliminated candidate was the top choice has its vote transferred to the highest ranking remaining candidate on the ballot. The process of elimination continues until one candidate has more than half the first place votes (a "majority"), in which case that candidate is declared the winner.
- *Dictatorship*: In a dictatorship, one voter is specially designated so that the social choice is always the alternative that this voter has at the top of his or her ballot.

For example, suppose that there are 100 voters in an election, and three candidates (*A*, *B*, and *C*). Suppose that the voters express their votes as shown in the following table:

Number of Voters	**40**	**35**	**25**
1st Choice	*A*	*B*	*C*
2nd Choice	*C*	*C*	*B*
3rd Choice	*B*	*A*	*A*

Note that 40 of the voters prefer *A* as their top choice, 35 prefer *B* as their top choice, and 25 prefer *C* as their top choice. If this election were decided using the plurality method, then candidate *A* would win with 40 first place votes (with *B* and *C* earning 35 and 25 first place votes, respectively). Using the Borda count, *A* would tally $40\times 2=80$ points, *B* would earn $(35\times 2)+25=95$ points, and *C* would win with $(25\times 2)+75=125$ points. Using the Hare system, candidate *C* would be eliminated in Round 1, and *C*'s votes would transfer to candidate *B*, because *B* is second on all 25 ballots. In Round 2, *B* has 60 first place votes to *A*'s 40, so *B* is the winner.

This example demonstrates that different methods can yield different results. As Donald Saari writes, "Rather than reflecting the voters' preferences, the outcome may more accurately reflect which election procedure was used."

It should be noted that there are other methods of voting that do not require preference ballots. In a system called "approval voting," a voter may vote for as many candidates as desired. The winner is the candidate receiving the most votes. No distinction is made among the candidates of which the voter "approves," and the voter can vote for any combination of the candidates.

Fairness

By aggregating voters' preferences and producing a social choice, an election method should reflect, in some way, the will of the people. Given the vast library of possible election methods, it is natural to ask whether there is a method that captures this will in an ideal way. Social choice experts have developed different ways of assessing the quality of voting methods, and the notion of "fairness" has emerged as a prime consideration. When there are two alternatives, it can be expected that any reasonable voting method will be anonymous (all voters are treated equally), neutral (the two candidates are treated equally), and monotonic (if a voter changes his vote from candidate *A* to candidate *B*, then that should not hurt candidate *B*). Mathematician Kenneth May proved in 1952 that if the number of voters is odd and ties are not allowed, then only one voting method is anonymous, neutral, and monotonic: "majority rule," the procedure where the candidate with more than half the first place votes is declared the winner.

When there are three or more alternatives, there are many desirable properties for voting methods. The following list of criteria is far from exhaustive:

Majority criterion: This method requires that when some alternative is the first choice on more than half the ballots, that alternative should be the social choice. The plurality method satisfies this criterion, for if a candidate has a majority, no other candidate can have as many first place votes. On the other hand, the Borda count violates the majority criterion, for there are elections where a candidate can have a majority but still lose.

Condorcet winner criterion: This is a slightly weaker condition: if an alternative is preferred head-to-head over every other alternative in a one-on-one matchup that ignores the other alternatives, then that candidate should win the election. The example above shows that the plurality method violates this criterion. While candidate *C* is preferred over *A* on 60 of the ballots, and *C* is preferred over B on 65 of the ballots, *C* loses the plurality election to *A*. The Hare system and the Borda count fail the Condorcet winner criterion as well.

Pareto condition: This method asserts that for every pair *x* and *y* of candidates, if all voters prefer *x* to *y*, then *y* should not be a social choice. This is a relatively weak criterion, and all of the methods described above satisfy it.

Monotonicity criterion: According to this method, if *x* is a social choice and someone changes a ballot in such a way that *x* is moved up one spot (in other words, *x* exchanged with the alternative immediately above *x* on the ballot), then *x* should still be a social choice. In other words, making a change to a ballot that is favorable only to a winning candidate should not hurt the candidate. The plurality method and the positional voting methods satisfy monotonicity, but the Hare system does not.

Independence of Irrelevant Alternatives: Also called "binary independence," this method states that if *x* is a social choice while *y* is not, and if a voter changes a ballot in a way that does not change the relative positions of *x* and *y* on the ballot, then *y* should still not be a social choice. In other words, changing the positions of other "irrelevant" candidates on a ballot should not affect the relative position of *x* over *y* or *y* over *x* in the outcome. This is precisely the difficulty that occurred in the 2000 U.S. presidential election, where Nader's presence in the election affected the relative rankings of Bush and Gore.

Although each of these criteria is, in turn, a reasonable expectation of a voting method, Kenneth Arrow, in 1952, proved the mutual exclusivity of them. In his "impossibility theorem," Arrow showed that if there are at least three alternatives and a finite number of voters, then the only social welfare function that satisfies both the Pareto condition and "independence of irrelevant alternatives" is a dictatorship. This profound result, which earned Arrow the Nobel Prize in Economics in 1972, argues against the possibility of a theoretically perfect democracy. Nevertheless, Arrow himself encourages continuing to search for voting methods that work well most of the time. He writes:

> My theorem is not a completely destructive or negative feature any more than the second law of thermodynamics means that people don't work on improving the efficiency of engines. We're told that you'll never get 100% efficient engines . . . It doesn't mean you wouldn't like to go from 40% to 50%.

Sincere and Strategic Voting

Strategic voting is the practice of voting against one's true preferences in order to achieve a better outcome in an election. This contrasts with sincere voting, where one votes according to one's true preferences. Strategic voting most often occurs in situations where a voter's preferred candidate has little chance of winning, or where the voter's top candidate is most threatened by his second or third candidate. While strategic voting can affect the outcome of an election, its effects can be disastrous. Election results should reflect the aggregate will of the people, and if voters do not express their individual preferences truthfully, then the voting method has little hope of determining the socially desired outcome. Therefore, voting methods that tend to encourage strategic voting are unattractive. It should be noted that for strategic voting to be at all effective, there must be at least three candidates in the election, and the voters need a thorough understanding of both the voting method being used and the preferences of other voters.

For example, in the 2000 election, exit polls in Florida indicated that Nader voters widely supported Gore as their second choice, far beyond both the margin of error of the polls and Bush's margin of victory. Had these voters instead voted strategically for Gore, Gore would likely have carried Florida and its 25 electoral votes, thereby winning the presidency. The U.S. electoral college notwithstanding, this shows how powerfully the plurality method encourages strategic voting. Had the 2000 election been decided by the Borda count, one could imagine that a conservative voter might have Bush as the top choice, Gore as the second choice, and Nader last, but might insincerely rank the candidates in the sequence Bush, Nader, Gore in an attempt to maximize the point differential between Bush and Gore.

Some voting methods prove resistant to strategic voting. One of the major advantages of the Hare system is that it tends to encourage sincere voting. In the 2000 election, for example, a Nader supporter would have less reason to vote strategically for Gore if it is known that the vote will transfer to Gore should Nader be eliminated. Nevertheless, there are situations where even with the Hare system, strategic voting can prove beneficial to a voter.

In the 1970s, Allan Gibbard and Mark Satterthwaite proved that no voting method is completely immune to strategic voting. Any non-dictatorial system that uses preference ballots and allows at least the possibility of any candidate winning will necessarily lead to situations, however hypothetical, where strategic voting can be beneficial. This proof serves as a result analogous to Arrow's, but in the realm of strategic voting. As with Arrow's result in fairness, it is important to note that the degree to which a voting method encourages sincerity still serves as an important criterion for selection.

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Stephen Szydlik

See Also: Elections; Government and State Legislation; Rankings.

W

Water Distribution

Category: Architecture and Engineering.
Fields of Study: Algebra; Data Analysis and Probability; Geometry; Measurement; Number and Operations; Problem Solving.
Summary: Mathematicians have long studied issues related to optimizing water distribution.

Water distribution has two separate but interrelated meanings: the natural physical distribution of water in the world and the way in which people choose to distribute available water. In some regions, accessing and distributing fresh water for human needs, like drinking and irrigation, can be a significant challenge. Roughly 70% of the Earth's surface is covered with water but most is saline (salty). Much of Earth's fresh water is in glaciers or underground. Some is polluted from human activities. In the early twenty-first century, approximately 20% of Earth's population lived in areas with insufficient fresh water because of climate or geography. About the same number lived in areas in which water existed but where technological or economic barriers limited effective distribution. Many systems have been devised throughout history and in different societies to access and distribute water. It is so valuable a resource that armed conflicts been fought over water. Mathematicians, scientists, and others who work on water distribution problems use mathematical techniques to design, build, optimize, and monitor water distribution and associated wastewater systems. For example, graph theory is used to model water distribution networks. Graph edges may represent pipes and nodes represent intersections, junctions, and access points. Statistical and topological methods can be used to compare networks in terms of capacity and reliability against failure.

Irrigation

Irrigation is an ancient practice that allows food to be grown where it might otherwise not thrive. Evidence shows that it was used as early as the sixth millennium B.C.E. in Mesopotamia, Egypt, and Persia, and the fifth millennium B.C.E. in South America. In the early twenty-first century, agriculture is still globally the greatest consumer of fresh water, though it varies widely by location. For example, the United Kingdom's abundant rainfall means that it requires almost no irrigation. Mexico and India, on the other hand, use it extensively. The green revolution of the twentieth century, which greatly increased the agricultural yield of many developing countries, relied in part on irrigation. One criticism was that the increased food production in these areas resulted in accelerated population growth that placed further burdens on scarce water resources. This criticism is supported by some statistics and mathematical models, which show that the demand for water grew at rates that exceeded population increases, raising per-capita water requirements.

Mathematicians and others who study ancient systems of irrigation in order to better understand them (and perhaps improve modern methods) have noted that some societies appear to have created and implemented complex and efficient water distribution methods without using mathematical methods for planning. Others have sought to build mathematical models of irrigation systems. The paddy field system used for growing rice generally requires the creation of intricate structures of terraces, canals, and reservoirs in order to ensure that all fields receive adequate water. It is believed to have been used as early as 4000–3500 B.C.E. in China and Korea. Researchers who have investigated mathematical models to describe a paddy field system have noted that it may not be possible to create a reliable model by including only variables based on physical measures such as amount of water available and rate of evaporation. A variable describing an ethic of cooperation among owners of the various fields, a factor that is difficult to quantify, was also required to ensure that water would be used fairly. For example, if owners on the upstream end of a water source took more than their fair shares, the owners farther downstream would not receive sufficient water for their crop, regardless of the values of some other variables.

Industry

Industry is the second largest category of global water use. Most industrial processes need water in some way, though some are more readily visible, such as hydropower generation of electricity and water extraction of minerals in mining. At the start of the twenty-first century, per capita water use is typically higher in industrialized nations than in developing countries, though this gap is closing. Some economists use the term "virtual water" to refer to the water that is used in the entire chain of manufacturing a product or growing an agriculture commodity. Similar to a carbon footprint, which is often used to quantify the quantity of greenhouse gasses emitted by a process, a water footprint represents the total amount of water used to create a good or service. Calculating water footprints provides an additional metric for assessing and comparing the environmental impact of competing products and services. For example, in 2010, the Water Footprint Network estimated that production of 1 kilogram of beef required about 16,000 liters of water, while one kilogram of rice required 3000 liters of water, and one liter of milk required 1000 liters of water.

Sanitation

The creation of sanitary systems of water supply and wastewater disposal or treatment is a major factor in the general improvement of public health from about the mid-nineteenth century onwards. Large cities, such as London, New York, and Boston, were among the first to establish municipal water supply systems. They were motivated in part by data collected by statisticians and others such as physician John Snow, who demonstrated via statistical methods that an 1854 cholera outbreak in London could be traced to the local water pump.

Mathematical methods may be used to model different aspects of supply systems. The fluid pressure necessary for water to flow through a system is affected by variables like gravity. Water stored in a rooftop tank will deliver water at a higher pressure to lower floors versus higher ones. Mathematical calculations show that a vertical foot of water exerts a pressure of 0.433 pounds per square inch (psi) at its bottom surface. The flow of water through the system is a function of the cross-sectional area of the pipe: $Q = A \times V$, where Q is the flow of water through the system, A is the cross-sectional area of the pipe, and V is the velocity of the water.

Municipal water systems tend to be quite complex, involving massive networks of storage tanks, pipes, pumps, and valves. Mathematical models are used to describe and manage these systems. Navier–Stokes equations, named for mathematicians Claude-Louis Navier and George Gabriel Stokes, are partial differential equations that describe fluid flow and velocity, while the Reynolds number, named for mathematician Osborne Reynolds, quantifies "laminar" (smooth) and turbulent fluid flow through a pipe. Contamination is an ever-present risk because of the natural physical deterioration of system components over time (such as corroded pipes) as well as the possibility of accidental or deliberate introduction of contaminants. Researchers are developing systems that can sense when a contaminant has been introduced into the water distribution system, allowing for rapid identification of the time and location of its introduction. For example, experiments done by the U.S. Environmental Protection Agency showed statistically that chlorine and total organic carbon, which are routinely monitored in municipal water systems, were sensitive and reliable predictors of contamination.

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Sarah Boslaugh

See Also: Canals; Carbon Footprint; Farming; Floods; Tides and Waves; Water Quality.

Water Quality

Category: Weather, Nature, and Environment.
Fields of Study: Algebra; Data Analysis and Probability; Measurement.
Summary: Water quality standards and data are mathematically modeled and analyzed to help keep drinking water safe.

Water is fundamental for human life. Approximately 70% of the Earth's surface is covered by water but only a very small fraction is consumable fresh water, and much of that has chemical or biological contaminants. Drinking water comes from a variety of sources. Underground water, such as aquifers or springs, may be tapped by wells; surface water, such as rivers and streams, are diverted for use; precipitation may be collected or allowed to flow into other sources; and plants may be processed for moisture. Desalinization (the process of removing salt from water) makes seawater drinkable. Waterborne diseases in open water sources like rivers are endemic to many parts of the developing world. Natural disasters may spread contamination via flooding. Some global warming researchers predict that increased rainfall, flooding, and warmer weather will result in more waterborne disease worldwide. In developed countries, water is commonly piped to end users and may be recycled via sewage treatment. The standards for potable water in many countries are set by government agencies, though the regulation of bottled water differs from piped and well water. Even in nations with extensive closed water distribution systems and sewage treatment, contamination occurs in a number of ways, including agricultural runoff, dumping of manufacturing byproducts into streams and rivers, and degradation of systems that may contain outdated materials such as lead. One of the Millennium Development Goals adopted by the United Nations and other international organizations is to cut in half the proportion of people that do not have reliable access to safe drinking water by 2015. Mathematicians and mathematical methods contribute significantly to the discovery, testing, and delivery of potable water.

How Safe is Your Drinking Water?

The Environmental Protection Agency (EPA) sets the standards for drinking water in the United States. For each potentially harmful substance, the EPA identifies the maximum contaminant level (MCL) allowed and the maximum contaminant level goal (MCLG). The MCLG is the level below which there is zero expected risk to human health. While it would be best to have levels of a substance like arsenic at or below the MCLG, the EPA sets MLC requirements at concentrations that can be higher. U.S. citizens who receive water from a community water system should receive a Water Quality Report each year.

Providing access to safe drinking water around the world is one of the goals of the United Nations.

Those curious about water quality at work may request a copy from the building owner. Each report includes the source of the water (such as a river or lake); a list of all detected regulated contaminants and their levels; potential health effects of contaminants detected that violate the standards; information for people with weakened immune systems; and contact information for the company or agency that supplies the water. The report will alert the public to violations of the EPA safe drinking water standards and, equally important, will list information about potentially harmful substances that are below the legal limit. For example, a report may list arsenic, describe that it is measured in parts per billion (ppb), give the highest level measured, and list the range measured in the water. The report will also provide the MCLG (0.0 ppb for Arsenic) and the MCL (10.0 ppb). If the report states that the water ranges from 0.5 to 2 ppb for arsenic, water consumers will know that it is safe to drink according to EPA standards. However, upon comparing the MCL and MCLG, consumers may consider drinking water from other sources or request additional information from the water company since 0.5 ppb is higher than the 0.0 ppb MCLG.

Mathematical Analysis and Modeling

The management of water resources is increasingly reliant on mathematical modeling and analysis. For example, the dynamics and kinetics of surface water, along with distributions and dispersal over time of contaminants, have been extensively modeled and simulated. Reactive transport (RT) models use coupled equations to examine particle transportation through porous surfaces, which are widely used to model infiltration of contaminants into ground water. They may utilize mathematical and statistical concepts such as stochastic differential equations, which can be traced in part to physicist Paul Langevin's work on the mathematical theories of dynamic molecular systems. Animal behavioral responses to variables like water quality have been successfully modeled using the Eulerian–Lagrangian–Agent Method (ELAM). The Eulerian framework, named for mathematician Leonhard Euler, mathematically models environment factors affecting the animal agents, while the Lagrangian framework, named for mathematician Joseph Lagrange, governs the perception and movement of individual agents.

Near-continuous water quality monitoring provides a wealth of data and facilitates time series analyses and other statistical models of water quality as functions of variables like land use and precipitation patterns, as well as other measurable human behaviors and natural occurrences. Model calibration, verification, and sensitivity analysis often require comparing mathematical equations and simulation results with observed data. Mathematicians, engineers, and scientists have improved systems for remote water quality monitoring and assessment using data, mathematical methods, and theories from many sciences. Some applications include remote automated stations with the ability to wirelessly network and transmit data, artificial intelligence algorithms that can adaptively sample in response to problems or concerns, and satellite or aircraft observation and analysis of large areas.

These analyses also influence public policy and legislation, such as the U.S. Safe Drinking Water Act and the Clean Water Act. Scientists in many fields continue to seek methods to provide easily accessible clean water for everyone.

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Christine Klein

See Also: Farming; Floods; Water Distribution.

Waves

See *Tides and Waves*

Weather Forecasting

Category: Weather, Nature, and Environment.
Fields of Study: Connections; Data Analysis and Probability; Problem Solving; Representations.

Summary: Accurate weather forecasting requires the use of advanced mathematical models and powerful supercomputers to handle the vast number of calculations.

Weather prediction, or forecasting, is the application of science and technology to predict the future state of the atmosphere at a given location using available past and present data from the surrounding area. The word "weather" describes the state of the atmosphere at a particular time, or short time period, while the word "climate" is an average of these conditions over long time periods—often months or years. The weather is typically described in terms of temperature, wind speed, wind direction, air pressure, density, and atmospheric composition (for example, water vapor, liquid water, or carbon dioxide content). The intensity of solar and terrestrially emitted radiations is also a fundamental determining factor. A forecast typically includes the prediction of these meteorological variables and helps people make more informed daily decisions that may be affected by the weather. Moreover, it helps predict dangerous weather phenomena, such as hurricanes, which might endanger human life.

History

People have tried to forecast the weather for thousands of years and throughout history, farmers, hunters, warriors, shepherds, and sailors understood the importance of accurate weather predictions for planning daily activities. Ancient civilizations appealed to the gods of the sky: the Egyptians looked to Ra, the sun god; the Greeks sought out Zeus; and in the ancient Nordic culture, Thor was believed to govern the air with its thunder, lightning, wind, rain, and fair weather. The Aztecs used human sacrifice to satisfy the rain god, Tlaloc, while Native American and Australian aborigines performed rain dances.

The Babylonians were predicting the weather from cloud patterns as well as astrology by 650 B.C.E., but the earliest scientific approach to weather prediction occurred circa 340 B.C.E. when Aristotle described his theories about the earth sciences and weather patterns in *Meteorologica*. The ancient Greeks invented the term "meteorology," which derives from the Greek word *meteoron* which refers to any phenomenon in the sky. The Greek philosopher Theophrastus, one of Aristotle's successors, compiled the ultimate weather text *The Book of Signs*, which contained a collection of weather lore and forecast signs and served as the definitive weather book for over 2000 years.

Weather forecasting advanced little from these ancient times to the Renaissance. Beginning in the fifteenth century, Leonardo da Vinci designed an instrument for measuring humidity, Galileo Galilei invented the thermometer, and his student Evangelista Torricelli came up with the barometer. With these tools, people could objectively monitor the atmosphere. In 1687, Sir Isaac Newton published the physics and mathematics that govern the motion of all bodies and can be used to accurately describe the atmosphere. To this day, his principles are the foundation for modern mathematical analysis and computer prediction of weather.

However, scientifically accurate weather forecasting was not feasible until the early twentieth century, when meteorologists were able to collect and organize data about current weather conditions from observation stations in a timely fashion. Vilhelm and Jacob Bjerknes developed a weather station network in the 1920s that allowed for the collection of regional weather data. The data collected by the network could be transmitted nearly instantaneously by use of the telegraph, invented in the 1830s by Samuel F. B. Morse. This system allowed knowledge of the weather conditions upwind to be incorporated into downwind forecasts, improving their quality.

Great progress was made in the science of meteorology during the twentieth century. The possibility of numerical weather prediction was proposed by Lewis Fry Richardson in 1922, although computers did not yet exist. It was consequently impossible to perform the vast number of calculations required to produce a forecast before the predicted events actually occurred. Practical use of numerical weather prediction began in 1955, spurred by the development of programmable electronic computers.

Numerical Weather Prediction

Numerical weather prediction is the science of forecasting weather using computer simulations built from mathematical models. In this process, the atmosphere is divided into a three-dimensional lattice of grid points, and at each point the various atmospheric variables of interest are represented. These values are initialized with a state determined through analysis of past and present conditions. This state is then evolved

Weather forecasters prepare their forecasts at PC workstations with weather analysis software.

forward into the future by solving, at each grid point, the classical laws of (fluid) mechanics and thermodynamics, which are known to accurately approximate the behavior of the atmosphere. The output from the model provides the basis of the weather forecast.

The equations that govern how the state of a fluid changes with time contain many variables and require a great deal of computer processing resources to solve. Weather prediction centers have access to supercomputers containing thousands of processors on which to run a forecasting model. The required calculations are shared among the processors and computed simultaneously to produce a complete forecast in a fraction of the time possible with a single computer. This system is essential to ensure that an accurate prediction can be made within a useful time frame.

Good weather forecasts depend upon an accurate knowledge of the current state of the weather system, also called the "starting point" or "initial condition." The initial conditions are determined from global measurements of the state of the atmosphere. Surface weather observations of atmospheric pressure, temperature, wind speed, wind direction, humidity, and precipitation are made near the Earth's surface by trained observers, automatic weather stations, or buoys. The initial state has a degree of uncertainty since there are an insufficient number of measurements to initialize all meteorological variables at every grid point. Furthermore, the locations of the measurements do not usually coincide with the numerical grid points and there is also a degree of error in the actual measurement. The problem of determining the initial conditions for a forecast model is very important, highly complex, and has become a science in itself (known as "data assimilation").

The atmosphere is an incredibly complex dynamical system and the approximation of its behavior is only compounded by the inability to measure its state at each and every grid point in the model. The limit on useful weather forecasts using present technology is typically one week. The forecast errors are initially localized, leading to incorrect predictions in small regions, but are generally accurate enough to be useful in most of the forecast area. The longer the simulation is run, the more the measurement and model approximation errors begin to dominate the calculation. However, steady improvements in computer power and prediction models in the twenty-first century have led to a three-day forecast being as accurate as a two-day forecast from the 1990s. Weather forecasting centers are constantly reviewing the accuracy of their forecasts and set themselves annual targets for accuracy improvements.

The raw output from the simulation is often modified before being presented as a forecast. Modifications include either the use of statistical techniques to remove known biases in the model or adjustments to take into account consensus among other numerical weather predictions. Accurate forecasts of precipitation for a specific location are particularly challenging because of the chance that the rainfall may fall in a slightly different place (such as several kilometers away) or at a slightly different time than the model forecasts, even if the overall quantity of precipitation is correct. Therefore, daily forecasts give fairly precise temperatures but put probabilistic values on quantities such as rain, based on knowledge of the uncertainty factors in the forecast.

Probability of Precipitation

A Probability of Precipitation (PoP) is a formal measure of the likelihood of precipitation that is often published from weather forecasting models, although its definition varies. In U.S. weather forecasting, PoP is the probability that greater than 1/100th of an inch of precipitation will fall in a single spot, averaged over the forecast area. For instance, if there is a 100% probability of rain covering one side of a city and a 0% probability of rain on the other side of the city, the PoP would be 50%. A 50% chance of a rainstorm covering the entire city would also lead to a PoP of 50%. The mathematical definition of

PoP is defined as PoP = $C \times A \times 100$, where C is the confidence that precipitation will occur somewhere in the forecast area, and A is the percent of the area that will receive measurable precipitation, if it occurs at all.

For example, a forecaster may be 40% confident that precipitation will occur and that, should rain happen to occur, it will happen over 80% of the area. This results in a PoP of 32%: $0.4 \times 0.8 \times 100 = 32\%$.

The Future

Over the years, the quality of the models and methods for integrating atmospheric observations has improved continuously, resulting in major forecasting improvements. The power of supercomputers has increased dramatically, allowing for the use of much more detailed numerical grids and fewer approximations in the operational atmospheric models. Small-scale physical processes (such as clouds, precipitation, turbulent transfers of heat, moisture, momentum, and radiation) have been more accurately represented within the model. Finally, the use of increasingly accurate methods of data assimilation and the integration of satellite and aircraft observations has resulted in improved initial conditions for the models, which ultimately lead to a better forecast.

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Silvia Liverani

See Also: Climate Change; Data Analysis and Probability in Society; Forecasting; Parallel Processing; Statistics Education; Temperature; Weather Scales.

Weather Scales

Category: Weather, Nature, and Environment.
Fields of Study: Algebra; Connections; Measurement.

Summary: Weather scales and tools are used to help measure and classify atmospheric conditions.

Weather affects virtually every aspect of human life, including afternoon showers that might inconvenience commuters; tremendously destructive episodes, like hurricanes; and long-term occurrences, like drought, which impact agriculture and increase the likelihood of other events like wildfires. Meteorology is an interdisciplinary science that focuses on weather and short-term forecasts, typically up to a few weeks. Climatology is a science that looks at long-term average weather. In fact, many define the word "climate" in terms of the average of weather over time, both locally and globally. Mathematics plays a critical role in weather science, enabling people to quantify, compare, model, and predict weather. Valid and reliable comparisons are facilitated by the development of scales and standard systems of quantification, along with mathematical and statistical models that use those measures.

It is thought that some ancient peoples had methods for predicting the weather, though historical evidence is mixed. In the early twentieth century, mathematician Vilhelm Bjerknes and colleagues examined several measurable variables of weather and derived equations to connect them to one another. Mathematician Lewis Richardson, who contributed significantly to mathematical weather prediction and pioneered the use of finite differences in the field, reformulated the Bjerknes equations. However, they remained impractical for rapid forecasting until the introduction of computers. Another product of his work, the Richardson number, is a function of density and velocity gradients that helps predict fluid turbulence in weather and other applications. Mathematicians continue to contribute and modern forecasting involves a wide variety of mathematical techniques and models, drawing in depth from such areas as chaos theory, data assimilation, statistical analyses, scale cascades of error (related to the so-called butterfly effect), numerical analysis, vectors, fluid dynamics, and entropy. Climatologists, scientists, and mathematicians also research related phenomena like geomagnetic and solar storms.

Temperature, Pressure, and Humidity

One of the most pervasive and intuitively obvious variables used to characterize the weather is air temperature—along with air pressure and humidity in most modern reports and forecasts. Strictly speaking, air temperature is a measure of the average kinetic energy of the air molecules, measured by a variety of types of thermometers. The most common scales used to quantify temperature are the Celsius (or centigrade) scale used throughout most of the world and the Fahrenheit scale used primarily in the United States. Atmospheric pressure is measured by a barometer, whose invention is attributed to various sources including Galileo Galilei and mathematicians Gasparo Berti and Evangelista Torricelli.

There are many common units for pressure, including inches of mercury, pounds per square inch, pascals, named for mathematician Blaise Pacsal, and atmospheres. One atmosphere is defined as the mean atmospheric pressure at mean sea level, originally measured with respect to the latitude of Paris, France. Millibars are often used in weather reports and forecasts. A hygrometer measures the amount of water vapor in the air. How much water vapor the air can hold is a function of temperature and relative humidity expresses the quantity of water vapor as a unitless fraction or percentage of the possible amount of water for a given temperature. Humidity can be used in probability models to predict precipitation, dew, and fog. Further, high humidity changes the subjective feeling of the air temperature for people because high humidity reduces the evaporation of sweat. This effect is quantified as a heat index, with assumptions about many variables such as wind speed, body mass, clothing, physical activity, and exposure to sunlight. A similar concept is wind chill, which relates the subjective perception of cold. Scientist Robert Steadman has researched and mathematically modeled both of these effects and they have become a common part of weather forecasts.

Wind

Another weather variable is wind speed. In 1805, Sir Francis Beaufort, an Irish hydrographer, developed what is now called the Beaufort scale to describe and categorize the strength of the wind. The scale has 13 points ranging from zero (calm air) to 12 (hurricane-force winds). On the scale, the Beaufort number two is identified as a "light breeze," with wind speed 6–11 kilometers per hour (km/hr) producing wind that is felt on the face, leaves that rustle, movement of a wind vane, and on the water, small, short wavelets that do not break. Further along the scale is Beaufort number five, a "fresh breeze," with wind speeds between 29 and 38 km/hr. At this point, small, leafy trees will sway, moderate waves become longer, and there are many whitecaps and some spray. Wind speeds between 62 and 74 km/hr are classified as a "gale," Beaufort number eight. Twigs and small branches break off trees. At sea, there are moderately high waves of greater length. Beaufort number 10 is used when wind speeds are between 89 and 102 km/hr and are "storm-force" winds. Trees are broken and uprooted and structural damage occurs. At sea, there are very high waves with overhanging crests and visibility is reduced.

Table 1: Fujita scale of tornado strength.

Scale	Wind Speed (km/hr)	Damage
F-0	65–118	Light
F-1	119–181	Moderate
F-2	182–253	Considerable
F-3	254–332	Severe
F-4	333–419	Devastating
F-5	420–513	Incredible

Table 2. Saffir-Simpson scale of hurricane strength.

Scale Number	Wind Speed (km/hr)	Storm Surge (meters)	Central Pressure (millibars)	Damage
1	121–154	1–2	≥ 980	Minimal
2	155–178	2–3	965–979	Moderate
3	179–210	3–4	945–964	Extensive
4	211–250	4–6	920–944	Extreme
5	>250	>6	<920	Catastrophic

The terms and descriptions make it clear that as wind speed rises so does its destructive power. In fact, the force exerted by wind increases as the square of the velocity such that a doubling of the wind's velocity leads to a quadrupling of the force: $F \sim V^2$. Some of the most powerful winds experienced on Earth are found in hurricanes and tornadoes. Their destructive power can be astounding and has been the subject of much study and research. The Fujita scale, presented in Table 1, is used to categorize tornado strength in terms of rotational wind speed (given in km/hr) and damage inflicted by the wind. While tornadoes are generally associated with severe thunderstorms and are seldom more than 1.5 km in diameter, hurricanes can involve whole systems of thunderstorms and may be several hundred kilometers in diameter. The Saffir–Simpson scale, used to categorize hurricanes, is presented in Table 2.

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See Also: Clouds; Doppler Radar; Hurricanes and Tornados; Temperature; Weather Forecasting.

Weightless Flight

Category: Travel and Transportation.
Fields of Study: Algebra; Geometry.
Summary: The forces to experience the sensation of weightlessness, or zero-*G,* can be calculated and achieved in a variety ways.

Gravity is the mutual attraction of two masses. Important aspects of the mathematics and the theory of gravity were described centuries ago by Galileo Galilei and Isaac Newton. Albert Einstein's work was critical to the modern understanding of gravity and weightlessness. Mass is the measure of the amount of matter in an object. For living beings, weight can be thought of as the subjective experience of muscles resisting the pull of the much larger Earth on their smaller masses. On the Earth's surface, gravitational acceleration is about 9.8 meters per second (one gravity or *g*). Other planets have different gravity. For example, an Earth person would feel about 2.5 times heavier on Jupiter. Infants learn to accommodate gravity's pull when performing the activities of daily life until the force feels natural and largely unnoticed. However, sometimes people experience other forces acting on their bodies that counter the pull of gravity and change their perceptions of weight. For example, the quick start or stop of an elevator can make a person feel heavier or lighter. Roller coasters purposely induce similar effects for amusement. Parabolic drops, turns, and loops exert temporary linear or angular forces on a moving body, some of which act along a different directional vector than gravity and combine mathematically to alter the body's perception of weight. Mathematicians, scientists, and engineers precisely calculate the net effect of gravity and other forces on objects for a wide range of applications, such as banked curves on racetracks and highways, the movement of subatomic particles, launching spacecraft to the moon, and of course, ever more thrilling amusement park rides.

Zero-*G*

The planet's mass exerts a strong gravitational pull even on objects in space. This force is what keeps satellites in position. However, many people have seen video images of astronauts who are floating around as if they are weightless. This effect is known as *zero-G* or, more accurately, "microgravity" (about 1×10^{-6} *g*). Like roller coasters, this effect results from a combination of forces acting on the body. At any given instant in time, the astronauts are accelerating freely toward the Earth inside an object that is accelerating freely at the same rate. They can be visualized in that instant as falling on a straight line drawn from the spaceship to the Earth, perpendicular to a tangent line drawn at the ship's current position in its curved orbit. However, the ship's directional vector is constantly changing because of its curved orbit, so it perpetually "falls" in a

Astronauts aboard an aircraft that flies a parabolic pattern to provide weightlessness training.

new direction—around the Earth, instead of toward it. The spacecraft's precisely calculated inertial trajectory effectively counters the astronauts' constant "falling." As a result, the astronauts do not move with respect to their immediate surroundings, so they look and feel as if they are floating weightlessly. A spacecraft lands by altering its curved orbit so that the gravity is no longer sufficiently opposed.

Free-fall or zero-*G* can be achieved in several ways without leaving Earth's atmosphere. NASA's Neutral Buoyancy Simulator uses the world's largest indoor pool, containing over six million gallons of water, to simulate weightlessness without flying or falling, while their Zero Gravity Research Facility can achieve just over five seconds of free fall in a 467-foot long steel vacuum chamber, which is used to test microgravity effects on phenomena such as combustion and fluid physics. As part of a series of experiments in the 1960s, Air Force Captain Joseph Kittinger parachuted from a gondola at an altitude of almost 103,000 feet. He achieved a speed of over 600 miles per hour on his descent but he reported having no real subjective sensation of the incredible speeds. Standard aircraft can be used to create brief periods of weightlessness, about 30 seconds, by flying in a parabolic pattern or "Kepler curve," named for Johannes Kepler. NASA uses this method to train astronauts, and the weightless effects seen in the 1995 movie *Apollo 13* were produced using parabolic flight. Several commercial companies also offer the experience to the general public. A privately funded experimental "spaceplane" called SpaceShipOne achieved suborbital flight in 2004. A revised commercial version called VSS Enterprise flew for the first time in 2010 and is taking reservations for future commercial flights that will launch passengers into suborbital space.

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Julian Palmore

See Also: Airplanes/Flight; Gravity; Interplanetary Travel; Planetary Orbits; Ride, Sally; Spaceships.

Wheel

Category: Travel and Transportation.
Fields of Study: Algebra; Geometry.
Summary: Wheels help humans perform work and travel by providing a mechanical advantage.

Circles are present in many places in nature and mathematicians studied them long before the common use of the wheel. A wheel is traditionally a cylinder rotating around an axle. Together, a wheel and an axle form a simple machine that can change direction and magnitude of forces. Wheels are widely used in transportation as gears, as handles and knobs, and for converting the energy of water, animals, or people into work. The notion of curvature is of interest to many mathematicians, scientists, engineers, and others. In geometry, wheels are often modeled as circles or as concentric circles. In addition to standard circles or cylinders, mathematicians have explored the properties of wheels of other shapes along with varying surfaces. Aristotle's Wheel paradox, named for Aristotle of Stagira, is an interesting mathematical problem involving the paths traced by a wheel made of two concentric circles. It seems to imply that the circumferences of different sized circles are equal. This is one of many mathematical questions that arise from rotating concentric circles or exploring the curves generated by wheels.

History and Mechanical Advantage

Wheeled vehicles were invented about 6500 years ago, but they were not used widely until the rise of large, organized, road-building societies. This discrepancy between the discovery and its wide adoption, because of the lack of infrastructure, is frequent in science. Using wheels as levers to change the magnitude of force for applications like grinding grain was more widespread in many societies. The force advantage that a wheel provides is equal to the radius of the wheel divided by the radius of the axle. For example, a ship's capstan with the radius of eight feet and the axle radius of one foot multiplies the force of sailors using it by eight. This relationship is the reason that water wheels on small, weak streams that do not provide much force have to be larger than on fast-moving streams—a weak stream will not provide enough force to turn a small wheel. Rotating handles or knobs, grinders, drills, and old-fashioned water wells all use the wheel's mechanical advantage.

Geometry and Physics of Rolling: Work Smart, Not Hard

Rolling vehicles on wheels save work compared to dragging the same weight along the ground. Friction between the ground and a dragged object occurs along the length of the path. The work needed to overcome this friction is proportional to the friction coefficient, which depends on the surfaces of the object and the path. On smooth surfaces, such as ice, the friction coefficient is lower than on rough surfaces, such as rock. Work is also proportional to the weight of the object and the length of the path. When an object is rolled, its weight presses the axles to the wheels. Instead of the object-road friction, the force to overcome is now the axle-wheel friction, which is also proportional to the weight. When a wheel turns around, the vehicle travels the distance equal to the wheel's circumference. If the radius of the axle is one-tenth of the radius of the wheel, then the distance the axle slides within the wheel is one-tenth of the distance the vehicle travels and the required work is divided by 10. It is relatively easy to reduce axle-wheel friction many times by using smooth surfaces, oil, and ball bearings. Vehicles for heavier loads usually have more wheels to distribute the force of the load.

Reinventing the Wheel

Since wheels are essential to most human endeavors, there are many wheel-related sayings. "Reinventing the wheel" means "needlessly duplicating a well-known method." Ironically, wheels themselves are being constantly reinvented. For example, roller bearings first appeared in Leonardo da Vinci's drawings in the sixteenth century but were patented and used widely only in the nineteenth century. Magnetic bearings reduce axle-wheel friction to essentially zero and, therefore, promise huge increases in machine efficiency; their development started in 1980s. In the 1990s, mathematics and science museums began to feature bikes with square wheels that move smoothly over special surfaces consisting of "catenaries," which are hyperbolic shapes resembling hanging lengths of chains.

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Maria Droujkova

See Also: Bicycles; Curves; Pi; Street Maintenance; Windmills.

Wiles, Andrew

Category: Mathematics Culture and Identity.
Fields of Study: Algebra; Connections; Geometry.
Summary: Over 350 years after its conjecture in a marginal comment, Fermat's Last Theorem was finally proven by British mathematician Andrew Wiles.

Andrew Wiles is most well-known for solving Fermat's Last Theorem, and he has received many awards, including the prestigious MacArthur Fellowship. For seven years, Wiles worked in unprecedented secrecy, struggling to solve Fermat's Last Theorem, a problem that had perplexed and motivated mathematicians for three centuries. Wiles's solution of Fermat's Last Theorem brought him both fame and personal satisfaction. He said of his accomplishment, "I had this very rare privilege of being able to pursue in my adult life what had been my childhood dream." This work also brought him pain when a subtle but fundamental error was discovered in his proof. Wiles eventually fixed the mistake, solidifying his magnificent achievement and permanent place in history.

Fermat's Last Theorem

Fermat's Last Theorem states that the equation $x^n + y^n = z^n$ has no positive whole number solutions for $n > 2$. In other words, while the Pythagorean Theorem $x^2 + y^2 = z^2$ has whole number solutions (such as $x = 3$, $y = 4$, and $z = 5$), similar equations with larger exponents, like $x^3 + y^3 = z^3$ and $x^4 + y^4 = z^4$, have no positive whole number solutions. French mathematician Pierre Fermat (1601–1665) wrote in the margin of a book that he had discovered a remarkable proof for this theorem, but that the margin was too small to contain it. For the next three centuries, the best mathematicians in the world sought a solution to this problem, and these attempts inspired many new mathematical ideas and theories.

Wiles's Proof

As a 10-year-old, Andrew Wiles already loved solving mathematical problems. He read about the history of Fermat's Last Theorem in a library book about mathematics. Despite its long history, this problem was simple enough for him to understand, and it fascinated and motivated him. As his mathematical knowledge became more advanced, he realized that there were no new techniques available to solve Fermat's Last Theorem. When Fermat's Last Theorem became linked to modern mathematical methods in algebraic geometry, he resumed his work. The quest to find a proof of Fermat's Last Theorem finally came to an end when Wiles announced his results in 1993. Wiles had worked in isolation on the problem for many years while on the faculty at Princeton University, and his announcement came as a surprise to the mathematics community. Wiles's work combined two fields of mathematics, elliptical functions and modular forms, to solve the elusive problem.

Wiles directly proved what is known as the Taniyama–Shimura Conjecture. Goro Shimura and Yutaka Taniyama were two Japanese mathematicians who, in the 1950s, conjectured that there was a relationship between elliptical equations and modular forms. Later, thanks to the earlier work of mathematicians Gerhard Frey, Ken Ribet, and Barry Mazur, it was shown that if the Taniyama–Shimura Conjecture were true, then so was Fermat's Last Theorem. His results were presented in a dramatic series of lectures at a conference in Cambridge, England.

However, not long after Wiles announced his discovery, an error was found in one section of the long and difficult proof. With the help of one of his former students, Richard Taylor, Wiles was able to make the necessary changes. However, these corrections took over a year to complete, illustrating the complexity of the proof that Wiles had constructed.

Methods

Many people may wonder how Andrew Wiles was able to solve a problem that had eluded so many others skilled mathematicians. Wiles himself has said that he does not always know exactly where his new techniques come from, but he defines a good mathematical problem by the mathematics it generates, not by the problem itself. He never uses a computer in his work, preferring to doodle, scribble, or find patterns via calculations. As do most scholars, he also reads previous research for methods that he can adapt to his work. When he gets stuck working on a problem, he reportedly tries to change it into a new version that he can solve or steps away from it entirely to relax and allow his subconscious to work. He has described his personal process by the following analogy:

> Perhaps I could best describe my experience of doing mathematics in terms of entering a dark mansion. One goes into the first room, and it's dark, completely dark. One stumbles around bumping into the furniture, and gradually, you learn where each piece of furniture is, and finally, after six months or so, you find the light switch. You turn it on, and suddenly, it's all illuminated. You can see exactly where you were. Then you move into the

next room and spend another six months in the dark. So each of these breakthroughs, while sometimes they're momentary, sometimes over a period of a day or two, they are the culmination of—and couldn't exist without—the many months of stumbling around in the dark that proceed them.

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See Also: Cubes and Cube Roots; Mathematics, Theoretical; Proof; Pythagorean Theorem.

Wind and Wind Power

Category: Weather, Nature, and Environment.
Fields of Study: Data Analysis and Probability; Geometry; Measurement.
Summary: Wind and wind power have been mathematically studied for centuries as an energy source and promise to be increasingly important energy sources.

Wind is omnipresent. There are few parts of the world that are not affected by the wind, from the pleasant breezes off a lake to the terrifying destruction of hurricanes and tornados. Historically, wind was one of the most important sources of energy; it drove sailing ships and was key to driving some pre-industrial revolution machines, such as windmills. Being able to master the wind was a key component in the fate of empires. For example, in 1588, it is said that the Spanish Armada of Catholic King Philip II was defeated by a "strong Protestant wind" that forced his fleet off course and prevented a vulnerable England under the reign of Queen Elizabeth I from being invaded. In the wake of the steam engine, developed by James Watt in the 1760s, and the emergence of coal-powered machines during the Industrial Revolution, the age of wind and sail began to decline for much of the industrialized world. Many cite this shift to fossil fuel sources as a cause of the rise in carbon dioxide, other greenhouse gasses (GHGs), and the global warming phenomenon, and there is a movement toward returning to wind as one source of clean energy.

Mathematicians and scientists have long been involved in the study of wind and wind energy. Posidonius of Rhodes (c. 135–51 B.C.E.) theorized about clouds, mist, wind, and rain. Francis Beaufort (1774–1857) developed a mathematical scale to describe wind speed. Twenty-first-century engineer Michael Klemen has explored mathematical issues of wind data acquisition as a function of time and estimated wind resource availability for power generation. Mathematicians continue to contribute to these fields and to the exploration of related phenomena like solar winds, which are believed to have first been observed by astronomer John Herschel during his observations of Halley's comet in 1835.

History

Seventeenth-century mathematician Evangelista Torricelli was reputed to be skilled in making instruments and he is often credited with inventing the barometer. He also conducted research about weather and is believed to have given the first correct explanation of wind when he said, "winds are produced by differences of air temperature, and hence density, between two regions of the earth." In the seventeenth and eighteenth centuries, mathematician Philippe de La Hire studied instruments to measure climate, including temperature, pressure, and wind speed. He went on to collect data using these instruments at the Paris Observatory. In the nineteenth century, William Ferrel proposed a model for wind circulation, which was the first recorded theory to explain the westerly winds in the middle latitudes of both the northern and southern hemispheres. Ferrel cells are phenomena where air flows eastward and towards the pole near the Earth's surface, but westward and toward the equator at higher altitudes. The Beaufort wind scale was also named in the nineteenth century after Francis Beaufort, a British Rear Admiral who reportedly extended the work of many individuals in trying to standardize wind measurement and description. The invention of the cup anemometer by astronomer and physicist John Robinson in the middle of the same century aided in measuring winds and reputedly helped popularize the measure. The Beaufort wind scale was later revised by meteorologist George Simpson in the early twentieth century. Mathematician Lewis Richardson is widely considered a pioneer of mathematical weather prediction. He applied the method of finite differences and other mathematical methods in his Weather Prediction by Numerical Process in 1922. Wind is often mathematically modeled as a fluid, and some of Richardson's work was an extension of studies regarding water flow in peat. The Richardson number is a function involving gradients of temperature and wind velocity. Edward Milne, his contemporary, studied wind and sound, helping to refine huge binaural listening trumpets used to detected aircraft at night during World War I. In the twenty-first century, mathematicians often model various aspects of wind and wind power, including the wind movement through plant canopies using first and second order closure techniques; the probability of bird collisions with wind turbine rotors using statistical methods and calculus; descriptions and predictions of surface wind in mountainous terrain using statistical methods, geometry, vectors, and other mathematical functions; and the wind flow or turbulence over many types of surfaces, including turbine blades, ocean waves, automobiles, and structures.

Wind Tunnels

Wind tunnels allow scientists and mathematicians to create wind under controlled conditions to test theories and applications. Mathematicians Benjamin Robins and George Cayley constructed simple spinning devices to model drag and other aerodynamic forces in the nineteenth century but the flow is difficult to control under such conditions. Engineer Francis Wenham is credited with the invention of the first enclosed wind tunnel, in 1871, with colleague John Browning. Wind tunnels were used by Orville and Wilbur Wright in developing their airplane prototypes as well as by German scientists at the famous World War II Peenemünde research facility. With advances in computer technology, the properties of wind are often modeled using computational fluid dynamics rather than physical data collection in wind tunnels, or the two methods are used to compare and cross-validate results. The foundations of these methods are the Navier–Stokes equations, which are systems of nonlinear partial differential equations developed by mathematicians Claude-Louis Navier and George Stokes.

U.S. Wind Research and Applications

The first wind system to generate electricity in the United States was built by Charles Brush in the late nineteenth century. However, there was relatively little development in that area until the energy crises of the 1970s, which motivated people to seek alternative sources of electricity, such as wind. The 1990s and the 2000s saw technological advances, decreasing turbine costs, and the emergence of popular and political support for wind energy. At the start of the twenty-first century, the U.S. government aimed to have 20% of all electricity generated by wind by 2030. Moreover, statistical studies and other data suggest that wind should be able to compete on a cost-effective basis with traditional fossil fuel sources. Some reports even estimate that wind will account for 26% of the increase in renewable energy production by 2035, though this extrapolation may not be reliable. Wind has shown a number of advantages compared to other forms of electricity production: it does not emit greenhouse gasses while in operation, it is freely available, it is not subject to energy security concerns, there are no waste products, and the maintenance costs are relatively low compared to traditional or nuclear generating facilities. For energy sources such as wind and nuclear, the emissions occur during the construction phase and tend to be associated with the amount of concrete and steel used in the facilities. Wind energy also faces technological problems with intermittency, as electricity can only be produced while the wind is blowing and this problem had been studied by mathematicians.

For example, the Weibull correlation model, based on the Weibull distribution named for mathematician (Ernst) Waloddi Weibull, estimates energy outputs with reduced uncertainty versus previous models, which is potentially useful for preventative operation and maintenance strategies. The National Renewable Energy Laboratory offers both wind data sets and has developed many mathematical models to explore wind energy grids, economic impact of wind energy, and even a model called Village Power Optimization Model for Renewables (ViPOR), which is a computational tool that facilitates the design of a village electrification system using the lowest cost combination of centralized and isolated power generation. Beyond land-based power generation, scientists and engineers like Maximillian Platzer and Nesrin Sarigul-Klijn are exploring the potential benefits of a return to wind energy as a supplement for large, ocean-going ships.

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See Also: Hurricanes and Tornadoes; Tides and Waves; Weather Forecasting; Weather Scales.

Wind Instruments

Category: Arts, Music, and Entertainment.
Fields of Study: Geometry; Number and Operations; Representations.

Summary: The frequency and pitch of wind instruments are determined by their shape, length, and other factors.

Wind instruments convert the energy of moving air into sound energy—vibrations that are perceptible to the human ear. Under this definition, wind instruments include the human voice; pipe organs; woodwind instruments, such as the clarinet, oboe, and flute; and brass instruments, like the trumpet. The nature of this vibration and the associated resonator tube are responsible for the unique timbre of each type of wind instrument.

Sources of Vibrations

In the human voice, the flow of air from the lungs causes the vocal cords (also called "vocal folds") in the larynx to open and close in rapid vibration. This periodic stopping of the air stream creates oscillatory pulses of air pressure, or sound. The frequency of this vibration and the pitch of the resulting sound are determined by the length and tension of the cords. A singer or speaker controls these factors using the musculature of the larynx.

The rapid open-close vibration of the vocal cords is present in many wind instruments. In brass instruments, such as the trumpet, trombone, French horn, and tuba, the lips of the musician form a small aperture that opens and closes in response to air pressure. Brass instruments are sometimes called "lip-reed" instruments. In single-reed instruments, like the clarinet and saxophone, a thin cane reed vibrates in oscillatory contact with a specially shaped structure (the mouthpiece) to bring about the open-close effect. The oboe and bassoon utilize two cane reeds held closely together with a small space between them that opens and closes in response to flowing air, controlled by the muscles of the lips.

A third important mechanism for converting the energy of moving air into vibration is utilized in the flute and the so-called flue pipes of the pipe organ. In these instruments, vibration occurs when flowing air passes over an object with a distinct edge that splits the airstream. The resulting turbulence gives rise to oscillatory vibration. With the modern flute, the flutist's lip muscles actively control the interaction between the airstream and the edge. With the recorder and other whistle-type instruments, as well as flue pipes of the organ, the interaction is controlled by the mechanical design of the instrument alone.

Tube Resonators and Overtones

With the exception of the human voice, all wind instruments are constructed with a tube resonator enclosing a column of air that functions in much the same way as the vibrating string. Oscillations in air pressure inside the tube reflect from the ends, resulting in significant feedback with the primary vibrating medium. The relationship between the vibration frequency and length of a string fixed at both ends is explained by the concept of "harmonics." In idealized settings, changing the string length by small integer factors (for example, 1/2, 1/3, or 1/4) results in frequency changes that are recognizable as musical intervals (for example, an octave, an octave plus a fifth, or two octaves). The resonating air column in wind instruments behaves similarly to a vibrating string.

An important performance practice on most wind instruments is overblowing. Not to be confused with simply playing overly loudly, the term "overblowing" refers to the fact that changes in the airflow can cause the resonating air column to vibrate at an overtone above its fundamental frequency. Overblowing allows performers on modern instruments to achieve a large range of pitches (often two octaves or more) from a relatively compact resonating tube. Instruments with cylindrical tubes open at both ends, such as in some flutes, overblow at the octave, as do conical instruments that are closed at one end, such as the oboe and saxophone. On the other hand, cylindrical tubes closed at one end, such as the clarinet, overblow at the twelfth—an octave plus a fifth. The relative weakness of the overtone at the octave and other even-numbered overtones account for the particular timbre of the clarinet.

Altering the Tube Length in Performance

Just as the length of a vibrating string determines the frequency or pitch of the vibration, the length of the resonating air column accounts for the pitch of notes played by a wind instrument. In reed instruments, the resonating tube is perforated along its length with holes. By systematically covering some of the holes but not others, the musician effectively changes the length of the resonating column. This change, in turn, causes the vibrating reed assembly to assume the frequency of the air column. Most brass instruments have secondary lengths of tubing that are brought into play by mechanical valves by which the performer alters the length and the fundamental frequency of the vibrating air column. The exception to this is the slide trombone, which features a concentric tube arrangement by which the outer tube can move to lengthen the air column resonator.

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Eric Barth

See Also: Geometry of Music; Harmonics; Percussion Instruments; Pythagorean and Fibonacci Tuning; Scales; String Instruments.

Windmills

Category: Architecture and Engineering.
Fields of Study: Algebra; Geometry; Measurement.
Summary: The amount of power that a windmill can harness can be determined mathematically according to its size and design.

For centuries, windmills have captured peoples' imaginations through their form, function, and romantic appeal. Immortalized by Miguel de Cervantes in his book *Don Quixote*, windmills have transformed over the years from broad, short structures with an even number of sails to tall, sleek, three-sailed structures equipped with turbines for capturing energy from the wind. Windmills utilize natural power sources to perform a variety of functions, including energy production and food processing. Wind-driven prayer wheels have been used since the fourth century in Tibet and China. Historians believe that people in ancient Persia built the first practical windmills for both grinding grain and pumping water. From there, they spread through the Middle East and parts of Asia, as well as to India. They can be documented in Europe by the twelfth century. Wind turbines developed primarily in the twentieth century. Mathematics has been important for both the design of windmills and in calculating and modeling their output. Interestingly, English mathematician and physicist George Green was also a miller, and he is believed to have done much of his mathematics work in his windmill.

Designs

Windmills have had a wide variety of designs and appearances. Some of the earliest windmills rotated along a vertical axis, with the main rotor placed vertically in relation to the ground and giving a look similar to a helicopter. Some modern wind turbines have retained this engineering design in areas where wind direction is variable. This design is advantageous because vertical-axis windmills have an axis of rotation perpendicular to the ground, so the sails react similarly to all wind directions. On the other hand, horizontal-axis windmills have an axis of rotation that is parallel to the ground, resembling the more common image of a windmill such as that found in *Don Quixote*. The structure of horizontal-axis windmills gives the advantage of allowing their potential work to be maximized with respect to a specific wind direction. It is important to place a horizontal-axis windmill in line with the prevailing wind.

Windmills have traditionally been designed symmetrically, including an even number of sails. Historically, workers would place food and other substances in special locations inside the windmill to be ground by stones or other clashing materials. The grinding materials were sometimes connected to a system of gears and pulleys to increase the power beyond the mere rotation of the sails. Most modern wind turbines continue to have a sleek, symmetric design but have three sails. The insides of these turbines are devoted mostly to the attainment of electric power.

Number of Blades

The number of blades on a windmill is in direct correlation to the power generated, although the coefficient is quite small. The amount of power generated increases nearly linearly with each additional blade but the increase in power beyond just two or three blades is quite small for modern wind turbines. Physicists have determined that the power generated by a wind turbine is proportional to the cube of the wind speed and can be found algebraically by

$$P = EA\frac{1}{2}dv^3$$

where E is the power efficiency of the rotor, A is the swept area, d is air density, and v is wind speed. The swept area relates to the circle created by a rotation of a sail, calculated by

$$A = \frac{1}{2}\pi l^2$$

where l is the length of the sail. The theoretical maximum of E, known as the "Betz limit," is 0.59. The Betz limit is named for Albert Betz, a German physicist who was also interested in wind power. However, this theoretical value is reduced significantly when common physical constraints, including friction and drag on the rotors, are considered. One can calculate the maximum power produced by a windmill algebraically as

$$P_{\max} = \frac{8}{27}A\frac{1}{2}dv^3.$$

It is difficult to put tight parameters on the variables that determine the amount of power produced by a wind turbine. However, a good estimate of the production of power for a 10-foot diameter sail in 12 miles per hour average winds is 2300 kilowatts of power. In a wind farm, several turbines are interconnected by a power collection system and communications network to pool their output and connect to a power grid. Probabilistic mathematical models are used to estimate and describe the output of networks of wind turbines.

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See Also: Carbon Footprint; Electricity; Energy; Wind and Wind Power.

Wireless Communication

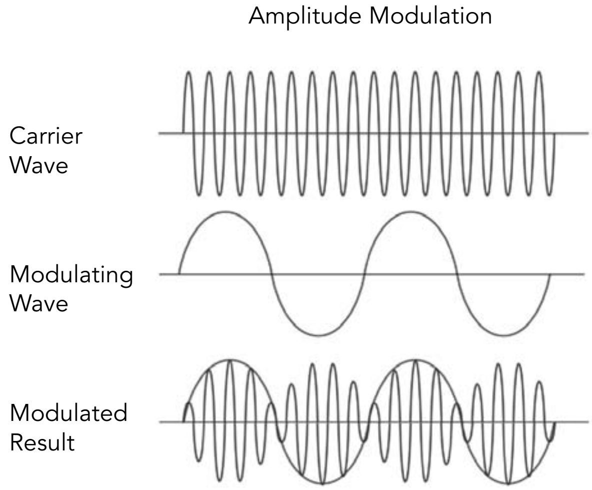
Wireless communication has become ubiquitous in the twenty-first century. Consider all of the aspects of one's life that are impacted by wireless communications, including text messaging and voice calls over a cellular network, and e-mail and Web surfing over a wireless Internet connection. Wireless communication consists of encoding information onto radio waves and passing them through the atmosphere—not unlike how an amplitude modulation (AM) or frequency modulation (FM) radio signal is sent and received. Wireless communication would not be possible without mathematics, and mathematicians contribute in many ways to creating, sustaining, and studying wireless processes and technologies.

Information theory plays a central role in wireless communications; its origins are attributed to mathematician Claude Shannon in the mid-twentieth century. Sergio Verdu, who is cited as a world-renown researcher in wireless communications noted, "Claude Shannon was the archetypical seamless combination of mathematician and engineer. . . . Shannon's theory has been instrumental in anything that has to do with modems, wireless communications, multi-antenna and so on."

Many other theoretical and applied mathematical methods have also been fundamental in wireless communication. For example, methods like stochastic calculus, stochastic modeling, control theory, graph theory, game theory, signal processing, wavelets, simulation and optimization, and multivariate statistical analysis have been used to develop communication networks, quantify or predict performance characteristics like network traffic, and to create protocols for signal transmission, encryption, and compression. Some mathematical models have been used by developers to quantify and compare wired versus wireless communication systems.

Mathematicians and engineers working in wireless communications must consider the properties of the waves and how the information is encoded. Information, whether an e-mail, telephone, video, or other data, is encoded onto the sinusoidal waveform by combining changes in frequency, amplitude, and phase. This encoding is accomplished by modifying various properties of a periodic sinusoidal function—the carrier wave—to embed information or message wave on the carrier. Figure 1 shows a simple example for the case of AM. The height or amplitude of the carrier wave is modified to represent or information or modulating wave.

Figure 1. Amplitude modulation (AM).



Source: M. Qaissaunee.

Researchers also consider the variety of factors that can affect the strength and quality of the signal. A communications engineer or technician is most often concerned with behaviors that will affect the propagation of the radio wave through the air. These include absorption, attenuation, diffraction, free space path loss, gain, reflection, refraction, and scattering. A combination of these factors will impact the signal quality and determine the likelihood of a successful transmission.

One common number associated with a wireless signal is the frequency. Frequency is a measure of how many cycles occur for a given time period. A signal cycle occurs every time a waveform repeats. Frequency is measured in cycles per second, which are also called "hertz" (Hz) after German physicist Heinrich Hertz. A waveform that repeats once every second has a frequency of 1 hertz. Waves used in communications are at much higher frequencies, so some prefixes must be used to measure radio frequencies. The wireless networks used for laptops and smartphones at the beginning of the twenty-first century often operate at the 2.4 GHz and 5 GHz frequencies of the spectrum. AM and

FM radio are in the kHz or MHz frequencies, while satellites operate at very high frequencies—often in the hundreds of GHz.

Michael Qaissaunee

See Also: Cell Phone Networks; Satellites; Telephones.

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Women

Category: Mathematics Culture and Identity.
Fields of Study: Communication; Connections.
Summary: Historically, women have been underrepresented in mathematics careers and professions.

Questions are raised periodically about women's participation or lack thereof in mathematics. This issue has been investigated from the perspectives of various disciplines, among them, history, psychology, neuroscience, economics, and statistics. Each of these perspectives has strengths and weaknesses and sheds light on different aspects of the issue.

Differences of era, place, and culture can affect findings; thus, results for one population do not always extend to others and findings from one decade may not hold for the next. In all cases, various forms of bias may affect the selection and interpretation of the information presented—on the part of newspapers, journals, researchers, and writers, as well as their audiences.

Pre-College and College Participation

Historical research has documented how the proportions of women in mathematics and other fields have waxed and waned with changes in societal norms, institutional policies, and mathematical practices.

In the antebellum, nineteenth-century United States, schooling was not compulsory, and most adolescents did not attend school. Mathematics, other than arithmetic, was not a college prerequisite and the adolescent girls enrolled in school did often not study the Greek and Latin required of college-bound boys. By the 1890s, about 7% of 14–17-year-olds attended high school. Girls outnumbered boys in mathematics courses at public high schools, sometimes outperforming them.

The proportion of adolescents attending high school increased rapidly. By 1940, almost three-quarters of 14–17-year-olds attended high school. However, many high schools de-emphasized or eliminated mathematics requirements and smaller proportions of students enrolled in advanced courses. The percentages of girls in these courses declined to parity in the early 1900s and decreased further until the 1950s. By the 1970s, their proportions had increased and 2005 statistics showed them at or above parity.

In every epoch on record, girls have predominated in high school, but before 1900 and between 1930 and 1980, women were a minority of undergraduates. Women's share of mathematics and statistics baccalaureates was similar to their share of all baccalaureates in 1950 but later lagged, remaining at 40% to 50%, although their overall share has since risen.

Recent Research on College and Pre-College Populations

Cognitive factors such as spatial abilities have been analyzed independently and with respect to mathematical performance. A 1985 meta-analysis by Marcia Linn and Anne Petersen grouped spatial abilities into three categories: spatial perception, spatial visualization, and mental rotation. They found little evidence of gender differences for the first two categories but found large gender differences on mental rotation tasks for which scores depend on speed and accuracy. Subsequent research reports that these differences have diminished and training studies conducted by Nora Newcombe, Sheryl Sorby, and others show this ability can be improved. Mental rotation appears more important for careers such as engineering and fashion design than mathematics.

Another line of research has focused on mathematical aptitude, often as measured by the mathematics section of the Scholastic Aptitude Test (SAT or SAT-M). One finding, frequently cited as evidence for innate gender differences in mathematical aptitude, concerns the SAT-M scores from "talent searches" among middle school student volunteers. Between 1980 and 1982, the ratio of boys to girls scoring 700 or above was 13:1. Later, larger samples have yielded different, smaller ratios; a 2005 ratio is 2.8:1.

Although the first finding received extensive media coverage and is widely cited, the drop has received little publicity and few citations. Underlying causes may be related to those of the file-drawer effect—the tendency for findings that fail to reject a null hypothesis to remain unpublished.

Use of the SAT-M as a measure of mathematical aptitude or ability has been criticized on the grounds of construct validity and predictive validity. Studies of the latter find that the SAT-M underpredicts women's undergraduate mathematics course grades and overall grade point averages relative to those of men.

Possible reasons for gender gaps in SAT-M scores include differences in strategies (documented by Ann Gallagher and her collaborators) and the phenomenon of stereotype threat identified by Claude Steele and Joshua Aronson. An individual may be vulnerable to stereotype threat in a particular context if the individual is a member of a group that is stereotyped as performing poorly in such contexts. For example, reminding a woman of such stereotypes can hamper her mathematical performance, particularly when she cares about doing well in mathematics.

Using imaging techniques, researchers have found gender differences in brain areas used for processing when subjects were asked to calculate or solve mathematics problems. These have been popularly interpreted as "hard-wired" gender differences. However, the subjects of these studies are adults. Thus, these differences may result from differences in experience. Moreover, the studies are small in scale, and their findings are not always consistent.

International assessments for primary and secondary education are administered by the Trends in Mathematics and Science Survey and the Programme for International Student Assessment. Scores on these assessments, representing 493,495 students, were analyzed in 2010 by Nicole Else-Quest and her colleagues. They concluded that, on average, males and females differ little in mathematical achievement, despite more positive attitudes toward mathematics among males and substantial variability across nations. The most powerful predictors of cross-national variability in gender gaps were gender equity in school enrollment, women's share of research jobs, and women's parliamentary representation.

Graduate and Faculty Participation

Since the nineteenth century, a standard credential for professors at four-year academic institutions has been a Doctor of Philosophy degree (Ph.D.). In the United States, a Ph.D. is a terminal degree—the highest degree given in scientific fields. Thus, for modern times, Ph.D. attainment is a frequently used measure of women's participation in mathematics.

The first American woman to be awarded a Ph.D. in mathematics was Winifred Edgerton Merrill, in 1886. (A decade earlier, Christine Ladd-Franklin had completed a dissertation in mathematics at Johns Hopkins University. However, her Ph.D. was not awarded until 1926.) Before 1890, most Ph.D. programs in the United States did not allow women to enroll, making them less likely to frequent many mathematics departments. Other obstacles were quotas and professors who refused to have women as Ph.D. students. Reflecting societal norms, qualified women were sometimes not considered for academic positions, paid less, promoted more slowly or not at all, or expected to quit their positions if they married. University anti-nepotism rules were often used to exclude wives from paid employment at a spouse's institution (except during extreme circumstances, such as World War II). Such policies were likely to have affected women in mathematics more than women in many other disciplines. Then, as now, husbands and partners of female mathematicians and scientists tended to also be mathematicians and scientists. Unlike experimental scientists, female mathematicians had few opportuni-

Winifred Edgerton Merrill

ties for professional employment outside academia or as laboratory researchers within academia.

Despite these factors, the numbers and percentages of women earning Ph.D.s in mathematics increased until the 1940s. Between 1950 and 1970, women's numbers stalled while the numbers of men earning Ph.D.s in mathematics and science increased. Part of this increase was because of the influx of World War II veterans whose college and graduate tuition was supported by the Servicemen's Readjustment Act of 1944, known as the GI Bill. The lack of any corresponding increase in women's numbers may have been because of neglect of the female veterans who were nominally beneficiaries of the GI Bill together with changes in social norms and science policy. Consistent with these factors, women who were called to teach at colleges and universities during the war were displaced by men returning from war projects.

Changes in science policy and views of science may have had an especially damping effect on women's participation in mathematics, intensifying what was often seen as a dichotomy between teaching (associated with women) and research (associated with men). Margaret Murray writes that the "myth of the mathematical life course" became the prevailing model of how a mathematical career should unfold—a trajectory more compatible with societal expectations of men than women. In this view, mathematical talent emerges in childhood—creative achievements begin early and are quickly recognized. The mathematician focuses on research, ignoring distraction or shielded by a spouse or relative. Accomplishments continue, without interruption, until the mathematician's early 40s.

Faculty Participation After 1970

With the women's movement of the 1970s, percentages of women in mathematics and other fields increased. In 1971, the American Association of University Professors and the Association of American Colleges issued official policy statements urging that anti-nepotism rules be rescinded. However, the absence of anti-nepotism policies does not always solve a "two-body problem"—finding appropriate professional employment in the same geographical area for two Ph.D.s.

Another important event was the passing of the Educational Amendments Act of 1972. Its Title IX prohibits discrimination against women at educational institutions that receive federal funding and mandates periodic reviews of these grantees by federal agencies.

Elimination of anti-nepotism policies and prohibition of sex discrimination were major changes. However, for two decades, proportions of women had been very small in many mathematics departments and elsewhere in academia. Changes in institutional policies and federal regulations were no guarantee of change in individual expectations and departmental policies.

Individual expectations may be affected by evaluation bias. One example is a study conducted by Linda Fidell in 1970. Sets of 10 fictitious "résumés" of psychologists were sent to psychology department heads with the request to indicate the appropriate professorial rank at which each person described should be hired. Six of the résumés carried a male's name and the others female names. These were rotated so that the same résumé would sometimes carry a female name and sometimes a male name. The department heads assigned different ranks to identical qualifications, depending on the names they carried. Those with female names received lower ranks than those with male names. Later research suggests that this phenomenon is more complex than originally hypothesized because ratings are affected by social context. An explanatory mechanism identified by Virginia Valian is the notion of "gender schemas"—implicit hypotheses, usually unarticulated, that affect expectations and evaluations of women and men.

Although the percentage of women earning Ph.D.s in mathematics has continued to increase by at least 5% every decade since the 1970s, the presence or absence of departmental policies, such as family leave, may weigh more heavily on women. Moreover, sociological research suggests that women in science have fewer professional interactions within their departments or workplaces and are thus less likely to be aware of expectations conveyed informally. For example, a study of science departments found that departments with written guidelines for graduate students about courses of study, exams, and other expectations tended to have a larger percentage of women who earned Ph.D.s.

A variety of empirical findings suggest that, since the 1970s, the cumulative effects of individual actions, departmental practices, and institutional policies have changed to filter out fewer women. One factor may have been individual and class-action lawsuits brought on the grounds of Title IX violation. In contrast, a 2004 Government Accountability Office study found that federal agencies that fund scientific and mathematical research had not conducted the compliance reviews of their grantees mandated by Title IX.

In 2006, a National Academies report recommended that Title IX and other federal antidiscrimination laws be enforced and that federal agencies work with scientific societies to host mandatory workshops on gender bias. In 2007, the Gender Bias Elimination Act was introduced in Congress, which would have authorized such workshops and directed funding agencies to better enforce federal antidiscrimination laws. This bill did not pass, and similar bills were introduced in 2008 and 2009.

Recent Survey Findings

Every five years, the Conference Board of the Mathematical Sciences surveys a representative sample of two- and four-year academic institutions. The 2005 survey found that women were 50% of the full-time permanent mathematics faculty at two-year colleges (up from 34% in 1990 and 40% in 1995). At four-year institutions, the percentages of women in tenure-track (entry-level) and tenured (permanent) positions also increased, with the exception of tenure-track positions at B.A.-granting institutions (see Table 1).

Table 1. Percentages of Women on Mathematics Faculties of Four-Year Institutions

	1995	2000	2005
Tenured women (% of tenured faculty)			
Ph.D.-granting departments	317 (7%)	346 (7%)	427 (9%)
M.A.-granting departments	501 (15%)	608 (19%)	532 (21%)
B.A.-granting departments	994 (20%)	972 (20%)	1373 (24%)
Tenure-track women (% of tenure-track faculty)			
Ph.D.-granting departments	158 (20%)	177 (22%)	220 (24%)
M.A.-granting departments	235 (29%)	276 (32%)	337 (33%)
B.A.-granting departments	748 (43%)	517 (32%)	693 (28%)

Source: Conference Board of the Mathematical Sciences 2000 and 2005 Surveys.

The Survey of Doctorate Recipients has collected longitudinal data about 40,000 science and engineering Ph.D. recipients who earned their degrees from institutions within the United States. Recent analysis of data from this survey and the National Survey of Postsecondary Faculty—together with results from surveys of research-intensive departments and faculty members conducted in 2005—found few gender differences on key measures such as grant funding and salary for faculty members. In mathematics, women published fewer articles than men, and the proportions of women applying for jobs were slightly smaller than the proportion earning Ph.D.s. Overall, for the six scientific fields surveyed, the likelihood that a position would have female applicants was affected by institutional characteristics, the presence of family-friendly policies, the proportions of women on search committees, and the gender of the search committee chair. On average, men and women in research-intensive departments reported similar allocations of time on research and teaching but differences in professional interactions. Women were more likely to have mentors; men were more likely to engage with their colleagues on a wide range of topics from research to salary. Women were less satisfied with their jobs, and indirect evidence suggests that women were more likely than men to leave before tenure consideration.

Organizations

Organizations such as the Association for Women in Mathematics (AWM), European Women in Mathematics, and Korean Women in the Mathematical Sciences are dedicated to supporting and promoting women and girls in the mathematical sciences. Student organizations at colleges and universities include AWM chapters and Noetherian Ring groups, the latter named for mathematician Emmy Noether, who is well-known for her pioneering work in abstract algebra.

These and other organizations document women's participation in mathematics. Biographies of past and present women in mathematics are available online at the MacTutor History of Mathematics Archive, Biographies of Women Mathematicians at Agnes Scott College, and Mathematicians of the African Diaspora. The biographies of 228 women who earned Ph.D.s in mathematics at U.S. institutions before 1940 are maintained at the Web site for the book *Pioneering Women in American Mathematics*.

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Cathy Kessel

See Also: Educational Testing; Measurement in Society; Minorities.

World War I

Category: Government, Politics, and History.
Fields of Study: All.
Summary: World War I saw an increased emphasis on applied mathematics but ultimately disrupted mathematics research.

Although mathematicians were not as heavily involved with the conduct of World War I as they would be with World War II, the four years of conflict impacted the field of mathematics in two main ways: they severed international ties among researchers, thus slowing collaborative research efforts; and the war provided the circumstances for applied mathematics to develop more fully through military research. Many mathematicians contributed their knowledge and abilities to the war effort. At the same time, others published papers unrelated to the military, worked to encourage reconciliation among mathematicians of warring nations, or strove to end the war outright. World War I, which was fought from 1914 to 1918, was precipitated by the assassination of Archduke Franz Ferdinand of Austria. After the initial declaration of war on Serbia by Austria, countries with various political alliances joined the fighting, with the result that more than 30 countries on five continents were ultimately named as combatants. The massive scope of this first truly global war led U.S. President Woodrow Wilson to refer to it as the "war to end all wars."

Mathematics Applied to Military Research

Some mathematicians turned their attention to more practical and applied uses of the field. World War I saw extensive use of both trench warfare, which the United States had already experienced somewhat during the U.S. Civil War; and potent chemical weapons, like mustard gas. In the United States and in Europe, mathematicians researched ballistics and aeronautics as the warring countries sought advantages in firepower on land and began to realize the potential of air power. Mathematician John Littlewood performed research on ballistics and improved tables for the British Royal Garrison Artillery. In the United States, important figures such as Gilbert Bliss, Oswald Veblen, Norbert Wiener, and Forest Ray Moulton worked at the U.S. Army's Aberdeen Proving Ground, Maryland, in ordnance and improvement in ballistics calculations. The American Mathematical Society published, in 1919, a list of over 175 mathematicians working in some capacity to support the war effort. The National Advisory Committee on Aeronautics also began construction of the Langley Laboratory in 1917, although research did not fully get underway until a few years later.

Similarly, Europeans conducted research with the aim of improving military operations. The British mathematician Frederick William Lanchester devised a formula to calculate the likely outcome of a battle between opponents of different strengths. He also published a series of articles on the military potential of aeronautics, which in 1916 were collected into a book. At Göttingen, Germany, Felix Klein and others instituted the Aerodynamic Proving Ground in 1917. In Italy, Mauro Picone investigated new methods for calculating ballistics tables, and Vito Volterra proposed using helium in airships.

As was the case in many wars dating back into antiquity, codes and cyphers played an important role. For example, "trench codes" consisting of three number or letter groups were used for rapid communications of tactical situations but they were fairly easily cracked and were quickly supplanted by more complex structures. The Germans widely employed the ADFGVX cypher, so named because only those six letters were used in coded messages. They had been chosen to minimize operator error because when those letters are sent by Morse code, they sound very different from one other. The code was a fractionating transposition cipher using a modified Polybius square, named for second-century B.C.E. historian Polybius of Megalopolis, with a single columnar transposition.

The cypher keys were typically changed every few days and the code was broken in only a few isolated cases during the war. A general solution was found in the 1930s by William Friedman, who is often referred to as the "father of modern U.S. cryptography." The Germans also used some double transposition cyphers, which applied the same transposition key horizontally and vertically to the same matrix. In addition, they proved to be skillful in deciphering the codes of others, and the U.S. Army began to experiment with using Native American languages as military code. Several Choctaw soldiers served in the U.S. Army in Europe during World War I and are credited with helping to win some major battles.

The goal of war-related mathematicians was to improve the efficiency of military action. In the United States, this goal also applied to the home front. Allyn A. Young, the president of the American Statistical Association, proposed in a December 1917 address that a central statistical office or commission be established to aid the coordination of various boards and agencies then gathering statistics related to the war.

A greater division between mathematics research and teaching concerns also occurred around the time of World War I, as evidenced by the founding and branching off of the Mathematician Association of America in 1915 and the National Council of Teachers of Mathematics in 1920 from the more research-focused American Mathematical Society.

Suspension of International Cooperation

The war ended or made much more difficult the international relations among mathematicians that had developed in previous decades. National organizations of mathematicians publicly condemned their colleagues in enemy countries. International meetings were abandoned. Even after the war, an international congress did not fully accept German members again until 1928. A mathematician of one nation working in or visiting a hostile country might run the risk of being stranded, or worse, face arrest and imprisonment. As a whole, there were few mathematicians who made efforts during the war to maintain relations with their counterparts and such efforts were sometimes limited to individual statements of protest against a severing of ties among nations. The division of researchers slowed the development of some fields, like topology and set theory.

Non-Military Research During the War Years

Although much mathematical work from 1914 to 1918 related to improving military capability, there were many other notable advances that did not have immediate effects on war power. For instance, Albert Einstein published his general theory of relativity in 1915. David Hilbert also published field equations about that time. While a prisoner of war in Russia, the Polish mathematician Waclaw Sierpinski published a paper on his fractal triangle. Together with Godfrey Harold Hardy, after arriving at Cambridge University on Hardy's invitation, Srinivasa Iyengar Ramanujan published a series of papers on number theory during the war.

Efforts for Peace and Reconciliation

At the same time, some mathematicians focused not on improving the conduct of war or other research, but instead on ending the conflict and reconciling with their colleagues in the peace that would follow. Perhaps the most famous case is that of the British mathematician Bertrand Russell, who soon after the turn of the century had identified a paradox that challenged assumptions of set theory and in the years immediately before the war had co-authored *Principia Mathematica* with Alfred

North Whitehead. Repulsed by the battlefield slaughters and the general support of his countrymen for the war, Russell became an increasingly active pacifist, eventually taking part in public demonstrations and spending six months in prison for his antiwar writings.

Less dramatically, but still forcefully, the German David Hilbert made a point of recognizing the accomplishment of colleagues in enemy countries. The Dutch geometer Luitzen Egbertus Jan "L. E. J." Brouwer worked after the war to bring German mathematicians back into recognition. Gosta Mittag-Leffler, a Swedish mathematician, deliberately published English, French, and German papers in his journal *Acta Mathematica*. After the war, he and Godfrey Harold Hardy worked to encourage reconciliation with German researchers.

Approaching the cause of peace from another angle, the Quaker mathematician Lewis Fry Richardson, who had served in an ambulance unit in France during the war, worked to understand the causes of wars so as to better prevent them. A limited printing of his first paper on the subject, "The Mathematical Psychology of War," appeared in 1919. In later decades, as World War II loomed, Richardson would return to the subject.

Conclusion

The death of possible future contributors to the field of mathematics during World War I as a whole was, of course, an incalculable loss. By disrupting the continuity of research and discovery, the war also delayed advances in areas of mathematics such as topology and set theory. At the same time, however, the possible applied uses of mathematics began to receive more attention and appreciation. In addition, national governments became more aware of the military value of mathematicians—a value that they would exploit much more thoroughly and effectively in World War II.

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See Also: Airplanes/Flight; Artillery; Mathematics, Applied; Predicting Attacks; Professional Associations; World War II.

World War II

Category: Government, Politics, and History.
Fields of Study: All.
Summary: World War II saw significant mathematical advances in cryptography, operations research, and navigation.

World War II was fought between two major alliances of countries, the "Axis" and the "Allies." The beginning might be traced to pacts signed in 1936 and 1937 by the three primary Axis powers: Germany, which came to control much of the European continent; Italy, which influenced the Mediterranean; and Japan, which governed much of East Asia and the Pacific. The ultimately victorious Allies coalition, led by Great Britain, the United States, and the Soviet Union, gained the surrender of Italy in 1943 and Germany and Japan in 1945. Well over 50 countries participated in the war, and there were millions of military and civilian deaths, some of the most controversial being those that resulted from the United States' use of the atomic bomb in Japan. Mathematics played a critical role in many aspects of the war effort, notably in coding and encryption, which achieved levels unseen in previous wars and led to additional developments in the subsequent cold war era, such as mathematician Claude Shannon's ideas on information theory. New areas of applied mathematics, such as operations research, also emerged from technologies and problems created during or inspired by the war. Many mathematicians served in the military or worked for military agencies, such as the U.S. Aberdeen Proving Grounds. An Applied Mathematics Panel was formed in 1942 to solve war-related mathematical problems. Mathematicians were involved in the Manhattan Project to develop the atomic bomb, a matter that is widely discussed even in the twenty-first century with regard to the ethics of mathematics research and social obligations of mathematicians as citizens of the world. The immediate prewar era and wartime would also result in a flood of mathematicians and scientists emigrating to the United States and many other Allied countries, fleeing religious or political persecution, particularly in Nazi-controlled Europe. It also likely accelerated the growth of participation of women in mathematical and scientific careers. These individuals would shape both research and teaching for decades to come.

Codes and Cyphers

Through World War I, most encrypted messages either used a paper-and-pencil cipher or a "book code" in which the enciphered version of each word was looked up in a codebook. Between the world wars, two new types of cryptography emerged: superencypherment and rotor machines.

With superencypherment, the text to be enciphered was converted into a string of digits. Then, a string of random digits (known as "additives") was added with non-carrying addition. If the additives were never used again, the result was the "one-time pad" cipher. However, if the string of additives digits is reused, it is possible for code-breakers to break the cipher. In the 1930s, American cryptographer William Friedman developed the "kappa test," a statistical test to determine when a superencypherment string was being reused.

The Japanese Navy used a codebook to convert plain text into numeric code groups, which were then superencyphered using a book of 50,000 random digits. During wartime, the number of encrypted messages sent was such that any string of these digits was reused, and the U.S. Navy was able to break the Japanese code.

The main technique was to search for so-called double hits. Suppose two encrypted messages read:

… 77899 45616 27249 31464 68461 …
… 77899 81957 27249 81279 59138 …

The double hit is underlined. It could be because of chance but the cryptographer assumes that it is because of the same code words being enciphered by the same stretch of additive. With enough double hits, the cryptographer can recover portions of the additive and start decoding the underlying code words, as well as locating the so-called indicator (numbers hidden in the message to tell the recipient where in the book of additives the sender started). It took months of traffic for enough double hits to appear to break the Japanese naval code, which was changed several times a year. The kappa test could also be used to locate re-used stretches of additive. In 1943, in a project later code-named VENONA, the U.S. Army spotted seven double hits in 10,000 Soviet diplomatic messages. The Soviets, who used the unbreakable one-time pad system, had blundered by re-issuing some 30,000 pages of random additive, and VENONA succeeded in breaking some 2900 Soviet messages.

The Germans and Italians used the "Engima" cipher machine, which consisted of three rotors plus a steckerboard (a plugboard), which added a monalphabetic substitution to the polyalphabetic generated by the rotors. A rotor was a disc with 26 electrical contacts (for the Roman alphabet) on each side. Wiring inside the rotor connected the contacts. Such a rotor creates a monalphabetic cipher—each letter would always be replaced with the same letter. If the rotor is allowed to rotate one contract between letters, it generates a polyalphabetic cipher with a period of 26. If two rotors are connected together, so that the second one advances one space after the first one completes a rotation (in the same way as the rotating numbers in a mechanical car odometer), then the two rotors generate a polyalphabetic cipher with a period of 26×26 (sometimes 26×25, depending on how the two rotors were geared together). Three rotors generate a period of $26 \times 26 \times 26$, and so on. The operator had up to eight rotors available, giving up to

$$\frac{8!}{5!} = 336$$

possibilities for the rotors. For each day, there was a prearranged rotor selection and steckerboard setup and the operator would choose at random an initial rotation for each of the three rotors of the day. An "indicator" giving this random initial position had to be inserted into the message.

In the 1930s, three mathematicians, Marian Rejewski, Zerzy Rozycki, and Henyrk Zygalski of the Polish *Biuro Szyfrow* (Cypher Bureau) had figured out the wiring of the rotors in the Enigma, had worked out techniques for deciphering this indicator; which had been enciphered using the same Enigma, and had invented a machine called a "bomby," which automated much of the work. With these tools and techniques, they were able to read German Enigma messages until the Germans introduced changes in 1938 that defeated the Polish techniques.

The Poles then turned over their work to the British and French. The British took over an estate north of London called Bletchley Park and brought in mathematicians to work on the Enigma and other ciphers. The first four mathematicians were Alan Turing (whose Turing Machine, of 1936 formed the theoretical basis of later computers), Gordon Welchman, John Jeffreys, and Peter Twinn. Bletchley Park's main method for breaking Enigma was to find a crib (a word or words that were highly likely to be in a particular place in the message). Despite the features of Enigma that were supposed to hide any evidence of the plain text, there were certain relationships among the letters of the cyphertext that had to occur when the crib was enciphered. A machine called a "Bombe" then ran through all 26^3 positions of the three rotors, finding the very few that would produce these relationships. Multiple runs would be required for different choices of rotors but Bletchley also developed a statistical technique that—with luck—would eliminate numerous rotor choices.

Searching for a code that would be difficult to break using mathematically based cryptography methods, the U.S. government recruited native Navajo speakers. The Navajo language is very complex with unique phonetics, grammar, and syntax and no written or symbolic alphabet, making it nearly impossible for someone without substantial exposure to understand (no Axis linguists had such exposure) and providing no written cypher that could be analyzed. Several hundred Navajo code talkers served with the U.S. Marines, most in the Pacific theater.

Computers

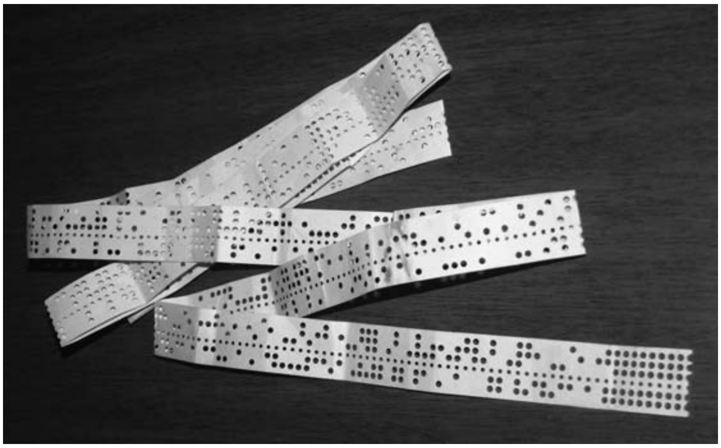
While general-purpose electronic computers did not exist until after World War II, work during the war helped lead to their development. By 1940, analog computers of considerable sophistication existed. However, there were only a handful of digital computers, all of them electromechanical and not differing much in concept from Babbage's analytical machine of the nineteenth century. At that time, the only design for an electronic computer was from John V. Atanasoff of Iowa State College (now Iowa State University), who with Clifford Berry designed the Atanasoff–Berry Computer (ABC). It was not a general-purpose computer, limited to the solution of sets of linear equations.

In Germany, Konrad Zuse began working on computers in 1936. In 1941, he constructed the electromechanical Z3, which was the first general-purpose programmable computer. It was used for calculations for aircraft design and was destroyed by Allied bombing in 1943. After the war, Zuse built computers commercially and also developed the first programming language, Plankalkül.

In 1941, the Germans invented a new type of cypher for high-level communications. Instead of replacing or scrambling letters, a machine was developed that worked on the bits of the five-bit teletype (Baudot–Murray) code. In principle, this process was a superencypherment in which the bits of the teletype code were superenciphered by a string of binary additives. The additives were not random but were produced by a set of 10 wheels that rotated with different periods.

To solve this cipher, Bletchley Park constructed an electronic device called the "Colossus." Ten were built, each having from 1500 to 2500 vacuum tubes apiece. It was not a general-purpose computer since it could solve only one particular problem but the experience with electronic circuits and the knowledge that a device with thousands of vacuum tubes would work inspired, after the war, three successful British efforts (Turing's ACE, Cambridge University's EDSAC, and Manchester University's Mark I) to build general-purpose electronic computers. This kept the United Kingdom competitive in computer design with the United States through the beginning of the 1960s.

The Ordnance Department of the U.S. Army had the task of computing large numbers of range tables for artillery. Its Ballistic Research Laboratory, in cooperation with the Moore School of Engineering at the University of Pennsylvania, had the foresight—and ambition—to contract for an electronic computer, to be known as Electrical Numerical Integrator and



A paper tape with holes from the five-bit teletype cyphered in the early 1940s German "Baudot Code."

Computer (ENIAC). The principal designers of the ENIAC were John Mauchly and John Presper Eckert (later developers of the UNIVAC line of computers), although many of the ideas of the design came from Atanasoff's ABC. The ENIAC did not become operational until 1945. One of its first uses was in designing the hydrogen bomb.

By 1944, the shortcomings of this pioneering design had been realized. It could not handle the workload required for numerical solution of partial differential equations and plans were started for a more advanced computer to be known as EDVAC. In 1945, John von Neumann combined his own ideas, those of Alan Turing, and those of the ENIAC developers into the paper, "First Draft of a Report on the EDVAC," which laid out the principles of the modern computer. This paper led to the "Von Neumann machine" model, still used in the twenty-first century, although most of the ideas came from Turing.

Operations Research

In June 1941, Coastal Command (that portion of the Royal Air Force that operated over the seas from land bases) brought in physicist Patrick M. S. Blackett as an advisor. Blackett decided that instead of designing new weapons, his duty was to analyze how Coastal Command performed its operations and see what he could recommend to improve them. Hence, his work became known to the British as "operational research" (also called "operations research").

Blackett and his colleagues investigated a wide variety of submarine and anti-submarine operations. In one such project, the group figured out that a submarine attacked by an aircraft would not have time to dive very deep (indeed, it might still be on the surface), and that a setting of 25 feet for the depth charges the aircraft dropped had the best chance of lethality to the submarine. Another project was to figure out the optimum size of a convoy. It turned out that the larger the convoy was, the better. A convoy, even a large one, had almost the same chance of avoiding being seen by a submarine as a single ship did. What mattered was not the area of sea the convoy covered but its perimeter, where the escorts were stationed. The perimeter increased much slower than did the number of ships, so if both the number of ships and the number of escorts were doubled, each escort had a smaller length of the perimeter to cover, which gave it a better chance to catch enemy submarines trying to penetrate its portion of the perimeter.

The success of Blackett's original group led to operational research's extension to many other parts of the British forces. In April 1942, the U.S. Navy founded its own Anti-Submarine Warfare Operations Research Group, originally for antisubmarine warfare and later for work throughout the Navy. As Admiral King reported:

> The knowledge . . . made it possible to work out improvements in tactics which sometimes increased the effectiveness of weapons by factor or three or five, to detect changes in the enemy's tactics in time to counter them before they became dangerous, and to calculate force requirements for future operations.

Navigation

World War II presented navigation problems not seen in prewar flying, such as how to find a target at night from the air. In the Battle of Britain, the Germans first used the "Knickebein" system for target location at night. Knickebein and it successor "X-gerät" used narrow radio beams that crossed over the target. Later, the Germans introduced "Y-gerät," which used a single ground station, with the aircraft transmitting a return signal from which the distance from the aircraft to the transmitter could be determined by the ground station.

The Allies also developed targeting systems. One was the British "OBOE" in which two stations broadcast signals to which the aircraft responded, allowing each station to determine the distance to the aircraft. The aircraft flew a fixed distance in a circular arc from the first station until it was at a specified distance from the second station. The intersection of these two arcs was the target location. This Y-gerät/OBOE technique, except with the aircraft transmitting and the ground station responding, is still used in the twenty-first century in the Distance Measuring Equipment (DME) system widely used by both military and civilian aircraft for navigating over land.

The British also developed the "GEE" system, which used a different mathematical technique. There was no transmitter on the aircraft. Instead, there was a "primary" or "master" transmitter and at least two "secondary" or "slave" transmitters on the ground. The primary would broadcast a signal, and each secondary would broadcast its own signal as soon as it received the signal from the primary. Any given difference between the arrival times of the signal from a primary and secondary defined one branch of a hyperbola (since a hyperbola is the locus of all points the difference of whose distance from two foci is constant and whether the primary or secondary signal arrived first tells which branch of the hyperbola). The second primary-secondary pair defined one branch of a second hyperbola, and these two branches intersect in exactly two points. Either dead reckoning or a third pair could then be used to determine which of these two intersection points was the aircraft's position.

GEE was soon developed into the Long Range Navigation (LORAN) system, which is still used worldwide for navigation at sea within approximately 1000 kilometers of the LORAN stations. Beyond that distance, the ionospheric bounce of the signals interferes with the ground wave.

The Mathematics Community in World War II

Mathematicians participated in both military service and multiple civilian roles during World War II. Some enlisted voluntarily or were drafted, such as Herman Goldstine, who worked as the army liaison to the ENIAC project. Many stayed in their academic positions, continuing to prepare students and working on war-related training programs in mathematics. Others left their colleges and universities to work for government programs related to the war effort, including the growing area of operations research, such as G. Baley Price, who worked on applications like bomber accuracy and Philip Morse, who is sometimes referred to as the "father of U.S. operations research" and is credited with organizing the U.S. Anti-Submarine Warfare Operations Research Group. Companies like the Radio Corporation of America (RCA), Westinghouse Electric Corporation, Bell Laboratories, Bell Aircraft Corporation, Grumman Aircraft Engineering Corporation, and Lockheed Corporation recruited mathematicians to help fulfill war contracts. The government also widely recruited nonmilitary mathematicians for groups like the Office of Scientific Research and Development, which had branches conducting medical research, fuse research, and a multi-application area looking at problems like submarine warfare, radar, and rocketry. This body came to include the Applied Mathematics Panel in 1942.

Mathematician and scientist Warren Weaver, a pioneer in the field of machine translation, headed the panel. Some of the problems investigated included gas dynamics and compressible fluids, underwater ballistics and explosions, shock waves in air and water, mechanics and damage in air-to-air combat and anti-aircraft fire, ballistics and firing tables, torpedo spread angles, land mine clearance techniques, and statistical methods. In this time period, women also experienced increasing opportunities to pursue and contribute to a diverse range of careers, including science and mathematics. Hunter College professor Mina Rees took a leave of absence during World War II to contribute to the war effort, working with the Applied Mathematics Panel. Following the war, she became head of the mathematics branch of the Office of Naval Research. The American Mathematical Society said

> . . . the whole postwar development of mathematical research in the United States owes an immeasurable debt to the pioneer work of the Office of Naval Research and to the alert, vigorous and farsighted policy conducted by Miss Rees.

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James A. Landau

See Also: Atomic Bomb (Manhattan Project); Coding and Encryption; Intelligence and Counterintelligence; Pearl Harbor, Attack on; Radio; Strategy and Tactics; World War I.

Wright, Frank Lloyd

Category: Architecture and Engineering
Fields of Study: Geometry; Measurement; Representations.
Summary: Frank Lloyd Wright is one of the world's most renowned architects and he revolutionized architecture and design.

Considered one of the greatest American architects of all time, Frank Lloyd Wright was also an interior designer, writer, and educator. Born in Richland Center, Wisconsin, in 1867, he died in Taliesin West, Arizona, in 1959. His mother, who had always expected her son to become an architect, gave him a set of Froebel gifts after visiting the 1876 Centennial Exhibition in Philadelphia. Developed by Friedrich Fröbel in the 1830s, the kindergarten maplewood building blocks allow children to learn the elements of geometric form, mathematics, and creative design while playing. In his autobiography, Wright attests to their influence on his professional career.

Career

After taking engineering courses at the University of Wisconsin, he started working as a draftsman for architect J. Lyman Sielbee and, later, for Louis Sullivan, one of the most prominent members of the Chicago School who coined the famous modernist slogan "form ever follows function." In 1893, Wright established his own practice and in the early 1900s he initiated the series of the Prairie Houses. Rejecting the traditional vocabulary and ornaments of classical architectural styles, he revolutionized the U.S. home by focusing on geometry and the design of volumetric spaces, allowing a free spatial flow between the main living areas. The Robie House, with its low horizontal lines, nearly flat roof, overhanging eaves, central hearth, clerestory windows with delicate geometrical patterns, and open interior spaces is one Wright's finest examples of Prairie architecture.

Convinced of the critical role played

Taliesin West was Frank Lloyd Wright's winter home and school in Scottsdale, Arizona. It currently houses the Frank Lloyd Wright School of Architecture, the Frank Lloyd Wright Foundation, and hosts tours year-round.

by architecture in promoting democracy, Wright used similar design principles to develop affordable homes he called "Usonian" during the Great Depression. Simultaneously, he proposed the utopian planning concept of Broadacre City, a low-density, automobile-based, surburban community where each U.S. household would live in a Usonian house on one acre of land.

Fascinated by the integration of the natural world, Wright argued that "form and function are one" and he promoted organic architecture as the modern ideal. He strived to reinterpret the patterns and principles of nature into an architectural language respecting the properties of building materials and the harmonious relationship between the form and function of the structure. Organic architecture is the outcome of an inclusive design process that aims at integrating the various spaces into a coherent aesthetic and functional whole. Wright believed that a building is a unified organism that has an intrinsic relationship not only with people but also with both its site and its time. With such concerns in mind, he designed architectural projects down to their smallest external and internal details including custom-made furniture, stained glass, rugs, light fixtures, and other decorative elements. Fallingwater, the Kaufman house outside Pittsburgh, Pennsylvania, cantilevered over a waterfall, and Taliesin West, which was built with the sand, gravel, and native boulders from the magnificent Arizona desert and mountain setting, exemplify Wright's theories of organic architecture. Another structure reflecting Wright's increased sensitivity to building materials and methods was the 14-story tall Johnson Wax headquarters, whose dendriform columns echoed inside the edifice and clerestories transformed the modern office building into a cathedral of the future.

Later Life

Until the end of his life, Wright increased his range of geometrical and structural themes and after World War II his nonresidential projects gained more significance. To the rectangular forms characteristic of the earlier decades, he added more complex geometries of the plan based on 30 degree and 60 degree angles, polygons, circles, hemicycles, and spirals that he developed in three dimensions. The Guggenheim Museum, Wright's last major work, is also one of the twentieth century's most important architectural landmarks. Its continuous upward spatial helix with sloping walls capped by a glass dome dramatically contrasts with the urban grid of the city of New York and offers a unique spatial experience to the visitor.

Wright left a rich legacy of truly American modern architectural projects unifying art and geometry and an architectural tradition of respect for the natural environment.

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Catherine C. Galley
Carl R. Seaquist

See Also: City Planning; Educational Manipulatives; Green Design; Interior Design; Skyscrapers.

Writers, Producers, and Actors

Category: Arts, Music, and Entertainment.
Fields of Study: Communication; Connections.
Summary: Some actors, screenwriters, and producers are also mathematicians or consult with them.

Mathematical scenes can be found in many scripted and unscripted productions. Some of these references are created by mathematically educated people, including writers, producers, or mathematical consultants. Mathematical references can shape society's views of mathematics and some writers or producers have noted that they have this goal in mind during the creation process. Other times, mathematics and mathematicians serve purely as entertainment value and so stereotypes, such as the nerd or mad scientist, proliferate. Actors and actresses may also have mathematical training and some use their popularity to encourage students to succeed in mathematics. Mathematicians and educators showcase these people and their mathematical references or accomplishments in order to interest and motivate students and to highlight the importance, beauty, and usefulness of mathematics, as well as the diverse career options that are available to mathematically talented individuals. Mathematicians also work with writers, producers, and actors in order to increase the realism of the representations.

Similarities Between Production and Mathematics

Numerous writers and producers have likened their work to mathematical processes. As theatrical producer Oscar Hammerstein described:

> A producer is a rare, paradoxical genius: hard-headed, soft-hearted, cautious, reckless, a hopeful innocent in fair weather, a stern pilot in stormy weather, a mathematician who prefers to ignore the laws of mathematics and trust intuition, an idealist, a realist, a practical dreamer, a sophisticated gambler, a stage-struck child. That's a producer.

A producer oversees the script, the hiring process, the budget, editing, music, and advertising. Ronald Bean is a hip-hop producer who uses the name Allah Mathematics. Jeff Westbrook has a bachelor's degree in physics and the history of science from Harvard University and a Ph.D. in computer science from Princeton University. He was an associate professor at Yale University and also worked at AT&T Labs before becoming a television writer and producer for the shows *Futurama* and *The Simpsons*. He noted the similarity between working with a team of people on computer science and mathematics problems and writing:

> Solving story problems is very similar in some ways. Given a problem, how can you fit all the pieces together to make it work? There are a lot of analytical parts to writing and analytical ability is as useful in that as in any field. That's the plus about mathematics. Nothing trains you better and gives you more analytical skills than mathematics. That skill is useful in the craziest places you might imagine: writing a TV show, writing a cartoon, and lawyering perhaps.

Actress Danica McKellar

Actress Danica McKellar is well-known for her role on the television show *The Wonder Years* (1988–1993) and her other acting projects since then. She obtained her bachelor's degree in mathematics from the University of California, Los Angeles, in 1998. She continues to be interested in mathematics and mathematics education, saying:

> I'd like to show girls that math is accessible and relevant, and even a little glamorous! Math is a fabulous mind strengthener—it's like going to the gym, for your brain. . . . I want them to feel empowered; if they can do math, they can do anything! Math is the only place where truth and beauty mean the same thing.

With that goal in mind, she has written three mathematical books as of 2010: *Math Doesn't Suck: How to Survive Middle School Math Without Losing Your Mind or Breaking a Nail*, *Hot X: Algebra Exposed*, and *Kiss My Math: Showing Pre-Algebra Who's Boss*. Her books have achieved a wide readership and appeared on best-seller lists like the *New York Times*' children's books category.

Other Mathematician Writers, Producers, and Actors

In addition to Danica McKellar and Jeff Westbrook, there have been numerous other mathematically trained writers, producers, and actors. Stewart Burns obtained a master's degree in mathematics and has worked for *The Simpsons*. Shane Carruth was an engineer with a degree in mathematics who wrote, produced, directed, and acted in the movie *Primer*, which won numerous awards including an Alfred P. Sloan Prize, which is awarded for science, technology, or mathematical content. David X. Cohen received a bachelor's degree in physics and a master's degree in theoretical computer science, and he published an article on pancake sorting before working for *The Simpsons* and co-developing *Futurama*.

Gioia De Cari is an actress and playwright who has a master's degree in mathematics. She wrote and performed the autobiographical play *Truth Values: One Girl's Romp Through M.I.T.'S Male Math Maze*. Jane Espenson double-majored in computer science and linguistics as an undergraduate student and was a graduate student at Berkeley in linguistics. She has worked as a writer and producer for shows such as *Buffy the Vampire Slayer, Battlestar Galactica*, and *Caprica*. Al Jean earned an undergraduate degree in mathematics and he has been the head writer for *The Simpsons.* Mike Judge was a graduate student in mathematics before developing shows such as *Beavis and Butt-Head* and *King of the Hill*. He has also performed as a voice actor in *King of the Hill* and as an actor in the *Spy Kids* movie franchise. Ken Keeler has a Ph.D. in applied mathematics. He worked for Bell Labs and published an article with Jeff Westbook. He wrote for David Letterman, *The Simpsons*, and *Futurama.* Writer Guillermo Martínez has a Ph.D. in mathematics and was in a postdoctoral position at Oxford University. His novel *The Oxford Murders* was a 2008 movie. There has also been a grant program designed to train mathematicians and scientists to become screenwriters. Robert J. Barker of the U.S. Air Force, who is noted as having approved the grant, justified the program by explaining that: "a crisis is looming, unless careers in science and engineering suddenly become hugely popular."

Goals and Impact

Some writers, producers, and directors state as their motivation the desire to positively impact people's responses to mathematics. Many people learn about mathematicians and scientists from representations in popular culture, and the importance of role models has been well-documented. *Flatland the Movie* film producer Seth Caplan noted, "Our goal is to create a movie that not only entertains, but also inspires. Flatland will help create the next generation of innovative mathematicians and scientists by demonstrating the wonders hidden throughout our universe." Nick Falacci and Cheryl Heuton, writers, producers, and creators of *NUMB3RS* explained: "Our goal first and foremost is to intrigue and tantalize the non-math people out there in TV land. We want people who have never given mathematics a second thought to stop and consider the role that math plays in society and day-to-day life." David X. Cohen hoped that those that appreciated the mathematical references would become die-hard fans of *Futurama*. He also has expressed concern that some of the popular culture portrayals of genius mathematicians with floating numbers that make it look like a magic power could discourage children who need to see that it takes hard work to become good at mathematics. Research has shown that stereotypical representations of mathematicians can discourage students from pursuing more mathematics. Cohen also apologized for inaccurate references:

> One thing I worry about is that when we purposely present inaccurate science in *Futurama* in the name of entertainment, that viewers may hold it against us. We do have genuine respect for science, and we're trying, when we can, to raise the level of discussion of science on television. If we fail sometimes, I hope people still appreciate the frequent attempts to bring real science into the show. I apologize in advance for any failures in the future, because I'm sure there will be many more, hopefully entertaining, failures.

Consultants

Writers or producers sometimes elicit help from mathematical consultants on mathematical references in a script or a blackboard scene. Some consultants are credited as such or acknowledged in interviews or DVD commentaries, while others remain anonymous. Some consultants provide feedback for just one line or scene while others work with a producer or writer for years. Producers, directors, and writers have used consultants in a wide variety of movies, plays, and television shows with mathematical content, including the following examples:

Antonia's Line. In the 1985 movie, the main character's granddaughter was a mathematics professor who lectured about mathematics and homology theory. Wim Pudshoorn was listed as a mathematical consultant.

Arcadia. Teenage mathematics genius Thomasina Coverly worked on Fermat's Last Theorem, named for Pierre de Fermat, Fourier's heat equation, named for Joseph Fourier, and chaos theory in this 1993 play by Tom Stoppard. Mathematician Manil Suri was listed as the production mathematics consultant.

A Beautiful Mind. The 2001 movie explored the life and work of Nobel-Prize–winning mathematician John Nash. Mathematician Dave Bayer was a consultant and his hand appeared in the movie for written blackboard scenes.

Big Bang Theory. The television series debuted in 2007. Young physicists and engineers often discuss their work as well as mathematics. Physicist David Saltzberg has been acknowledged as a consultant.

Bones. The television series first aired in 2005 and the forensic team sometimes engages in mathematical discussions. In addition, the main character was listed as belonging to both a chemistry club and mathematics club in high school. Donna Cline has been acknowledged as a forensic consultant.

Caprica. The television series debuted in 2009 as a spinoff of *Battlestar Galactica*. Among other references on both shows, Dr. Philomon obtained a bachelor's in applied mathematics in addition to other degrees. Physicist Kevin Grazier was a consultant on the original show, and engineer Malcom MacIver has been a consultant on the spinoff.

Cube. The 1997 movie explored the escape attempts by those trapped in interconnected cubes, and some of the plot twists in the movie were also mathematical. Mathematician David Pravica consulted.

Contact. In this 1997 movie based on the novel by Carl Sagan, the main character explained how prime numbers could be used to communicate with aliens. Mathematician Linda Wald and physicist Tom Kuiper were consultants.

Donald in Mathmagic Land. In the 1959 short film, Donald Duck entered a mathematical world filled with references to numbers, geometric objects, and the connections between mathematics and music, architecture, and nature. Physicist Heinz Haber was the chief scientific consultant to Walt Disney productions.

Eureka. The television series began airing in 2006 and focused on scientists in a town where almost everyone worked at a research facility. There have been numerous mathematical references, including mention of a Nobel Prize by scientist and mathematician Nathan Stark, and work by his mathematical savant stepson. Physicist Kevin Grazier consulted.

Futurama. This animated science fiction television series aired 1999–2003 and was brought back to life beginning in 2007. There have been hundreds of references to science and mathematics, written mostly by the scientific writing staff. Astrophysicist David Schiminovich and mathematician Sarah Greenwald consulted on some scenes.

Flatland the Movie. This 2007 movie was based on the well-known work on dimensions by Edwin Abbott. Mathematicians Tom Banchoff, Jonathan Farley, and Sarah Greenwald and mathematics educators L. Charles Biehl and Jon Benson consulted.

Fringe. The television series first aired in 2008. The team sometimes discusses mathematics such as in the episode titled "The Equation." Neuroscientist Ricardo Gil da Costa has consulted.

Good Will Hunting. The main character in this 1997 movie was gifted in mathematics and worked as a janitor at MIT. Physicist Patrick O'Donnell and mathematician Daniel Kleitman were consultants.

Hard Problems: The Road to the World's Toughest Math Competition is a 2008 documentary about the 2006 United States International Mathematical Olympiad Team. The idea for the video was credited to mathematician Joseph Gallian, who also served as an executive producer.

House. Although the television show debuted in 2004, intern Martha Masters, who also had a Ph.D. in applied mathematics, joined the medical team in 2010. Internist Harley Liker has been a consultant.

It's My Turn. In the 1980 romantic comedy, the main character was a mathematician and she proved what is known as the "snake lemma" in the movie. Mathematician Benedict Gross was a consultant.

Madame Curie. Physicist Rudolph Langer consulted in this 1943 movie about physicist Marie Curie.

Medium. This television series aired from 2005 to 2011. The husband of the main character was an applied mathematician. Mathematician Jonathan Farley consulted.

The Mirror has Two Faces. One of the main characters in the 1996 movie is a mathematics professor. Mathematician Henry Pinkham was a consultant.

N is a Number: A Portrait of Paul Erdos. This 1993 documentary listed Donald J. Albers, Gerald L. Alexanderson, Ronald Graham, Reuben Hersh, Charles L. Silver, and Joel Spence as mathematics consultants.

NUMB3RS. This television show aired from 2005 to 2010. Charlie Eppes was a mathematics professor who consulted for the FBI. Each episode featured mathematics as a significant part of the plotline. The mathematics helped with the crime solving. The producers used many mathematical consultants but the most well-publicized were mathematician Gary Lorden and a team from Wolfram Research: Michael Trott, Eric Weisstein, Ed Pegg, Jr, and Amy Young.

The Price Is Right. The television game show aired from 1956 to 1965 and again starting in 1972. Some of the games involved mathematics and mathematicians Bill Butterworth and Paul Coe consulted.

Proof. The 2005 movie was based on David Auburn's Pulitzer Prize winning play. The lead character and her father were both talented mathematicians who also wrestled with the notion of mental illness. Mathematician Timothy Gowers was a consultant.

The Simpsons. This long-running animated television series debuted in 1989. The show's many mathematically talented writers and producers created most of the mathematical references, which have often connected to astrophysics, number theory, geometry, innumeracy, or women in mathematics. Physicist David Schiminovich consulted on some blackboard scenes.

Square One. The mathematics educational television series aired from 1987 to 1994 and featured popular culture parodies. Edward T. Esty was a mathematical consultant.

Sneakers. In this 1992 movie, a mathematician lectures on cryptography. Computer scientist Leonard Adleman consulted.

Team Umizoomi. In this mathematics educational television program, which premiered in 2010, Christine Ricci is listed as an educational consultant.

Watchmen. In this 2009 movie, Dr. Manhattan discusses mathematics. Physicist James Kakalios consulted and is also noted for his *Science of Watchman* video, which also contains mathematical elements.

Some consultants have remarked that the producers and writers were very responsive to their efforts to make the mathematics more realistic. Others have commented that advice was ignored at times in order to focus on entertainment value. Mathematicians and scientists are also members of a Hollywood Math and Science Film Consulting firm and a program run by the National Academy of Sciences called the Science and Entertainment Exchange, which matches scientists with entertainment professionals. In addition to consulting, mathematicians Thomas Banchoff, Sarah Greenwald, and Gary Lorden appeared on mathematical featurettes on movie and television DVDs. In 2003, Scott Frank estimated that approximately 20% of the highest money-making films had scientific or technical consultants.

Connections to Education

Producers of *NUMB3RS* and *Fringe* worked with mathematicians and educators to create worksheet programs based on references in the show. The CBS Network, Texas Instruments, and the National Council of Teachers of Mathematics co-sponsored an educational Web site for *NUMB3RS*. Worksheet authors received a summary of all or part of an episode and designed lesson plans to complement them. Some critiqued the blurred line between entertainment and curricula and questioned the appropriateness of violent representations for middle-grade students or the relationship between the character of Amita and her thesis advisor Charlie. The Fox network partnered with the Science Olympiad organization to create a Science of Fringe Web site of lesson plans.

Actor Portrayals

Actors that portray mathematically talented individuals are sometimes asked about their portrayals in interviews and they have expressed a wide variety of viewpoints regarding mathematics. *Flatland: The Movie* actress Kirsten Bell, who played Hex, noted: "I really enjoyed math when I was growing up. . . . When you actually figure out the solution to a problem it's very rewarding." Martin Sheen acted as Arthur Square in the same movie and stated: "Nothing can happen without math. You can't do anything. You can't build anything. You can't go anywhere without math." *NUMB3RS* actor David Krumholtz, who played the main mathematician Charlie noted: "What's great is that because math is such a universal language, really, our fans come in all shapes and sizes, all ages and genders and races and backgrounds and cultures. . . . I've been more than thrilled to meet a lot of younger people, even as young as 6 years old, who tell me they're inspired by the math and they just think it's a really cool concept." Judd Hirsch, who played his father, stated: "I don't think anybody has to understand all the mathematics in this in order to be interested in it." Navi Rawat, who played a graduate student of Charlie and his eventual wife noted, "Having the chance to help to educate people about the importance of math through the character of Amita makes my job even more rewarding." Lindsay Lohan, who portrayed a mathematically talented high school student in *Mean Girls* stated, "I'm not bad at math. It just wasn't my favorite subject. I just did it just to do it."

Professional Organizations

The professional mathematical community has interacted with writers, producers, actors, and mathematical consultants in a number of ways. They have invited them to speak at conferences or showcase their mathematical work. For example, there have been sessions on mathematics and Hollywood, on using mathematical references in the classroom, and some mathematical films like *Flatland: The Movie* and *Hard Problems: The Road to the World's Toughest Math Competition* have held premiers for the mathematical community at conferences. Mathematicians have also written reviews, columns, articles, and books about the references.

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See Also: Movies, Mathematics in; Musical Theater; Plays; Popular Music; Television, Mathematics in; Women.

Z

Zero

Category: History and Development of Curricular Concepts.
Fields of Study: Communication; Connections; Number and Operations.
Summary: The concept of zero took time to be accepted and was explicitly rejected when first introduced to Greek and Roman culture.

Numbers initially served to count property, such as livestock. The numbers needed to count 1, 2, 3, 4, . . . became known as "counting" or "natural" numbers. The number zero is not found among these because one cannot count zero objects. Early civilizations existing over millennia used numbers only to count and so had no need for zero. The word "zero" has various linguistic origins: the French *zéro* and Venetian *zero*, which likely evolved from the Italian *zefiro*. This word came in turn from Arabic *sifr*, meaning "zero or nothing," derived from word *safira*, meaning "it was empty."

Early Development

The ancient Babylonians first introduced zero. With a base-60 system and initially two symbols (a wedge to represent "1" and a double wedge to represent 10), the Babylonians left empty spaces between groups of symbols. The fact that the spaces were not standardized in length made it difficult at times to distinguish between numbers because place value could not always be determined. To remedy this situation, the Babylonians developed zero but the zero was not a number in and of itself. It was rather a placeholder used to denote place values that had been skipped.

Independently and across the ocean, the Mayans developed a base-20 number system that included zero. Here, zero was used as a number to mean the absence of something. Zero also appeared in the Mayans' calendar. There was a year zero, and each month had a day zero in it as well. Because of the vast distance between the Mayans and the old world, Mayans' use and understanding of zero did not spread to these other areas.

Rejection by the Greeks and Romans

Despite the Babylonians use of zero, the Greeks and Romans initially rejected its use. Zero was considered dangerous spiritually as it represented the opposite of god and unity. It was associated with the void and chaos. Mathematically, zero presented many dilemmas. While any of the natural numbers (1, 2, 3, 4, . . .) when added to itself yields a larger number, zero added to itself does not. This characteristic violated Archimedes's principal that repeatedly adding a number to itself tends to a sum that is infinitely large. Additionally, a natural number plus any other natural number yields a sum larger than the initial natural number but again zero added to a natural number does not yield a number larger than the original natural number. Finally, multiplication of any number by zero yields zero and division by zero was outside the acceptable norms for these civilizations. The Greeks, known for geometry, often associated geometric figures to the natural numbers but zero could be associated with no figure. They preferred to reject zero as a number altogether.

Zero in India

Indian mathematicians in the fifth century C.E. took ideas from the Babylonians, including the concept of zero. They treated zero as a number that was found in the number line between −1 and 1. They also introduced negative numbers and, in 700, Brahmagupta introduced the idea that $1/0 = \infty$. Thus, infinity and unity depend upon the void and chaos. This idea was troubling to many civilizations, and the Hindu-Arabic numerals commonly used through the twenty-first century were not fully accepted until Leonardo de Pisa (also known as Fibonacci) introduced them to the Western world in his 1202 work *Liber Abaci*. One of the earliest recorded references to the mathematical impossibility of assigning a value to 1/0 occurred in George Berkeley's 1734 work *The Analyst*, which criticizes the foundations of calculus.

Calendars

Zero also caused confusion with the calendar system. Dionysius' calendar, created in 525 C.E., introduced the notation of BC and AD. However, it did not include a year zero. Thus, 1 BC is followed by 1 AD. This omission of zero causes confusion into the twenty-first century. Consider a person born in 1 AD. This person would have to go through 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10 to have lived 10 years, and a new decade would begin at the end of this first decade (10 years). That is, it would begin in 11. Thus, the next decade would begin in 21. The first century would end in 100, and the new one would begin in 101. Thus, the twenty-first century technically began in 2001, not in 2000 when most everyone celebrated it. This confusion rears its head at the start of every decade and century all a result of the omission of a year zero.

Division by Zero

One way in which mathematicians interpret division by zero is to reframe division in terms of other arithmetic operations. Using standard rules for arithmetic, division by zero is undefined, since division is defined to be the inverse operation of multiplication. While division by zero cannot reasonably be resolved with real numbers and integers, it can be defined using other algebraic structures or analytical extensions.

Zero in the Physical Sciences

Zero is an important value for many physical quantities or measurements. In some cases, zero means "nothing" or an absence of the characteristic, such as in most units of length and mass. However, in some cases, zero represents an arbitrarily chosen starting point for counting or measuring, such as in the Fahrenheit and Celsius temperature scales (though on the Kelvin scale, zero is the coldest possible temperature that matter can reach).

Other more advanced examples can be found in chemistry and physics. Zero-point energy is the lowest possible energy that a quantum mechanical physical system may possess. This energy level is called the "ground state" of the system and is important for investigating concepts such as entropy and perfect crystal lattices. Professor Andreas von Antropoff introduced the term "neutronium" for theoretical matter made solely of neutrons. As early as 1926, he redefined the periodic table with the atomic number zero, rather than the standard hydrogen (Atomic Number 1) in the initial position. More recent investigations suggest that the hypothesized element tetraneutron, a stable cluster of four neutrons with no protons or electrons, could have this atomic number zero.

Zero and Computers

In 1997, the naval vessel USS *Yorktown*'s propulsion system was brought to a dead stop by a computer network failure resulting from an attempt to divide by zero. Mathematical operations like these are problematic for computers, leading to various methods to avoid errors. The floating-point standard used in most modern computer processors has two distinct zeroes: a +1 (positive zero) and a −0 (negative zero). They are considered equal in numerical comparisons but some mathematical operations will have different results depending on which zero is used. For example, 1/−0 yields negative infinity, while 1/+1 gives positive infinity, though a "divide by zero" warning is usually issued in either case. Integer division by zero is usually handled differently from floating point, as there is no integer representation for the answer. Some processors generate an exception for integer division by zero, although others will simply generate an incorrect result for the division.

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Lidia Gonzalez

See Also: Babylonian Mathematics; Infinity; Number and Operations; Number Theory.

Chronology of Mathematics

30,000 B.C.E.: System of tallying by groups; an impressive example is a notched wolf shinbone of uncertain date found in Czechoslovakia in 1937. In addition to bone, stones and wood marked with notches have been used for tallying. There is archaeological evidence of counting as early as 50,000 B.C.E. and of primitive geometric art as early as 25,000 B.C.E.

17,500 B.C.E.: The notched Ishango bone, dating from this period, was found at Ishango along the shore of Lake Edward, one of the headwater sources of the Nile River.

2200 B.C.E.: Mythical date of the Chinese *lo-shu* magic square, a square array of numbers in which any row, column, or main diagonal have the same sum.

1850 B.C.E.: Moscow (or Golenischev) papyrus, an Egyptian mathematical text containing 25 numerical problems, dates from this period.

1750 B.C.E. (± 150 years): Plimpton 322, a Babylonian clay tablet containing Pythagorean triples (actually the smallest and largest of the three numbers of each triple) and a column of squares of ratios of the numbers not appearing in the table over the largest number of the triple (leg over hypotenuse), is from this period.

1650 B.C.E.: The Rhind papyrus, an Egyptian mathematical text containing 85 numerical problems copied by the scribe Ahmes from an earlier work, dates from this period.

1600 B.C.E.: Approximate date of the "oracle bones," which is the source of our knowledge of early Chinese number systems.

600 B.C.E.: The Greek mathematician Thales of Miletus is traditionally credited with the beginnings of demonstrative geometry.

540 B.C.E.: Pythagoras of Samos (b. ca. 572 B.C.E.) and the Pythagorean school did considerable work in arithmetic (i.e., number theory) and geometry. Among the accomplishments of the Pythagoreans were several discoveries related to the properties of numbers, work on the Pythagorean theorem, discovery that irrational numbers exist, solution of algebraic equations geometrically, and work with some of the regular solids.

450 B.C.E.: Zeno's paradoxes of motion is attributed to this date.

440 B.C.E.: Hippocrates of Chios made progress in the duplication of the cube problem.

440 B.C.E.: Anaxagoras of Clazomenae (ca. 500–ca. 488 B.C.E.) was the first Greek known to be connected with the quadrature of the circle problem.

430 B.C.E.: Antiphon the Sophist made early important contributions to the problem of squaring the circle with a method that contained the germ of the Greek method of exhaustion.

425 B.C.E.: Hippias of Elis (b. ca. 460 B.C.E.) invented a curve (the quadratrix) that solves the trisection and quadrature problems.

425 B.C.E.: Theodorus of Cyrene (b. ca. 470 B.C.E.) showed the irrationality of several numbers after $\sqrt{2}$ was shown to be irrational.

410 B.C.E.: Democritus of Abdera's work was a forerunner of Bonaventura Cavalieri's method of indivisibles.

400 B.C.E.: Archytas of Tarentum (428–347 B.C.E.) gave a higher geometry solution to the duplication of the cube problem and applied mathematics to mechanics.

380 B.C.E.: Plato (429–347 B.C.E.) founded Plato's Academy around 385 B.C.E that drew scholars from all over the Greek world. Advances toward solving the problems of duplicating the cube and squaring the circle and toward dealing with incommensurability and its impact on the theory of proportion were achieved partly because of Plato's Academy. Much of the important mathematical work of the fourth century B.C.E. was done by friends or pupils of Plato. Plato studied philosophy under Socrates and mathematics under Theodorus of Cyrene.

375 B.C.E.: Theaetetus of Athens (ca. 415–ca. 369 B.C.E.) contributed to the study of incommensurables and the regular solids. Some of his work later became a part of Euclid of Alexandria's *Elements*.

370 B.C.E.: Eudoxus of Cnidus (408–ca. 355 B.C.E.) contributed to incommensurables, duplication of the cube, the method of exhaustion, and the theory of proportion.

350 B.C.E.: Menaechmus did early work on conics. His brother, Dinostratus, also worked in geometry.

340 B.C.E.: Aristotle (384–322 B.C.E.) did important work in systematizing deductive logic. He was the author of *Metaphysics*. Aristotle studied at Plato's Academy.

335 B.C.E.: Eudemus of Rhodes wrote a history of early Greek mathematics that is lost but was referenced by later writers; the *Eudemian Summary* of Proclus is a brief outline of Greek geometry from the earliest times to Euclid.

320 B.C.E.: Aristaeus the Elder did early work on conics and regular solids.

306 B.C.E.: Ptolemy I Soter (d. 283 B.C.E.) of Egypt and his successor Ptolemy II Philadelphus founded the museum and library at Alexandria.

300 B.C.E.: Euclid wrote a number of mathematical works with the most important mathematical text of Greek times, and probably of all times, being his *Elements*. The *Elements* is comprised of 13 books devoted to geometry, number theory, and elementary (geometric) algebra.

280 B.C.E.: Aristarchus of Samos (ca. 310–230 B.C.E) applied mathematics to astronomy. He put forward the heliocentric hypothesis of the solar system.

240 B.C.E.: Nicomedes invented a higher plane curve that will solve the trisection problem.

230 B.C.E.: Eratosthenes of Cyrune served as chief librarian at the University of Alexandria. His most scientific work was a measurement of the earth. He developed a device known as the sieve for finding all prime numbers less than a given number.

225 B.C.E.: Apollonius of Perga (ca. 262–ca. 190 B.C.E.) is most famous for his *Conic Sections*, an extraordinary work that thoroughly examines these curves.

225 B.C.E.: Archimedes of Syracuse (287–212 B.C.E.) is recognized as the greatest mathematician of the ancient world. He worked in numerous areas including measurement of the circle and the sphere, computation of π, area of a parabolic segment, the spiral of Archimedes, infinite series, method of equilibrium, mechanics, and hydrostatics.

140 B.C.E.: Hipparchus of Rhodes (ca. 180–ca. 125 B.C.E.) was an eminent astronomer who played an important part in the development of trigonometry.

75 C.E.: Heron of Alexandria developed a formula for finding the area of a triangle in terms of the sides, now known as Heron's formula. His many works include a detailed work on indirect measurement, a book on mechanics, a handbook of practical mensuration, extraction of roots, and formulas for calculating the volumes of many solids.

100: Nicomachus of Gerasa's *Introduction to Arithmetic*, one of his two works to survive, is devoted to the classification of integers and their relations.

100: Menelaus of Alexandria's *Sphaerica* sheds considerable light on the development of Greek trigonometry.

100: *Nine Chapters on the Mathematical Art*, the most important of the ancient Chinese mathematical texts, was compiled during the Han period of 206 B.C.E.–221 C.E. Our knowledge of very early Chinese mathematics is limited and uncertain. Legend holds that the emperor Qin Shi Huangdi in 213 B.C.E. ordered the burning of all books to suppress dissent, but there is some reason to doubt that this was carried out. Very little work of a primary nature is known to us from the early Chinese civilizations.

150: Claudius Ptolemy (ca. 85–ca. 165) is especially known for his work in trigonometry and astronomy. His definitive Greek work on astronomy is the *Syntaxis mathematica*, better known by its later title the *Almagest*. In the *Almagest*, he gives the value of π as 377/120, or 3.1417.

250: Diophantus of Alexandria played a major role in the development of algebra and exerted influence on later European number theorists.

300: Pappus of Alexandria wrote commentaries on Greek mathematics and did original work in mathematics. Probably his greatest work is *Mathematical Collection*, a combined commentary and guidebook of the existing geometrical works of his time, with propositions, improvements, extensions, and comments.

390: The Greek commentator Theon of Alexandria edited Euclid's *Elements*, the revision that is the basis for modern editions of the work. After Pappus, Greek mathematics ceased to be creative, and its memory was perpetuated by writers and commentators with Theon being one of the earliest.

410: Hypatia of Alexandria (d. 415), Theon's daughter, is the first woman mentioned in the history of mathematics. She wrote commentaries on Diophantus' *Arithmetica* and Apollonius's *Conic Sections*.

460: Proclus Diadochus (410–485) wrote one of our principal sources of information on the early history of elementary geometry, *Commentary on Euclid, Book I*. Proclus had access to historical works now lost to present-day mathematicians.

476: Aryabhata the Elder (b. 476) is the earliest identifiable Indian mathematician. His main work concentrated mainly on astronomy but also contained a wide range of mathematical topics, including, for example, the methods of calculating square and cube roots and what amounts to a special case of the quadratic formula.

480: The mathematician and astronomer Tsu Ch'ung Chih found π to be between 3.1415926 and 3.1415927 and gave the rational approximation 355/113, which is correct to six decimal places.

500: Metrodorus assembled one of the best sources of ancient Greek algebra problems in a collection known as the "Greek Anthology."

505: Varahamihira made contributions to Indian trigonometry and astronomy.

510: The writings of Anicius Manlius Severinus Boethius (ca. 475–524) on geometry and arithmetic became standard texts in the monastic schools.

530: Simplicius wrote commentaries on Aristotle, the first book of Euclid's *Elements*, accounts of Antiphon's attempt to square the circle, of the lunes of Hippocrates, and of a system of concentric spheres invented by Eudoxus to explain the apparent motions of the members of the solar system.

560: Eutocius of Ascalon wrote commentaries on Archimedes's *On the Sphere and Cylinder, Measurement of a Circle*, *On Plane Equilibriums*, and *On Apollonius' Conic Sections*.

Seventh century: The Bakhshali manuscript, a mathematical manuscript discovered in 1881 in northwestern India, has numbers written using the place value system and with a dot to represent zero. The date of the manuscript is uncertain, but the best evidence available is that the manuscript dates from the seventh century.

625: A work by Wang Xiaotong contained cubic equations without a method of solution given except for a reference to solve according to the rule of cube root extractions.

628: Brahmagupta developed theorems dealing with cyclic quadrilaterals, gave us the well-known dissection proof of the Pythagorean theorem as well as at least one other proof, and did some early work in algebra.

775: Many Indian works had been brought to the Arabian world and they were translated into Arabic, from which they were translated into Latin and other languages.

820: The earliest extant Arabic algebra text was written by Muhammad ibn Musa al Khwarizmi (ca. 780–850). Al Khwarizmi's algebra was ultimately even more influential than his important arithmetical work. The title of al Khwarizmi's algebra work has the word *al-jabr* in it; the word algebra is a corrupted form of *al-jabr*. The earliest extant Arabic geometry is a separate section of al Khwarizmi's algebra text. The work on geometry was not influenced by theoretical Greek mathematics; the geometry work has no axioms or proofs.

850: Mahavira worked in arithmetic and algebra, including giving an explicit algorithm for calculating the number of combinations. Several problems from Mahavira are similar to "word" problems in elementary algebra today.

870: Thabit ibn Qurra (836–901) translated some Greek works, including the first really satisfactory Arabic translation of the *Elements* and especially important versions of some of Apollonius's *Conics*. He also wrote on astronomy, the conics, elementary algebra, magic squares, and amicable numbers.

900: Egyptian mathematician Abu Kamil ibn Aslam (ca. 850–930) wrote an algebra text and wrote a commentary on al-Khwarizmi's algebra that was later drawn upon by Leonardo Fibonacci.

920: Abu Abdallah Mohammad ibn Jabir Al-Battani (ca. 855–929) was an astronomer who also contributed to trigonometry.

980: Abu al-Wafa' (940–998) is known for his translation of *Diophantus*, his introduction of the tangent function into trigonometry, his computation of a table of sines and tangents for 15' intervals, and geometric constructions with compasses of fixed opening.

1000: Gerbert d'Aurillac (945–1003), who became Pope Sylvester II in 999, started a revival of interest in mathematics toward the end of Europe's Dark Ages of about 476–1000. Gerbert's work has the first appearance in the Christian West of the Hindu-Arabic numerals, although the absence of the zero and the lack of suitable algorithms for calculating showed that he did not understand the full significance of the Hindu-Arabic system. Gerbert wrote on astrology, arithmetic, and geometry.

1000: Abu Bakr al-Karaji (d. 1019) was one of the Arabian mathematicians who was instrumental in showing that the techniques of arithmetic could be fruitfully applied in algebra and, reciprocally, that ideas originally developed in algebra could be important in dealing with numbers. Little is known of his life other than that he worked in Baghdad around the year 1000.

Twelfth Century: Many of the major works of Greek mathematics and a few Islamic works were translated from the Arabic into Latin. Some of the translators and a sampling of their translations were Adelard of Bath (fl. 1116–1142; first translation from the Arabic of Euclid's *Elements*), Plato of Tivoli (fl. 1134–1145; Archimedes's *Measurement of a Circle* and Theodosius' *Spherica*), John of Seville and Domingo Gundisalvo (fl. 1135–1153; a work that was an elaboration of al-Khwarizmi's *Arithmetic*), Robert of Chester (fl. 1141–1150; al-Khwarizmi's *Algebra*), Gerard of Cremona (fl.

1150–1185; Euclid's *Elements*, Archimedes' *Measurement of a Circle*, Ptolemy's *Almagest*, and al-Khwarizmi's *Algebra*).

1100: Omar Khayyam (1050–1123), who is best known in the West for his collection of poems known as the "Rubaiyat," is noted in mathematics for systematically classifying and solving cubic equations. He also headed a group that worked to reform the calendar.

1115: An important edition of *Nine Chapters of the Mathematical Art* was printed.

1130: Jabir ibn Aflah did early Islamic work on spherical trigonometry.

1150: Bhaskara II (1114–ca. 1185; called Bhaskara II to distinguish him from an earlier prominent mathematician of the same name) is most noted for his *Lilavati* and *Vijaganita*, which deal with arithmetic and algebra, respectively. Much of our knowledge of Indian arithmetic stems from the *Lilavati*. Among other things in algebra, Bhaskara dealt with indeterminate equations and affirmed the existence and validity of negative as well as positive roots. He gave several approximations for π. The proof of the Pythagorean theorem known as Bhaskara's dissection proof actually appeared much earlier in China.

1202: Leonardo of Pisa, also known as Fibonacci (ca. 1170–1240), wrote several works dealing with arithmetic, algebra, geometry, and statistics. He was one of the earliest European writers on algebra. A trivial problem (the rabbit problem) in his most famous work, the *Liber Abaci*, gives rise to the sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377 . . . that Leonardo lists in the margin and notes can be continued indefinitely; this sequence, calculated recursively, is known today as a Fibonacci sequence. The *Liber Abaci* is devoted to arithmetic and elementary algebra and did much to aid the introduction of Hindu-Arabic numerals into Europe. The book contains problems in such practical topics as calculation of profits, currency conversions, and measurement, supplemented by the now standard topics of current algebra texts such as mixture problems, motion problems, container problems, the Chinese remainder problem, and problems solvable by quadratic equations.

1225: Jordanus de Nemore wrote on arithmetic, geometry, astronomy, mechanics, and algebra and was one of the first mathematicians to make some advances over the work of Leonardo.

1250: Nasîr ed-din wrote the first work on plane and spherical trigonometry considered independently of astronomy.

1250: Ch'in Chu-shao (ca. 1202–1261) published his *Mathematical Treatise in Nine Sections* in 1247. *Nine Sections* is the oldest extant Chinese mathematical text to contain a round symbol for zero and is the first in which numerical equations of degree higher than three occur. Ch'in began the custom of printing negative numbers in black type and positive ones in red.

1250: Li Ye (1192–1279) made an original contribution to Chinese mathematical notation by indicating negative quantities by drawing a diagonal stroke through the last digit of the number in question, an improvement over the earlier use of red and black colors, which became the accepted notation in printed works.

1260: Johannes Campanus (d. 1296) made a Latin translation of the *Elements* from the Arabic that became the basis for the first printed edition of the *Elements* in 1482.

1260: Yang Hui gave the earliest extant presentation of Pascal's arithmetic triangle and worked with decimal fractions by essentially our present methods.

1303: Chu Shih-chieh (fl. 1280–1303) wrote works that gave the most accomplished presentation of Chinese arithmetic-algebraic methods that has come down to us and employed familiar matrix methods of today. Chu speaks of what is now known as Pascal's arithmetic triangle as being ancient in his time, so the binomial theorem would appear to have been known in China for a long time.

1325: Thomas Bradwardine (1290–1349) wrote four mathematical tracts on arithmetic and geometry and developed some of the properties of star polygons.

1360: Probably the greatest mathematician of the fourteenth century was Nicole Oresme (ca. 1323–1382), who was associated with the University of Paris. He wrote five mathematical works and translated some of Aristotle. In one of his tracts, he has the first use of fractional exponents (not in modern notation) and in another tract he locates points by coordinates.

1435: Persian astronomer Ulugh Beg (1393–1449) calculated sine and tangent tables for every minute of arc correct to eight or more decimal places.

1450: Nicholas Cusa (1401–1465) was a minor German mathematician who is known primarily for his work on calendar reform and his attempts to square the circle and trisect the general angle.

1460: Georg von Peurbach (1423–1461) wrote an arithmetic and some works on astronomy, and compiled a table of sines. His main work was in Vienna and he made the university there the mathematical center of his generation.

1470: Johann Müller (1436–1476) is more generally known from the Latinized form of his birthplace of Königsberg as Regiomontanus. He wrote *De triangulis omnimodis*, which was the first European exposition of plane and spherical trigonometry considered independently of astronomy.

1478: First printed arithmetic, in Treviso, Italy.

1482: First printed edition of Euclid's *Elements.*

1484: Nicolas Chuquet (d. 1487) wrote an arithmetic known as *Triparty en la science des nombres* in 1484, a work on arithmetic and algebra in three parts. The *Triparty* was the first detailed algebra in fifteenth-century France. Chuquet recognized positive and negative integral exponents and syncopated some of his algebra.

1489: Johann Widman (ca. 1462–1498) wrote an influential German arithmetic that was published in 1489. Here appears for the first time our present + and – signs but not as symbols of operation; they were used to indicate excess and deficiency.

1491: Italian Filippo Calandri wrote one of the less important arithmetics, but it does contain the first printed example of today's modern process of long division.

1494: Italian Luca Pacioli (1445–1509) compiled from many sources the most comprehensive mathematics text of the time. His *Suma de arithmetica, geometrica, proportioni et proportionalita* contained little that was original, but its comprehensiveness and the fact that it was the first such work to be printed made it quite influential. The 600-page book contained practical arithmetic, algebra, and geometry. The *Suma* also contained the first published treatment of double entry bookkeeping.

1510: The German artist-mathematician Albrecht Dürer (1471–1528) wrote the earliest geometric text in German, published in 1525. Dürer felt that German artists needed to know elementary geometrical ideas before they could approach perspective in drawing.

1515: Scipione del Ferro (1465–1526), a professor of mathematics at the University of Bologna, solved algebraically the equation $x^3 + mx = n$. Antonio Maria Fiore (ca. 1506) was del Ferro's pupil who famously challenged Tartaglia to a contest of solving cubic equations.

1518: Adam Riese (ca. 1489–1559) wrote an especially influential German commercial arithmetic, published in 1522. The phrase *nach Adam Riese* (according to Adam Riese) is used even today in Germany.

1525: Christoff Rudolff (ca. 1500–ca. 1545) wrote his *Die Coss*, the first comprehensive German algebra, in the early 1520s.

1530: Nicolaus Copernicus (1473–1543) of Poland was a prominent astronomer who stimulated mathematics; his work necessitated the improvement of trigonometry, and Copernicus himself contributed an arithmetic treatise on the subject. Copernicus's theory of the universe in *De revolutionibus* was completed by about 1530 but was not published until 1543.

1544: Michael Stifel (1487–1567) is perhaps the greatest German algebraist of the 16th century. His best-known mathematical work is *Arithmetica integra*, published in 1544. The book was divided into three parts devoted to rational numbers, irrational numbers, and algebra. He foreshadowed the invention of logarithms by pointing out the advantages of associating an arithmetic progression with a geometric one. He gives the binomial coefficients up to the seventeenth order. Like most of his contemporaries, Stifel did not accept negative roots of an equation. The signs +, −, and $\sqrt{\ }$ are used, and often the unknown is represented by a letter.

1545: Cubic and quartic equations were solved by Italian mathematicians in the 16th century. Ludovico Ferrari (1522–1565) solved quartic equations by reducing the complete quartic to a form that could be reduced to a cubic that could then be solved by methods already known. Nicolo Fontana of Brescia (ca. 1499–1557), commonly known as Tartaglia (the stammerer) because of a childhood injury that affected his speech, discovered an algebraic solution to $x^3 + px^2 = n$ and also found an algebraic solution for cubics lacking a quadratic term. Girolamo Cardano, a brilliant but unprincipled mathematician, published his *Ars Magna*, a great Latin treatise on algebra, and in it appeared Tartaglia's solution of the cubic, despite an apparent promise of secrecy when Cardano wheedled the key to the cubic from Tartaglia.

1550: The Teutonic mathematical astronomer George Joachim Rheticus (1514–1574) was the first to define the trigonometric functions as ratios of the sides of a right triangle. He also formed tables of trigonometric functions.

1550: Johannes Scheubel (1494–1570) was one of the German authors to use Pascal's triangle to find roots.

1550: Italian geometer Federigo Commandino (1509–1575) prepared Latin translations of almost all of the known works of many Greek mathematicians.

1556: The first work on mathematics was printed in the New World.

1557: Robert Recorde (ca. 1510-1558) was the most influential English textbook writer of the 16th century. His first book was on arithmetic and he also wrote on astronomy, geometry, and algebra.

1570: The first complete English translation of Euclid's *Elements* was by Henry Billingsley (d. 1606), with a remarkable preface by English scientist and mystic John Dee (1527–1608) that gave detailed descriptions of some 30 different fields that need mathematics and the relationships among them.

1572: Italian mathematician Rafael Bombelli (ca. 1526–1572) wrote an algebra text that began with elementary material and gradually worked up to the solving of cubic and quartic equations. In his *Algebra*, Bombelli introduced a different kind of cube root that comes in cubic equations of the form $x^3 + mx = n$ when $(n/2)^2 + (m/3)^2$ is negative. Bombelli was the first mathematician to accept the existence of imaginary numbers and presented laws of multiplication for these new numbers.

1575: William Holzmann (1532–1576), also known as Xylander, translated Diophantus's *Arithmetica* into Latin and translated major portions of *Elements* into German.

1580: French mathematician François Viète (1540–1603) wrote a number of works on trigonometry, algebra, and geometry. In his trigonometry book, he developed systematic methods for solving plane and spherical triangles with the aid of all six trigonometric functions. Viète's most famous work is his *In artem analyticam* that did much to aid the development of symbolic algebra. In another work, he gave a systematic process for successively approximating to a root of an equation, and in general contributed to the theory of equations. Viète showed that the trisection and duplication problems both depend upon the solution of cubic equations.

1583: Christopher Clavius (1537–1612) was a German scholar who added little of his own to mathematics, but wrote highly esteemed textbooks on arithmetic and algebra. He also wrote on trigonometry and astronomy and played an important part in the Gregorian reform of the calendar.

1590: Italian mathematician Pietro Antonio Cataldi (1548–1626) wrote a number of mathematical works and is credited with taking the first steps in the theory of continued fractions.

1590: Simon Stevin (1548–1620) is best known in mathematics for his contribution to the theory of decimal fractions. He was born in Belgium, but spent much of his adult life in Holland.

1595: German clergyman Bartholomaus Pitiscus (1561–1613) invented the term "trigonometry" in his treatise on the subject.

1600: Thomas Harriot (1560–1621) is usually considered the founder of the English school of algebraists. His great work in the field, *Artis analyticae praxis*, deals largely with the theory of equations.

1600: Swiss instrument maker Jobst Bürgi (1552–1632) conceived and constructed a table of logarithms independently of Napier, but published after Napier.

1600: Italian astronomer Galileo Galilei (1564–1643) contributed notably to mathematics. Among other contributions, Galileo founded the mechanics of freely falling bodies and laid the foundation of mechanics in general, realized the parabolic nature of the path of a projectile in a vacuum and speculated on laws involving momentum, invented the first modern-type microscope, and made several excellent telescopes (the telescope was invented about 1608 in Holland).

1610: Johann Kepler (1571–1630) discovered the laws of planetary motion, used a crude form of integral calculus to find volumes, and made contributions to the subject of polyhedral and other areas of mathematics.

1612: The Frenchman Bachet de Méziriac (1581–1638) translated Diophantus's *Arithmetica* into Latin; many of Pierre de Fermat's contributions to number theory occur in the margins of his copy of Bachet's work.

1614: Logarithms were invented by Scottish mathematician John Napier (1550–1617). Other contributions by Napier were the "rule of circular parts," a mnemonic for reproducing the formulas used in solving right spherical triangles; "Napier's anologies," useful in solving oblique spherical triangles; and "Napier's rods" or "bones," used for mechanically multiplying, dividing, and taking square roots of numbers.

1619: Savilian professorships in geometry and astronomy were established at Oxford University by mathematician Henry Savile.

1624: Englishman Henry Briggs (1561–1631) constructed a large table of logarithms with base 10, published in his *Arithmetica Logarithmica*, after he and Napier had agreed that logarithms would be more useful with a base of 10. Briggs was the first person to hold the Savilian Chair in astronomy at Oxford University.

1630: French number theorist Marin Mersenne (1588–1648) is especially known in mathematics for what are now called Mersenne primes, or prime numbers of the form $2^p - 1$, which he discussed in his *Cogitata physico-mathematica* of 1644.

1630: William Oughtred (1574–1660) was one of the most influential of the seventeenth-century English writers on mathematics. His *Clavis mathematicae* on arithmetic and algebra helped spread mathematical knowledge in England. Oughtred placed emphasis on mathematical symbols, but only a few of them are still in use. He and another Englishman, Richard Delamain (ca. 1630), independently created a physical version of a logarithm table in the form of a circular (later rectilinear) slide rule.

1630: Albert Girard (1595–1632), who spent much of his life in Holland, gave the first explicit statement of the fundamental theorem of algebra.

1635: Frenchman Pierre de Fermat (1601–1665) made important contributions to analytic geometry and probability, but of his varied contributions to mathematics, the most outstanding is the founding of the modern theory of numbers.

1635: Italian Bonaventura Cavalieri (1598–1647) developed a complete theory of indivisibles, an important pre-calculus development.

1637: Frenchman René Descartes (1596–1650) shares with Fermat early work on analytic geometry that was important in the beginnings of the subject. The work of the two was different in that, to oversimplify a bit, Descartes, in his *La géométrie*, began with a locus and then found its equation whereas Fermat did the reverse. Also in *La géométrie*, Descartes stated without proof the result known today as Descartes's rule of signs, a rule for determining limits to the number of positive and the number of negative roots possessed by a polynomial.

1640: Frenchman Gérard Desargues (1591–ca.1662) did original work on conic sections that was important in the early development of synthetic projective geometry.

1640: Italian Evnagelista Torricelli (1608–1647) is best known for his work in physics and is probably most famous for his discovery of the principle of the barometer in 1643. In mathematics, he did some work with pre-calculus indivisibles and showed that an infinite area, when revolved about an axis in its plane, can sometimes yield a finite volume for the solid of revolution; he used a method similar to the cylindrical shell method of calculus but expressed in terms of indivisibles.

1640: Frenchman Gilles Persone de Roberval (1602–1675) and Torricelli were both accomplished geometers and physicists. Roberval did work in mathematics similar to Torricelli's with questions of priority difficult to settle. Roberval successfully employed the method of indivisibles to find a number of areas, volumes, and centroids.

1650: Frenchman Blaise Pascal (1623–1662) had significant accomplishments in his short life, among them the invention of a calculating machine and the investigation of the action of fluids under the pressure of air. Pascal's triangle appeared in his *Traité du triangle arithmétique*, but he was not the first to exhibit the arithmetic triangle as Chinese writers had anticipated such a triangle several centuries earlier; the work also is famous for its explicit statement of the principle of mathematical induction. The problem of the points, stated by Pacioli in his *Suma* of 1494 and considered by several mathematicians, was important in the origin of probability theory; there was a remarkable correspondence between Pascal and Fermat that largely laid the foundation of this theory.

1650: John Wallis (1616–1703) was appointed Savilian professor of geometry at Oxford in 1649, and occupied this position for 54 years. While at Oxford, Wallis wrote his mathematical works including tracts on algebra, conic sections, mechanics, and of special interest his *Arithmetica infinitorum* that systematized and extended the methods of Descartes and Cavalieri. The *Arithmetica infinitorum* was important for its early calculus work, especially integration; Isaac Newton read Wallis's work and expanded upon what Wallis had done. Wallis was the first to fully explain the significance of zero, negative, and fractional exponents and he introduced the symbol ∞ for infinity.

1650: Dutchman Frans van Schooten the Younger (1615–1660) edited Descartes and Viète.

1650: Belgian mathematician Grégoire de St. Vincent (1584–1667) applied pre-calculus methods to various quadrature problems.

1650: Nicolaus Mercator (1620–1687) lived most of his life in England. He edited Euclid's *Elements* and wrote on trigonometry, astronomy, the computation of logarithms, and cosmography.

1650: Englishman John Pell (1611–1685) extended the factor tables of J. H. Rahn (1622–1676), which had numbers up to 24,000, to 100,000. Pell is incorrectly credited with the Pell equation, actually due to his countryman Lord William Brouncker (1620–1684), the first president of the Royal Society of London.

1650: Belgian René François Walter de Sluze (1622–1685) wrote numerous tracts on mathematics in which he discussed spirals, points of inflection, and the finding of geometric means.

1650: Italian mathematician Vincenzo Viviani (1622–1703) had a number of geometric accomplishments, but is especially noteworthy for setting forth a challenge problem that led to the beginnings of the subject of double integrals in Leibniz's solution to the problem.

1662: The Royal Society was founded in London, followed by the French Academy in Paris in 1666. These were centers where scholarly papers could be presented and discussed.

1663: Lucasian professorship in mathematics was established at Cambridge University, named for donor Henry Lucas.

1670: Englishman Isaac Barrow (1630–1677) gave a near approach to the modern process of differentiation in his *Lectiones opticae et geometricae*. Barrow was probably the first to realize in full generality the fundamental theorem of calculus, that differentiation and integration are inverse operations, which he stated and proved in his *Lectiones*. Barrow was the first occupier of the Lucasian chair at Cambridge, a position he held from 1664 to 1669.

1670: Scottish mathematician James Gregory (1638–1675) was one of the first to distinguish between convergent and divergent series. He expanded functions into series and a series for arctan(x) that played a part in calculations of π that is know by his name. Gregory is also known for his work in astronomy and optics.

1670: Dutchman Christiaan Huygens (1629–1695) wrote the first formal treatise on probability in 1657, basing his work on the Pascal-Fermat correspondence. He introduced the concept that is now called mathematical expectation.

1670: Sir Christopher Wren (1632–1723) was a famous architect who might have been remembered as a mathematician had it not been for the Great Fire of London in 1666. Wren was Savilian professor of astronomy at Oxford and taught geometry there from 1661 to 1673.

1672: Danish mathematician Georg Mohr (1640–1697) showed that all the constructions of Euclid's *Elements* can be done with a straightedge and a compass of fixed opening.

1680: Englishman Isaac Newton (1642–1727) made numerous contributions to mathematics and physics and is especially noted in mathematics for inventing the calculus.

1680: Dutchman Johann Hudde (1633–1704) gave a rule for finding multiple roots of an equation.

1682: German mathematician Gottfried Wilhelm Leibniz (1646–1716) made numerous contributions to mathematics and shares with Newton credit for the invention of the calculus; the two men worked independently of each other. Leibniz's notation was superior to Newton's and is still in use today.

1690: French nobleman the Marquis de l'Hospital (1661–1704) wrote the first calculus textbook, based on the lectures of his teacher, Johann Bernoulli. The so-called l'Hospital's rule appears in the text.

1690: Edmund Halley (1656-1742), successor of Wallis as Savilian professor of geometry, made major original contributions in astronomy. In mathematics, he restored the lost Book VIII of Apollonius's *Conic Sections* by inference, edited various works of the ancient Greeks with translations of some of them from the Arabic, and compiled a set of mortality tables of the kind now basic in life insurance.

1690: Swiss mathematicians and brothers Jakob (Jacques, or James) Bernoulli (1654–1705) and Johann (John, or Jean) Bernoulli (1667–1748) were among the first in Europe to understand the new techniques of Leibniz and to apply them to solve new problems. They made numerous contributions to mathematics and are part of the famous Bernoulli family of mathematicians.

1691: Frenchman Michel Rolle (1652–1719) is known for the theorem in beginning calculus that bears his name.

1700: Antoine Parent (1666–1716) first systematically developed solid analytic geometry in a paper presented to the French Academy.

1706: Englishman William Jones (1675–1749) first used the symbol π for the ratio of the circumference to the diameter.

1715: Englishman Brook Taylor (1685–1731) and Scotsman Colin Maclaurin (1698–1746) made important contributions to mathematics. They are best known for Taylor's well-known expansion theorem $f(a+h)=f(a)+hf'(a)+h^2f''(a)/2!+\ldots$ with Maclaurin's later expansion being the special case with $a=0$.

1720: Frenchman Abraham De Moivre (1667–1754) is especially known for his work *Annuities upon Lives*, which played an important role in actuarial mathematics; his *Doctrine of Chances*, which contained much new material in probability; and his *Miscellanea analytica*, which contributed to recurrent series, probability, and analytic trigonometry.

1731: Frenchman Alexis Claude Clairaut (1713–1765) did important work on differential equations. He made a systematic attempt to calculate volumes of certain regions as well as the areas of their bounding surfaces. His definitive work was his *Théorie de la figure de la Terre*, published in 1743.

1733: Italian Girolamo Saccheri (1667–1733) wrote *Euclid Freed of Every Flaw* in which he purported to prove the parallel postulate (Euclid's fifth postulate) by the method of *reductio ad absurdum*.

1734: Irish philosopher Bishop George Berkeley (1685–1753) made one of the ablest criticisms of the faulty foundation of early calculus in his tract *The Analyst*.

1740: Gabrielle Émilie Le Tonnelier de Breteuil, Marquisse du Châtelet (1706–1749) translated Newton's *Principia* into French.

1748: Maria Gaetana Agnesi (1718–1799) contributed to mathematics education by writing a two-volume work, *Instituzioni Analitiche*, in her native Italian instead of the customary Latin. The first volume deals with arithmetic, algebra, trigonometry, analytic geometry, and mainly calculus. The second volume deals with infinite series and differential equations. Included in her work was a cubic curve, $y\left(x^2+ a^2\right) = a^2$ that had been studied by others and is now known, due to a mistranslation, as the "witch of Agnesi."

1750: Swiss mathematician Leonhard Euler (1707–1783) was the most prolific writer ever in mathematics with contributions too numerous to mention in detail here. He made original contributions to almost every branch of elementary and advanced mathematics. Just in elementary mathematics, he conventionalized much of our notation, gave us the formula $e^{ix} = \cos(x) + i\sin(x)$, contributed the method for solving quartic equations that is known as Euler's method, and made significant contributions in elementary number theory.

1770: Johann Heinrich Lambert (1728–1777) was born in Alsace and moved to Switzerland in 1748. Lambert attempted to improve upon Saccheri's work on the parallel postulate in his *Die Theorie der Parallellinien*, a work that places him among the forerunners of non-Euclidean geometry. Like Saccheri, Lambert used an indirect approach but considered a quadrilateral with three right angles and made three hypotheses as to the nature of the fourth angle (right, acute, or obtuse) whereas Saccheri had considered a quadrilateral ABCE in which angles A and B are right angles with sides AD and BC equal; the hypotheses concerning the other two angles are then the same, as were Lambert's for the one angle. Among Lambert's other accomplishments were his rigorous proof that π is irrational and his systematic development of the theory of hyperbolic functions.

1777: Georges Louis Leclerc, Comte de Buffon (1707–1788) devised his needle problem by which π may be approximated by probability methods.

1788: Italian-born Joseph Louis Lagrange (1736–1813) spent his later years in France. His most important work was *Mécanique analytique*, in which Lagrange extended the mechanics of Newton, the Bernoullis, and Euler and emphasized the fact that problems in mechanics can generally be solved by reducing them to the theory of ordinary and partial differential equations.

1794: French mathematician Gaspard Monge (1746–1818) created descriptive geometry and is considered the father of differential geometry. His work entitled *Application de l'analyse à la géométrie* was one of the most important of the early treatments of the differential geometry of surfaces.

1794: The French *Journal de l'École Polytechnique* was launched. The journal is perhaps the oldest of the current journals devoted chiefly or entirely to advanced mathematics. The nineteenth century saw the rise of a number of mathematical societies and journals devoted to current mathematical research.

1797: Italian Lorenzo Mascheroni (1750–1800) discovered that all Euclidean constructions, insofar as the given and required elements are points, can be made with compasses alone.

1797: Norwegian surveyor Caspar Wessel (1745–1818) presented for the first time the association of the complex numbers with the real points of a plane.

1799: France adopted the metric system of weights and measures.

1800: German mathematician Carl Friedrich Gauss (1777–1855) gave the first wholly satisfactory proof of the fundamental theorem of algebra. His *Disquisitiones arithmeticae* was a work of fundamental importance in the modern theory of numbers. Gauss made the first systematic investigation of the convergence of a series. Gauss was the first to suspect that the parallel postulate is independent of the other axioms and worked with the Playfair form of the parallel postulate by considering the three possibilities: through a given point can be drawn more than one, or just one, or no line parallel to a given line; he shares with Bolyai and Lobachevsky the honor of discovering the geometry that results from having no line parallel to a given line. These are but a few of the ground-breaking results because of Gauss.

1803: French geometer Lazare Nicolas Marguerite Carnot (1753–1823) first systematically employed sensed magnitudes in synthetic geometry.

1805: Frenchman Pierre-Simon Laplace (1749–1827) did his most outstanding work in the fields of celestial mechanics, probability, differential equations, and geodesy. Adrien-Marie Legendre (1752–1833) is known in elementary mathematics for his *Élements de géométrie,* which attempted to improve pedagogically on Euclid's *Elements* by rearranging and simplifying many of the propositions. Both Laplace and Legendre contributed significantly to advanced mathematics.

1806: Swiss bookkeeper Jean-Robert Argand (1768–1822) published a geometric interpretation of the complex numbers that was similar to the one that had been put forth earlier by Caspar Wessel. The delay in general recognition of Wessel's accomplishment is why the complex number plane came to be called the Argand plane.

1816: Frenchwoman Sophie Germain (1776–1831) was awarded a prize by the French Academy for a paper on the mathematics of elasticity. She later proved that for each odd prime $p < 100$, the Fermat equation $x^p + y^p = z^p$ has no solution in integers not divisible by p. She introduced into differential geometry the idea of the mean curvature of a surface at a point of the surface in 1831.

1819: Englishman William George Horner (1786–1837) is known for the numerical method of solving algebraic equations that goes by his name, although a similar method had been used by the Chinese much earlier.

1822: French mathematician Jean Baptiste Joseph Fourier (1768–1830) is known for his mathematical theory of heat and especially for Fourier series. Fourier believed that any function can be resolved into a sum of sine and cosine functions. While it is not true that any function can be represented by trigonometric series, the class of functions so representable is very broad and Fourier series are useful in the study of many functions.

1824: Scotsman Thomas Carlyle (1795–1881) made an especially important English translation of Legendre's *Géométrie*.

1826: The principle of duality, important in the development of projective geometry, was enunciated by French mathematician Jean-Victor Poncelet.

1826: The theory of elliptic functions was independently and simultaneously established by German mathematician Carl Gustav Jacobi (1804–1851) and Norwegian mathematician Niels Henrik Abel (1802–1829). In abstract algebra, commutative groups are now called Abelian groups. Both Jacobi and Abel made many other contributions to mathematics.

1827: French mathematician Augustin-Louis Cauchy (1789–1857) strengthened the rigorization of analysis that got underway with the work of Lagrange and Gauss. Cauchy's numerous contributions include researches in convergence and divergence of infinite series, real and complex function theory, differential equations, determinants, and probability. In a paper of 1846, Cauchy introduced the concept of a line integral in n-dimensional space (with the incidental notion of a space higher than three included) and of a theorem today generally known as Green's theorem (George Green, 1793–1841).

1829: Russian mathematician Nicolai Ivanovitch Lobachevsky (1793–1856) published findings on non-Euclidean geometry similar to those of Gauss published later and Hungarian Janos Bolyai (1802–1860) published in 1832. Lobachevsky's publication was first, but all three of these mathematicians share credit for the creation of the geometry that comes from accepting the hypothesis of the acute angle, now known as Lobachevskian or hyperbolic geometry.

1830: French mathematician Siméon-Denis Poisson (1781–1840) had numerous mathematical publications. He applied probabilities to social areas where significant statistical information was available to him.

1830: George Peacock (1791–1858) worked on reforming mathematical study in England. In his *Treatise on Algebra,* he attempted to give algebra a logical treatment comparable to that of Euclid's *Elements*.

1830: English mathematician Charles Babbage (1792–1871) was one of the early mathematicians to work on machines to automatically do a series of arithmetic operations.

1831: German Julius Plücker (1801–1868) developed a coordinate system for the projective plane to deal with points at infinity with his introduction of homogeneous coordinates.

1831: Scotswoman Mary Fairfax Somerville (1780–1872) wrote a popular exposition of Laplace's *Traité de mécanique céleste.*

1832: Frenchman Évariste Galois (1811–1832) essentially created the study of groups that was carried out by his successors. In 1830, he was the first to use the term "group" in its technical sense. He also made contributions to theory of equations. Galois died in a duel at age 21.

1834: Swiss geometer Jacob Steiner (1796–1863) made numerous original contributions to higher synthetic geometry.

1837: Trisection of an angle and duplication of a cube were proved impossible.

1841: *Archiv der Mathematik und Physik* was founded and *Nouvelles annales de mathématiques* was founded a year later, the earliest permanent periodicals devoted to teachers' interests rather than mathematical research.

1843: Czechoslovakian Bernhard Bolzano (1781–1848) produced a function continuous in an interval that has no derivative at any point of the interval, although Karl T. W. Weierstrass (1815–1897) was credited with the first example of this kind. Both men were proponents for rigorization in analysis. Weierstrass is known for being an outstanding teacher of advanced mathematics.

1843: Irish mathematician William Rowan Hamilton (1788–1856) invented an algebra in which the commutative law of multiplication does not hold, his quaternions.

1844: German Herman Günther Grassman (1809–1877) was the first mathematician to present a detailed theory of spaces of dimension greater than three.

1847: German geometer Karl Georg Christian von Staudt (1798–1867) freed projective geometry of any metrical basis in his *Geometrie der Lage*.

1847: English mathematician George Boole (1815–1864) published a pamphlet entitled *The Mathematical Analysis of Logic* in which he maintained that the essential character of mathematics lies in its form rather than in its content; mathematics is not merely the science of measurement and number, but is any study consisting of symbols along with precise rules of operation upon those symbols, the rules being subject only to the requirement of inner consistency.

1849: German mathematician Peter Gustav Lejeune Dirichlet (1805–1859) analyzed the convergence of Fourier series, which led him to generalize the function concept. He also facilitated the comprehension of some of Gauss's more abstruse methods and contributed notably to number theory.

1850: Frenchman Amédée Mannheim (1831–1906) standardized the modern slide rule.

1852: French mathematician Michel Chasles (1793–1880) contributed notably to synthetic geometry.

1854: German mathematician Georg Friedrich Bernhard Riemann (1826–1866) contributed notably to analysis and non-Euclidean geometry. Riemann showed that a consistent geometry can be developed from the hypothesis of the obtuse angle; this geometry is known as Riemannian or elliptic geometry today.

1854: English mathematician George Boole expanded and clarified an earlier pamphlet of 1847 into a book entitled *Investigation of the Laws of Thought*, in which he established both formal logic and a new algebra, the algebra of sets known today as Boolean algebra.

1857: English mathematician Arthur Cayley (1821–1895) devised a noncommutative algebra, the algebra of matrices, which is not commutative under multiplication.

1865: The London Mathematical Society was founded and published the *Proceedings of the London Mathematical Society*. It was the earliest of a number of large mathematical societies that were formed in the second half of the nineteenth century that had regular official periodicals. These became important because they provided forums in which mathematicians could congregate, publish, and set policies.

1872: German mathematician Felix Klein (1849–1925) set forth a definition of "a geometry" that served to codify essentially all the existing geometries of the time and pointed the way to promising geometrical research. The program is known as the Erlanger Programm.

1872: German mathematician Richard Dedekind (1813–1916) published his idea of "Dedekind cuts" as a way of providing an arithmetic definition of the real numbers (he had come up with the idea in 1858). Dedekind, along with Georg Cantor, showed how to construct the real numbers from the rational numbers, and Dedekind completed the process of arithmetizing analysis by characterizing the natural numbers, and hence rational numbers, in terms of sets in a work published in 1888. Dedekind gave a useful definition of an infinite set as one that is equivalent to some proper subset of itself.

1873: French mathematician Charles Hermite (1822–1901) proved that *e* is transcendental.

1874: The Birth of Set Theory. German mathematician Georg Cantor (1845–1918) published a paper in *Crelle's Journal* in which he showed, among other things, that the set of algebraic numbers can be placed in one-to-one correspondence with the natural numbers (countable in later terminology) but that the set of real numbers is not countable. This established for the first time the fact that there are different orders of infinity. Cantor proceeded during the latter quarter of the 19th century to develop naïve (non-axiomatic) set theory.

1877: In 1850, English mathematician James Joseph Sylvester (1814–1897) coined the term "matrix" in the sense that it is used today. Sylvester made important contributions to modern algebra. He came to America in 1877 to chair the mathematics department at the newly opened Johns Hopkins University in Baltimore and helped develop a tradition of graduate education in mathematics in the United States.

1878: Sylvester founded the *American Journal of Mathematics*. It is the oldest mathematics journal in the western hemisphere that has been in continuous publication.

1881: Josiah Willard Gibbs (1839–1903) in America and Oliver Heaviside (1850–1925) in England independently realized that the full algebra of quaternions was not necessary for discussing physical concepts. Gibbs published his version of vector analysis in 1881 and 1883 and Heaviside published his methods in papers on electricity in 1882 and 1883.

1882: German mathematician Ferdinand Lindemann (1852–1939) proved that π is transcendental. From this fact, the impossibility of squaring the circle with Euclidean tools easily follows.

1888: The American Mathematical Society (AMS) was founded (under the name of the New York Mathematical Society) and the *Bulletin of the American Mathematical Society* was begun. In 1900, the society added its *Transactions* and in 1950 its *Proceedings*.

1888: Russian mathematician Sonja Kovalevsky (1850–1891) was awarded the prestigious Prix Bordin for her memoir *On the Problem of the Rotation of a Solid Body about a Fixed Point*.

1889: Italian mathematician Giuseppe Peano 1858–1932 attempted to deduce the truths of mathematics from pure logic in a small tract that contains his famous postulates for the natural numbers.

1892: *Jahresbericht*, the professional journal of Deutsche Mathematiker-Vereinigung (organized in 1890), was founded. The *Jahresbericht* contained a number of extensive reports on modern developments in different fields of mathematics; these reports may be regarded as forerunners of the later large encyclopedias of mathematics.

1895: French mathematician Jules Henri Poincaré (1854–1912) contributed to virtually every area of mathematics. His "Analysis situs" (1895) is the first significant paper devoted wholly to topology.

1896: The French and Belgian mathematicians J. Hadamard (1865–1963) and C. J. de la Vallée Poussin (1866–1962) independently proved the prime number theorem: Let A_n denote the number of primes less than n. Then $\left(A_n / \ln(n)\right)/n$ approaches 1 as n becomes larger and larger.

1899: German mathematician David Hilbert (1862–1943) made highly important contributions in many areas of mathematics. In his *Grundlagen der Geometrie* (1899), Hilbert sharpened the mathematical method from the material axiomatics of Euclid to the formal axiomatics of today. Hilbert founded the formalist school of mathematics.

1904: Henri Lebesgue (1875–1941) generalized the Hiene–Borel theorem to arbitrary infinite collections.

1906: English mathematicians Grace Chisholm Young (1868–1944) and her husband William Henry Young wrote the first comprehensive textbook on set theory and its applications to function theory, *The Theory of Sets of Points*. In 1895, Grace Chisholm became the first woman to receive a German doctorate through the regular examination process (women were not admitted to graduate schools in England at that time).

1906: French mathematician Maurice Fréchet (1878–1973) inaugurated the study of abstract spaces with his introduction of the concept of a metric space.

1908: Dutch mathematician L. E. J. Brouwer (1881–1966) originated the intuitionist school about this time, although some of the intuitionist ideas had been enunciated earlier. The intuitionist thesis is that mathematics is to be built solely by finite constructive methods on the intuitively given sequence of natural numbers.

1910: English mathematicians Bertrand Russell (1872–1970) and Alfred North Whitehead (1861–1947) wrote *Principia Mathematica*. The basic idea of the *Principia* is the identification of much of mathematics with logic by the deduction of the natural number system, and hence of the great bulk of existing mathematics, from a set of premises or postulates for logic itself.

1915: The Mathematical Association of America (MAA) was founded. Although the AMS and MAA are both concerned with university mathematics, the AMS leans more toward research and the MAA more toward teaching. The two organizations together sponsor the Joint Mathematics Meetings every January.

1916: Albert Einstein (1879–1955) introduced his general theory of relativity.

1917: British number theorist G. H. Hardy (1877–1947) and Indian mathematician Srinivasa Ramanujan (1887–1920) reached penetrating results in number theory. Ramanujan had an uncanny ability to see quickly and deeply into intricate number relations. Hardy's efforts brought Rananujan to England to study at Cambridge University, and a remarkable mathematical association resulted between the two of them.

1920: The International Mathematical Union was founded, which was to become a prominent society in the twentieth century and beyond.

1922: German mathematician Amalie Emmy Noether (1882–1935) became extraordinary professor at Göttingen and kept the position until 1933 when she left Germany to accept a professorship at Bryn Mawr College in Pennsylvania and to become a member of the Institute for Advanced Study at Princeton. Her studies on abstract rings and ideal theory have been important in the development of modern algebra. Her father, Max Noether (1844–1921), was also an algebraist.

1923: Polish mathematician (1892–1945) Stefan Banach introduced the notion of what is now called a Banach space, a vector space possessing a norm under which all Cauchy sequences converge.

1931: Austrian logician Kurt Gödel (1906–1978) showed that it is impossible for a sufficiently rich formalized deductive system to prove consistency of the system by methods belonging to the system (Incompleteness theorem). Later, about 1940, Gödel showed that the continuum hypothesis (that the cardinal number of the reals is the next cardinal number after the cardinal number of the natural numbers) is consistent with a famous postulate set of set theory (Zermelo–Fraenkel), provided these postulates themselves are consistent, and conjectured that the denial of the continuum hypothesis is also consistent with the postulates of set theory.

1934: Bourbaki's works started. Nicolas Bourbaki is the collective pseudonym employed by a group of French mathematicians who met in a Paris café to discuss writing a new calculus textbook for French university students. From that beginning, the project grew into the more ambitious undertaking of developing with rigor the essentials of modern French mathematics. The membership has varied over the years. Members must leave the group at age 50.

1940s: IBM's automatic sequence controlled calculator (ASCC) was debuted in 1944, and may be cited as the beginning of the computer age. The electronic numerical integrator and computer (ENIAC), which debuted in 1945, was the first general purpose, completely electrical computer. It used vacuum tubes rather than electromechanical switches. Many mathematicians played a role in computer development, and computers would come to play an important role in many areas of mathematics research and education.

1940: *Mathematical Reviews*, containing abstracts and reviews of the current mathematical literature in the world, was organized by mathematical groups both in the United States and abroad to help researchers keep abreast of mathematical work in their fields. The increase of mathematical specialization in the twentieth century led to the formation of many journals focused on specific subfields of mathematics.

1957: The Soviet Union launched the first satellite, *Sputnik*, into space. The shock in the United States of the unexpected venture into space caused Congress to establish the National Aeronautics and Space Administration (NASA) in 1958, which led to men on the moon and huge breakthroughs in computers. There was a renewed emphasis on mathematics and science education with many government-sponsored programs to support graduate work in these fields.

1960s: The era of so-called new math that emphasized understanding over rote memorization was ushered in at the urging of the National Council of Teachers of Mathematics and other groups concerned with teaching mathematics. The impact was perhaps larger at the elementary school level than at the high school level. In high school, there was consideration of what to do after the traditional two years of algebra and year of geometry. For many years, the courses above these three years of high-school mathematics were a semester each of solid geometry and trigonometry. Experimental courses such as functions and matrices were added as the final year of high school. Many of the reforms fell out of favor after a decade or so with criticism of such things as the emphasis on vocabulary in elementary school mathematics and lack of emphasis on memorizing addition and multiplication tables. Taking algebra I before high school became common resulting in more courses needed at the upper levels of high school. Today, advanced placement calculus is common at the senior year; the study of calculus for decades before had been strictly a college-level course.

1963: Paul J. Cohen (1934–2007) followed up on a conjecture of Gödel some 25 years earlier that a denial of the continuum hypothesis in Zermelo–Fraenkel set theory would not lead to contradictions in the theory. Cohen was able to show that both the continuum hypothesis and the axiom of choice (given a collection of mutually disjoint, nonempty sets, there exists a set which has as its elements exactly one element from each set in the given collection of sets) are independent of Zermelo–Fraenkel set theory (named for mathematicians Ernst Zermelo and Abraham Fraenkel) without the axiom of choice; this makes the situation analogous to that of the parallel postulate in Euclidean geometry.

1969: The National Association of Mathematicians (NAM) was founded to address the needs of the minority mathematical community.

1971: The first pocket calculator was offered for sale in the consumer market. Pocket calculators quickly became cheaper and more sophisticated. Hungarian mathematician John von Neumann (1903–1957) was the person most responsible for initiating the first fully electronic calculator and for the concept of a stored program digital computer. Von Neumann migrated to America in 1930 and became a permanent member of the Institute for Advanced Study at Princeton in 1933. Computers were initially designed to solve military problems, but are now pervasive in the form of personal computers for use in education and business.

1971: The Association for Women in Mathematics was founded.

1976: The four-color theorem, first conjectured in 1852, was established by Kenneth Appel (b. 1932) and Wolfgang Haken (b. 1928) of the University of Illinois. The four-color theorem of topology states that any map on a plane or sphere needs at most four colors to color it so that no two countries sharing a common boundary will have the same color. The Appel–Haken solution of the four-color problem depended on intricate computer-based analysis and raised philosophical questions of just what should be allowed to constitute a proof of a proposition in mathematics.

1985: Supercomputers came into general use.

1994: Princeton mathematician Andrew Wiles (b. 1953) completed the proof of Fermat's Last Theorem after correcting a flaw in his 1993 work that had taken seven years to complete. Fermat's Last Theorem states that there do not exist positive integers x, y, z, n such that $x^n + y^n = z^n$ when $n > 2$.

2002: Russian mathematician Grigori Perelman (b. 1966) posted a proof of the long-standing Poincaré conjecture in three installments on the Internet. The Poincaré conjecture states essentially that any closed three-dimensional manifold in which every closed curve can be shrunk to a point is homeomorhic to the three-dimensional sphere. In 2006, the International Mathematical Union awarded Perelman its prestigious Fields Medal. Perelman declined to accept the medal, which also included a million-dollar prize, saying that "everyone understood that if the proof is correct then no other recognition is needed." Subsequently, Perelman decided to drop out of mathematics entirely.

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Resource Guide

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Biometrics
Chance
The College Mathematics Journal
Experimental Mathematics
The Fibonacci Quarterly
Historia Mathematica
IMU-Net
Involve
Journal of Humanistic Mathematics
Journal of Integer Sequences
Journal of Recreational Mathematics
Journal of Statistics Education
Loci
MAA FOCUS
Math Horizons
Mathematics Magazine
Mathematics Teacher
NAM Newsletter
Notices of the American Mathematics Society
The Pentagon
Pi Mu Epsilon Journal
Plus Magazine
PRIMUS
Rose-Hulman Undergraduate Mathematics Journal
SIAM Review
Scholastic Math
Significance
Teaching Children Mathematics
Undergraduate Mathematics and Its Applications

Internet

American Institute of Mathematics
www.aimath.org

The Algebra Project
www.algebra.org

AMATYC
www.amatyc.org

American Mathematical Society
www.ams.org

American Statistical Association
www.amstat.org

Association for Women in Mathematics
www.awm-math.org

CryptoKids
www.nsa.gov/kids

Datamath Calculator Museum
www.datamath.org

Illuminations
illuminations.nctm.org

MacTutor History of Mathematics
www-history.mcs.st-and.ac.uk

Mathematical Fiction
http://kasmana.people.cofc.edu/MATHFICT

Math for America
www.mathforamerica.org

Math Forum
www.mathforum.com

Math Fun Facts!
www.math.hmc.edu/funfacts

MathDL
mathdl.maa.org/mathDL

Mathematical Association of America
www.maa.org

Mathematical Science Research Institute
www.msri.org

The Museum of Mathematics
www.momath.org

National Association of Mathematicians
www.nam-math.org

National Council of Teachers of Mathematics
www.nctm.org

RadicalMath
www.radicalmath.org

Society for Industrial and Applied Mathematics
www.siam.org

We Use Math
www.weusemath.org

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www.mathworld.wolfram.com

Glossary

absolute value
For a real number x, the absolute value of x, written $|x|$, is the "unsigned" version of the number. If x is nonnegative, then $|x| = x$; if x is negative, then $|x| = -x$. For a complex number $x = a + bi$, the definition is slightly more complicated:

$$|x| = \sqrt{a^2 + b^2} .$$

In both cases, $|x|$ represents the magnitude of x, and $|x - y|$ represents the distance between x and y.

acute angle
A nonzero angle that is smaller than a right angle. The measure of an acute angle is between 0 and 90 degrees.

algebra
A branch of mathematics dealing with the formal properties and behavior of symbolic operations, relations, and structures. Mathematicians use the word "algebra" much, much more broadly than it is used in everyday usage; nonmathematicians often use the term specifically for what is taught in the secondary school curriculum under the heading "algebra," which is only the most elementary part of algebra.

algebraic number
A complex number is called algebraic if it is the root of some polynomial with integer coefficients. For example, all rational numbers are algebraic, as are $\sqrt{19}$, i,

$$\sqrt[3]{5 + 2\sqrt{5}},$$

and all five roots of $x^5 - x + 1 = 0$. Though any number that can be expressed in terms of arithmetic operations and roots must be algebraic, not all algebraic numbers can be written in this way (in particular the roots of the given polynomial cannot be so written).

analysis
Elements of analysis are part of the curriculum under the name "calculus." The most elementary part of analysis is part of the curriculum under the name "calculus."

antiderivative
If $f(x)$ is the derivative of a function $F(x)$, then we say that $F(x)$ is an antiderivative (integral) of $f(x)$.

Arabic numerals
The familiar base-ten number system, with digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 and a positional place value system based on powers of 10.

arithmetic mean
The arithmetic mean of a set of numbers $a_1, a_2, \ldots, a_n$ is

$$\frac{a_1 + a_2 + \cdots + a_n}{n}.$$

The arithmetic mean is often called the "average" or just the "mean."

arithmetic progression
A sequence (finite or infinite) such that the difference of any two consecutive terms is the same; equivalently, a sequence in which each term (except the first and last) is the arithmetic mean of the terms immediately preceding and following.

associativity
A binary operation is said to be associative, or (less frequently) to associate, if the (chronological) order in which it is evaluated does not affect the answer. In symbols, ~ is associative if $(x \sim y) \sim z = x \sim (y \sim z)$ for all x, y, z for which the operation is defined.

base *b*
See *number bases*.

bijection
A function is a bijection (also called a "one-to-one correspondence") if it is both injective and surjective.

binary (notation system)
Base two. (See *number bases*.)

binomial coefficient
The binomial coefficient

$$\binom{n}{r}$$

(often read "n choose r"), defined for integers $n \geq r \geq 0$ by the formula

$$\frac{n!}{r!(n-r)!};$$

this binomial coefficient counts the number of combinations of r elements from a set of size n.

Binomial Theorem
A theorem of basic algebra which connects the binomial coefficients to the numbers appearing in the expansion of a binomial raised to a power (indeed this is the reason they are called binomial "coefficients"). The precise statement is

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

calculus
A branch of mathematics that focuses on (1) evaluating the rate at which a function changes and (2) evaluating the rate at which a function accumulates. If we plot a single-valued function $y = f(x)$, these correspond to (1) the slope of $f(x)$ and (2) the area under the curve $f(x)$.

Cartesian product
If A and B are any sets whatsoever, then the Cartesian product $A \times B$ is the set of all ordered pairs (a, b) with a in A and b in B.

circle
In a plane, the set of all points at a fixed distance (the "radius") from a given point (the "center").

claim
See *proposition*.

closed form
An expression or formula is in closed form if it is written explicitly in a way that could be directly evaluated. Recursions, summation (or product) notations, and ellipses to indicate omitted terms cannot be part of closed-form expressions.

codomain
The set where a function's values ("outputs") live. Specifying the codomain is part of defining a function.

combination
One of the fundamental enumeration problems. The combinations of r objects from a set of n objects are the "unordered" sets of r distinct objects from the set. There are

$$\frac{n!}{r(n-r)!}$$

combinations of r objects from a set of size n.

commutativity
A binary operation is said to be commutative, or to commute, if the left-to-right order of its arguments does not matter. That is, ~ is commutative if $x \sim y = y \sim x$ for all x, y for which the operation makes sense. In the context of group theory, a commutative group operation is usually called “abelian.”

complex conjugate
The (complex) conjugate of a complex number $z = x + iy$ is $\overline{z} = x - iy$; that is, complex conjugation changes the sign of the imaginary part. Complex conjugation preserves the structure of the complex number system; in particular $\overline{x + y} = \overline{x} + \overline{y}$ and $\overline{xy} = (\overline{x})(\overline{y})$.

complex numbers
The set of all numbers of the form $a + ib$, where a and b are real numbers and i is the imaginary unit, the square root of −1 (a is sometimes called the real part, and b the imaginary part). Complex numbers can be added, subtracted, multiplied, and divided (except by zero). Geometrically, the set of complex numbers can be visualized as a plane. The complex plane is the complex analogue of the number line. The set of complex numbers is traditionally denoted by a blackboard-bold $\mathbb{C}$.

composite
A positive integer n is composite if it can be written as a product $n = ab$, where a and b are positive integers greater than 1. (For positive integers *greater than* 1, “composite” means “not prime.”)

congruent
In geometry, one object is congruent to another object of the same type if they are “the same size and shape.” For line segments, this means they have the same length; for angles, this means they have the same angle measure. For triangles (and other figures, though the term is used most frequently to refer to triangles), congruence means that corresponding sides have the same length and corresponding angles have the same measure.

In modular arithmetic, the term “congruent” is often used as the analogue of “equal.” For example 2 and 16 are congruent modulo 7. Equations in modular arithmetic are thus often called congruences.

conic section
Any geometric figure that can be realized as the intersection of a full (double) cone with a plane. Such a figure can be defined by an equation of the form $ax^2 + bxy + cy^2 + dx + ey + f = 0$. Setting aside degenerate cases, the conic sections have three types: “ellipses,” “hyperbolas,” and “parabolas.”

conjecture
An assertion that is believed to be true (at least by some people) but that has not yet been rigorously demonstrated. The term “conjecture” is not generally applied to any wild guess; a conjecture is typically justified (but not proven) by a heuristic argument or experimental evidence.

continuous function
Intuitively, a function is continuous if whenever x is “near” y, then $f(x)$ is “near” $f(y)$. How exactly this is formalized depends on context, especially the domain and codomain of the function. For real-valued functions of a single real variable, it is often said that a function is continuous if its graph could be drawn “without picking up your pencil” (which may be a helpful heuristic, but is not, strictly, true). The formal definition in that case is that a function is continuous at x_0 if, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that, $|x - x_0| < ä$ implies $|f(x) - f(x_0)| < \varepsilon$. Continuity is conceptually very close to limits. Continuity is a central notion to analysis and to topology.

contrapositive
If a statement takes the form “If P, then Q,” then the contrapositive is the statement “If not Q, then not P.” For example, the contrapositive of “If I have meditated, then I am relaxed” is “If I am not relaxed, then I have not meditated.” A statement and its contrapositive are either both true or both false. It is often more straightforward to prove the contrapositive form of a theorem than to prove the theorem directly.

coordinate geometry
A technique for studying Euclidean geometry by identifying points in the plane with ordered pairs of real numbers (or points in space with ordered triples, and generally points in n-dimensional space with ordered n-tuples) based on their position relative to a family of axes (sing. axis). This allows us to understand

geometric objects in terms of numbers and equations, using algebra and arithmetic knowledge to solve geometric problems.

corollary
A proposition that is an easy consequence of a previous theorem.

countable (set)
A set is countable if it is finite or if it is "the smallest sort of infinite," if its elements can be listed $a_0, a_1, a_2, \ldots$. There are countably many integers, countably many rational numbers, and countably many polynomials with integer coefficients. However, the set of real numbers is uncountable.

decimal (notation system)
The familiar notation for writing real numbers, base ten. (See *number bases*.)

definite integral
The definite integral

$$\int_a^b f(x)\,dx$$

represents the "accumulation" of the function $f(x)$ over the interval $a \le x \le b$. Geometrically, this is the signed area between the graph of the function and the x-axis. ("Signed" indicates that area above the x-axis is weighted positively, and area below is weighted negatively.)

degrees
One of the two most commonly used units of angle measure, denoted by the symbol °. There are 360 degrees in one full circle, and 90 degrees in a right angle. The primary advantage of this unit (and the historical reason for its use) is that 360 has many factors, so that many important angles are a whole number of degrees.

derivative
See *differentiation*.

differential equations
Equations or systems of equations relating one or more functions to their derivatives. A wide array of specialized techniques have been developed to solve such equations exactly or to approximate solutions. Such equations have myriad uses in mathematics (pure and applied) and in science and engineering.

differentiation
The process of finding the "derivative" of a function, which is intuitively the "rate of change" of the value of a function with respect to the input.

discrete mathematics
A very wide branch of mathematics dealing with finite or countable objects and structures. This includes counting (enumeration) problems, partitions, graph theory, matroids, designs, and Ramsey theory.

domain
The set of valid arguments ("inputs") to a function. Specifying the domain is part of defining a function.

***e* (exponential or Euler's constant)**
Arguably the most "natural" base for exponential and logarithmic functions. The constant e can be defined by

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

or by the formula

$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

(or numerous other formulae). It has the approximate value 2.71828. . . . See also *exponential function* and *logarithm*.

ellipse
The set of points at a fixed sum of distances from two given points. That is, if A and B are two points (called the *foci* – sing. *focus*) and r is any positive constant greater than the distance AB, then the set of all points P such that $AP + BP = r$ is an ellipse. After suitably rotating and translating the coordinate axes, any ellipse can be described by an equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

enumeration
Counting and/or listing the objects or structures of a particular kind of type. Questions of the shapes "How

many X have the properties Y?" or "What are all the X that have the properties Y?" lead to enumeration problems.

exponential function
For any positive base b, there is a base-b exponential function, traditionally written $f(x) = b^x$.

factorial
If n is a nonnegative integer, then $n!$ (read "n factorial") is defined by $0! = 1$, $1! = 1$, and generally $n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$. Factorials show up frequently throughout mathematics, especially in combinatorics or in answers to problems where combinatorics is "in the background." Using more advanced methods, it is possible to define $x!$ for some values of x that are not nonnegative integers. This generalization is called the "gamma function."

Fields Medal
Officially called the "International Medal for Outstanding Discoveries in Mathematics," the Fields Medal is widely considered the mathematician's analogue of the Nobel Prize. The prize is awarded by the International Mathematical Union once every four years to two, three, or four mathematicians no older than 40 years old.

finite (set)
A set is finite if its elements can be put in one-to-one correspondence with the elements of a set $\{1, 2, 3, \ldots, n\}$ for some n.

function
Formally, a function f has three parts: a domain set D, a codomain set C, and a rule which corresponds each domain element to a unique codomain element. More formally, there is a subset S_f of the product $D \times C$, such that each element of D is the first member of exactly one of the ordered pairs in the subset. If (x, y) is the unique element of S_f beginning with x, we say that $f(x) = y$. Informally, the domain is the set of inputs, and the codomain is the set of potential outputs. Functions are most often specified by an algebraic expression such as

$$f(x) = x^3 + x^2$$

but this is not necessary.

Fundamental Theorem of Algebra
Every polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

with coefficients in the complex numbers has at least one root in the complex numbers. Furthermore, if roots are counted with multiplicity, an nth-degree polynomial will always have exactly n roots.

Fundamental Theorem of Arithmetic
Every integer greater than 1 can be written as a product of primes; furthermore, the prime factorization is unique except for the order in which the factors are written.

Fundamental Theorem of Calculus
Any of several important theorems of elementary analysis relating differentiation and integration as (in some suitable sense) inverse processes. The most commonly given are the following two:

1. If a function F is an antiderivative of a function f on an interval $[a, b]$, then

   $$\int_a^b f(x)\,dx = F(b) - F(a)$$

   (We can easily compute definite integrals given an antiderivative.
2. If f is a continuous function on some interval containing a, then the function

   $$F(x) = \int_a^x f(t)\,dt$$

   is an antiderivative of f. (Definite integrals can be used to define antiderivatives.)

gamma function ($\tilde{A}(s)$)
A generalized version of the factorial function which makes sense for non-integral arguments, and even nonreal arguments. The gamma function is defined (for complex numbers with positive real part) by

$$\tilde{A}(s) = \int_0^\infty t^{s-1} e^{-t}\,dt$$

and satisfies $\tilde{A}(s) = (s-1)!$ and the factorial-like identity $\tilde{A}(s+1) = s\tilde{A}(s)$. (It is possible to extend this definition through complex analysis techniques to all complex numbers except nonpositive integers.)

geometric mean

The geometric mean of a set of numbers $a_1, a_2, \ldots a_n$ is

$$\sqrt[n]{a_1 a_2 \cdots a_n}.$$

This is the geometric mean of 2 and 8 is 4.

geometric progression

A sequence (finite or infinite) such that the ratio of any two consecutive terms is the same; equivalently, a sequence in which each term (except the first and last) is the geometric mean of the terms immediately preceding and following.

geometry

A branch of mathematics dealing with shapes, sizes, lengths, angles, areas, volumes, and so on. Because many contemporary mathematics curricula stress proofs and axiomatic reasoning for the first (and often last) time in secondary-school geometry class, geometry and axiomatic reasoning are sometimes conflated in the mind of the general public.

golden ratio

The number $\varphi = \dfrac{1+\sqrt{5}}{2} \approx 1.6180339887$, or roughly 8/5.

The golden ratio can be defined as the unique positive solution to the equation

$$\frac{1}{\varphi} = \varphi - 1.$$

This number appears throughout mathematics, nature, music, and art. The golden ratio also has the interesting continued fraction representation

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}}$$

golden rectangle

A rectangle for which the ratio of its side lengths is equal to the golden ratio. Such a rectangle has the notable property that it can be dissected into a square and a smaller rectangle which is similar to the original. Golden rectangles have been considered the most aesthetically pleasing rectangles, and they play a role in classical art and architecture.

gradians

A somewhat obscure unit of angle measure, still occasionally seen in certain texts and certain calculators. There are 400 gradians in a full circle.

harmonic mean

The harmonic mean of a set of positive numbers $a_1, a_2, \cdots, a_n$ is

$$\frac{n}{\dfrac{1}{a_1} + \dfrac{1}{a_2} + \cdots + \dfrac{1}{a_n}}.$$

harmonic progression

A sequence (finite or infinite) in which each term (except the first and last) is the harmonic mean of the terms immediately preceding and following. Equivalently, a sequence whose reciprocals form an arithmetic progression.

harmonic series

The series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots.$$

This series famously diverges to infinity, that is, has no finite value. Its partial sums

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{N}$$

are sometimes called the "harmonic numbers" and are well-approximated by $\log N$.

hexadecimal (notation system)

Base sixteen (the digits used are traditionally 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F). (See *number bases*.)

homomorphism

A function from one mathematical structure to another mathematical structure that respects all the operations and properties of that type of structure. For example, a homomorphism of rings (structures in which addition and multiplication make sense) must preserve addition and multiplication. For example, the function that maps each polynomial to its constant term is a ring homomorphism from the set of polynomials to the set of real numbers.

hyperbola
The set of points at a fixed difference of distances from two given points. That is, if A and B are two points (called the *foci* – sing. *focus*) and r is any positive constant, then the set of all points P such that $|AP - BP| = r$ is a hyperbola. After suitably rotating and translating the coordinate axes, any hyperbola can be described by an equation of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

iff
A commonly used mathematician's abbreviation for "if and only if," used to indicate that two statements logically imply one another.

image
The set of all values realized by a function. In symbols, if $f{:}D \to C$ is a function with domain D and codomain C, then the image of f is $\{y \in C : \exists x \in D \ s.t.\ f(x) = y\}$ ("$\exists$" stands for "there exists").

imaginary number
This term is used inconsistently. Some use it to refer to any complex number that is not real, others only for a *pure* imaginary number of the form bi, where b is real.

imaginary unit
The symbol i, whose defining property is that $i^2 = -1$.

indefinite integral
The indefinite integral $\int f(x)\,dx$ is the family of all antiderivatives of $f(x)$ (usually written in the form $F(x) + C$, where F is a particular antiderivative and C is a general constant).

induction
A proof technique used to prove that a statement or property holds in an infinite number of cases (e.g., to prove that $1 + 2 + 3 + \ldots + n = n(n+1)/2$ for all positive integers n). First, one checks the simplest cases individually (the "base case(s)"); then, one proves that, if the claim holds in the first n cases, it will also hold in the $(n + 1)$-st case. In this way, it is possible to prove an infinite collection of statements with a finite proof. (Note that this is not the same as "inductive reasoning.")

infinite set
A set is infinite if its elements cannot be put in one-to-one correspondence with the elements of the set $\{1, 2, 3, \ldots n\}$ for any n. Equivalently, a set is infinite if its elements can be put in one-to-one correspondence with a proper subset of itself.

injective
A function $f{:}D \to C$ is injective (also called "one-to-one") if distinct inputs give distinct outputs. In symbols, $f(x) = f(y)$ implies $x = y$, that is, passes the horizontal line test—a test used to determine if a function is one-to-one. If no horizontal line intersects a function's graph more than once, the function is said to be "one-to-one."

integer
The set of integers includes the counting numbers 1, 2, 3, 4, 5, 6, . . . and their negatives, as well as 0. The set of integers is traditionally denoted by a blackboard-bold $\mathbb{Z}$ (from the German word *Zahl*, meaning "number").

integral
See *integration* for the broad concept. See *definite integral* and *indefinite integral* for particular uses common in classroom usage. (An adjective form of "integer.")

integration
Intuitively, the operation of "adding up" or "accumulating" the values of a function over part or all of its domain. More concretely but still informally, the integral of a real-valued single-valued function is the area under its graph. There are numerous types of integration, at varying degrees of complexity and technicality. The simplest is the Riemann integral, which is part of the standard calculus curriculum. The most commonly used in professional mathematics is arguably the Lebesgue integral.

irrational number
A number (usually the term is only used for real numbers) is called irrational if it is not rational, that is, if

it cannot be expressed as the ratio p/q of two integers. The decimal expansion of an irrational number neither terminates nor repeats.

isomorphism
A homomorphism that is also a bijection. Two objects related by an isomorphism are called "isomorphic." Isomorphic objects are in a deep sense "the same." This notion is fundamental to modern mathematics.

lemma
A proposition which is proven primarily so that it can be used in the proof of another proposition, which is presumably considered more important.

limit
Intuitively, if whenever x is "near" a, then $f(x)$ is "near" L, we say $\lim f(x) = L$. How exactly this is formalized depends on context; there are even versions of the definition that allow variables to approach infinity or allow for infinite limit values. For real-valued functions of a single real variable, the formal definition is that $\lim f(x) = L$ if, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that, $0 < |x - a| < ä$ implies $|f(x) - L| < \varepsilon$. Derivatives and integrals are both defined in terms of limits. The limit is the foundational concept on which calculus and analysis are built.

linear algebra
In its more narrow meaning, linear algebra is the study of systems of linear equations, matrices, vectors, and their operations. More broadly and abstractly, linear algebra is the study of linear operators, functions which respect addition and multiplication by scalars.

logarithm
The inverse operation to exponentiation. For any positive base $b \neq 1$, we can define a continuous function $\log_b x$ for all $x > 0$ with the following formal properties:

$$\log_b (xy) = \log_b x + \log_b y$$
$$\log_b 1 = 0 \quad \log_b b = 1$$
$$\log_b \left(x^n\right) = n \log_b x.$$

We have $\log_b x = k$ if and only if $b^k = x$. If the base is not specified, then the base e is implied in standard mathematical usage; among nonmathematicians and in most secondary school textbooks, the default base is the less natural 10.

logic
The formal study of valid reasoning and inference. Logic is both a branch of mathematics (particularly symbolic logic) and part of its architecture.

matrix
A rectangular array of numbers or mathematical symbols. Two matrices of the same shape can be added or subtracted. In order to multiply two matrices, the number of columns of the first must match the number of rows of the second. The algebraic properties of these operations are a bit different from operations on numbers; for example, matrix multiplication is not commutative, and it is possible to multiply two nonzero matrices together and get zero. There are many applications of matrices and matrix algebra.

modular arithmetic
The arithmetic of congruences in number theory. When working *modulo n*, one considers two integers to be congruent if their difference is a multiple of n (equivalently, if they give the same remainder when divided by n). Working modulo 2 is just keeping track of whether numbers are even or odd (even + odd = odd, odd + odd = even, etc.). Working modulo 12 is so-called "clock arithmetic."

multinomial coefficient
The multinomial coefficient

$$\binom{n}{n_1, n_2, \ldots, n_k}$$

defined for nonnegative integers

$$n = n_1 + n_2 + n_3 + \cdots + n_k \text{ by the formula}$$

$$\frac{n!}{n_1!\, n_2!\, n_3! \cdots n_k!}.$$

This multinomial coefficient counts the number of ways to divide n objects into k piles so that the first pile contains n_1 objects, the second contains n_2, and so on.

Multinomial Theorem
A theorem of basic algebra which connects the multinomial coefficients to the numbers appearing in the expansion of a sum raised to a power (indeed this is the reason they are called multinomial "coefficients"). The precise statement is

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{\substack{n_1, n_2, \ldots, n_k \geq 0 \\ n_1 + n_2 + \cdots + n_k = n}} \binom{n}{n_1, n_2, \cdots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$$

natural number
A natural number is a "counting number": 1, 2, 3, 4, 5, ... Some sources include 0 as a natural number, while others do not. The set of natural numbers is traditionally denoted by a blackboard-bold $\mathbb{N}$.

number bases
For any integer $b > 1$, we can define a system for writing down real numbers using the digits $0, 1, 2, \ldots, (b-1)$. A string of digits $a_k \cdots a_2 a_1 a_0 .a_{-1} a_{-2} \ldots$ represents the number

$$a_k b^k + a_{k-1} b^{k-1} + \cdots + a_2 b^2 + a_1 b + a_0 + a_{-1} b^{-1} + a_{-2} b^{-2} + \cdots .$$

For example, if $b = 7$, then 123.4 (base 7) represents the number

$$49 + 2 \times 7 + 3 + \frac{4}{7} = 66\frac{4}{7}$$

(written in familiar base 10). This is called base b notation, and several of the more commonly-used bases have other names, such as "binary" for base 2.

number line
A commonly used device for visualizing the real number system in which every real number is represented by a point. Typically the number line is drawn so that the numbers increase toward the right.

number theory
A branch of mathematics concerned with the properties of number systems, especially integers. This includes modular arithmetic, prime numbers, Diophantine equations (an indeterminate polynomial equation that allows the variables to be integers only), and modular forms.

The concepts of "integer," "prime," and so on, can be generalized to much wider contexts than just the "ordinary" integers.

obtuse angle
An angle that is larger than a right angle but less than half a circle. The measure of an obtuse angle is between 90 and 180 degrees.

octal (notation system)
Base eight. (See *number bases*.)

one-to-one
See *injective*.

one-to-one correspondence
See *bijection*.

onto
See *surjective*.

open problem
An unsolved problem, an opportunity for mathematical research. A mathematical question is said to be "open" if it has not been answered in the existing mathematical literature. (compare *conjecture*)

opposite
The opposite of a number x is $-x$. Also called "additive inverse."

ordered pair
A pair of numbers or other mathematical objects, denoted (a, b), where the "order matters," so that (a, b) and (b, a) are different as ordered pairs (unless $a = b$).

ordered triple
A list of three numbers or other mathematical objects, denoted (a, b, c), where the "order matters," so that

$$(a, b, c), (a, c, b), (b, a, c), (b, c, a), (c, a, b), (c, b, a)$$

are different as ordered triples (if a, b, c are distinct).

parabola
The set of points that are at the same distance from a fixed point (the "focus") and a fixed line (the "directrix"). The graph of any quadratic function $f(x) = ax^2 + bx + c$ with a nonzero is a parabola, and every parabola can be described in this way after suitably rotating the coordinate axes.

parity
Whether a number is even or odd. (Considering numbers based only on parity is just another name for working modulo 2.)

perfect number
A positive integer is said to be perfect if it is equal to the sum of all its proper divisors. For example, 6 is perfect because 6 = 1 + 2 + 3. The next smallest perfect number is 28 = 1 + 2 + 4 + 7+ 14. It is not known whether there are infinitely many perfect numbers, nor whether there exists even one odd perfect number.

permutation (enumeration)
One of the fundamental enumeration problems. The permutations of r objects from a set of n objects are the "ordered" r-tuples of distinct objects from the set. There are

$$\frac{n!}{(n-r)!}$$

permutations of r objects from a set of size n.

permutation (function)
A bijection from a set to itself. Intuitively, a permutation is a rearrangement of a set of objects. There are n! permutations of a set of n elements, including the trivial permutation, which leaves the order unchanged.

phi (φ)**.** See *golden ratio*.

pi (π)
A constant defined as the ratio of the circumference of any circle to its diameter, or the number of radians in half a circle. The approximate value of π is 3.1415926536. The constant π is ubiquitous in mathematics and physics.

polygon
Traditionally a plane figure bounded by a closed path made up of a sequence of line segments. These line segments are called the "sides" of the polygon, and the places where one edge ends and the next starts are called "vertices" (sing. "vertex"). A polygon with three sides is usually called a "triangle." A four-sided polygon is a "quadrilateral," a five-sided polygon is a "pentagon," and so on. More generally an n-sided polygon is called an "n-gon." Depending on context, we may want to think of the boundary itself as the polygon, or we might prefer to consider the boundary and its interior together as the polygon.

polyhedron
The three-dimensional analogue of a polygon. The boundary of a polyhedron consists of a collection of polygons in space ("face") with common sides ("edges" of the polyhedron) and "vertices." The faces form a closed figure in space.

polynomial
A polynomial in variables $x_1, x_2, \ldots x_n$ is any expression that can be constructed from the variables and from constants by addition and multiplication. The expressions $2xy + z^3$ and $x^2 + y^2 + z^2 - 1$ are polynomials in x, y, z,

$$\text{but } x + y + \frac{1}{z} \text{ and } \sqrt{xyz} \text{ are not.}$$

polytope
The analogue of polygons and polyhedra in dimensions greater than three.

prime factorization
An expression of a positive integer as a product of primes (possibly raised to powers). For example, $42 = 2 \times 3 \times 7$ and $525 = 3 \times 5^2 \times 7$.

prime (number)
A positive integer is prime if it has exactly two factors, itself and 1. The first few primes are 2, 3, 5, 7, 11, 13, 17, ... (In particular, note that the number 1 is not considered prime.)

probability
A numerical measure taking values between 0 and 1 quantifying the likelihood of an event to occur (or beliefs about that likelihood). Also, the general theory for understanding and working with such measures of likelihood and expectation.

proof

Any rigorous demonstration of the validity of a proposition. Proofs may be written in formal or informal language and may consist of any proportion of words and symbols.

proof by contradiction

A proof technique in which one assumes that the intended conclusion of the theorem is false, and derives from this supposition and from the other hypotheses of the theorem and known facts some statement which contradicts something already known. This shows that one of our assumptions must have been wrong, and the only questionable one was the assumption that our theorem was false. This in turn shows that the conclusion of the theorem does in fact hold.

proposition

A proposition is a declarative statement. Depending on the importance of a statement and/or its role in a larger argument, a proposition may be called a "lemma," a "corollary," a "claim" or a "theorem," depending on its perceived importance.

Q.E.D.

Abbreviation for *quod erat demonstrandum* (Latin for "which was to be demonstrated"), classically used by mathematicians to indicate the conclusion of a proof, signifying that the claims may have now been fully justified. While Q.E.D. itself is less commonly used than it once was, it is still standard practice to include an "end-of-proof symbol" of some kind. The most commonly used end-of-proof symbol today is probably □, called the "Halmos tombstone" and named for mathematician Paul R. Halmos (1916–2006).

quadratic formula

The equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

which expresses the solutions of the polynomial equation $ax^2 + bx + c = 0$ in terms of its coefficients. Analogous formulas exist for cubic equations

$$ax^3 + bx^2 + cx + d = 0$$

and quartic equations $ax^4 + bx^3 + cx^2 + dx + e = 0$, but they are less well-known because they are much more complicated. Notably, it is known that no such formulas exist for polynomials of degree five or larger.

radians

One of the two most commonly-used units for angle measure. Among mathematicians, radians are the standard unit of angle measure, and radian measure is considered implied unless some other unit is indicated. There are 2π radians in a circle. If a circle is drawn with center at the vertex of an angle, then the radian measure of the angle is the ratio of the length of the arc inside the angle to the radius of the circle. The trigonometric functions have the simplest properties (particularly in the context of calculus) if radians are used.

range

The set of values that a function $y = f(x)$ may take corresponding to the values of the input over the specified domain of x. Also called "codomain."

rational function

A rational function in variables $x_1, x_2, \ldots, x_n$ is any expression that can be constructed from the variables and from constants by addition, multiplication, and division. A rational function can always be written as a ratio of two polynomials.

rational number

The set of rational numbers (from the word "ratio") consists of those numbers which can be expressed as a fraction m/n, where m and n are integers with n nonzero. The set of rational numbers is traditionally denoted by a blackboard-bold $\mathbb{Q}$ (for "quotient").

real number

This is what is typically meant when one says "number" without further explanation, including all the integers, all the rational numbers, and many more numbers "between" the rational numbers. A real number can always be described by a (generally infinite) decimal expansion. Real numbers exclude imaginary numbers, complex numbers, and the square root of minus one. The set of real numbers is traditionally denoted by a blackboard-bold $\mathbb{R}$.

reciprocal

The reciprocal of a nonzero number x is $1/x$. Also called "multiplicative inverse."

recursion
A formula or equation relating each term of a sequence or object from a family to the previous terms or objects. This (usually together with one or more initial values) implicitly determines all the terms or objects, but it may be difficult or impossible to get a closed-form description of all the terms.

reductio ad absurdum
Latin for "reduction to absurdity." See *proof by contradiction.*

relation
Formally, a relation R on sets $X_1, X_2, \ldots X_n$ is a subset of the product $X_1 \times X_2 \times \cdots \times X_n$. If $(x_1, x_2, \ldots x_n)$ belongs to the subset we say that $R\,(x_1, x_2, \ldots x_n)$ is true or the relation holds. If not, the statement is false. Most often, $n = 2$ and we write the relation symbol between the arguments, as in "$x_1 < x_2$." Examples include "is less than," "is a factor of," and "does not equal." Examples with $n = 1$ include things like "is prime" or "is positive." Intuitively, a relation is a property that a combination of objects may or may not possess. (One can alternately think of a relation as a special kind of function whose output is "true" or "false.")

relatively prime
A set of integers is relatively prime if there is no number larger than 1 that is a common factor of all of them. For example 18 and 25 are relatively prime. So are 6, 10, and 15. A set of integers is "pairwise" relatively prime if every pair of number in the set is relatively prime. So 6, 10 and 15 are *not* pairwise relatively prime, but 69, 11, and 35 are.

right angle
An angle that is congruent to its own supplement; that is, a quarter of a full circle. The measure of a right angle is 90 degrees.

root
The nth roots of a number a, sometimes written

$$\sqrt[n]{a}\,,$$

are the numbers x such that $x^n = a$. Also, a root of a polynomial

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

is a solution of the equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

that is, a *zero* of the function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

scalar
A numerical quantity (as opposed to a vector or function) that may take one of several forms; for example, a real or complex number.

sequence
A finite or infinite list of terms (which can be numbers or any other type of object). A sequence can be given by listing its terms explicitly separated by commas or by giving a closed-form or recursive formula for the terms.

series
A sum of a finite or infinite (but more frequently the latter) list of terms (usually numbers or mathematical expressions). The terms of a series can be listed as the terms of a sequence can, but separated by plus signs, not commas. In the case of an infinite series, this goes beyond the ordinary concept of addition, and requires the notion of limit to make sense.

set theory
A branch of mathematics studying *sets*, which can be thought of as collections of objects. It also includes a vocabulary for making precise certain ideas about infinite sets and their relative sizes. Set theory sits close to the core of mathematical theory, and much of the architecture of mathematics is traditionally built on the foundations of set theory; set theory is also an object of study for its own sake.

surjective
A function $f\colon D \to C$ is surjective (also called "onto") if the image is the whole codomain. That is, for all y an element of C there exists some x an element of D such that $f(x) = y$.

symmetry
A symmetry of an object is a transformation or change of perspective after which the object is "the same as it was before." A figure has reflective symmetry if it

looks the same when reflected across a certain line; the human face has at least approximate reflective symmetry. A figure has rotational symmetry if it looks the same when rotated around a certain point by a certain angle (traditional crossword puzzle grids have rotational symmetry). Symmetry is by no means a purely geometric concept; a unifying concept of much of modern mathematics is the problem of identifying and describing symmetries of all types.

tau (τ)
A circle constant defined as the ratio of the circumference of any circle to its radius, or the number of radians in a full circle. The approximate value of tau is 6.2831853072. While π is in much more common usage (for chiefly historical reasons), some consider τ to be the more mathematically significant constant.

theorem
A theorem is a mathematical statement that can be demonstrated to be true provided that the set of axioms and other theorems from which this theorem is derived is true. It is usually a general component of some larger theory. Its significance is often a subjective decision.

theory
A collection of related definitions and theorems on a particular topic, such as "number theory," "knot theory," or "graph theory"; an area of study or research in mathematics. Note that "theory" as used in mathematics means something like "formal study" and it definitely does not have the unproven, conjectural connotation present in scientific and everyday usage.

topology
A branch of pure mathematics dealing with those properties which are preserved by continuous deformations (stretching, twisting, enlarging, shrinking, and so on).

transcendental number
A complex number is called transcendental if it is not algebraic, that is, if it is not the root of any polynomial with integer coefficients. Famously transcendental numbers include π and *e*.

trichotomy
The principle that, if x, y are real numbers, then of the statements $x<y$, $x>y$, and $x=y$, "exactly" one is true. Some other sets and order relations have trichotomy properties as well.

tuple
General term for "ordered pair," "ordered triple," "ordered quadruple," and "ordered n-tuple" for any larger *n*. The adjective "ordered" is usually regarded as implied from context and omitted.

variable
A symbol, often x (though just about any symbol can be used), which is meant to represent some unspecified object of a certain type (e.g., a real number). Sometimes a variable stands for a specific (unknown) number, as in "If $2x+3=7$, then what is x?," while in other contexts variables are used formally without a particular value. A variable is to be contrasted with a "constant," which does not change within a given problem.

vector
Often defined as a quantity with a magnitude and a direction. Geometrically, a vector can be thought of as an arrow with a defined head and tail. More generally, the name "vector" is sometimes applied to any element of any vector space; the term is most commonly used for *n*-tuples of real numbers.

whole number
Some people use "whole number" as a synonym for "integer," others use it as a synonym for "positive integer," and still others as a synonym for "nonnegative integer."

zero (number)
A number that serves as the additive identity, characterized by the property that $x+0=x$ for all x. Many mathematical structures have additive identities (zero matrices, zero functions, and so on), and the term zero (and even the symbol 0) are often used for these. In contexts where multiplication makes sense, zero is also characterized by the property that $0 \times a = x$ for all a.

zero (of a function)
The zeroes of a function $f(x)$ are the domain values x_0 such that $f(x_0)=0$.

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