Kurt Fischer

Relativity for Everyone

How Space-Time Bends

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ISBN 978-3-319-00586-7 ISBN 978-3-319-00587-4 (eBook)
DOI 10.1007/978-3-319-00587-4
Springer Cham Heidelberg New York Dordrecht London

Library of Congress Control Number: 2013939093

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

To Yukiko:
You edited the text to become readable.

Preface

Dear Reader,

I encountered the theory of relativity at first while in my teens, in the public library of my hometown. There were two kinds of books: The easy ones did not really explain, but displayed large, colored pictures of the science-fiction kind. The serious books seem to explain something, but that was hidden in a mass of mathematical symbols, and *those* they did not explain, so I was back to square one. Nevertheless, they aroused my interest in physics. It also incited me to fill the gap. The result is this book.

This book is about light, energy, mass, space, time, and gravity: I explain to you how the theory of special relativity and the theory of general relativity work out.

We will use many **thought experiments**, and show you how physicists create and solve models. This method is the one used by Einstein himself: He understood the theory in *physical* and *geometrical* pictures. We will follow Einstein's original train of thought as closely as possible, using even some of his own thought experiments.

I will present some involved arguments, so you need your imagination, but no complicated mathematics to understand the essence of it all. However, the beauty of physics is, that we *can* calculate how the numbers work out. Therefore I prepared the equations of the theory of relativity together with the most important *exact* solutions, using only elementary mathematics. Even the Einstein equation of gravity, showing the bending of space and time, we present *in detail*, in common language. We will see why it is the *simplest theory* of gravity. We will not be content with analogies like "space bends like the surface of a sphere!"

In the first four chapters, I explain what is called the theory of special relativity: I describe the relation between light, matter, space, and time.

1. In Chap. 1, I describe after introducing the basics, that mass and energy are the opposite site of the same coin.
2. In Chap. 2, we will see why time and length are "relative".
3. In Chap. 3, we will see why any electric wire shows relativity in everyday life.
4. In Chap. 4, we learn that while riding a merry-go-round, school-geometry ceases to be true.

In the Chaps. 5 to 9 I describe gravity, that is the theory of general relativity.

5. In Chap. 5, I show you that earth's gravity does not pull at you at all, and that it bends space *and* time.
6. In Chap. 6, we will see in detail which effects the bending space and time are causing.
7. In Chap. 7, we explain thoroughly the meaning of the famous Einstein equation of gravity, and why it is the simplest possible way of describing gravity.
8. In Chap. 8, we introduce the most famous *exact* solution of the Einstein equation, that is the Schwarzschild solution, in simple terms.
9. In Chap. 9, we use the solutions to the Einstein equation, and explore the famous predictions of the theory of general relativity, as for example how much a light beam bends while passing at the sun, how large and heavy black holes are, why the orbits of the planets around the sun turn slowly around the sun, and why the universe had a big bang, but why its future is unclear to us.

One word about highlighted text: Indexed words appear in **boldface**. Such you can easily find them on their page, searching from the index. Before going on with describing the strange properties of light, we better tell in what units we measure and compare the size of things.

Units and Symbols

In physics, we use certain **units** to measure things. Lengths we measure only in meters, time only in seconds, *not* in minutes, hours, or days. Mass we measure only in kilograms. Any *other* unit we use in this book is a combination of these units. For example, speed we measure in meters per second. Other units like pounds or inches or such like we never use. The merit of this is that we can *leave the units away* in all calculations, because we know anyway what units to add *afterwards*, just because we *fixed* them at the beginning.

We will encounter often really large or very small numbers. For example, while numbers like "one thousand" we can write down as 1000, the number of one billion two hundred fifty-two million seven hundred eighty-four 1,250,000,784 is much harder to read. Mostly we are only interested in the first three or so digits, for a rough estimate of how large things are. Here physicists count the number of digits after the first one, that is nine in this case, and write

$$1.25 \times 10^9$$

In the same way, a very small number like 0.000145 we write as 1.45×10^{-4} by counting the number of leading zeros. Then we can easily multiply such numbers: We multiply $1.25 \times 10^9 \times 1.45 \times 10^{-4}$ by multiplying at first 1.25 and 1.45 which is roughly 1.81, and adding the exponents $9 - 4 = 5$, so that the result is $\approx 1.81 \times 10^5$. Here the symbol $\approx$ means "**is roughly equal to**".

For example, we use for the speed of light mostly the rough value

$$\text{speed of light} \approx 3.00 \times 10^8 \text{ meters per second}$$

We will encounter other important numbers of nature in the text. We collected them in Table A.1 for reference.

Special Expressions

At last, one word about how we use phrases like "far enough away from something", "fast enough", and the like. For example, we say

> "if the astronaut is far enough away from earth, the astronaut can neglect earth's gravity."

We are aware of the fact, that gravity is *never* exactly zero, even *very* far away from earth. The point is here, that the astronaut wants to measure some effect, with gravity spoiling this effect, to, say, one percent of the result that the astronaut would get in *really* empty space. So if the astronaut finds that gravity is still spoiling his experiment too much, he is *free to move* to a place which is *so far away* from earth that there, gravity really *does* spoil his experiment only up to the desired one percent at the most. Of course, if he wants to measure even more accurately, he must move even farther away from earth. That is why we say, in short, that if he moves "far enough away" from earth, he always can neglect earth's gravity *to the extent that he wants to* neglect it.

Tokuyama, Japan Kurt Fischer

Contents

Chapter 1
Light, Matter, and Energy

1.1 Light Beams

An astronaut floating in space switches on a torch or a laser beamer. The emerging light beam travels at the speed

$$299{,}792{,}458 \text{ meters per second} \tag{1.1}$$

This is the **speed of light in vacuum**. *Exactly*.

How can we verify this? First of all, we need an apparatus to send a light beam, such as the box on the left in Fig. 1.1. It is open to the right. This box symbolizes the torch, or the laser, or the *sender* for short. The black horizontal arrow stands for the light beam. It travels through the gray box on the right, which is some speedometer, *measuring* the speed of the light beam in meters per second. We will *not* discuss what constitutes such an apparatus: We just assume that there *are* such devices.

1.2 First Law of Relativity: Straight, Steady Speed Is Relative

Does the speed of light change if we move the torch while sending a light beam? This prompts the question: Moving *relative* to what?

If we are moving in a fast train, then we do *not* feel the steady speed of the train, but only feel a slight tremble, that is the *non-steady* part of the speed. For example, in Fig. 1.2 a table is standing inside a train. The train is moving straightly and steadily. Then the black ball will not begin to move on the table. Do you feel the tremendous speed while you are sitting at home at a table, feeling yourself at rest? Which speed? The tremendous speed with which the earth is moving around the sun, for example. This speed is at least during a few minutes, say, nearly straight and steady. Or the speed with which the whole solar system is moving along the galaxy. And not to forget the speed with which the galaxy is moving—to where?

Since the outgoing middle ages we know that we *cannot* detect if we are moving steadily, straight ahead in some direction, by *no means whatsoever*. We *always* can

K. Fischer, *Relativity for Everyone*, DOI 10.1007/978-3-319-00587-4_1,

Fig. 1.1 Light beam from *left* to *right*, sketched as *black horizontal arrow*. The *box* on the *left* stands for some sender like a torch or laser, the *gray box* on the *right is* the light-speedometer, displaying the speed of light

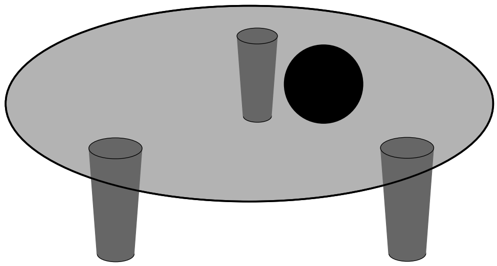


Fig. 1.2 The ball on the glass table in a train will not begin to move on the table, if the train moves straightly and steadily

take the point of view, that *we are at rest*, in the same way as we do when we “sit at a table”.

Then even while sitting at a table in a steadily, straight ahead moving train, we can say that *we* are at rest, and the whole station plus the surrounding area we pass through is moving *towards* us. At the same time, a friend standing on the platform of the station, objects: Of course the train is moving towards the station, and the station together with the earth around it are at rest!

Then who is correct? Answer: *Both* we and our friend are correct in insisting to be at rest: The train is moving only *relative* to the train station and the earth around it. This is the **first law of the theory of relativity**, formulated by **Galilei** a few hundred years ago:

> The speed of straight, steady motion of a body we can only measure *relative* to other bodies. The laws of nature do *not* depend on a straight and steady speed at which we may move relative to other bodies.

1.3 Measuring the Speed of Light

We measure the speed of the bodies in Fig. 1.3 *relative* to the ground. While sending the light beam, we move the torch towards the speedometer with, say, 10000 meters per second, *relative* to the ground, while the speedometer rests relative to the ground. In order that nothing should disturb the light beam, we prepared the situation such that there is no air above the ground.

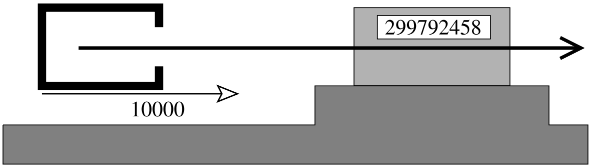


Fig. 1.3 Now the sender moves towards the speedometer with 10000 meters per second relative to the ground. The ground we indicated in *dark gray*

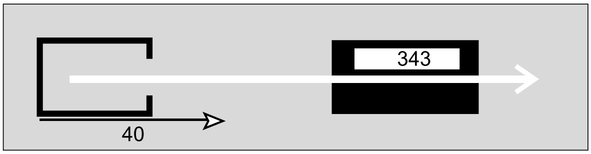


Fig. 1.4 A loudspeaker moves towards the speedometer with 40 meters per second. The speed of sound remains the same as if the loudspeaker would rest

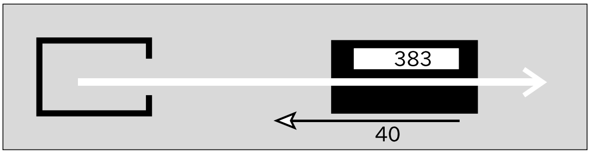


Fig. 1.5 If the speedometer is moving towards the loudspeaker which is resting in the air, we measure a larger speed of sound

Nevertheless, the speedometer shows the very same speed of light! Now, this does not sound so astonishing. Here is an analogy: Replace the torch with a loudspeaker, and the speedometer for light with a speedometer for sound, as in Fig. 1.4. The sound we drew as white arrow. The calm air we depicted as light gray background. The sound travels through the calm air with a speed of about 343 meters per second.

The sound travels through calm air, and so its movement does not feel the moving loudspeaker: We will measure the same 343 meters per second, even if the loudspeaker travels towards the speedometer with 40 meters per second. So is this not a very similar situation to the light beam?

No, because next we will put the loudspeaker at rest inside calm air, and move the *speedometer* towards the loudspeaker, with the same 40 meters per second.

Because the sound moves with 343 meters per second *relative* to calm air to the right, and the detector moves with 40 meters per second *relative* to the calm air as well, to the left, we will measure a sound speed of

$$343 + 40 = 383 \text{ meters per second.}$$

This we sketched in Fig. 1.5.

1.4 Second Law of Relativity: Speed of Light Is Absolute

Next we repeat that thought experiment with light, as in Fig. 1.6.

The astonishing result is, that for light the speedometer *still* shows the same speed of light! That means that light does *not* need any medium like “air” to travel: It

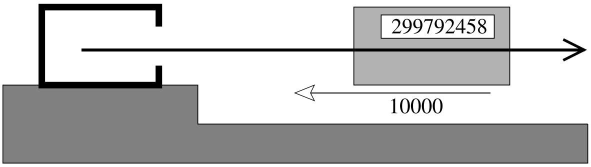


Fig. 1.6 If the speedometer moves towards the sender, the speed of light does not change

travels *as such* through the empty space. As one of the ***c***onstants of nature it got its own name, **c**. Therefore, to simplify calculations, physicists recently fine adjusted the length of the meter, such that the speed of light has exactly the value of Eq. (1.1).

> Light travels always at the same speed $c = 299{,}792{,}458$ meters per second through the vacuum. That is, *the speed of light is absolute*.

This phenomenon of nature is the **second law of the theory of relativity**. Physicists have tested and confirmed it in many experiments, with growing precision, over the last hundred years or so. It is the starting point for the **theory of relativity**.

We think about some consequences in the next section, qualitatively, and introduce the more detailed physical concepts later on.

1.5 Faster than Light?

Clearly, there are things moving faster than sound. When we hear the cracking sound of a whip in the circus, for example, the tip of the whip is moving faster than sound! For another example, using a fast airplane we can overtake the sound so that in the setup of Fig. 1.7 the sound does not reach the speedometer any more.

However, because light travels always at the same speed, we cannot overtake it: Suppose that in Fig. 1.8 we travel with $299{,}792{,}458 - 1$ meters per second to the right away from the torch.

Nevertheless we will see the light in the speedometer moving with the same speed as before. This means that we cannot escape from the light beam:

> We *cannot* travel **faster than light**!

You think that this is too weird, that it cannot be true? Here is another example: In everyday life, we see that whatever is moving, will come to rest, if we do not sustain its movement: balls, cars, airplanes, and so on. However, in reality, bodies will move *forever,* with steady speed, along a straight line, if nothing hampers them. What hampers them on earth is that by friction they heat up other bodies and *such* lose their energy, and come to rest. From an every-day-life point of view this is weird, but we know since a few hundred years or so that this is correct. So we should not judge from our everyday experience, because that may be not enough.

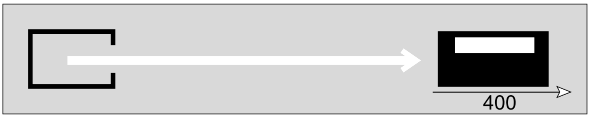


Fig. 1.7 Escaping from sound

1.6 Theory and Practice

Any engineer should take the statement that we cannot travel faster than light as a challenge. Let us built a powerful rocket, which, say, converts nearly all its fuel to thrust. The payload shall be just a black peppercorn. Now start the rocket and observe the peppercorn on top of the rocket accelerating, moving faster and faster away from us. In Fig. 1.9, we sketched the peppercorn on top of the rocket larger than life-size, for better view.

All the same, the faster the peppercorn becomes, the more difficult it becomes for us to **accelerate** it further. If it has got, say, 99 percent of the speed of light, that is, if it is moving with this speed away from us, then it seems hardly to react at all to the thrust of the rocket.

1.7 Mass and Inertia

This is the time to ask *what* has changed within the peppercorn? Its **inertial mass** has grown! "Inertial mass", sometimes called for short **mass**, or **inertia**, you can experience by putting a polished stone on a skating rink, as sketched in Fig. 1.10: Even if the friction from contact with the ice is small, it resists being "pushed", that is, being accelerated. If the stone was twice as big, you have to use twice the effort

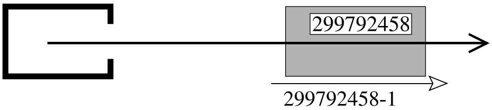


Fig. 1.8 Even if we try to escape with nearly the speed of light from a light-beam, it eventually passes us at the speed of light c

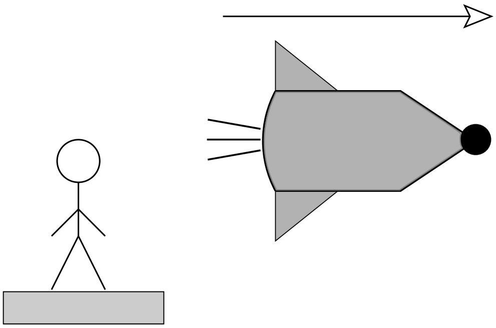


Fig. 1.9 Trying to accelerate a peppercorn by putting it on top of a rocket

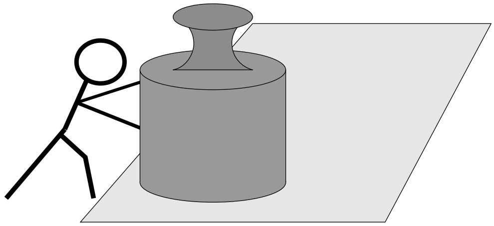


Fig. 1.10 Pushing inertial mass on an ice skating rink

you have to put up to move the lighter stone. This is because the more stone there is, the more mass it contains. Clearly inertia has the connotation of "laziness": Matter resists being pushed, because all kinds of matter have mass, to a different extent. Experience tells us that having mass is *the* fundamental property of matter.

1.8 Inertia and Weight: A First Glance

Usually we measure mass of a, say, stone, by *weighing* it. However, why on earth *heaviness* should have anything to do with *massiveness*?

The massier a stone is, the *less* it wants to move. The *heavier* a stone is, the *more* it wants to move, that is to say, to fall down!

What is more: On earth a stone has some weight, but out in space—and most matter is out in space—we *cannot* weigh our stone, because there is no planet conveniently at hand! But we can push it, try to **accelerate** it, and measure how much it resists.

Nevertheless there *is* a deep connection between mass and weight, which we will explore later on in Chap. 5. For the time being, we continue to use the inertial mass.

Now back to our peppercorn: The faster it passes us, the more it resists being accelerated to a higher speed. This means that its *mass* must have increased. In general:

The faster a material body passes us, the more *mass* it contains.

However, nothing comes from nothing, so what did we add to the peppercorn that it became massier?

1.9 Energy

We burned fuel in the rocket, and such provided the peppercorn with what is called **motion energy**, or **kinetic energy**, that is, energy of motion. There are many kinds of energy, like heat energy, or electrical energy, or motion energy, but it turns out by careful experiments that no matter which type of energy there is, we can *convert* it into any other type. For example, while driving a car we convert the electrical energy sitting in the chemical bonds of the gasoline molecules to motion energy of the molecules by burning the gasoline, and this into motion energy of the car, for that is what we want. However, we carry away with us air which breaks away as air whirls behind the car, so that part of our motion energy converts to motion energy of air whirls. These whirls break up further and further until the energy converts to heat, by air friction. However, heat is nothing but disordered movement of molecules. . . .

It took physicists a long time to establish per experiment, that we can convert *any* form of energy into *any* other, *without loss*. In fact, it took a few centuries before the **concept of energy** established itself: Why heat, electricity, or a fast car should contain basically the same stuff, that is to say, *energy*? Nevertheless, physicists found in every experiment so far, that energy may convert from one from into another, but it never disappears, or springs up from nothing.

We can take one form of energy, say, motion energy, to specify the unit of energy, the **Joule**: Two kilograms of mass traveling at the speed of 1 meter per second have about 1 Joule of motion energy.

Let us convert 1 Joule of energy to different forms back and forth. In the end, there is still just this 1 Joule of energy, just enough to accelerate the kilogram as described before. We stress that this is an experimental fact, not something that you can prove, only suggest by experience. Physicists say that

> **Energy is conserved**: We can only convert energy into different forms, but not create or destroy it.

To come back to our rocket: Clearly the rocket has converted the main part of the energy sitting originally in its fuel, to motion energy of the peppercorn. If we burn out our rocket, we have supplied the peppercorn with some energy.

1.10 Mass and Motion Energy

How can we measure the energy of the peppercorn? Simply put a wall into its path, and assess the damage: The larger the hole, the more energy the peppercorn brought along. A double as powerful rocket will load the peppercorn with double as much energy, roughly. However, if the original rocket accelerated the peppercorn to 99 percent of the speed of light, then a double as strong rocket will achieve the peppercorn to fly at about 99.75 percent of c, which is not much faster.

So where the energy is getting into, when the peppercorn cannot become much faster? Answer: The more matter approaches the speed of light, the more its motion energy shows up as mass.

In other words: The more motion energy matter carries, the more mass it has. That suggests that mass itself is a form of energy: Can we imagine a stone to be a kind of frozen-in motion energy?

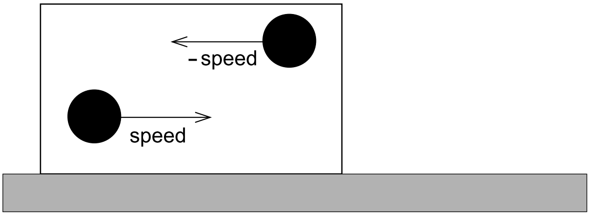


Fig. 1.11 Two internal bouncing masses increase the **resting-mass** of the box

1.11 Resting-Mass and Motion Energy

We can convert the different types of energy into each other. What then about mass of a resting body, the **resting-mass**? Up to know, we imagined resting-mass as rigid, like a stone. However, we know that mass consists of electrons, protons and the like which all the time move around each other. These small constituents are not at rest, even when the "stone" rests.

1.11.1 Internal Motion-Energy

We know that heat means disordered motion of atoms, that is, motion energy of atoms. Hence if we heat a stone, it should contain more motion energy, and hence more mass: The hotter, the massier! Here is a graphic example of internal motion: Take a box where inside two identical balls bounce back and forth, as in Fig. 1.11.

Inside the box, they will bounce back at exactly the same time at the left and right wall, so that the box remains resting on the ground. They carry motion energy, and hence they carry *more* than just their resting-mass. The faster they bounce back and forth, the more *resting-mass* the box gains!

We will calculate how much more mass a moving body has in Sect. 2.6.

Coming back to our resting stone: How much of the resting stone is actually motion energy of its atoms? We see that by and by, the concept of "mass" and "energy" blend in: They seem to be like two sides of the same coin!

1.11.2 Pure Energy

Motion energy is attached to a mass that can rest. This prompts the question: Is there **pure energy**, without **resting-mass**? If there is, it never can rest beside us, because what would be left over of resting energy without mass? However, we learned already of such a thing: In fact, *light* is pure energy: It never rests, because it moves always at the speed c, *no matter how much we change our speed*. This experimental fact was the starting point of the theory of relativity, and we see that the more

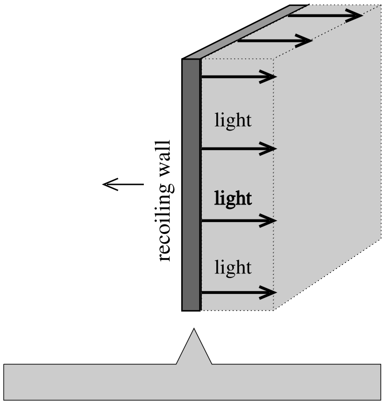


Fig. 1.12 The packet of light has built up completely and starts to leave the wall. The *triangular bump* in the floor indicates the center of mass. The wall starts to recoil

we probe it, think about it, the more astonishing details it reveals. To coin a catch-phrase:

> **Light is pure energy**, without resting-mass,
> *rest-less*, always moving at speed c.

If mass and pure energy are really two sides of the same coin, then we would like to have a thought experiment connecting the two. In fact, Einstein himself provided one thought experiment, the essence of which we present now.

1.12 Inertia of Pure Energy

Think of a wall, standing on an ice skating rink, as in Fig. 1.12. The wall contains some light bulbs, so that light emits from the wall to the right. It builds up a package of light, which we sketched in light gray.

When it has reached some width, we switch off the light bulbs, so that the package of light starts to leave the wall. After some time, the light package is at some distance from the wall, as we see in Fig. 1.13.

While energy is building up inside the light-gray volume, it presses against the wall, because that **pressure** *is* energy per volume. How can we understand that? Think of a pressure cooker. If it is under some pressure, it certainly contains energy: The energy you free by opening the lid of the cooker. Twice as much pressure means twice as much energy in the cooker, and the same pressure in a twice as large cooker means twice as much energy as well. Hence energy is pressure times volume, because this doubles if either we double the pressure or the volume. In other words: Pressure is energy per volume.

Now we know that the package of light presses against the wall before leaving it. In other words: The wall *recoils* from the leaving light package, and moves to the left at some speed.

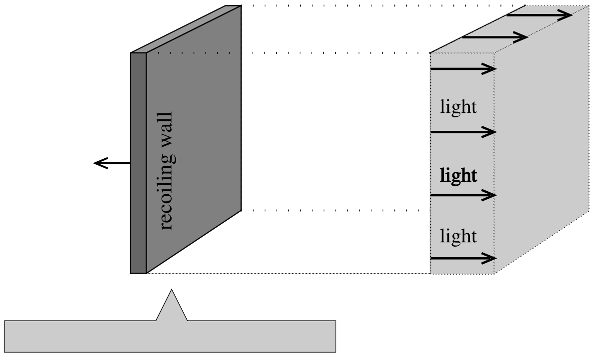


Fig. 1.13 The light packet has left the wall, moving at the speed of light to the *right*. The wall is slowly moving to the *left*

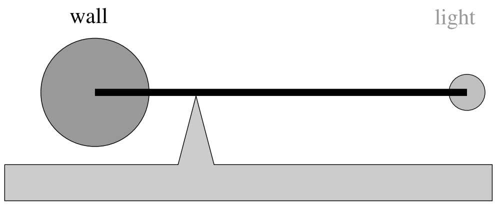


Fig. 1.14 The center of mass does not move without outside influence

However, this is weird: The massy wall begins suddenly to move to the left on the ice skating rink, and nothing, that is, no other mass is moving to the right? This cannot be: The center of mass will not move without outside influence. There is only one way out: The light carries some mass to the right!

We sketched in Fig. 1.14, that if the large mass of the wall recoils a little to the left, then the light must carry a "light" mass over a longer distance to the right, so that their masses are balanced again. This is similar to balanced weights on a scale, where their center of mass remains at the triangular bump.

How much mass does the light carry away? The wall got pushed by the light, which is pure energy. If there goes twice as much light out to the right, then the wall recoils twice as much, so that we expect that the light carries twice as much mass, or inertia, with it. We conclude that even a light package of pure **energy** E carries mass m, and that the two are in proportion. The constant of proportion must be a constant of nature, because we talk about a law of nature relating mass and energy. The detailed discussion in physical terms you find in Sect. A.2, showing that the constant of proportion is the inverse of the square of the speed of light:

$$m = \frac{E}{c^2} \tag{1.2}$$

Said **Einstein** in 1905:[1]

[1] A. Einstein. Ist die Trägheit eines Körpers von seinem Energieinhalt abhängig? Annalen der Physik, Volume 18, page 639, 1905.

A. Einstein. Das Prinzip von der Erhaltung der Schwerpunktsbewegung und die Trägheit der Energie. Annalen der Physik, Volume 20, page 627, 1906.

Wenn die Theorie den Tatsachen entspricht, so überträgt die Strahlung Trägheit [...].

That is to say:

If the theory of relativity is correct, then radiation carries inertia.

By the way: In the original thought experiment Einstein made a mistake, which we also explain in Sect. A.2.

1.13 Mass Is Energy Is Mass

Let us sum up: The theory of special relativity *suggests*, that

Inertial mass is a form of energy, and **energy has mass**. Consequently there should be a way of converting resting-mass into energy and vice versa.

Physicists say "mass and energy are equivalent". Equation (1.2) we can rewrite as the famous formula

$$E = mc^2 \tag{1.3}$$

connecting energy, mass, and the speed of light.

We can see this formula (1.3) at work in Fig. 1.15, when light hits an atom. Then the atom recoils from the light, that is, the energy of the light converts to motion energy of the atom. If the light carries enough energy, then sometimes some of its energy converts into an electron and a positron, which leave the much slower atom. Such we can observe directly the conversion of energy into mass. By the way, the process works also in opposite direction: Then we can see how the matter of the electron and positron convert into pure energy.

1.14 Information Needs Energy

We saw that material bodies like mass or pure energy cannot travel faster than light. However, what if we do not send any energy or mass, but only the *information* about it? Then maybe we can send pure **information** faster than light, using some kind of telephone to send this information, to someone very far away, and he puts together the mass or energy after our specification.

However, it is just the other way round: Information needs to travel mass or energy! In order to handle information, you have to store it somehow, to remember

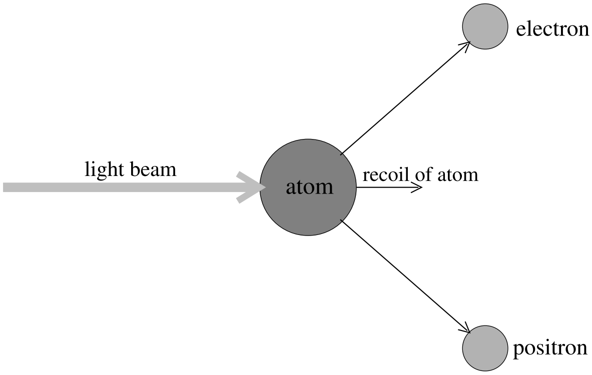


Fig. 1.15 Pure energy converts into matter in form of an electron and a positron, by hitting an atom

it, and for this you need some mass or energy. Hence even pure information cannot travel faster than light.

Chapter 2
Light, Time, Mass, and Length

2.1 Light and Time

Let us think a little bit more about light sender and speedometer resting *relative* to each other, as in Fig. 2.1. Here we omit the speedometer. We put the whole setup into some transparent elevator, like you find sometimes in department stores. We depicted it as light-gray background box.

The man inside the elevator rests relative to the light-sender. He sees the light beam traveling horizontally. This is because the speed of the elevator has no absolute meaning, as we saw. He measures the time it takes for the light beam to arrive at the tip of the arrow with the clock resting beside him.

We, standing at the right, outside the elevator will see the light starting at the sender. While the light moves to the right, for us the elevator moves up, together with the light beam. For example, at half the time it will be half the way up, so it just travels along the diagonal upwards.

This diagonal is *longer* than the horizontal distance. But remember that light travels always at the same speed c along the line, whether measured inside, or from outside the elevator. We see that the **direction** of the light beam changes, but the *magnitude* of the speed of light is for both of us the same. Hence we see the light traveling a *longer* time than the observer resting with the light source:

$$\begin{pmatrix} \text{time of clock} \\ \text{moving relative to us} \end{pmatrix} > \begin{pmatrix} \text{time of clock} \\ \text{resting relative to us} \end{pmatrix} \tag{2.1}$$

That means that light and time are connected: For us outside, time inside the elevator evolves *slower* than ours. When the observer sitting inside the elevator says: "1 second has passed", we outside say: "No, *more* than 1 second has passed". In other words:

> Because the magnitude of the **speed of light is absolute**, the pace of time is *relative* to the speed of the observer.

K. Fischer, *Relativity for Everyone*, DOI 10.1007/978-3-319-00587-4_2,

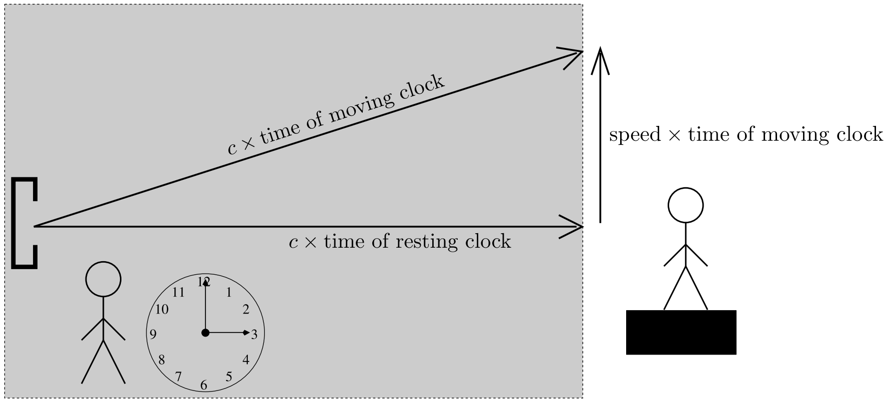


Fig. 2.1 We see the elevator moving upwards. Because speed is distance per time, we can write the distances as speed times time

We use the Greek letter **gamma** "γ" to describe this: It denotes a number between zero and one and connects the larger time of the moving clock with the smaller time of the resting clock:

$$\begin{pmatrix} \text{time of clock} \\ \text{moving relative to us} \end{pmatrix} \times \gamma = \begin{pmatrix} \text{time of clock} \\ \text{resting relative to us} \end{pmatrix}$$

We see: For speed zero, γ is one, because then we agree with the observer in the elevator on our times. The larger the speed of the elevator, the longer the diagonal becomes, so the smaller the factor γ.

Please have again a look at Fig. 2.1. During the time of the moving clock in which the light travels along the longer diagonal with speed c, the elevator travels the shorter vertical line with its speed. Hence its speed is always *less* than the speed of light. We see again:

Matter cannot travel **faster than the speed of light**, relative to us.

2.2 The Gamma Factor

We can actually compute the γ factor. We need only the **Pythagoras theorem** to get the answer. Please have a look at Fig. 2.2. We took the triangle from Fig. 2.1. The theorem of Pythagoras tells us that

$$\begin{aligned}&(c \times \text{time of moving clock})^2\\ &\quad = (c \times \text{time of resting clock})^2 + (\text{speed} \times \text{time of moving clock})^2\end{aligned}$$

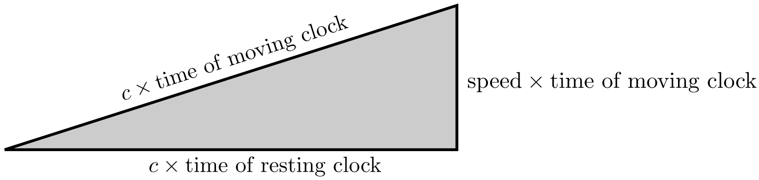


Fig. 2.2 The Pythagoras theorem tells us how the times of the resting and moving clock depend on each other

We want the time of the resting clock, looking from the outside, so

$$c^2 \times (\text{time of moving clock})^2 - \text{speed}^2 \times (\text{time of moving clock})^2$$
$$= c^2 \times (\text{time of resting clock})^2$$

Take the common factor of the "time of the moving clock" on the left out,

$$(\text{time of moving clock})^2 \times \left(c^2 - \text{speed}^2\right) = c^2 \times (\text{time of resting clock})^2$$

Then divide by c^2:

$$(\text{time of moving clock})^2\left(1 - \frac{\text{speed}^2}{c^2}\right) = (\text{time of resting clock})^2$$

Because the square of the moving time and the resting time are never negative, the factor $(1 - \frac{\text{speed}^2}{c^2})$ must be at least zero as well. Hence we see here again, that the speed cannot be larger than the speed of light. Further, we can take the square root, to get the relation between our moving time and the resting time inside the elevator:

$$(\text{time of moving clock}) \times \sqrt{1 - \frac{\text{speed}^2}{c^2}} = \text{time of resting clock} \qquad (2.2)$$

The square root term is the γ **factor**,

$$\gamma = \sqrt{1 - \left(\frac{\text{speed}}{c}\right)^2} \qquad (2.3)$$

We can also get the γ factor in a graphic way: We use again the triangle of Fig. 2.2, and express the lengths of all three sites as multiples of the longest side, that is, we divide all three sides by (c × speed of moving clock). Then the longest side has length one, by construction. The lower side becomes

$$\frac{\not{c} \times (\text{time of resting clock})}{\not{c} \times (\text{time of moving clock})} = \gamma$$

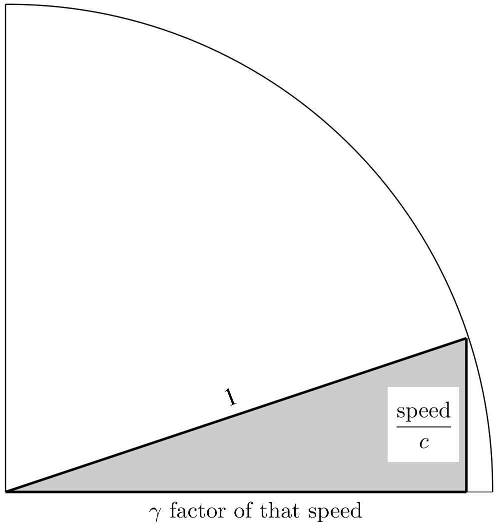


Fig. 2.3 The Pythagoras theorem tells us how the γ factor depends on the speed

and the right side becomes

$$\frac{\text{speed} \times \cancel{(\text{time of moving clock})}}{c \times \cancel{(\text{time of moving clock})}} = \frac{\text{speed}}{c}$$

In other words, γ and $\frac{\text{speed}}{c}$ are the coordinates of a point on a circle of radius one, as you can see in Fig. 2.3. Hence we can read off the size of the γ factor also from this figure. We see again that as the speed decreases to zero, γ increases to one.

This factor γ appears nearly everywhere in the theory of relativity, once you start to calculate things. Sometimes it is easier to use γ than to use the speed.

How much differs the γ factor from one for ordinary speeds on earth? An airplane travels at roughly 1000 kilometers per hour, that are one million meters per 3600 seconds, or roughly 300 meters per second, that is 3×10^2. The speed of light is 3×10^8, so the airplane travels roughly at 10^{-6}, that is one part of a million of the speed of light

$$\frac{\text{speed}}{c} \approx 10^{-6}$$

Now square this: It becomes practically zero, namely one part in a million-million, that is 10^{-12}. Hence the gamma factor is practically one for ordinary speeds on earth. Can we conclude from this, that the theory of relativity does not play any role for nature around us on earth, as sometimes people state? No! We will see in Chap. 3, that even at speeds of less than one millimeter per second we can easily observe the time slip!

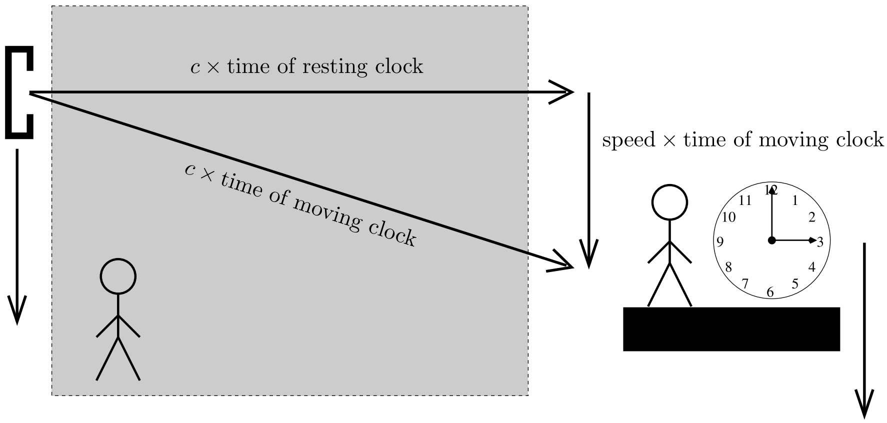


Fig. 2.4 We inside the elevator see the observer outside moving down, together with the light-sender

For small speeds, we can estimate the γ factor, as you will find in Sect. A.3. The result is

$$\gamma = \sqrt{1 - \left(\frac{\text{speed}}{c}\right)^2} \approx 1 - \frac{1}{2}\left(\frac{\text{speed}}{c}\right)^2 \tag{2.4}$$

We list the most important **properties of the gamma factor**:

1. For zero speed, γ is one.
2. The larger the speed, the smaller γ.
3. For nearly the speed of light, γ is nearly zero.
4. For very small speed, γ is smaller than one by a factor which is in proportion to the square of the speed.

2.3 Whose Clock Is Running Slower?

Let us now place the light-sender *outside, left of* the elevator, and let us be inside, as in Fig. 2.4. Let the elevator move upwards. Then the outside observer will see the light beam passing horizontally, and we see it passing downwards. For us, the light travels the longer, diagonal path, so we conclude that for us the *time of the observer standing outside of the elevator* is running slower. However, in the last section we saw that for the outside observer, the time *inside* the elevator runs slower than his own!

Compare this with Fig. 2.1. Then whose time is *really* running slower?

Answer: The question is wrong! It is the kind of question like: “When is the air feeling colder: At night, or outside?” We cannot decide, because we cannot *compare*

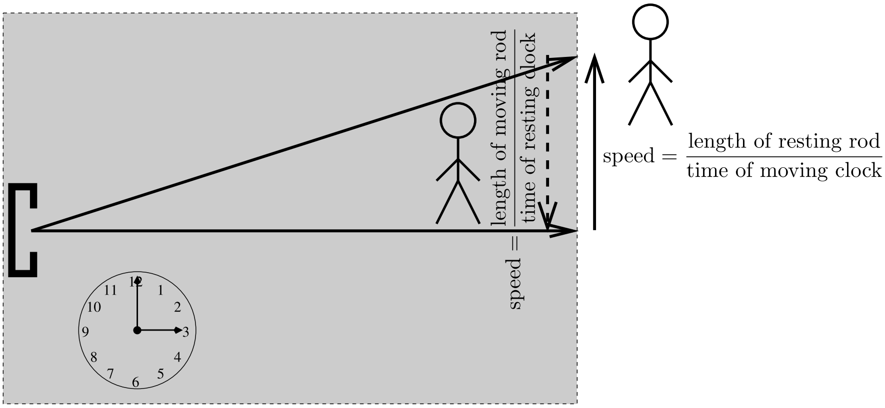


Fig. 2.5 Measuring the relative speed in terms of own length and time differences

the two situations. The same goes here: If the inside and outside observer want to compare their clocks, one of them, or both, have to *change* their speed to stop near each other. *Then* they can compare their clocks. However, by changing speed our clocks will change their pace! We will see in Chap. 4 what will happen then. For the time being, the observers pass at each other and *continue* to travel at a steady speed. Hence *both* are correct in their statements, that the other one's clock is going slow.

2.4 Light, Time, and Length

2.4.1 Length in Direction of the Speed

How do the observers measure the speed of the setup in Fig. 2.1? At first, we as the outside observer: We put down some rod in front of us. We sketched the rod as the solid black arrow pointing upwards in Fig. 2.5. Then we measure the length of the rod.

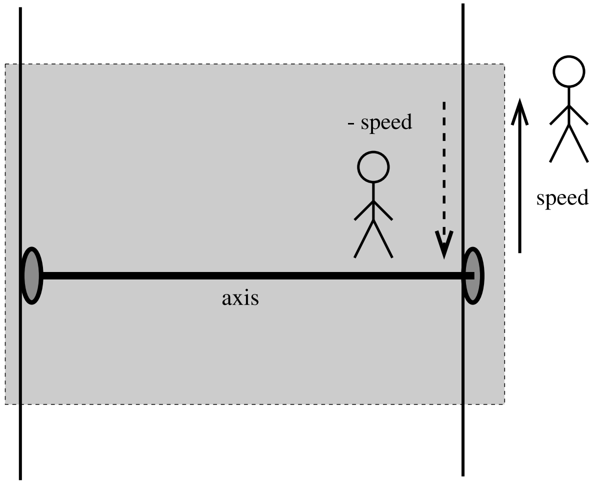
Because the rod rests relative to us, we call this the "length of resting rod". Then we measure the time it takes the setup to pass the rod. We chose the time of the *moving* clock, measured by us, resting. Hence we measure the speed

$$\text{speed} = \frac{\text{length of resting rod}}{\text{time of moving clock}}$$

Let us say that the resting rod has a length of one meter and that the elevator moves in 1 second one meter upwards, so that the speed is one.

What does the observer inside the setup measure? Because speed is relative, he can insist that *he* rests, but the rod outside and we travel with the same speed *downwards*. We indicated this with the dashed downwards pointing arrow. For him, the

Fig. 2.6 The length of a rod at right angles to the movement does not change



speed is

$$\text{speed} = 1 = \frac{\text{length of moving rod}}{\text{time of resting clock}}$$

If for us outside one second on the moving clock is passing, for the observer inside the elevator, on his resting clock only less than one second, namely $\gamma < 1$ second has passed. Hence for him, the length of the moving rod must be *shorter* than for us, by the same γ factor, to get the same speed "one":

$$\left(\begin{array}{c}\textbf{length of rod moving}\\ \textbf{in direction of speed}\end{array}\right) = (\text{length of resting rod}) \times \gamma \tag{2.5}$$

2.4.2 Length at Right Angles to Speed

What happens at right angles to the movement? In Fig. 2.6, we drew also the rails on which the elevator moves up and down. We sketched two wheels and one axis. Both observers measure the *same* length for the axis.

Why? For the outside observer, the rails always rest. Their distance is just the length of the *resting* axis. If the elevator moves up, then for the observer inside, the axis has still the same length, because speed is only relative, and he can insist that he rests. If the outside observer would measure another length of the axis than the resting length, this would mean that for him, the axis would be longer or shorter than the distance of the still resting two rails: The elevator would derail! This is absurd. Hence:

Lengths at right angles to speed do not change.

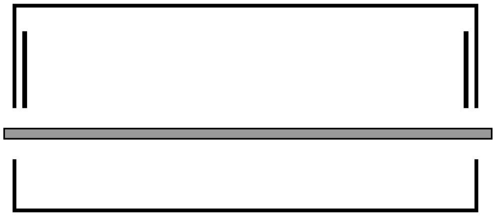


Fig. 2.7 The resting pole is longer than the barn

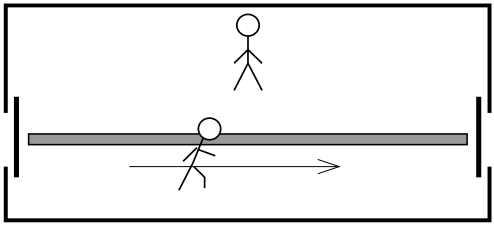


Fig. 2.8 The moving pole is shorter than the barn, for us standing inside the barn

2.5 At the Same Time?

Let us see how the relativity of time and length play hand in hand, with the help of a famous thought experiment called “The pole and the barn”. Here at first light does not enter, only a pole enters a barn, and apparently contracts:

A friend carries a gray pole, slightly longer than the barn in which he wants to place it. The barn has an electrical front door at the left, and an electrical back door at the right. At first, we check that the resting pole is really longer than the barn, as in Fig. 2.7.

Then the friend takes the pole out of the barn, and runs very fast through it, from the left to the right. Because of the contraction we described in Sect. 2.4, the pole looks for us standing in the barn *shorter* than the barn, if the pole is only fast enough. We can arrange in advance that both doors close at the *same instant of time*, and open shortly afterwards. If we chose the right moment for closing and opening the doors, the barn is for a short moment *completely inside the barn*, as we see in Fig. 2.8.

However, what does our running friend see? For him the *barn* is moving towards him, so the *barn* is shorter for him. Hence the pole *does not fit* into the barn at all! By which miracle can the pole for some time be completely *inside* the barn, with both doors closed at least for a moment?

The point is here, that for the running friend *at first* the right door closes and opens again, and *afterwards* the left door closes and opens again, as you see in Fig. 2.9.

Why is that? Enter light: Suppose that as we pressed the switch to close both doors, some light flashes at the doors. For us standing in the center of the barn, the flashes occur at the same time. Hence for us, the doors are closing *at the same time*.

What does our running friend see? He also sees the light of both flashes traveling with the speed of light c, as we know. However, the flash from the right door moves a shorter distance to the runner, because he runs *towards* it, while he moves away

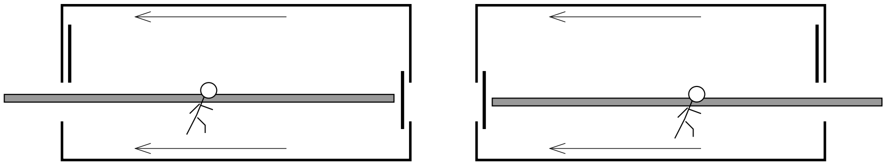


Fig. 2.9 For our running friend, the moving barn is shorter than the resting pole, but for him, the right door closes and opens *before* the left door does

from the flash coming from the left door. Hence he sees the right flash *before* the left flash: He sees that the right door is closing *before* the left door. The reason is twofold: First, the speed of light is absolute, and second, the two events „the backdoor is closing" and „the front door is closing" happen at *different* places. We conclude:

> The statement "**At the same time**" is not an absolute true statement for things happening at different places: It depends on the relative speed of the clock and the observer.

2.6 Time and Mass

While we were standing outside the moving elevator in Sect. 2.1, we saw that time itself slows down for bodies inside that elevator. What consequences has this for the bodies moving inside? It means that all movements slow down by exactly the same amount. So some property shared by all the bodies inside, must change, at least from our standpoint. The obvious candidate is the inertia of the body: If they all become more inert, they move more sluggishly, as if in slow motion. That fits with our observation before: For the same reason that time slows down, the bodies will never be able to mover faster than light from us away. To coin a catch-phrase:

> For bodies moving relative to us, time evolves slower than they experience themselves. Therefore they look more inert from our point of view: Their **inertia increases as their time** slows down, relative to us.

Let us see how much the inertia grows with speed: Suppose a ball with some resting-mass bounces at right angles, *slowly* at a wall and bouncing back elastically, as in Fig. 2.10.

When the ball bounces back, it changes from its original downwards speed to the same speed, but upwards. It makes no difference if the ball had, say, three times as much mass, or if the same mass was three times as fast, so the "push" the wall receives, only depends on the product

$$\text{push} = \text{mass} \times \text{speed}$$

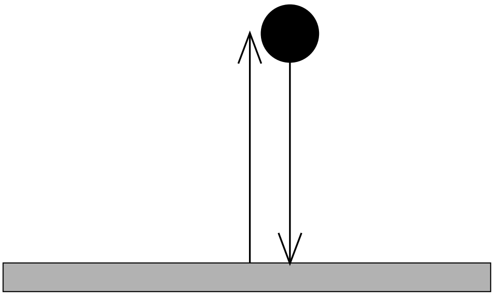


Fig. 2.10 A ball bounces from a resting wall

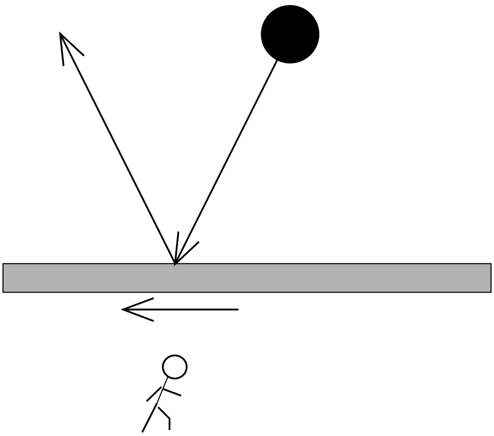


Fig. 2.11 The same ball bouncing at right angles from the same wall, seen by us, moving relative to the wall, along the wall

The wall receives *twice* this push: Once when the wall absorbs the push from the ball, and a second time when the wall pushes the ball elastically *back* with the same speed.

By the way, physicists call this push **momentum**.

Further, the ball was slow enough, so that its mass is, by our experience, nearly unchanged, that is, nearly the resting-mass,

$$\text{push} = (\text{resting mass}) \times \text{speed}$$

Next, we imagine ourselves traveling fast *along* the wall, that is, parallel to it, as in Fig. 2.11. That is, relative to us, the wall travels from the right to the left.

All the same, the wall will receive the same push, at right angles. Also the distance between the wall and the ball does not change for us, because we learned in Sect. 2.4.2 that lengths at right angles to the motion do not change. The only difference is now, that we see the time of the ball proceeding more slowly, by the factor γ. Hence we see that the speed of the ball is *slower* by this factor. Therefore, to maintain the *same* push, the mass of the ball must be larger by the same amount that its

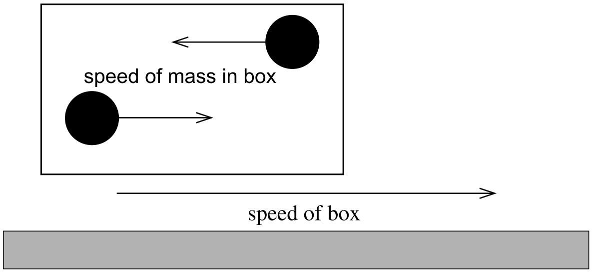


Fig. 2.12 Masses moves inside a light moving box

speed is slower, that is by the same amount that its time is going slower:

$$\textbf{total mass} \text{ moving at speed } = \frac{\textbf{resting mass}}{\gamma} \tag{2.6}$$

We learned in Sect. 1.13 that this growing mass is due to the motion energy which the mass has gained. Therefore this **mass has the total energy**

$$\begin{aligned} &\text{total energy of mass moving at some speed} \\ &= \text{mass moving at speed} \times c^2 = \frac{\text{resting mass}}{\gamma} \times c^2 \end{aligned} \tag{2.7}$$

2.7 Speed Addition

We know already that we cannot travel faster than light. We demonstrated that in Sect. 1.6 with a peppercorn, driven by a rocket. However, maybe there is a detour, *without* needing to accelerate things? Here comes a thought experiment, shown in Fig. 2.12. In a very light, yet strong box there are two balls with equal resting-mass moving to the left and the right, so that we can ignore the mass of the box and concentrate on the mass of the two balls.

This box moves with, say, 70 percent of the speed of light to the right, relative to the ground. Let the upper ball move with the speed of, say, 70 percent of the speed of light to the left, in the box, that is, relative to the box. Let the lower ball move with the speed of, say, 70 percent of the speed of light to the right relative to the box. Then it seems that the lower ball is moving relative to the ground with 140 percent of the speed of light?

Let us count the masses: If we stand inside the box, the masses of both balls are equal, and because of Eq. (2.6) larger than their resting-masses by the factor $1/\gamma$ belonging to the speed of 70 percent of c. These two masses add up to the **resting-mass** of the box, if we look at it from the outside, because we assumed the box itself to have nearly no mass.

Now look at the masses while standing on the ground: The box moves with the same speed to the right as the lower ball inside the box, so its moving mass is again larger by the inverse gamma factor $1/\gamma$. Hence the total mass of the box is twice the resting-mass of one ball, divided by γ^2.

Next look from the ground at each ball separately. The right ball moves with the same speed as the box, but to the left, so it rests relative to the ground. Hence its mass is now just its resting-mass. What is the mass of the left ball? The left ball has just such more mass as the right ball has less, because their masses should add up to the same total mass of the box as before. However, we know that mass increases the more the higher the speed. Hence the left ball must *gain less* speed than the upper ball has lost. In other words, the left ball must be *slower* than twice the speed he has in the box. In fact, it is so much slower that again it is not faster than light! In other words:

Relativistic speed addition
If some box moves relative to the ground, with some speed, and some mass moves inside that box with some speed in the same direction, then that mass moves relative to the ground with a speed which is *so much less* than the sum of the speeds, so that it never can be faster than light.

For the actual calculation, see Sect. A.4. The result is, in terms of fractions of the speed of light

$$\frac{\text{total speed}}{c} = \frac{\frac{\text{speed of box}}{c} + \frac{\text{speed of mass in box}}{c}}{1 + \frac{\text{speed of box}}{c} \times \frac{\text{speed of mass in box}}{c}} \tag{2.8}$$

For our example speeds we get

$$\frac{\text{total speed}}{c} = \frac{0.7 + 0.7}{1 + 0.7 \times 0.7} \approx 0.94$$

which is of course less than the speed of light, because it *must come out* that way!

Chapter 3
Light, Electricity, and Magnetism

3.1 Electric Charge and Speed

We have seen that the theory of relativity provides us with a wealth of new insight about the relations between light, time, space, mass, energy, and other quantities. Some quantities were before supposed to be invariable. For example, before relativity, the total mass involved in some experiment was supposed not to change, or **conserved**, as physicists call it. Another example is the conservation of energy, which we discussed in Sect. 1.9. Relativity knocked both from its pedestal, only to merge energy and mass as different forms of the same underlying thing, and so, as Einstein put it in 1906:[1]

> Nach der in dieser Arbeit entwickelten Auffassung ist der Satz von der Konstanz der Masse ein Spezialfall des Energieprinzipes.

That is to say:

> Mass in the capacity of being energy, is conserved as energy.

It was certainly hard to reexamine such basic concepts as space and time from the very start, things that the philosopher Kant thought to be built into our brain to be able to grasp the world around us. However, there *was* a road map at hand, a blueprint of what was to be expected: That was, and still is, the theory of moving electrical charges, that is, **electrodynamics**. The reason is simple:

> While mass, time, and length change with speed relative to the observer, **electrical charge does not change** at all!

[1] A. Einstein. Das Prinzip von der Erhaltung der Schwerpunktsbewegung und die Trägheit der Energie. Annalen der Physik, volume 20, page 627, 1906.

K. Fischer, *Relativity for Everyone*, DOI 10.1007/978-3-319-00587-4_3,

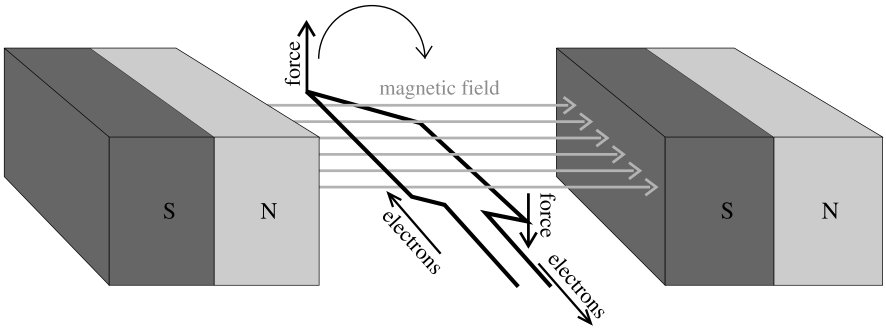


Fig. 3.1 The magnetic field of magnets exerts a force on electrical charges moving relative to it. The magnetic field itself is invisible. This is the basic model for the electric motor

There is even more to it: In Sect. 2.2, we got the impression that relativistic effects are important only at large speeds, because the γ factor differs from one only less than a part in a million-million, for ordinary speed on earth. This is incorrect!

We will see in this chapter that the theory of relativity enables us to explain *magnetism*: It is caused by electric charges, which move much less than a millimeter per second, obeying the laws of relativity. In other words, magnetism is relativity visible at really *low* speed.

3.2 Electric Charges and Magnets

Maxwell created the theory of electromagnetism in the nineteenth century. Before that, electricity and magnetism were considered separate things. However, then people observed things like

> An **electric current** in a wire makes a nearby *magnetic* compass needle move.

and began to have a more closer look, which yielded:

> In an **electric motor**, electric current through wires, makes magnets move relative to those wires. The other way round, in an **electricity generator** magnets moving relative to wires, create electric current.

We sketched the principle of the electric motor in Fig. 3.1. Electric charges, that are in this case electrons, move through the black wire loop. In other words: An *electric current* runs through the wire. The magnetic field exerts a force on these moving electrons.

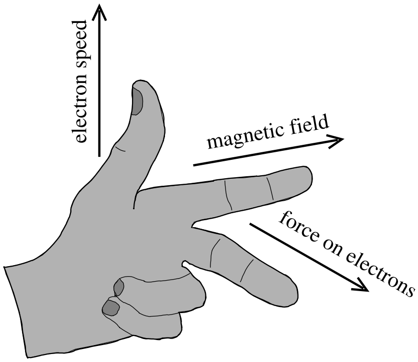


Fig. 3.2 Left-hand rule for negative charges. Spread the thumb, the index finger, and the middle finger of your left hand, such that they stand at right angles to each other. Then turn the left hand such that the negative charge, electrons in our case, move in direction of the thumb, and that the magnetic field points in direction of the index finger. Then there will be a force acting on the moving electrons in the direction of the middle finger. Please try this rule now at the situation shown in Fig. 3.1, by turning your left hand accordingly!

This force is called the **Lorentz force**. The direction of the force we get from the **left-hand rule**,[2] which we explain below Fig. 3.2.

Let us see how the Lorentz force acts for our model of the electric motor in Fig. 3.1. The electrons enter the magnetic field and move to the rear, and the magnetic field points from the north pole of the left magnet to the south pole of the right magnet, that is, to the right. Hence according to the left-hand rule, a force is pushing the electrons *upwards*. Therefore this force pushes the left part of the wire loop upwards.

Then the electrons turn back behind the magnetic field and move to the front. The left-hand rule tells us that there is a force pushing the electrons, and therefore the right part of the wire loop, *downwards*. In total, we see that the wire loop will begin to rotate clockwise. This thought experiment shows, how an **electric motor** works.

In Fig. 3.3, we see the opposite effect at work: Now no current is flowing through the black wire. *We* turn the wire loop clockwise. Then the electrons sitting in the left part of the wire, move upwards. The left-hand rule tells us that there is a force pushing the electrons to the *front*. Likewise, the electrons in the right part of the wire move downwards. The left-hand rule tells us that a force is pushing these electrons to

[2]Some textbooks specify the electrical current to flow in *opposite direction* to the flow of the electrons. Then the left-hand rule becomes a right-hand rule, but of course the physical phenomena do not change.

Some other textbooks use the *Fleming left hand rule*. However, this rule uses the fingers in a different order, and would become a right hand rule in our case: The electrons move in the direction of the right middle finger, and the magnetic field points in the direction of the right forefinger, so that the force on the moving electrons points in the direction of the thumb, that is, again upwards.

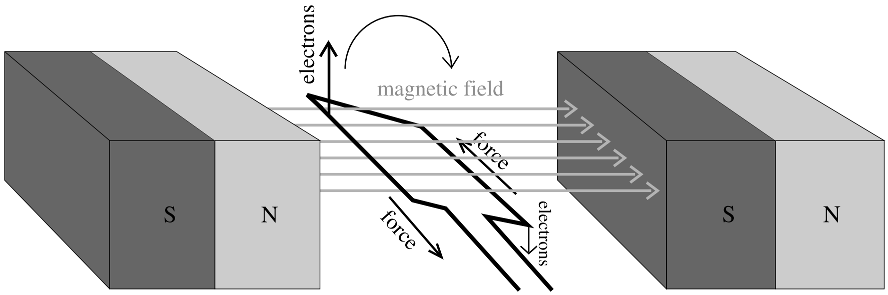


Fig. 3.3 Again, the magnetic field of magnets exerts a force on electrical charges moving relative to it. This time the same setup as before serves as the basic model for the electricity generator

the *rear*. Hence an electric current will begin to flow through the wire. This thought experiment shows, how an **electricity generator** works.

3.3 Electric and Magnetic Fields

What is more, by carefully observing and interpreting the experiments, Maxwell concluded that the energy of a magnet or electric charge spreads out in space, as an **electric field** or **magnetic field**. Indeed, the wire in Figs. 3.1 and 3.3 does not touch the magnet, but acts or reacts on the invisible magnetic field of the magnet.

If an electric field varies with time, it creates around it a certain amount of magnetic field, and vice versa. He summarized his findings in his **Maxwell equations**.

Actually, the Maxwell equations *alone* do not tell us how a mass carrying an electric charge *reacts* to an electric or magnetic field. This law is logically independent of the Maxwell-equations. The force which an electromagnetic field is exerting on a charged mass is called the **Lorentz force**. The Maxwell equations together with the Lorentz force together make up **electrodynamics**.

We saw the Lorentz force in action in Figs. 3.1, 3.2, and 3.3.

After convincing himself that his equations described the experiments so far correctly, Maxwell pushed his new theory to the extreme: Even if there are *no* electric charges or magnets around, in empty space, suitably varying electric and magnetic fields can sustain themselves. However, they *cannot rest* at the same place, but must travel at a certain fixed speed. Maxwell could even calculate this speed from his equations and found that it was the same as the speed of light! Hence light is nothing but a wave of varying electric and magnetic fields, called an **electromagnetic wave**.

Maxwell's great achievement was not only to unify electric and magnetic phenomena under one roof, but also to show that light itself is an electromagnetic phenomenon. Therefore he actually unified *three* types of phenomena: Electric, magnetic, and optic.

What is equally important: The absolute, fixed speed of light comes out of the Maxwell equations as constant of nature *without further ado*. We already mentioned

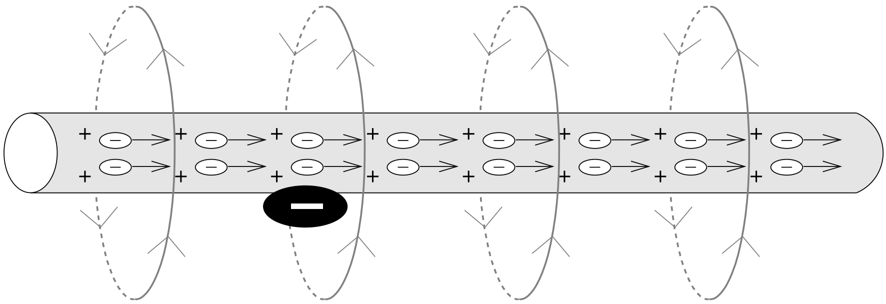


Fig. 3.4 A steady current of electrons moves in a straight, long wire from the left to the right. The negative, black charge with the white minus-sign rests in front of the wire, and remains there

that electrical charge does also not depend on the speed of the charge, and therefore is absolute. In short, electrodynamics fits in with the theory of relativity *from the start*.

Therefore we can use electrodynamic phenomena as road map to build the theory of relativity. We do so using a thought experiment, with an electrical current running through a wire, creating a magnetic field. In fact, this thought experiment was the starting point for Einstein in his article about the theory of special relativity. Its title is "**On the electrodynamics of moving bodies**", and now you understand why!

3.4 Magnetic Field from Electric Current

In Fig. 3.4 we sketched part of a long, straight wire made of some metal. The wire itself rests relative to us. Through the wire flows a steady electric current, that is, electrons from the left to the right. We chose for our thought experiment a wire of very low electrical resistivity, for example by freezing the wire to a very low temperature, so that the atoms nearly freeze out and do not disturb the electrons moving. We assume from now on that there is no electrical voltage needed in the wire to sustain the flow of the electrons.

By the way: The electrons in a typical metal wire are really slow: They move at less than a tenth of a millimeter per second!

In Fig. 3.4 we sketched the electrons of the electrical current as white ovals with a black minus-sign. In the wire some of the atoms of the metal provide for the moving electrons, so that they now miss an electron. The charge of these atoms is therefore now positive. We sketched these atoms as black plus-signs. They rest in the wire. The total wire is electrically neutral, that is, it carries no *net* charge.

We can check that by putting a, say, negative charge *resting* in front of the wire, like the one we sketched as black oval with white minus-sign. Nothing happens: The black charge will not begin to move.

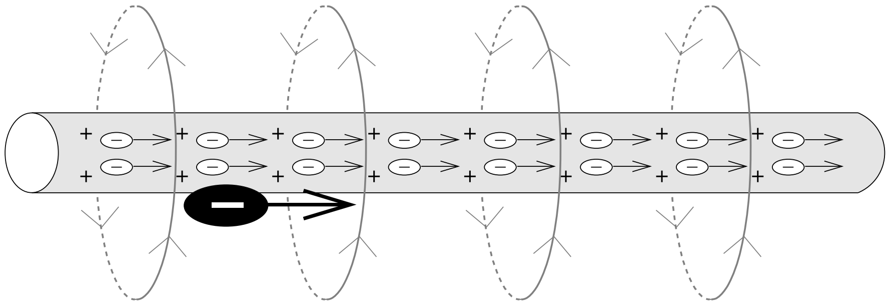


Fig. 3.5 The negative black charge moves to the *right*. Then the magnetic field of the wire attracts this charge

However, the electric current in the wire produces a magnetic field around the wire, the direction of which we sketched by the lines going around the wire. Then the **left-hand rule** of Fig. 3.2 tells us that the magnetic field *attracts* the black electrical charge if that charge *moves relative* to the field *to the right*, as in Fig. 3.5. Experiment tells us, that this **Lorentz force** grows in proportion to the speed of the black charge. Experiment tells us also, that this force grows in proportion to the current as well, that is, in proportion to the speed of the electrons. Therefore *if* the black negative charge is moving with the *same* speed as the electrons to the right, then the magnetic field of the wire attracts the negative black charge with a force which grows in proportion to the *square* of this speed.

3.4.1 The Faraday Paradox

Next, we move at the same speed as the outside negative charge and the electrons to the right. Hence for us now the electrons of the current in the wire as well as the black negative charge outside are *resting*. However, the positively charged atoms move with the same amount of speed to the *left*. In other words: The wire itself moves to the left, as sketched in Fig. 3.6.

Let us at first only use the **first law of relativity**: The physical effects do not depend on whether the observer moves with steady speed in one direction. We see now a current of *positive* charges of the same size moving at the same speed to the *left*, as we saw before moving to the *right* as *negative* charges. Hence these positive moving charges produce the same magnetic field. Thus we have the same situation as before. In particular, we are again *resting* relative to the magnetic field!

In other words: In the situation of Fig. 3.5, we rested relative to the wire, and relative to the magnetic field. However, in the situation of Fig. 3.6, we move along the wire, but still rest relative to the same magnetic field as before! This raises the question: Is this "magnetic field" we drew around the wire, a real, physical quantity?

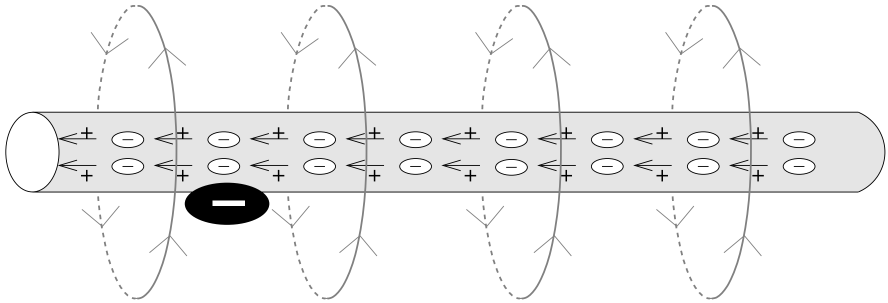


Fig. 3.6 Now the positive charges of the wire move to the *left*. We and the outside negative charge rest again with the magnetic field

After all, it is invisible, and seems to rest even if we move along the wire! This is called the **Faraday paradox**.[3]

In order to check the magnetic field, we now look at the force it exerts on the black negative charge.

3.4.2 No Attraction Without Relativity

Let us again at first only use the first law of relativity: In Fig. 3.5, the outside negative charge moves relative to the magnetic field, so that the magnetic field attracts it. However, in Fig. 3.6, the outside negative charge does not move relative to the magnetic field, so the magnetic field will *not* attract it! This is weird, and exactly this was the situation when Einstein published his article about the "electrodynamic of moving bodies".

3.4.3 Attraction with Relativity

Next we use also a consequence of the **second law of relativity**, that is to say, the relativity of lengths in direction of speed of Sect. 2.4.1. We know that the resting wire was electrically neutral. This means that per meter, say, the number of positive charges on the atoms and the number of negative charges were the same. As moving observer we see now:

1. The charge *per* electron or atom does *not* change. That is an experimental fact, suggesting already that relativity and electrodynamics fit.

[3]Many textbooks use a rotating magnet for this thought experiment.

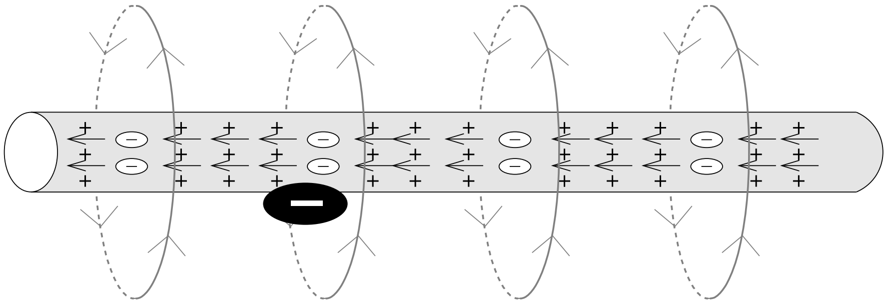


Fig. 3.7 The moving wire has now a net positive charge

2. Suppose that we put a rod parallel to the wire, resting with it. Because we now move relative to this rod, the rod looks *shorter* for us, so the same number of positive charges fits into a shorter distance. Hence per meter, relative to us, there are now *more* positive charges than before.
3. Suppose that a rod moved with the same speed as the electrons to the right. Because we now rest relative to this rod, it looks *longer* for us, so that the same number of negative charges fits into a longer distance. Hence per meter, relative to us, there are now *less* negative charges of the electrons than before.

Altogether the wire has now per meter a net positive charge. We sketched this in Fig. 3.7, overdoing it for better view.

We also know from Eq. (2.4) that the γ factor for slow motion differs from one by an amount which is in proportion to the square of the speed. Hence the net positive charge on the wire is also in that proportion. Therefore now the positively charged wire *attracts* the outer negative charge, because that is what charges of the opposite sign do, in proportion to their charges. Hence the wire attracts the positive charge with a force in proportion to the square of the speed.

Compare with the resting wire of Fig. 3.5: In both cases the force attracts, and in both cases the force grows in proportion to the square of the speed! Hence we can guess already, without any calculation, that the two situations are identical.

We see that with the help of the theory of relativity we can understand electrodynamics: We just *explained* the magnetic Lorentz force and the left-hand rule! Magnetic fields are nothing but electric fields from moving charges, acting upon moving charges.

Without knowing quantum theory, we can guess already that in a permanent magnet, there must be moving charges to sustain the magnetic field!

When estimating the γ factor we assumed that the electrons or atoms only move slowly, but that is not necessary: From both points of view of the observer, the wire attracts the outside negative charge in *exactly* the same amount.

What about the Faraday paradox? Because there are in Fig. 3.7 more positive charges moving with the same speed to the left, than there were in Fig. 3.5 negative charges moving to the right, the electric current is now stronger. Hence the magnetic field is *not* the same as before: It has *increased*!

We have seen in this chapter that relativity is *not* only about large speeds and spaceships: It is also about *very slow* bodies—for example, the electrons in a wire move with less than a millimeter per second! And it is about everyday electrical appliances: Magnetism itself *is* a consequence of the theory of relativity.

Chapter 4
Acceleration and Inertia

Let us come back to the theme of the first chapters: Sending light at different speed. Up to now, the sender and observer moved steadily, with speed both constant in magnitude and direction. The part of the theory of relativity dealing with these phenomena is the **theory of special relativity**.

However, to arrive at some speed, or to change the direction, we must **accelerate**. We ask: How does acceleration of a body affect time, length, mass, or energy? This will eventually help us understanding gravity.

4.1 Rotating Motion: Twin Paradox 1

We begin with the simplest case, in which speed changes its direction, but not its magnitude, as on the merry-go-round in Fig. 4.1. The merry-go-round has the form of a disk, with a hole in its center. We placed the merry-go-round on such a small planet, that we can neglect its gravity.

Suppose we ride at the rim of the rotating merry-go-round. The merry-go-round pulls us constantly inwards, along a circle. As it is, our speed *relative* to the ground always changes, that is, it changes its direction: We **accelerate by rotation**. We feel our inertia, because mass itself resists acceleration. If we let go, we would slip off the merry-go-round, along a *straight line*, as our inertial mass prefers to do: We would change into an **inertial state**, in which we continue to move steadily, with speed both constant in magnitude and direction.

What if we place ourselves with a clock at the center of the merry-go-round, on the ground? Then we do not turn around. Nothing accelerates us. We are in a state of *inertia*. For this state we know by *experience* that time moves on steadily. We see a friend with his clock riding on the rim of the merry-go-round with some speed. For a very short time interval, we can think of our friend at the rim going nearly straight ahead. We use our knowledge about moving clocks and conclude that the clock at the rim will *run slow*, and this by the γ factor of Eq. (2.3).

K. Fischer, *Relativity for Everyone*, DOI 10.1007/978-3-319-00587-4_4,

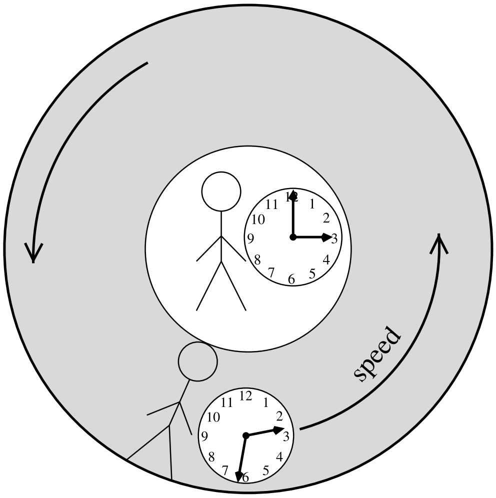


Fig. 4.1 Measuring time on a merry-go-round

Because the clock at the rim does not move away from us, we can wait as long as we wish, seeing the clock at the rim slowing down against our clock in the center more and more. This leads to a weird thought experiment: Suppose that both observers were identical *twins*. At first, they both rest in the center of the merry-go-round. Then one of them steps on the merry-go-round and moves to the rim. He rides there some time, and then returns back to the center. Certainly his clock will react somehow during the time in which he moves towards the rim, and afterwards when he returns. However, that does not depend on what happens while he is riding on the rim which he can do as long as he wishes. In the end, when he has returned after a long time riding on the rim, he is *younger* than the twin who waited all the time in the center!

This is the famous **twin paradox** or **clock paradox**. In fact, it is no paradox, but measured reality!

For example, in 1977 **muons** were sent along a circular tube of diameter 14 m, at the European Organization for Nuclear Research **CERN** in Geneva. Muons with negative electric charge are a kind of heavy version of an electron. They decay into electrons and other particles. A muon *resting* near us has a lifetime of about $2 \cdot 10^{-6}$ s.

However, inside the "merry-go-round", that is the circular tube, the muons circled with a speed of $v = 0.99942 \cdot c$. Hence the γ-factor is about $\frac{1}{29}$, so that they should live about 29 times *longer* than when resting near the observer. This is precisely what was found![1]

[1]You can download the original paper here: http://cds.cern.ch/record/929453/files/ep63_001.pdf. Their γ is our γ^{-1}.

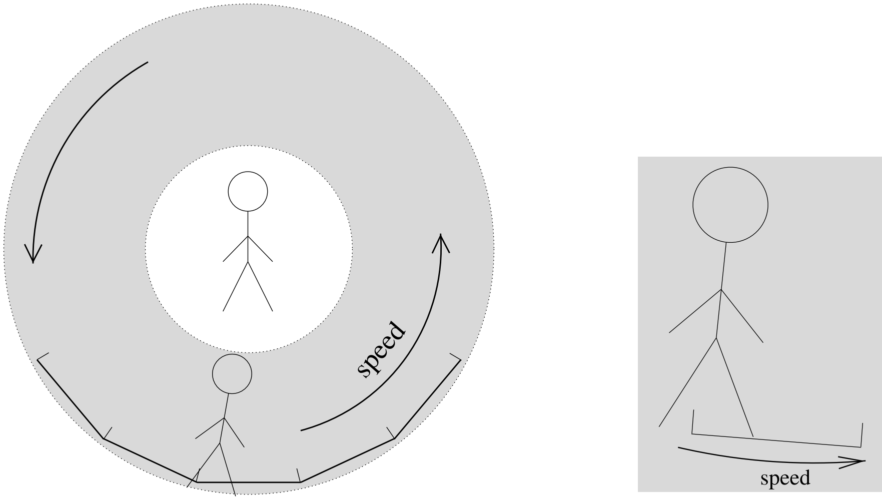


Fig. 4.2 Measuring lengths on a merry-go-round. Short lengths along the rim look nearly straight, as we see in the closeup on the *right*

4.2 Rotating Motion: Not the School Geometry

Next, our friend at the rim wants to check out how lengths are effected by the rotation. We in the center produce many small rods and give them to our friend at the rim. The rods are so small that by joining them he can estimate the length of the rim precisely enough.

We know from Sect. 2.4.2 that lengths at right angles to speed do not change. Therefore we agree with our friend that the diameter of the merry-go-round does not change as it turns around.

Then our friend measures the length of the rim of the merry-go-round, as in Fig. 4.2. A small part of the rim looks nearly straight, if we only zoom in enough, as in the right part of the figure. Hence for small enough rods our friend can estimate the length of the rim, in terms of the number of rods it takes to exhaust the rim.

Suppose at first that the merry-go-round is at rest. Our friend sends a light beam along the rod as his feet. The light beam passes his feet at the speed c. This speed is the length of the rod, divided by the "very short time" it takes for the light to pass along the rod. Therefore the length of the rod is c times this "very short time". This "very short time" he reads off a clock he is carrying, and such he knows the length of the rod. Because he counted the number of rods that exhaust the rim, he will conclude that the length of the rim is π times the length of the diameter, as we expect from **school-geometry**.

Next, for the rotating merry-go-round, the light beam will still pass along the rod at his feet, because the rod lies in direction of the rotation speed. The light beam passes at the rod still at the same speed c, because the speed of light is absolute,

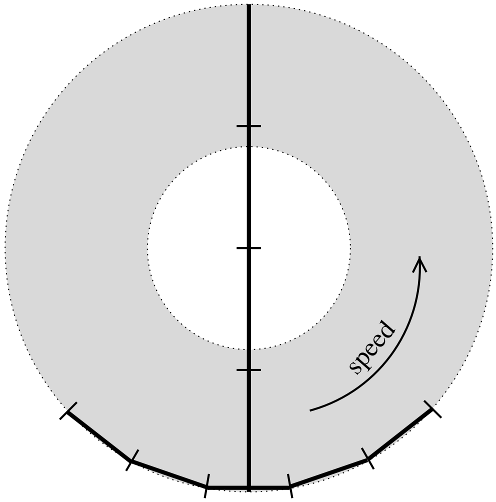


Fig. 4.3 The rim is longer than π times the diameter, when measured as number of rods

and for the "very short time" our friend travels nearly straight ahead, along the rim. He reads off his clock the "very short time" and finds that the light beam has not made it along the rod, but traveled only along some part of it! Why is that? For our rotating friend, time is passing *slower*, that is, in *shorter* intervals than for us resting in the center, shorter by the γ factor of the rotation speed. Hence the light beam will not make it during the "very short time" he reads off the clock. However, for our friend the length of the rod is still the speed of light c times the time it takes the light beam to travel along the rod. In other words, for him, the rod has become *longer*.

However, there are still the same number of rods exhausting the rim. Therefore the *ratio* between the length of the rim and the diameter is for him *larger* than π! To be precise: The ratio is by a factor $1/\gamma$ larger. Hence our friend decides to shorten the rods by the factor γ. Then he needs now *more* rods to exhaust the rim of the disk. In Fig. 4.3, we overdid the effect, for better view.

Up to now, when bodies moved with constant speed straight ahead, we saw that time and length among others changed relative to the speed of the observer. However, at least we still could use **school geometry**, or what mathematicians call **Euclidean geometry**. For example, we calculated the γ factor using the **Pythagoras theorem** in Sect. 2.2, which is part of Euclidean geometry. This geometry is a rigid systems of theorems, so when one of them ceases to be true, many others will fail as well. One of these theorems is that the ratio of the length of the boundary and the diameter of a circle is always π. In our case of the rotating merry-go-round, we saw already that this is not any longer true: The rim of a rotating merry-go-round is longer than its diameter times π! We only can conclude, that when we are under **acceleration**, the **school geometry is incorrect**!

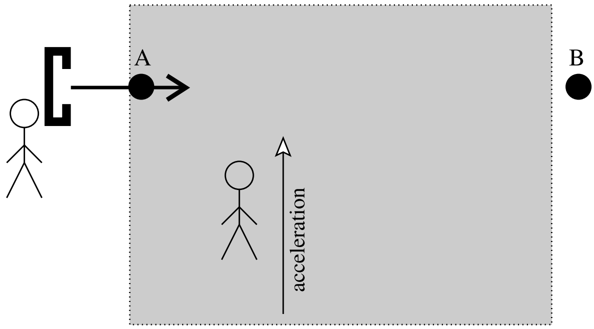


Fig. 4.4 We send a light beam through an accelerating, transparent elevator

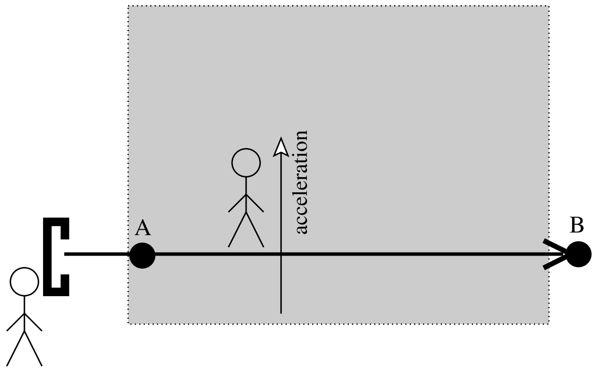


Fig. 4.5 We outside are in an inertial state, so that light travels along a *straight line*

4.3 Straight Motion

Under acceleration, the weirdest things happen. This is not limited to rotating motion: Let us now put the light sender to the left of a transparent elevator, as in Fig. 4.4. The upwards pointing arrow shows that the elevator is **accelerating upwards**. We stand outside, and see the elevator passing by. At this instant, we send a light beam through point A.

We see this beam passing through the glass walls of the elevator, in a horizontal, straight line, from point A to the point B, as in Fig. 4.5. This is the fastest way some body can travel from A to B, because nothing is faster than light.

What does our friend in the elevator see? Please have a look at Fig. 4.6. It shows the moment when the light is leaving the elevator. As the light passes through the elevator from point A on, the elevator gathers speed, upwards, so that for our friend, the light *bends* downwards. He sees the light beam leaving the elevator at some point B further downwards.

We drew a straight, dotted line on the wall of the elevator between the points A and B. Is this path not *shorter* than the path the light has chosen? If so, can we not send some signal along that shorter, straight path, *faster than light*?

The answer is: The question was wrong! We drew the straight, dotted line *before* the elevator began to accelerate. Then for us time proceeds at an even pace. Once the elevator is accelerating, time inside the elevator slows down more and more.

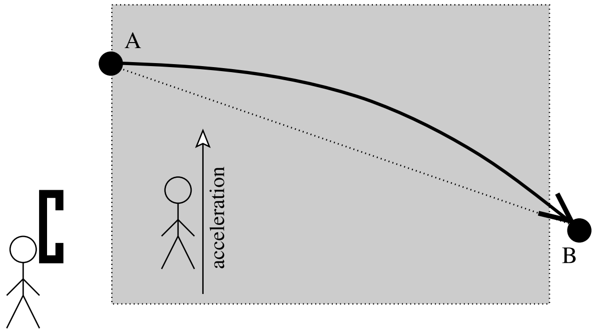


Fig. 4.6 Light bends for the observer under acceleration

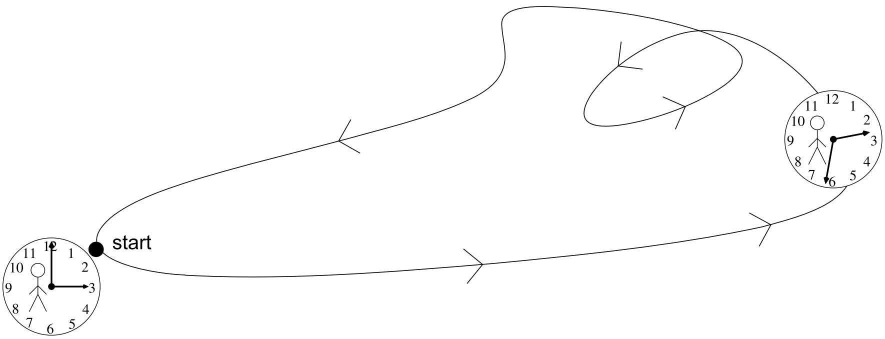


Fig. 4.7 No matter what course the twin takes: After returning home, he finds that his clock is running slow

Straight lines will deform, because vertical lengths will shrink, while horizontal lengths will remain the same. Further, we *cannot* even measure "the" length of the dotted line, because while we measure, time and lengths will distort even more! We sit in **space-time**, not only in space!

We see that the easiest way to handle rods and clocks, is while in a state of inertia: Time passes steadily on, and lengths do not shrink. Once you begin to accelerate, lengths can change as time proceeds, Euclidean geometry becomes wrong, so that things get complicated.

4.4 Proper Time and Inertia: Twin Paradox 2

The time for the twin riding the merry-go-round in Sect. 4.1, proceeded slower than the time for the resting twin. This time, measured by one's own clock, we call the **proper time** of the observer or a body. Hence we can say: The proper time for the resting twin passes steadily on, *faster* than the proper time of the twin who left the center of the merry-go-round.

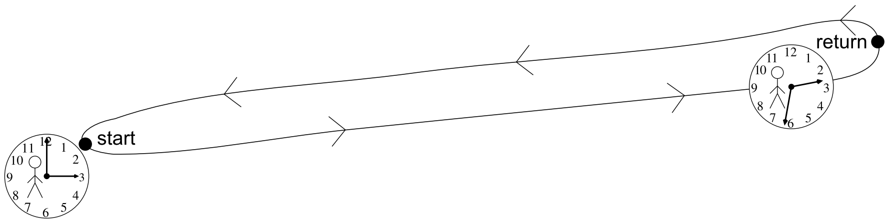


Fig. 4.8 The clock of the left twin remaining at the start, remains in the same state of inertia and therefore is running faster

Now let us repeat this thought experiment in another way, as shown in Fig. 4.7. At first, both twins rest near each other, at the start, not feeling any acceleration. Their proper time passes steadily on.

Then one twin decides to *leave* the place. Hence he has not only to leave the *place*, but also this **state of inertia**: He must change his speed to get away. He travels on a route like the one in the figure, and after he returned, he finds that his clock is running slow. We can push this thought experiment to its extreme:

In Fig. 4.8, the traveling twin accelerates at first to a certain speed, then he travels a long way with this speed, straight away, in an inertial state. Then he gently returns, again traveling nearly all the way home at the same constant speed straight away, in an inertial state, and finally accelerates to stop at the start beside the first twin. Certainly his clock will react somehow during his initial and final acceleration, or when he returns. However, that does not depend on what happens when he is traveling at constant speed back and forth. The longer the distance to the point of return, the longer he travels straightly at constant speed back and forth, and the more his clock slows down.

Therefore we can neglect these accelerations: The time of the traveling twin advances less by the γ-factor of his mainly constant speed, in comparison to the time of the twin who remained at the start.

However: Is it really during these long trip in the inertial state, that the traveling twin's clock is going slow? He may say: "I only saw my twin brother going away from *me* to the left. I certainly was in a state of inertia for a long time, during which I saw my twin brother moving away from me at constant speed, straight away, and later approaching me, again at constant speed. So during that time my state of inertia is as good as his, so from what I learned in Chap. 2, I reckon that *his* clock slowed down, not mine!"

This is again the **clock paradox** or **twin paradox**.

However, his opinion is in error: We compare the clocks *after the traveling twin returned, and is again resting with the other twin*. It is *irrelevant* what he *thinks* would happen in the meantime, because that was not the question, only which time he accumulated in the end.

In fact, *even* the traveling twin *can observe* that his clock does only advance the fraction γ of the remaining twin's clock! Why is that? Let us think about the *distance* between the start and return points in Fig. 4.8. The twin remaining at the start measures this distance, for example by sending a light beam to the return point, where a mirror reflects the light back to the start point. From the time the light was traveling, he calculates the distance. And from this distance and the speed of the traveling twin, he finds the time it takes the traveling twin to return to the start. And of course, that is the time that passes *for the twin remaining at the start*.

Now, what distance will the traveling twin measure? We saw already in Sect. 2.4.1, that distances in direction of speed will *shrink* by the γ-factor of that speed. And because the clock of the traveling twin does advance always at the same pace *for him*, he indeed needs only the fraction γ of the time which the twin at the start observes, because he has to travel *less a distance* than what the remaining twin measured.

In short: For the twin remaining at the start, the *time* of the traveling twin is passing slower, by the factor γ, while for the traveling twin, the *distance* is shorter by this factor γ. So there is no paradox: It all fits!

Let us sum up: The only, but crucial difference between the two twins is, that the traveling twin *changed* his state of inertia. Turning tables, this means that as long as we drift freely in space, in the same state of inertia, our proper time evolves always *fastest*. To coin a catch-phrase:

> Because matter wants to remain inert, it moves without force such that it takes the **longest proper time possible**.

Chapter 5
Inertia and Gravity

Matter drifting in empty space resists being accelerated: it has inertia. However, near large masses like the earth, for example, mass seems to accelerate towards the large mass. Starting from this simple observation will help us to understand better how to deal with acceleration and gravity.

Gravity we feel everyday, beginning when we get up in the morning, when we go upstairs or downstairs, or drop a cup, or such like. It is too common to be astonishing. And yet gravity is one of the most mysterious things in the universe, if we think a little bit about it. For this we need pure gravity, not marred by air drag, so let us make a thought experiment, similar to what high school teachers really do: We put a bird feather and a heavy metal ball into a glass tube, and close the tube at both ends with some plug. Through one of the plugs we drive a thin tube connected to a vacuum pump, and pump the air out of the glass tube, and seal it. Then we hold the air void tube vertically, and overturn it fast. The result will be as shown in Fig. 5.1.

The ball and the feather fall *equally fast down*! On earth we have to get rid of the air to see this, but on the moon, astronaut David Scott from the Apollo 15 mission did exactly that: He dropped at the same time a hammer from his right hand and a feather from his left hand, and both reached the ground at the same time. Just search the Internet for the keywords "hammer feather Apollo", and you even can see a movie from the original television capture.

Physicists repeated this kind of experiment, with more and more accuracy, and all kinds of materials including even subatomic particles, over the last hundred years or so, and always found this result:

> All matter *reacts* in the same way to the same gravity.

This is astonishing, because it shows that inertia and gravity are somehow connected. Let us sort it out: Inertia of a 1 kg-stone, say, you can measure on an ice skating rink, by pushing the mass, as we already saw in Fig. 1.10. This has nothing to do with gravity: You can do the same in empty space, away from earth, and still mass will resist to being pushed.

K. Fischer, *Relativity for Everyone*, DOI 10.1007/978-3-319-00587-4_5,

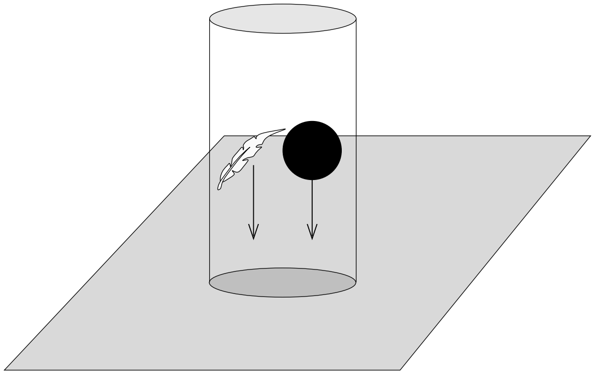


Fig. 5.1 A bird feather falls as fast as a metal ball inside a void tube

Or we can *weigh* the stone. Again you will find “1 kg”, this time as weight, or **gravitational mass**. Here the stone on the scale is at rest, so it does not resist any acceleration, rather it *wants* to **accelerate**, to fall towards the center of earth.

The point is now, that when you take a 1 kg-piece of some material and another 2 kg-piece of another material, as measured on the ice skating rink, then the first material will *weigh* exactly 1 kg, and the second will *weigh* exactly 2 kg. In other words, their inertial mass and their gravitational mass are *exactly* equal. Why, if the second materials would weigh, say, 3 kg, then in the glass tube it would fall *faster* than the first material. Nobody has seen such a thing up to now:

Inertial and gravitational mass are exactly equal for all materials.

5.1 Gravity Is No Force

So what exactly happens when the feather and the ball fall down the tube? The gravitational mass of the ball, say, tries to accelerate the ball downwards. At the same time, the equally large inertial mass resists this acceleration. The same goes for the feather. Now comes the point: Because inertial and gravitational mass are exactly equal, the ball *does not accelerate at all*! The ball is **free-falling** and therefore in an inertial state.

However, if you are standing nearby, you may argue: But I *see* the ball accelerating towards the earth! Answer: No, *you* are accelerating away from earth! While standing on the ground, you *yourself* feel that upward **acceleration**, and therefore your imagination is hampered by all kinds of weird things which can happen if we are in an accelerated state, as described for the merry-go-round and the elevator in Chap. 4.

So in order to really sort it out, we had better remain in a state of inertia. To clarify things, we make a number of thought experiments, using *elevators*, in a similar way as Einstein originally did. Please have a look at Fig. 5.2.

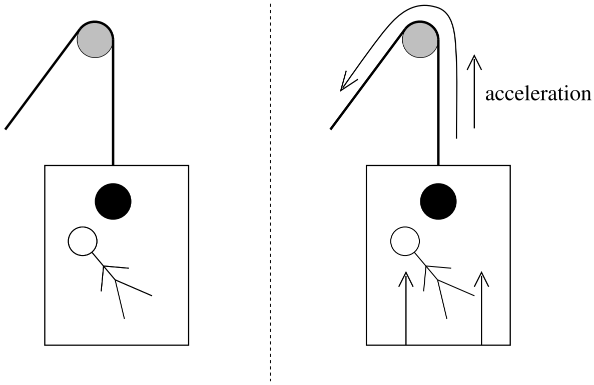


Fig. 5.2 Suddenly someone pulls the elevator with the rope

In the picture on the left, we sketched an elevator floating in empty space, far away from any planet, star or other large mass, and no air inside. The astronaut inside the elevator in his spacesuit and the ball nearby float freely as well. Then suddenly someone pulls the elevator, so that in the picture on the right the astronaut sees the floor of the elevator accelerating up towards him. However, the astronaut as well as the ball *still are floating freely*. No force is pulling at them! Hence he reckons that someone is pulling the rope, accelerating the elevator upwards.

Or does he? What if in reality it all happens as in Fig. 5.3? This time nobody pulls the rope, but suddenly a planet sneaked under the elevator. Still the astronaut does not feel *any acceleration*, and the nearby ball does not move relative to him. This is because the inertial and gravitational mass of the astronaut and the ball are exactly equal. He and the ball are still in an inertial state, free-falling! Rather he sees again the floor of the elevator accelerating towards *him* and the ball. However, the observer standing on the surface of the planet, sees the astronaut and the ball falling downwards.

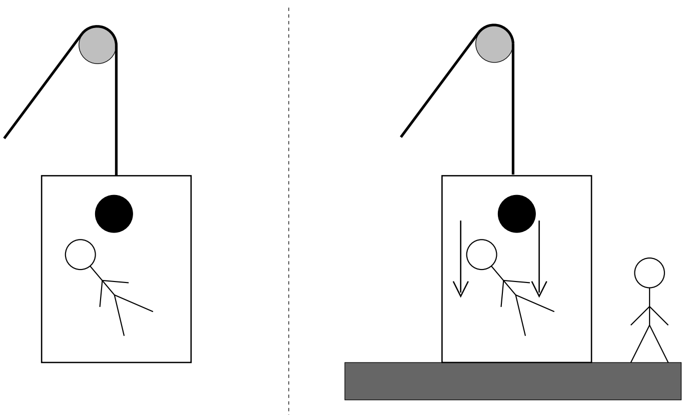


Fig. 5.3 Suddenly a planet appears under the elevator

By measuring acceleration *alone* the astronaut in the elevator cannot decide from inside the elevator which of the two cases are true. He has to *look outside* the elevator to find out.

This is the famous *equivalence principle* by Einstein:

Equivalence Principle

By measuring our acceleration alone we can in no way decide whether gravity acts upon us. In particular, **free-falling** masses are in an **inertial state**, as if they would float freely in empty space.

The equivalence principle will be our handy tool to understand gravity. Because free-falling masses are in a state of inertia, their proper time is proceeding steadily, and we can use our gathered knowledge about this state to explore the surroundings of large masses, to find out how gravity works: The equivalence principle links inertia with gravity.

Let us begin by again considering the merry-go-round of Sect. 4.1: For an observer riding on the rim of the merry-go-round, Euclidean geometry does not work, and his clock is running slower than the clock of someone resting outside in an inertial state.

We saw that this was the case because he constantly is changing the direction of its speed relative to the center of the merry-go-round. Now compare with what happens on the surface of earth: We on the surface of the earth constantly accelerate upwards, that is, away from the center of earth. In other words: We are not in an inertial state. Therefore we can expect that for us under gravity Euclidean geometry does not work as well, and that time will run slower than for an observer resting far away from earth.

However: The acceleration we feel both while riding on the merry-go-round or while standing on earth, *as such cannot be* the reason for the clocks to slow down! For we saw in Sect. 4.1 that in case of the merry-go-round, for example, the clock riding on the rim goes slow depending on its rotation *speed*, *and not its acceleration:* A smaller merry-go-round with the same rotation speed would produce the same effects, but an observer riding it would feel a *larger* acceleration.

We have at first to understand in more detail how gravity works.

5.2 Gravity Bends Space-Time

Image a rocket going around earth, with its engine switched off. The rocket is **free-falling**, not towards earth, but around it, as sketched in Fig. 5.4. We place ourselves resting far enough away from earth, so that we are nearly in an inertial state (for the meaning of "far enough", see Preface). By the equivalence principle, the rocket is in a state of inertia as we are. However, we see that the rocket does not travel along the dashed straight line. What forces the rocket around the earth, along the bended path, drawn as solid line?

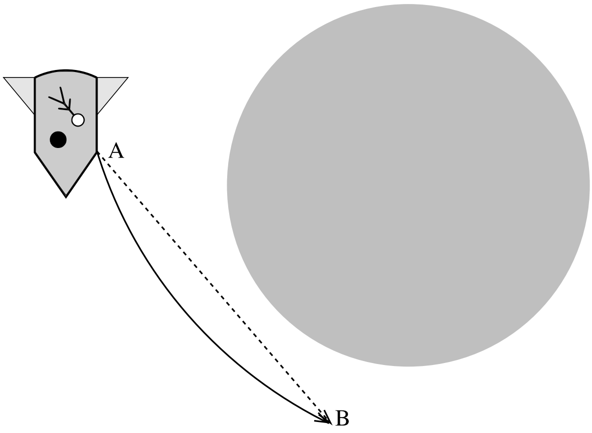


Fig. 5.4 Rocket free-falling around earth

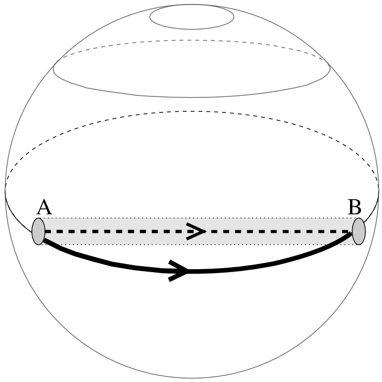


Fig. 5.5 On a bended surface, the shortest path between two points usually bends as well

5.2.1 Bended Surface

To find the answer to that question, think of an analogy: Chose two points A and B, somewhere in empty space, far away from any planet or star or any other large mass. Try to find the shortest path between these points. Clearly this is the straight path connecting them. How can we trace this path? Start at A, head towards B, and keep going *straight ahead*.

Next, suppose the two points A and B are along the equator in the Pacific Ocean, say, with A being west of B. Again try to find the shortest path between the two points. The shortest path between the two points is a tunnel along the straight dashed line connecting them, as sketched in Fig. 5.5.

If we restrict ourselves to the earth's surface, the shortest path is the solid line, which is part of the equator: It *bends*. How can we trace this path? Start at A and keep *straight ahead*, eastwards.

For short distances, the ocean surface looks *flat*. Then keeping straight ahead means for short distances, to the horizon perhaps, to follow a straight path. Hence

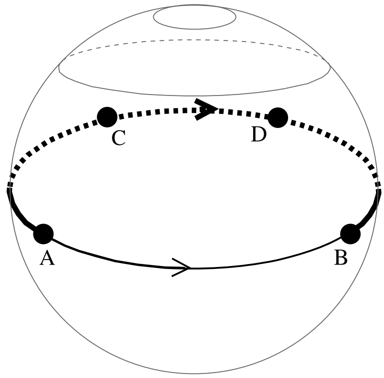


Fig. 5.6 On a bended surface, the straightest path between two points is sometimes not the shortest path of all

the shortest path connecting two points on a bended surface is not straight, but the **straightest path**.

Such shortest paths are always straightest paths, but on a bended surface only short enough straightest paths are necessarily the shortest paths: Let us again start at A and keeping *straight ahead*, moving *westwards*, along the equator, following the fat line in Fig. 5.6. This path is a straight line for short distances, and up to some not-too-far away intermediate point C, it is the shortest of all possible paths connecting A and C. This you can see in Fig. 5.7, by comparing it for example with the gray path connecting A and C. If we follow this straightest path further on, it is again the shortest path between points C and D, and afterwards between points D and the endpoint B. Hence this straightest path is the shortest of all paths which cross it a short-enough intervals, like the total gray path in Fig. 5.7.

Hence the total path is the straightest path from point A to point B, if we start in a westward direction. However, this path is of course longer than the eastward path we chose in Fig. 5.5. We will encounter such a straightest path in space-time in Sect. 6.2.

Such a straightest path is the natural analog to the straight line in a plane. Its name is a **geodesic**. The name tells: *geodesy* is the science of measuring and mapping the surface of earth...

5.2.2 Bended Space-Time

In the same way, the free-falling rocket travels over short distances nearly along a straight path, with constant speed. For longer distances, the path can bend, and the speed can change, as we have seen. However, this path from A to B in space is usually *not* the straightest path in space, as is clear from Fig. 5.4. Then what does gravity bend?

> Gravity does not only bend space, gravity **bends** space *and* time, called **space-time**.

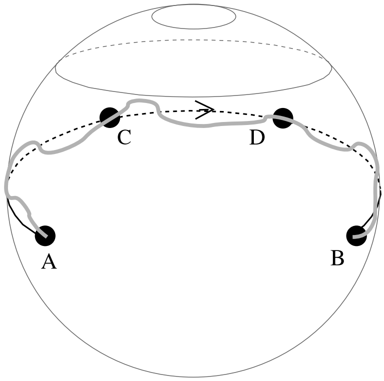


Fig. 5.7 The straightest path is shorter than all **nearby paths**, that is, which cross it again and again at short enough intervals

Let us see how.

From Sect. 4.4 we know that when we rest at some point without accelerating, that then our clock is running *fastest* of all clocks which at first rest with us, then move away a little bit and return to us, resting again with us, after some time. Now comes the equivalence principle into play: We in the rocket in Fig. 5.4 do not feel any force pulling at us, while free-falling along the solid curved line from A to B. Hence we can insist that we *rest*. Therefore any clock which after resting with us, has moved a little bit away at point A and returned at point B, again resting with us, will run slower than our clock. Likewise we in the rocket *can* move along the dashed line, but then we must switch on the rocket motor and *accelerate*: We must leave our present state of inertia. Before leaving, we place a clock next to our rocket, which continues to free-fall from A to B, along the solid curved line. When we meet again and rest relative to this clock in B, then our clock is running *slow* against this clock which traveled along the bended path free-falling in an inertial state.

Next we need to know what a "nearby" path in space-time means, in analogy to Fig. 5.7. It must be a path which crosses the original path in short-enough intervals. If we move on the original path, and our friend along such a nearby path, then in short enough intervals, he crosses us. It means that relative to us, he *stops over* at us. Such a path we call a **space-time nearby path**.

Therefore we can use as "length" of a path in space-time the *proper time* it takes to pass through it. The *straightest* path between A and B is the path which takes *most* proper time of all *space-time nearby* paths, with the same starting and arriving speed.

Can it happen, that while we travel on such a straightest path, that our proper time is running *slower* than for our friend, who left us some time ago, and just joined us again, in analogy to a straightest but not shortest path on a surface in Sect. 5.2.1? This is possible, if our friend does not follow a space-time nearby path, as we will see in Sect. 6.2.

Such space and time are connected to space-time: Just drawing some line between two points A and B in *space* does not mean anything: In addition, we must fix the starting speed and at least in a thought experiment *travel* from A to B to find out what the *straightest* path is. It will be a *path* on which we can travel force-free,

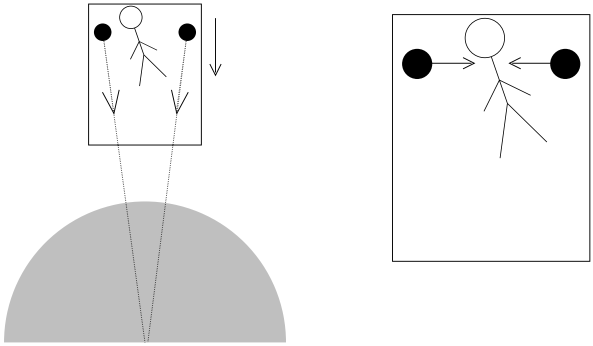


Fig. 5.8 Balls inside a large enough, free-falling spaceship begin to move *towards* each other

that is, **free-falling**. The time it takes depends also on the starting speed in bended space-time, and we use it as the "length" of the path in space-time.

All matter is free-falling in the same way, if it is light enough, so that it practically does not influence other masses with its gravity, and if it is small enough, so that its movement traces out a line. Such a mass we call a **test-mass**, like a for example a small clock. Under gravity, the test-masses will move relative to each other and show us how a piece of space *deforms* as time proceeds. Such we can see how space-time is bending. We show in the next section how this works out.

Let us sum up:

> Gravity bends space-time. The **straightest path** between two points is when a test-mass is **free-falling** between them, that is, when the **proper time** for a test-mass traveling between the points with given initial speed, is passing **fastest, compared with space-time nearby paths**. This path is a **geodesic in space-time**.

5.3 Measuring the Bending of Space-Time

We said in Sect. 5.1 that to detect gravity we have to look outside the elevator. All the same, we can use a large enough spaceship, free-falling to earth, without air inside, as we sketched in the left picture of Fig. 5.8.

On the left and right hand of the astronaut are two balls, which are falling together downwards towards the center of earth, that is, parallel to him. However, gravity acts in direction of the center of earth. Hence the two balls both begin to head towards the center of earth, approaching each other. In the picture on the right, we see that for the astronaut, the two balls will begin to move *towards* each other.

Again, we can make the analogy to the straight-ahead motion on the bending surface of the earth. In Fig. 5.9, two observers starting at the equator and moving straight-ahead northwards, will meet at the north-pole, so they move towards each other.

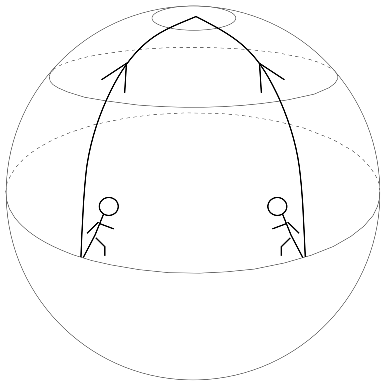


Fig. 5.9 Moving parallel along two straightest lines on a ball, two observers can move towards each other

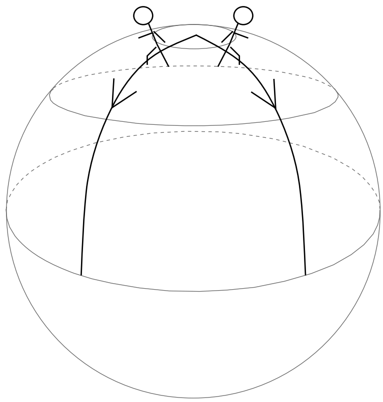


Fig. 5.10 Moving parallel along two straightest lines on a ball, two observers can also move away from each other

However, there is more to this analogy: In Fig. 5.10, two observers move straight-ahead towards the equator. Hence they move away from each other. An analog thing happens in the spaceship: In the left picture of Fig. 5.11, we placed the balls in the free-falling spaceship on top of each other. The upper ball is farther away from the center of earth than the lower ball, so that gravity acts more strongly on the lower ball.

Inside the spaceship, in the right picture of Fig. 5.11, we observe therefore that the upper and lower ball will begin to *move away* from each other!

How does the whole picture fit together? Let us place even more balls as test-masses inside the spaceship, resting relative to each other, so that they mark the corners of an *imagined* shoe-box. Again we suppose for simplicity, that inside the spaceship there is no air. We sketched one side of the box in the left picture of Fig. 5.12 in light gray. In the picture in the center, we see both effects working at the same time: The left and right balls move towards each other, while the upper and lower balls move away from each other. In other words, the imagined box will gain height, but lose width, and in the third dimension depth, as we see in the picture on the right.

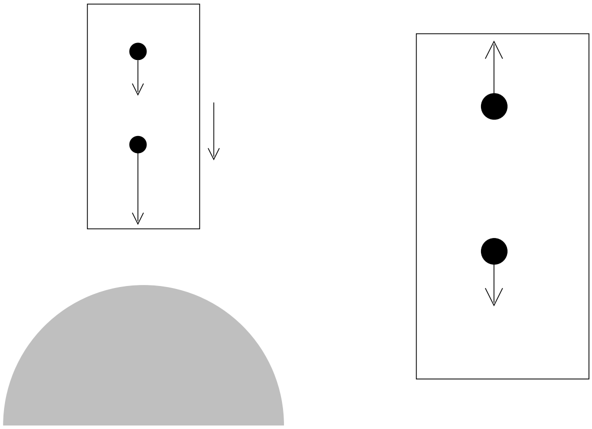


Fig. 5.11 Balls inside a large, free-falling spaceship on top of each other, begin to move *away* from each other

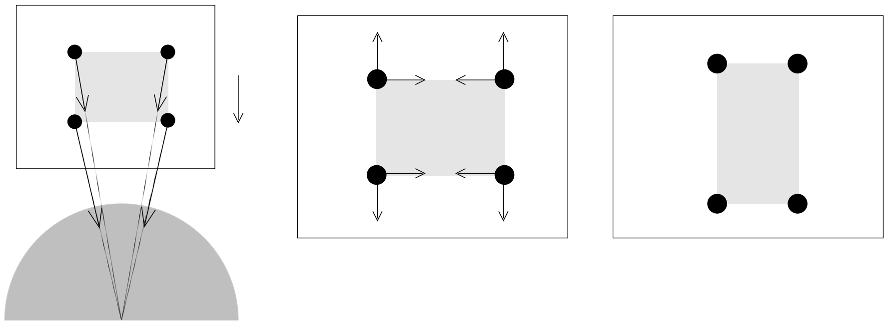


Fig. 5.12 Inside a large, free-falling spaceship the form of a small piece of space begins to change, but not its volume

Now if you measure this carefully, you will find that the *volume* of the box does not change! Can we understand this intuitively? If the imagined box would contain mass, then the box would *begin to shrink*, because gravity tends to clump mass together. However, we marked a box with *test-masses* that do *not* disturb the neighborhood. Hence the box *contains no mass*. The gravity acting from *outside* masses like earth will *deform* the box, but not *shrink* it.

This is the essence of what gravity does. We will show how exactly mass *creates* gravity in Chap. 7. However, at first we have to understand how mass *reacts* on gravity in more detail.

Chapter 6
Equivalence Principle in Action

6.1 Time and Gravity

How time reacts on gravity? Please have a look at Fig. 6.1. We put one clock on the surface of a planet. To make things simpler, we chose a planet which does not spin. We put the other, left clock far enough away (for the meaning of "far enough away", see Preface), where we can neglect the gravity of the planet. We place it resting relative to the planet, so that this left clock is nearly in a state of inertia. We arrange that at some instant, both clocks show, say, two o'clock, as you see in Fig. 6.1. We ask: Does the clock on the planet advance at the same pace as the left clock far away from the planet?

In order to compare the pace of the two clocks, we move the left clock to the right clock, *without* changing its state of inertia. This is possible because the equivalence principle tells us that a free-falling clock *is* in an inertial state. Therefore we chose to free-fall together with the left clock towards the planet, starting at, say, two o'clock. Then for us, the planet with its clock on it, is moving *towards* us. The horizontal arrows show the speed. In Fig. 6.1, the planet with the right clock resting on it, just started to move *relative* to us. In Fig. 6.2, the right clock nearly reached us. When we pass near the right clock, we can compare the two clocks. Our clock was always in the same inertial state, so it proceeded at the *same* pace as it did far away from the planet. However, we see that the clock on the planet is moving relative to us at a certain speed, when we pass it. Hence we conclude that the clock on the planet *is running slow* against our clock.

In other words:

> **Gravity of a large mass slows down nearby clocks**. If we let another clock start from a place resting far away from the large mass, free-falling towards it, and passing a clock resting near that large mass, then the γ factor of this speed is the rate at which the clock near the large mass is running slow.

We use this thought experiment to *calculate* the slowing rate in Sect. 8.5, when we solve the Einstein equation of gravity.

K. Fischer, *Relativity for Everyone*, DOI 10.1007/978-3-319-00587-4_6,

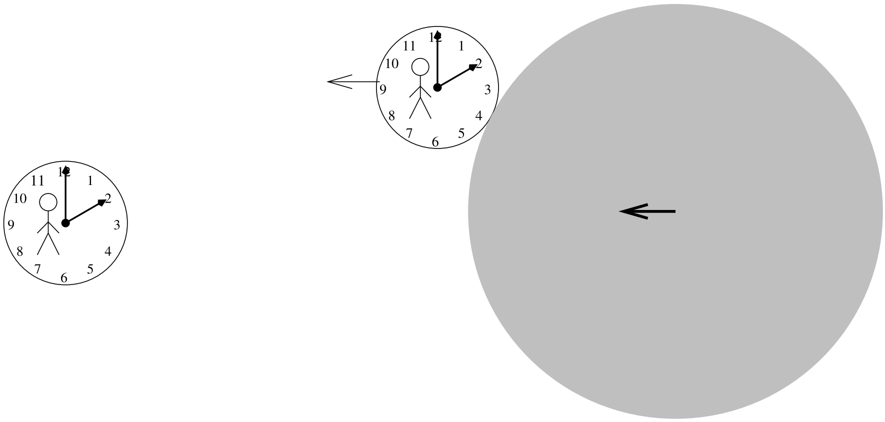


Fig. 6.1 The *left* clock begins to free-fall from far away towards the planet

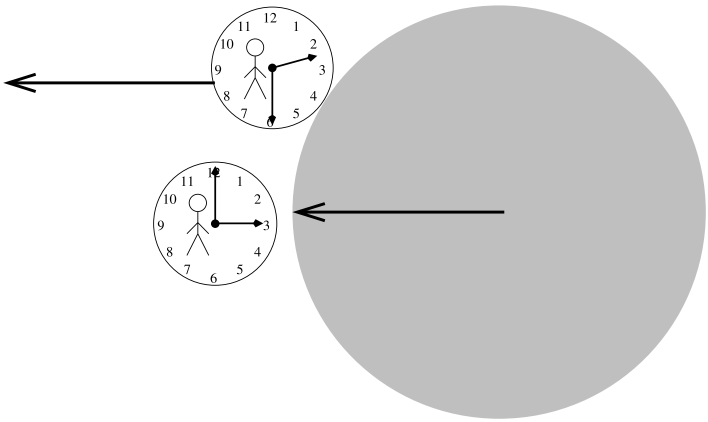


Fig. 6.2 The *left*, free-falling clock passes the *right* clock which rests on the planet

In practice, we just use three very precise clocks, plus a high rise building. We adjust the clocks, and put one clock on the ground floor of the building on this planet, and one on the upper floor. Then we free-fall again from far away, and pass at the high-rise building, as in Fig. 6.3. We pass the upper clock with some speed, and the lower clock with some larger speed. Hence we see that the lower clock is proceeding at a slower pace: Time on the upper floor of the high-rise building is proceeding faster than on the ground floor![1]

[1]This has been checked in 2010 by physicists C.W. Chou, D.B. Hume, T. Rosenband, and D.J. Wineland from the National Institute of Standards and Technology, USA. See for example http://www.sciencedaily.com/releases/2010/09/100923142436.htm.

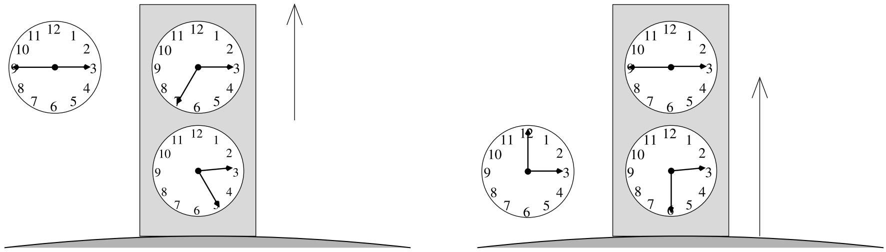


Fig. 6.3 Clocks in a high-rise building

Some popular texts claim, that it is the *acceleration* we feel on the planet, that slows the clocks down, by comparing it with clocks in a starting rocket. This is incorrect. Compare with the merry-go-round: The clock of the observer riding on the rim slows down according to his rotation *speed*. We can build two merry-go-rounds, one with a smaller diameter, and one with a larger diameter. We let them both rotate with the *same* speed. Then the *acceleration* on the small merry-go-round is *larger* than on the larger merry-go-round, but the time delay of the rotating clocks is *the same*. Likewise, we can find two planets, one with roughly double the mass and diameter of the other, such that they have *different* acceleration due to gravity, but the *same* time delay of clocks. The exact condition we will learn in Chap. 8.

6.2 Proper Time in Bended Space-Time: Twin Paradox 3

In Sect. 5.2.2 we learned that the proper time of a free-falling test-mass is passing fastest, compared with *space-time nearby* paths. However, what can happen if two twins who were at first together, move along completely different paths in bended space-time? For this please have a look at Fig. 6.4. At first, the two twins free-fall around a planet, as you can see in the left picture of Fig. 6.4. Then the white-headed twin accelerates and stops at the platform, as in the right picture.

The white-headed twin sees again and again the dark-headed twin passing at him, encircling free-falling the planet. Finally, when the dark-headed twin passes the platform again, the white-headed twin rejoins the dark-headed twin. While the dark-headed twin continued to free-fall, the white-headed twin was always accelerating: He clearly accelerated in order to depart from and to rejoin the dark-headed twin. What is more, while he was standing on the platform, the white-headed twin constantly felt the gravity of the planet, that is, he constantly accelerated also there.

Let us compare the proper time of the two twins. We carry a clock and start free-falling vertically from a resting place far away from the planet. We fall such that we pass at the same time both the dark-headed and the white-headed twin, as you see in Fig. 6.5. We see the white-headed twin moving upwards towards us with some

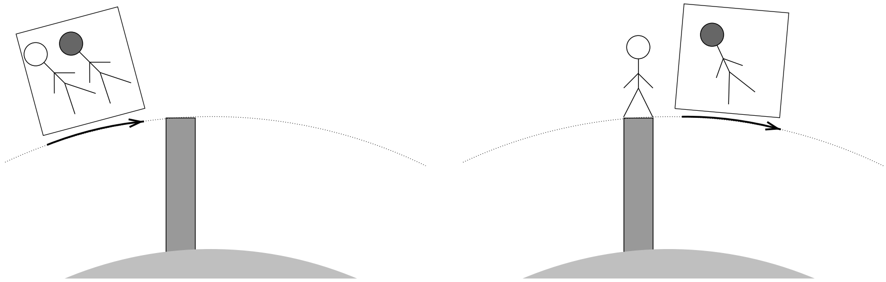


Fig. 6.4 The dark-headed twin continues to move free-falling in a circle around a planet, while the white-headed twin stops at the platform

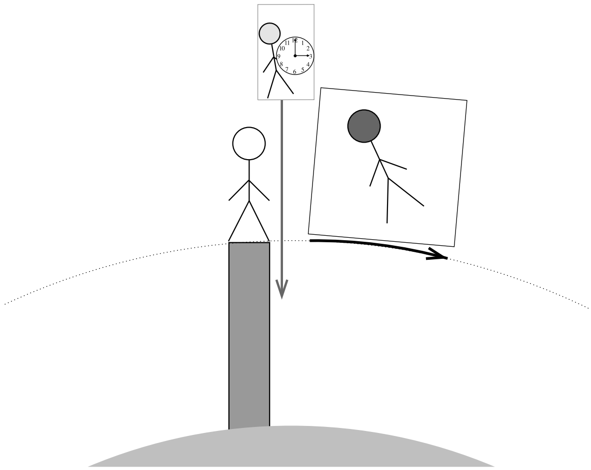


Fig. 6.5 We are carrying a clock and are passing at the same time vertically the white-headed twin standing on the platform, and the free-falling dark-headed twin. The *arrows* show the speed, as seen by the white-headed twin

speed. However, the dark-headed twin moves in addition to the vertical speed also at some horizontal speed, that is, *faster* than the upwards moving white-headed twin, relative to us. Hence the proper time of the dark-headed twin passes *more slowly* than the proper time of the white-headed twin standing on the platform.

During the stopping and starting from the platform the proper time of the white-headed twin will change somehow, but because the white-headed twin can wait on the platform as long as he wishes, he can make sure that when he joins the dark-headed twin again, his proper time proceeded *more* than the dark-headed twin's proper time.

This is then again an example of the **twin paradox** or **clock paradox**. However, here the time of the twin remaining in the inertial state proceeds *slower* than the time of the twin who changes his state of inertia, in contrast to what happened in Sects. 4.1 and 4.4.

In other words: Although the dark-headed twin moved along the *straightest* path in space-time between leaving and joining the white-headed twin, his proper time did not pass fastest of all paths. This is in analogy to the straightest westward

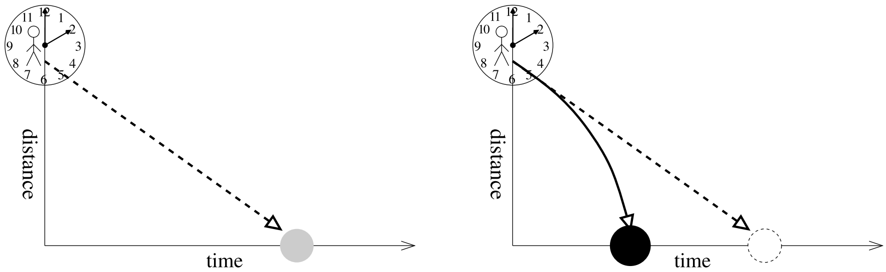


Fig. 6.6 Bended path in space-time

path along the equator in Fig. 5.6. Similar to that figure, the dark-headed twin does not move on a **space-time nearby path** to the white-headed twin, but "westwards".

6.3 Moving Straightly in Bended Space-Time

In Sect. 6.1, the clock free-fell along a straight line. However, in bended space-time this should be a curved line, in analogy to Fig. 5.4. Let us see why: In the left picture of Fig. 6.6 we sketched a very light planet as a gray disk. This planet creates nearly no gravity. For better view, we drew the planet smaller than the clock. The two directions in this picture are not width and height, but *time* and distance in space-time. We see that the clock moves towards the light planet, along a *straight* line in space-time, at an even pace: When one third of the time-span until impact time has elapsed, the clock traveled one-third of the distance to the planet. When two-thirds of the time until impact time has elapsed, the clock traveled two-thirds of the distance and so on.

In the picture on the right-hand side, we have a situation similar to the one in Fig. 6.1: Now the planet has mass, and we sketched this as dark black disk. We let the clock start at some initial speed, which is the same initial speed as in the left picture. The clock will **accelerate relative** to the planet while approaching it. It follows now the curved solid line, instead of the straight dashed line, reaching the heavy planet in less time than the very light planet before.

We see although the clock free-falls along a straight line in *space* towards the center of the planet, the clock moves along a bended path in *space-time* near a gravitating mass.

6.4 Length Under Gravity of a Perfect Ball

In order to see how lengths change under gravity, let us **model a planet or star**. Physicists try to construct for physical phenomena the simplest possible models,

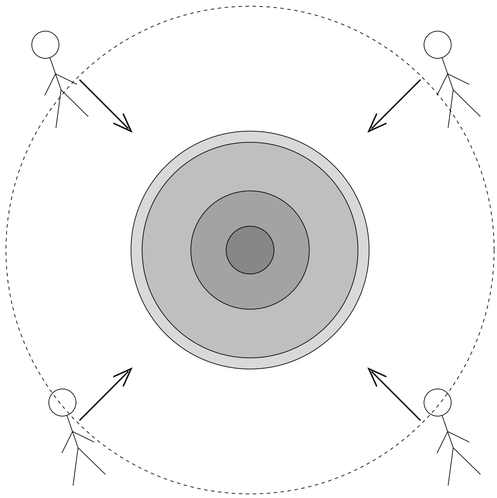


Fig. 6.7 In a perfect ball, mass density depends maybe on depth, but not on direction. *Darker colors* mean layers of higher mass density

which still contain the essence of the phenomenon. Let us do the same: We know that planets or stars are nearly balls. Their mass may vary with depth, but in all directions there is nearly the same amount of mass. This we sketched in Fig. 6.7. For real planets and stars, this is a good assumption. We call such a body a **perfect ball**.

Let us imagine in empty space the surface of a ball, which has the same center as the perfect ball. We sketched it in Fig. 6.7 as dashed line. Next, we let several observers free-fall towards the perfect ball, from far away, were they rested relative to the ball, at the same large distance. They measure time and length in this state of inertia. Because the mass of the perfect ball is the same in all direction, gravity acts the same in all direction, and the observers free-fall at the same pace. Therefore they pass with the same speed through the surface of the imagined dashed ball. Let us free-fall with one of the observers. Then *relative* to us, this dashed ball moves towards us. A friend resting at the imagined dashed ball surface has placed rods along its surface. The lengths *along* the surface of the dashed ball are at right angles to our speed relative to the perfect ball. Hence when we pass such a rod, the length of the rod does not change with this speed, as we saw in Sect. 2.4.2. All observers agree on that: Lengths on the surface of *any* ball with the same center as the perfect ball do not change. In particular, the length of *any* circle with the same center as the perfect ball is the same as it would be without gravity.

Our friend resting at the dashed imagined ball also placed identical rods along the diameter of the ball. When we pass such a rod, it is in the direction of our speed, which grows, as we approach the perfect ball. Hence relative to us, this rod shrinks by the γ factor of our passing speed, as we saw in Sect. 2.4.1. We gained this speed while free-falling in from a far away point resting with the perfect ball. We sketched this in Fig. 6.8. However, we are still in the same inertial state as when we started resting far away from the planet. Hence a rod resting with the planet along

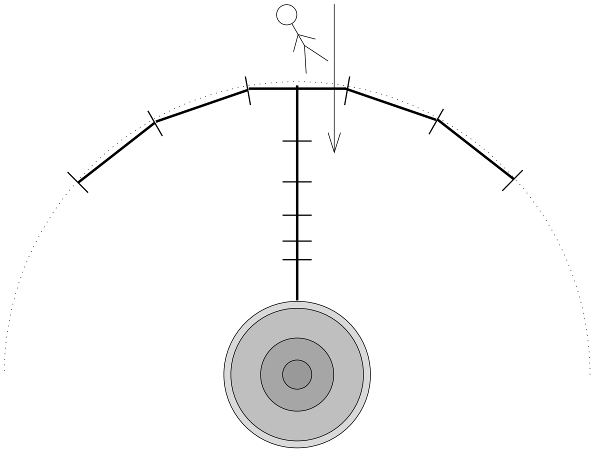


Fig. 6.8 Around a perfect ball, rods along the diameter shrink, but rods at right angles do not, according to an observer free-falling in from a resting-place far enough outside

the diameter, shrinks relative to us resting outside the planet the more, the nearer it is to the center.

What does our friend resting near the perfect ball observe? He will need *more* rods to exhaust the diameter than he would need without the gravity of the perfect ball. Hence the ratio of the length of any circle with the same center as the perfect ball, and its diameter will be *less* than π because of gravity. Hence near a perfect ball, space *itself* is bending!

It will turn out that there is a more practical way to keep track of the bending space. In Fig. 6.9, the observer rests between two circles with the same center as the perfect ball. He counts the number of identical rods needed to trace the outer circle, and subtracts the number of rods needed to trace the inner circle. If space does not bend, the number of rods on the outer circle is the number of rods needed to trace its radius $\times\ 2\pi$. Likewise, the number of rods on the inner circle is the number of rods on its radius $\times\ 2\pi$. Hence the number of rods fitting into the solid line between the two circles, in front of the observer in the figure, is the difference of the number of rods on the large and small circles, divided by 2π. The observer wants this ratio to remain *the same* even when a perfect ball in its center is bending space. We know that the rods on the circles will not change their length, even with the perfect ball in its center now gravitating. However, the rod in front of the observer in the figure, drawn as solid line, will shrink by the factor γ, corresponding to the speed of the bypassing clock, free-falling from a resting place sufficiently far away from the perfect ball.

Hence the observer decides to *stretch* the rods in front of him by the factor $1/\gamma$ so that now the number of *stretched* rods is again the difference of the number of rods on the large and smaller circle, divided by 2π. Such the observer keeps track of the bending space. Hence for us free-falling from a resting place sufficiently far away from the perfect ball, the rod shrinks by the same γ factor, so that for us, the rod has now the *same* length as the *original* rods far away outside. This is why this way of measuring is practical.

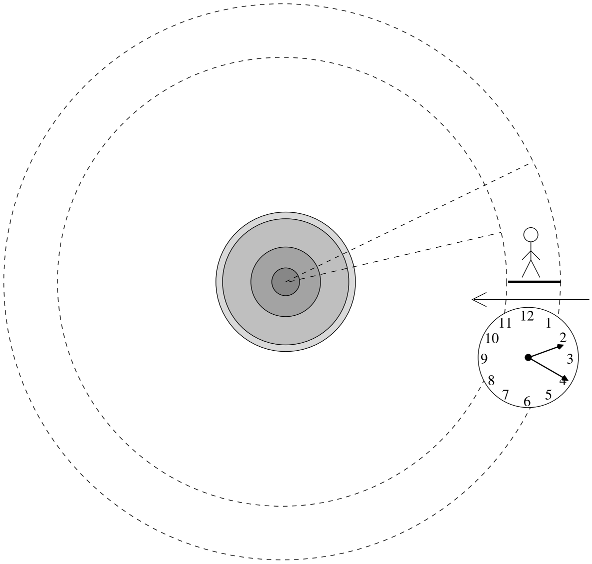


Fig. 6.9 Our friend standing between the two nearly equally large *dashed circles* measures a length which is shorter than when there is no gravity, by the factor γ of the speed of the clock free-falling in from a point resting far away from the perfect ball

6.5 Gravity Around a Perfect Ball

Let us sum up what we learned about **gravity around a perfect ball** of mass. Space-time bends in the following way:

1. At a place resting near the perfect ball, time will run slow by a factor γ, relative to a place resting sufficiently far away from the perfect ball.
2. This γ factor belongs to the speed which the clock reaches here, after free-falling in vertically from a place at rest sufficiently far away from the perfect ball.
3. This factor γ depends only on the radius. Far outside, γ is one. The smaller the radius outside the perfect ball, the smaller γ.
4. Lengths at right angles to the diameter will not change: Geometry on the surface of a ball with the same center as the perfect ball does not change.
5. The ratio of the boundary and radius of any circle with the same center as the perfect ball, will be as in school geometry 2π, and the surface area of a ball with the same center as the perfect ball, will be as in school geometry 4π times the *squared* radius.
6. The distance between two nearby points on the same diameter will be *larger* than what the measuring tape shows, by the factor $1/\gamma$. However, for an observer free-falling in from a resting place sufficiently far away from the perfect ball, the measuring tape shows the correct distance.

This way in which rods and time change locally is called a **metric**. It allows us to *measure* how space-time is bending. We see that the equivalence principle gives us nearly the complete picture of how gravity is bending space-time around a

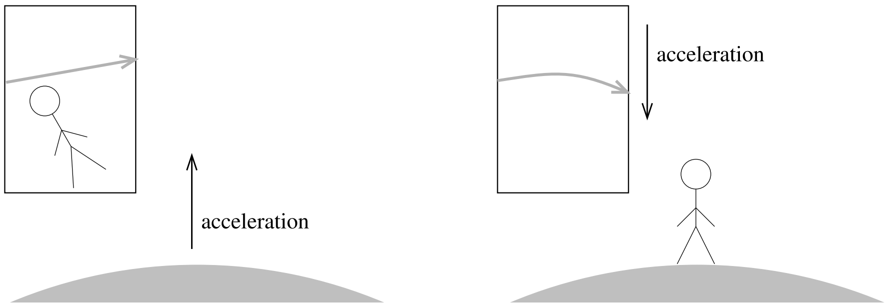


Fig. 6.10 Light, depicted as *light-gray arrow*, bending under gravity

perfect ball. The only information we still are missing, is how exactly the shrinking factor γ is depending on the radius! We will calculate this factor in Chap. 8, when we solve the **Einstein equation of gravity** *exactly* for the perfect ball. This bending of space-time is the **Schwarzschild solution** of the Einstein equation of gravity, or **Schwarzschild metric**. It is named after **Karl Schwarzschild**, a German astronomer and physicist. It is the most important exact solution of the Einstein equation of gravity, because planets and stars are nearly perfect balls after all.

6.6 Mass Under Gravity

We saw learned in Sect. 2.6 that if clocks slow down, then inertial mass increases. Hence a gravitating mass increases the **inertial mass of a test-mass nearby**. This is remarkable: There is so much mass in the universe, acting on a test-mass with its gravity, so that we may speculate:

> What if *all* inertial mass of bodies comes from the gravity of the other masses in the universe?

This is not a strict physical law, but only an idea, called the **Mach principle**. In fact, already before the theory of relativity was created, Ernst Mach speculated about this possibility. It is interesting because it would allow to state a *reason* why matter has mass. However, it never led to an exact theory. The equivalence principle shows that at least some part of the inertial mass of a body may come from nearby gravitating masses.

6.7 Light Under Gravity

Suppose that a light beam passes through a transparent, free-falling box, as in the left picture of Fig. 6.10. Box and observer in the box are in a state of inertia. Therefore

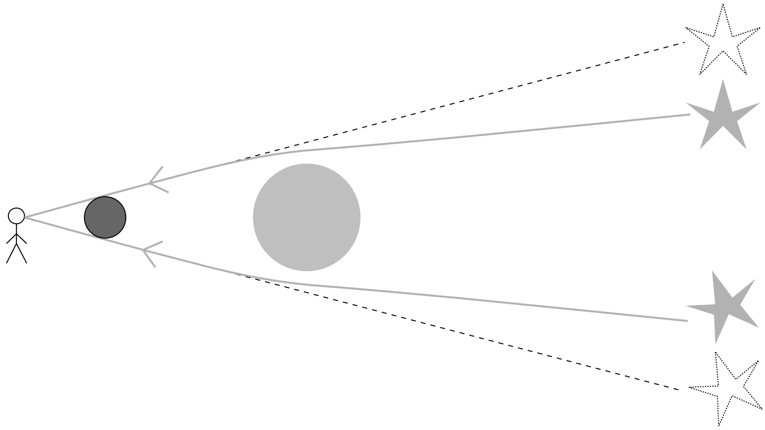


Fig. 6.11 Light bending along the sun

the observer in the box sees the light passing straight away through the box. Then the equivalence principle tells us that in the picture on the right, the observer on the planet will see the box accelerating towards him, together with the light beam: The light beam *bends*, that is, it **accelerates** towards the planet. This effect is strongest if the light passes near the surface of the planet or star, because there gravity is strongest.

The sun has enough mass so that we can see this effect: We take a photograph of distant stars behind the sun, so that their light bends near the surface of the sun. However, the sun is much too bright, so that we have to wait for the moon interrupting the light from the sun, at a solar eclipse, as in Fig. 6.11. Here the small, dark moon stands between us and the gray, large sun in the center. The light-gray drawn stars really stand more closer together than they look on the photograph, as "white" stars.

To verify this, we wait half a year. Then, at night, the sun is on the left behind us as you see in Fig. 6.12. We take the same photograph again. This time we see the stars at the place of the "gray" stars. We see that the stars are really nearer together than they seemed when the sun was in-between:

Light bends near mass, because space-time bends.

In this setup, the stars are very far away from the bending body, and the observer is relatively nearby.

Another possible setup is when the bending body, a yellow galaxy for example, is far away and a blue star is behind the galaxy, as you see in Fig. 6.13. The galaxy acts as **gravity lens**. Then we can see the light from the blue star as a *ring*. This is the so-called **Einstein ring**. In Fig. 6.14, you see a photograph of such a ring.[2]

[2]Credit: NASA, ESA, A. Bolton (Harvard-Smithsonian Center for Astrophysics) and the Sloan Lens Advanced Camera for Surveys Team.

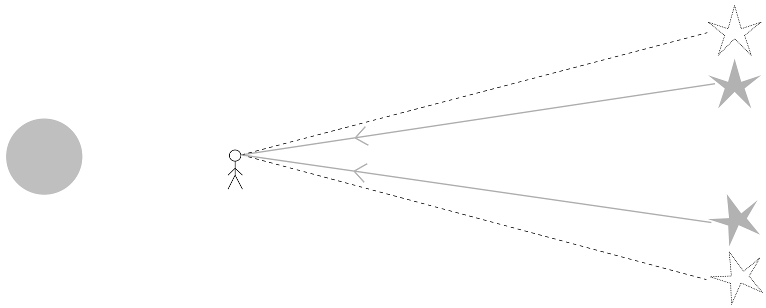


Fig. 6.12 The original position of the stars, seen half a year later

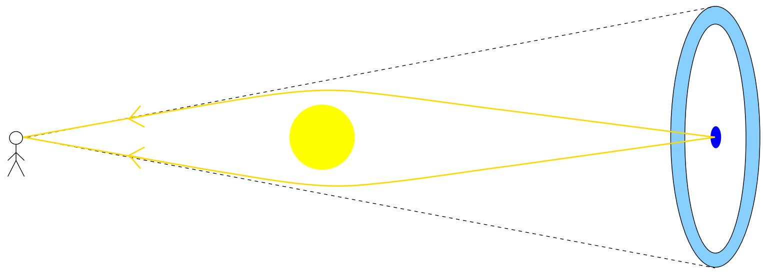
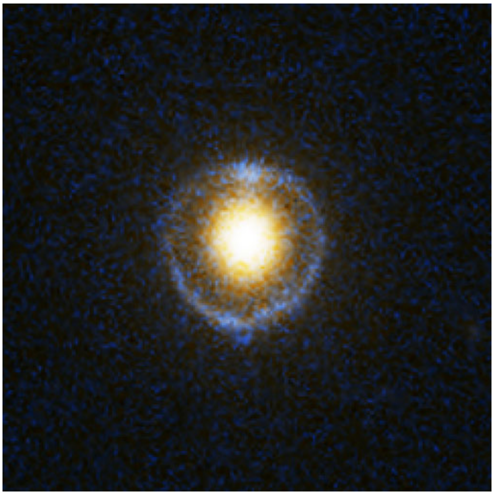


Fig. 6.13 A galaxy acts as lens for the light from a distant star

Fig. 6.14 Einstein ring with the number SDSS J162746.44-005357.5, photographed by the Hubble space telescope. The resolution you see here, is the resolution of the camera



In Sect. 9.2 we calculate the angle at which light bends while passing a star, using the Schwarzschild exact solution.

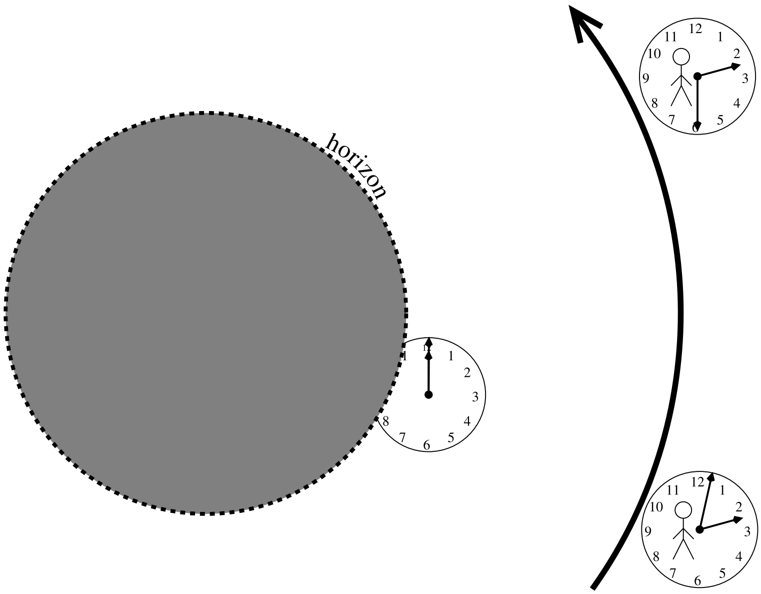


Fig. 6.15 Clocks run slow near large masses, and freeze at the horizon of a black hole, relative to outside observers

6.8 Black Holes: A First Look

The more mass a star contains, the more gravity acts on its neighborhood: Light bends more, time runs more slowly, inertial masses get more inertia. Hence, maybe from a certain mass on, light rays bend so much, that even light sent upwards from its center is falling back: The star emits no light. Such a body is a **black hole**. Around it there is a ball-shaped imaginary surface, the **horizon**. Any body can travel from the outside through this horizon, but as for a trapdoor, there is no way out of a black hole.

If some spaceship approaches the horizon from the outside, then for a far away observer time in the spaceship is running slower and slower. When it reaches the horizon, time is running *infinitely slow* as you can see in Fig. 6.15. Hence we never *see* the spaceship traveling *through* the horizon. However, the spaceship is free-falling, so it does not notice anything special in its neighborhood: It will travel through the horizon into the interior of the black hole. Why, the equivalence principle tells us that free-falling test-masses act like floating in empty space.

There is no contradiction: An astronaut in the spaceship *cannot tell* the outside world that he has passed inside the horizon, because *not even light* can pass through the horizon from the interior.

We will calculate the necessary mass of a black hole from the Schwarzschild exact solution in Sect. 9.1.

6.9 Equivalence Principle: Summary

If we look at gravity just as a force which pulls masses together, then why on earth the inertia should be the same as the weight of the mass? There is no *reason* for it. Einstein saw that this equality of inertial and gravitational mass is the key to understand gravity more deeply. All small enough mass reacts in the same way to gravity. Because the inertia resists acceleration and the heaviness tends to accelerate, but both are of exactly the same size, test-masses in gravity will not accelerate at all, but move in a state of inertia, force-free, that is, **free-falling**.

Moving force-free, that is, free-falling, does not depend on the material, so this explains why inertia and heaviness are always of the same size.

However, we know that test-masses move under gravity along *bended* paths, with varying speed, so gravity must *bend* space and time. Gravity does not bend space alone, because the free-falling test-masses do not follow the shortest path, but the path that takes the *longest proper time*, compared with nearby paths in space-time. This path is called a **geodesic in space-time**. We can produce a geodesic, by fixing the direction *and* the starting speed of the test-mass, and letting it free-fall. Different starting speeds give different geodesics, so again it is not only space that bends.

With such *free-falling* test-masses, we can use our knowledge about the theory of special relativity, to understand how mass is **reacting to gravity**.

Then *why* does matter create gravity? Nobody knows! We only know *how* matter creates gravity: It does so consistent with the equivalence principle, and in the *simplest possible* way. This we will see in the next chapter.

Chapter 7
How Mass Creates Gravity

We know from experience, and used in the preceding chapter, that mass creates gravity. We saw that gravity bends space-time, so mass *itself* should bend space-time. What is the simplest possible way that mass *can* bend space-time?

7.1 Gravity in a Lonely Cloud

We simplify the situation as much as possible: We drive with our spaceship into some empty region of space, so that no large mass is nearby, and we are in an inertial state. Then we place outside the spaceship a small cloud of dust, carefully, such that the dust particles are resting near each other. In Fig. 7.1 we sketched the dust particles as black balls. Then we gently move away and rest near the cloud. Hence the cloud as well as we are in an inertial state.

From experience, we expect that gravity tends to move mass together. In fact, as soon as we left the dust particles alone, the cloud **begins to shrink**. What is the simplest quantity to describe shrinking? This is how much the **volume** of the cloud decreases per *time*, that is, at which speed the volume decreases. What is the simplest quantity to describe the *beginning* of shrinking? That is how much the shrinking speed starts to change from zero, that is, the *acceleration* of the volume decrease, that is, change of volume per time, *per time*. That shrinking beginning rate should depend on how much mass is in the cloud. In the simplest case, the rate should grow *in proportion* to the mass in the cloud: Twice as much mass in the same volume should produce a twice as large shrinking rate.

Indeed, this is what nature has chosen as the law of gravity!

Let us give a numerical example of how a volume may begin to shrink. Suppose that we prepare a dust cloud of the size $10 \times 10 \times 10$ meters, that is a volume of 1000 m^3. When this cloud is beginning to shrink, we measure its volume each second, say. In Fig. 7.2, we see in the column of boxes on the left how much volume is left after 0, 1, 2, 3, 4 seconds. The rate at which the volume shrinks per time, that is from second to second, stands in the middle column of boxes. Finally, the rate at

K. Fischer, *Relativity for Everyone*, DOI 10.1007/978-3-319-00587-4_7,

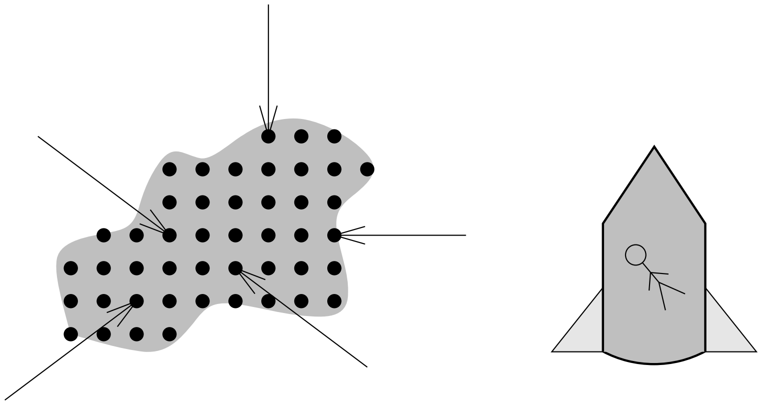
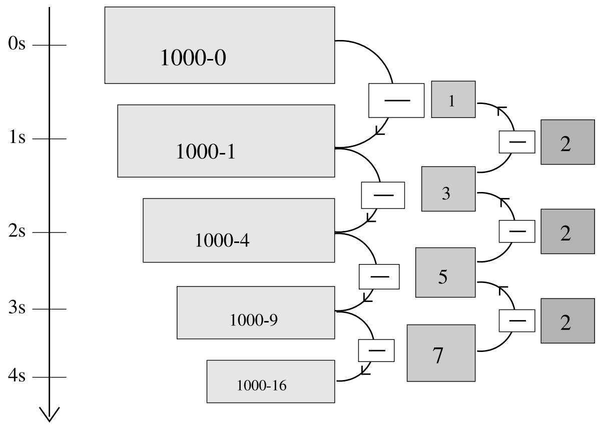


Fig. 7.1 A dust cloud begins to shrink under its own gravity

Fig. 7.2 Example of how a volume may begin to shrink



which this change of the volume *itself* is changing per second, that is the change per second, *per second*, we find in the right column. We see that the volume *begins* to shrink at a rate of 2 m^3, per second, per second, that is 2 $\frac{m^3}{s^2}$.

However, we wanted to know how mass bends space-time, not just how mass clumps together. Here the equivalence principle comes in again:

We know that test-masses, especially dust-like small particles, will *react* to gravity in the *same* way, no matter what their matter consists of, or how much mass they carry, if only not too much. Therefore we can think of the dust particles as just *probing* the space-time around them. While the cloud is beginning to shrink, the test-masses *accelerate* towards each other. However, remember that nothing pulls the test-masses together: Only because inside the cloud, space *itself* is beginning to shrink in time, the test-masses accelerate towards each other. Hence when the cloud begins to shrink, this means that the **small volume of space** itself begins to shrink, as **time** proceeds a little bit. This is how mass is bending a small piece of space-time.

We have found the Einstein law of gravity!

7.2 Einstein Equation of Gravity

The **Einstein law of gravity**, or with other name the **Einstein equation of gravity** is:

> The rate at which a small enough, resting cloud of matter begins to shrink, is in proportion to the *mass* in that cloud. The constant of proportion is 4π times the **gravity constant**.

Why not just "gravity constant" but 4π times it? This has purely historical reasons, nothing else. The value of the gravity constant is about 6.67×10^{-11} in appropriate units. You find it for reference in the Table A.1.

The **mass density**, that is the mass per volume, is nearly constant if we look at a small enough volume. It often is more practical to ask for the shrinking rate *per volume*, that is the **relative shrinking rate**. The **Einstein equation of gravity in terms of the mass density**, that is in terms of the mass per volume, reads

> The *relative* rate at which a small enough, resting cloud of matter begins to shrink, grows in proportion to the *mass density* in that cloud. The constant of proportion is 4π times the gravity constant.

However, mass is energy, divided by the square of the speed of light. In terms of the energy density the **Einstein equation of gravity for energy** of gravity reads

> The *relative* rate at which a small enough, resting cloud of matter begins to shrink, grows in proportion to the *energy density* in that cloud. The constant of proportion is 4π times the gravity constant, divided by the square of the speed of light.

7.3 Enter Pressure

We assumed that we can always place our test-masses inside the cloud such that at least at the beginning they rest with each other. What happens if our cloud does also contain **pure energy**, that is, light? Pure energy moves always at the speed of light, so we cannot place it like our masses. This light behaves much like a gas: Suppose we put our masses inside a gas. The simplest case is when at least near the cloud the gas looks everywhere the same. Therefore the gas particles, or light in the case of pure energy, are constantly entering and leaving the cloud we marked with

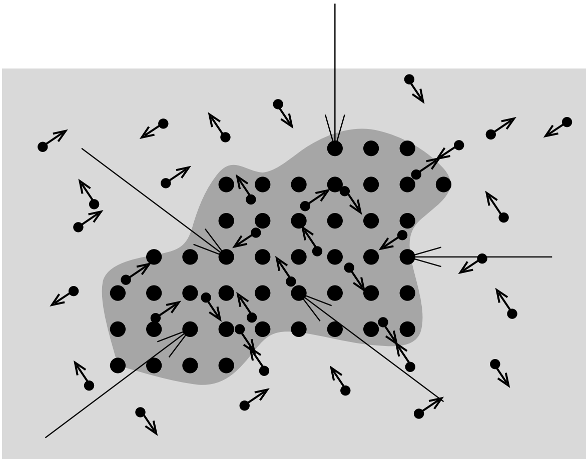


Fig. 7.3 A dust cloud inside a gas. We sketched the gas-particles as *small black disks*, and show their momentary speed by *small arrows*. The dust cloud contains also the energy coming from the disordered moving particles in the gas. This energy also creates gravity

test-masses, but the number of gas particles which leave per second the cloud, is the same as the number of particles which enter the cloud during this second. Still, the particles of the gas always quiver and bump against each other, so that they never rest relative to the test-masses in the cloud. This you can see in Fig. 7.3.

Because there enter per second as many gas-particles the cloud as there are leaving the cloud, we can imagine that for every gas particle that wants to leave the cloud, a gas particle from the outside bounces against it, to keep the gas inside the cloud together. In other words: the outside gas **presses** from any of the three directions of space. But pressure is energy density, as we saw in Sect. 1.12. This means that we have to add to the energy density inside the cloud the sum of the three pressures in the three directions of space.

We get the **complete Einstein equation of gravity**:

> The *relative* rate at which a small enough, resting cloud of matter begins to shrink, grows in proportion to the *energy density* plus the *pressures* in each of the three directions in that cloud. The constant of proportion is 4π times the gravity constant, divided by the square of the speed of light.

However, inside the cloud, the gas particles are not only moving around, but also bumping against our test-masses. If pressure is constant in all direction, the test-mass will not move on the average. However, under a microscope, we see the test-mass vibrating under the bombarding small gas-particles. Hence we cannot place our test-masses *perfectly* at rest. The very concept of **pressure** does only make sense if we do not look *too* closely, that is, if our cloud of masses is not *too* small.

Einstein himself was aware of this:[1]

[1] A. Einstein. *The meaning of relativity*. Princeton University press, 1956.

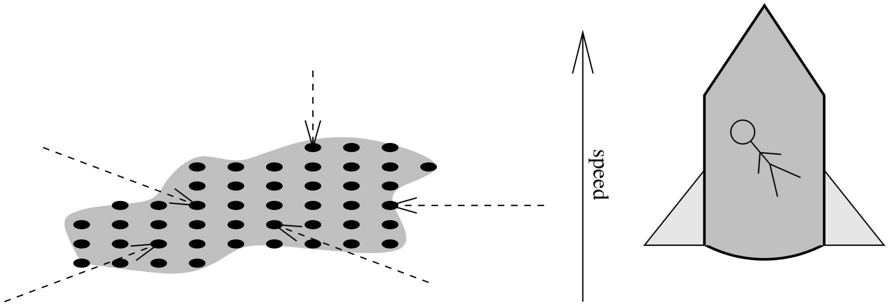


Fig. 7.4 A moving dust cloud begins to shrink

> We know that matter is built up of electrically charged particles, but we do not know the laws which govern the constitution of these particles. In treating mechanical problems, we are therefore obliged to make use of an inexact description of matter, which corresponds to that of classical mechanics. The density [...] of a material substance and the hydrodynamical pressures are the fundamental concepts upon which such a description is based.

If you like to see the mathematical expression for the Einstein equation of gravity, please have a look at Sect. A.5. With the exception of Sect. 9.9, we deal in the following with matter whose pressure is so small that we can ignore it, so that we can still use the model of the dust-cloud of the Sect. 7.1.

7.4 Enter Speed

The law of how mass creates gravity should fit with the theory of special relativity. That is: If we are in a state of inertia, and the cloud is passing us free-falling with some speed, then the Einstein equation of gravity should not change. However, we know that all kind of things change: First of all, the cloud has more mass by a factor $1/\gamma$.

Therefore it creates more gravity! However, also the reaction of the volume changes: The length of the cloud in direction of the speed is now smaller by this γ factor, while its sizes which are vertically to the speed do not change, as we can see by comparing the Figs. 7.1 and 7.4. Hence the *volume* of the moving cloud is smaller by this factor γ. However, the volume begins to shrink in *less* time: We know from Sect. 2.1 that our time runs faster than the proper time of the moving cloud by the inverse factor $1/\gamma$. Hence the cloud shrinks faster by this amount, and it *begins* to shrink at the even faster rate of $(1/\gamma) \times (1/\gamma)$. In total the volume of

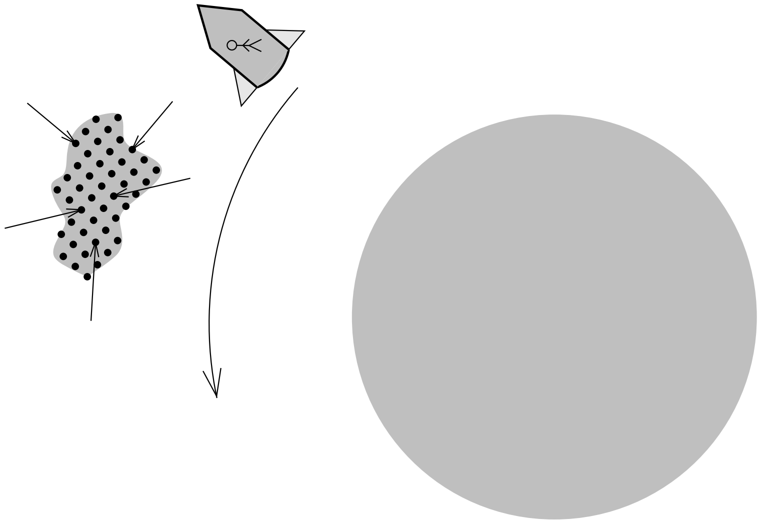


Fig. 7.5 A shrinking small cloud acts like a test-mass and free-falls under gravity of a nearby mass

the cloud begins to shrink faster by the factor

$$\underbrace{\gamma}_{\text{Volume shrinks}} \times \underbrace{(1/\gamma) \times (1/\gamma)}_{\text{shrinking begins faster}} = 1/\gamma$$

which is precisely the amount by which the mass in the cloud became larger. Hence the Einstein equation of gravity holds also for free-falling clouds.

7.5 Enter Outside Masses

In Sect. 7.2 we described how mass creates gravity for a carefully prepared small cloud of dust, far away from other large masses. In reality there are stars, planets and the like *outside* the cloud. Let us consider them. Because our cloud is small enough, it moves like a test-mass under the gravity of outside masses, that is, it *free-falls*, as you can see in Fig. 7.5. Then by the equivalence principle, the cloud is free-falling and reacts *as if* there would be no large mass nearby: The shrinking rate does only depend on how much mass is *inside* the dust cloud. The law of gravity does *not* change at all!

The outside masses can only change the *form* of the small free-falling cloud of mutually resting test-masses, as we already saw in Fig. 5.12.

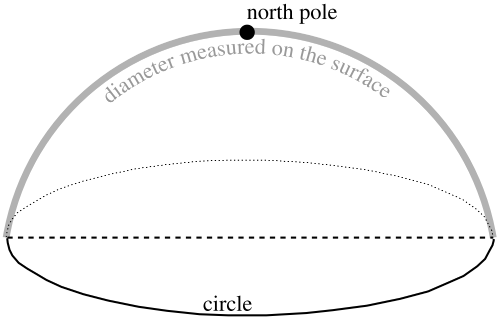


Fig. 7.6 Measuring how much the surface of the earth is bending near the north pole: Draw a circle around the north pole, on the surface of earth. The boundary of the circle is π times the *black dashed* diameter, which runs inside the earth. However, the diameter measured *on the bended surface* of earth, is the *gray arc* spanning from *left* to the *right*, and this is *longer* than the *dashed straight line*. Hence on the surface of the earth, the ratio of the boundary and the diameter of a circle is *less* than π, similar as in Fig. 6.8 for a circle around a **perfect ball**

7.6 Local and Global Space-Time

The Einstein law of gravity is formulated for a *small enough* piece of space (our "cloud") and a *small enough* period of time, in which the cloud "begins" to shrink, that is altogether, in *small enough* piece of space-time, that is, a *local* piece of space-time. How do we get the *global* picture of what happens under gravity? Take for example the sun. We know that the sun will bend space-time, and that far away from it, space-time will be *flat*. We insert here and there small clouds of test-masses to probe space-time. Then we *patch* these local pictures to a smooth map of space-time. It is a little like building a globe from patches showing parts of the surface of earth: Only when we have *connected* them smoothly, we see that earth indeed has the form of a globe. We get the *global* picture of how the globe is bending.

However, in space-time, things are more involved than for a surface: If the volume begins to shrink near a mass, then space-time must bend to connect the nearby space, where there is no mass. What is more, not only does the volume of a cloud shrink in time, but time *itself* depends on the relative speed of the free-falling masses in a cloud, so we have to follow how *space and time* evolve.

For a bended surface in ordinary three-dimensional space, it is enough to know one number per surface point, the **Gauss curvature**, to know how a surface bends. This curvature measures basically, how much the ratio of the boundary and the diameter of a small circle around a point differ from π, as in Fig. 7.6. In other words, it measures to what extend Euclidean geometry is wrong. However, space-time has more directions in which it can bend, so one number is not enough. Careful thinking shows that for each point in space-time we need *twenty* numbers to render the bending rate.

It was Riemann who generalized the theory of bended surfaces of Gauss to three and more dimensional space. The twenty numbers are named after him the **Riemann curvature tensor**. Einstein used and developed this theory to describe bended

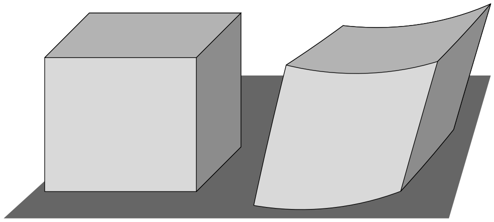


Fig. 7.7 A material bends

space-time. The mathematical toolbox for calculations is **tensor analysis**. The word "tensor" comes from the Latin word for "tension". In fact, engineers use tensors to describe how materials of bridges and the like bend under *tension*. Materials bend *inside* space-time, which we can easily show, as in Fig. 7.7. However, it is much harder to imagine how space-time *itself* bends. We will show in Sect. 9.7 that there is a fundamental difference between a bending material and bending space-time.

7.7 How to Solve the Einstein Equation of Gravity

Not only getting the global picture is difficult: Focus on one small cloud: What happens *after* that cloud has begun to shrink? The test-masses inside have just begun to move against each other, free-falling. To follow them, we must look into even smaller parts of the cloud, adjust our speed and time, look for the shrinking rate and so on. This is because the masses in the cloud at the same time *act* on space-time by bending it, and *react* on space-time by free-falling. We cannot easily separate action and reaction of mass. In other words, not every pattern of masses that we can think of, we may be able to realize in space-time. That makes gravity unique among the interactions in nature. This is also what makes solving the Einstein equation of gravity so difficult.

In fact, we know only a few exact solutions to the Einstein equation of gravity. In order to solve it *exactly*, we must place mass in a balanced way, so that we can handle the self-interaction.

1. For the **perfect ball** of mass, as in Sect. 6.5, the same amount of mass is sitting in every direction from its center. Hence mass balances itself out. Gravity depends only on the distance to the center of the ball, not the direction. All that remains to do, is to find the γ factor, that is, with which speed test-masses will vertically free-fall in at a certain distance from the center. We will do so in Chap. 8 and get the **Schwarzschild exact solution**, and the **Newton law of gravity** will follow from it for weak gravity. This case is most important because stars and planets are to a good degree perfect balls.
2. Observations show that over large enough distances, there is more or less the same amount of mass *everywhere* in the universe. Then again mass is balanced, because the space-time bending must be the same everywhere in *space*, and can only depend on *time*. We thus will see in Sect. 9.8 how it comes to the big-bang of the universe.

And basically, that was that! There are some slightly more general solutions, like for a perfect, rotating ball, or a perfect ball with electric charges, or even a rotating perfect ball with electric charges, and the like, but for an *arbitrary* collection of masses, we have to use tensor analysis and solve the Einstein equation of gravity on a computer approximately.

Make no mistake: We cannot *prove* that the law of gravity must be the Einstein equation. We only said that the Einstein gravity law is the *simplest possible*. Physicists constructed other theories, in which mass changes the space-time around it in a more complicated way. However, then it becomes more and more difficult to accommodate the law of gravity with the equivalence principle. In other words: It is hard to construct a theory which correctly describes how mass *creates* gravity *and* how mass *reacts* on gravity. What is more, experiments and observations again and again showed that only the Einstein gravity law plus the equivalence principle seem to give the correct law of gravity.

Look how beautiful everything falls into place:

1. We fixed positions and speeds for test-masses inside a small cloud, such that they rest relative to each other, and then let them loose. Then the simplest way mass can bend space-time, is by the *rate* at which the volume *begins* to shrink, while *free-falling*.
2. This shrinking rate does *only* depend on the mass *inside* the volume.
3. It does so in the simplest manner, that is, in proportion to the mass inside.
4. The gravity law fits with the theory of special relativity, because it does not change even if we move free-falling relative to the free-falling cloud.
5. *One physical* quantity, that is to say the mass *inside* the cloud, determines how another, *geometrical* quantity, that is to say its *volume*, changes with time. In other words: The Einstein equation of gravity does *not* regulate directly how the *form* of the small volume changes in time!
6. This is done by masses *outside* the cloud: They *deform* the cloud, but do not change the *volume*, to fit the global picture of the bending space-time.

This theory, using the Einstein equivalence principle and the Einstein gravity law, is the **theory of general relativity**.

Next we want to see the gravity law in action. How does the motion of the planets around the sun follow from it? How does this law fit in with the classic Newton way of looking at gravity, where the sun seems to "pull" earth around it? And what new effects can we find?

Chapter 8
Solving the Einstein Equation of Gravity

Having stated and discussed the Einstein equation of gravity, let us solve them!

8.1 Gravity Causes Law of Motion

The equivalence principle told us that test-masses move free-falling under gravity, that is, how they *react* on bended space-time. This is the **law of motion**. The Einstein equation of gravity tells us how mass *acts* on space-time, how it bends space-time. Surprisingly the Einstein equation of gravity also tells us how test-masses must *react*! A thought experiment will tell us why: Suppose that we rest inside a small cloud of test-masses, in an inertial state. The test-masses are also resting relative to us, as sketched in Fig. 8.1.

Then according to the Einstein equation of gravity, the cloud will begin to shrink at a rate which is in proportion to the mass inside the cloud. Now let us look at an even smaller volume, inside the cloud, directly around us: The smaller this volume, the less mass is inside. The less mass is inside, the less the volume will begin to shrink.

What is more, the cloud can begin to change its form as in Fig. 5.12, due to the gravity of outside masses. However, the volume we showed there, is quite large, compared with the planet nearby. Again, the smaller volume we mark around us, the less it will begin to deform.

Therefore let us chose a tiny volume inside the cloud, that contains only the white test-mass beside us, and us. Then this volume will neither begin to shrink, nor begin to change its form. However, this is only possible if the white test-mass beside us will *not* begin to move relative to us, that is, will not *accelerate* relative to us. That means that the white test-mass is also in an inertial state, moving along a geodesic in space-time.

In other words: We have found the law of motion for this white test-mass!

Compare this with the Newton theory of gravity: Gravity of earth, say, creates a force pulling at a test-mass. This is the Newton law of gravity, telling us how matter

K. Fischer, *Relativity for Everyone*, DOI 10.1007/978-3-319-00587-4_8,

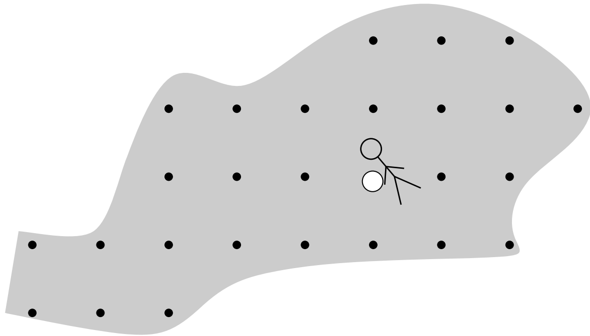


Fig. 8.1 The white test-mass directly beside us will not begin to accelerate relatively to us

creates gravity. Then the test-mass *reacts* by resisting the acceleration because of its *inertia*. This reaction of the test-mass we *cannot* infer from the Newton law of gravity. What is more, it is a riddle why weight and inertia should be equal at all. In fact, the Newton law of gravity is only an estimate of what really happens.

Or compare gravity with electrodynamics: Electric charges create an electromagnetic field around them, obeying the Maxwell-equations. But these *alone* do not tell us how another charged mass will *react* to the field. We know that the **Lorentz force** of Chap. 3 describing this reaction, fits in with both the theory of relativity and the Maxwell equations. However, the Lorentz force is not the only *possible* force fitting with the Maxwell equations.

Incidentally, an electromagnetic field carries energy and hence bends space-time, so that using the Einstein equation of gravity and a similar argument as above, we can derive the Lorentz force!

8.2 Gravity Inside a Perfect Ball of Mass

A star or planet is nearly a perfect ball. In a thought experiment, we drill thin vertical shafts into the ball, through its center, in many directions. We sketched for better view only four directions in Fig. 8.2. We place test-masses in the shafts, at the same fixed distance from the center, marking the small black ball in the center. At some time we release the test-masses and let them loose, so that they now fall freely.

If the black volume is small enough, and if the test-masses rest relative to each other when we release them, we can use the Einstein equation of gravity for it: The rate by which the black volume marked the free-falling test-masses *begins* to shrink, is in proportion to the mass inside, that is, the mass of the black ball. Because the mass spreads in all directions in the same way, the test-masses fall all with the same speed, inside the vertical shafts, and mark at all times a ball.

Make no mistake: The mass of the perfect ball *itself* does *not* move at all! Only the *test-masses* begin to move, because of the gravity of the *motionless* mass *inside the perfect ball*.

We measure now the space-time inside and around the perfect ball. However, we know from Sects. 6.1 and 6.4, that under gravity time itself runs different for

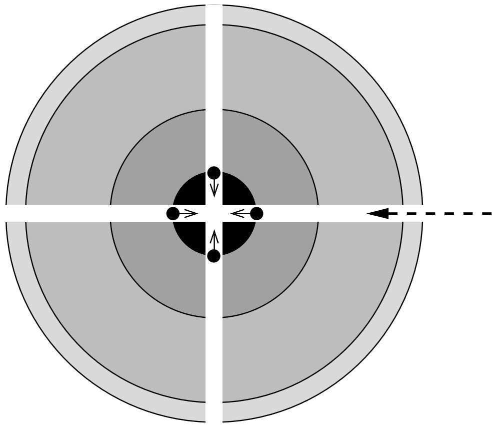


Fig. 8.2 The *small black disks* at the surface of the *central black mass* are the test-masses, which start to free-fall in vertical shafts of the perfect ball. The *dashed arrow* marks us, free-falling into one of the shafts, having started from a position where we were resting far away enough from the perfect ball

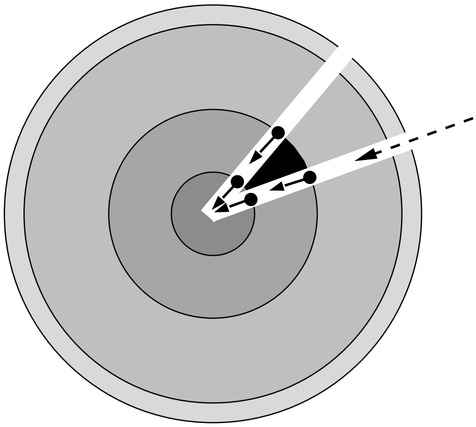


Fig. 8.3 A small, thin *black* volume sandwiched between two balls begins to shrink

different distances to the center, and vertical lengths also change under gravity. We also know from Sect. 6.6 that even the mass of a section of the perfect ball itself is changed by the gravity of other parts of the perfect ball.

We account for that, using the equivalence principle: We start at a point at the right, nearly resting in an inertial state far away relative to the perfect ball, and free-fall in from the right. In Fig. 8.2, we sketched this by the horizontal dashed arrow. We enter the shaft, and at the time when we pass the center, all the test-masses begin to free-fall. Because we are free-falling, our proper time is the same time as if resting far outside the perfect ball.

We know from Sect. 7.4 that the Einstein equation of gravity does not change, if we measure mass and volume of the small ball marked by the test-masses as well as time from our point of view.

Next, we repeat this thought experiment with a small black volume which is sandwiched between the ball at the center, and some slightly larger ball with the same center, as in Fig. 8.3. The shell between the two balls should be very thin. We only drew it thicker for better view.

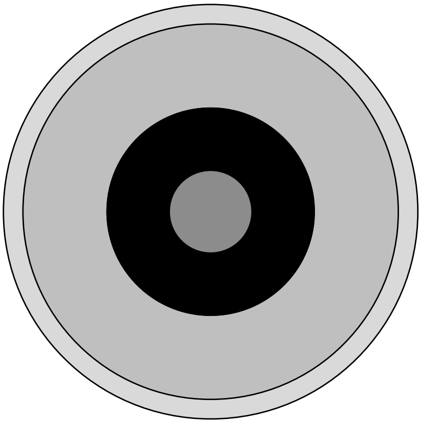


Fig. 8.4 A thin *black* shell inside the perfect ball. We drew the shell not so thin for better view

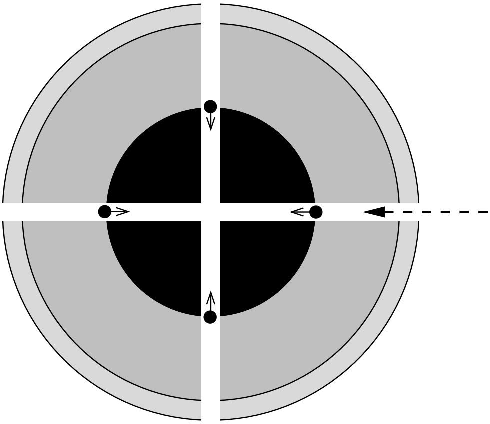


Fig. 8.5 A slightly larger ball, marked by test-masses, begins to shrink

Again we free-fall in from our resting-place far outside. Again the volume marked by the test-masses is very small. Again the test-masses do not move relative to each other. Again the test-masses begin their free-fall when we pass through that volume. Therefore, we can use the Einstein equation of gravity: The volume marked by the test-masses, begins to shrink in proportion to the mass in the black volume. The same is true for any other small volume inside this shell in any other direction. Hence it is true for the total thin shell, sketched as black region in Fig. 8.4. Again we drew the thin shell not so thin, for better view.

In both Figs. 8.2 and 8.4 we free-fall in. Hence the time in which we measure the beginning of the shrinking, is proceeding at the same pace as for bodies resting far outside the perfect ball. Hence the *total* ball marked by the masses on the outside of the thin shell, begins to shrink as well in proportion to *its* mass, as *we* measure it. This total mass is the black volume in Fig. 8.5.

Such we mark piece by piece towards the outside adjoining thin shells, like the skins of an onion. Repeating the argument above, we conclude that the volume of

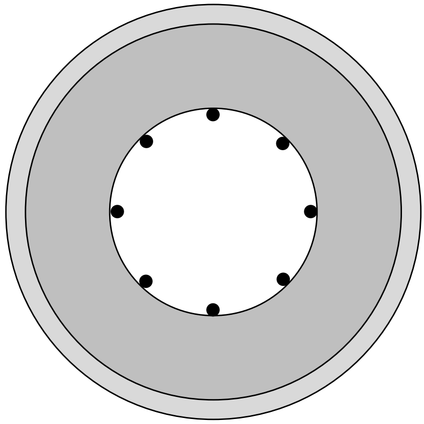


Fig. 8.6 Within the interior of a hollow **perfect ball** there is no gravity

the total perfect ball marked by test-masses, begins to shrink at a rate in proportion to the mass inside the ball.

8.3 Flat Space-Time Inside a Ball-Shaped Hollow

Now let us think of a weird planet, which has a hole in its center, as sketched in Fig. 8.6. We can use the arguments of the last section for the ball-shaped hollow itself: Place test-masses at the rim of the hollow. Then the ball marked by this test-masses will begin to shrink in proportion to the mass it contains, that is zero. In other words: It will not begin to shrink at all. However, it also must remain ball-shaped, because the mass on the outside spreads equally in all directions. Hence the test-masses do not begin to move at all: Inside the hollow there is *no* gravity.

This is our first **exact solution** of the **Einstein equation of gravity**! It is called the **Birkhoff theorem**:

> **Birkhoff Theorem**
>
> If a **perfect ball** contains a ball-shaped **hollow** with the same center as the ball, then inside that hollow **space-time does not bend**.

8.4 Gravity Outside a Perfect Ball of Mass

We continue the argument of Sect. 8.2 to the outside of the perfect ball, by adding also on the outside of the perfect ball thin shells piece by piece. **Outside the perfect ball** there is no more mass, so that we conclude,

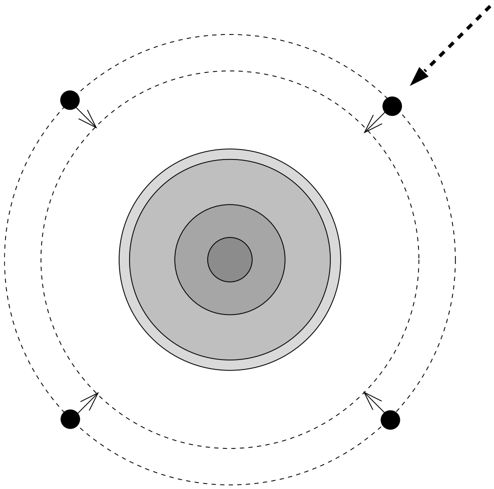


Fig. 8.7 The marked ball around the perfect ball begins to shrink, as we pass it. In the figure, we sketched our passing by the *dashed arrow*

> The volume of any ball with the same center as the perfect ball, but larger as it, marked by test-masses resting to each other, will begin to shrink at a rate which is 4π times the gravity constant times the mass of the perfect ball.
> Time and lengths we measure free-falling in from a resting place far away from the perfect ball.

Let us put this differently, in terms of the *radius* of such a ball, as sketched in Fig. 8.7. The test-masses rest relative to each other, and mark the volume of the larger of the dashed balls, outside the perfect ball. At some time, they are released and begin to free-fall. We started free-falling from a resting-place far away from the perfect ball and pass, say, the right-upper test-mass just as it begins to free-fall. During a short time, the larger dashed ball shrinks a bit, becoming the smaller dashed ball. We saw in Sect. 6.5, point 6, that then the distance of the two balls is for us just the amount that the radius has shrunk during that short time.

When the larger dashed ball begins to shrink to the slightly smaller dashed ball, it loses the volume which is between the two balls. As you see from Fig. 8.7, this is nearly the surface of the larger ball, times the distance between the two balls, or the difference of the radiuses of the two balls, as we said above. The surface area of a ball of some radius is the same as in school geometry, that is 4π times the square of the radius, from what we saw in Sect. 6.5, point 5. Hence the rate at which the volume of the dashed ball begins to *shrink*, is nothing but the rate at which the radius begins to shrink, times 4π times the square of the radius.

In other words: According to the Einstein equation of gravity,

$$4\pi \times (\text{radius})^2 \times (\text{acceleration of the radius})$$
$$= -4\pi \times (\text{gravity constant}) \times (\text{mass of the perfect ball})$$

The "minus" we need because the radius is shrinking, so its acceleration is negative. The factor 4π drops out, and we divide by the squared radius, so that the change of the radius accelerates at a rate of

$$\begin{pmatrix}\text{acceleration of radius}\\ \text{in proper time of observer}\\ \text{free-falling from resting place}\\ \text{far away from perfect ball}\end{pmatrix} = -\frac{\binom{\text{gravity}}{\text{constant}} \times \binom{\text{mass of}}{\text{perfect ball}}}{(\text{radius})^2} \tag{8.1}$$

Even if our friend passes us at the given radius, free-falling with *less speed*, the equivalence principle tells us that we and our friend are both in an inertial state, so that we do *not* accelerate relative to each other. Hence for a given radius, Eq. (8.1) remains valid for a free-falling test-mass of *any* possible speed relative to the perfect ball.

We see that although the change of the radius accelerates, the observer does *not* feel any acceleration. Again, this is because space-time *bends*: In the same way, an astronaut in the spaceship of Fig. 5.4 does not feel any acceleration, despite going along a bended path around earth: This **acceleration** is only **relative** to the perfect ball.

8.5 Schwarzschild Exact Solution

We let a test-mass free-fall from a place resting far away from the perfect ball. Let us calculate at which speed the test-mass is approaching the perfect ball, to complete the picture of Sect. 6.5. The test-mass starts far away from the perfect ball, at zero speed, relative to the perfect ball. The rate at which the speed is growing, is the acceleration. This we know for any radius from Eq. (8.1). Hence we can calculate the γ factor for any radius, bit by bit. This is enough to know how space-time bends, after what we learned in Sect. 6.4. First we tell the result for the speed, at a given radius, both in capital letters,

$$(\text{SPEED})^2 = \frac{2 \times \binom{\text{gravity}}{\text{constant}} \times \binom{\text{mass of}}{\text{perfect ball}}}{(\text{RADIUS})} \tag{8.2}$$

Let us now go backwards and confirm that this speed must have come from the **acceleration of the radius** of Eq. (8.1). First we see from Eq. (8.2) that for a very large radius, the speed becomes very small, so that we really start to free-fall at nearly zero speed, if we rested only far enough away from the perfect ball. Next, we study the free-fall during a short time. We measure that time as our proper time. After the short time, the speed will grow by a small additional amount. During that short time, the radius shrinks by a small amount. For us free-falling, this is just the small *distance* we traveled during that short time, as we saw in Sect. 6.5, point 6.

Hence we have at this later time the equation

$$\left(\text{SPEED} + \begin{matrix}\text{small additional}\\ \text{speed}\end{matrix}\right)^2 = \frac{2 \times \begin{pmatrix}\text{gravity}\\ \text{constant}\end{pmatrix} \times \begin{pmatrix}\text{mass of}\\ \text{perfect ball}\end{pmatrix}}{\left(\text{RADIUS} - \begin{matrix}\text{small}\\ \text{distance}\end{matrix}\right)} \tag{8.3}$$

The small additional speed, per short time, is the acceleration we search for. We expand the square of the speed,

$$(\text{SPEED})^2 + 2 \times (\text{SPEED}) \times \begin{pmatrix}\text{small additional}\\ \text{speed}\end{pmatrix} + \begin{pmatrix}\text{small additional}\\ \text{speed}\end{pmatrix}^2$$
$$= \frac{2 \times \begin{pmatrix}\text{gravity}\\ \text{constant}\end{pmatrix} \times \begin{pmatrix}\text{mass of}\\ \text{perfect ball}\end{pmatrix}}{\left(\text{RADIUS} - \begin{matrix}\text{small}\\ \text{distance}\end{matrix}\right)}$$

The term in the middle on the left hand side, as the product of the original and the small additional speed, is much larger than the product of the small additional speed with itself on the right. Hence we can neglect this term if the short time is short enough, and have

$$(\text{SPEED})^2 + 2 \times (\text{SPEED}) \times \begin{pmatrix}\text{small additional}\\ \text{speed}\end{pmatrix}$$
$$= \frac{2 \times \begin{pmatrix}\text{gravity}\\ \text{constant}\end{pmatrix} \times \begin{pmatrix}\text{mass of}\\ \text{perfect ball}\end{pmatrix}}{\left(\text{RADIUS} - \begin{matrix}\text{small}\\ \text{distance}\end{matrix}\right)}$$

For the squared speed on the left hand side we replace the right hand side of Eq. (8.2).

$$\frac{\cancel{2} \times \begin{pmatrix}\text{gravity}\\ \text{constant}\end{pmatrix} \times \begin{pmatrix}\text{mass of}\\ \text{perfect ball}\end{pmatrix}}{(\text{RADIUS})} + \cancel{2} \times (\text{SPEED}) \times \begin{pmatrix}\text{small}\\ \text{additional}\\ \text{speed}\end{pmatrix}$$
$$= \frac{\cancel{2} \times \begin{pmatrix}\text{gravity}\\ \text{constant}\end{pmatrix} \times \begin{pmatrix}\text{mass of}\\ \text{perfect ball}\end{pmatrix}}{\left(\text{RADIUS} - \begin{matrix}\text{small}\\ \text{distance}\end{matrix}\right)}$$

The factor 2 we cancel on both sides. In order to get the small additional speed, we subtract now the first term on both sides,

$$(\text{SPEED}) \times \begin{pmatrix}\text{small additional}\\ \text{speed}\end{pmatrix}$$
$$= \frac{\begin{pmatrix}\text{gravity}\\ \text{constant}\end{pmatrix} \times \begin{pmatrix}\text{mass of}\\ \text{perfect ball}\end{pmatrix}}{\left(\text{RADIUS} - \begin{matrix}\text{small}\\ \text{distance}\end{matrix}\right)} - \frac{\begin{pmatrix}\text{gravity}\\ \text{constant}\end{pmatrix} \times \begin{pmatrix}\text{mass of}\\ \text{perfect ball}\end{pmatrix}}{(\text{RADIUS})} \tag{8.4}$$

If for example the original radius was 100,000 meters and we approached the perfect ball during the short time by the small distance of 3 meters, then the difference of the *inverse* radiuses is

$$\frac{1}{99{,}997} - \frac{1}{100{,}000} = \frac{100{,}000}{99{,}997 \times 100{,}000} - \frac{99{,}997}{99{,}997 \times 100{,}000}$$
$$= \frac{3}{99{,}997 \times 100{,}000} \approx \frac{3}{100{,}000^2}$$

Hence this is roughly the small distance, divided by the *square* of the radius. Multiplying this with $\left(\substack{\text{gravity}\\ \text{constant}}\right) \times \left(\substack{\text{mass of}\\ \text{perfect ball}}\right)$, we can rewrite the right hand side of Eq. (8.4) as,

$$(\text{SPEED}) \times \begin{pmatrix}\text{small additional}\\ \text{speed}\end{pmatrix} = \begin{pmatrix}\text{gravity}\\ \text{constant}\end{pmatrix} \times \begin{pmatrix}\text{mass of}\\ \text{perfect ball}\end{pmatrix} \times \frac{\left(\substack{\text{small}\\ \text{distance}}\right)}{(\text{RADIUS})^2}$$

Now, the small additional speed per short proper time is the *acceleration*, and the small distance per short time is the *negative* speed, because the radius shrinks by that amount. Hence per short time we have

$$(\text{SPEED}) \times (\text{acceleration}) = -\begin{pmatrix}\text{gravity}\\ \text{constant}\end{pmatrix} \times \begin{pmatrix}\text{mass of}\\ \text{perfect ball}\end{pmatrix} \frac{(\text{SPEED})}{(\text{RADIUS})^2}$$

We divide both sides of the equation by the speed, and end up with Eq. (8.1). This shows us that Eq. (8.2) really gives the correct speed of a test-mass free-falling in from a place resting far away from the perfect ball. From it we get the γ factor,

$$\gamma = \sqrt{1 - \frac{\text{speed}^2}{c^2}} = \sqrt{1 - \frac{2 \times \left(\substack{\text{gravity}\\ \text{constant}}\right) \times \left(\substack{\text{mass of}\\ \text{perfect ball}}\right)}{(\text{radius}) \times c^2}} \tag{8.5}$$

Hence we have **solved exactly** the **Einstein equation of gravity**, that is, we got the **Schwarzschild exact solution**:

Schwarzschild Exact Solution

1. A **perfect ball** of mass bends space-time such that time is running slow at a certain radius by the factor γ of Eq. (8.5), relative to a clock resting far enough away from the perfect ball.
2. The vertical distance between two nearby points will be larger than the difference of their radiuses, by the factor $1/\gamma$.
3. Horizontal lengths do not change.
4. The ratio of the boundary and radius of any circle with the same center as the perfect ball is 2π, and the surface area of a ball with the same center as the perfect ball is 4π times the squared radius.

8.6 Newton Law of Gravity

The Newton law of gravity follows from the Einstein gravity law, if the gravitating mass is small. Hence test-masses resting far away from the gravitating mass, will not reach large speeds while free-falling towards it. For example, according to Table A.1, the radius of the sun is 7×10^8 and its mass is 2×10^{30}. The gravity constant is about 7×10^{-11}. According to Eq. (8.2) we find that such a test-mass free-falling from a resting place far away into the sun reaches at most 0.2 percent of the speed of light,

$$\frac{\text{speed}}{c} = \sqrt{\frac{2 \times (\text{gravity constant}) \times \text{mass}}{(\text{radius}) \times c^2}}$$

Let us insert the above numbers:

$$\frac{\text{speed}}{c} \approx \sqrt{\frac{2 \times (7 \times 10^{-11}) \times (2 \times 10^{30})}{(7 \cdot 10^8) \times (3 \cdot 10^8)^2}} \approx \sqrt{\frac{2^2}{10^6} \times \frac{10}{9}} \approx \frac{2}{1000} \tag{8.6}$$

Then the γ factor is nearly one,

$$\gamma = \sqrt{1 - \frac{\text{speed}^2}{c^2}} \approx 0.999998$$

Therefore time runs everywhere nearly at the same pace, and the length of the same rod is nearly everywhere the same. The radius is now nearly the distance from the center: School geometry is nearly valid. The radius entering Eq. (8.1) is now practically the distance of the test-mass to the center. Hence the Einstein equation of gravity tells us that a test-mass falling in vertically towards a perfect ball, will **accelerate relative** to the perfect ball with a rate

$$\begin{pmatrix}\text{acceleration of} \\ \text{distance to center}\end{pmatrix} = -\frac{\binom{\text{gravity}}{\text{constant}} \times \binom{\text{mass of}}{\text{perfect ball}}}{(\text{distance to center})^2} \tag{8.7}$$

This looks already very much like the Newton law of gravity. In fact, this law merges *two* laws of the original Newtonian mechanics: The first law states that there is a force coming somehow from the center of the perfect ball, acting infinitely fast at any distance, pulling a test-mass of some heaviness towards the center of the perfect ball in proportion to its heaviness,

$$\begin{pmatrix}\text{force on} \\ \text{test-mass}\end{pmatrix} = -\frac{\binom{\text{gravity}}{\text{constant}} \times \binom{\text{mass of}}{\text{perfect ball}}}{(\text{distance to center})^2} \times \begin{pmatrix}\text{heaviness of} \\ \text{test-mass}\end{pmatrix} \tag{8.8}$$

This is the **Newton law of gravity**. At the same time, the inertia of the test-mass resists the force such that the test-mass accelerates in inverse proportion to its inertia

$$\text{acceleration} = \frac{\binom{\text{force on}}{\text{test-mass}}}{\binom{\text{inertia of}}{\text{test-mass}}} \tag{8.9}$$

This is the **Newton law of motion**. Dividing both sides of the Newton law of gravity (8.8) by the "inertia of test-mass", we see how the acceleration depends on the distance to the center of the perfect ball:

$$\text{acceleration} = \frac{\binom{\text{force on}}{\text{test-mass}}}{\binom{\text{inertia of}}{\text{test-mass}}} = -\frac{\binom{\text{gravity}}{\text{constant}} \times \binom{\text{mass of}}{\text{perfect ball}}}{(\text{distance to center})^2} \times \frac{\binom{\text{heaviness of}}{\text{test-mass}}}{\binom{\text{inertia of}}{\text{test-mass}}}$$

In the Newton theory, the heaviness and the inertia of a test-mass are *accidentally* equal, so they cancel each other on the right hand side of the equation, and we have again the law (8.7).

However, there is no *reason* why inertia and heaviness should be the same within the Newtonian mechanics. We see that in general relativity, inertia and heaviness are *equal because* gravity is *no* force, but all kinds of test-masses move *in the same way* free-falling through space-time bended by masses. We also see how the classic Newton law of gravity *and* the Newton law of motion emerge as good estimate from the Einstein equation of gravity, if the gravitating masses are not too large! The Newton law of motion emerges because of what we said in Sect. 8.1, that the Einstein equation of gravity also determines the law of motion of a test-mass.

Chapter 9
General Relativity in Action

9.1 Black Holes

A nearly perfect ball as the sun has mass 2×10^{30} and radius 7×10^8. These two numbers determine how *strongly* this star bends space-time, via the following ratio

$$S = \frac{2 \times \binom{\text{gravity}}{\text{constant}} \times \binom{\text{mass of}}{\text{perfect ball}}}{c^2} \tag{9.1}$$

The letter **S** stands for the **Schwarzschild radius** of the perfect ball. For the sun, we have in meters

$$S = \frac{2 \times (6.67 \times 10^{-11} \, \frac{\text{m}^3}{\text{kg} \times \text{s}^2}) \times (2 \times 10^{30} \text{ kg})}{(3 \times 10^8 \, \frac{\text{m}}{\text{s}})^2}$$
$$\approx 2.96 \times 10^3 \text{ m} \approx 3000 \text{ m} \tag{9.2}$$

which is much smaller than its radius of about 7×10^8 m. The vertical speed of Eq. (8.2) with which a test-mass resting far away from a perfect ball, free-falls towards the perfect ball, becomes in terms of the Schwarzschild radius

$$\frac{\text{speed}^2}{c^2} = \frac{\frac{2 \times \binom{\text{gravity}}{\text{constant}} \times \binom{\text{mass of}}{\text{perfect ball}}}{c^2}}{\text{radius}} = \frac{S}{\text{radius}} \tag{9.3}$$

The Schwarzschild radius enters the γ factor of the Schwarzschild exact solution (8.5) as

$$\gamma = \sqrt{1 - \frac{2\binom{\text{gravity}}{\text{constant}} \times \binom{\text{mass of}}{\text{perfect ball}}}{(\text{radius}) \times c^2}} = \sqrt{1 - \frac{S}{\text{radius}}} \tag{9.4}$$

which is for the sun about

$$\gamma \approx 0.999998$$

K. Fischer, *Relativity for Everyone*, DOI 10.1007/978-3-319-00587-4_9,

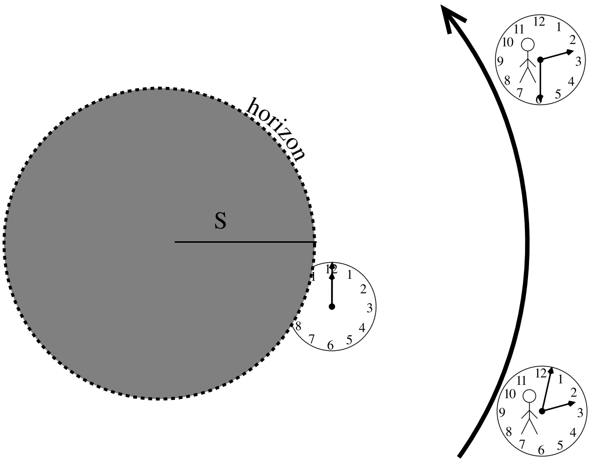


Fig. 9.1 Clocks run slow near large masses, and freeze at the horizon of a black hole, relative to outside observers

Please have a look at Fig. 9.1. We know from the Schwarzschild solution that a clock of an observer on the surface of the ball runs slow for a distant, resting observer by the factor γ. For the same radius, the larger the mass of the ball, the larger the ratio $\frac{S}{\text{radius}}$, and the stronger gravity acts on the surface of the perfect ball.

When the mass in the perfect ball clumps so much together that its radius is *smaller* than its Schwarzschild radius, then a strange thing happens: A clock at the Schwarzschild radius is still outside the perfect ball, with a γ factor of

$$\gamma = \sqrt{1 - \frac{S}{S}} = 0$$

That is: An outside observer sees that the time of the clock is *freezing*! Such a star is a **black hole**. We discussed this already in Sect. 6.8, but here we see how the Schwarzschild exact solution predicts that black holes are possible. The horizon which we discussed in Sect. 6.8, is therefore the surface of the ball, which has its center at the black hole, and its radius is the Schwarzschild radius.

Outside the black hole, its gravity acts like any other perfect ball. Equation (9.2) tells us that if the sun could shrink to less than about 3000 m, it would become a black hole. However, the planets would move on exactly the same paths around the sun as they do now!

The theory allows in principle *any* body to become a black hole, if it only shrinks enough. This depends of course on the inner structure of the body: You cannot just "like this" squeeze a stone. For collapsing stars this depends on how rigid its elementary particles are. With the help of **quantum theory** we can predict that a star must have at least roughly one and a half times the mass of the sun to be able to collapse eventually into a **black hole**. Up to now (2013), the smallest *observed* black holes have a mass of about 3 times the mass of the sun. Large black holes should sit in the center of a galaxy, "feeding" on stars which came to close. It seems that in the center of our galaxy there sits a monster black hole which has a mass of over three million sun masses.

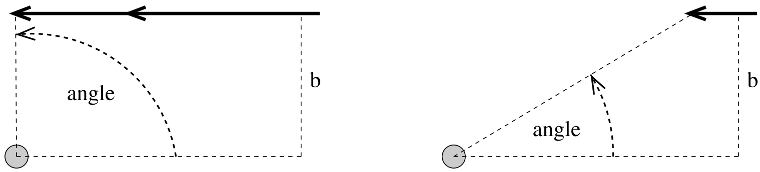


Fig. 9.2 Light passes near a light perfect ball nearly along a straight line

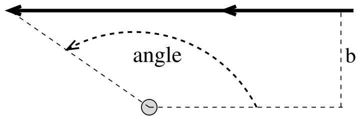


Fig. 9.3 The total angle between the incoming and the outgoing direction will grow eventually to 180 degrees

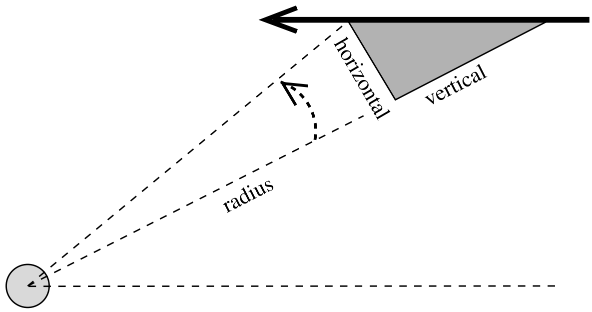


Fig. 9.4 The *dashed arrow* shows the growth of the angle during some short time like, say, one millisecond. We draw the *gray triangle* much larger than in reality, for better view

9.2 Light Bending: Weak Gravity 1

In Fig. 6.10 we saw already that a light beam should *bend* near a large mass. How much does a light beam bend?

A light beam passes a perfect ball, from the right to the left, as sketched in Fig. 9.2. The light beam is the solid black arrow, and all other, dashed lines are only a guide for the eye. The gray circle at the bottom of the figure shows a very light perfect ball. We aim not at the center of the ball, but at some distance b from it. Because the ball is very light, it nearly has no gravity. Hence the light beam will pass along a straight line. We measure how far the light has gone by the angle it has with the dashed line which runs from the left to the right. At the far right, the angle is practically zero. It grows as the light beam approaches the perfect ball. When it is nearest, the angle is 90 degrees, as in the left picture of Fig. 9.2. When the light has passed far to the left, the angle grows eventually to 180 degrees, as you can see in Fig. 9.3.

First we need to know at which rate the angle is growing when a very light perfect ball is there. In Fig. 9.4 we see the light passing during some short time of, say, one millisecond. During this time, the angle grows as much as the dashed arrow shows.

How much does the angle grow? The ratio of the growth and the full angle of 360 degrees is the ratio of the horizontal side of the gray triangle and the circumference

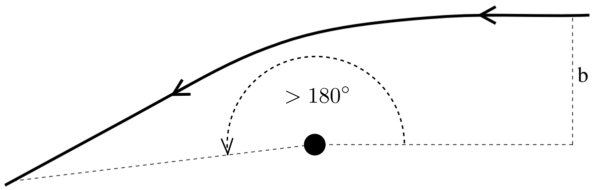


Fig. 9.5 A light beam bends near a heavy perfect ball

of the circle with the radius,

$$\frac{\text{angle growth}}{360\text{ degrees}} = \frac{\text{horizontal}}{2\pi \times \text{radius}} \tag{9.5}$$

Beware: "Horizontal" does *not* mean "along the lines of the text in the book", but "at right angles to the respective radius"! Next, we replace the light by a heavy perfect ball. Then as the light passes it to the left, the angle will grow *beyond* 180 degrees, as you see in Fig. 9.5: The light beam *bends*. We estimate now this additional, **bending angle of the light beam**, using the Schwarzschild exact solution.

The Schwarzschild solution (8.5), point 2, tells us that the small gray triangle in Fig. 9.4 will distort, because space is bending: The horizontal side will not change, but the vertical side will grow by the $1/\gamma$ factor of the radius at which the triangle sits. We flatten out space near the triangle by growing the horizontal distance in the same manner,

$$\text{horizontal} \longrightarrow \text{horizontal}/\gamma \tag{9.6}$$

Time also "bends": The light beam passes along the longest side of the small triangle with the speed c. However, we observe the light from a place far away "above" the perfect ball, resting relatively to the perfect ball in an inertial state. Relative to us, time slows down by the factor γ.

$$\text{short time} \longrightarrow (\text{short time}) \times \gamma \tag{9.7}$$

The ratio of the horizontal side per time, per radius, is in proportion to how much the angle will grow, per short time,

$$\begin{aligned}\frac{\text{horizontal}}{(\text{short time}) \times \text{radius}} &\longrightarrow \frac{\text{horizontal}/\gamma}{(\text{short time} \times \gamma) \times \text{radius}} \\ &= \frac{\text{horizontal}}{(\text{short time}) \times (\gamma^2 \times \text{radius})}\end{aligned} \tag{9.8}$$

We see: If the perfect ball is heavy, the angle grows more by the factor γ^{-2}. In other words: The light beam bends. However, then Fig. 9.4 cannot be correct, because the light will not follow a straight line! Nonetheless we can use this figure, if the perfect ball is not too heavy, as for example the sun. Then the light beam bends only a little. Hence it does not make much difference if we assume that the light beam came in along a straight line, from the right. We make only a small mistake by adding up these *bending angles* for each short time. In other words: Once we have flattened

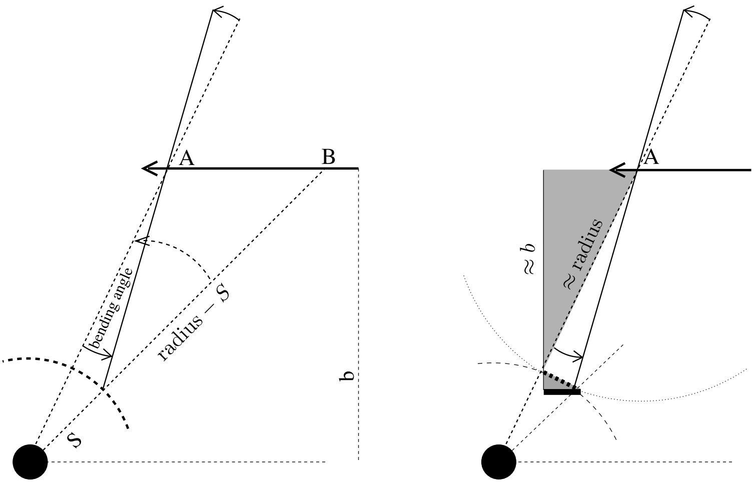


Fig. 9.6 Light bends because it feels only the radius minus the Schwarzschild radius. For better view, we drew the Schwarzschild radius S larger than the radius of the black perfect ball. For a not to heavy perfect ball, the Schwarzschild radius is much smaller than either b or the radius. Hence to a very good approximation, the left side of the *larger gray triangle* has length b, and its right side is nearly the radius

out space-time near where the light beam passed, we can use school geometry to calculate the *bending* angle. Physicists call such a **perturbative calculation**.

The second line of Eq. (9.8) tells us that instead of changing the horizontal side of the triangle and time, we also can shrink the radius by the amount γ^2 and leave all other distances and times as they are. In this way we can flatten space-time around a not-too heavy perfect ball with sufficient accuracy.

How much do we have to shrink the radius? We know the square of the γ factor from the Schwarzschild exact solution (9.4),

$$\text{radius} \longrightarrow (\text{radius}) \times \gamma^2 = (\text{radius})\left(1 - \frac{S}{\text{radius}}\right) = \text{radius} - S \qquad (9.9)$$

In words: In order to calculate the *bending* angle, we shrink the radius by the Schwarzschild radius. Please have a look at the left picture of Fig. 9.6. For a very light perfect ball, the light beam travels during a short time along the arrow from point B to point A. The additional angle during that short time is the dashed angle, measured from the center of the perfect ball, in the same way as in Fig. 9.4.

For a more heavy perfect ball, we use the reduced radius, "radius − S". That is, we measure the angle not from the center of the perfect ball, but from the Schwarzschild radius on. We get the thin solid line through A. T his line turns against the left dashed line by the solid *bending* angle.

We show now that the *bending* angle grows in proportion to the solid **horizontal fat** side of the small gray triangle in the right picture.

The solid **horizontal fat** side of the small gray triangle stands at right angles to the solid side b of the larger gray triangle. Its largest side is the **dashed fat** line, which is nearly part of the dotted circle around A, and therefore is standing at right angles to the radius which is the largest side of the larger gray triangle. Its smallest side stands at right angles to the smallest side of the larger gray triangle. Hence if we turn the small gray triangle by 90 degrees clockwise and enlarge it, we could fit the larger triangle. Therefore the sides of the small and large gray triangle are in proportion. In particular,

$$\frac{\textbf{horizontal fat}\text{ line}}{b} = \frac{\textbf{dashed fat}\text{ line}}{\text{radius}} \tag{9.10}$$

The **dashed fat** line is also nearly that part of the dotted circle with center in A, belonging to the *bending* angle in the left picture. Hence the ratio of the **dashed fat** line and the circumference of $2\pi \times$ radius of the dotted circle equals the ratio of the *bending* angle and the full angle,

$$\frac{\textit{bending}\text{ angle}}{360\text{ degrees}} = \frac{\textbf{dashed fat}\text{ line}}{2\pi \times \text{radius}}$$

We multiply both sides with 2π,

$$2\pi \times \frac{\textit{bending}\text{ angle}}{360\text{ degrees}} = \frac{\textbf{dashed fat}\text{ line}}{\text{radius}} \tag{9.11}$$

Because the right hand sides of the two equations (9.10) and (9.11) are equal, the left hand sides are also equal: Indeed the *bending* angle is growing in proportion to the **horizontal fat** line,

$$\frac{\textbf{horizontal fat}\text{ line}}{b} = 2\pi \times \frac{\textit{bending}\text{ angle}}{360\text{ degrees}}$$

Next have a look at Fig. 9.7. As the light beam travels from the right to the left, the **horizontal fat** line is growing all the way from the right to the left of the dashed circle, filling eventually its *diameter* $2S$. Hence the total *bending* angle becomes

$$\frac{2S}{b} = 2\pi \frac{\text{total } \textit{bending}\text{ angle}}{360\text{ degrees}}$$

The factors 2 we cancel. After multiplying both sides of the equation with $\frac{360}{\pi}$ degrees, we get finally how much a light beam is bending around a not-too-heavy perfect ball,

$$\frac{360}{\pi} \times \frac{S}{b}\text{ degrees} = \text{total } \textit{bending}\text{ angle} \tag{9.12}$$

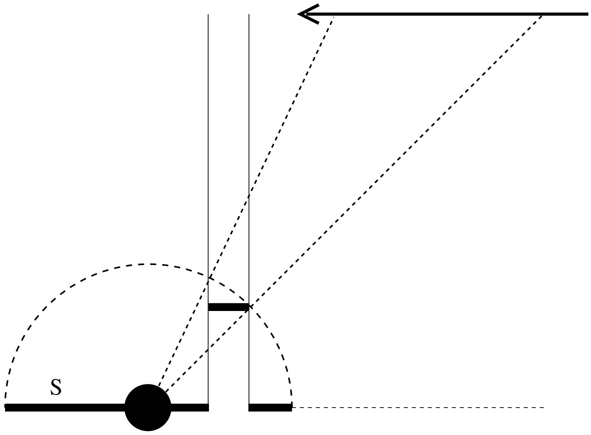


Fig. 9.7 The *bending* angle grows in proportion to the **horizontal fat** *line*

To get the largest effect, for a given star as the sun, we should make b as small as possible: We should use light beams just touching the sun, so that b is then the radius of the sun. Multiplying with 60 gives this in **arc minutes**, and again with 60 gives the bending angle in **arc seconds**, and filling in the numbers from Table A.1,

$$\text{bending angle} \approx \frac{360}{\pi} \times \frac{2.96 \times 10^3}{6.96 \times 10^8} \times 60 \times 60 \approx 1.75 \text{ arc seconds} \tag{9.13}$$

This is what many experiments have shown over the years!

Observe that to get the correct bending angle, we needed three effects:

1. Space bends around the perfect ball.
2. Time slows down near the perfect ball.
3. A light beam passes near the observer with speed c.

If only space would bend, then the above argument shows, that we would have to ignore Eq. (9.7), and hence have to shrink the radius in Eq. (9.8) not by the factor γ^2, but only by the factor γ. Hence our calculation would predict *less* (in fact, about one-half) of the observed bending. So it is really again space-*time* which is bending.

9.3 Kepler Laws

We saw in Sect. 8.6 that for weak gravity, the Newton law of gravity is nearly correct. The Newton law of gravity can explain the famous three **Kepler laws**. Planets going around the sun move nearly according to these three laws:

1. A planet moves around the sun on an **ellipse**, with the sun sitting in one focus of the ellipse.
2. A straight line extending from the center of the sun to the center of the planet, moves in equal small enough times over an equally small area. See Fig. 9.8.

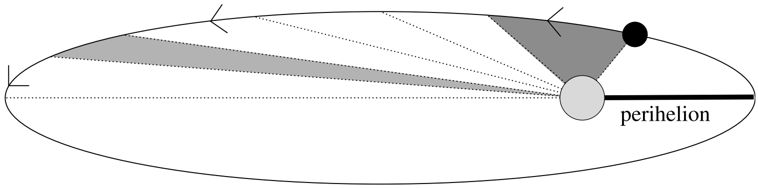


Fig. 9.8 The sun is the light-gray ball sitting in one focus point of the ellipse. The planet is the *black ball* on the *ellipse*. The **perihelion** is the line at which the planet comes nearest to the sun. The *dotted straight lines* show where the planet is after a fixed time span has passed. The *light gray* and the *darker gray triangles* have the same area, so that the planet is faster near the sun than far away from it

3. The square of the year of the planet is in proportion to the third power of the *largest* radius of the ellipse. This largest radius is half the length of the horizontal diameter line in Fig. 9.8. The constant of proportion is $4\pi^2$, divided by the gravity constant and the mass of the sun.

We explain here only the simplest case in which the planets move in ellipses which are nearly circles. This case still shows all the interesting physical effects. Then the "largest radius" is just "the" radius to the sun. The **third Kepler law** simplifies to:

> The square of the year of the planet is in proportion to the third power of its radius to the sun. The constant of proportion is $4\pi^2$, divided by the gravity constant and the mass of the sun.

Example: Earth is at distance 1.5×10^{11} to the sun, which has mass 2×10^{30}, and the gravity constant is 6.67×10^{-11}. Gravity of the sun is not too strong, so that we can use the distance of earth to the sun as radius. Hence one year should be about

$$\text{year} = \sqrt{4\pi^2 \frac{(1.5 \times 10^{11})^3}{(6.67 \times 10^{-11}) \times (2 \times 10^{30})}} \approx 3.16 \times 10^7 \text{ seconds}$$

Check: Including the leap years, one year is about 365.25 days, that are 365.25×24 hours, or $365.25 \times 24 \times 60 \times 60 \approx 3.16 \times 10^7$ seconds, as it should be.

How do the Kepler laws follow from the Einstein equation of gravity? First of all, the planets are much smaller than the sun, at least the ones nearer to the sun. Therefore we assume that the planets themselves are just test-masses. Then we know that the sun is nearly a perfect ball. How such a test-mass can move around the perfect ball in a circle, and at the same time free-fall? Please have a look at the left picture of Fig. 9.9. The box free-falls vertically. Inside the box, a test-mass moves horizontally with steady speed, as sketched by the horizontal thick arrow. The vertical arrow shows the perfect ball **accelerating relative** towards the box.

In the picture on the right hand side, we stand on a platform which reaches up to the height where the test-mass is. We see the box falling down faster and faster.

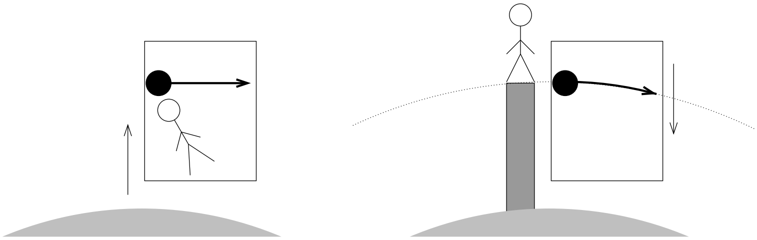


Fig. 9.9 A test-mass moves free-falling in a circle around a perfect ball, if we tune its horizontal speed: We need no force for that, just the bended space-time

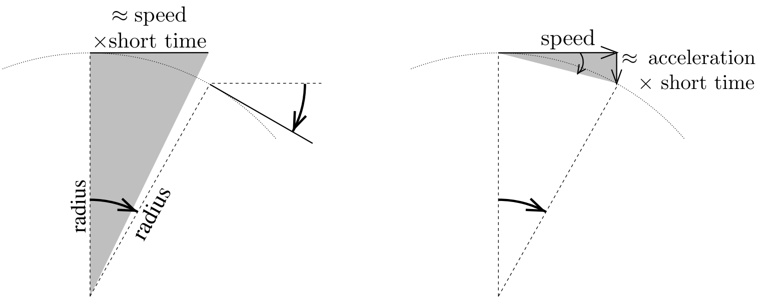


Fig. 9.10 During a short time the place and the speed of the planet turn at nearly the same angles, so that the planet remains on the circle around the sun

Hence the test-mass moves more and more downwards, while moving to the right: It moves along a *bended* path, much in the same way as the light beam in Fig. 6.10. If we adjust the horizontal speed, we can arrange that the bended path continues to be the dotted circular path around the perfect ball. Hence we see here how just by bending space-time, the sun makes a planets moving around it, without any force pulling at the planet. We show now how large this speed is, using the Einstein equation of gravity, *exactly*, so that we do *not* need the Newton law of gravity at all!

We look from a place far enough above the plane in which the planet moves, nearly resting relatively to the perfect ball in an inertial state. We observe the planet during a short time interval, as in Fig. 9.10. During this short time, it moves a certain distance along the circle to the right. This is nearly the horizontal distance in the left picture of Fig. 9.10. Distance per time is speed, so it moves at a distance which is speed times the short time.

During this short time the place of the planet turns by a small angle. We learned in Sect. 6.5 that the circumference of the circle with the same center as the perfect ball is 2π times the radius, even within bended space-time. Hence for a small angle, the ratio of horizontal distance and radius is in proportion to the angle.

Next let us have a look at how the speed is changing its direction during the short time, as in the right picture. The additional speed is nearly vertically during the short time. Acceleration is additional speed per short time, so the additional speed is just the acceleration times the short time, as we see in the right picture.

Now comes the point: To remain on the circle also in the future, the direction of the speed must turn by the *same* small angle during the short time. Because both the left and the right gray triangles have therefore the same small angle and are nearly right triangles, the ratio of their smallest side and the side at right angles to it, must be nearly the same, and this the better, the smaller the time interval is:

$$\frac{\text{speed} \times (\text{short time})}{\text{radius}} = \frac{\text{acceleration} \times (\text{short time})}{\text{speed}}$$

The short time we can cancel, and see how the speed depends on the acceleration and the radius:

$$\text{speed}^2 = (\text{acceleration}) \times (\text{radius})$$

The acceleration we have from the Einstein equation of gravity (8.1), for us looking from a place far above the plane in which the planet moves, resting relatively to the perfect ball in an inertial state. The minus sign in that equation we can ignore here, because here we viewed the vertical speed in Fig. 9.10 as positive number. Hence a planet moving along a circle with a certain radius, moves at the speed

$$\text{speed}^2 = \frac{\binom{\text{gravity}}{\text{constant}} \times \binom{\text{mass of}}{\text{perfect ball}}}{(\text{radius})^2} \times (\text{radius})$$

We cancel the radius on the right against one radius in the denominator,

$$\text{speed}^2 = \frac{\binom{\text{gravity}}{\text{constant}} \times \binom{\text{mass of}}{\text{perfect ball}}}{(\text{radius})} \tag{9.14}$$

In terms of the Schwarzschild radius (9.1) this is

$$\begin{aligned} \frac{\text{speed}^2}{c^2} &= \frac{1}{2} \times \frac{2 \times \binom{\text{gravity}}{\text{constant}} \times \binom{\text{mass of}}{\text{perfect ball}}}{(\text{radius}) \times c^2} \\ \frac{\text{speed}^2}{c^2} &= \frac{1}{2} \times \frac{S}{\text{radius}} \end{aligned} \tag{9.15}$$

This is again an exact result of the theory of general relativity!

Now, speed is distance per time. During one year the test-mass moves once around the circle. This distance is 2π times the radius, as we know from Sect. 6.5. Hence

$$\frac{4\pi^2 \times (\text{radius})^2}{(\text{year})^2} = \frac{\binom{\text{gravity}}{\text{constant}} \times \binom{\text{mass of}}{\text{perfect ball}}}{\text{radius}}$$

Solving for the length of the year we get

$$\frac{4\pi^2 \times (\text{radius})^3}{\left(\substack{\text{gravity}\\\text{constant}}\right) \times \left(\substack{\text{mass of}\\\text{perfect ball}}\right)} = (\text{year})^2 \qquad (9.16)$$

This is just the **third Kepler law**. We see that if we use the radius as in Sect. 6.5, and measure the time from a point resting far away relative to the perfect ball, that then for a circular orbit, the third Kepler law remains *exactly* correct in the theory of general relativity.

9.4 Planet Orbits Rotate: Weak Gravity 2

In the previous section we stated the Kepler laws, in particular that the planets move around the sun on an ellipse. As a matter of fact, the planets do *not* follow an ellipse to perfection. The reason is mainly, that the planets act with their *own* gravity on each other. As one result, the ellipse itself slightly deforms. This is hard to observe. However, one effect *piles up* as the years go by, so that astronomers can easily measure it, just by waiting long enough: Very slowly, the ellipses themselves *rotate* along the sun, as sketched in Fig. 9.11.

Astronomers use the line where the planet comes closest to the sun, as reference line. It is called the **perihelion**, and you see it in Fig. 9.8. "Peri" stands for "near" and "helios" for the sun. Every time the planet comes nearest to the sun, this perihelion has turned at some angle. This angle per one turn astronomers call the **perihelion advance**.

Mathematicians and physicists developed *calculus* among other things to *calculate* such tiny effects, using the Newton law of gravity. One way is to *take off* the influence of the other planets. Then a lonely planet should move along a perfect ellipse, because that is what the Newton law of gravity predicts for a single planet going around the nearly perfect ball of the sun.

However, what we really see is that the ellipse of such an imagined lonely planet *still* rotates by some angle per one turn around the sun! Now we know that the Newton law is out of line with the correct Einstein equation of gravity. So Einstein asked himself if this left-over *perihelion advance* is an effect which General Relativity can explain?

First let us use the Einstein equation of gravity only in the approximation of the Newton law of gravity, which leads to the Kepler laws. Then a test-mass moves around a perfect ball on a fixed ellipse. So to make things simpler, assume that the ellipse differs from a circle only by a *tiny bit*.

Such an ellipse has to a very good approximation the form of a circle, as the solid ellipse in the left picture of Fig. 9.12. However, the perfect ball does not sit quite in the center of the ellipse, but rather somewhat shifted in the center of the *dashed* circle.

A test-mass starts at the perihelion at point 1, heading parallel to the dashed circle, along the solid ellipse. Then it will *swing* about the dashed circle: At point 2,

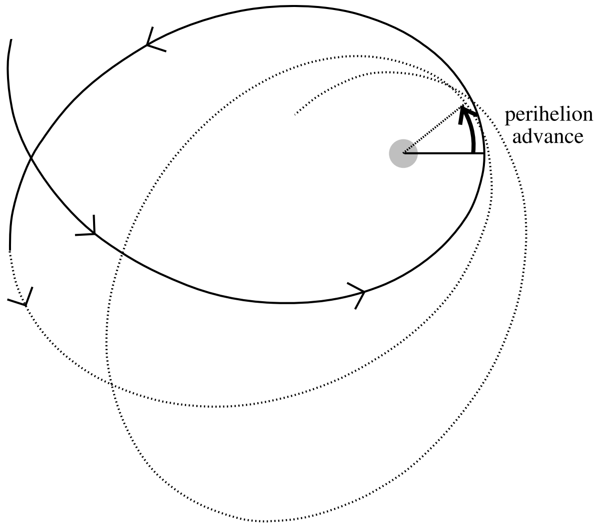


Fig. 9.11 A planet moves along a slowly rotating ellipse, so that the orbit does not close properly. For better view, we calculated the orbit for a planet which has roughly the same distance to the star as earth has to the sun, but the star here is one-million times heavier than the sun

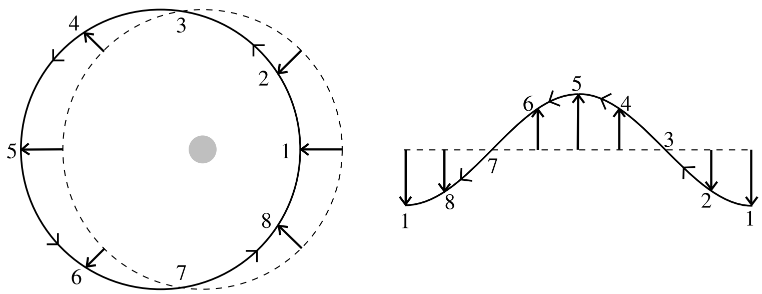


Fig. 9.12 *Left picture*: The perfect ball sits at the center of the *dashed circle*. The *solid ellipse* is to a very good approximation a circle. The *arrows* show how much the test-mass is departing from the circular orbit. Point 1 is the perihelion. *Right picture*: The *dashed circle* expanded to a *straight line*, so that we see how much the test-mass is departing from the *circle*

it already approaches the dashed circle. From point 3 on, it moves away from the dashed circle, and so on. After one year it passes through the starting point 1 with the same speed as it started, heading again parallel to the dashed circle.

We saw in Sect. 9.3 that Kepler's third law determines the period of a planet moving along the dashed circle. In other words: If the Newton law of gravity would be exactly correct, then Kepler's third law (9.16) would determine also the period of this swing.

We estimate now how much the bending space-time changes the speed of the swing, when we see it from an inertial state far away from the central mass, as we did in Sect. 9.3 for Kepler's third law. We assume that the mass of the perfect ball is not too large, so that we can **calculate perturbatively**, similar to Sect. 9.2. For this purpose we flatten the bended space-time near the dashed circle.

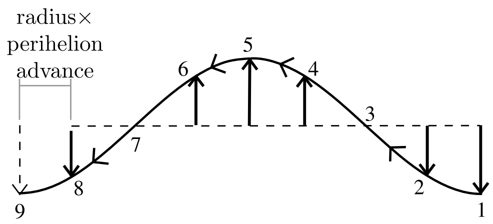


Fig. 9.13 A test-mass moving along the *solid* ellipse swings around the *dashed* circular orbit slower than Kepler's third law predicts. Therefore it turns more than 360 degrees between the perihelion, and the next perihelion

The arrows in Fig. 9.12 point in direction of the radius. We know from the Schwarzschild exact solution (6.5) that the length of the vertical arrows will be *larger* than what the measuring tape shows, by the factor γ^{-1} of the radius of the dashed circle. Time is running slower by the γ factor, relative to us resting far away from the perfect ball in an inertial state. The speed at which the arrow changes its length, is change of length *per time*, and hence is larger by the factor $y^{-1}/\gamma = \gamma^{-2}$, than in the Newtonian theory.

We can get the same effect by enlarging *all* vertical distances near the dashed circle by this factor γ^{-2}, calculated for the radius of the dashed circle, but measuring time with our clock. This will lead of course to the incorrect space-time in general, but near the dashed circle we have such flattened out space-time. Because we measure with our clock, and gravity is not so strong, we can use again Kepler's third law (9.16) to get the period of the swing, but with the radius enlarged by the factor γ^{-2}. The third power of the radius will then grow by the factor $(\gamma^{-2})^3 = (\gamma^{-3})^2$. Because this is in proportion to $(\text{period})^2$, the period itself will be larger than one year by the factor γ^{-3}.

Hence the test-mass arrives again at a perihelion, starting from one perihelion after about $360 \times \gamma^{-3}$ degrees, as you see in Fig. 9.13. We know from the Schwarzschild exact solution (9.4) that the square of the γ factor is smaller than one by

$$\gamma^2 = 1 - \frac{S}{\text{radius}}$$

We also know that for the solar system, this γ factor is nearly one. Hence we can use Sect. A.3 and see that the γ factor itself is about half of that amount smaller than one, that is, by $1/2 \times S/\text{radius}$, and the inverse of its third power is about three times of that amount *larger* than one. Hence the **perihelion** of an ellipse will **advance** each year by the angle

$$\text{perihelion advance} = 360 \text{ degrees} \times \frac{3}{2} \times \frac{S}{\text{radius}} \qquad (9.17)$$

The nearer the planet is to the sun, the larger is the effect.

Let us check this for earth. Earth moves nearly in a circle around the sun, so we can use the formula (9.17). Earth is at the distance 1.5×10^{11} from the sun. This is nearly the radius, because gravity of the sun is weak. Hence the perihelion of the ellipse advances per year about

$$360 \times \frac{3}{2} \times \frac{2.96 \cdot 10^3}{1.5 \times 10^{11}} \approx 1.066 \times 10^{-5} \text{ degrees}$$

Per hundred years, this piles up hundred times to 1.066×10^{-3} degrees. Multiplying with 60 gives this in **arc minutes**, and again with 60 gives the perihelion advance in **arc seconds**. Hence the ellipse of earth should rotate by

3.8 arc seconds per century

This is what astronomers observe!

9.5 Strong Gravity Near Black Holes

In Sects. 9.2 and 9.4, we have seen how the theory of general relativity acts, if gravity is weak. The reason was in both cases, that the Schwarzschild radius $S \approx 3000$ m is much smaller than radius of the sun $\approx 6.96 \cdot 10^8$ m, and therefore much smaller than the smallest radius of the light beam or the planet moving around the sun. We can estimate the weakness of the gravity also from the speed of the test-mass moving free-falling in a circle around the sun: According to the Kepler law (9.15), the squared speed in fractions of the speed of light, is just half the fraction of the Schwarzschild radius and the radius of the circular path of the test-mass,

$$\frac{\text{speed}^2}{c^2} = \frac{1}{2} \times \frac{S}{\text{radius}} \tag{9.18}$$

For example, the speed of the earth around the sun is, using for the radius as approximation its distance $1.50 \cdot 10^{11}$ m from the sun,

$$\frac{\text{speed}}{c} \approx \frac{1}{10000}$$

which is really much slower than the speed of light.

We see also from the Kepler law (9.18), that the nearer the test-mass circles around the perfect ball, the faster it will be. If the perfect ball is a **black hole**, then how near can the test-mass circle free-falling around the black hole?

The speed of the test-mass entering the Kepler law is in terms of the time of us resting relatively to the perfect ball in an inertial state. We know that no mass can pass an observer at a speed faster than light. To let the test-mass pass us in our inertial state, we free-fall in, as usual, from our place resting far away from the perfect ball, as you see in Fig. 9.14.

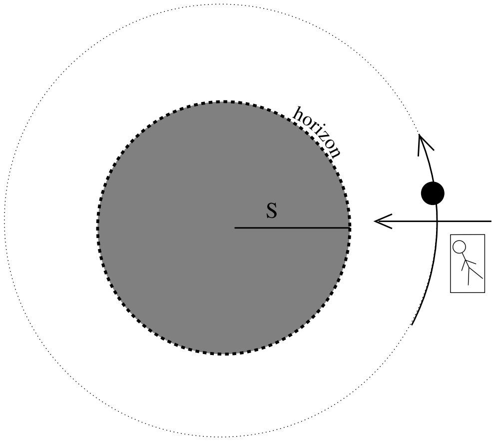


Fig. 9.14 Testing how near to a black hole the black test-mass can free-fall around it in a circle

The horizontal part of the speed, at which the black test-mass is passing us, is the speed of the Kepler law (9.18), at which the test-mass is going around the black hole,

$$\frac{(\text{horizontal speed})^2}{c^2} = \frac{1}{2} \times \frac{S}{\text{radius}} \tag{9.19}$$

Its vertical part is our speed relative to the black hole, given by Eq. (9.3),

$$\frac{(\text{vertical speed})^2}{c^2} = \frac{S}{\text{radius}} \tag{9.20}$$

Our time runs at the same pace as the time in which we measured the Kepler law. Hence we get the total speed with which the black test-mass is passing us, simply by using the **Pythagoras theorem**, as in Fig. 9.15,

$$\frac{(\text{total speed})^2}{c^2} = \frac{1}{2} \times \frac{S}{\text{radius}} + 1 \times \frac{S}{\text{radius}} = \frac{3}{2} \times \frac{S}{\text{radius}} \tag{9.21}$$

The test-mass can pass us with no more than the speed of light. Hence the left hand side of Eq. (9.21) is never larger than one, and so is therefore the right hand side. In other words: Any test-mass free-falling on a circle around the black hole, must do so at a radius which is at least 1.5 times larger than the Schwarzschild radius. At exactly 1.5 times the Schwarzschild radius, Eq. (9.21) tells us that a **light beam** will **go around** the black hole in a **circle**!

We can say even more: Any free-falling test-mass approaching the black hole nearer than 1.5 times the Schwarzschild radius, will fall into the black hole. Why is that? If the test-mass does not fall into the black hole, it will keep a smallest distance. Let us assume that the black test-mass in Fig. 9.16, is just passing at its smallest distance to the center of the black hole. Hence it must move at least with the horizontal speed, which it needs to continue on the dotted circular path. However, this dotted circle has a radius which is smaller than 1.5 times the Schwarzschild radius, so that

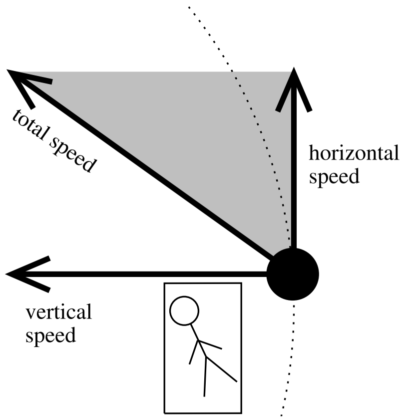


Fig. 9.15 The total speed relative to the free-falling observer we get from the Pythagoras theorem

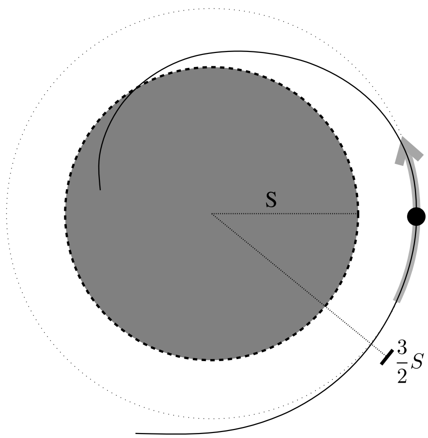


Fig. 9.16 A test-mass free-falling nearer to the center of the black hole than 1.5 times the Schwarzschild radius, will pass the **horizon**, that is, free-fall into the black hole

is impossible. Hence the black test-mass can only escape if it leaves its free-falling state, and fires for example a rocket motor, accelerating it to an orbit which is at least this 1.5 times the Schwarzschild radius away from the center of the black hole.

9.6 Gravity Waves

No body can influence its neighborhood faster than light, not even with the help of gravity *itself*. We make a thought experiment to see what happens, if the bended space-time around two bodies is changing with time. For simplicity, let two *equal* masses swing back and forth, under their gravity, and bouncing at each other *elastically*, like billiard balls. We sketched their movement in Fig. 9.17. Observe that they both are *no* test-masses: *Both* act with their gravity upon each other.

In the first line of the figure, the two balls start to move towards each other, because of their gravity. We illustrated the gravity as a spring, pulling the balls together.

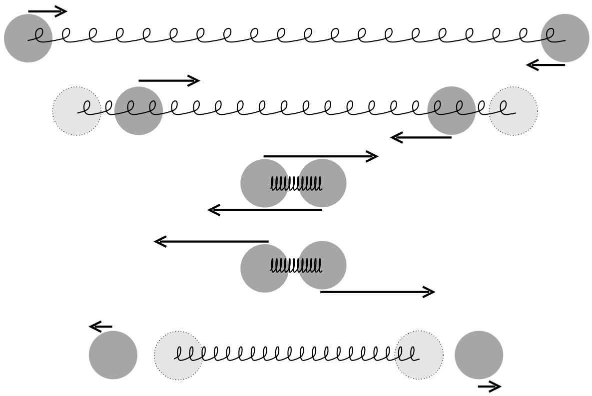


Fig. 9.17 Gravity acts weaker when the balls approach each other, than when the move away

In the second line, the balls move already at some speed. Now, gravity *itself* can only move at most with the speed of light. Hence in order to reach the right ball, the left ball must have acted with its gravity already a little *earlier*, in order that the gravity can reach the right ball at its position *right now*, and the same the other way round. We sketched the balls exercising their gravity in the past as dotted balls, and let the spring begin and end there. However, because the balls were then further apart, gravity acts *weaker* than at their actual position.

In the third line, the balls collide, and in the fourth line they bounce elastically. That is, they depart with the *same* speed as they have collided some moments ago. In other words, they do *not* lose any of their motion energy due to the bouncing.

In the last line, gravity acts again a little bit earlier than at their actual position, because it has to travel to influence the other ball. However, now gravity acts when the balls were nearer than at their actual position, so it is *stronger* than at their actual position.

In a nutshell: The balls use more motion energy to overcome their gravity while departing from each other, than they gain from their gravity while approaching each other. In total, they have *less* energy than before, when they are in rest again: They come nearer to each other any time they begin to approach each other, because of **delayed gravity**.

Where has the energy gone? By bending space-time between the balls, gravity itself has not stopped when it reached the other ball, but spread: It formed **gravity waves**, because gravity acts at a distance *delayed*.

How can we measure a gravity wave which is passing us? We saw in Fig. 5.12 that in vacuum, gravity does not contract sufficiently small volumes, but rather stretches them in one direction, and shrinks them in the other direction. Hence when a gravity wave is passing a volume marked with test-masses, it should react as shown in Fig. 9.18. After the volume has deformed, some of the energy of the gravity wave sits in the springs between the test-masses. This means that a gravity wave is carrying energy.

Up to now, 2013, nobody has detected gravity waves directly, but Hulse and Taylor found in 1974 that two very compact stars, called PSR-1913+16, move around

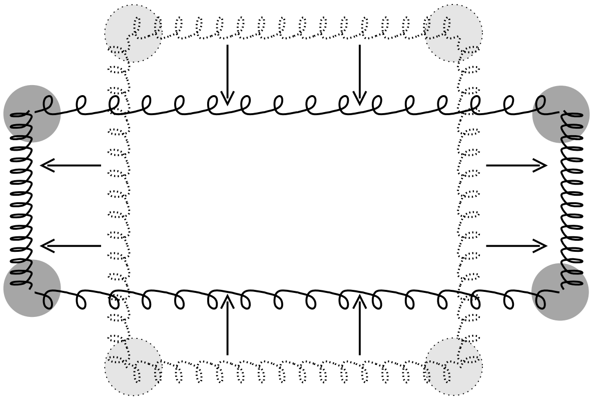


Fig. 9.18 As the gravity wave is passing, the volume marked by the test-masses is expanding in one direction, and shrinking in the other direction

each other at a very narrow distance. They have both about the same mass of roughly 1.4 times the sun mass. They must have come together from somewhere. Hence they lost energy by delayed gravity, and therefore *continue* to lose energy, by spiraling ever faster towards each other. One of the two stars is a so-called **pulsar**, that is, it rotates very fast and emits electromagnetic waves, like the rotating mirror of a lighthouse, acting as a timer. Physicists calculated with the help of the Einstein equation of gravity how much energy the two stars should lose to gravity waves. Now, after 30 years of gathering data, we know that the theory of general relativity describes this effect to at least 99 percent correctly.

9.7 Where Is the Gravity Energy?

Let us again free-fall vertically towards a perfect ball, as sketched in Fig. 9.19.

We start free-falling from the position at the far left. When we pass through the dashed imaginary circle, we already travel at some speed towards the perfect ball. At this instant, our dark-headed friend on the right who rested at this distance from the perfect ball, begins to free-fall. The perfect ball moves with much more speed towards *us* than it does towards our friend. Hence for *us*, the perfect ball has *more* mass than for him, as we know from Sect. 2.6. However, according to the equivalence principle, we are in the same state of inertia as we were when we started far away from the perfect ball. Hence we conclude that the perfect ball contains *more* mass when looked upon from a place resting far away from it, than when looked upon from resting at some smaller distance. The energy of this mass seems to be sitting between us and the perfect ball. The question is only, where exactly is this energy sitting?

Compare with the case of *elastic* energy. In Fig. 9.17 we used the picture that gravity attracts masses like a spring does. Hence maybe the space outside the perfect ball is bending like an elastic body, as the body sketched in Fig. 7.7 does? Inside the material we can imagine little springs holding the atoms together. When the material bends, the springs expand or contract, and such contain the bending energy.

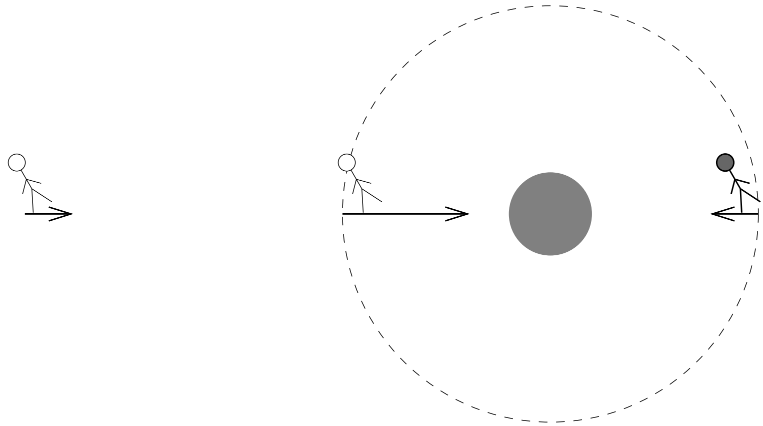


Fig. 9.19 Falling from different distances towards a perfect ball. The *arrows* show the direction and magnitude of the speed *relative* to the perfect ball

Therefore let us check a small volume of space outside the perfect ball, and search for the "gravity energy" in it. We mark the small volume by test-masses, which are resting relative to each other. However, when we release them, we know from the Einstein equation of gravity of Sect. 7.2, that the small volume will *not* begin to shrink, because we are outside the mass of the perfect ball. Hence there cannot be any gravity energy inside this small volume, which would have caused gravity.

However, in the case of gravity waves in Sect. 9.6, we saw that test-masses marking a small volume as in Fig. 9.18 *can* pick up energy from gravity in form of a gravity wave, even if it does *not* shrink the volume! We repeat this thought experiment now: We put test-masses connected by springs, in flat space-time, marking a small volume, far away from other masses, as you see in Fig. 9.20. We are in the middle of the test-masses. Then we let a perfect ball pass near us. While the perfect ball is passing, the test-masses will react to the bending of space-time more or less as in the Fig. 5.12. When the springs have picked up some energy, we lock the springs until the perfect ball has passed.

Then the test-masses and springs have more energy than before! What kind of energy did we pick up? The perfect ball moves *slower* away from us, than it approached us. Hence we did *not* pick up some energy of bending space around the perfect ball, but rather some small part of the *motion energy* of the perfect ball, relative to us.

To sum up: Precisely *because* bending space-time outside of a perfect ball does *not* shrink a small volume marked by test-masses which are resting relative to each other, we cannot find the gravity energy in a small volume, because we know that energy inside a small volume *would shrink* that volume. The gravity energy sits in the *total* bended space-time around the gravitating mass, but we *cannot* place it! Why, *where* do you want to place it? In *space* and *time*. That is the big difference: In Fig. 7.7, the *material* bends *in space-time*, whereas gravity bends *space-time itself*.

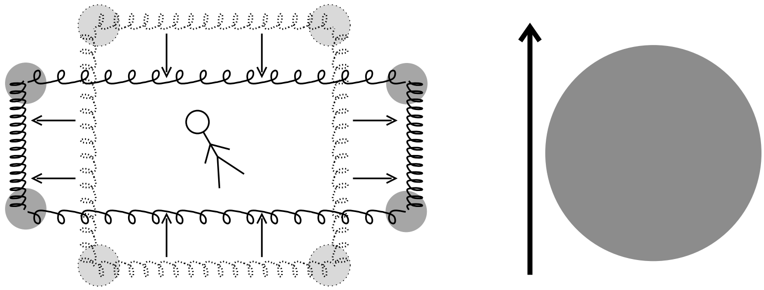


Fig. 9.20 As the perfect ball is passing, the volume marked by the test-masses is expanding in one direction, and shrinking in the other directions

9.8 Big Bang of the Universe

If we look with our naked eyes at the night sky, we see planets, stars, and galaxies. Between them, there is much void. However, to all our knowledge, the universe looks pretty much the same in all directions, if we look only at distances which are much larger than between galaxies. On such scales, there is everywhere the same amount of mass per volume. This observation is called the **cosmological principle**. What is more, only gravity seems to act over such large distances.

Hence we construct the following *simplest model* of the universe:

1. The energy, including mass, in the universe spreads everywhere, and in every direction, to the same extend.
2. Gravity dominates between the masses, but no other forces, and nearly no **pressure**.
3. The number of stars and galaxies in the universe does not change with time.

Then in what way the energy in the universe can bend space-time? In this model, time must run everywhere at the same pace, on a coarse scale, because mass spreads everywhere the same. Lengths can change, but again, lengths must change by the same amount *everywhere*, and on top of this, lengths must change by the same amount in *any direction*. The only way that this can happen, is that lengths only *change with time*. This whole picture is called the **Friedman model** of the universe.

We illustrate this in Fig. 9.21. A typical length is the distance of some not too near galaxies, which we sketched as stars. *All* bodies move away from each other, if time proceeds as we pass on from the left to the right picture, or move towards each other, if time proceeds as we pass from the right to the left.

The universe does not need to be infinitely large: Suppose that the universe had only two dimension, such as the surface of a balloon. The stars and galaxies we realize by adhering small pips to it. If we blow up the balloon, all pips on the balloon will move away from each other, and there is no center, just as in Fig. 9.21. However, the pips *themselves* do not grow! In a similar way, if three dimensional space around us will grow, all galaxies will move away from us, from a certain large average distance on, but the stars and galaxies themselves do not grow.

Then we like to get an answer to the following questions:

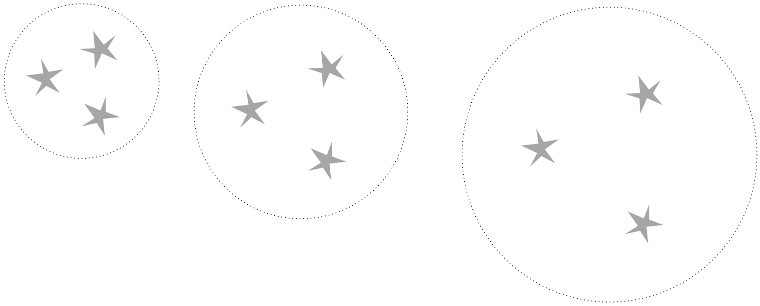


Fig. 9.21 The universe can expand or shrink, depending on whether time is proceeding as we pass from the left to the right, or vice versa

1. Can we not have a static universe, of must it shrink or expand?
2. It so, can we *calculate* the expansion or shrinking rate?

We start by looking in our neighborhood around us, and generalize to larger regions of the universe later on.

9.8.1 Small Ball of Mass in the Universe

We chose a small ball around us. In the real universe, "small" means "still large enough, so that the mass and energy spreads more or less equally in all directions". We sketched the ball in Fig. 9.22.

Then the rest of the universe does *not* influence our small ball at all! Why is that? We assumed in the Friedman model, that only gravity acts among the galaxies. What is more, mass spreads *in any direction* in the same way. Hence by the **Birkhoff theorem** of Sect. 8.3, the outside masses do not gravitate inside our small ball.

How does then the gravity of the masses acts inside the ball? As in Sect. 8.2, we place black test-masses at the surface of the ball, which rest there relative to each other, and let them loose, starting to free-fall. Then the Einstein equation of gravity (7.2) tells us:

> The *relative* rate at which the *volume* of the small ball marked by the test-masses begins to shrink, is 4π times the gravity constant times the *density* of the mass in the ball.

Galaxies, stars and the like at the surface of the ball may move relative to the resting test-masses, but they do not **accelerate relative** to the nearby test-masses, because they both free-fall. Hence we can let the ball move with the galaxies on its surface. Then all the same, the rate at which the growth of the ball begins to

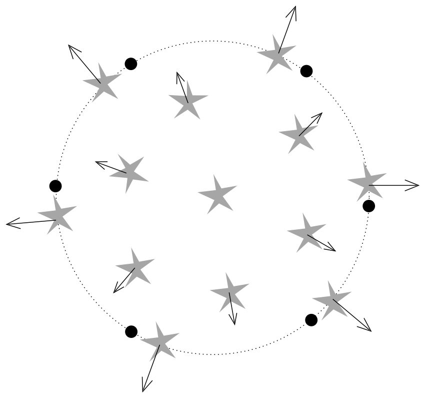


Fig. 9.22 We placed black test-masses at the rim of the ball, resting relative to each other. We just released them, so that they free-fall, as the nearby stars do, and therefore do *not* accelerate relative to the *nearby* stars

slow down, or the shrinking of the ball is quickening, is the same as for the test-masses. In other words: Forget about the test-masses. The Einstein equation reads now

> The *relative* rate at which the *volume* of the small ball marked by the galaxies, and moving with the galaxies on its surface, begins to slow down its growth, or begins to accelerate its shrinking, is 4π times the gravity constant times the *density* of the mass in the ball.

We express this law now in terms of the *radius* of the small ball. In the Friedman model universe, we can make the ball as small as we wish, because we assumed that mass spreads *everywhere* in the same way. This means also that because there is *per volume* everywhere the same amount of mass, that the smaller the ball, the less mass is in it, and the less the space-time is bending. Hence for a sufficiently small ball, it is nearly perfectly correct to use school-geometry to calculate its volume in terms of its radius.

In school-geometry, the volume of the small ball is in proportion to the *third* power of its radius. If for example the radius of the ball shrinks from 1 to 0.999, that is *relatively* by one per thousand, then its volume shrinks by $0.999^3 \approx 0.997$, that is *relatively* by three per thousand, three times as much as the *relative* shrinking of the radius.

Therefore the relative rate at which the growth of the volume of the small ball of galaxies begins to slow down, or the shrinking begins to accelerate, is *three* times the negative *acceleration* of the radius, per radius, and the Einstein equation of gravity (7.2) tells us that this is

$$-3 \times \frac{\text{acceleration of radius}}{\text{radius}} = 4\pi \times \begin{pmatrix}\text{gravity} \\ \text{constant}\end{pmatrix} \times (\text{mass density}) \qquad (9.22)$$

This acceleration is in the case of an expanding ball the rate at which its expansion is slowing down.

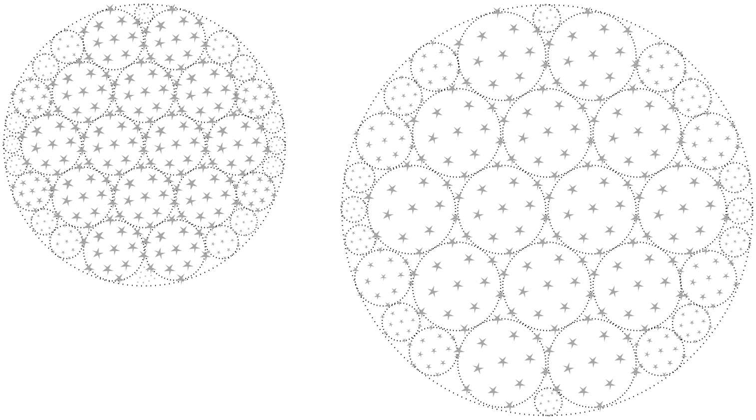


Fig. 9.23 The larger ball grows or shrinks *relatively* in the same manner, as all the small balls grow or shrink

9.8.2 Large Ball of Mass in the Universe

We really wanted to know how much larger portions of the universe behave under gravity. In the Friedman model, mass spreads to the same extent everywhere, so that even for a large ball of mass, as before, only the gravity of the masses *inside* this ball determine how much the ball will begin to shrink.

By the cosmological principle, the mass density on the right hand side of Eq. (9.22) is everywhere the same. Hence the radius of a ten times as large ball must begin to change *relatively* in the same way, because we can fill it with smaller balls, all of which begin to change by the same *relative* amount, as you see in Fig. 9.23. Therefore we can use this equation for a ball of *any size* in the universe!

We want to rewrite the gravity law for the universe in terms of the *mass* inside a large ball, because in our model, the mass of the stars and galaxies inside a ball like in Fig. 9.22 does not change in time because the ball is growing or shrinking with the distance of the galaxies. However, the mass *density* will change, if the ball grows or shrinks. Mass density is mass per volume, so we can use that instead of the right hand side of the Einstein equation (9.22). However, inside a large ball there is so much mass, that space-time, and in particular space, will bend. Hence we cannot use school geometry: The volume of that large ball, at a given time, is *not* in proportion to the third power of its radius, but there is a much more complicated law connecting the radius and the volume of a large ball.

However, there is a way out: Please have again a look at Fig. 9.23. We see that if the volume of the large ball grows by, say, 2 percent, that then the volumes of the small balls also grow or shrink *relatively* by that amount. Hence the *volume* of the large ball grows or shrinks *relatively* by the same amount as the volumes of the small balls. We also learned at the end of the last section, that the radius of a large and a small ball grow or shrink *relatively* by the same amount.

Finally, we know that because school geometry holds nearly perfectly for the small balls, that the volumes of the small balls shrink or grow *relatively* three times as much as their radius. Hence also the volume of the large ball will grow or shrink *relatively* three times as much as *its* radius,

$$\text{volume of ball} = \frac{4\pi}{3}\,\frac{(\text{radius})^3}{\text{constant}}$$

For very small balls, the constant is one, and we get again the formula from school geometry.

The constant does depend on the radius of the ball at *some* given time, but does *not* change with time. Hence the mass density of the ball is in terms of the radius of the ball,

$$\begin{aligned}\text{mass density} &= \frac{\text{mass of ball}}{\text{volume of ball}}\\ &= \frac{\text{mass of ball}}{(\text{radius})^3} \times \frac{\text{constant}}{\frac{4\pi}{3}}\end{aligned} \tag{9.23}$$

Hence we can use the Einstein equation of gravity (9.22), which we write here again,

$$-3 \times \frac{\text{acceleration of radius}}{\text{radius}} = 4\pi \times \begin{pmatrix}\text{gravity}\\ \text{constant}\end{pmatrix} \times (\text{mass density})$$

and replace the "mass density" in terms of the mass and radius of the large ball of (9.23),

$$-\cancel{3} \times \frac{\text{acceleration of radius}}{\cancel{\text{radius}}} = \frac{\cancel{4\pi}\binom{\text{gravity}}{\text{constant}} \times (\text{mass of ball})}{(\text{radius})^{\cancel{3}\,2}} \times \frac{\text{constant}}{\frac{\cancel{4\pi}}{\cancel{3}}}$$

One radius and the factors 3 and 4π we can cancel on both sides, so

$$\text{acceleration of radius} = -\frac{\binom{\text{gravity}}{\text{constant}} \times (\text{mass of ball})}{(\text{radius})^2} \times \text{constant}$$

The radius we can replace with the distance to the center of the ball, because in the Friedman model both are in proportion. Then the constant changes its value, but the equation remains in the same form

$$\begin{pmatrix}\text{acceleration of}\\ \text{distance to center}\end{pmatrix} = -\frac{\binom{\text{gravity}}{\text{constant}} \times (\text{mass of ball})}{(\text{distance to center of the ball})^2} \times \text{constant} \tag{9.24}$$

This is the **Friedman-equation**. We have seen this equation before: Because time in the universe runs everywhere at the same pace, the equation looks up to the constant

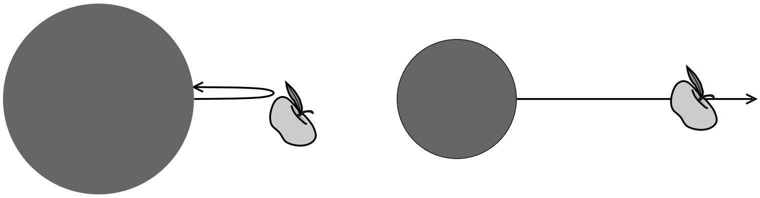


Fig. 9.24 The universe is expanding in the same way as an apple thrown vertically upwards from a small planet is slowing down

on the right, just like the *Newton law of gravity* for a perfect ball (8.7). Let us coin a catch-phrase:

For the universe as a whole, the law of gravity looks like the Newton law!

We can solve this equation in the opposite way as Newton did: Not by looking at an apple falling from a tree, but by *throwing* an apple *vertically* into the sky. To get rid of the air drag, we do it as the little prince of Saint-Exupéry would do, on the surface of several small planets of different mass, with the same initial speed. In Fig. 9.24 we sketched what happens: The planet on the left has *enough* mass, so that the apple will eventually fall back. The planet on the right is so light that the apple is leaving it, moving forever away.

We have found again an **exact solution** of the **Einstein equation of gravity**!

Now we can answer the first question of p. 109: A *static* universe is *not stable*: It will immediately start to collapse, just as the apple resting above the planet will start to fall down. The universe has only two choices: To expand, or to collapse. In fact, astronomers know since about 80 years ago, that the distant galaxies all move away from each other. Hence they all must have been much more closer in the past: There must have been a **big bang**, where all energy of the universe suddenly started to expand.

In the same way as for the apple on the small planet, if the universe has enough mass, then the expansion caused by the big bang will eventually stop and the universe will collapse in the future. If not, then the universe will forever expand, but in any case, the expansion will *slow down*.

Therefore we should observe that the distant galaxies are moving *faster* away from us than the nearer galaxies, because in the past the universe should have expanded faster, and slowed down its expansion up to today. However, in the last ten years or so astronomers found the *opposite* to be true: The universe *expands ever faster*!

This is still a riddle, but there is one possibility that we neglected: The *energy and pressure of the vacuum*.

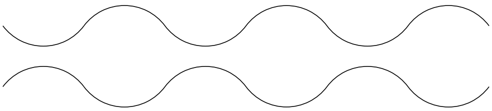


Fig. 9.25 Two light waves wiping each other out. In reality they are on top of each other, but we drew them beside each other, for better view

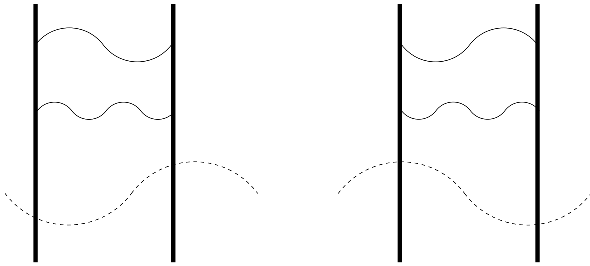


Fig. 9.26 Two parallel metal plates in vacuum, with swinging light waves in between: We took the *right* picture at some later time than the *left* picture

9.9 Vacuum Energy and Gravity

We have a definite idea about the **vacuum**: It is what is left over, when we removed everything. Hence nothing should be left, especially no energy. However, experiments show that energy *is* left over! Let us see how. Suppose two light waves travel through empty space, through the same path, but swinging in the opposite way, as sketched in Fig. 9.25. Then in the same way as water waves, they wipe each other out: No wave is left over. However, both waves carry a *positive* amount of energy. Therefore no wave, but **vacuum energy** is left.

If such waves wipe each other out, how can we see them anyway? We can, and we now describe how they have been seen. Put two metal plates near to each other in vacuum. Metal does not allow light waves to pass, at least no waves with small enough wave lengths. Also, the light waves cannot swing inside the metal. In Fig. 9.26, we show in the left picture swinging light waves at one time instant, and in the right picture some time later. Compare the two pictures: The upper two, solid waves fit between the plates, because near the plates they do not swing. However, the dashed wave does swing near the plates, so it cannot fit. This means that there are *more* possible waves and vacuum energy *without* metal plates than *with* metal plates.

Now, move the metal plates nearer together, say, to one-fourth of their distance, as you see in Fig. 9.27. Then the upper wave does *not* fit any longer. Only the wave in the middle still fits. Hence *the more* the metal plates approach each other, *the less* energy fits between them. Therefore the metal plates begin to approach each other *by themselves*, to lower their energy. This is what physicists observe! It is the **Casimir effect**.[1]

[1]It has been observed in 2001 by physicists G. Bressi, G. Carugno, R. Onofrio, and G. Ruoso. You can download the original paper here: http://arxiv.org/abs/quant-ph/0203002.

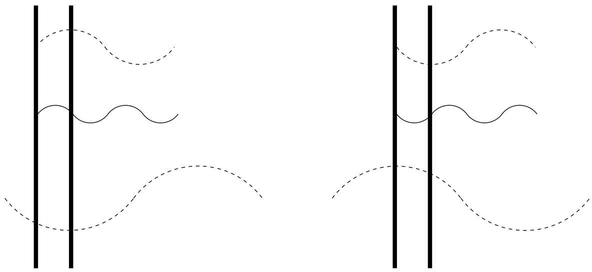


Fig. 9.27 The nearer the metal plates comes together, the less light waves fit in-between

How much energy does such a light wave carry? Its energy is in proportion to its *inverse* wave length. The constant of proportion is the **Planck constant** times the speed of light. It is one of the most important phenomenons of the subatomic world, and we would need **quantum theory** to estimate the Casimir effect. This is out of the scope of this book. In fact Casimir has done the calculation, and indeed quantum theory predicts the correct attraction between the two plates.

However, we can take the Casimir effect as experimental fact, showing that these "ghost" waves are real. By the way, they are called **vacuum fluctuations**. We see that the more we look into small pieces of space, the more such waves with smaller and smaller wave lengths fit into that space. Because their energy goes as the inverse of their wavelength, this energy grows more and more! And *any* type of energy will gravitate!

Hence we have the same situation as we already discussed shortly in Sect. 7.3: The vacuum *itself* acts as a kind of gas in which matter immerses. We see from the Casimir effect that the vacuum is carrying energy, so it must exert **pressure**. Then what pressure does the vacuum energy exert? For this, let us do a thought experiment that we *never* can do for real. In Fig. 9.28, we sketched a lightly shaded volume containing vacuum, with *nothing, not even a vacuum* around. That is of course why we never can realize this experiment, but never mind. We saw that vacuum has some definite positive energy. Hence to create more vacuum, we need to add energy. That means, we have to do some work to pull the piston to the right.

Compare to positive air pressure: We have to do work to *compress* the tire of a bicycle. So for vacuum it is the other way round: It has **negative pressure**. Hence in contrast to what we learned about the energy density in a pressure cooker in Sect. 1.12, we conclude that the pressure is here just the *negative* vacuum energy density itself. This pressure is the same in any of the three directions of space. The complete Einstein equation of gravity (7.3) tells us that instead of the energy density of the vacuum we have to use

$$\begin{pmatrix}\text{energy density}\\ \text{of vacuum}\end{pmatrix} + 3\begin{pmatrix}\text{pressure} = \text{negative}\\ \text{energy density}\\ \text{of vacuum}\end{pmatrix} = (-2)\begin{pmatrix}\text{energy density}\\ \text{of vacuum}\end{pmatrix}$$

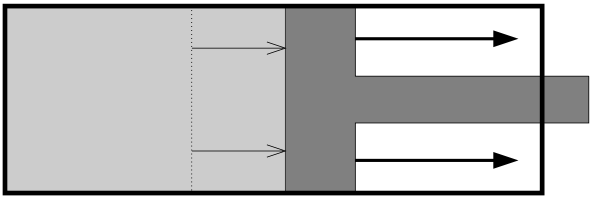


Fig. 9.28 Positive vacuum energy means negative pressure

This is a *constant*, *negative* mass density, if we divide by c^2 using Eq. (1.2). Hence our equation for the universe expansion (9.22) gets a new source of acceleration,

$$-3\frac{\binom{\text{extra acceleration}}{\text{of radius}}}{\text{radius}} = 4\pi\binom{\text{gravity}}{\text{constant}} \times (-2)\frac{\binom{\text{energy density}}{\text{of vacuum}}}{c^2}$$

Because the distance from the center is in proportion to the radius, we can use it as well, and use the constant to correct for larger balls, as before:

$$-3\frac{\binom{\text{extra acceleration}}{\text{of distance to center}}}{\text{distance to center}} = 4\pi\binom{\text{gravity}}{\text{constant}} \times (-2)\frac{\binom{\text{energy density}}{\text{of vacuum}}}{c^2} \times \text{constant}$$

In other words: This extra acceleration is positive and in proportion to the distance to the center:

$$\binom{\text{extra acceleration}}{\text{of distance to center}} = \frac{8\pi}{3c^2}\binom{\text{gravity}}{\text{constant}}\binom{\text{energy density}}{\text{of vacuum}} \times \binom{\text{distance}}{\text{to center}}$$

Hence the more the universe expands, the *larger* is this extra acceleration, while the original slowing down of Eq. (9.24) is becoming *smaller*. Therefore already today the vacuum energy should dominate the fate of the universe!

We say "should" because there is a snag: You can already guess that many, many types of ghost waves fit in the universe. So there should be a huge amount of vacuum energy, actually so large, that its huge negative pressure should make the universe explode right now! So there must be something else, some mechanism which keeps the vacuum energy much smaller. Nobody has any convincing explanation why the vacuum energy is so small, that we see the universe expanding so *slowly*. Or to put it another way:

It seems that at present we do not understand how the effects of the subatomic world fit into bended space-time, or how bending space-time influences the subatomic world.

Chapter 10
Epilogue

We saw that the theory of relativity is based on four principles:

1. Light passes near an observer always at the same speed c.
2. Time, lengths, and all other steady speed have only a meaning relative to some observer.
3. The equivalence principle: A small enough mass moves under gravity, that is in bended space-time, free-falling, with steady proper time.
4. Mass bends space-time in the simplest possible way fitting with the first three principles: The rate at which the volume of a small enough, resting cloud of small pieces of matter begins to shrink, grows in proportion to the *mass* in that cloud.

The first two principles are the foundation of the theory of special relativity, and all together form the theory of general relativity.

The theory does not ask, what *technically* makes masses move, or what atomic structure mass has, but explores how mass, momentum, energy on one hand, and time and space on the other hand, interact. It even absorbs one force of nature, gravity, into bended space-time, so that gravity turns out to be no force at all. Einstein created this theory using only meticulous physical reasoning. All that makes the theory of relativity so beautiful.

We saw that especially the *equivalence principle* is a powerful tool, to see how mass is *reacting* on bended space-time. It is not only the famous elevator allegory of Sect. 5.1, repeated in many popular books: Because mass acts *and* reacts on bended space-time, the equivalence principle helps to solve the Einstein equation of gravity, or to derive the Friedman model of the universe, for example.

Checked again and again over a period of over hundred years, the theory of relativity serves today as a *frame* in which more detailed theories of matter have to *fit in*. For example, electrodynamics, describing all electrical and magnetic processes, fits naturally in from the beginning.

K. Fischer, *Relativity for Everyone*, DOI 10.1007/978-3-319-00587-4_10,

It took a long time to fit **quantum theory**, that is the theory of the small, at least into the frame of *special relativity*, that is, the first two of the four principles above. This beared much fruit, but it would need at least one more book to describe its weird, yet real phenomena, such as anti-matter.

It remains for future generations to merge this theory of the small with the theory of **general relativity** describing the largest structures in our world.

Appendix

A.1 Important Numbers

The numbers in Table A.1 are in fact known to higher precision, but given here rounded to two decimal places.

A.2 Inertia of Pure Energy in Detail

Please have a look at Fig. A.1. While building up, the light package **presses** against the wall. Pressure itself is force per area of the wall. So let us fix the area of the right side of the wall to be just one square-meter, so that pressure and force are the same. During the time the light is pressing against the wall, the wall receives a "push", that is, **momentum**. This recoil momentum is mass times speed of the wall, as we saw in Sect. 2.6. Double the time, or double the pressure, and you will get twice as much recoil from the wall. Hence the recoil is the pressure of the light, times the time it takes to build up the package of light,

$$\text{(wall mass)} \times \text{(wall speed)} = \text{pressure} \times \text{time}$$

By the way, nowadays we can measure this light **pressure** in the laboratory. We saw in Sect. 1.12 that this light pressure is energy per volume. This volume is the surface area of the wall, which we set to one, times the width of the light package. Hence the pressure times the time it acts, is the energy of the light, per width of the light package, times the time it acts on the wall

$$\text{pressure} \times \text{time} = \frac{\text{energy}}{\text{width}} \times \text{time}$$

The package of light builds up with the speed of light, so that $\frac{\text{width}}{\text{time}}$ is just c,

$$\text{pressure} \times \text{time} = \frac{\text{energy}}{c}$$

K. Fischer, *Relativity for Everyone*, DOI 10.1007/978-3-319-00587-4,

Table A.1 Important quantities and their units

Name of the number	Value	Unit
speed of light c	$2.99792458 \times 10^8 \approx 3.00 \times 10^8$	m/s
gravity constant G	6.67×10^{-11}	$m^3/(kg\,s^2)$
sun's mass	1.99×10^{30}	kg
sun's radius	6.96×10^8	m
sun's Schwarzschild radius	2.96×10^3	m
earth's mean distance from sun	1.50×10^{11}	m

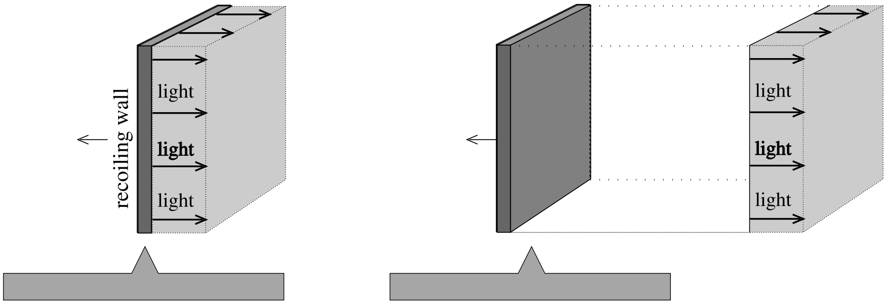


Fig. A.1 A wall recoils to the *left*, because pure energy left the wall to the *right*

Hence the wall will move to the left with speed

$$\begin{aligned}(\text{wall mass}) \times (\text{wall speed}) &= \text{pressure} \times \text{time} \\ (\text{wall mass}) \times (\text{wall speed}) &= \frac{\text{energy}}{c^2} \times c\end{aligned} \tag{A.1}$$

After "some time" has elapsed, the wall traveled to the left to some distance, which is the product (wall speed) × (some time). During this "some time", the light package traveled some distance c × (some time) to the right. Hence we multiply Eq. (A.1) for the wall speed with this "some time", and we know how far the wall has traveled in the right picture of Fig. A.1,

$$(\text{wall mass}) \times (\text{wall distance}) = \frac{\text{energy}}{c^2} \times (\text{light package distance})$$

The center of the total mass is still at rest. Hence while the wall carries its mass to the "wall distance" to the left, the light, that is the **pure energy**, carries some **mass** to the "light package distance". Hence we read off the previous equation that this is the energy of the light package, divided by c^2,

$$\text{mass of light package} = \frac{\text{energy}}{c^2} \tag{A.2}$$

This is just Eq. (1.2).

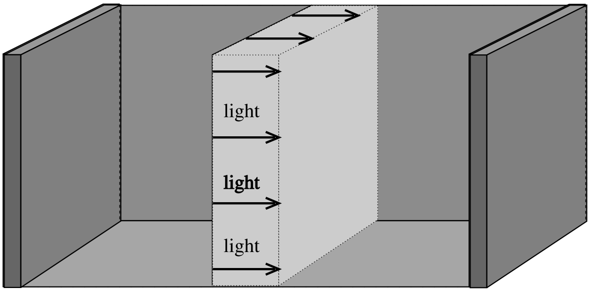


Fig. A.2 A packet of light moves from the *left* to the *right* wall of a box

We made several small mistakes: If the light carries away some mass from the wall, the wall must have lost this mass. However, we can make this error as small as we want by making the wall as massy as we wish. Likewise, it takes the wall some time to react, because the left part of the wall cannot know that light has left the right part already. However, we can think of the wall as being as thin and dense as we wish.

Said Einstein in 1905:

> Wenn die Theorie den Tatsachen entspricht, so überträgt die Strahlung Trägheit zwischen den emittierenden und absorbierenden Körpern.

That is to say:

> If the theory of relativity is correct, then radiation carries inertia between the emitting and adsorbing body.

Apropos "adsorbing body": In the original setup by Einstein the wall was the left wall of a box, and the right wall of the box absorbed the light again, as in Fig. A.2. He made basically the same calculation as we did here.

However, then Einstein made the mistake to assume that this *whole* box recoiled *without delay* from the emitted light. This contradicts the theory of relativity, because the right wall must then react *before* the light package reached it, that is, it must have received the forces making it react with a speed *faster* than light! You see that sometimes even Einstein himself had the wrong intuition about relativity.

A.3 Relativity for Small Speeds

We know that relativistic effects nearly always depend on the speed via the γ factor. For small speed, we know that the γ factor is nearly one. We want to know to what extend it is different from one. Check by hand or with a pocket calculator that

$$\left(1 - \frac{1}{2} \times 0.01\right)^2 = 0.995^2 = 0.990025 \approx 0.99 = 1 - 0.01$$

$$\left(1-\frac{1}{2}\times 0.001\right)^2 = 0.9995^2 = 0.99900025 \approx 0.999 = 1-0.001$$

so that by taking roots on both sides we have the estimate

$$1-\frac{1}{2}\times 0.001 \approx \sqrt{1-0.001} = (1-0.001)^{\frac{1}{2}}$$

and we see that this estimate becomes the better, the less the γ factor differs from one. Therefore we can now estimate the γ factor at small speed

$$1-\frac{1}{2}\left(\frac{\text{speed}}{c}\right)^2 \approx \sqrt{1-\left(\frac{\text{speed}}{c}\right)^2} = \gamma \tag{A.3}$$

The same pattern holds for other powers: If a number like 0.999 differs from one only by a very small amount -0.001, then the power of that number, differs from one by that amount -0.001 times that power, if the power is not too large. For example, for the minus first or minus third power, we have

$$(1-0.001)^{-1} = \frac{1}{0.999} \approx 1.001001\cdots \approx 1.001 = 1+(-1)\times(-0.001)$$
$$(1-0.001)^{-3} = \frac{1}{0.999^3} \approx 1.0030006\cdots \approx 1.003 = 1+(-3)\times(-0.001)$$

and so on. Using the previous estimate (A.3), we have an estimate for the inverse γ factor at small speed,

$$\begin{aligned}\gamma^{-1} &= \frac{1}{\gamma} \approx \frac{1}{1-\frac{1}{2}(\frac{\text{speed}}{c})^2} \approx 1+\frac{1}{2}\left(\frac{\text{speed}}{c}\right)^2 \\ \gamma^{-3} &= \frac{1}{\gamma^3} \approx \frac{1}{1-\frac{3}{2}(\frac{\text{speed}}{c})^2} \approx 1+\frac{3}{2}\left(\frac{\text{speed}}{c}\right)^2\end{aligned} \tag{A.4}$$

A.4 Speed Addition from Growing Mass

From Fig. A.3 we see that the two balls of **resting-mass** m_0 have *relative to the box* the total mass

$$M_0 = \frac{2m_0}{\sqrt{1-\frac{u^2}{c^2}}}$$

because the square of both speeds $u^2 = (-u)^2$. This mass M_0 is the *resting-mass* of the box, because we assumed the box itself to be very light, so that the balls make

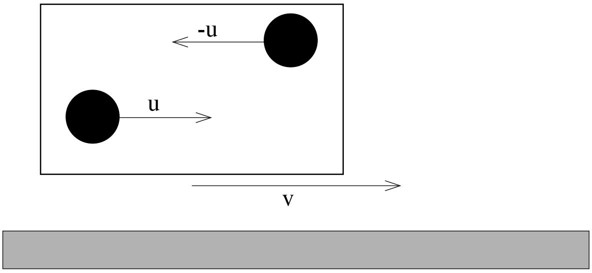


Fig. A.3 The *left ball* moves in the *light box* with speed u to the *right*, and the *right ball* with speed $-u$ to the *left*, *relative* to the *box*. The *box* itself moves with speed v to the *right*, *relative* to the ground

up nearly all of its mass. Hence, when the box moves with speed v, its mass should be

$$\frac{M_0}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{2m_0}{\sqrt{1-\frac{u^2}{c^2}} \times \sqrt{1-\frac{v^2}{c^2}}} \tag{A.5}$$

We check this mass, adding up the masses of the balls one at a time. If one ball moves in the box with a fraction $\frac{u}{c}$ of the speed of light, and the box moves relative to the ground with a fraction $\frac{v}{c}$ of the speed of light, then we claim that the ball moves relative to the ground not with a fraction $\frac{u}{c} + \frac{v}{c}$ of the speed of light, but with a smaller **total speed** w, whose fraction $\frac{w}{c}$ of the speed of light is,

$$\frac{w}{c} = \frac{\frac{u}{c} + \frac{v}{c}}{1 + \frac{u}{c} \times \frac{v}{c}} \tag{A.6}$$

To see this, we calculate the γ factor for w:

$$1 - \left(\frac{w}{c}\right)^2 = 1 - \left(\frac{\frac{u}{c} + \frac{v}{c}}{1 + \frac{u}{c} \times \frac{v}{c}}\right)^2 = \frac{(1 + \frac{u}{c} \times \frac{v}{c})^2}{(1 + \frac{u}{c} \times \frac{v}{c})^2} - \frac{(\frac{u}{c} + \frac{v}{c})^2}{(1 + \frac{u}{c} \times \frac{v}{c})^2}$$

We rearrange the numerator, crossing out the factors that cancel each other. We find that we can write it as product of the γ factors of u and v:

$$\begin{aligned}
&\left(1 + \frac{u}{c} \times \frac{v}{c}\right)^2 - \left(\frac{u}{c} + \frac{v}{c}\right)^2 \\
&= 1 + \cancel{2\frac{u}{c} \times \frac{v}{c}} + \left(\frac{u}{c}\right)^2 \left(\frac{v}{c}\right)^2 - \left(\frac{u}{c}\right)^2 - \cancel{2\frac{u}{c} \times \frac{v}{c}} - \left(\frac{v}{c}\right)^2 \\
&= \left(1 - \frac{u^2}{c^2}\right) \times \left(1 - \frac{v^2}{c^2}\right)
\end{aligned}$$

Hence, taking roots,

$$\sqrt{1 - \left(\frac{w}{c}\right)^2} = \frac{\sqrt{1-\frac{u^2}{c^2}} \times \sqrt{1-\frac{v^2}{c^2}}}{1 + \frac{u}{c} \times \frac{v}{c}}$$

Therefore the ball traveling with speed u inside the box traveling with speed v gets the total mass

$$\frac{m_0}{\sqrt{1-\frac{w^2}{c^2}}} = \frac{m_0(1+\frac{u}{c}\times\frac{v}{c})}{\sqrt{1-\frac{u^2}{c^2}}\times\sqrt{1-\frac{v^2}{c^2}}}$$

Replacing u by $-u$ we arrive at the γ factor for the ball moving left with speed $-u$ inside the box. This changes only the sign in the numerator from $1+\frac{u}{c}\times\frac{v}{c}$ to $1-\frac{u}{c}\times\frac{v}{c}$. Therefore the two moving masses add to

$$\frac{m_0(1+\cancel{\frac{u}{c}\times\frac{v}{c}})}{\sqrt{1-\frac{u^2}{c^2}}\times\sqrt{1-\frac{v^2}{c^2}}} + \frac{m_0(1-\cancel{\frac{u}{c}\times\frac{v}{c}})}{\sqrt{1-\frac{u^2}{c^2}}\times\sqrt{1-\frac{v^2}{c^2}}} = \frac{2m_0}{\sqrt{1-\frac{u^2}{c^2}}\times\sqrt{1-\frac{v^2}{c^2}}}$$

that is, to the same mass as in Eq. (A.5). Hence Eq. (A.6) is the correct way to add parallel speeds.

A.5 Einstein Equation of Gravity in Terms of Tensors

The complete Einstein equation of gravity of Sect. 7.3 reads as follows:

The *relative* rate at which a small enough, resting cloud of matter begins to shrink, grows in proportion to the *energy density* plus the *pressures* in each of the three directions in that cloud. The constant of proportion is 4π times the gravity constant G, divided by the square of the speed of light, or in short,

$$\begin{pmatrix}\text{relative shrinking}\\ \text{beginning rate}\end{pmatrix} = \frac{4\pi G}{c^2}\left[\begin{pmatrix}\text{energy}\\ \text{density}\end{pmatrix} + \begin{pmatrix}\text{sum of pressures}\\ \text{in each direction}\end{pmatrix}\right] \qquad (A.7)$$

In order to make contact to text books, we write this equation in terms of tensors. We cannot give here a course on tensor analysis, but at least explain what the symbols mean. For matter resting relative to each other and us, energy density is one component of the **energy tensor** T^m_n. This tensor has sixteen components, which are labeled by two indices m and n, both of which can take the values 0, 1, 2, or 3. Energy density is the component T^0_0. The *negative* of the component T^1_1 is the pressure in one direction of space, and $-T^2_2$ and $-T^3_3$ in the other two directions. For resting matter without internal stress, like a gas or a liquid, the other components are zero.

The relative volume shrinking beginning rate is the component R^0_0 of the **Ricci tensor**, times c^2. Hence the complete Einstein equation of gravity is

$$R^0_0 = \frac{4\pi G}{c^4}\left(T^0_0 - T^1_1 - T^2_2 - T^3_3\right)$$

We need still a tensor U whose component U_0^0 gives the right hand site of the equation. There is the tensor δ_n^m which is one if $m = n$ and zero else. Using the tensor

$$T = T_0^0 + T_1^1 + T_2^2 + T_3^3$$

which has only this one component T we can built the tensor $T\delta_n^m$ and from it the tensor

$$U_n^m = 2T_n^m - T\delta_n^m$$

Its ${}_0^0$ component is

$$U_0^0 = 2T_0^0 - \left(T_0^0 + T_1^1 + T_2^2 + T_3^3\right) = T_0^0 - T_1^1 - T_2^2 - T_3^3$$

Hence the Einstein equation of gravity in terms of the energy tensor and Ricci tensor is

$$R_0^0 = \frac{4\pi G}{c^4}\left(2T_0^0 - T\delta_0^0\right) \tag{A.8}$$

In Sects. 7.1 and 7.2, we chose the simplest case in which pressure is zero, for a small cloud of test masses which are resting relatively to each other. In this case, all components of the energy tensor are zero, except the component T_0^0, which gives the energy density. For more complicated distributed and moving masses, we need the equations all for the components of the Ricci and energy tensor. This is the **Einstein equation of gravity in terms of tensors**,

$$R_n^m = \frac{4\pi G}{c^4}\left(2T_n^m - T\delta_n^m\right) \tag{A.9}$$

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K. Fischer, *Relativity for Everyone*, DOI 10.1007/978-3-319-00587-4,