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Contents

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Preface

Overview

Why another introductory textbook on partial differential equations when so many are already available? The question is rightly asked, and the justification is in order.

We have found that every year an increasing number of students enter advanced courses involving boundary value problems, which deal mostly with numerical techniques, such as finite difference, finite element, or boundary element methods. At the same time they are introduced to partial differential equations as graduate students, although a few do manage to acquire some knowledge of the subject through other courses in engineering and physics. It is a pity than an opportunity to learn this subject at undergraduate level is lost, because the students encounter textbooks that are graded strictly for graduate level courses. Even when the textbooks are written for undergraduate students, quite often that may not be the case for a majority of students. Most of these textbooks, though written with quality material, are generally based on hard analysis. Our textbook, written for a two–semester course, is aimed at attracting junior and senior undergraduate students, so they get an early training in the subject and do not miss out on elementary techniques and simple beauty of the subject. In the pedagogical spirit of moderation we have avoided the extreme situation where a beginner's course is so advanced and severe that it is likely to break the spirit of even mature students in an attempt to cover practically everything in the subject. On the other hand, one should encourage textbooks on this subject which in pace and thought are graded to undergraduate levels.

Accordingly, the authors have striven to produce a beginner's textbook which is mature, challenging, and instructive, and which, at the same time, is reasonable in its demands. Certainly, it is not claimed that partial differential equations can become easy and effortless. However, the authors' combined classroom experience over a number of years justifies the effort that the subject can be made reasonably easy to understand despite its complexity, provided that the student has a thorough background in multivariate calculus and ordinary differential equations. It can impart understanding and profit even to the undergraduate juniors and seniors who take it only for one semester before their graduation. The goal, then, has been to produce a textbook that provides both the basic concepts and the methods for those who will take it only for a semester, and a textbook which also provides adequate training and encouragement for those who plan to continue their studies in the subject itself or in applied areas. The distinctive features and the scope of the book can be determined from the table of contents.

Audience

Most of the material in this textbook, especially the first six chapters, is developed for a beginner's course on partial differential equations. These chapters are designed primarily for junior/senior level undergraduate students in mathematics, physics, and engineering who have completed at least the courses on multivariate calculus and ordinary differential equations, and possess some working knowledge of Mathematica in case they opt to use its versatility in symbolic manipulation and graphics capabilities. Adequate material on other topics from mathematical analysis is provided in the text as and when needed.

The book represents a two semester course. The first six chapters can be taught at the earliest after the completion of multivariate calculus and ordinary differential equations, while the remaining part definitely requires some degree of maturity. An important consideration at this point is the need for engineering and physics majors to learn the subject at an earlier stage. In most cases they start using partial differential equations and their solutions prior to any formal training in the subject. As a result, their understanding of applied technical areas is hampered by the lack of familiarity with the theory and methods of

partial differential equations. It is our hope that mathematics, physics and engineering majors will take this course at the beginning of their junior years and thus learn and enjoy other technical subjects with better understanding.

The book provides a comprehensive and systematic coverage of the basic theory and applications that can readily be followed by undergraduate students at the junior or senior level.

Salient Features

This textbook has evolved out of lecture notes developed while teaching the course to undergraduate seniors, and graduate students in mathematics, engineering, and physics at the University of New Orleans during the past three years. Much effort has gone into the organization of the subject matter in order to make the course attractive to students and the textbook easy to read. Although there is a large number of classical textbooks available on the subject, there has been a need for an introductory textbook with Mathematica. This publication, based on classroom experience, fulfills such a need. The mathematical contents of the book are simple enough for the average student to understand the methodology and the fine points of theory and techniques of partial differential equations. The Mathematica component has been presented in detail to bring out the salient features of different methods. The chapters present a balanced two–semester course material, which can be tailored to the needs of different levels of instructions. The following table outlines some suggested curricula at the three levels.

Elementary/Juniors	Beginning Seniors	Beginning Graduates
Chapter 1	Chapter 1	Chapter 2
Chapter 3	Chapter 2	Chapter 7
Chapter 4	Chapter 3	Chapter 8
Chapter 5	Chapter 4	Chapter 9
Chapter 6	Chapter 5	Chapter 10
	Chapter 6	

Chapter 1 provides some useful definitions, classification of second order partial differential equations, some well-known equations, and the superposition principle. The method of characteristics for first and second order partial differential equations is studied in Chapter 2. This topic is usually ignored in most of the textbooks, or delayed toward the end of the book. However, it is our opinion and experience that it helps the students understand the nature of the solutions and form a guide for higher order partial differential equations, provided the topic is handled with clarity and ample geometrical presentations. Mathematica is found to be very useful in achieving this perspective. An old technique of inverse operators, borrowed from the theory of ordinary differential equations, has been used in Chapter 3 to solve homogeneous and nonhomogeneous partial differential equations with constant coefficients. Chapter 4 puts together the concepts of orthogonality, orthonormality, orthogonal polynomials, series of orthogonal functions, trigonometric Fourier series, eigenfunction expansions, and the Bessel functions. This material is needed in Chapter 5 which deals with the method of separation of variables for boundary and initial value problems. These problems involve the wave, heat and Laplace equations, with homogeneous and nonhomogeneous boundary conditions, in the Cartesian, polar cylindrical and spherical geometries. The integral transforms, especially the Laplace and Fourier transforms, are presented in Chapter 6. These techniques are powerful tools to solve different types of boundary value problems with initial conditions.

The advanced material consists of the Green's functions (Chapter 7); weighted residual methods based on the theory of the variational calculus (Chapter 8); perturbation methods (Chapter 9) applied to problems involving partial differential equations only; and lastly the numerical methods based on finite differences, where Mathematica unfolds the intricate details and the beauty of these methods. We have decided not to include other numerical methods, like the finite element and boundary element methods, which are now fully developed into individual courses with many fine textbooks.

Although the text is rich in developing the underlying mathematical analysis with sufficient theorems and proofs, the emphasis is basically on the development of methods. A large number of examples in every chapter presents the techniques that are representative of virtually every concept in the book. There are over 130 examples solved with meticulous detail. Besides, there are over 170 exercises, spread chapterwise throughout the book. Unlike most textbooks, the answers, hints, or sometimes detailed solution of all exercise are provided on the spot. The authors feel that this will enhance the interest of both the students and the instructors in the subject.

The icon [www] highlights the Mathematica material throughout the book, and details are provided on the spot as how Mathematica works in the respective situations, or the reader is directed to the corresponding material on the web site, which are identified by the chapter number, section number, and/or example number, with platform–independent presentation.

Besides a general introduction to Mathematica in the very beginning (prior to Chapter 1), which describes the Mathematica style and important concepts, Mathematica functions, a glossary of Mathematica functions used in the book are provided in Appendix C.

Although Mathematica occupies a major portion in the form of notebooks and packages available on the CRC web server as mentioned below, it opens up an opportunity to use the symbolic manipulation and graphics facilities. The textbook is, however, independent of the Mathematica interface and can be used with the same ease and advantage without Mathematica if this facility is not available at an institution.

A word of caution: It has been our experience that some students get so enthralled with Mathematica that they fail to learn analytical techniques and underlying theory. This is our hope that the instructors will avoid this kind of entrapment by an excessive infusion of the technology into their introductory course material.

Mathematica Interface

Mathematica packages,with detailed instructions, are available via the World Wide Web *http://www.crcpress.com/books/isbn/0-8493-7853-2*. In the sequel this web site is referred to as the CRC web server. Details about the Mathematica material can be also found in Appendix D toward the end of the book. This material requires at least Mathematica 2.0. The Mathematica interface based on Mathematica version 3.0 shall be available on the above web site soon after this version is marketed.

At times the reader will find that Mathematica code embedded in the text is a little awkward to read. This is due to the priority for typesetting. Such situations can be found in long Mathematica sessions presented in the book, but do not exist in the Mathematica files and Notebooks available on the CRC web server.

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Kythe, Puri, and Schäferkotter
New Orleans, Louisiana

Introduction to Mathematica

Introduction

Mathematica is a powerful mathematical programming environment. Mathematica provides numeric, symbolic, and graphical tools in order to assist one in the mathematical aspects of problem solving. Significant uses have been found for Mathematica in investigating and analyzing problems in engineering, mathematics, and physics, as well as economics and other sciences. Mathematica can also be used as a high level programming language. Mathematica will run on most of the major platforms, from Cray supercomputers to desktop systems and laptops.

Mathematica is comprised of two parts, called the Front End and the Kernel. The Kernel is the computation engine, which does all the calculations. The Front End takes the form of either a notebook interface (advanced Front End) or a command line interface, allowing the user to communicate with the Kernel.

Mathematica Notebooks allow one to add notes, explanations, and conclusions to the work in a similar fashion to a word processor. In version 3.0 one will be able to import and export graphics in most graphic formats. Presentations can be prepared and also documents for electronic publishing. Items may be cut and pasted within notebooks or between notebooks in order to reuse or modify text, graphics, and calculations.

The Notebook Front End is a file that organizes text, graphics, and calculations in *cells*. Many different kinds of cells comprise a Mathematica Notebook. Input cells contain Mathematica commands and may be evaluated by pressing SHIFT-RETURN or simply RETURN when the cell is selected. Text cells contain text information and are not evaluated by the Kernel. Graphic cells contain pictures of plots and graphs.

Cells can be formatted with various attributes, such as the font to use in displaying text, font size, and color. The cells may be grouped in an outline fashion in order to organize a document into sections containing titles, headings, and subheadings.

A Mathematica Notebook can be transferred from one platform to another without losing information or formatting, though some consideration should be given in order to successfully port the notebooks. For example, one should use the Uniform Style command to eliminate font variation within cells when translating a Macintosh Notebook for Windows. For more information on this subject, see the note "Notebook Conversion Tips", which is available on MathSource (see section below). The Notebooks may also be sent via electronic mail since the notebooks are ASCII text files.

Conventions

Reserved words in the Mathematica programming environment always begin with a capital letter. The arguments of functions are delimited by brackets ([ ]), while parentheses are used to effect grouping. Lists, which are the primary data structure of Mathematica are delimited with braces ({ }) with the elements of the list separated by commas.

Certain symbols should be pointed out. The multiplication symbol is represented by $*$ or by a space as in $a * b$ or $a\,b$. The symbol $=$ stands for substitution, as in $t = 1$, while equal is denoted by $==$, as in Equal[x,t] or $x == t$ yields True only if x and t have the same value. The Mathematica command Not can be rendered !, as in $x! = t$, which is True if x and t do not have the same value. The last input is denoted %, while %n stands for In[n], which is the input cell number n. Finally, never type the prompt In[n] := that begins each line. Mathematica automatically puts the In and Out prompts. Type only the text that follows the In prompt.

Getting Started

After the program is running either a command line interface will appear or a Notebook will appear. To begin, one just types and Mathematica will put the characters into an *input* cell. With cursor in the input cell, press the ENTER key or SHIFT-RETURN keys together to evaluate the input cell and generate on output cell as shown below. The Mathematica Notebook Intro2Mma.ma, found on the CRC web server mentioned in the Preface, is a notebook for new users. The notebook provides explanation and examples in order to get started.

In[1]:=

```
1+1
```

Out[1]=

```
2
```

In[2]:=

```
x := Table[Sin[k],{k,1,5}]//N
x
```

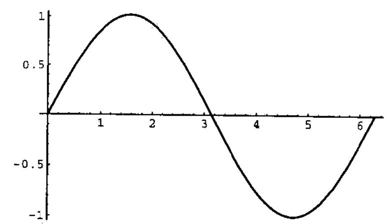
Out[3]=

```
{0.841471, 0.909297, 0.14112, -0.756802, -0.958924}
```

In[3]:=

```
Plot[Sin[x],{x,0,2Pi}]
```

Out[4]=




```
-Graphics-
```

Occasionally, it will be necessary to interrupt or abort a Mathematica calculation. On most systems there is a command key sequence to interrupt and abort. For GUI (Graphical User Interface) systems, there is a menu choice, under the Action menu, that will enable an interrupt or abort of a calculation.

File Manipulation

Mathematica can be used for file manipulation on many different computers as well as non-Unix systems. The advantage is that one never has to learn the file manipulation commands of different computer systems. A few of the commands for file and directory manipulation follow. See the book *Mathematica* (Wolfram, 1991) for other commands and examples.

In[4]:=

```
(* Give the current working directory *)

Directory[]
```

Out[5]=

```
Macintosh HD:Applications:Mathematica 2.2
```

In[5]:=


```
(* List all files in the current working directory *)

FileNames[]
```

Out[6]=

```
{Mathematica, Mathematica Kernel, MathLive, Packages}
```

In[6]:=

```
(* List all packages containing ''Plot'' in the two lev-
els of subdirectories below the current directory *)

FileNames["*Plot*.m", "*", 2]
```

In[7]:=

```
{Packages:Graphics:ContourPlot3D.m,
Packages:Graphics:FilledPlot.m,
Packages:Graphics:ImplicitPlot.m,
Packages:Graphics:MultipleListPlot.m,
Packages:Graphics:ParametricPlot3D.m,
Packages:Graphics:PlotField.m,
Packages:Graphics:PlotField3D.m,
Packages:Miscellaneous:WorldPlot.m,
Packages:ProgrammingExamples:ParametricPlot3D.m}
```

Ordinary Differential Equations

Differential equations are used in many areas of natural science in order to study processes that are continuous in space or time. The Mathematica command **DSolve** computes solutions to ordinary differential equations, as well as systems of ordinary differential equations, and also first order partial differential equations.

DSolve is a collection of algorithms that allows Mathematica to solve a wide range of equations. Mathematica can solve various types of equations including linear homogeneous and inhomogeneous equations, second order variable coefficient equations, second order non-linear equations, and first order partial differential equations. A numerical approximation may be obtained using the **NDSolve** function.

In[8]:=

```
DSolve[y'[x] == y[x],y[x],x]
```

Out[7]=

```
{{y[x] -> E^x C[1]}}
```

A question that very often appears at this stage is, "How can the results from DSolve be used in other calculations?" The answer is that the output from **Solve**, **NSolve**, **DSolve**, or **NDSolve** is a list in which each element is a list of rules. One can assign a name to the solution and easily check that the solution is correct.

In[9]:=

```
f[x_] := y[x]/.First[%]
f[x]
```

Out[9]=

```
E^x C[1]
```

In[10]:=

```
f'[x] == f[x]
```

Out[10]=

```
True
```


To the Instructor

The Mathematics Department at the University of New Orleans realized the value in using Mathematica for the teaching of Calculus. The authors recognized that the symbolic, numerical, and graphical capabilities of Mathematica were well suited to augment the teaching of partial differential equations. Subsequently, the first author taught a course on Ordinary Differential Equations with Mathematica, and the first two authors recently gave a course in Partial Differential Equations in which Mathematica was used. The third author taught Calculus and Mathematica, and Vector Calculus using Mathematica.

The notebook concept creates a manageable interface for the student, without all the headaches of programming I/O (input and output), and creating sophisticated graphics, as well as simplifying complicated algebraic expressions. It should be understood by the students that, although Mathematica will provide the power to perform mathematical tasks, Mathematica should not be used as a crutch to solve the problems.

We suggest that the instructor work through the examples to discover what is to be emphasized. We also ask that the instructor realize that only representative examples are presented in the text material. Other examples and exercises can be worked out in an analogous fashion. Sometimes the nature of a problem may require some variation and/or modification of the given Mathematica code. In all cases the student should be encouraged to explore possibilities.

It is also suggested that notebooks be downloaded from the World Wide Web via *http://www.crcpress.com/books/isbn/0-8493-7853-2*, and be made such that they cannot be erased or modified. The students should copy the originals and modify a copy. The instructor should also become aware of the hardware capabilities in terms of memory and speed. Mathematica is quite capable of using substantial amounts of memory in evaluating expressions.

To the Student

Currently there is a revolution involving the use of computer technology to facilitate the learning of mathematics as well as other sciences. The technology, in this case takes the form of Mathematica and the associated Notebooks. With this technology, the student can easily explore many of the graphical, numerical, and symbolic aspects of any number of problems. You will find that using Mathematica and the computer to learn partial differential equations can be both exciting and frustrating at the same time! A few suggestions will follow that will help you to maximize your experience.

Although Mathematica is a programming environment, you do not need to learn how to program Mathematica. You can learn by example and easily adapt the examples to solve most of the problems. We encourage you to copy, paste, and edit whenever possible. Besides, if you have a working example that can be slightly modified to solve your problem, then there will be less chance for a typing mistake if you let the computer do the typing by cutting and pasting. We recommend that you first cut and paste, and then modify the example.

Mathematica is capable of making mistakes. So the student should be able to verify and check some results by hand. At times, you will have to do the entire calculation by hand in order to verify that Mathematica is providing you with the correct answer to your problem.

Some of you will, no doubt, be interested in learning how to program Mathematica, or have a question about Mathematica. The moderated newsgroup *comp.soft-sys.math.mathematica* is a forum which offers the opportunity to ask questions and receive answers regarding Mathematica related issues. Note that the standard rules of list netiquette apply.

MathSource

MathSource is a well-organized and easily accessible online database for Mathematica materials. For information on MathSource commands, send an email with Help Intro in the body to mathsource@wri.com.

1

Introduction

In many mathematical modeling formulations, partial derivatives are required to represent physical quantities. These derivatives always involve more than one independent variable, generally the space variables $x, y, \ldots$ and the time variable t. Such formulations have one or more dependent variables, which are the unknown functions of the independent variables. The resulting equations are called *partial differential equations*, which, together with the initial and/or boundary conditions, represent physical phenomena.

1.1. Notation and Definitions

Definitions about order, linearity, homogeneity, and solutions for partial differential equations resemble those in the case of ordinary differential equations and are as follows: The *order* of a partial differential equation is the same as the order of the highest derivative appearing in the equation. The partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y \partial x}$ and $\frac{\partial^2 u}{\partial y^2}$ are sometimes denoted by u_x, u_y, u_{xx}, u_{xy} and u_{yy}, or p, q, r, s and t respectively. The most general first order partial differential equation with two independent variables x and y is written in the form

$$F(x, y, u, p, q) = 0, \qquad p = u_x, \quad q = u_y. \tag{1.1}$$

The most general second order partial differential equation is of the form

$$F(x, y, u, p, q, r, s, t) = 0, \qquad r = u_{xx}, \quad s = u_{xy}, \quad t = u_{yy}. \tag{1.2}$$

A partial differential equation is said to be *linear* if the unknown function u and all its partial derivatives appear in an algebraically linear form, i.e., of the first degree. For example, the equation

$$a_{11}\, u_{xx} + 2a_{12}\, u_{xy} + a_{22}\, u_{yy} + b_1\, u_x + b_2\, u_y + c\, u = f, \tag{1.3}$$

where the coefficients $a_{11}, a_{12}, a_{22}, b_1, b_2$, and c and the function f are functions of x and y, is a second order linear partial differential equation in the unknown $u(x, y)$. An operator L is a linear differential operator iff $L(\alpha u + \beta v) = \alpha L u + \beta L v$, where α and β are scalars, and u and v are any functions with continuous partial derivatives of appropriate order.

A partial differential equation $Lu = 0$ is said to be *homogeneous*, whereas $Lu = g$, where L is any differential operator and $g \neq 0$ is a given function of the independent variables, is said to be *nonhomogeneous*. For example,

$$(x + 2y)u_x + x^2 u_y = \cos(x^2 + y^2)$$

is a nonhomogeneous first order linear equation, whereas

$$(x + 2y)u_x + x^2 u_y = 0$$

is homogeneous. Thus, a linear homogeneous equation is such that whenever u is a solution of the equation, then cu is also a solution where c is a constant. A function $u = \phi$ is said to be a *solution* of a partial differential equation if ϕ and its partial derivatives, when substituted for u and its partial derivatives occurring in the partial differential equation, reduce it to an identity in the independent variables. The *general* solution of a partial differential equation is a linear combination of all solutions of the equation with as many arbitrary functions as the order of the equation; a partial differential equation of order k has k arbitrary functions. A *particular* solution of a partial differential equation is one that does not contain arbitrary functions or constants.

A partial differential equation is called *quasi–linear* if it is linear in all the highest order derivatives of the dependent variable. For example, the most general form of a quasi–linear second order equation is

$$A(x, y, u, p, q)u_{xx} + B(x, y, u, p, q)u_{xy} + C(x, y, u, p, q)u_{yy} + \\ + f(x, y, u, p, q) = 0. \tag{1.4}$$

It is assumed that the reader is familiar with the theory and methods of ordinary differential equations. Since the subject of partial differential equations is broad, we shall discuss certain well–known equations of second order in detail.

1.2. Initial and Boundary Conditions

A partial differential equation subject to certain conditions in the form of initial or boundary conditions is known as an initial value or a boundary value problem. The initial conditions, also known as *Cauchy conditions*, are the values of the unknown function u and an appropriate number of its derivatives at the initial point.

The boundary conditions fall into the following three categories:

(i) *Dirichlet conditions* (also known as boundary conditions of the first kind) are the values of the unknown function u prescribed at each point of the boundary ∂D of the domain D under consideration.

(ii) *Neumann conditions* (also known as boundary conditions of the second kind) are the values of the normal derivatives of the unknown function u prescribed at each point of the boundary ∂D.

(iii) *Robin conditions* (also known as boundary conditions of the third kind, or mixed boundary conditions) are the values of a linear combination of the unknown function u and its normal derivative prescribed at each point of the boundary ∂D.

The following problems are examples of each category, respectively:

$$\begin{aligned} u_t &= k\, u_{xx}, \quad 0 < x < l,\ t > 0, \\ u(x,0) &= f(x), \quad u_t(x,0) = g(x),\ 0 < x < l, \\ u(0,t) &= T_1, \quad u(l,t) = T_2,\ t > 0; \end{aligned} \tag{1.5}$$

$$\begin{aligned} u_t &= k\, u_{xx}, \quad 0 < x < l,\ t > 0, \\ u(x,0) &= f(x), \quad u_t(x,0) = g(x),\ 0 < x < l, \\ u_x(0,t) &= T_1, \quad u_x(l,t) = T_3,\ t > 0; \end{aligned} \tag{1.6}$$

$$\begin{aligned} u_t &= k\, u_{xx}, \quad 0 < x < l,\ t > 0, \\ u(x,0) &= f(x), \quad u_x(x,0) = g(x),\ 0 < x < l, \\ &\left.\begin{aligned} u(0,t) + \alpha\, u_x(0,t) &= 0, \\ u(l,t) + \beta\, u_x(l,t) &= 0, \end{aligned}\right\} \quad t > 0. \end{aligned} \tag{1.7}$$

1.3. Classification of Second Order Equations

If $f = 0$ in Eq (1.3), the most general form of a second order homogeneous equation is

$$a_{11}\,u_{xx} + 2a_{12}\,u_{xy} + a_{22}\,u_{yy} + b_1\,u_x + b_2\,u_y + c\,u = 0. \tag{1.8}$$

In order to show a correspondence with an algebraic quadratic equation, we replace u_x by α, u_y by β, u_{xx} by α^2, u_{xy} by $\alpha\beta$, and u_{yy} by β^2. Then Eq (1.8) reduces to a second degree polynomial in α and β:

$$P(\alpha,\beta) = a_{11}\alpha^2 + 2a_{12}\alpha\beta + a_{22}\beta^2 + b_1\alpha + b_2\beta + c. \tag{1.9}$$

It is known from analytical geometry and algebra that the polynomial equation $P(\alpha,\beta) = 0$ represents a *hyperbola, parabola,* or *ellipse* according as its discriminant $a_{12}^2 - a_{11}a_{22}$ is positive, zero, or negative. Thus, Eq (1.8) is classified as hyperbolic, parabolic, or elliptic according as $a_{12}^2 - a_{11}a_{22} \gtreqless 0$.

An alternate approach to classify the types of Eq (1.8) is based on the following theorem:

THEOREM 1.1. *The relation $\phi(x,y) = C$ is a general integral of the ordinary differential equation*

$$a_{11}\,dy^2 - 2a_{12}\,dx\,dy + a_{22}\,dx^2 = 0 \tag{1.10}$$

iff $u = \phi(x,y)$ is a particular solution of the equation

$$a_{11}\,u_x^2 + 2a_{12}\,u_x\,u_y + a_{22}\,u_y^2 = 0. \tag{1.11}$$

PROOF. Since the function $u = \phi(x,y)$ satisfies Eq (1.11), then

$$a_{11}\left(\frac{\phi_x}{\phi_y}\right)^2 - 2a_{12}\left(-\frac{\phi_x}{\phi_y}\right) + a_{22} = 0 \tag{1.12}$$

holds for all x, y in the domain of definition of $u = \phi(x,y)$ and $\phi_y \neq 0$. In order that the relation $\phi(x,y) = C$ be the general solution of Eq (1.12), we must show that the function y defined implicitly by $\phi(x,y) = C$ satisfies Eq (1.12). Suppose that $y = f(x,C)$ is such a function. Then

$$\frac{dy}{dx} = -\left[\frac{\phi_x(x,y)}{\phi_y(x,y)}\right]_{y=f(x,C)}.$$

Hence, in view of Eq (1.11),

$$a_{11}\left(\frac{dy}{dx}\right)^2 - 2a_{12}\left(\frac{dy}{dx}\right) + a_{22} = \left[a_{11}\left(-\frac{\phi_x}{\phi_y}\right)^2 - 2a_{12}\left(-\frac{\phi_x}{\phi_y}\right) + a_{22}\right]_{y=f(x,C)} = 0. \tag{1.13}$$

Thus, $y = f(x,C)$ satisfies Eq (1.12).

Conversely, let $\phi(x,y) = C$ be a general solution of Eq (1.11). We must show that for each point (x,y)

$$a_{11}\,\phi_x^2 + 2a_{12}\,\phi_x\,\phi_y + a_{22}\,\phi_y^2 = 0. \tag{1.14}$$

If we can show that Eq (1.14) is satisfied for an arbitrary point (x_0, y_0), then Eq (1.14) will be satisfied for all points. Since $\phi(x,y)$ represents a solution of Eq (1.14), we construct through (x_0,y_0) an integral of Eq (1.11) where we set $\phi(x_0,y_0) = C_0$, and consider the curve $y = f(x, C_0)$. For all points of this curve we have

$$a_{11}\left(\frac{dy}{dx}\right)^2 - 2a_{12}\left(\frac{dy}{dx}\right) + a_{22} = \left[a_{11}\left(-\frac{\phi_x}{\phi_y}\right)^2 - 2a_{12}\left(-\frac{\phi_x}{\phi_y}\right) + a_{22}\right]_{y=f(x,C_0)} = 0.$$

If we set $x = x_0$ in this equation, we get

$$a_{11}\,\phi_x^2(x_0,y_0) + 2a_{12}\,\phi_x(x_0,y_0)\,\phi_y(x_0,y_0) + a_{22}\,\phi_y^2(x_0,y_0) = 0,$$

where $y_0 = f(x_0, C_0)$. ■

Eq (1.10) or (1.11) is called the *characteristic equation* of the partial differential equation (1.3) or (1.8); the related integrals are called *characteristics*.

Eq (1.13), regarded as a quadratic equation in dy/dx, yields two solutions:

$$\frac{dy}{dx} = \frac{a_{12} \pm \sqrt{a_{12}^2 - a_{11}\, a_{22}}}{a_{11}}.$$

The expression under the radical determines the type of the differential equation (1.3) or (1.8). Thus, Eq (1.3) or (1.8) is of the hyperbolic, parabolic, or elliptic type according as $a_{12}^2 - a_{11}a_{22} \gtreqless 0$.

EXAMPLE 1.1. The Tricomi equation $u_{xx} + x u_{yy} + u = 0$, for which $a_{12}^2 - a_{11}a_{22} = -x$, is hyperbolic if $x < 0$, parabolic if $x = 0$, and elliptic if $x > 0$. ■

The general form of a linear second order partial differential equation in n variables $x_1, \dots, x_n$ is

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu + f = 0, \tag{1.15}$$

where the coefficients a_{ij}, b_k, c, and d are real constants or functions of $x_1, \dots, x_n$. If we assume that the second order partial derivatives of u are continuous, then the terms involving the highest order derivatives, i.e., those in the first summation in (1.15), can be arranged such that $a_{ij} = a_{ji}$. If we consider the quadratic form

$$\sum_{i,j} a_{ij}\, v_i v_j,$$

then at a fixed point $P^0 = \left(x_1^0, \dots, x_n^0\right)$ the coefficients a_{ij} are constants. This quadratic form can always be transformed by an affine transformation into the canonical form

$$Q = \sum_{i=1}^{n} \alpha_i\, w_i^2,$$

where not all α_i vanish. Then the partial differential equation (1.15) is

elliptic	if all α_i have the same sign;
hyperbolic	if all α_i except one have the same sign;
ultrahyperbolic	if two or more α_i have different signs; and
parabolic	if one or more α_i vanish.

For quasi-linear second order partial differential equations, the above criteria still hold, since only the highest order terms are considered for this classification.

EXAMPLE 1.2. For the partial differential equation $4u_{xx} + u_{yy} + 4u_{yz} + 4u_{zz} = 0$ in R^3, the quadratic form is

$$4v_1^2 + v_2^2 + 4v_2\, v_3 + 4v_3^2 = 0,$$

which, by setting $2v_1 = w_1$, $v_2 + 2v_3 = w_2$, and $v_2 - 2v_3 = w_3$, reduces to $w_1^2 + w_2^2 = 0$. Hence the given equation is parabolic because the coefficient of w_3 is zero. ■

EXAMPLE 1.3. Consider $u_{xx} - x^2 y u_{yy} = 0$, $y > 0$. Here $a_{12}^2 - a_{11}a_{22} = x^2 y > 0$, so the partial differential equation is hyperbolic. ■

EXAMPLE 1.4. Consider $e^{xy} u_{xx} + u_{yy} \sinh x + u = 0$. Here $a_{12}^2 - a_{11}a_{22} = -e^{xy} \sinh x$, and the partial differential equation is hyperbolic if $x < 0$, parabolic if $x = 0$, and elliptic if $x > 0$. ■

www The classification of a given second order partial differential equation into its type can be achieved by loading the Mathematica package `EquationType.m`, and the Notebook `EquationType.ma` found on the CRC web server mentioned in the Preface.

1.4. Some Known Equations

The following equations appear frequently during the analysis of physical phenomena:

1. *Heat equation* in R^1: $u_t = k u_{xx}$, where u denotes the temperature distribution and k the thermal diffusivity.

2. *Wave equation* in R^1: $u_{tt} = c^2 u_{xx}$, where u represents the displacement, e.g., of a vibrating string from its equilibrium position, and c the wave speed.

3. *Laplace equation* in R^2: $\nabla^2 u \equiv u_{xx} + u_{yy} = 0$, where $\nabla^2 = \nabla \cdot \nabla$ denotes the Laplacian.

4. *Transport (Traffic) equation*: $u_t + a(u)\, u_x = 0$.

4a. *Transport equation* in R^1: $u_t + a u_x = 0$, where a is a constant.

5. *Berger's equation* in R^1: $u_t + u\, u_x = 0$, which arises in the study of a stream of particles or fluid flow with zero viscosity.

6. *Eikonal equation* in R^2: $u_x^2 + u_y^2 = 0$, which arises in geometric optics.

7. *Poisson's equation* in R^n: $\nabla^2 u = f$, also known as the nonhomogeneous Laplace equation in R^n; it arises in various field theories and electrostatics.

8. *Helmholtz equation* in R^3: $(\nabla^2 u + k^2) = 0$, which arises, e.g., in underwater scattering.

9. *Klein–Gordon equation* in R^3: $u_{tt} - c^2 \nabla^2 u + m^2 u = 0$, which arises in quantum field theory, where m denotes the mass.

10. *Telegrapher's equation* in R^3: $u_{tt} - c^2 + \alpha u_t + m^2 u = 0$, where α is the damping coefficient; it arises in the study of electrical transmission in telegraph cables when the current may leak to the ground.

11a. *Schrödinger equation* in R^3: $u_t = i[\nabla^2 u + V(x)u]$, where $V(x)$, $x \in R^3$, denotes the potential; it arises in quantum mechanics.

11b. *Cubic Schrödinger equation* in R^3: $u_t = i[\nabla^2 u + \sigma u\, |u|^2]$, where $\sigma = \pm 1$; this is a semilinear version of 11a.

12. *Sine–Gordon equation* in R^3: $u_{tt} - c^2 \nabla^2 u + m^2 u = 0$, which arises in quantum field theory.

13. *Semilinear heat equation* in R^3: $u_t - k \nabla^2 u = f(x, t, u)$.

14a. *Semilinear wave equation* in R^3: $u_{tt} - c^2 \nabla^2 u = f(x, t, u)$.

14b. *Semilinear Klein–Gordon equation* in R^3: $u_{tt} - c^2 \nabla^2 u + m^2 u + \gamma u^p = 0$, where γ denotes a coupling constant, and $p \geq 2$ is an integer.

14c. *Dissipative Klein–Gordon equation* in R^3: $u_{tt} - c^2 \nabla^2 u + \alpha u_t + m^2 u + u^p = 0$.

14d. *Dissipative sine–Gordon equation* in R^3: $u_{tt} - c^2 \nabla^2 u + \alpha u_t + \sin u = 0$.

15. *Semilinear Poisson's equation* in R^3: $\nabla^2 u = f(x, u)$.

16. *Porous medium equation* in R^3: $u_t = k\, \nabla^2 (u^a\, \nabla u)$, where $k > 0$ and $a > 1$ are constants; it is a quasi-linear equation, and arises in the seepage flows through porous media.

17. *Biharmonic equation* in R^3: $\nabla^4 u \equiv \nabla^2(\nabla^4 u) = 0$; it arises in elastodynamics.

18. *Korteweg de Vries (KdV) equation* in R^1: $u_t + c u\, u_x + u_{xxx} = 0$, which arises in shallow water waves.

19a. *Euler's equations* in R^3: $\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\rho} \nabla p = 0$, where $\mathbf{u}$ denotes the velocity field, and p the pressure.

19b. *Navier–Stokes equations* in R^3: $\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\rho} \nabla p = \nu \nabla^2 \mathbf{u}$, where ν denotes the kinematic viscosity of a fluid.

20. *Maxwell's equations* in R^3: $\mathbf{E}_t - \nabla \times \mathbf{H} = 0$, $\mathbf{H}_t + \nabla \times \mathbf{E} = 0$, where $\mathbf{E}$ and $\mathbf{H}$ denotes the electric and the magnetic field, respectively; they are a system of six equations in six unknowns.

Origins of these and other equations of mathematical physics are related to some interesting physical problems. We shall present derivation of some of them as examples which will also bring out certain aspects of mathematical modeling of these problems.

EXAMPLE 1.5. (*One–dimensional wave equation for vibrations of a string*) Consider a stretched string of length l which is fixed at both ends. It is assumed that (i) the string is thin and flexible, i.e., it offers no resistance to change of form except a change in length, and (ii) the tension T_0 in the string is much larger than the force due to gravity acting on it so that the latter can be neglected. Let the string in its equilibrium state be situated along the x–axis. Let $u(x, t)$ denote the displacement of the string at time t from its equilibrium position. The shape of the string at a fixed t is represented in Fig. 1.1.

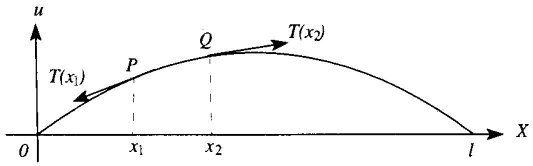


Fig. 1.1. Vibrations of a string.

Let us further assume that the vibrations are small, which implies that the displacement $u(x,t)$ and its derivative u_x are small enough so that their squares and products can be neglected. As a result of vibrations, let a segment (x_1, x_2) of the string be deformed into the segment PQ. Then at time t the length of the arc $\overset{\frown}{PQ}$ is given by

$$\int_{x_1}^{x_2} \sqrt{1+u_x^2}\, dx \approx x_2 - x_1, \tag{1.16}$$

which simply means that under small vibrations the length of the segment of the string does not change. By Hooke's law, the tension T at each point in the string is independent of t, i.e., during the motion of the string any change in T can be neglected in comparison with the tension in equilibrium. We shall now show that the tension T is also independent of x. In fact, it is evident from Fig. 1.1 that the x-component of the resulting tension at the points P and Q must be in equilibrium, i.e.,

$$T(x_1)\cos\alpha(x_1) - T(x_2)\cos\alpha(x_2) = 0,$$

where $\alpha(x)$ denotes the angle between the tangent at a point x and the positive x-axis at time t. Since the vibrations are small,

$$\cos\alpha(x) = \frac{1}{\sqrt{1+\tan^2\alpha(x)}} = \frac{1}{\sqrt{1+u_x^2}} \approx 1,$$

which implies that $T(x_1) \approx T(x_2)$. Since x_1 and x_2 are arbitrary, the magnitude of T is independent of x. Hence, if T_0 denotes the tension at equilibrium and T the tension in the vibrating string, then $T \approx T_0$ for all x and t.

Now, the sum of the components of tension $T(x_1)$ at P and $T(x_2)$ at Q along the u-axis must be zero, i.e.,

$$\begin{aligned} 0 &= T_0\left[\sin\alpha(x_2) - \sin\alpha(x_1)\right] \\ &= T_0\left[\frac{\tan\alpha(x_2)}{\sqrt{1+\tan^2\alpha(x_2)}} - \frac{\tan\alpha(x_1)}{\sqrt{1+\tan^2\alpha(x_1)}}\right] \\ &= T_0\left[\frac{u_{x_2}}{\sqrt{1+u_{x_2}^2}} - \frac{u_{x_1}}{\sqrt{1+u_{x_1}^2}}\right] \approx T_0\left[\frac{\partial u}{\partial x_2} - \frac{\partial u}{\partial x_1}\right] \\ &= T_0\left[\left.\frac{\partial u}{\partial x}\right|_{x=x_2} - \left.\frac{\partial u}{\partial x}\right|_{x=x_1}\right] \\ &= T_0\int_{x_1}^{x_2} \frac{\partial^2 u}{\partial x^2}\, dx, \end{aligned} \tag{1.17}$$

using (1.16). Let $g(x,t)$ denote the external force per unit mass acting on the string along the u-axis. Then the component of $g(x,t)$ acting on the segment $\overset{\frown}{PQ}$ along the u-axis is given by

$$\int_{x_1}^{x_2} g(x,t)\, dx. \tag{1.18}$$

Let $\rho(x)$ be the linear density of the string. Then the inertial force on the segment $\overset{\frown}{PQ}$ is

$$-\int_{x_1}^{x_2} \rho(x)\,\frac{\partial^2 u}{\partial t^2}\, dx. \tag{1.19}$$

Hence the sum of the components (1.17), (1.18), and (1.19) must be zero, i.e.,

$$\int_{x_1}^{x_2}\left[T_0\frac{\partial^2 u}{\partial x^2} + g(x,t) - \rho(x)\frac{\partial^2 u}{\partial t^2}\right] dx = 0. \tag{1.20}$$

Since x_1 and x_2 are arbitrary, it follows from (1.20) that the integrand must be zero, which gives

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} + g(x,t). \tag{1.21}$$

This represents the partial differential equation for the vibrations of the string.

If $\rho = \text{const}$, then (1.21) reduces to

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} + f(x,t), \tag{1.22}$$

where $a = \sqrt{T_0/\rho}$, $f(x,t) = g(x,t)/\rho$. In the absence of external forces, Eq (1.22) becomes

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \tag{1.23}$$

which is the wave equation for free vibrations (oscillations) of the string. ∎

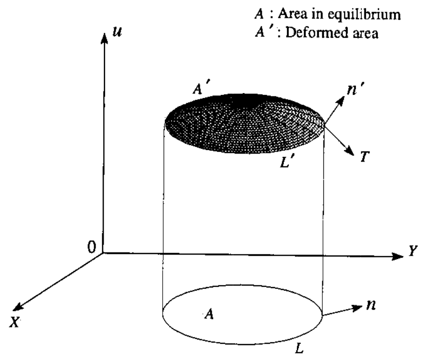


Fig. 1.2. Vibrations of a membrane.

EXAMPLE 1.6. (*Two-dimensional wave equation for oscillations of a membrane*) Suppose that a membrane which is a perfectly flexible thin stretched sheet occupies a region D in the xy-plane in its equilibrium state. Further, let the membrane be subjected to a uniform tension T applied on its boundary ∂D. This means that the force acting on an element ds of the boundary ∂D is equal to $T\,ds$. We shall examine the transverse oscillations of the membrane, which move perpendicular to the xy-plane at each point in the direction of the u-axis. Thus, the displacement u at a point $(x, y) \in D$ is a function of x, y and t. Assuming that the oscillations are small, i.e., the functions u, u_x, and u_y are so small that their squares and products can be neglected, let A denote an arbitrary area of the membrane ($A \in D$) situated in equilibrium in the xy-plane and bounded by a curve L. After the membrane is displaced from its equilibrium position, let the area A be deformed into an area A' bounded by a curve L' (see Fig. 1.2.), which at time t is defined by

$$A' = \iint_A \sqrt{1 + u_x^2 + u_y^2}\, dx\, dy \approx \iint_A dx\, dy = A.$$

Thus we can neglect the change in A during the oscillations, and the

tension in the membrane remains constant and equal to its initial value T.

Note that the tension T which is perpendicular to the boundary L' lies at all points in the tangent plane to the surface area A'. Let ds' denote an element of the boundary L'. Then the tension acting on this element is $T\,ds'$, and $\dfrac{\partial u}{\partial n} = \cos\alpha$, where α is the angle between the tension vector $\mathbf{T}$ and the u-axis, and n is the outward normal to the boundary L. Then the component of the tension acting on the element ds' in the direction of the u-axis is $T\dfrac{\partial u}{\partial n}\, ds'$. Hence the component of the resultant force acting on the boundary L' along the u-axis is

$$\begin{aligned} T \int_{L'} \frac{\partial u}{\partial n}\, ds' &\approx T \int_L \frac{\partial u}{\partial n}\, ds \\ &= \iint_A \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) dx\, dy, \end{aligned} \tag{1.24}$$

by Green's identity, where, in view of small oscillations, we have taken $ds' \approx ds$, and replaced L' by L. Let $g(x, y, t)$ denote an external force per unit area acting on the membrane along the u-axis. Then the total force acting on the area A' is given by

$$\iint_A g(x, y, t)\, dx\, dy. \tag{1.25}$$

Let $\rho(x, y, t)$ be the surface density of the membrane. Then the inertial force at all times t is

$$\iint_A \rho(x, y, t) \frac{\partial^2 u}{\partial t^2}\, dx\, dy. \tag{1.26}$$

Since the sum of the inertial force and the total force is equal and opposite to the resultant of the tension on the boundary L', we find from (1.24)–(1.26) that

$$\iint_A \left[\rho(x, y, t)\frac{\partial^2 u}{\partial t^2} - T\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - g(x, y, t)\right] dx\, dy = 0,$$

or, since A is arbitrary,

$$\rho(x, y, t)\frac{\partial^2 u}{\partial t^2} = T\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + g(x, y, t). \tag{1.27}$$

This is the partial differential equation for small oscillations of a membrane. If the density $\rho =$ const, then Eq (1.27) in the absence of external forces reduces to

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad a = \sqrt{T/\rho}. \ ∎ \tag{1.28}$$

EXAMPLE 1.7. (*Heat transfer equation for a uniform isotropic body*) Let $u(x, y, z, t)$ denote the temperature of a uniform isotropic body at a point (x, y, z) and time t. If different parts of the body are at different temperatures, then heat transfer takes place within the body. Consider a small surface element δS of a surface S drawn inside the body. Under the assumption that the amount of heat δQ passing through the element δS in time δt is proportional to δt, δS, and the normal derivative $\dfrac{\partial u}{\partial n}$, we get

$$\delta Q = -k \frac{\partial u}{\partial n}\, \delta S\, \delta t = -k\, \delta S\, \delta t\, \nabla_n u, \tag{1.29}$$

where k is the thermal conductivity of the body which depends only on the coordinates (x, y, z) of points in the body but is independent of the direction of the normal to the surface S, and ∇_n denotes the gradient in the direction of the outward normal to the surface element δS. Let Q denote the *heat flux* which is the amount of heat passing through the unit surface area per unit time. Then Eq (1.29) implies that

$$Q = -k \frac{\partial u}{\partial n}. \tag{1.30}$$

Now, consider an arbitrary volume V bounded by a smooth surface S. Then, in view of (1.30), the amount of heat entering through the surface S in the time interval $[t_1, t_2]$ is

$$\begin{aligned} Q_1 &= -\int_{t_1}^{t_2} dt \iint_S k(x, y, z) \frac{\partial u}{\partial n}\, dS \\ &= \int_{t_1}^{t_2} \iiint_V \nabla \cdot (k \nabla u)\, dV, \end{aligned} \tag{1.31}$$

by divergence theorem, where n is the inward normal to the surface S. Let δV denote a volume element. The amount of heat required to change the temperature of this volume element by δu in time δt is

$$\delta Q_2 = [u(x, y, z, t + \delta t) - u(x, y, z, t)]\, \rho(x, y, z)\, \delta V, \tag{1.32}$$

where $c(x, y, z)$ and $\rho(x, y, z)$ are the specific heat and density of the body, respectively. Integrating (1.31) we find that the amount of heat required to change the temperature of the volume V by $\delta u = u(x, y, z, t+ \delta t) - u(x, y, z, t)$ is given by

$$\begin{aligned} Q_2 &= \iiint_V [u(x, y, z, t + \delta t) - u(x, y, z, t)]\, c\rho\, dV \\ &= \int_{t_1}^{t_2} dt \iiint_V c\rho \frac{\partial u}{\partial t}\, dV. \end{aligned} \tag{1.33}$$

We shall now assume that the body contains heat sources, and let $g(x, y, z, t)$ denote the density of such heat sources. Then the amount of heat released by or absorbed in V in the time interval $[t_1, t_2]$ is

$$Q_3 = \int_{t_1}^{t_2} dt \iiint_V g(x, y, z, t)\, dV. \tag{1.34}$$

Since $Q_2 = Q_1 + Q_3$, we find from (1.31)–(1.34) that

$$\int_{t_1}^{t_2} dt \iiint_V \left[c\rho \frac{\partial u}{\partial t} - \nabla \cdot (k \nabla u) - g(x, y, z, t) \right] dV = 0,$$

or, since the volume V and the time interval $[t_1, t_2]$ are arbitrary,

$$\begin{aligned} c\rho \frac{\partial u}{\partial t} &= \nabla \cdot (k \nabla u) + g(x, y, z, t) \\ &= \frac{\partial}{\partial x}\left(k \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial u}{\partial z}\right) + g(x, y, z, t), \end{aligned} \tag{1.35}$$

which is the required heat transfer equation for a uniform isotropic body. If c, ρ, and k are constant, Eq (1.35) becomes

$$\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + f(x, y, z, t), \tag{1.36}$$

where $a = \sqrt{k/c\rho}$ is known as the thermal diffusivity, and $f = g/c\rho$ denotes the heat source (sink) function. In the absence of heat sources (i.e., when $g(x, y, z, t) = 0$), Eq (1.36) reduces to the homogeneous heat conduction equation

$$\frac{\partial u}{\partial t} = a^2 \nabla^2 u = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right). \tag{1.37}$$

In the case when the temperature distribution throughout the body reaches the steady state, i.e., when the temperature becomes independent of time, Eq (1.37) reduces to the Laplace equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{1.38}$$

For the derivation of the heat conduction equation in $R^1 \times R^+$, consider a laterally insulated rod of uniform cross section with area A and constant density ρ, constant specific heat c, and constant thermal conductivity k. We shall assume that the temperature $u(x, t)$ is a function of x and t only, $t > 0$, and use the law of conservation of energy to derive the heat conduction equation. Consider a segment PQ of the rod, with coordinates x and $x + \Delta x$ (Fig. 1.3). Let R denote the rate at which the heat is accumulating on the segment PQ. Then, assuming that there are no heat sources or sinks in the rod, R is given by

$$R = \int_x^{x+\Delta x} \frac{\partial c\rho A u(\xi, t)}{\partial t}\, d\xi.$$

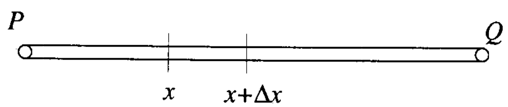


Fig. 1.3. Segment PQ on a thin uniform rod.

Note that R can also be evaluated as the total flux across the boundaries of the segment PQ, which gives $R = kA[u_x(x+\Delta x, t) - u_x(x, t)]$. Now, using the mean–value theorem for integrals, we get

$$c\rho A\, u_t(x + h\Delta x, t)\, \Delta x = kA[u_x(x + \Delta x, t) - u_x(x, t)], \quad 0 < h < 1.$$

After dividing both sides by $c\rho A \Delta x$ and taking the limit as $\Delta x \to 0$, we get

$$u_t(x, t) = a^2\, u_{xx}(x, t). \blacksquare$$

EXAMPLE 1.8. (*One–dimensional traffic flow problem*) Let $\rho(x, t)$ denote the traffic density which represents the number of vehicles per mile at time t at an arbitrary yet fixed position x on a roadway. Let $q(x, t)$ denote the traffic flow which is a measure of number of vehicles per hour passing a fixed position x. Consider a section of the roadway bounded by the positions $x = x_1$ and $x = x_2$, and assume that there are no exits or entrances between these two positions. Then the number N of vehicles in the segment $[x_1, x_2]$ is given by $N = \int_{x_1}^{x_2} \rho(x, t)\, dt$. The rate of change of N with respect to time t is equal to the difference between the number of vehicles per unit time entering the position at $x = x_1$ and that leaving at the position $x = x_2$, i.e.,

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t} \int_{x_1}^{x_2} \rho(x, t)\, dx = q(x_1, t) - q(x_2, t). \tag{1.39}$$

As in the case of heat conduction (Example 1.7), Eq (1.39), which is also known as the integral representation of conservation of vehicles, can be written as

$$q(x_1, t) - q(x_2) = -\int_{x_1}^{x_2} \frac{\partial}{\partial x} q(x, t)\, dx = \frac{\partial}{\partial t} \int_{x_1}^{x_2} \rho(x, t)\, dx,$$

which after taking the partial derivative $\partial/\partial t$ inside the last integral and noting that x_1 and x_2 are arbitrary leads to the required partial differential equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0. \tag{1.40}$$

Let $u(x, t)$ denote the velocity of a vehicle. Then, since the number of vehicles per hour passing a given position is equal to the density of vehicles times the velocity of vehicles, we obtain

$$q(x, t) = \rho(x, t) u(x, t).$$

If we assume that the velocity u depends only on the density ρ, $u = u(\rho)$, i.e., the vehicles slow down as the traffic density increases, then $\dfrac{\partial u}{\partial \rho} \le 0$. This inequality implies that the traffic flow depends only on the traffic density, i.e., $q = q(\rho)$, and Eq (1.40) then reduces to

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial \rho}\frac{\partial \rho}{\partial x} = 0,$$

or

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0, \tag{1.41}$$

where $c(\rho) = \partial q/\partial \rho$. Eq (1.41) is a first order homogeneous quasi–linear partial differential equation. ■

1.5. Superposition Principle

Let L denote a linear differential operator of any order and any kind. The superposition principles for homogeneous and non-homogeneous linear differential equations are represented by the following two theorems:

THEOREM 1.2. *Let $Lu = 0$ be a differential equation. Suppose u_1 and u_2 are two linearly independent solutions. Then $c_1u_1 + c_2u_2$ is also a solution.*

PROOF. By hypotheses $Lu_{1,2} = 0$. By definition $L(c_1u_1 + c_2u_2) = c_1Lu_1 + c_2Lu_2 = 0$. ∎

THEOREM 1.3. *If $Lu = \sum_1^n c_i f_i$ be a non-homogeneous linear differential equation and if $Lg_k = f_k$, then $\sum_1^n c_i g_i$ is a solution of the above differential equation.*

PROOF. $L\sum_1^n c_i g_i = \sum_1^n c_i L g_i = \sum_1^n c_i f_i$. Thus $\sum_1^n c_i g_i$ satisfies the differential equation. ∎

It is obvious from these two principles that if v is a solution of an equation $Lu = 0$ and if F a solution of $Lu = f$, then $v + F$ is also a solution of $Lu = f$. A generalized superposition principle is defined as follows:

THEOREM 1.4. *If the functions u_i, $i = 1, 2, \dots$, are separately the solutions of a linear homogeneous differential equation $L(u) = 0$, then the series $u = \sum_{i=1}^{\infty} C_i u_i$ is also a solution of the differential equation, provided that the derivatives appearing in $L(u)$ can be differentiated term-by-term.*

PROOF. In fact, if the derivatives of u appearing in $L(u) = 0$ can be differentiated term-by-term, we have

$$\frac{\partial^n u}{\partial x^m \partial t^{n-m}} = \sum_{i=1}^{\infty} C_i \frac{\partial^n u_i}{\partial x^m \partial t^{n-m}},$$

and since the equation $L(u) = 0$ is linear and a convergent series can be added term-by-term, we can write

$$L(u) = L\left(\sum_{i=1}^{\infty} C_i u_i\right) = \sum_{i=1}^{\infty} C_i L(u_i) = 0.$$

The sufficient condition for term-by-term differentiability is the uniform convergence of the series $\sum_{i=1}^{\infty} C_i \frac{\partial^n u_i}{\partial x^m \partial t^{n-m}}$. ∎

1.6. Exercises

Classify the partial differential equation as hyperbolic, parabolic, or elliptic:

1.1.	$u_{xx} - 3u_{xy} + 2u_{yy} = 0$	ANS. Hyperbolic
1.2.	$4u_{xx} - 7u_{xy} + 3u_{yy} = 0$	ANS. Hyperbolic
1.3.	$u_{xx} + a^2 u_{yy} = 0,\ a \neq 0$	ANS. Elliptic
1.4.	$a^2 u_{xx} + 2a u_{xy} + u_{yy} = 0,\ a \neq 0$	ANS. Parabolic
1.5.	$4u_{tt} - 12u_{xt} + 9u_{xx} = 0$	ANS. Parabolic
1.6.	$2u_{xt} + 3u_{tt} = 0$	ANS. Hyperbolic
1.7.	$u_{xx} + 2u_{xy} + 5u_{yy} = 0$	ANS. Elliptic
1.8.	$8u_{xx} - 2u_{xy} - 3u_{yy} = 0$	ANS. Hyperbolic

For what values of x and y are the following partial differential equations hyperbolic, parabolic, or elliptic?

1.9. $u_{xx} - x u_{yy} = 0$.
ANS. Hyperbolic for $x > 0$, parabolic for $x = 0$, elliptic for $x < 0$.

1.10. $u_{xx} - 2x u_{xy} + y u_{yy} = 0$.
ANS. Hyperbolic for $x^2 > y$, parabolic for $x^2 = y$, elliptic for $x^2 < y$.

1.11. $u_{xx} + 2x u_{xy} + (1 - y^2) u_{yy} = 0$.
ANS. Hyperbolic for $x^2 + y^2 > 1$, parabolic for $x^2 + y^2 = 1$, elliptic for $x^2 + y^2 < 1$.

1.12. $x u_{xx} + x u_{xy} + y u_{yy} = 0$.
ANS. Hyperbolic for $x^2 > 4xy$, parabolic for $x^2 = 4xy$, elliptic for $x^2 < 4xy$.

1.13. $(1 + y^2) u_{xx} + (1 + x^2) u_{yy} = 0$.
ANS. Elliptic for all x and y.

1.14. $u_{xx} + xu_{xy} + yu_{yy} - xyu_y = 0$.
ANS. Hyperbolic for $x^2 > 4y$, parabolic for $x^2 = 4y$, elliptic for $x^2 < 4y$.

1.15. $u_{xx} + x^2 u_{yy} = 0$.
ANS. Elliptic for $x \neq 0$.

1.16. $u_{xx} - 2\sin x\, u_{xy} - \cos^2 x\, u_y = 0$.
ANS. Hyperbolic for all x.

1.17. $yu_{xx} + u_{yy} = 0$.
ANS. (Tricomi) Hyperbolic for $y < 0$, parabolic for $y = 0$, and elliptic for $y > 0$.

1.18. $u_{xx} - 2\cos x\, u_{xy} - (3 + \sin^2 x)\, u_{yy} - y\, u_y = 0$.
ANS. Hyperbolic for all x.

1.19. $(1 + x^2)u_{xx} + (1 + y^2)u_{yy} + xu_x + yu_y = 0$.
ANS. Elliptic for all x and y.

1.20. $x^2 u_{xx} - y^2 u_{yy} = 0\ x > 0, y > 0$.
ANS. Hyperbolic.

1.21. $y^2 u_{xx} + x^2 u_{yy} = 0,\ x > 0, y > 0$.
ANS. Elliptic.

1.22. $x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} = 0$.
ANS. Parabolic for all nonzero x and y.

1.23. $(1 - x)u_{xx} - u_{yy} - u_x = 0,\ 0 < x < 1$.
ANS. Hyperbolic for $x < 1$, parabolic for $x = 1$, and elliptic for $x > 1$.

1.24. $x^2 u_{xx} - y^2 u_{yy} - 2yu_y = 0$.
ANS. Hyperbolic for all nonzero x and y.

1.25. $x^2 u_{xx} - 2xyu_{xy} + y^2 u_{yy} + xu_x + yu_y = 0$.
ANS. Parabolic for all nonzero x and y.

1.26. $u_{xx} - yu_{yy} - \frac{1}{2}u_y = 0$.
ANS. Hyperbolic for $y > 0$, parabolic for $y = 0$, and elliptic for $y < 0$.

[www] All of the above exercises are solved in the Mathematica Notebook `EquationType.ma` found on the CRC web server mentioned in the Preface.

2

Method of Characteristics

It is customary in modern texts on partial differential equations either to completely ignore first order partial differential equations or postpone their discussion to a much later chapter. Since first order partial differential equations are important from both physical and geometrical standpoints, their study is essential to understand the nature of solutions and form a guide to the solutions of higher order partial differential equations.

First Order Equations

First order partial differential equations occur in a variety of situations. Some of the common ones are traffic flow, conservation laws, Mainardi–Codazzi relations in differential geometry, and shock waves. In order to solve first order linear, quasi–linear, or nonlinear partial differential equations, the method of characteristics is very useful. This method is explained in the next six sections. Second order equations, confined to linear equations, are discussed in §2.7. We shall limit our discussion to problems in R^2. An extension to higher dimensions, though routine, is more complicated.

2.1. Linear Equations with Constant Coefficients

The most general form of first order linear partial differential equations with constant coefficients is

$$a\,u_x + b\,u_y + k\,u = f(x,y). \tag{2.1}$$

If $u(x,y)$ is a solution of (2.1) then

$$du = u_x dx + u_y dy. \tag{2.2}$$

Comparing (2.1) and (2.2) one gets the auxiliary system of equations

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{f(x,y) - k\,u}. \tag{2.3}$$

The solution of the left pair is $bx - ay = c$. The other pair

$$\frac{dx}{a} = \frac{du}{f(x,y) - ku}$$

can be reduced to an ordinary linear differential equation with u as the dependent variable and x as the independent variable. This equation is given by

$$\frac{du}{dx} + \frac{ku}{a} = \frac{f(x,(bx-c)/a)}{a},$$

the integrating factor for which is $e^{kx/a}$.

This observation leads us to introduce a new dependent variable $v = ue^{kx/a}$, reducing Eq (2.1) to

$$a\,v_x + b\,v_y = f(x,y)e^{kx/a} = g(x,y).$$

Note that a reduction can also be obtained by substituting $v = u\,e^{ky/b}$. This substitution will lead to $av_x + bv_y = f(x,y)\,e^{ky/b}$. Thus, we need to consider only the formal reduced form

$$a\,u_x + b\,u_y = f(x,y). \tag{2.4}$$

The auxiliary system of equations for Eq (2.4) is

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{f(x,y)}. \tag{2.5}$$

The solution of $\dfrac{dx}{a} = \dfrac{dy}{b}$ is $bx - ay = c$, which when solved for x gives $x = \dfrac{ay+c}{b}$; a substitution of this value into $\dfrac{dy}{b} = \dfrac{du}{f(x,y)}$ yields

$$\frac{dy}{b} = \frac{du}{f\left(\dfrac{ay+c}{b}, y\right)},$$

which reduces to $du = F(y,c)dy$. Its solution is $u = G(y,c) + c_1$, where $G_y(y,c) = F(y,c)$. Thus, the general solution is obtained by replacing c_1 by $\phi(c)$ and c by $bx - ay$, thereby yielding

$$u(x,y) = G(y, bx - ay) + \phi(bx - ay).$$

Eqs (2.3) are known as the *equations of the characteristics*. The system (2.3) has two independent equations, with two solutions of the form $F(x,y,u) = 0$ and $G(x,y,u) = 0$. Each of these solutions represents a family of surfaces. The curves of intersection of these two families of surfaces are known as the *characteristics* of the partial differential equation. The projections of these curves in the (x,y)–plane are called the *base characteristics*, which are often called characteristics for brevity when there is no ambiguity. The general solution represents a family of surfaces, which are called *integral surfaces*.

Thus, the equation $bx - ay = c$ represents a family of planes. The intersection of any one of these planes with an integral surface is a curve whose projection in the (x,y)–plane will again be given by $bx - ay = c$, but this time this equation represents a straight line and is the base characteristic. The solution u on a base characteristic $bx - ay = c$ is therefore given by $u = G(y,c) + c_1$, and the general solution is the same as above.

An alternate procedure is to introduce a new set of coordinates

$$\xi = bx - ay, \quad \text{and} \quad \eta = bx + ay. \tag{2.6}$$

This substitution reduces Eq (2.4) to

$$u_\eta = F(\xi,\eta), \quad F(\xi,\eta) = \frac{f\left((\eta+\xi)/2b, (\eta-\xi)/2a\right)}{2ab}, \tag{2.7}$$

and the solution of (2.6) is

$$u(\xi,\eta) = \phi(\xi) + G(\xi,\eta), \tag{2.8}$$

where $G_\eta(\xi,\eta) = F(\xi,\eta)$. If $f(x,y) = 0$ in Eq (2.4), then the auxiliary system of equations is $dx/a = dy/b, du = 0$. The solutions of these equations are $bx - ay = c$, and $u = c_1 = \phi(c) = \phi(bx - ay)$. This procedure can also be regarded as a problem in rotation of axes (see Exercise 2.22).

Note that Eq (2.8) is

$$u(x,y) = \phi(bx - ay) + G\left(bx - ay, bx + ay\right).$$

If an initial condition $u(x,\psi(x)) = \mu(x)$ is prescribed, then

$$u\left(x,\psi(x)\right) = \mu(x) = \phi(bx - a\psi(x)) + G(bx - a\psi(x), bx + a\psi(x))$$

can be used to determine $\phi(x)$ uniquely. Thus, the existence of a unique solution u for the partial differential equation (2.1) subject to the above initial condition is established. (For the existence and uniqueness of the solution, see end of §2.7.)

EXAMPLE 2.1. Consider

$$2u_x - 3u_y = \cos x.$$

The auxiliary system of equations is

$$\frac{dx}{2} = \frac{dy}{-3} = \frac{du}{\cos x}.$$

The first solution is then given by $3x + 2y = c$. The other equation is $\dfrac{dx}{2} = \dfrac{du}{\cos x}$. Its solution is $u = c_1 + \dfrac{1}{2}\sin x$. Noting that $c_1 = f(c)$ and $3x + 2y = c$, the general solution becomes

$$u = f(3x + 2y) + \frac{1}{2}\sin x.$$

Alternately, the substitution $\xi = 3x + 2y$, $\eta = 3x - 2y$ reduces the equation to

$$u_\eta = \frac{1}{12}\cos\frac{(\xi+\eta)}{6},$$

which yields

$$u = f(\xi) + \frac{1}{2}\sin\frac{(\xi+\eta)}{6}.$$

On replacing ξ and η by their values in x and y, one gets the above general solution.

In this problem the characteristics are given by the curves of intersection of the planes $3x + 2y = c$ and the integral surfaces $u = \dfrac{1}{2}\sin x + c_1$. The projections of these curves on the (x,y)–plane $u = 0$

are the base characteristics. Graphs of these characteristics are shown in Fig. 2.1 for $c = 1$ and $c_1 = 0$.

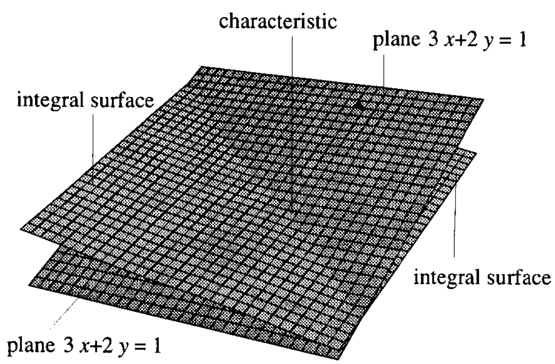


Fig. 2.1.

If a linear partial differential equation is of the form $P(x,y)\,u_x + Q(x,y)\,u_y = 0$, then the base characteristics and the characteristics are the same curves. We will now develop solutions for some specific conditions prescribed on initial curves. For example, if $u = 1$ on the initial curve $y = 0$, then

$$u = 1 + \frac{1}{2}\left(\sin x - \sin\frac{3x+2y}{3}\right),$$

which is an integral surface denoted in Fig. 2.2 by S_1. Also, if $u = x^2$ on the initial curve $y = x$, then

$$u = \frac{1}{2}\sin x + \frac{(3x+2y)^2}{25} - \frac{1}{2}\sin\frac{3x+2y}{5},$$

which is another integral surface denoted by S_2. The graphs of the integral surfaces S_1 and S_2 and the characteristics are shown in Fig. 2.2. Note that an initial curve (or initial line) is a curve where an initial condition on u is prescribed. ■

WWW A complete Mathematica solution for this example and the individual plots of the integral surfaces S_1 and S_2 are available in the Mathematica Notebook `Example2.1.ma`.

It is obvious that the solution of a first order linear partial differential equation represents a surface and contains an arbitrary function and not an arbitrary constant. Clearly, the solution represents a family of surfaces. A unique surface is obtained if u is prescribed on an initial curve, which is not a characteristic. The reader can observe that the existence and uniqueness of the solution of a first order equation are closely related to the existence and uniqueness of the solutions of the auxiliary system (2.5) of ordinary differential equations (see end of §2.7).

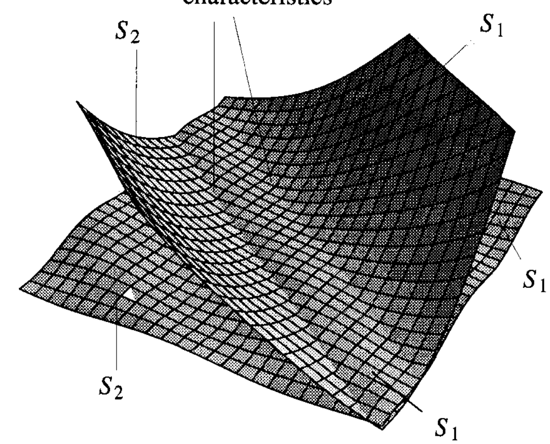


Fig. 2.2.

EXAMPLE 2.2. Consider

$$4u_x + u_y = x^2 y.$$

The auxiliary system of equations is

$$\frac{dx}{4} = \frac{dy}{1} = \frac{du}{x^2 y}.$$

The first solution is then given by $x - 4y = c$, and the solution, following the method of the previous example, is

$$u = c_1 + \frac{3x^4 - 4cx^3}{48} = f(c) + \frac{3x^4 - 4cx^3}{48}.$$

On replacing c by $x - 4y$, one gets the general solution

$$u = f(x-4y) + \frac{3x^4 - 4(x-4y)x^3}{48} = f(x-4y) - \frac{x^4}{48} + \frac{x^3 y}{3}. \;■$$

2.2. Linear Equations with Variable Coefficients

The general form of first order linear partial differential equations with variable coefficients is

$$P(x,y)u_x + Q(x,y)u_y + f(x,y)u = R(x,y). \tag{2.9}$$

Once again our attempt is to eliminate the term in u from Eq (2.9). This can be accomplished by substituting

$$u = ve^{-\xi(x,y)},$$

where $\xi(x,y)$ satisfies the equation

$$P(x,y)\,\xi_x(x,y) + Q(x,y)\,\xi_y(x,y) = f(x,y).$$

Hence, Eq (2.9) is formally reduced to

$$P(x,y)u_x + Q(x,y)u_y = R(x,y), \tag{2.10}$$

where P, Q, R in (2.10) are not the same as in (2.9). The method for solving these equations, known as Lagrange's method, is essentially the same as in the previous section except that now the auxiliary system of equations is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{du}{R}, \tag{2.11}$$

which becomes more complicated. This system has two solutions of the type

$$g(x,y,u) = c_1, \quad \text{and} \quad h(x,y,u) = c_2,$$

representing two families of surfaces. The curves of intersection of these surfaces are called *characteristics* of the equation. The projection of a in the plane $u = 0$ is called a base characteristic.

If $R(x,y) = 0$, then there is no difference in the base characteristics and the characteristics. Frequently the word *base* is omitted from the term *base characteristic*. These characteristics are clearly a one–parameter family of curves. In some cases it is convenient to introduce a parameter, say s, in the auxiliary system of equations, which are then expressed in the form

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{du}{R} = ds. \tag{2.12}$$

EXAMPLE 2.3. Consider

$$3u_x + 4u_y + 14(x+y)u = 6x\,e^{-(x+y)^2}.$$

The first step is to find a function $\zeta(x,y)$ which satisfies the equation

$$3\zeta_x + 4\zeta_y = 14(x+y).$$

A particular solution of this equation is $\zeta(x,y) = (x+y)^2$. Then the substitution $u = ve^{-\zeta(x,y)}$ reduces the given equation to

$$3v_x + 4v_y = 6x,$$

with auxiliary equations

$$\frac{dx}{3} = \frac{dy}{4} = \frac{dv}{6x}.$$

The solution of $\dfrac{dx}{3} = \dfrac{dy}{4}$ is $4x - 3y = c_1$, and solution of $\dfrac{dx}{3} = \dfrac{dv}{6x}$ is $v = x^2 + c_2$. Now, as before, v can easily be found to be $v = x^2 + f(4x - 3y)$. Let us examine these solutions a little more. If we write $c_1 = 4x - 3y = g(x,y,v)$ and $c_2 = v - x^2 = h(x,y,v)$, and consider an expression of the form $F(g,h) = 0$, then

$$F_x = F_g(g_x + g_v v_x) + F_h(h_x + h_v v_x) = 0,$$

which reduces to

$$4F_g + (v_x - 2x)F_h = 0.$$

Similarly the expression for F_y yields

$$-3F_g + v_y\,F_h = 0.$$

Eliminating F_g and F_h from these two equations, one gets

$$\det\begin{vmatrix} 4 & v_x - 2x \\ -3 & v_y \end{vmatrix} = 0,$$

i.e.,

$$3v_x + 4v_y = 6x.$$

Thus, $F(g,h) = 0$ is also a solution of the differential equation. Of course, we have to replace v by $u\,e^{(x+y)^2}$ to obtain the solution of the original problem. ∎

Solutions of the type $F(g,h) = 0$ or $g = F(h)$ or $h = F(g)$ are known as *general solutions*.

DEFINITION 2.1. Two C^1 functions g and h are said to be *functionally independent* if $\nabla g \times \nabla h \neq \mathbf{0}$.

We shall state an important theorem.

THEOREM 2.1. *Let $g = c_1$ and $h = c_2$ be any two functionally independent solutions of Eq (2.9). Then $g = F(h)$, or $h = F(g)$, or $F(g, h) = 0$ represents the general solution of Eq (2.9).*

COROLLARY 2.1. *If $g = c_1$ is a solution of Eq (2.9), then $F(g) = 0$ is also a solution of Eq (2.9).*

EXAMPLE 2.4. Let us further consider the reduced equation $3v_x + 4v_y = 6x$ from Example 2.3. The auxiliary equations in the parametric form are

$$\frac{dx}{3} = \frac{dy}{4} = \frac{dv}{6x} = ds.$$

The solutions are

$$x = 3s + c_1, \quad \text{and} \quad y = 4s + c_2, \tag{2.13}$$

and, therefore,

$$v = 9s^2 + 6c_1 s + c_3. \tag{2.14}$$

The last solution is obtained by first substituting $x = 3s + c_1$ in $\dfrac{dv}{6x} = ds$. The values of x, y, v in terms of s represent the parametric form of the equation of the characteristics of the given partial differential equation. In order to find a specific characteristic, we need to have an initial condition, e.g., $x = x_0$, $y = y_0$, $v = v_0$ at $s = s_0$. By eliminating s from (2.13) and (2.14) we get

$$4x - 3y = 4c_1 - 3c_2 = \alpha \quad \text{and} \quad v = x^2 - c_1^2 + c_3 = x^2 + \beta,$$

from which we get

$$v = x^2 + f(4x - 3y). \blacksquare$$

We note here that, except for singular cases, a unique characteristic will, in general, pass through a point in space. Thus, if a continuous initial curve is prescribed, then a unique characteristic will pass through every point of the initial curve. The locus of these characteristics will form the integral surface. Hence, if the initial curve is a characteristic itself, then the existence of an integral surface cannot be guaranteed (see end of §2.7).

EXAMPLE 2.5. Consider

$$u_x + e^x u_y = y, \quad u(0, y) = 1 + y.$$

The auxiliary system of equations is

$$\frac{dx}{1} = \frac{dy}{e^x} = \frac{du}{y}. \tag{2.15}$$

Thus, the solutions of the auxiliary system are given by $dy = e^x\,dx$, which yields $y = e^x + c$. From the second set of Eq (2.15) one gets $du = y\,dx$ which results in $du = (e^x + c)\,dx$. Its solution is $u = e^x + cx + c_1$; hence

$$u = e^x + cx + f(c), \quad \text{or} \quad u = e^x + (y - e^x)x + f(y - e^x).$$

Now, in view of the initial condition, $u(0, y) = 1 + f(y - 1) = 1 + y$, which yields $f(y) = y + 1$. Hence the solution is

$$u(x, y) = e^x + (y - e^x)x + y - e^x + 1 = 1 + y + xy - xe^x. \blacksquare$$

[www] The plots of the surfaces $y = e^x + c$ and $u = e^x + (y - e^x)\,x$ for $c = 1$ are available in the Mathematica Notebook `Example2.5.ma`. The intersection of these surfaces is a characteristic.

EXAMPLE 2.6. Consider

$$2y\,u_x + u_y = x, \; u(0, y) = f(y).$$

The auxiliary system of equations is

$$\frac{dx}{2y} = \frac{dy}{1} = \frac{du}{x}.$$

In this example, a slightly different procedure will be demonstrated. We will first find the equation of the characteristics through an arbitrary point (x_0, y_0) on the integral surface. The left pair in the auxiliary equations is $dx = 2y\,dy$, whose solution is the family of surfaces

$$S_1: \quad x = y^2 + c.$$

When a surface S_1 passes through the point (x_0, y_0), its equation becomes $x = y^2 + x_0 - y_0^2$. If the initial curve is $x = 0$, where $u(0, y) = f(y)$, then the value of y on the initial curve is given by

$\hat{y}_0^2 = y_0^2 - x_0$, where $\hat{y}_0$ is the value of y on the integral surface through (x_0, y_0) at $x = 0$. The right pair of auxiliary equations is $du = x\,dy$, which also represents a family of surfaces. Let us denote the integral surface through $(0, \hat{y}_0)$ by S_2. The curve of intersection of S_1 and S_2 is a characteristic. The differential equation of this characteristic is given by $du = (y^2 + x_0 - y_0^2)\,dy$. Its solution is

$$u = \frac{y^3}{3} + (x_0 - y_0^2)y + c_1.$$

At $x = 0$,

$$u(0, \hat{y}_0) = \frac{\hat{y}_0^3}{3} + (x_0 - y_0^2)\,\hat{y}_0 + c_1 = f(\hat{y}_0),$$

and substituting the value of $\hat{y}_0$, one gets

$$c_1 = f((y_0^2 - x_0)^{1/2}) + \frac{2}{3}(y_0^2 - x_0)^{3/2}.$$

Using this value of c_1 in u, the value of u at (x_0, y_0) is given by

$$u(x_0, y_0) = \frac{2}{3}(y_0^2 - x_0)^{3/2} - \frac{2}{3}y_0^3 + x_0 y_0 + f((y_0^2 - x_0)^{1/2}).$$

Since (x_0, y_0) is an arbitrary point, the expression for $u(x_0, y_0)$ is generalized to

$$u(x, y) = \frac{2}{3}(y^2 - x)^{3/2} - \frac{2}{3}y^3 + xy + f((y^2 - x)^{1/2}). \blacksquare$$

EXAMPLE 2.7. Consider

$$(x + 2y)u_x + (y - x)u_y = y.$$

The auxiliary equations are

$$\frac{dx}{x + 2y} = \frac{dy}{y - x} = \frac{dv}{y}.$$

The first two equations can be expressed as

$$\frac{dy}{dx} = \frac{y - x}{x + 2y}.$$

This is a homogeneous ordinary differential equation of the first order. A standard substitution for such problems is $y = vx$, which leads to a first order ordinary differential equation with separable variables

$$\frac{(1 + 2v)dv}{1 + 2v^2} = -\frac{dx}{x}.$$

This equation can be solved to give

$$\sqrt{2}\tan^{-1}\frac{\sqrt{2}y}{x} + \ln(x^2 + 2y^2) = c_1.$$

The other solution can be obtained by observing that $dx + dy = 3\,du$, thus yielding $u = \frac{1}{3}(x + y) + c_2$. Hence the general solution becomes

$$u = \frac{1}{3}(x + y) + f\left[\sqrt{2}\tan^{-1}\frac{\sqrt{2}y}{x} + \ln(x^2 + 2y^2)\right]. \blacksquare$$

WWW The plots of the surfaces

$$\sqrt{2}\tan^{-1}\frac{\sqrt{2}y}{x} + \ln(x^2 + 2y^2) = 1, \quad \text{and} \quad u = \frac{1}{3}(x + y)$$

are available in the Mathematica Notebook `Example2.7.ma`. Note that the intersection of these two surfaces gives a particular characteristic.

2.3. First Order Quasi–linear Equations

If the coefficients P, Q, and R in Eq (2.9) are functions of x, y, and u, but not of u_x and u_y, then the equation is known as quasi–linear. In these equations the first order derivatives occur only in the first degree, although the equation need not be linear in u. Such equations occur in shock waves of various kinds, e.g. traffic flow, water waves. The basic technique is the same as for first order linear equations. The starting point is still the system of auxiliary equations (2.11).

EXAMPLE 2.8. Consider the quasi–linear equation

$$u_x + u\,u_y = 0, \quad u(0, y) = f(y).$$

The system of auxiliary equations is

$$\frac{dx}{1} = \frac{dy}{u}, \quad du = 0.$$

Here $du = 0$ implies $u = c$. Using this value of u in $\frac{dx}{1} = \frac{dy}{u}$ one gets $y = cx + c_1$. Thus, the characteristics are given by the curves of intersection of the surfaces $y - ux = c_1$ and $u = c$, and the general solution can be expressed as $u = g(y - ux)$. Applying the initial condition, we get $f(y) = g(y)$. Hence the solution to the problem is $u = f(y - ux)$.

If $f(y) = y$, the solution becomes $u = \frac{y}{1+x}$.

If $f(y) = y^2$, then we have $u = (y-ux)^2$, which after some algebraic simplification yields

$$u = \frac{1 + 2xy \pm \sqrt{1+4xy}}{2x^2}.$$

A careful examination by checking the limit as $x \to 0$ shows that the valid solution is

$$u = \frac{1 + 2xy - \sqrt{1+4xy}}{2x^2}. \blacksquare$$

WWW In the Mathematica Notebook `Example2.8.ma`, the following plots are given:
(i) The graphs of $y - cx = c_1$ for $c = 1, 2, 3$, and $c_1 = 0, 1, 2$. These curves are characteristics and also base characteristics.
(ii) The graph of $u = \frac{y}{1+x}$ represents an integral surface.
(iii) The graph of $u = \frac{1 + 2xy - \sqrt{1+4xy}}{2x^2}$ represents another integral surface.

EXAMPLE 2.9. Consider

$$u_x + g(u)u_y = 0, \quad u(0, y) = f(y).$$

The system of auxiliary equations is

$$\frac{dx}{1} = \frac{dy}{g(u)}, \quad du = 0.$$

As in Example 2.7, $u = c$, and $y = g(c)x + c_1$. The general solution is $u = h(y - xg(u))$. After applying the initial condition, we get $f(y) = h(y)$. Hence the solution to the problem is $u = f(y - xg(u))$. $\blacksquare$

EXAMPLE 2.10. Consider

$$u_x + u_y = u^2 + 1, \quad u(0, y) = f(y).$$

The auxiliary equations are then

$$dx = dy = \frac{du}{u^2+1}.$$

The solutions are $y = x + c$, and $\tan^{-1} u = x + c_1$. Thus, the general solution is $u = \tan(x + g(y - x))$, and the particular solution for the problem is

$$u = \frac{\tan x + f(y-x)}{1 - f(y-x)\tan x}. \blacksquare$$

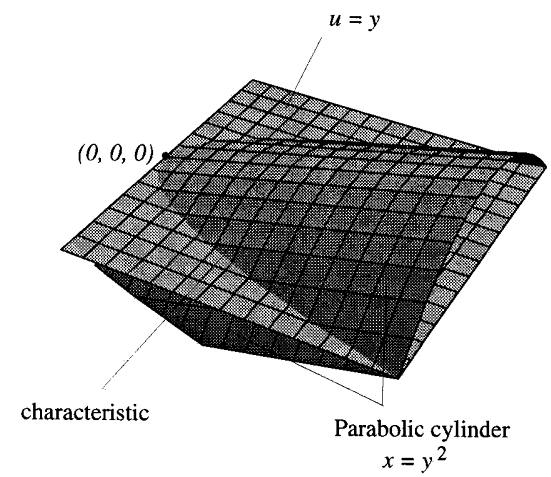


Fig. 2.3.

EXAMPLE 2.11. Consider the partial differential equation

$$2y\,u_x + u_y = 1,$$

where $u = 1$ is prescribed on the initial curve $y = 0$ (x–axis) for $0 \le x \le 1$. The auxiliary system (2.11) for this partial differential equation is

$$\frac{dx}{2y} = dy = du.$$

The solution of $dx = 2y\,dy$ is the parabolic cylinder $x = y^2 + A$, while that of $du = dy$ is the plane $u = y + B$, where the parameters A and B are constant for each characteristic. For any point (x_0, y_0, u_0) we find that $A = x_0 - y_0^2$, and $B = z_0 - y_0$. If the point (x_0, y_0, u_0) is taken, e.g., as the origin of the the coordinate system, then

$$x = y^2, \quad \text{and} \quad u = y. \tag{2.16}$$

The equation of the characteristic is the intersecting curve (in this case, the parabola) of the solution (2.16), as shown in Fig. 2.3.

The characteristic through the point $(1, 0, 1)$ is the intersection of the parabolic cylinder $x = y^2 + 1$ and the plane $u = y + 1$. Moreover, if $u = x^2$ on the initial curve $y = 0$, then $u = y + (x - y^2)^2$ represents the integral surface. ■

EXAMPLE 2.12. Consider

$$u_x + 2uu_y = 1.$$

The auxiliary equations are

$$dx = \frac{dy}{2u} = du.$$

The solutions are $u = x + c$, and $u^2 = y + c_1$. Hence the general solution is

$$u = x + f(u^2 - y), \quad \text{or} \quad u^2 = y + g(u - x).$$

It is important to choose the appropriate general solutions for the given initial conditions. Thus, for example, if the initial line (curve) is $x = y$, and the value of u on this initial line is $u(y, y) = y$, then from the first solution we get $u = x$. But the second solution gives $y^2 = y + g(0)$, which does not yield a value for the function g.

If the initial line is $y = x$, and $u(y, y) = y^2$, then the second solution yields

$$u^2 = y + (u - x)^2 + 2(u - x) \pm (u - x)\sqrt{1 + 4(u - x)},$$

where the plus sign corresponds to $x > \frac{1}{2}$, $y > \frac{1}{2}$, and the minus sign to $x < \frac{1}{2}$, $y < \frac{1}{2}$. Substituting $y = x$ and $u = y^2$ in the second solution, we have $y^4 = y + g(y^2 - y)$. Now let $z = y^2 - y$, then

$$y = \frac{1}{2}\left(1 \pm \sqrt{1 + 4z}\right), \quad \text{and} \quad y^4 - y = z^2 + 2z \pm z\sqrt{1 + 4z} = g(z),$$

which gives the solution of the equation. But if we use the first solution we get $y^2 = y + f(y^4 - y)$, which is more difficult to resolve. ■

EXAMPLE 2.13. Consider

$$(y + u)\,u_x + y\,u_y = x - y.$$

For this problem, it is convenient to use the auxiliary equations in parametric form (2.12), i.e.,

$$\frac{dx}{y + u} = \frac{dy}{y} = \frac{du}{x - y} = ds,$$

which can be rewritten as

$$\frac{dx}{ds} = y + u, \quad \frac{dy}{ds} = y, \quad \frac{du}{ds} = x - y.$$

The solution of the middle equation is $y = Ae^s$. Addition of the first and third equations yields

$$\frac{d(u + x)}{ds} = u + x.$$

Its solution is $u + x = Be^s$. Subtracting the first equation from the third results in

$$\frac{d(u - x)}{ds} = x - u - 2y,$$

which can be expressed as

$$\frac{d(u - x)}{ds} + (u - x) = -2Ae^s.$$

This equation is linear in $u - x$. Its solution is

$$u - x = Ce^{-s} - Ae^s.$$

Replacing e^s by $\dfrac{y}{A}$ in the two solutions, we get

$$u + x = \frac{B}{A}y, \quad \text{and} \quad u - x + y = \frac{CA}{y}.$$

Noting that B/A and CA can be replaced by c_1 and c_2, and that $c_2 = f(c_1)$, we have

$$(u - x + y)y = f\left(\frac{u + x}{y}\right) \tag{2.17}$$

as the general solution of the given equation. ■

EXAMPLE 2.14. Consider

$$x\,y\,u\,u_x + (x^2 - u^2)\,u_y = x^2 y.$$

The auxiliary equations are

$$\frac{dx}{xyu} = \frac{dy}{x^2 - u^2} = \frac{du}{x^2 y}.$$

From

$$\frac{dx}{xyu} = \frac{du}{x^2 y},$$

we get $x^2 - u^2 = c_1$, and then using this solution in

$$\frac{dy}{x^2 - u^2} = \frac{du}{x^2 y},$$

we obtain

$$y\,dy = \frac{c_1 du}{u^2 + c_1}.$$

This equation yields

$$y^2 = \begin{cases} 2a\tan^{-1}\dfrac{u}{a} + c_2, & \text{if } c_1 = a^2, \\ a\ln\dfrac{u-a}{u+a} + c_2, & \text{if } c_1 = -a^2. \end{cases}$$

The general solution can now be found. ■

2.4. First Order Nonlinear Equations

The general form of a first order nonlinear equation is

$$F(x, y, u, u_x, u_y) = 0. \tag{2.18}$$

Consider the two–parameter family of surfaces

$$f(x, y, u, a, b) = 0. \tag{2.19}$$

Then

$$f_x + f_u u_x = 0, \quad \text{and} \quad f_y + f_u u_y = 0. \tag{2.20}$$

Equations (2.19) and (2.20) form a set of three equations in the two parameters a and b. If a and b are eliminated from these equations, one gets an equation of the type (2.18). Therefore, it is reasonable to assume that the solution of (2.18) is of type (2.19). It is clear that any envelope* of this family will also be a solution of equation (2.18). At this point we state the *Cauchy problem*: Determine $u(x, y)$ such that u and its partial derivatives satisfy

$$F(x, y, u, u_x, u_y) = 0,$$

subject to the condition $u(0, y) = \phi(y)$. In this case $u(x, y)$ is prescribed on the initial line $x = 0$ (y–axis). Initial data can, however, be prescribed on any simple curve which is not a characteristic of Eq (2.18). Thus, for example, if the parametric form of the curve is $x = x(s), y = y(s)$, then the initial data can be written as

$$u(x(s), y(s)) = \psi(s).$$

We will now distinguish between different kinds of solutions of equation (2.18).

Complete integral: A two–parameter family of solutions of the type $f(x, y, u, a, b) = 0$ is known as a complete integral.

General integral: If $b = g(a)$, where g is an arbitrary function and an envelope of the family of solutions of the complete integral is found, then the envelope whose equation contains an arbitrary function is known as the general integral corresponding to the solution (2.19).

Singular integral: If the two–parameter family of solutions (2.19) has an envelope, then the equation of this envelope is known as the singular integral of (2.18).

Cauchy's method of characteristics: This method is similar to the method of characteristics discussed earlier for linear and quasi–linear partial differential equations.

Consider a first order nonlinear equation (2.18). For convenience, we will use the notation

$$u_x = p \quad \text{and} \quad u_y = q. \tag{2.21}$$

*An envelope of the family of surfaces $f(x, y, u, a, b)$ is a surface which touches some member of this family at every point.

If $u(x,y)=c$ is a solution of (2.18), then u and its partial derivatives satisfy the equation (2.18). But the total derivative of u is given by

$$du = p\,dx + q\,dy. \tag{2.22}$$

Differentiating (2.18) and (2.22) with respect to p, we get

$$dx + \frac{dq}{dp}\,dy = 0, \tag{2.23}$$

and

$$F_p + F_q\,\frac{dq}{dp} = 0. \tag{2.24}$$

Equations (2.23) and (2.24) yield

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{p\,dx+q\,dy}{p\,F_p+q\,F_q} = \frac{du}{p\,F_p+q\,F_q} = dt. \tag{2.25}$$

These are the characteristic (or auxiliary) equations of Eq (2.18). It can be easily verified that they reduce to the characteristic equations of a linear or a quasi-linear partial differential equation according as Eq (2.18) is linear or quasi-linear. The parameter t introduced in (2.25) is such that

$$\frac{dx}{dt} = F_p, \quad \frac{dy}{dt} = F_q \quad \text{and} \quad \frac{du}{dt} = p\,F_p + q\,F_q. \tag{2.26}$$

Since p is a function of t, it follows that

$$\frac{dp}{dt} = p_x\frac{dx}{dt} + p_y\frac{dy}{dt} = p_xF_p + p_yF_q = p_xF_p + q_xF_q, \ (q_x = p_y). \tag{2.27}$$

Differentiating Eq (2.18) with respect to x, we have

$$F_x + F_u u_x + F_p p_x + F_q q_x = 0.$$

Using this equation and inserting the value of $F_p p_x + F_q q_x$ in (2.27), we get

$$\frac{dp}{dt} = -(F_x + F_u u_x) = -(F_x + pF_u). \tag{2.28}$$

Similarly,

$$\frac{dq}{dt} = -(F_y + qF_u). \tag{2.29}$$

Combining equations (2.25), (2.28), and (2.29) we get

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{du}{pF_p+qF_q} = \frac{dp}{-(F_x+pF_u)} = \frac{dq}{-(F_y+qF_u)} = dt. \tag{2.30}$$

This system of auxiliary equations (also known as the characteristic equations) is used to solve a nonlinear equation by Cauchy's method.

The difference between the equations (2.25) and the corresponding equations (2.11) for a linear partial differential equation is that the equations (2.25) contain p and q explicitly and, therefore, in order to solve them we need additional equations which are included in (2.30). The solution is found by eliminating p and q from the solutions of (2.30) and the given equation. The eliminant will, in general, contain two arbitrary constants and will represent a complete integral of the equation. We demonstrate the method by the following examples.

EXAMPLE 2.15. Consider

$$u = 4p\,q.$$

The auxiliary system (2.30) for this equation is

$$\frac{dx}{dt} = -4q, \quad \frac{dy}{dt} = -4p, \quad \frac{du}{dt} = -8pq, \quad \frac{dp}{dt} = -p, \quad \frac{dq}{dt} = -q. \tag{2.31}$$

Since

$$\frac{dx}{dt} = 4\frac{dq}{dt}, \quad \text{and} \quad \frac{dy}{dt} = 4\frac{dp}{dt},$$

we get

$$x + c_1 = 4q \quad \text{and} \quad y + c_2 = 4p. \tag{2.32}$$

Substituting these values of p and q into the given equation, we get the complete solution as

$$u = \frac{1}{4}(x+c_1)(y+c_2). \tag{2.33}$$

However, if we demand that the solution pass through a given curve, then (2.33) may or may not yield the required solution. For example, since the solution of

$$\frac{d^2x}{dt^2} = -4\frac{dq}{dt} = 4q = -\frac{dx}{dt}$$

is $x = c_1 + c_2\,e^{-t}$, we can require that $u = y^2$ be the initial condition on the initial curve $x = 0$, which corresponds to $t = 0$. Then the solution given by (2.33) fails to yield the required solution. To avoid this situation we follow an alternate approach. The initial values for

x, y, and u can be taken to be $x_0 = 0$, $y_0 = \nu$, and $u_0 = \nu^2$, where ν is a parameter. Let the initial value of q be $q_0 = \left(\frac{\partial u}{\partial y}\right)_0$. Thus,

$$q_0 = \left(\frac{\partial u}{\partial y}\right)_0 = 2y_0 = 2\nu,$$

and since $u_0 = 4p_0q_0$, we find that the initial value of p is $p_0 = \frac{\nu}{8}$. But from equations (2.31) we note that p and q can be solved in terms of t as

$$p = Ae^{-t}, \quad \text{and} \quad q = Be^{-t}. \tag{2.34}$$

Substituting the initial values in (2.32) and (2.34) we find that

$$c_1 = 8\nu, \quad c_2 = -\nu/2, \quad A = \nu/8, \quad B = 2\nu.$$

Then, from (2.32) and the given equation, x, y and u can be expressed in terms of ν and t as

$$x = 8\nu(e^{-t} - 1), \quad y = \frac{\nu}{2}(e^{-t} + 1), \quad u = \nu^2 e^{-2t}. \tag{2.35}$$

The required solution can now be found by eliminating ν and t from(2.35) as

$$u = \left(\frac{x}{16} + y\right)^2. \blacksquare$$

WWW In the Mathematica Notebook `Example2.15.ma`, the plot of the above solution represents an integral surface.

We will now consider some special cases of equation (2.18).

EXAMPLE 2.16. Consider

$$u = px + qy + f(p, q), \quad \text{or} \quad F = px + qy + f(p, q) - u = 0. \tag{2.36}$$

This is a special type of nonlinear equation. It always has a complete solution which can be obtained in a simple manner. The last two of the characteristic equations are

$$dp = 0, \quad \text{and} \quad dq = 0,$$

which yield $p = a$ and $q = b$, where a and b are arbitrary constants. Substituting these values in (2.36), one gets $u = ax + by + f(a, b)$ as the complete solution. Equations of the type (2.36) are known as *Clairaut equations*. There are other special types of partial differential equations which yield the complete solution in a relatively easy manner. ■

EXAMPLE 2.17. Consider

$$f(p, q) = 0. \tag{2.37}$$

Note that the characteristic equations will yield $dp = 0$ and $dq = 0$. So $p = a$, and solving $f(p, q) = 0$ for q will give $q = g(a)$; then observing that

$$du = a\,dx + g(a)\,dy,$$

we get

$$u = ax + g(a)y + c.$$

As an example, let $f(p, q) = p^2 + q^2 - 1 = 0$. The auxiliary equations (2.30) are

$$\frac{dx}{dt} = p, \quad \frac{dy}{dt} = q, \quad du = dt, \quad dp = 0, \quad dq = 0.$$

Using $dp = 0$, we get $p = a$ and $q = \sqrt{1 - a^2}$, and these two combined with $du = p\,dx + q\,dy$ yield

$$u = ax + y\sqrt{1 - a^2} + c.$$

This is a complete solution. Another complete solution can be obtained by using $p = a$ in $\frac{dx}{dt} = p$ and noting that $du = dt$, thus getting $du = \frac{dx}{a}$, which gives

$$u = \frac{x}{a} + \alpha.$$

Similarly, $du = \frac{dy}{\sqrt{1 - a^2}}$ implies that

$$u = \frac{y}{\sqrt{1 - a^2}} + \beta.$$

Thus, we can write $au = x + a\alpha$, and

$$u\sqrt{1 - a^2} = y + a\beta.$$

Replacing $a\alpha$ and $a\beta$ by $-c$ and $-d$, respectively, and eliminating a we get

$$u^2 = (x - c)^2 + (y - d)^2,$$

which is another complete solution. ■

EXAMPLE 2.18. Consider

$$F(u,p,q) = 0. \tag{2.38}$$

The last three terms of the system (2.30) yield

$$\frac{dp}{-pF_u} = \frac{dq}{-qF_u} = dt, \quad \text{or} \quad \frac{dp}{p} = \frac{dq}{q},$$

i.e., $p = a^2 q$, where a^2 is an arbitrary constant. This equation together with (2.38) can be solved for p and q, and then we proceed as in the previous example. Thus, let $F(u,p,q) = u^2+pq-4 = 0$. Then following the above procedure we have $p = a^2 q$, which gives

$$q = \pm\frac{1}{a}\sqrt{4-u^2} \quad \text{and} \quad p = \pm a\sqrt{4-u^2}.$$

Therefore, since $du = p\,dx + q\,dy$, we have

$$du = \pm\sqrt{4-u^2}\left(a\,dx + \frac{1}{a}\,dy\right),$$

or

$$\frac{du}{\sqrt{4-u^2}} = \pm\left(a\,dx + \frac{1}{a}\,dy\right),$$

which gives

$$\sin^{-1}\frac{u}{2} = \pm\left(a\,x + \frac{1}{a}\,y + c\right).$$

Hence

$$u = \pm 2\sin\left(a\,x + \frac{1}{a}\,y + c\right). \blacksquare$$

WWW In the Mathematica Notebook `Example2.18.ma`, the plots of

$$u = 2\sin\left(a\,x + \frac{1}{a}\,y + c\right)$$

for $c = 0, 1, 2$, and $a = 1, 2$ represent some particular integral surfaces.

EXAMPLE 2.19. If $F(x,y,u,p,q) = 0$ is independent of u and can be expressed as $\phi(x,p) = \psi(y,q)$, then each of these functions must be constant.Thus, if $\phi(x,p) = c$ and $\psi(y,q) = c$ can be solved for p and q, then a complete integral can be obtained. For example, consider

$$F(x,y,u,p,q) = p^2(1-x^2) - q^2(4-y^2) = 0.$$

Then

$$p^2(1-x^2) = q^2(4-y^2) = a^2,$$

which gives

$$p = \frac{a}{\sqrt{1-x^2}}, \quad \text{and} \quad q = \frac{a}{\sqrt{4-y^2}},$$

where we have ignored the negative solutions. Now since $du = p\,dx + q\,dy$, we have

$$du = \frac{a}{\sqrt{1-x^2}}\,dx + \frac{a}{\sqrt{4-y^2}}\,dy,$$

which can be integrated to give

$$u = a\left(\sin^{-1}x + \sin^{-1}\frac{y}{2}\right) + b. \blacksquare$$

WWW In the Mathematica Notebook `Example2.19.ma`, the plots of the above solution for $a = 1$ and $b = 0, 1$ represent some particular integral surfaces.

EXAMPLE 2.20. Consider

$$F(x,y,u,p,q) = 2pqy - pu - 2a = 0, \tag{2.39}$$

for which

$$F_x = 0, \quad F_y = 2pq, \quad F_u = -p, \quad F_p = 2qy - u, \quad \text{and} \quad F_q = 2py.$$

The auxiliary equations are

$$\frac{dx}{2qy-u} = \frac{dy}{2py} = \frac{du}{2pqy - pu + 2pqy} = \frac{du}{pu+4a} = \frac{dp}{p^2} = \frac{dq}{-pq}.$$

The last pair reduces to

$$\frac{dp}{p} + \frac{dq}{q} = 0,$$

which gives $pq = \alpha$. Using this value of pq in (2.39), we get

$$2\alpha y - pu - 2a = 0,$$

which yields

$$pu = 2\alpha y - 2a, \quad \text{and} \quad q = \frac{\alpha u}{2(\alpha y - a)}.$$

These equations after integration yield

$$u^2 = 4(\alpha y - a)(x - \beta). \blacksquare$$

2.5. Geometrical Considerations

Before we discuss the geometrical interpretation of the partial differential equation (2.18), i.e., $F(x, y, u, p, q) = 0$, let us recall the geometrical interpretation of a first order ordinary differential equation $y' = f(x, y)$. Here $f(x, y)$ represents the slope of any integral curve at the point (x, y). This slope is unique at every point. If we graph $f(x, y) = c$, then the curve so obtained is known as an isocline or a curve of constant slope. Of course, the curve itself does not have constant slope, but every integral curve which intersects $f(x, y) = c$ has the slope c at the point of intersection. Since the correspondence between integral curves and the points of an isocline is one–to–one, the number of integral curves is, in general, equal to the number of points on the isocline, i.e., there exists a single infinity of them. However, the exception to this rule occurs when isoclines intersect at a point, in which case the point is a singular point of the differential equation, or when the isocline is also a solution curve, in which case the isocline is a straight line with slope c and the isocline is an envelope of the integral curves, except for the equation $\dfrac{dy}{dx} = \dfrac{y}{x}$ whose isoclines and the integral curves are the same.

The situation for a partial differential equation is somewhat complicated. In this case the values of p and q are not unique at a fixed point (x, y, u). If an integral surface is $g(x, y, u) = 0$, then p and q represent the slopes of the curves of intersection of the surface with the planes $u =$ const. Moreover, p, q, -1 represent the direction ratios of the normal to the surface at the point (x, y, u). The derivatives p and q are constrained by Eq (2.18). Obviously, at a fixed point, p and q can be represented by a single parameter. Hence, there are infinitely many possible normals and consequently infinitely many integral surfaces passing through any fixed point. So, unlike the case of ordinary differential equations, we cannot determine a unique integral surface by making it pass through a point.

Cauchy established that a unique integral surface can be obtained by making it pass through a continuous twisted space curve, also known as an initial curve, except when the curve is a characteristic of the differential equation. Now, the infinity of normals passing through a fixed point generates a cone known as the normal cone. The corresponding tangent planes to the integral surfaces envelope a cone known as the Monge cone. In the case of a linear equation, the normal cone degenerates into a plane since each normal is perpendicular to a fixed line.

Consider the equation $a\,p + b\,q = c$. Then the direction $p, q, -1$ is perpendicular to the direction ratios a, b, c. This direction is fixed at a fixed point. The Monge cone then degenerates into a coaxial set of planes known as the Monge pencil. The common axis of the planes is the line through the fixed point with direction ratios a, b, c. This line is known as the Monge axis.

2.6. Some Theorems on Characteristics

Suppose $u(x, y) = f(x, y)$ is an integral surface S of the partial differential equation Eq (2.18). Then the set of numbers

$$\left(x_0, y_0, \, u_0 = u(x_0, y_0), \, p_0 = \left(\frac{\partial f}{\partial x}\right)_{(x_0, y_0)}, \, q_0 = \left(\frac{\partial f}{\partial y}\right)_{(x_0, y_0)}\right),$$

which represents a plane with normal $(p_0, q_0, -1)$ and passing through the point (x_0, y_0, u_0), is called a *plane element*. If the point (x_0, y_0, u_0) lies on S, then the element $(x_0, y_0, u_0, p_0, q_0)$ satisfies Eq (2.18) and is called an *integral element* of the surface. Let R be a neighborhood of (x_0, y_0) in the plane $u = 0$. If the functions f_x and f_y are continuous in R, then the element $(x_0, y_0, u_0, p_0, q_0)$ is called a *tangent element* of S.

A curve Γ with parametric equations $x = x(t), y = y(t), u = u(t)$ lies on the surface S if $u(t) = f(x(t), y(t))$ for all admissible values of t. If a point P_0 on Γ corresponds to the value t_0 of the parameter t, then the direction of the tangent line is given by $\left(\dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{du}{dt}\right)_{t=t_0}$. This direction is normal to $\left(p_0 = (\frac{\partial u}{\partial x})_{t_0}, q_0 = (\frac{\partial u}{\partial y})_{t_0}, -1\right)$ if

$$\left(\frac{du}{dt}\right)_{t_0} = p_0 \left(\frac{dx}{dt}\right)_{t_0} + q_0 \left(\frac{du}{dt}\right)_{t_0}.$$

Thus, a set of five functions $x(t), y(t), u(t), p(t), q(t)$, which satisfy the condition

$$\frac{du}{dt} = p(t)\,\frac{dx}{dt} + q(t)\,\frac{dy}{dt},$$

defines a *strip* on the curve Γ. If this strip is an integral element, then it is an integral strip of the partial differential equation. If this integral strip at each point touches a generator of the Monge cone, then the integral strip is a *characteristic strip*.

We will state some theorems on the characteristics. The proofs can be found in the references cited below.

THEOREM 2.2. *A necessary and sufficient condition for a surface to be an integral surface of a partial differential equation is that at each point its tangent element should touch its elementary cone (tangent cone or Monge cone) of the equation.*

THEOREM 2.3. *The function $F(x, y, u, p, q)$ is constant along every characteristic strip of the equation $F(x, y, u, p, q) = 0$.*

THEOREM 2.4. *If a characteristic strip contains at least one integral element of $F(x, y, u, p, q) = 0$, it is an integral of the equation $F(x, y, u, u_x, u_y) = 0$.*

For the linear partial differential equation $a\,p + b\,q = c$, we have

THEOREM 2.5. *Every surface generated by a one-parameter family of characteristic curves is an integral surface of the partial differential equation.*

THEOREM 2.6. *Every characteristic curve which has one point in common with an integral surface lies entirely on the integral surface.*

THEOREM 2.7. *Every integral surface is generated by a one-parameter family of characteristic curves.*

THEOREM 2.8. *If*

$$D = \frac{dx}{ds}\frac{dy}{dt} - \frac{dy}{ds}\frac{dx}{dt} = a\frac{dy}{dt} - b\frac{dx}{dt} \neq 0$$

everywhere on an initial curve C, then the initial value problem has one and only one solution. If, however, $D = 0$ everywhere along C, the initial value problem cannot be solved unless C is a characteristic curve, and then the problem has an infinity of solutions.

Proofs of Theorems 2.2, 2.3 and 2.4 can be found in Sneddon (1957, pages 62–64), and of Theorems 2.5, 2.6, 2.7, and 2.8 in Courant and Hilbert (1965, pages 64–66).

Second Order Equations

2.7. Linear and Quasi–linear Equations

For a linear or quasi–linear partial differential equation of second or higher order, the characteristic equation is determined by the highest order terms in the partial differential equation. These terms are known as the *principal part* of the partial differential equation. While the solution of the characteristic equation leads to the solution of the first order partial differential equation, the solution of the characteristic equation of a second order partial differential equation leads to a coordinate transformation which when applied reduces the second order partial differential equation to a simpler form. This simpler form is called the *canonical form*. Consider a second order partial differential equation

$$a_{11}\,u_{xx} + 2a_{12}\,u_{xy} + a_{22}\,u_{yy} + F(x, y, u, u_x, u_y) = 0, \qquad (2.40)$$

where a_{11}, a_{12} and a_{22} are functions of x and y only, and F is a function of x, y, u, u_x, and u_y. The different canonical forms of Eq (2.40) are

$$\begin{aligned}
&\text{Elliptic} &&: u_{\xi\xi} + u_{\eta\eta} + G(\xi, \eta, u, u_\xi, u_\eta) = 0,\\
&\text{Hyperbolic} &&: \begin{cases} u_{\xi\xi} - u_{\eta\eta} + G(\xi, \eta, u, u_\xi, u_\eta) = 0,\\ u_{\xi\eta} + G(\xi, \eta, u, u_\xi, u_\eta), \end{cases}\\
&\text{Parabolic} &&: u_{\xi\xi} + G(\xi, \eta, u, u_\xi, u_\eta) = 0.
\end{aligned} \tag{2.41}$$

In order to reduce Eq (2.40) to a canonical form, we introduce a reversible transformation

$$\xi = \xi(x, y), \quad \text{and} \quad \eta = \eta(x, y), \tag{2.42}$$

with the condition that the Jacobian

$$J = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \xi_x \, \eta_y - \eta_y \, \xi_x \neq 0. \tag{2.43}$$

Using this transformation and noting that

$$\begin{aligned}
u_x &= u_\xi \, \xi_x + u_\eta \, \eta_x,\\
u_y &= u_\xi \, \xi_y + u_\eta \, \eta_y,\\
u_{xx} &= u_{\xi\xi} \, \xi_x^2 + 2u_{\xi\eta} \, \xi_x \, \eta_x + u_{\eta\eta} \, \eta_x^2 + u_\xi \, \xi_{xx} + u_\eta \, \eta_{xx},\\
u_{xy} &= u_{\xi\xi} \, \xi_x \, \xi_y + u_{\xi\eta} \, (\xi_x \, \eta_y + \xi_y \, \eta_x) + u_{\eta\eta} \, \eta_x \, \eta_y + u_\xi \, \xi_{xy} + u_\eta \, \eta_{xy},\\
u_{yy} &= u_{\xi\xi} \, \xi_y^2 + 2u_{\xi\eta} \, \xi_y \, \eta_y + u_{\eta\eta} \, \eta_y^2 + u_\xi \, \xi_{yy} + u_\eta \, \eta_{yy},
\end{aligned} \tag{2.44}$$

Eq (2.40) reduces to

$$A_{11} \, u_{\xi\xi} + 2A_{12} \, u_{\xi\eta} + A_{22} \, u_{\eta\eta} + G(\xi, \eta, u, u_\xi, u_\eta) = 0, \tag{2.45}$$

where G is a function of ξ, η, u, u_ξ, and u_η, and A_{11}, A_{12}, and A_{22} are functions of ξ and η, given by

$$\begin{aligned}
A_{11} &= a_{11} \, \xi_x^2 + 2a_{12} \, \xi_x \, \xi_y + a_{22} \, \xi_y^2,\\
A_{12} &= a_{11} \, \xi_x \, \eta_y + a_{12} \, (\xi_x \, \xi_y + \xi_y \, \eta_x) + a_{22} \, \xi_y \, \eta_y,\\
A_{22} &= a_{11} \, \eta_x^2 + 2a_{12} \, \eta_x \, \eta_y + a_{22} \, \eta_y^2.
\end{aligned} \tag{2.46}$$

The function G is linear or nonlinear according as F is linear or nonlinear.

If we now choose ξ and η such that both satisfy the condition

$$a_{11} \, \xi_x^2 + 2a_{12} \, \xi_x \, \xi_y + a_{22} \, \xi_y^2 = 0, \tag{2.47}$$

then $A_{11} = A_{22} = 0$. Eq (2.40) will then reduce to

$$2A_{12} \, u_{\xi\eta} + G(\xi, \eta, u, u_\xi, u_\eta) = 0.$$

It can be verified (after some tedious algebra) that

$$A_{12}^2 - A_{11}A_{22} = \left(a_{12}^2 - a_{11}a_{22}\right) \left(\xi_x \, \eta_y + \xi_y \, \eta_x\right)^2. \tag{2.48}$$

This means that the sign of the quantity $(a_{12}^2 - a_{11}a_{22})$ is invariant under the reversible transformation (2.42), and the quantity itself is invariant if $|J| = 1$. An important consequence of this result is that the partial differential equation does not change its classification under nonsingular transformations [see §1.3, where it was proved that if $\zeta = \phi(x, y)$ is a solution of Eq (2.47), then $\phi(x, y) = c$ is a solution of Eq (1.10)]. Both equations (1.10) and (2.47) are called characteristic equations of the partial differential equation (2.40). Equation (1.10) has two solutions given by

$$\frac{dy}{dx} = \frac{a_{12} \pm \sqrt{a_{12}^2 - a_{11}a_{22}}}{2a_{12}}. \tag{2.49}$$

If the partial differential equation is hyperbolic, then there are two solutions, resulting in two characteristics for the partial differential equation. If the partial differential equation is parabolic, then there is only one real solution, and hence only one characteristic. In the elliptic case there are no real solutions, and so there are no characteristics.

In order to transform the partial differential equation to its canonical form, we introduce two independent variables $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$, where ξ and η are solutions of Eq (2.49). In the case of a parabolic equation we have only one solution, so η is chosen arbitrarily except that it must satisfy the condition (2.43). In the case of an elliptic equation, the solutions are complex conjugates, and we can use the real and imaginary parts of the solutions as the new independent variables. It will be shown in Chapter 5 that canonical forms are frequently necessary to solve partial differential equations by the method of separation of variables.

We will now demonstrate the effectiveness of this technique for reducing second order partial differential equations to canonical forms by some examples.

EXAMPLE 2.21. Transform the partial differential equation

$$y^2 \, u_{xx} - 4x \, y \, u_{xy} + 4x^2 \, u_{yy} + (x^2 + y^2) \, u_x + u_y = 0$$

to its canonical form. This equation is parabolic and its characteristic equation, given by

$$y^2\,(dy)^2 + 4x\,y\,dx\,dy + 4x^2\,(dx)^2 = 0,$$

has only one solution

$$\frac{dy}{dx} = -\frac{2x}{y}, \quad \text{which yields} \quad 2x^2 + y^2 = c.$$

In this case there is only one characteristic curve, and so we make the substitution

$$\xi = 2x^2 + y^2, \quad \text{and} \quad \eta = x.$$

The substitution for η is arbitrary in this situation, the only condition being that the Jacobian should be nonsingular. Thus, we have

$$\begin{aligned}
u_x &= 4x\,u_\xi + u_\eta,\\
u_y &= 2y\,u_\xi,\\
u_{xx} &= 16x^2\,u_{\xi\xi} + 8x\,u_{\xi\eta} + u_{\eta\eta} + 4u_\xi,\\
u_{xy} &= 8x\,y\,u_{\xi\xi} + 2y\,u_{\xi\eta},\\
u_{yy} &= 4y^2\,u_{\xi\xi} + 2u_\xi.
\end{aligned}$$

Substitution of these values into the partial differential equation leads to the canonical form

$$\left(\xi - 2\eta^2\right)\,u_{\eta\eta} + \left[4\xi + 4\eta(\xi - \eta^2) + 2\sqrt{\xi - 2\eta^2}\right]\,u_\xi + \left(\xi - \eta^2\right)\,u_\eta = 0. \;\blacksquare$$

WWW The plots of the characteristic curves for this example are available in the Mathematica Notebook `Example2.21.ma`.

EXAMPLE 2.22. Transform the partial differential equation

$$y^2\,u_{xx} - 4x\,y\,u_{xy} + 3x^2\,u_{yy} - \frac{y^2}{x}\,u_x - \frac{3x^3}{y}\,u_y = 0$$

to the canonical form. The principal part in this partial differential equation is similar to the previous example, but it is hyperbolic and, therefore, its characteristic equation

$$y^2\,(dy)^2 + 4x\,y\,dx\,dy + 3x^2\,(dx)^2 = 0$$

has two independent solutions, namely,

$$\frac{dy}{dx} = -\frac{3x}{y}, \quad \text{which yields} \quad 3x^2 + y^2 = c_1,$$

and

$$\frac{dy}{dx} = -\frac{x}{y}, \quad \text{which yields} \quad x^2 + y^2 = c_2.$$

Hence, the new independent variables are

$$\xi = 3x^2 + y^2, \quad \text{and} \quad \eta = x^2 + y^2.$$

The partial derivatives of u with respect to the new variables are given by

$$\begin{aligned}
u_x &= 6x\,u_\xi + 2x\,u_\eta,\\
u_y &= 2y\,u_\xi + 2y\,u_\eta,\\
u_{xx} &= 36x^2\,u_{\xi\xi} + 24x^2\,u_{\xi\eta} + 4x^2\,u_{\eta\eta} + 6u_\xi + 2u_\eta,\\
u_{xy} &= 12x\,y\,u_{\xi\xi} + 16x\,y\,u_{\xi\eta} + 4x\,y\,u_{\eta\eta},\\
u_{yy} &= 4y^2\,(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) + 2\,(u_\xi + u_\eta)\,.
\end{aligned}$$

Substituting these values into the given partial differential equation we get the canonical form

$$(\xi - \eta)(\xi - 3\eta)\,u_{\xi\eta} = 0,$$

whose solution is

$$u = f(3x^2 + y^2) + g(x^2 + y^2). \;\blacksquare$$

WWW The plots of the characteristic curves for this example are available in the Mathematica Notebook `Example2.22.ma`.

We mention here an important property of hyperbolic partial differential equations. They are capable of transporting a discontinuity in the initial data along a characteristic. The solutions that are in C^2 are called strict solutions, whereas those with discontinuity in the function or its first two derivatives are called generalized solutions. We will demonstrate this idea by a simple example. For more details the reader is referred to the texts by John (1982) and by Courant and Hilbert (1965).

EXAMPLE 2.23. Solve the partial differential equation

$$u_{tt} - u_{xx} = 0, \quad 0 < x < \infty,$$

subject to the conditions

$$u(0,t) = H(t)e^{-t}, \quad u(x,0) = u_t(x,0) = 0,$$

where $H(t)$ is the Heaviside step function. By introducing the characteristic coordinates $\xi = x+t$, $\eta = x-t$, the partial differential equation is reduced to

$$u_{\xi\eta} = 0.$$

Its solution is given by

$$u = f(\xi) + g(\eta) = f(x+t) + g(x-t).$$

The term $f(x+t)$ represents a wave traveling with a negative velocity coming from infinity. Since there are no sources or boundaries at infinity, it is not possible for a wave to either emanate or be reflected from infinity. (This is also known as Sommerfeld's radiation condition.) Therefore, the function $f(x+t)$ must be taken to be zero. Thus, we have from the boundary condition

$$g(-t) = H(t)e^{-t},$$

which yields

$$u = H(t-x)e^{-(t-x)}.$$

The initial conditions are then automatically satisfied. In this case the discontinuity in u propagates along the characteristic $x = t$.

EXAMPLE 2.24. Transform the partial differential equation

$$y^2\, u_{xx} - 4x\, y\, u_{xy} + 3x^2\, u_{yy} = 0$$

to the canonical form. In this case the partial differential equation is of the elliptic type. The characteristic equation is

$$y^2\,(dy)^2 + 4x\,y\,dx\,dy + 8x^2\,(dx)^2 = 0,$$

and its solution is

$$\frac{dy}{dx} = (-1+2i)\,\frac{2x}{y},$$

which yields

$$y^2 + 2(1 \pm i)\,x^2 = c_{1,2}.$$

In this situation we define the new independent coordinates as

$$\xi = 2x^2 + y^2, \quad \text{and} \quad \eta = 2x^2.$$

Then, as in Examples 2.22 and 2.23, we get the canonical form

$$2\eta(\xi-\eta)\,(u_{\xi\xi} + u_{\eta\eta}) + (\xi+\eta)\,u_\xi + (\xi-\eta)\,u_\eta = 0. \blacksquare$$

WWW The plots of the characteristic curves for this example are available in the Mathematica Notebook `Example2.24.ma`.

It can be seen from these examples that canonical forms, though more of theoretical interest, also provide in some cases the general solution of the partial differential equations.

The well-known Cauchy–Kowalewsky theorem guarantees the uniqueness and existence of quasi-linear partial differential equations under certain specific conditions. A statement of this theorem for two independent variables is as follows:

THEOREM 2.9. *Consider a quasi-linear second order partial differential equation which can be solved for u_{xx}, i.e.,*

$$u_{xx} = F(x, y, u_x, u_y), \tag{2.50}$$

where F is an analytic function of x, y, u_x, and u_y in a domain $\Omega \subset R^2$. Let the Cauchy data on a curve $x = x_0$ be

$$u(x_0, y) = f(y), \quad \textit{and} \quad u_x(x_0, y) = g(y),$$

where f and g are analytic functions in a neighborhood of a point (x_0, y_0). Then the Cauchy problem has an analytic solution in some neighborhood of the point (x_0, y_0) and this solution is unique in the class of analytic functions.

Simply stated, this theorem guarantees a unique solution $u(x,y)$ in the form of a Taylor's series in a neighborhood of the point (x_0, y_0). The above statement is true if the second order partial differential equation can be solved for u_{yy} or u_{xy}. A similar statement holds for first order partial differential equations. There is, however, an exception. If the Cauchy data is prescribed on a characteristic, a unique solution may not exist. For example, consider $u_x = 0$. Its solution is $u = f(y)$. If the Cauchy data is $u(x,0) = \phi(x)$, where $\phi(x)$ is not constant, then no solution can be found.

For a general statement of this theorem for higher order partial differential equations, see Courant and Hilbert (1965) and Petrovskii (1967).

2.8. Exercises

2.1. Solve the equation $p + q + 3u = e^{-3x}\sin(x+2y)$, with the initial condition $u(x,0) = 0$.
ANS. $u(x,y) = (1/3)e^{-3x}[\cos(x-y) - \cos(x+2y)]$.

2.2. Solve the equation $2p + q + 2(2x-y)u = 6x^2 e^{y^2-x^2}$.
ANS. $2y - x = c_1$ and $ue^{x^2-y^2} - x^3 = c_2$;
general solution: $F(2y - x, ue^{x^2-y^2} - x^3) = 0$.

2.3. Solve the equation $p + q\sqrt{1-y^2} = 0$, with initial conditions (a) $u(0,y) = y$, (b)$u(x,0) = x^2$.
ANS. (a) $u(x,y) = y\cos x - \sqrt{1-y^2}\sin x$; (b) $u = (\sin^{-1} y - x)^2$.

2.4. Solve the equation $p + q = \frac{1}{3}u^{-2}$, with the initial condition $u(0,y) = \sin y$.
ANS. $u^3(x,y) = x + \sin^3(y-x)$.

2.5. Find the general solution of the equation $2p + uq = \frac{u^2}{y}$.

ANS. $\frac{u}{y} = c_1$ and $\frac{2y\ln y - xu}{u} = c_2$; and the general solution is $F\left(\frac{u}{y}, \frac{2y\ln y - xu}{u}\right) = 0$.

2.6. Find two functionally independent solutions of $(y-u)p + (u-x)q = x - y$.
ANS. $x + y + u = c_1$ and $x^2 + y^2 + u^2 = c_2$.

2.7. Find a complete solution of $u = \frac{1}{p} + \frac{1}{q}$.
ANS. $u = \pm\sqrt{2x+a} \pm \sqrt{2y+b}$, or $u^2 + b = 2a\left(x + \frac{y}{a-1}\right)$, where a and b are arbitrary constants. Discuss the relationship between the two solutions.
SOLUTION. We have

$$F_x = F_y = 0, F_u = -1, F_p = -1/p^2, F_q = -1/q^2.$$

The auxiliary equations are

$$\frac{dx}{dt} = -1/p^2, \quad \frac{dy}{dt} = -1/q^2, \quad \frac{du}{dt} = -\frac{1}{p} - \frac{1}{q} = -u,$$
$$\frac{dp}{dt} = p, \frac{dq}{dt} = q.$$

Noting that $\frac{dp}{p\,dt} = 1$, and $\frac{dq}{q\,dt} = 1$, we can express the first two equations as $dx = -\frac{dp}{p^3}$, $dy = -\frac{dq}{q^3}$. Solving these we get $p^2 = \frac{1}{2x+a}$, and $q^2 = \frac{1}{2y+b}$. Taking the positive square roots, we get $u = \sqrt{2x+a} + \sqrt{2y+b}$ as a solution.

An alternate approach is to note that solutions of the characteristic equations can be expressed as $u = Ae^{-t}, p = Be^t, q = Ce^t$. Hence, $up = AB = A_1, uq = AC = A_2$. Using these values in the given equation, we get

$$u = \frac{1}{p} + \frac{1}{q} = \frac{1}{A_1} + \frac{1}{A_2} = 1.$$

Integrating $up = AB = A_1, uq = AC = A_2$, we get $u^2 = 2A_1x + g_1(y)$, and $u^2 = 2A_2y + g_2(x)$. Comparing these two values of u^2, we find that

$$u^2 = 2(A_1x + A_2y + A_3) = 2\left(A_1x + \frac{A_1 - 1}{A_1}y + A_3\right).$$

Now we will discuss the relationship between the two solutions

$$u = \sqrt{2x+a} + \sqrt{2y+b}, \quad \text{and} \quad u^2 = \left(Ax + \frac{A-1}{A}y + B\right),$$

where we have suppressed the subscripts and replaced A_3 by B. Define

$$b = -a(A-1) + B(A-1)/A.$$

Then

$$\phi(x,y,u) = -u + \sqrt{2x+a} + \sqrt{2y - a(A-1) + B(A-1)/A} = 0.$$

Then

$$\phi_a = \frac{1}{2\sqrt{2x+a}} - \frac{A-1}{2\sqrt{2y-a(A-1)+B(A-1)/A}} = 0.$$

Thus

$$(A-1)\sqrt{2x+a} = \sqrt{2y - a(A-1) + B(A-1)/A},$$

which yields

$$u = \sqrt{2x+a} + (A-1)\sqrt{2x+a}.$$

Hence

$$u = A\sqrt{2x+a} \quad \text{or} \quad u^2 = 2A^2x + A^2a.$$

Now solving $(A-1)\sqrt{2x+a} = \sqrt{2y-a(A-1)+B(A-1)/A}$ for a and using the value so obtained in the expression for u^2, we get

$$u^2 = 2\left(Ax + \frac{A-1}{A}y + B\right).$$

So clearly the second solution is the envelope of the first.

2.8. Find a solution of $(1+q^2)u - px = 0$, which passes through the curve $2u = x^2, y = 0$.
ANS. $u^2 = x^2(y + x^2/4)$, or $4u^2 = x^2(4y^2 + x^2)$.

2.9. Find a solution of $F = p^2 + q^2 - 4u = 0$, which passes through $y = 0, u = x^2 + 1$.
ANS. $u = x^2 + (y+1)^2$.

2.10. Find a solution of $F = u - p^2 + q^2 = 0$, which passes through $y = 0$, $4u + x^2 = 0$.
ANS. $4u = -(x \pm \sqrt{2}y)^2$.

2.11. Find a general solution of the equation $xp - yq = x$.
ANS. $u = x + f(xy)$, or $xy = g(u - x)$.

2.12. Find a general solution of the equation $x^2p + q = xu$.
ANS. $u = xf\left(\frac{1+xy}{x}\right)$.

2.13. Find a complete solution of the equation $F = pqy - pu/2 - cq = 0$.
ANS. $(a^2y^2 - 2cu)^{3/2} - a^3y^3 = 3ac(2cx - uy) - b$.
Investigate if a solution exists at $x = 0$, for which the solution is a function of y which vanishes along with q for $y = 0$.
ANS. Yes, and $b = 0$ gives the required solution.

2.14. Find a general solution of the equation $F = pu - aq - x = 0$.
ANS. $f[(u+x)e^{y/a}, (u-x)e^{-y/a}] = 0$. Discuss the solution in the neighborhood of $x = 0$.

2.15. Find the solution of $(y+u)p + yq = x - y$, which passes through $x = 0, u = y$.
ANS. $u + x = y$.

2.16. Find a complete solution of the equation
$F = 2px^2y + 2qxy^2 + pq - 4uxy = 0$.
ANS. $u = ax^2 + by^2 + ab$.

2.17. Find a complete solution of the equation
$F = 1 + upx + uqy - u^2 = 0$.
ANS. $u^2 = 1 + ax^2 + by^2$.

2.18. Find the solution of the equation $F = px + qy - 2u = 0$,
(a) with the initial condition $u = ay^2 - b$ at $x = 1$;
(b) $u = a(1+y^2) + 2by$, at $x = 1$.
ANS. (a) $u = ay^2 - bx^2$; (b) $u = a(x^2+y^2) + 2bxy$.

2.19. Find a general solution of the equation $F = pxy + qy^2 - 2uy - 4q = 0$. Also find the particular solution subject to the initial condition $x = 1, au = 4 + b - y^2$.
ANS. $f\left(\frac{y^2-4}{x^2}, \frac{u}{x^2}\right) = 0$, $au - bx^2 + y^2 = 4$.
Note that both general and complete solutions are not unique.

2.20. Find a general solution of the equation $F = 2px + qy - u - x = 0$.
ANS. $y = (u-x)f\left(\frac{y^2}{x}\right)$.

2.21. Find a complete solution of the equation $F = u^2(p^2 + q^2 + 1) - a^2 = 0$.

ANS. $u^2 + \dfrac{(\alpha x + y - \beta)^2}{1+\alpha^2} = a^2.$

2.22. Find the general solution of the equation $3\,u_x + 4\,u_y - 5\,u = 10$, subject to the initial condition $u(x, 0) = x$.

SOLUTION. We first rotate the x, y axes through an angle θ; thus, the new axes ξ, η are related to the old axes x, y by the relations

$$\begin{aligned} \xi &= x\cos\theta + y\sin\theta, \qquad & x &= \xi\cos\theta - \eta\sin\theta,\\ \eta &= -x\sin\theta + y\cos\theta, \qquad & y &= \xi\sin\theta + \eta\cos\theta. \end{aligned}$$

Hence, we have

$$u(x, y) = u(\xi\cos\theta - \eta\sin\theta, \xi\sin\theta + \eta\cos\theta) = w(\xi, \eta),$$

$$\begin{aligned} \frac{\partial}{\partial x} &\equiv \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \cos\theta\,\frac{\partial}{\partial \xi} - \sin\theta\,\frac{\partial}{\partial \eta},\\ \frac{\partial}{\partial y} &\equiv \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} = \sin\theta\,\frac{\partial}{\partial \xi} + \cos\theta\,\frac{\partial}{\partial \eta}. \end{aligned}$$

The given partial differential equation then becomes

$$(3\,\cos\theta + 4\,\sin\theta)\,\frac{\partial w}{\partial \xi} + (4\,\cos\theta - 3\,\sin\theta)\frac{\partial w}{\partial \eta} - 5\,w = 10.$$

The coefficient of $\dfrac{\partial w}{\partial \eta}$ in the above equation vanishes if $\tan\theta = 4/3$, i.e., $\sin\theta = 4/5$, and $\cos\theta = 3/5$. With these values, the above equation reduces to

$$\frac{\partial w}{\partial \xi} - w = 2,$$

which has the general solution

$$w(\xi, \eta) = -2 + g(\eta)\,e^{\xi},$$

or

$$u(x, y) = -2 + g\,(3y/5 - 4x/5)\;e^{3x/5+\frac{4}{y}/5}.$$

Note that this is the general formal solution of the given partial differential equation. Now, to find the particular solution subject to the initial condition $u(x, 0) = x$, we have $\eta = -\dfrac{4x}{5}$ when $y = 0$, and then the above equation gives

$$x = -2 + g(\eta)\,e^{3x/5},$$

thus,

$$g(\eta) = (x+2)\,e^{-3x/5} = (2 - 4\eta/5)\;e^{3\eta/4}.$$

Substituting this value of $g(\eta)$ into the above general solution, we obtain

$$\begin{aligned} u(x, y) &= -2 + \left(2 - \frac{4}{5}\eta\right)\;e^{3\eta/4}\,e^{3x/5+4y/5}\\ &= -2 + \left[2 - \frac{5}{4}\left(\frac{3}{5}y - \frac{4}{5}x\right)\right]\,e^{3/4(3y/5-4x/5)+3x/5+4y/5}\\ &= -2 + \left[2 + \frac{1}{4}(4x - 3y)\right]\,e^{-5y/4}, \end{aligned}$$

which is the unique solution of the given partial differential equation subject to the given initial condition $u(x, 0) = x$.

WWW See the Mathematica Notebook `Exercise2.22.ma`.

In problems (2.23)–(2.30) find the characteristics where possible and reduce the partial differential equation to its canonical form.

2.23 $(a^2 + x^2)u_{xx} + (a^2 + y^2)u_{yy} + xu_x + yu_y = 0$. ANS. $u_{\xi\xi} + u_{\eta\eta} = 0$.

HINT: Solve the characteristic equation to determine that the substitution is

$$\xi = log(x + \sqrt{a^2 + x^2}), \quad \text{and} \quad \eta = log(y + \sqrt{a^2 + y^2}).$$

2.24 The Tricomi equation $u_{yy} - yu_{xx} = 0$ for $y > 0$.

ANS. $u_{\xi\eta} - \dfrac{u_\xi - u_\eta}{6(\xi - \eta)}.$

2.25 $(1 + \sin x)u_{xx} - 2\cos x u_{xy} + (1 - \sin x)u_{yy} + \dfrac{(1+\sin x)^2}{2\cos x}u_x u_x + \dfrac{1}{2}(1 - \sin x)u_y = 0.$

ANS. $4u_{\eta\eta} + u_\eta = 0, \xi, \eta = y \pm \log(1 + \sin x)$. The reduced form can now be solved to yield

$$u = f(\xi) + e^{\eta/4}g(\xi).$$

2.26 $e^{2x}u_{xx} - 2e^{x+y}u_{xy} + e^{2y}u_{yy} + e^{2x}u_x + e^{2y}u_y = 0.$
ANS. $u_{\eta\eta} = 0$, $\xi = e^{-x} + e^{-y}$, $\eta = e^{-x} - e^{-y}$.

2.27 $e^{2x}u_{xx} - 5e^{x+y}u_{xy} + 4e^{2y}u_{yy} + e^{2x}u_x + 4e^{2y}u_y = 0.$
ANS. $u_{\xi\eta} = 0$, $\xi = e^{-x} + e^{-y}$, $\eta = 4e^{-x} + e^{-y}$.

2.28 $9u_{xx} - 12u_{xy} + 4u_{yy} + 12u_x - 8u_y + 4u = 0.$
ANS. $36u_{\eta\eta} + 12u_\eta + u = 0, u = [f(\xi) + \eta g(\xi)]e^{\eta/6}$, $\xi, \eta = 2x \pm 3y$.

2.29 $3u_{xx} - 7u_{xy} + 4u_{yy} + 5u_x - u_y + 3u = 0.$
ANS. $5u_{\xi\eta} - u_\xi = 0$, $f(\xi) + g(\eta)e^{\eta/5}$, $\xi = 2x + y, \eta = x + 3y$.

2.30 $2u_{xx} + 6u_{xy} + 9u_{yy} + 2u_x + 3u_y - 2u = 0.$
ANS. $9(u_{\xi\xi} - u_{\eta\eta}) + 6u_\eta - 4u = 0$, $\xi = y - \frac{3}{2}x$, $\eta = \frac{3}{2}x$.

3

Linear Equations with Constant Coefficients

We will use the inverse operator method to solve homogeneous and nonhomogeneous partial differential equations with constant coefficients. This method, although basically developed and frequently used for solving ordinary differential equations, becomes useful for finding general solutions of partial differential equations with constant coefficients. The problem of finding the general solutions of second order partial differential equations with constant coefficients and determining their particular solutions under auxiliary (initial) conditions is also discussed in a later section. Before we discuss the partial differential equations with constant coefficients we will first review in §3.1 the technique of inverse operators from the theory of ordinary differential equations. This review should prove useful in discussing the homogeneous and nonhomogeneous partial differential equations with constant coefficients.

3.1. Inverse Operators

If D represents $\frac{d}{dx}$, then $\frac{1}{D}$ is defined as the inverse operator of D, i.e.,

$$\frac{1}{D}\phi(x) = \int \phi(x)\, dx.$$

If $f(D)$ represents a polynomial in D with constant coefficients, then $f(D)$

is a linear differential operator, and we define its inverse as $1/f(D)$. Thus,

$$f(D)\left[\frac{1}{f(D)}\phi(x)\right] = \phi(x).$$

Note that $\frac{1}{f(D)}[f(D)\phi(x)]$ is not necessarily equal to $\phi(x)$. However, if $\left[\frac{1}{f(D)}\phi(x)\right] = \psi(x)$, then $\psi(x)$ contains arbitrary constants, and $\psi(x) = \phi(x)$ for some value of these arbitrary constants. In the sequel we will ignore arbitrary constants. We will list some formulas for the operator pair $f(D)$ and $\frac{1}{f(D)}$:

1. $f(D)\left[\frac{1}{f(D)}\phi(x)\right] = \phi(x)$.

2. $\frac{1}{f_1(D)f_2(D)}\phi(x) = \frac{1}{f_1(D)}\left(\frac{1}{f_2(D)}\phi(x)\right) = \frac{1}{f_2(D)}\left(\frac{1}{f_1(D)}\phi(x)\right)$.

3. $\frac{1}{f(D)}[c_1\phi_1(x) + c_2\phi_2(x)] = c_1\frac{1}{f(D)}\phi_1(x) + c_2\frac{1}{f(D)}\phi_2(x)$.

4. $\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$, provided that $f(a) \neq 0$.

5. $f(D)\phi(x)e^{ax} = e^{ax}f(D+a)\phi(x)$.

6. $\frac{1}{f(D)}\phi(x)e^{ax} = e^{ax}\frac{1}{f(D+a)}\phi(x)$.

7. $\frac{1}{(D-a)^m}e^{ax} = \frac{x^m e^{ax}}{m!}$.

8. $\frac{1}{(D-a)^m f(D)}e^{ax} = \frac{x^m}{m!f(a)}e^{ax}, \quad a \neq 0$.

9. $\frac{1}{D^2+a^2}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} bx = \frac{\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} bx}{a^2-b^2}, \ |a| \neq |b|$.

10. $\frac{1}{f(D^2)}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} ax = \frac{\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} ax}{f(-a^2)}$, provided that $f(-a^2) \neq 0$.

11. $\frac{1}{(D^2+a^2)}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} ax = \pm\frac{\begin{Bmatrix}x\cos\\ x\sin\end{Bmatrix} ax}{2a}$.

12. $\frac{1}{aD^2+bD+c}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} \omega x = \frac{(c-a\omega^2)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\omega x \pm b\omega\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}\omega x}{(c-a\omega^2)^2+b^2\omega^2}$.

13. $\frac{1}{f(D)}x^n = \frac{1}{a_0[1+g(D)]}x^n$
$= \frac{1}{a_0}\left[1 - g(D) + g^2(D) - g^3(D) + \cdots + g^n(D) + \cdots\right]x^n,$
where the terms of degree $n+1$ or higher are ignored.

PROOF OF FORMULA 4. If $\phi(x) = e^{ax}$, we know that

$$De^{ax} = ae^{ax}, \quad D^2e^{ax} = a^2e^{ax}, \quad \ldots \quad D^ne^{ax} = a^ne^{ax}.$$

Thus

$$f(D)e^{ax} = f(a)e^{ax}.$$

If we take $\frac{1}{f(D)}$ as the inverse operator of $f(D)$, then obviously

$$\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax} \quad \text{provided } f(a) \neq 0.$$

Therefore, the particular integral y_p of the equation $f(D)y = Ae^{ax}$ is given by

$$y = \frac{A}{f(a)}e^{ax}, \quad f(a) \neq 0.$$

EXAMPLE 3.1. Consider $(D^2 + D + 1)y = e^{2x}$. Then

$$y_p = \frac{1}{D^2+D+1}e^{2x} = \frac{1}{2^2+2+1}e^{2x} = \frac{1}{7}e^{2x}. \blacksquare$$

EXAMPLE 3.2. Consider $(D^4+8)\,y = e^x$. Then

$$y_p = \frac{1}{D^4+8}e^x = \frac{1}{1^4+8}e^x = \frac{1}{9}e^x. \blacksquare$$

AN APPLICATION OF FORMULA 13. If $\phi(x) = x^n$, then

$$\begin{aligned} f(D) &= a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1}D + a_n \\ &= a_n\left[1 + \frac{a_{n-1}}{a_n}D + \frac{a_{n-2}}{a_n}D^2 + \cdots + \frac{a_0}{a_n}D^n\right], \end{aligned}$$

which gives

$$\frac{1}{f(D)} = \frac{1}{a_n(1+g(D))},$$

where

$$g(D) = \frac{1}{a_n}\left(a_{n-1}D + a_{n-2}D^2 + \cdots + a_0 D^n\right),$$

and

$$\frac{1}{f(D)} = \frac{1}{a_n}\left(1+g(D)\right)^{-1}.$$

Now, in order to find the particular integral y_p of $f(D)y = Ax^n$, we apply the inverse $\dfrac{1}{f(D)}$ of $f(D)$ to the ordinary differential equation and get

$$\begin{aligned} y &= \frac{1}{f(D)}(Ax^n) \\ &= A\cdot\frac{1}{a_n}\left(1+g(D)\right)^{-1}x^n \\ &= \frac{A}{a_n}\left[1 - g(D) + (g(D))^2 - (g(D))^3 + \cdots\right]x^n, \end{aligned}$$

where terms of degree $n+1$ and higher in D are ignored in the above expansion on the right side.

EXAMPLE 3.3. Consider $(D^2+D+2) = x^4$. Then

$$\begin{aligned} y_p &= \frac{1}{D^2+D+2}x^4 = \frac{1}{2\left(1+\frac{1}{2}D+\frac{1}{2}D^2\right)}x^4 \\ &= \frac{1}{2}\left[1+\left(\frac{1}{2}D+\frac{1}{2}D^2\right)\right]^{-1}x^4 \\ &= \frac{1}{2}\Big[1-\left(\frac{1}{2}D+\frac{1}{2}D^2\right)+\left(\frac{1}{2}D+\frac{1}{2}D^2\right)^2-\left(\frac{1}{2}D+\frac{1}{2}D^2\right)^3 \\ &\quad + \left(\frac{1}{2}D+\frac{1}{2}D^2\right)^4 - \cdots\Big]x^4 \\ &= \frac{1}{2}\Big[1-\frac{1}{2}D-\frac{1}{2}D^2+\frac{1}{4}D^2+\frac{1}{2}D^3+\frac{1}{4}D^4-\frac{1}{8}D^3-\frac{3}{8}D^4 \\ &\quad + \frac{1}{16}D^4 + O(D^5)\Big]x^4, \end{aligned}$$

where $O(D^5)$ means terms containing D^5 and higher powers in D. Thus,

$$\begin{aligned} y_p &= \frac{1}{2}\left[x^4 - 2x^3 - 6x^2 + 3x^2 + 12x + 6 - 3x - 9 + \frac{3}{2}\right] \\ &= \frac{1}{2}\left[x^4 - 2x^3 - 3x^2 + 9x - \frac{3}{2}\right]. \blacksquare \end{aligned}$$

EXAMPLE 3.4. Consider $(D^4+2D^3+D^2)\,y = x^3$. Then

$$\begin{aligned} y_p &= \frac{x^3}{D^2(D^2+2D+1)} = \frac{x^3}{D^2(1+D)^2} \\ &= \frac{1}{D^2}(1+D)^{-2}x^3 \\ &= \frac{1}{D^2}\left[1-2D+3D^2-4D^3+O\left(D^4\right)\right]x^3 \\ &= \frac{1}{D^2}\left[x^3-6x^2+18x-24\right] \\ &= \frac{1}{20}x^5 - \frac{1}{2}x^4 + 3x^3 - 12x^2. \blacksquare \end{aligned}$$

AN APPLICATION OF FORMULA 10. If $\phi(x) = \sin ax$ or $\cos ax$, then we know that

$$D^2\begin{cases}\cos ax\\ \sin ax\end{cases} = -a^2\begin{cases}\cos ax\\ \sin ax,\end{cases}$$

and

$$D^{2n}\begin{cases}\cos ax\\ \sin ax\end{cases} = (-1)^n a^{2n}\begin{cases}\cos ax\\ \sin ax.\end{cases}$$

Thus,

$$f(D^2)\begin{cases}\cos ax\\ \sin ax\end{cases} = f\left(-a^2\right)\begin{cases}\cos ax\\ \sin ax.\end{cases}$$

Even if the operator $f(D)$ is not an even function, we can still use the above formula; e.g.,

$$\begin{aligned}\left(D^3+2D^2+3D+1\right)\cos ax &= \left[\left(-a^2\right)D + 2\left(-a^2\right) + 3D + 1\right]\cos ax\\ &= \left[\left(3-a^2\right)D + 1 - 2a^2\right]\cos ax\\ &= -\left(3-a^2\right)a\sin ax + \left(1-2a^2\right)\cos ax.\end{aligned}$$

We, therefore, notice that when an operator of the type $f(D)$ is applied to $\cos ax$ or $\sin ax$, we can set $D^2 = -a^2$, and reduce $f(D)$ to a linear operator of first order. We shall use this observation to find particular integrals in this case.

If $f(D)$ happens to be of the form $f\left(D^2\right)$, i.e., if the ordinary differential equation is

$$f\left(D^2\right)y = \begin{cases}\cos ax\\ \sin ax,\end{cases}$$

then

$$y = \frac{A}{f\left(D^2\right)}\begin{cases}\cos ax\\ \sin ax,\end{cases} = \frac{A}{f\left(-a^2\right)}\begin{cases}\cos ax\\ \sin ax,\end{cases}$$

provided that $f\left(-a^2\right) \neq 0$.

If $f(D)$ is in general a polynomial containing both even and odd degree terms, then we let $D^2 = -a^2$, and reduce $f(D)$ to a linear operator in D of the form $\alpha D + \beta$, so that if

$$f(D)y = A\begin{cases}\cos ax\\ \sin ax,\end{cases}$$

then

$$y = \frac{A}{f(D)}\begin{cases}\cos ax\\ \sin ax.\end{cases}$$

Then we let $D^2 = -a^2$, which reduces $f(D)$ to $\alpha D + \beta$. Thus,

$$\begin{aligned}y &= \frac{A}{\alpha D + \beta}\begin{cases}\cos ax\\ \sin ax,\end{cases}\\ &= \frac{A(\alpha D - \beta)}{(\alpha D + \beta)(\alpha D - \beta)}\begin{cases}\cos ax\\ \sin ax,\end{cases}\\ &= \frac{A(\alpha D - \beta)}{\alpha^2 D^2 - \beta^2}\begin{cases}\cos ax\\ \sin ax,\end{cases}\\ &= \frac{A}{-\alpha^2 a^2 - \beta^2}\begin{cases}-\alpha a\sin ax - \beta\cos ax\\ \alpha a\cos ax - \beta\sin ax.\end{cases}\end{aligned}$$

EXAMPLE 3.5. Let $\left(D^2+1\right)y = 2\sin x$. Then

$$y_p = \frac{2}{D^2+1}\sin x.$$

Now, if we let $D^2 = -1$, then $D^2 + 1 = 0$, and the above method is not applicable. ■

EXAMPLE 3.6. Let $\left(D^4+D^2+1\right)y = 2\sin x$. Then

$$y_p = \frac{2}{D^4+D^2+1}\sin x.$$

Now, if we let $D^2 = -1$, then $D^4+D^2+1 = 1$, and we get $y_p = 2\sin x$. ■

EXAMPLE 3.7. Let $\left(D^3+2D^2+D+1\right)y = 3\cos 2x$. Then

$$\begin{aligned}y_p &= \frac{3}{-2^2D + 2\left(-2^2\right) + D + 1}\cos 2x\\ &= \frac{3}{-3D-7}\cos 2x = \frac{-3(3D-7)}{(3D+7)(3D-7)}\cos 2x\\ &= \frac{-3}{9D^2-49}\cos 2x = \frac{-3}{9\left(-2^2\right)-49}(3D-7)\cos 2x\\ &= -\frac{3}{85}[6\sin 2x + 7\cos 2x]. \blacksquare\end{aligned}$$

AN APPLICATION OF FORMULA 6. If $\phi(x) = V(x)e^{ax}$, where $V(x)$ is a function of x of the type x^n, $\cos bx$, $\sin bx$ or $x^n\cos bx$, $x^n\sin bx$, we will first give a heuristic justification for formula 6. Note that

$$D\left(Ve^{ax}\right) = e^{ax}DV + aVe^{ax} = e^{ax}(D+a)V,$$

which yields

$$D^2\left(Ve^{ax}\right) = D\left[e^{ax}\cdot(D+a)V\right] = e^{ax}(D+a)V.$$

Similarly,

$$D^n\left(Ve^{ax}\right) = e^{ax}(D+a)^n V,$$

which, in general, gives

$$f(D)\left(Ve^{ax}\right) = e^{ax}f(D+a)V.$$

We can, therefore, assume that

$$\frac{1}{f(D)}\left(Ve^{ax}\right) = e^{ax}\frac{1}{f(D+a)}V.$$

We shall use this formula to obtain the particular integral in this case, i.e., since for the ordinary differential equation $f(D)y = \phi(x) = Ve^{ax}$,

$$y_p = \frac{1}{f(D)}Ve^{ax} = e^{ax}\frac{1}{f(D+a)}V,$$

our problem reduces to finding $\frac{1}{f(D+a)}V$. If V is of the form x^n, $\cos ax$ or $\sin bx$, then we can use formula 13 or 10 to obtain the solution; if V is of the form $x^n\cos bx$ or $x^n\sin bx$, then we use de Moivre's theorem and write

$$e^{ibx} = \cos bx + i\sin bx,$$

or

$$\cos bx = \Re e^{ibx}, \quad \sin bx = \Im e^{ibx},$$

so that if $V = x^n\cos bx$, we write $V = \Re x^n e^{ibx}$, and if $V = x^n\sin bx$, we write $V = \Im x^n e^{ibx}$. Thus,

$$\frac{1}{f(D+a)}V = \begin{cases} \Re\frac{1}{f(D+a)}x^n e^{ibx} \\ \text{or} \\ \Im\frac{1}{f(D+a)}x^n e^{ibx}. \end{cases}$$

Now $f(D+a)$ is another polynomial in D, and we can write $f(D+a) = f_1(D)$. We consider

$$\frac{1}{f(D+a)}x^n e^{ibx} = \frac{1}{f_1(D)}x^n e^{ibx},$$

and using the formula (6) we get

$$\frac{1}{f_1(D)}x^n e^{ibx} = e^{ibx}\frac{1}{f_1(D+ib)}x^n,$$

where $\frac{1}{f_1(D+ib)}x^n$ can be evaluated by the use of formula 13. This discussion also covers the case when $f(x) = x^n\cos ax$ or $x^n\sin ax$.

EXAMPLE 3.8. Consider $\left(D^4 + D^3 - 3D^2 - D + 2\right)y = 4e^x$. Then

$$\begin{aligned} y_p &= \frac{4e^x}{D^4+D^3-3D^2-D+2} = 4\frac{1}{D^4+D^3-3D^2-D+2}e^x \\ &= 4\frac{1}{(D-1)^2}\frac{1}{D^2+3D+2}e^x = 4\frac{1}{(D-1)^2}\frac{1}{1+3+2}e^x \\ &= \frac{4}{6}\frac{1}{(D-1)^2}e^x = \frac{2}{3}e^x\frac{1}{(D+1-1)^2}\cdot 1 \\ &= \frac{2}{3}e^x\frac{1}{D^2}\cdot 1 = \frac{2}{3}e^x\frac{x^2}{2} = \frac{1}{3}x^2e^x. \blacksquare \end{aligned}$$

EXAMPLE 3.9. Consider $\left(D^3 - D^2 + 4D - 4\right)y = 2\sin 2x$. Then

$$y_p = \frac{1}{D^3-D^2+4D-4}2\sin 2x = 2\frac{1}{D^3-D^2+4D-4}\sin 2x.$$

If we put $D^2 = -4$, $f(D) = 0$, and, therefore, we write

$$\begin{aligned} y_p &= 2\frac{1}{D^3-D^2+4D-4}\sin 2x = 2\Im\frac{1}{D^3-D^2+4D-4}e^{2ix} \\ &= 2\Im\frac{1}{(D-1)\left(D^2+4\right)}e^{2ix} = 2\Im\frac{1}{(D-2i)(D-1)(D+2i)}e^{2ix} \\ &= 2\Im\frac{1}{D-2i}\left(\frac{1}{(D-1)(D+2i)}e^{2ix}\right) = 2\Im\frac{1}{(D-2i)}\left(\frac{1}{(2i-1)(4i)}e^{2ix}\right) \\ &= \frac{1}{2}\Im\frac{1}{(D-2i)}\frac{1}{(-2-i)}e^{2ix} = -\frac{1}{2}\Im\frac{1}{(2+i)}\frac{1}{(D-2i)}\left(e^{2ix}\cdot 1\right) \\ &= -\frac{1}{2}\Im\frac{2-i}{(2^2+1)}e^{2ix}\frac{1}{D}(1) \quad \text{(using the formula 6)} \\ &= -\frac{1}{2}\Im\frac{2-i}{5}e^{2ix}\frac{1}{D}(1) = -\frac{1}{10}\Im(2-i)e^{2ix}\cdot x \\ &= -\frac{1}{10}\Im x(2-i)(\cos 2x - i\sin 2x) = \frac{x}{10}(\cos 2x - 2\sin 2x). \blacksquare \end{aligned}$$

EXAMPLE 3.10. When $\phi(x) = e^{ax}$ and $f(a) = 0$, the ordinary differential equation is of the form

$$f(D)y = Ae^{ax}.$$

Since $f(a) = 0$, $(D-a)$ must be a factor of $f(D)$. Then $f(D) = (D-a)^n f_1(D)$, where $f_1(a) \neq 0$, and

$$\begin{aligned} y_p &= A\frac{1}{(D-a)^n f_1(D)}e^{ax} = A\frac{1}{(D-a)^n}\left(\frac{1}{f(D)}e^{ax}\right) \\ &= A\left(\frac{1}{(D-a)^n}\right)\frac{1}{f_1(D)}e^{ax} \\ &= \frac{A}{f_1(a)}\frac{1}{(D-a)^n}\left(e^{ax}\cdot 1\right), \end{aligned}$$

and using formula 6 we can write

$$\begin{aligned} \frac{A}{f_1(a)}\frac{1}{(D-a)^n}\left(e^{ax}\cdot 1\right) &= \frac{A}{f_1(a)}e^{ax}\frac{1}{D^n}(1) \\ &= \frac{A}{f_1(a)}e^{ax}\left(\frac{x^n}{n!}\right). \end{aligned}$$

If $f(D)y = \sin ax$ or $\cos ax$ and if $f(D)$ becomes zero by letting $D^2 = -a^2$, then it is convenient to consider $\cos ax$ and $\sin ax$ as real and imaginary parts of e^{iax} and deal with the problem as for e^{ax}. ∎

WWW The Mathematica package `InverseOperator.m` can be used to solve examples of the above type, and those in the sequel.

3.2. Homogeneous Equations

Let L be a linear partial differential operator with constant coefficients in two variables x and y. Then $Lu = f(x,y)$ is a partial differential equation with constant coefficients. If we define $D_x = \dfrac{\partial}{\partial x}$ and $D_y = \dfrac{\partial}{\partial y}$, so that, e.g., $D_x^i = \dfrac{\partial^i}{\partial x^i}$, then

$$L = \sum_{i,j=1}^{m,n} A_{ij}\, D_x^i\, D_y^j. \tag{3.1}$$

We shall first discuss the homogeneous case $Lu = 0$, and limit ourselves to the case where L can be expressed as a product of linear factors in D.

THEOREM 3.1. *If $u_1, u_2, \dots, u_n$ are solutions of $Lu = 0$, where L is a linear partial differential operator, then $\sum_{i=1}^{n} c_i u_i$ is also a solution of $Lu = 0$.*

PROOF. Since $u_1, u_2, \dots, u_n$ are solutions of $Lu = 0$, we have $Lu_1 = Lu_2 = \cdots = Lu_n = 0$. But then $L\left(\sum_1^n c_i u_i\right) = \sum_1^n c_i L u_i = 0$. ∎

THEOREM 3.2. *If the operator L of order n can be factored into n linearly independent factors of the type $a_i D_x + b_i D_y + c_i$, then the general solution of $Lu = 0$ is given by*

$$u = \sum_{i=1}^{n} f_i(a_i y - b_i x)\, e^{-c_i x/a_i}. \tag{3.2}$$

PROOF. It was established in §2.1 that the solution of $(a_i D_x + b_i D_y + c_i)u_i = 0$ is $u_i = \phi(b_i x - a_i y)\, e^{-c_i y/b_i}$. Then obviously u_i are n linearly independent solutions of $Lu = 0$ for $i = 1, 2, \dots, n$. Hence, by Theorem 3.1, $\sum_1^n u_i$ is a solution of $Lu = 0$.

THEOREM 3.3. *If $aD_x + bD_y + c$ is a factor of multiplicity k, then the corresponding solution is*

$$\sum_{i=0}^{k-1} x^i\, f_i(ay - bx)\, e^{-cx/a}. \tag{3.3}$$

A proof can be established by substitution.

EXAMPLE 3.11. Solve $(4D_x^2 - 16D_xD_y + 15D_y^2)u = 0$. This equation can be written as $(2D_x - 3D_y)(2D_x - 5D_y)u = 0$, so the solution is $u = f_1(3x+2y) + f_2(5x+2y)$. ∎

EXAMPLE 3.12. Solve $(2D_x^2 - D_xD_y - 6D_y^2 + 4D_x - 8D_y)u = 0$. This equation can be written as $(D_x - 2D_y)(2D_x + 3D_y + 4)u = 0$, so the solution is $u = f_1(2x+y) + e^{-2x} f_2(3x - 2y)$. ∎

EXAMPLE 3.13. Solve $(3D_x+7D_y)(2D_x-5D_y+3)^3u=0$. In this example the equation is already in the factored form and its solution is

$$u=f(7x-3y)+e^{-3x/2}[f_1(5x+2y)+xf_2(5x+2y)+x^2f_3(5x+2y)].\ \blacksquare$$

3.3. Nonhomogeneous Equations

If the partial differential equation is $Lu=f(x)$, then one finds the general solution u_c (complementary function) to the homogeneous equation $Lu=0$ and looks for any function $g(x)$ that satisfies $Lu=f(x)$, where $g(x)$ is also known as the particular solution and sometimes denoted by u_p. Thus, the solution of the partial differential equation is $u=u_c+g(x)$. In this section we will discuss methods for finding the particular solutions.

The operator technique for finding the particular solutions for ordinary differential equations is applicable for the cases where the method of undetermined coefficients for ordinary differential equations is used. This technique is useful in finding particular solutions of partial differential equations. Thus, if $f(D_x,D_y)$ is a linear partial differential operator, then the corresponding inverse operator is defined as $\dfrac{1}{f(D_x,D_y)}$. We will state some obvious results:

$$f(D_x,D_y)\left[\frac{1}{f(D_x,D_y)}\,\phi(x,y)\right]=\phi(x,y), \tag{3.4}$$

$$\begin{aligned}\frac{1}{f_1(D_x,D_y)f_2(D_x,D_y)}\,\phi(x,y)&=\frac{1}{f_1(D_x,D_y)}\left[\frac{1}{f_2(D_x,D_y)}\,\phi(x,y)\right]\\&=\frac{1}{f_2(D_x,D_y)}\left[\frac{1}{f_1(D_x,D_y)}\,\phi(x,y)\right],\end{aligned} \tag{3.5}$$

$$\begin{aligned}\frac{1}{f(D_x,D_y)}\,[c_1\,\phi_1(x,y)+c_2\,\phi_2(x,y)]&=c_1\frac{1}{f(D_x,D_y)}\,\phi_1(x,y)\\&\quad+c_2\frac{1}{f(D_x,D_y)}\,\phi_2(x,y),\end{aligned} \tag{3.6}$$

$$\frac{1}{f(D_x,D_y)}e^{ax+by}=\frac{1}{f(a,b)}e^{ax+by},\quad f(a,b)\neq 0, \tag{3.7}$$

$$f(D_x,D_y)\phi(x,y)e^{ax+by}=e^{ax+by}f(D_x+a,D_y+b)\,\phi(x,y), \tag{3.8}$$

$$\begin{aligned}\frac{1}{f(D_x,D_y)}\,\phi(x,y)\,e^{ax+by}&=e^{ax+by}\frac{1}{f(D_x+a,D_y+b)}\,\phi(x,y)\\&=e^{ax}\,\frac{1}{f(D_x+a,D_y)}\,e^{by}\,\phi(x,y)\\&=e^{by}\,\frac{1}{f(D_x,D_y+b)}\,e^{ax}\,\phi(x,y),\end{aligned} \tag{3.9}$$

$$f(D_x^2,D_y^2)\cos(ax+by)=f(-a^2,-b^2)\cos(ax+by), \tag{3.10}$$

$$f(D_x^2,D_y^2)\sin(ax+by)=f(-a^2,-b^2)\sin(ax+by). \tag{3.11}$$

EXAMPLE 3.14. For a particular solution of the equation

$$\left(3D_x^2+4D_xD_y-D_y\right)\,u=e^{x-3y},$$

note that

$$\begin{aligned}u_p&=\frac{1}{(3D_x^2+4D_xD_y-D_y)}\,e^{x-3y}\\&=\frac{1}{[3+4(-3)-(-3)]}\,e^{x-3y}=-\frac{1}{6}e^{x-3y}.\ \blacksquare\end{aligned} \tag{3.12}$$

EXAMPLE 3.15. For a particular solution of partial differential equation

$$(3D_x^2-D_y)u=\sin(ax+by),$$

we have

$$\begin{aligned}u_p&=\frac{1}{(3D_x^2-D_y)}\,\sin(ax+by)=\frac{1}{(-3a^2-D_y)}\,\sin(ax+by)\\&=-\frac{D_y-3a^2}{D_y^2-9a^4}\,\sin(ax+by)=\frac{b\cos(ax+by)-3a^2\sin(ax+by)}{b^2+9a^4}.\ \blacksquare\end{aligned} \tag{3.13}$$

EXAMPLE 3.16. To find a particular solution for the equation

$$(3D_x^2-D_y)u=e^x\sin(x+y),$$

we have

$$\begin{aligned}u_p&=\frac{1}{(3D_x^2-D_y)}e^x\,\sin(x+y)=e^x\frac{1}{(3(D_x+1)^2-D_y)}\,\sin(x+y)\\&=e^x\frac{1}{(3D_x^2+6D_x+3-D_y)}\,\sin(x+y)\end{aligned}$$

$$= e^x \frac{1}{(3(-1)_x^2 + 6D_x + 3 - D_y)} \sin(x+y)$$
$$= e^x \frac{1}{(6D_x - D_y)} \sin(x+y) = e^x \frac{(6D_x + D_y)}{(36D_x^2 - D_y^2)} \sin(x+y)$$
$$= e^x \frac{7\cos(x+y)}{-35}$$
$$= -\frac{1}{5} e^x \cos(x+y). \blacksquare \tag{3.14}$$

EXAMPLE 3.17. To solve $u_{tt} - c^2 u_{xx} = 0$, such that $u(x,0) = e^{-x}$, $u_t(x,0) = 1 + x$, note that the partial differential equation can be written as $(D_t - cD_x)(D_t + cD_x)u = 0$, which gives the solution as $u = f(x+ct) + g(x-ct)$. This solution is known as the d'Alembert's solution (see Eq (5.24)). Applying the initial conditions, we get

$$f(x) + g(x) = e^{-x}, \tag{3.15}$$

$$cf'(x) - cg'(x) = 1 + x. \tag{3.16}$$

On integrating (3.16) with respect to x we get

$$f(x) - g(x) = \frac{1}{c}(x + \frac{x^2}{2}) + c_1. \tag{3.17}$$

Eqs (3.15) and (3.17) yield

$$f(x) = \frac{1}{2} e^{-x} + \frac{1}{2c}(x + \frac{x^2}{2}) + \frac{c_1}{2},$$
$$g(x) = \frac{1}{2} e^{-x} - \frac{1}{2c}(x + \frac{x^2}{2}) - c_1/2.$$

Hence

$$u = \frac{1}{2} e^{-x}(e^{ct} + e^{-ct}) + (x+1)t. \blacksquare$$

A general scheme for initial value problems for the wave equation is as follows: Solve $u_{tt} = c^2 u_{xx}$, subject to the conditions $u(x,0) = \phi(x)$, $u_t(x,0) = \psi'(x)$. Then as in the above example

$$f(x) + g(x) = \phi(x),$$

$$cf'(x) - cg'(x) = \psi'(x).$$

Consequently

$$f(x) = \frac{1}{2}[\phi(x) + \frac{1}{c}\psi(x) + c_1],$$
$$g(x) = \frac{1}{2}[\phi(x) - \frac{1}{c}\psi(x) - c_1],$$

which yields

$$u(x,t) = \frac{1}{2}[\phi(x+ct) - \phi(x-ct)] + \frac{1}{2c}[\psi(x+ct) - \psi(x-ct)].$$

EXAMPLE 3.18. It is interesting to note that we can solve the Laplace equation by the above method. We will solve

$$u_{xx} + u_{yy} = 0,$$

such that $u(x,0) = \phi(x)$ and $u_y(x,0) = \psi'(x)$. We can express $u_{xx} + u_{yy} = 0$ as

$$(D_x + iD_y)(D_x - iD_y)u = 0,$$

and, therefore, its general solution is

$$u = f(x+iy) + g(x-iy).$$

Applying the initial conditions, we get $f(x)+g(x) = \phi(x)$, and $if'(x) - ig'(x) = \psi'(x)$. Consequently

$$f(x) = \frac{1}{2}[\phi(x) - i\psi(x) + c],$$
$$g(x) = \frac{1}{2}[\phi(x) + i\psi(x) - c].$$

Thus,

$$u(x,y) = \frac{1}{2}[\phi(x+iy) + \phi(x-iy)] + \frac{i}{2}[\psi(x-iy) - \psi(x+iy)].$$

The final value of u is real. If $\phi(x) = e^{-x}$, and $\psi' = \frac{1}{1+x^2}$, the solution is given by

$$u(x,y) = \frac{1}{2}[e^{(x+iy)} + e^{(x-iy)}] + \frac{i}{2}[\tan^{-1}(x-iy) - \tan^{-1}(x+iy)]$$
$$= e^{-x}\cos y - \frac{1}{4}\ln\left[\frac{x^2 + (1-y)^2}{x^2 + (1+y)^2}\right],$$

where we have used the formula

$$\tan\alpha = iz, \quad \text{or} \quad \alpha = \frac{i}{2}\ln\frac{(1+z)}{1-z},$$

with $\alpha = \tan^{-1}(x-iy) - \tan^{-1}(x+iy)$. $\blacksquare$

EXAMPLE 3.19. Consider

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < 1,$$

subject to the conditions $u(0,t) = u(l,t) = 0$, for $t \geq 0$, and $u(x,0) = x$, $u_t(x,0) = 0$. The general solution is

$$u = f(x + ct) + g(x - ct).$$

From the boundary conditions we find that

$$f(ct) + g(-ct) = 0, \quad \text{or} \quad f(z) + g(-z) = 0,$$

which yields $f(z) = -g(-z)$. Also $f(l+ct)+g(l-ct) = 0$ is equivalent to

$$f(ct + l) - f(ct - l) = 0,$$

which in turn gives $f(z) = f(z + 2l)$. This last equation implies that the function $f(x)$ is a periodic function of period $2l$. The solution, thus, reduces to

$$u = f(ct + x) - f(ct - x).$$

Applying the initial conditions, we get

$$f(x) - f(-x) = x, \quad \text{and} \quad f'(x) - f'(-x) = 0,$$

i.e., $f'(x)$ is an even function, which means that $f(x)$ is an odd function, i.e., $f(x) = -f(-x)$. Hence $2f(x) = x$. Since $f(x)$ is an odd periodic function of period $2l$, it can be expressed as a Fourier sine series. Thus

$$f(x) = \frac{x}{2} = \frac{l}{\pi} \sum_0^\infty \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l},$$

which yields

$$\begin{aligned} u(x,t) &= \frac{l}{\pi} \sum_0^\infty \frac{(-1)^{n+1}}{n} [\sin \frac{n\pi}{l}(ct + l) - \sin \frac{n\pi}{l}(ct - l)] \\ &= \frac{2l}{\pi} \sum_0^\infty \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}. \blacksquare \end{aligned}$$

Other techniques from ordinary differential equations such as the method of undetermined coefficients and the variation of parameters technique can also be extended to find the particular solution corresponding to the nonhomogeneous term in the partial differential equations.

3.4. Exercises

Evaluate (use the inverse operator method of §3.1):

3.1. $(D - 3)^{-1}(x^3 + 3x - 5)$.
ANS. $-\frac{1}{27}(9x^3 + 9x^2 + 33x - 34)$.

3.2. $(D - 1)^{-1}(2x)$.
ANS. $-2x$.

3.3. $(D - 1)^{-1}(x^2)$.
ANS. $-(x^2 + 2x + 2)$.

3.4. $(4D^2 - 5D)^{-1}(x^2 e^{-x})$.
ANS. $-\frac{e^{-x}}{729}(81x^2 + 234x + 266)$.

3.5. $(D^2 - 3D + 2)^{-1} \sin 2x$.
ANS. $\frac{3}{20} \cos 2x - \frac{1}{20} \sin 2x$.

3.6. $D^{-2}(2 \sin 2x)$.
ANS. $-\frac{1}{2} \sin 2x.$.

3.7. $D^{-3} x$.
ANS. $\frac{x^4}{24}$.

3.8. $D^{-2}(3e^{3x})$.
ANS. $\frac{e^{3x}}{3}$.

3.9. $D^{-1}(2x + 3)$.
ANS. $x^2 + 3x$.

3.10. $(D^3 - D^2)^{-1}(2x^3)$.
ANS. $-2\left(\frac{x^5}{20} + \frac{x^4}{4} + x^3 + 3x^2\right)$.

3.11. $(D^2 + 3D + 2)^{-1}(e^{ix})$.
ANS. $\frac{1 - 3i}{10} e^{ix}$.

3.12. $(D^2 - 3D + 2)^{-1}(3 \sin x)$.
ANS. $\frac{3}{10}(\sin x + 3 \cos x)$.

3.13. $(D^2 + 3D + 2)^{-1}(8 + 6e^x + 2 \sin x)$.

ANS. $4 + e^x + \frac{1}{5}(\sin x - 3\cos x)$.

3.14. $(D^5 + 2D^3 + D)^{-1}(2x + \sin x + \cos x)$.

ANS. $x^2 + \frac{x^2}{8}(\cos x - \sin x)$.

Find the general solution of the following partial differential equations:

3.15. $(3D_x^2 - 2D_xD_y - 5D_y^2)u = 3x + y + e^{x-y}$.

ANS. $u = f(5x + 3y) + g(x - y) + \frac{11}{54}x^3 + \frac{1}{6}x^2y + \frac{1}{8}xe^{x-y}$.

3.16. $(D_x^4 - 10D_x^2D_y^2 + 9D_y^4)u = 135\sin(3x + 2y)$.

ANS. $f_1(3x+y) + f_2(x-3y) + g_1(x+y) + g_2(x-y) - \sin(3x+2y)$.

3.17. $(D_x - 2D_y)^3u = 125e^x \sin y$.

ANS. $f_1(2x+y) + xf_2(2x+y) + x^2f_3(2x+y) - e^x(2\cos y + 11\sin y)$.

3.18. Find the particular solution for the following partial differential equations:

(a) $(D_x^2 - D_y)u = 17e^{x+y}\sin(x - 2y)$.

ANS. $-e^{x+y}\{\sin(x-2y) + 4\cos(x-2y)\}$.

(b) $(D_x^2 + D_y^2)u = 6xy + 25e^{3x+4y}$.

ANS. $x^3y + e^{3x+4y}$.

(c) $(D_x^2 + D_y^2 - D_x)u = 37e^{5y}\cos(3x + 4y)$.

ANS. $e^{5y}\sin(3x + 4y)$.

3.19. Show that $u = f(ay - bx)\,e^{-cy/b}$ is also a solution of

$$(aD_x + bD_y + c)\,u = 0.$$

3.20. Find the general solution of $3\,u_x + 4\,u_y - 2\,u = 1$, subject to the initial condition $u(x,0) = x^2$.

SOLUTION. Here $\tan\theta = 4/3$, thus

$$\frac{\partial w}{\partial \xi} - \frac{2}{5}w = \frac{1}{5},$$

whose general solution is

$$w(\xi,\eta) = -\frac{1}{2} + g(\eta)\,e^{2\xi/5},$$

or

$$u(x,y) = -\frac{1}{2} + g\left(\frac{3}{5}y - \frac{4}{5}x\right)\,e^{6x/25+8y/25}.$$

3.21. Find the general solution of $u_x - u_y + u = 1$, such that $u(x,0) = \sin x$.

SOLUTION. $\tan\theta = -1$, thus $\theta = 4\pi/4$, and

$$\frac{\partial w}{\partial \xi} - \frac{1}{\sqrt{2}}w = -\frac{1}{\sqrt{2}},$$

whose general solution is $w = 1 + g(\eta)\,e^{\xi/\sqrt{2}}$, or

$$u(x,y) = 1 + g\left(1 - \frac{x+y}{\sqrt{2}}\right)\,e^{(y-x)/2}.$$

Using the initial condition, we get $\sin x = 1 + G(-x/\sqrt{2})\,e^{-x/2}$, so that

$$g(\eta) = -(\sin\sqrt{2}\eta + 1)\,e^{-\eta/\sqrt{2}}.$$

Then

$u(x,y) = 1 - (\sin\sqrt{2}\eta + 1)\,e^{-\eta/\sqrt{2}}\,e^{\xi/\sqrt{2}} = 1 + [1 - \sin(x+y)]\,e^y$. ■

3.22. Solve $u_x + u_y - u = 0$, subject to the initial condition $u(x,0) = h(x)$.

SOLUTION. Here $\tan\theta = 1$, thus $\theta = \pi/4$, and $\sqrt{2}\frac{\partial w}{\partial \xi} = w$, whose general solution is $w = g(\eta)\,e^{\xi/\sqrt{2}}$, or

$$u(x,y) = g(\eta)\,e^{\xi/\sqrt{2}}.$$

The initial condition yields

$$h(x) = g(-x/\sqrt{2})\,e^{x/2} = g(\eta)\,e^{\eta/\sqrt{2}},$$

or $g(\eta) = h(-\sqrt{2}\eta)\,e^{\eta/\sqrt{2}}$. Hence

$$u(x,y) = h(-\sqrt{2}\eta)\,e^{\xi/\sqrt{2}} = h(x-y)\,e^y.$$

3.23. Solve $u_{tt} - c^2u_{xx} = 0$, subject to the conditions $u(x,0) = \ln(1 + x^2)$ and $u_t(x,0) = e^{-x}$.

ANS.

$$u(x,t) = \frac{1}{2}[ln\{1 + (x+ct)^2\} + ln\{1 + (x-ct)^2\}] + \frac{1}{c}e^{-x}\cosh ct.$$

4

Orthogonal Expansions

Unlike the ordinary differential equations, the general solution of a partial differential equation consists of one or more arbitrary functions. It is not easy to determine the particular form of these functions from the prescribed boundary and initial conditions even if the general solution is known. However, it is often possible to solve a specific boundary value or initial value problem in the form of an infinite series of functions known as eigenfunctions or characteristic functions. This chapter is devoted to developing orthogonal series, trigonometric Fourier series, eigenfunction expansions, and Bessel functions. Orthogonal expansions are important for the method of separation of variables which shall be discussed in the next chapter.

4.1. Orthogonality

The *inner product* of two real–valued functions f_1 and f_2 defined on an interval $a \le x \le b$ is given by

$$\langle f_1, f_2 \rangle = \int_a^b f_1(x) f_2(x)\, dx, \tag{4.1}$$

provided the integral in (4.1) exists.

DEFINITION 4.1. The functions f_1 and f_2 are said to be *orthogonal* on the interval $a \le x \le b$ if $\langle f_1, f_2 \rangle = 0$, i.e., the integral (4.1) vanishes.

DEFINITION 4.2. A set of real–valued functions $\{f_1(x), f_2(x), \cdots\}$ defined on the interval $a \le x \le b$ is said to be an *orthogonal set of functions* on the interval $a \le x \le b$ if for each m and n, $m \ne n$,

$$\langle f_m, f_n \rangle = \int_a^b f_m(x) f_n(x)\, dx = 0, \quad m, n = 1, 2, \cdots, \tag{4.2}$$

provided each integral exists. Thus, for example, the orthogonality relations for the function $\cos \frac{n\pi x}{l}$ are

$$\int_0^l \cos \frac{n\pi x}{l} \cos \frac{m\pi x}{l}\, dx = \begin{cases} 0, & n \ne m \\ \frac{l}{2}, & n = m \ne 0 \\ l, & n = m = 0. \end{cases} \tag{4.3}$$

DEFINITION 4.3. The *norm* of the functions $f_n(x)$ is denoted by $\|f_n\|$, and defined by

$$\|f_n\| = \sqrt{\langle f_n, f_n \rangle} = \left(\int_a^b f_n^2(x)\, dx \right)^{1/2} \ge 0. \tag{4.4}$$

DEFINITION 4.4. An orthogonal set of functions $\{f_n(x)\}$ is called an *orthonormal set of functions* on the interval $a \le x \le b$ if $\|f_n\| = 1$ for all $n = 1, 2, \cdots$.

Thus, if $\{f_n(x)\}$ is an orthonormal set of functions, then

$$\langle f_m, f_n \rangle = \int_a^b f_m(x) f_n(x)\, dx = \delta_{mn}, \tag{4.5}$$

where $\delta_{mn} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \ne n \end{cases}$ is the Kronecker delta. If an orthogonal set of functions $\{f_n(x)\}$ is defined on the interval $a \le x \le b$, with $\|f_n\| \ne 0$, we can always construct an orthonormal set of functions $g_n(x)$ by defining

$$g_n(x) = \frac{f_n(x)}{\|f_n\|}, \quad a \le x \le b. \tag{4.6}$$

In fact, in view of (4.5),

$$\langle g_m, g_n \rangle = \int_a^b \frac{f_m(x)}{\|f_m\|} \frac{f_n(x)}{\|f_n\|}\, dx = \frac{1}{\|f_m\| \|f_n\|} \langle f_m, f_n \rangle = \delta_{mn}, \tag{4.7}$$

and hence $\|g_n\| = 1$ for all n.

EXAMPLE 4.1. The functions $f_n(x) = \sin nx$, $n = 1, 2, \cdots$, make an orthogonal set of functions on the interval $-\pi \le x \le \pi$, since

$$\begin{aligned}\langle f_m, f_n\rangle &= \int_{-\pi}^{\pi} \sin mx \sin nx\, dx \\ &= \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m-n)x - \cos(m+n)x]\, dx \\ &= \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0,\ m \ne n,\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}\langle f_n, f_n\rangle &= \int_{-\pi}^{\pi} \sin^2 nx\, dx = \frac{1}{2}\int_{-\pi}^{\pi} (1 - \cos 2nx)\, dx \\ &= \frac{1}{2}\left[x - \frac{\sin 2nx}{2n}\right]_{-\pi}^{\pi} = \pi.\end{aligned}$$

Thus, $\|f_n\| = \sqrt{\pi}$, and the orthonormal set of functions is given by

$$g_n(x) = \frac{\sin nx}{\sqrt{\pi}}, \quad n = 1, 2, \cdots. \blacksquare$$

EXAMPLE 4.2. The set of functions

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \cdots\right\}$$

forms an orthogonal set on the interval $-\pi \le x \le \pi$. In Example 1.9, we have seen that $\dfrac{\sin nx}{\sqrt{\pi}}$ is orthonormal on the interval $-\pi \le x \le \pi$. Now, to verify the orthonormality of other functions,

$$\begin{aligned}&\int_{-\pi}^{\pi} \frac{\cos mx}{\sqrt{\pi}} \frac{\cos nx}{\sqrt{\pi}}\, dx \\ &= \begin{cases} \dfrac{1}{2\pi} \int_{-\pi}^{\pi} [\cos(m-n)x - \cos(m+n)x]\, dx = 0, & \text{for } m \ne n, \\ \dfrac{1}{2\pi} \int_{-\pi}^{\pi} [\cos(m-n)x - \cos(m+n)x]\, dx = 1, & \text{for } m = n. \end{cases}\end{aligned}$$

Also,

$$\begin{aligned}\int_{-\pi}^{\pi} \frac{\sin mx}{\sqrt{\pi}} \frac{\sin nx}{\sqrt{\pi}}\, dx &= \frac{1}{2\pi}\int_{-\pi}^{\pi} [\sin(m-n)x + \sin(m+n)x]\, dx \\ &= -\frac{1}{2\pi}\left[\frac{\cos(m-n)x}{m-n} + \frac{\cos(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0, \quad \text{for all } m, n.\end{aligned}$$

Finally,

$$\begin{aligned}\int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} \frac{\sin nx}{\sqrt{\pi}}\, dx &= -\frac{1}{\pi\sqrt{2}} \frac{\cos nx}{n}\Big|_{-\pi}^{\pi} = 0, \\ \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} \frac{\cos nx}{\sqrt{\pi}}\, dx &= \frac{1}{\pi\sqrt{2}} \frac{\sin nx}{n}\Big|_{-\pi}^{\pi} = 0, \\ \int_{-\pi}^{\pi} \left(\frac{1}{\sqrt{2\pi}}\right)^2 dx &= \frac{x}{2\pi}\Big|_{-\pi}^{\pi} = 1. \blacksquare\end{aligned}$$

EXAMPLE 4.3. The set of functions

$$1, \quad \cos x, \quad \sin x, \quad \cos 2x, \quad \sin 2x, \quad \ldots$$

forms an orthogonal set on the interval $-\pi \le x \le \pi$, for which

$$\begin{aligned}\int_{-\pi}^{\pi} \cos mx\, \sin nx\, dx &= \frac{1}{2}\int_{-\pi}^{\pi} \sin(m+n)\pi x\, dx \\ &\quad - \int_{-\pi}^{\pi} \sin(m-n)\pi x\, dx = 0\end{aligned}$$

for all $m, n = 0, 1, 2, \cdots$. The orthonormal set is

$$\frac{1}{\sqrt{2\pi}}, \ \frac{\cos x}{\sqrt{\pi}}, \ \frac{\sin x}{\sqrt{\pi}}, \ \frac{\cos 2x}{\sqrt{\pi}}, \ \frac{\sin 2x}{\sqrt{\pi}}, \ \cdots. \blacksquare$$

4.2. Orthogonal Polynomials

DEFINITION 4.5. The *weighted inner product* of two functions f and g, with weight $w > 0$, is defined by

$$\langle f, g\rangle_w = \int_a^b f(x)g(x)w(x)\, dx. \tag{4.9}$$

Some classes of orthogonal polynomials are defined for different values of a, b, $w(x)$ for $n = 0, 1, 2, \cdots$ as follows:

Orthogonal Polynomials	a	b	$w(x)$
Chebyshev* (1st kind) $T_n(x)$	-1	1	$(1-x^2)^{-1/2}$
Chebyshev (2nd kind) $U_n(x)$	-1	1	$(1-x^2)^{1/2}$
Hermite $H_n(x)$	$-\infty$	∞	e^{-x^2}
Jacobi $P_n^{(\alpha,\beta)}(x)$	-1	1	$(1-x)^\alpha(1-x)^\beta$
Laguerre $L_n(x)$	0	∞	e^{-x}
Legendre $P_n(x)$	-1	1	1

[www] The Mathematica package `orthonormality.m` is available on the CRC web server. It can be used to verify the orthonormality of sets of functions.

EXAMPLE 4.4. The Legendre polynomials $P_n(x)$ are the solutions of the Legendre equation

$$(1-x^2)\,y'' - 2x\,y' + n(n=1)\,y = 0.$$

These polynomials are also called the zonal harmonics of the first kind. The orthogonality relation for the Legendre polynomials $P_n(x)$ is

$$\int_{-1}^{1} P_n(x)\,P_m(x)\,dx = \begin{cases} 0 & \text{for } n \neq m \\ \dfrac{2}{2n+1} & \text{for } n = m \end{cases}.$$

We will present a Mathematica session to evaluate the integral

$$\int_{-1}^{1} P_n(x)\,P_m(x)\,dx$$

for $n, m = 0, \ldots, 10$.

In[1]:=

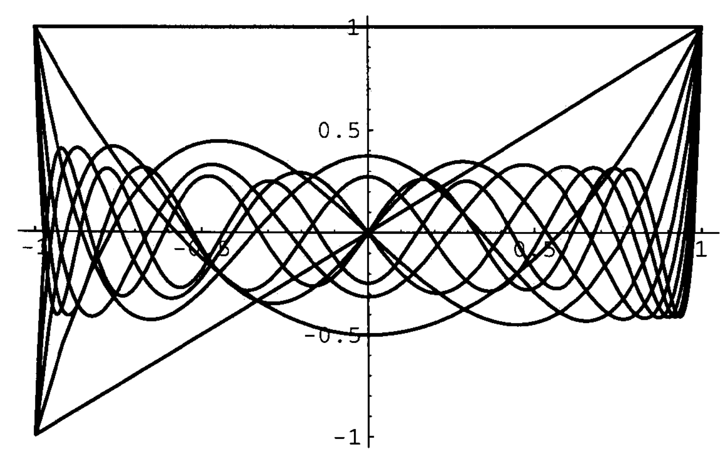
```
legendre = Table[LegendreP[n,x], {n, 0 , 10}];
```

* Also written as Tchebyscheff.

Out[1]=

$$1$$
$$x$$
$$\frac{-1+3\,x^2}{2}$$
$$\frac{-3\,x+5\,x^3}{2}$$
$$\frac{3-30\,x^2+35\,x^4}{8}$$
$$\frac{15\,x-70\,x^3+63\,x^5}{8}$$
$$\frac{-5+105\,x^2-315\,x^4+231\,x^6}{16}$$
$$\frac{-35\,x+315\,x^3-693\,x^5+429\,x^7}{16}$$
$$\frac{35-1260\,x^2+6930\,x^4-12012\,x^6+6435\,x^8}{128}$$
$$\frac{315\,x-4620\,x^3+18018\,x^5-25740\,x^7+12155\,x^9}{128}$$
$$\frac{-63+3465\,x^2-30030\,x^4+90090\,x^6-109395\,x^8+46189\,x^{10}}{256}$$

The plots of these Legendre polynomials are given below.



Graphs of $P_n(x)$, $n = 0, 1, \ldots, 10$.

```
-Graphics-
```

In[2]:=

```
orthogonality =
Table[Integrate[legendre[[i]] legendre[[j]],
   {x, -1, 1}], {i, 1, 10}, {j, 1 , 10}];

MatrixForm[orthogonality]
```

Out[2]=

$$\begin{matrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{7} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{9} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{15} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{17} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{19} \end{matrix}$$

4.3. Series of Orthogonal Functions

Some important types of series expansions are obtained from orthogonal sets of functions.

DEFINITION 4.6. Let $\{g_1(x), g_2(x), \cdots\}$ be an orthogonal set of functions on an interval $a \le x \le b$, and let a function $f(x)$ be represented in terms of the functions $g_n(x)$, $n = 1, 2, \cdots$, by a convergent series

$$f(x) = \sum_{n=1}^{\infty} c_n \, g_n(x). \tag{4.10}$$

This series is called a *generalized Fourier series* of $f(x)$, and the coefficients c_n, $n = 1, 2, \cdots$, are called the *Fourier coefficients* of $f(x)$ with

respect to the orthogonal set of functions $g_n(x)$, $n = 1, 2, \cdots$.

In view of Definition 4.4, it is easy to determine the coefficients c_n. If we multiply both sides of (4.10) by $g_m(x)$ for a fixed m, integrate with respect to x over the interval $[a, b]$, and assume term–by–term integration, which is justified in the case of uniform convergence, we obtain

$$\begin{aligned} \langle f, g_m \rangle = \int_a^b f g_m \, dx &= \int_a^b \left(\sum_{n=1}^{\infty} c_n \, g_n(x) \right) g_m(x) \, dx \\ &= \sum_{n=1}^{\infty} c_n \, \langle g_n(x), g_m(x) \rangle. \end{aligned} \tag{4.11}$$

Since in view of the relations (4.6) and (4.7), $\langle g_n, g_m \rangle = \delta_{nm}$, we find that $\langle g_n, g_m \rangle = \|g_n\|^2$ for $n = m$, and then (4.11) yields

$$\langle f, g_n \rangle = c_n \, \|g_n\|^2,$$

or

$$c_n = \frac{\langle f, g_n \rangle}{\|g_n\|^2} = \frac{1}{\|g_n\|^2} \int_a^b f(x) g_n(x) \, dx. \tag{4.12}$$

EXAMPLE 4.5. Using the orthogonal set of functions from Example 4.3 and formula (4.12), the representation (4.10) becomes

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \, \cos nx + b_n \, \sin nx)\,, \tag{4.13}$$

where, since $\|g_0\| = \frac{1}{\sqrt{2\pi}}$, $\|g_n\| = \frac{1}{\sqrt{\pi}}$ for $n = 1, 2, \cdots$, the coefficients are given by

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx, \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx, \end{aligned} \tag{4.14}$$

for $n = 1, 2, \cdots$. ■

We have introduced the *trigonometric Fourier series* of $f(x)$ in the above example, under the assumption that the series (4.13) converges and represents the function $f(x)$. The coefficients a_0, a_n and b_n for $n = 1, 2, \cdots$ are called the *Fourier coefficients*, and (4.14) are known as *Euler's formulas*.

4.4. Trigonometric Fourier Series

A function $f(x)$ is said to be *periodic* of period p if $f(x+p) = f(x)$, $c \le x \le c+p$, where c is a constant. For example, the functions $\sin x$ and $\cos x$ have period 2π. More generally, each of the functions $\sin \frac{2n\pi x}{p}$ and $\cos \frac{2n\pi x}{p}$ is periodic of period p where n is a positive integer. Hence, if the infinite series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2n\pi x}{p} + b_n \sin \frac{2n\pi x}{p} \right) \tag{4.15}$$

is convergent, then it represents a function of period p.

THEOREM 4.1. *(Fourier Theorem I, for periodic functions) Let $f(x)$ be a single-valued piecewise continuous periodic function of period p on a finite interval $I = [c, c+p]$, where c is a constant. Then the series (4.15) converges to $f(x)$ at all points of continuity and to*

$$\frac{1}{2}[f(x+) + f(x-)] \tag{4.16}$$

at the points of discontinuity (and also at all points of continuity). The coefficients a_0, a_n and b_n are given by

$$\begin{aligned} a_n &= \frac{2}{p} \int_c^{c+p} f(x) \cos \frac{2n\pi x}{p}\, dx, \quad (n = 0, 1, 2, \cdots), \\ b_n &= \frac{2}{p} \int_c^{c+p} f(x) \sin \frac{2n\pi x}{p}\, dx, \quad (n = 1, 2, \cdots). \end{aligned} \tag{4.17}$$

where c is a constant such that the interval $[c, c+p] = I$.

If we set $p = 2L$ and $c = -L$ in (4.17), then these formulas become

$$\begin{aligned} a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx, \quad (n = 0, 1, 2, \cdots), \\ b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx, \quad (n = 1, 2, \cdots). \end{aligned} \tag{4.18}$$

EXAMPLE 4.6. Let $f(x) = |x|$, $-\pi \le x \le \pi$ (see Fig. 4.1). Note that $f(x)$ is an even function, since $f(-x) = |-x| = |x| = f(x)$. Then, from (4.18), with $L = \pi$,

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} |x|\, dx = \frac{2}{\pi} \int_0^{\pi} x\, dx = \pi, \\ a_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx = \frac{2}{\pi} \int_0^{\pi} x \cos nx\, dx \\ &= \frac{2}{\pi} \left[\frac{\cos nx}{n^2} + \frac{x \sin nx}{n} \right]_0^{\pi} \\ &= \begin{cases} -\dfrac{4}{n^2\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \end{aligned}$$

and $b_n = 0$ for $n = 1, 2, \cdots$. Hence the Fourier series (4.15) is

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)\pi}{(2n-1)^2}, \quad -\pi \le x \le \pi. \blacksquare$$

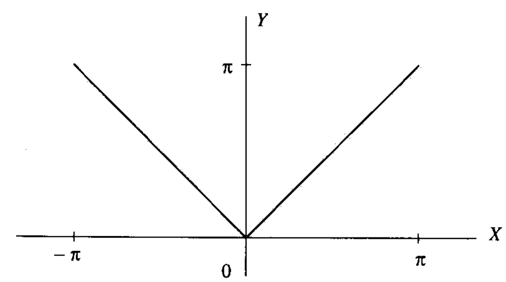


Fig. 4.1. Graph of $f(x) = |x|$, $-\pi \le x \le \pi$.

The following Mathematica session illustrates this example.

In[3]:=

```
Clear[f,a0,a,b,fourier];
f[x_]:= Abs[x]

Integrate[Abs[x],{x,-Pi,Pi}]

On::none:  Message SeriesData::csa not found.
```

Out[5]=

```
Integrate[Abs[x], {x, -Pi, Pi}]
```

Mathematica will *not* integrate $|x|$. Observe that the function $f(x) \cos nx$ is even and $f(x) \sin nx$ is odd. So we proceed as follows:

In[6]:=

```
a0 = (2/Pi) Integrate[x,{x,0,Pi}]
```

Out[6]=

```
Pi
```

In[7]:=

```
a = Table[2 Integrate[x Cos[n x],{x,0,Pi}]/Pi,{n,10}]
```

Out[7]=

$$\{\frac{-4}{Pi}, 0, \frac{-4}{9\,Pi}, 0, \frac{-4}{25\,Pi}, 0, \frac{-4}{49\,Pi}, 0, \frac{-4}{81\,Pi}, 0\}$$

In[8]:=

```
b = Table[(Integrate[-x Sin[n x],{x,-Pi,0}]
   + Integrate[x Sin[n x],{x,0,Pi}])/Pi,{n,10}]
```

Out[8]=

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

In[9]:=

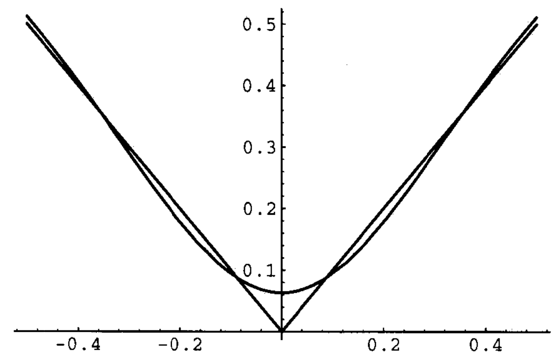
```
fourier[x_] = a0/2 +
Sum[a[[n]] Cos[n x]+ b[[n]] Sin[n x],{n,1,10}]
```

Out[9]=

$$\{\frac{Pi}{2} - \frac{4\,Cos[x]}{Pi} - \frac{4\,Cos[3\,x]}{9\,Pi} - \frac{4\,Cos[5\,x]}{25\,Pi} - \frac{4\,Cos[7\,x]}{49\,Pi} - \frac{4\,Cos[9\,x]}{81\,Pi}\}$$

In[10]:=

```
Plot[{f[x],fourier[x]},{x,-0.5,0.5}]
```



EXAMPLE 4.7. Consider

$$f(x) = \begin{cases} 2, & x = 0, \\ 3, & 0 < x < 2, \\ 2, & x = 2, \\ 1, & 2 < x < 4, \end{cases}$$

such that $f(x + 4) = f(x)$. This function is piecewise continuous and is of period 4 (see Fig. 4.2). Note that $f(2-) = 3$ and $f(2+) = 1$. In this example, $c = 0$, $p = 4$. Then, using (4.17) we get

$$\begin{aligned} a_0 &= \frac{1}{2}\left(\int_0^2 3\,dx + \int_2^4 1\,dx\right) = 4, \\ a_n &= \frac{1}{2}\left(\int_0^2 3\cos\frac{n\pi x}{2}\,dx + \int_2^4 1\cos\frac{n\pi x}{2}\,dx\right) = 0, \\ b_n &= \frac{1}{2}\left(\int_0^2 3\sin\frac{n\pi x}{2}\,dx + \int_2^4 1\sin\frac{n\pi x}{2}\,dx\right) \\ &= \frac{3 - 2\cos n\pi - \cos 2n\pi}{n\pi} = \begin{cases} 0, & \text{if } n \text{ is even}, \\ \dfrac{4}{n\pi}, & \text{if } n \text{ is odd}. \end{cases} \end{aligned}$$

Hence the Fourier series is

$$f(x) = 2 + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{(2n-1)\pi x}{2}.$$

Also, note that in view of (4.16), for example, $f(4) = \frac{1}{2}[f(4+)+f(4-)]$, i.e., $2 = \frac{1}{2}(3+1)$. ■

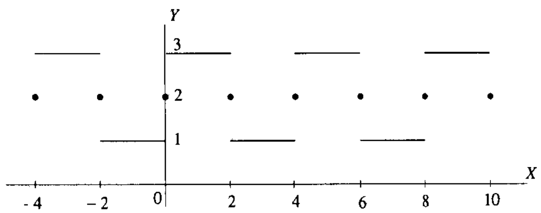


Fig. 4.2. Graph of $f(x)$.

The following Mathematica session illustrates this example.

In[11]:=

```
Clear[f,a0,a,b,fourier];

f[x_] := If[x==0,2,If[0<x<2,3,If[x==2,2,
If[2<x<4,1]]]]

Integrate[f[x],{x,0,4}]

On::none:  Message SeriesData::csa not found.
General::intinit:  Loading integration packages --
please wait.
```

Out[13]=

```
Integrate[If[x == 0, 2, If[0 < x < 2, 3, If[x == 2, 2,
If[2 < x < 4, 1]]]], {x, 0, 4}]
```

In[14]:=


```
a0 = (1/2) (Integrate[3,{x,0,2}] + Integrate[1,{x,2,4}])
```

Out[14]=

4

In[15]:=

```
a = Table[1/2 (Integrate[3 Cos[n Pi x/4],{x,0,2}] +
Integrate[Cos[n Pi x/4],{x,2,4}]),{n,16}]
```

Out[15]=

{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0}

In[16]:=

```
b = Table[1/2 (Integrate[3 Sin[n Pi x/4],{x,0,2}]+
Integrate[Sin[n Pi x/4],{x,2,4}]),{n,16}]
```

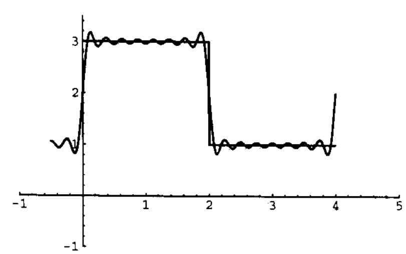
Out[16]=

$$\{\frac{4}{\text{Pi}}, 0, \frac{4}{3\,\text{Pi}}, 0, \frac{4}{5\,\text{Pi}}, 0, \frac{4}{7\,\text{Pi}}, 0, \frac{4}{9\,\text{Pi}}, 0, \frac{4}{11\,\text{Pi}}, 0, \frac{4}{13\,\text{Pi}}, 0, \frac{4}{15\,\text{Pi}}, 0\}$$

In[17]:=

```
fourier[x_]:= a0/2 + Sum[b[[n]] Sin[n Pi x/2],{n,1,16}]
Plot[{f[x],fourier[x]},{x,-0.5,4},
PlotRange->{{-1,5},{-1,3.5}}]
```

Out[18]=



EXAMPLE 4.8. Consider the wave equation $u_{tt} = c^2 u_{xx}$, where c is the wave velocity, subject to the boundary conditions $u(0,t) = 0 = u(l,t)$ for $t \geq 0$, and the initial conditions $u(x,0) = x$, and $u_t(x,0) = 0$. We shall assume the d'Alembert's solution of the wave equation (see Example 3.17 and (5.24))

$$u(x,t) = f(x+ct) + g(x-ct). \tag{4.19}$$

From the boundary condition $u(0,t) = 0$ we find that

$$f(ct) + g(-ct) = 0,$$

or, setting $ct = z$, we get $f(z) = -g(-z)$. Using the other boundary condition $u(l,t) = 0$ we have

$$f(l+ct) + g(l-ct) = 0,$$

which implies that $f(ct+l) - f(ct-l) = 0$, i.e., $f(z) = f(z+2l)$. This last equation means that f is a periodic function of period $2l$. The solution thus reduces to

$$u(x,t) = f(ct+x) - f(ct-x).$$

If we apply the initial conditions we get

$$f(x) - f(-x) = x, \quad \text{and} \quad f'(x) - f'(-x) = 0,$$

which means that $f'(x)$ is an even function, and therefore $f(x)$ an odd function, i.e., $f(x) = -f(-x)$. Hence $2\,f(x) = x$. Since $f(x)$ is an odd periodic function of period $2l$, it can be expressed as a Fourier sine series

$$f(x) = \frac{x}{2} = \frac{l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l},$$

which yields

$$\begin{aligned} u(x,t) &= \frac{l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left[\sin \frac{n\pi(ct+x)}{l} - \sin \frac{n\pi(ct-x)}{l} \right] \\ &= \frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}. \end{aligned}$$

This solution can be compared with (5.23). ■

EXAMPLE 4.9. Consider the wave equation $u_{tt} = c^2 u_{xx}$, where c is the wave velocity, subject to the boundary conditions (i) $u_x(0,t) = 0$, (ii) $u(l,t) = 0$, and the initial conditions (iii) $u(x,0) = x$, and (iv) $u_t(x,0) = 0$. Using the d'Alembert's solution (4.19) we find that

$$\frac{\partial u(x,t)}{\partial x} = \frac{\partial f(x+ct)}{\partial(x+ct)} \frac{\partial(x+ct)}{\partial x} + \frac{\partial g(x-ct)}{\partial(x-ct)} \frac{\partial(x-ct)}{\partial x},$$

which, by taking $ct = z$, in view of condition (i) yields $u_x(0,t) = f'(z) + g'(-z) = 0$, where $' \equiv d/dz$, i.e., $f'(z) = -g'(-z)$ which upon integration with respect to z yields $f(z) = g(-z) + c_1$, and hence

$$g(z) = f(-z) + c_1. \tag{4.20}$$

Condition (ii) gives $f(l+ct) + g(l-ct) = 0$, which by using the value of g from (4.20) becomes

$$f(l+ct) + f(ct-l) + c_1 = 0.$$

Condition (iii) , in view of (4.20), gives

$$f(x) + f(-x) + c_1 = x,$$

whereas condition (iv) gives

$$cf'(x) + cf'(-x) = 0,$$

i.e., $f'(x) = -f'(-x)$, which implies that f' is an odd function, and, therefore, $f(x)$ is an even function, i.e., $f(-x) = f(x)$. Hence, $2f(x) = x - c_1$. If we define

$$\psi(x) = f(x) + \frac{c_1}{2},$$

then we have

$$\psi(ct+l) + \psi(ct-l) = 0,$$

or, by taking $ct - l = v$, we get

$$\psi(v) + \psi(v+2l) = 0.$$

If we set $v = \zeta + 2l$, then

$$\psi(\zeta + 4l) = -\psi(\zeta + 2l) = \psi(\zeta).$$

Hence, ψ is a periodic function of period $4l$, and $\psi(x) + \psi(-x) = x$. Let

$$\psi(x) = A_n \sin \frac{n\pi x}{2l} + B_n \cos \frac{n\pi x}{2l}.$$

Then

$$\begin{aligned} u(x,t) &= \psi(ct+x) + \psi(ct-x) \\ &= A_n \left[\sin \frac{n\pi}{2l}(ct+x) + \sin \frac{n\pi}{2l}(ct-x)\right] \\ &\quad + B_n \left[\cos \frac{n\pi}{2l}(ct+x) + \cos \frac{n\pi}{2l}(ct-x)\right] \\ &= 2\left[A_n \sin \frac{n\pi ct}{2l} \cos \frac{n\pi x}{l} + B_n \cos \frac{n\pi ct}{2l} \cos \frac{n\pi x}{2l}\right]. \end{aligned}$$

Using the initial condition (iv), we find that $u_t(x,0) = 0$ gives $A_n = 0$. Again, condition (ii) yields $2B_n \cos \dfrac{n\pi ct}{2l} \cos \dfrac{n\pi}{2} = 0$; thus, $B_n = 0$ if $n = 2m$, and $B_n \neq 0$ if $n = 2m-1$, where m is a positive integer. This gives

$$u(x,t) = \sum_{m=1}^{\infty} 2B_{2m-1} \cos \frac{(2m-1)\pi ct}{2l} \cos \frac{(2m-1)\pi x}{2l}.$$

Since

$$u(x,0) = \sum_{m=1}^{\infty} 2B_m \cos \frac{(2m-1)\pi x}{2l} = x,$$

we find that

$$\begin{aligned} B_m &= \frac{1}{l} \int_0^l x \cos \frac{(2m-1)\pi x}{2l}\, dx \\ &= \left\{ \left[x \sin \frac{(2m-1)\pi x}{2l}\right]_0^l \frac{2l}{(2m-1)\pi} \right. \\ &\quad \left. - \frac{2l}{(2m-1)\pi} \int_0^l \sin \frac{(2m-1)\pi x}{2l}\, dx \right\} \\ &= \frac{2(-1)^m l}{(2m-1)\pi} - \frac{4l}{(2m-1)^2\pi^2}. \blacksquare \end{aligned}$$

DEFINITION 4.7. A function f defined on an interval $[a,b]$ is said to be of *bounded variation* on $[a,b]$ if its total variation $\text{var}(f)$ on $[a,b]$ is finite, i.e.,

$$\text{var}(f) = \sup_P \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|, \tag{4.21}$$

the supremum being taken over all partitions P

$$a = t_0 < t_1 < \cdots < t_n = b \tag{4.22}$$

of the interval $[a,b]$, and n is an arbitrary positive integer such that the choice of the values $t_1, t_2, \cdots, t_{n-1}$ satisfies (4.22). All functions of bounded variation form a vector space, with norm

$$\|f\| = |f(a)| + \text{var}(f).$$

A monotone function f on $[a,b]$ is of bounded variation, and its total variation is

$$\text{var}(f) = |f(b) - f(a)|.$$

A function f of the class C^1 is of bounded variation on $[a,b]$, with total variation

$$\text{var}(f) = \int_a^b |f'(x)|\, dx,$$

provided f' exists and is bounded in (a,b). A generalization of this result is: A function f is said to satisfy the *Lipschitz condition* if there exists a constant $M > 0$ such that

$$|f(x) - f(y)| < M\,|x-y|. \tag{4.23}$$

Obviously, such a function is uniformly continuous and is of bounded variation. Some well-known results are:
A function f is of bounded variation on $[a,b]$ is bounded on $[a,b]$.
If f and g are of bounded variation on $[a,b]$, then $f+g$ and $f\,g$ are of bounded variation on $[a,b]$.
If f is of bounded variation on $[a,b]$ and c is a real number, then the function cf is also of bounded variation on $[a,b]$.
If f is of bounded variation on $[a,b]$, then (a) $\text{var}(f)$ is an increasing function; (b) $\text{var}(f)$ is an increasing function, and (c) f is continuous at $x_0 \in [a,b]$ iff $\text{var}(f)$ is continuous at x_0. Moreover, a function f of bounded variation on $[a,b]$ can be represented as a sum of two monotone functions g and h:

$$f(x) = g(x) + h(x), \quad a \le x \le b, \tag{4.24}$$

such that g is nondecreasing and h is nonincreasing. Each such function f has the following properties:
(i) one-sided limits $f(x+)$ and $f(x-)$ from the interior of the interval exist at each point;
(ii) f has at most countably many discontinuities in the interval, and
(iii) f is bounded and integrable over the interval.

THEOREM 4.2. *(Fourier Theorem II) Let $f(x)$ denote a periodic function of period 2π whose integral from $-\pi$ to π exists. If that integral is improper, let it be absolutely convergent. Then at each point x which is interior to an interval on which f is of bounded variation, the Fourier series for the function f converges to the average value (4.16).*

The asymptotic behavior of the Fourier coefficients of a periodic function $f(x)$ is given by the following theorem:

THEOREM 4.3. *As $n \to \infty$, the Fourier coefficients a_n and b_n always approach zero at least as rapidly as α/n where α is a constant independent of n. If the function $f(x)$ is piecewise continuous, then either a_n or b_n, and in general both, decrease no faster than α/n. In general, if $f(x)$ and its first $k-1$ derivatives satisfy the conditions of the Fourier theorems I and II, then the Fourier coefficients a_n and b_n approach zero as $n \to \infty$ at least as rapidly as α/n^{k+1}. Moreover, if $f^{(k)}(x)$ is not everywhere continuous, then either a_n or b_n, and in general both, approach zero no faster than α/n^{k+1}.*

This theorem implies that the smoother the function f is, the faster its Fourier series converges. It should be noted that the Fourier series for two– and three–dimensional functions are similar to the above analysis for one–dimensional functions.

Proofs of these theorems are available in Davis (1963), Churchill and Brown (1978), and Walker (1988).

DEFINITION 4.8. Let f be a function defined on the interval $0 \le x \le L$ such that the integrals

$$\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx, \quad n = 1, 2, \cdots,$$

exist. Then the series

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \tag{4.25}$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx, \quad (n = 1, 2, \cdots), \tag{4.26}$$

is called the *Fourier sine series* of f 0n the interval $0 \le x \le L$.

Note that the series expansion (4.25) is identical to the trigonometric Fourier series (4.15) of an *odd* function defined on the interval $-L \le x \le L$, which coincides with $f(x)$ on the interval $0 \le x \le L$.

DEFINITION 4.9. Let f be a function defined on the interval $0 \le x \le L$ such that the integrals

$$\int_0^L f(x) \cos \frac{n\pi x}{L}\, dx, \quad n = 0, 1, 2, \cdots,$$

exist. Then the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \tag{4.27}$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx, \quad (n = 0, 1, 2, \cdots), \tag{4.28}$$

is called the *Fourier cosine series* of f on the interval $0 \le x \le L$.

Note that the series expansion (4.27) is identical to the trigonometric Fourier series (4.15) of an *even* function defined on the interval $-L \le x \le L$, which coincides with $f(x)$ on the interval $0 \le x \le L$.

EXAMPLE 4.10. To find the Fourier cosine series of period 2π which represents $f(x) = x$ on the interval $0 < x < \pi$, let $L = \pi$ in (4.28). Then

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^{\pi} x\, dx = \pi, \\ a_n &= \frac{2}{\pi} \int_0^{\pi} x \cos nx\, dx = \frac{2}{\pi} \left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right] \\ &= \frac{2(\cos n\pi - 1)}{\pi n^2} = \begin{cases} 0, & \text{if } n \text{ is even,} \\ -\dfrac{4}{\pi n^2}, & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Hence the Fourier cosine series is

$$x = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n-1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)x.$$

The graph of the right side of this series is given in Fig. 4.3. Note that at $x = 0$, the right side of the series equals

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi}{2} - \frac{4}{\pi}\frac{\pi^2}{8} = 0,$$

since $\sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. At $x = \pi$, the right side of the above series is obviously equal to $\frac{\pi}{2} + \frac{4}{\pi}\frac{\pi^2}{8} = \pi$. ∎

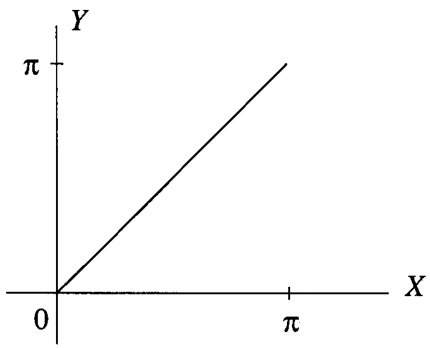


Fig. 4.3.

[www] **plotfourier.ma**, available in the Mathematica Notebook, generates a table of the elements of a trigonometric Fourier series and plots their graphs for a given function.

EXAMPLE 4.11. In the case of a jump discontinuity the Fourier series leads to what is known as the Gibbs phenomenon. Consider, e.g., the Fourier sine series for

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ -1, & -1 < x < 0. \end{cases}$$

Then, the coefficient b_n, defined by (4.23) are given by

$$b_n = \sum_{n=0}^{\infty} \frac{4}{2n+1} \sin(2n+1)\pi x.$$

The partial sums

$$M_k = \sum_{n=0}^{k} \frac{4}{2n+1} \sin(2n+1)\pi x$$

define the k harmonics, which approximate the jump as shown in Fig. 4.4. Notice the sharp peaks in the harmonics M_1, M_2, M_{10}, M_{20}, and M_{40} near 0 which is the discontinuity of $f(x)$. Gibbs showed that the height or overshoot of these peaks is greater than $f(0+)$ by about 9%. The width of the overshoot goes to zero as $k \to \infty$, but the height remains at 9% both at the top and the bottom such that

$$\lim_{k\to\infty} \max |f(x) - M_k(x)| \neq 0.$$

This phenomenon does not go away even when the number of harmonics is increased. ∎

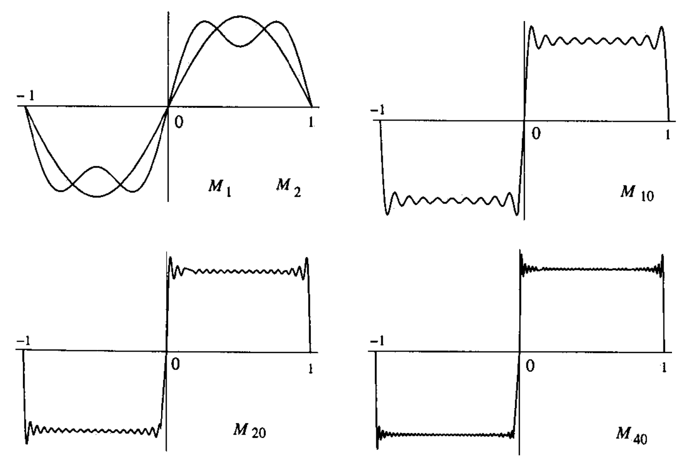


Fig. 4.4. Gibbs phenomenon.

4.5. Eigenfunction Expansions

The Sturm–Liouville problems arise in the solution of boundary value problems when one uses the method of separation of variables. This method, discussed in detail in the next chapter, is one of the most useful methods in solving boundary value problems involving partial differential equations. A Sturm–Liouville problem consists of the Sturm–Liouville equation

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda\, w(x)]\, y = 0, \tag{4.29}$$

which is a linear second order ordinary differential equation defined on a given interval $a \le x \le b$ and satisfies the boundary conditions of the form

$$\begin{aligned} a_1\, y(a) + b_1\, y'(a) &= 0, \\ a_2\, y(b) + b_2\, y'(b) &= 0, \end{aligned} \tag{4.30}$$

where λ is a real parameter, and a_1, a_2, b_1, b_2 are given real constants such that a_1 and b_1, or a_2 and b_2 are both not zero. It is obvious that (4.29)–(4.30) always has a trivial solution $y = 0$. The nontrivial solutions of this problem are called the *eigenfunctions* $\phi_n(x)$ and the corresponding values of λ the *eigenvalues* λ_n of the problem. The pair (ϕ_n, λ_n) is known as the *eigenpair*.

THEOREM 4.4. *Let the functions p, q, r and p′ in Eq (4.29) be real-valued and continuous on the interval $a \le x \le b$. Let $\phi_m(x)$ and $\phi_n(x)$ be the eigenfunctions of the problem (4.29)–(4.30) with corresponding eigenvectors λ_m and λ_n, respectively, such that $\lambda_m \ne \lambda_n$. Then*

$$\int_a^b \phi_m(x)\, \phi_n(x)\, w(x)\, dx = 0, \quad m \ne n, \tag{4.31}$$

i.e., the eigenfunctions ϕ_m and ϕ_n are orthogonal with respect to the weight function $w(x)$ on the interval $a \le x \le b$.

Proof of this theorem can be found in any standard textbook on ordinary differential equations, e.g., Ross (1964), Boyce and DiPrima (1992).

DEFINITION 4.10. The boundary conditions of the type

$$y(a) = y(b), \quad y'(a) = y'(b) \tag{4.32}$$

are known as the *periodic* boundary conditions. In this case the solution is of period $b - a$.

The eigenfunction expansion of an arbitrary function $f(x)$ in the interval $a \le x \le b$ is given by

$$f(x) = \sum_{n=1}^{\infty} c_n\, \phi_n(x), \tag{4.33}$$

where ϕ_n are the eigenfunctions, with corresponding eigenvalues λ_n of the Sturm–Liouville (or eigenvalue) problem

$$\frac{d^2y}{dx^2} + \lambda^2\, y = 0, \tag{4.34}$$

subject to the boundary conditions (2.25), and the coefficients c_n are determined by (4.12).

EXAMPLE 4.12. The set of orthogonal functions 1, $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, $\cdots$ of Example 4.3 are the eigenfunctions of the eigenvalue problem

$$y'' + \lambda\, y = 0, \quad y(-\pi) = y(\pi), \quad y'(-\pi) = y'(\pi).$$

The corresponding eigenvalues are $\lambda_n = n$, $(n = 1, 2, \cdots)$. ■

EXAMPLE 4.13. For the eigenvalue problem (4.34) defined on the interval $0 \le x \le L$, and (a) subject to the Dirichlet boundary conditions $y(0) = 0 = y(L)$, the eigenpair is

$$\phi_n(x) = \sin \lambda_n x, \quad \lambda_n = \frac{n\pi}{L}, \quad (n = 1, 2, \cdots);$$

and (b) with the Neumann boundary conditions $y'(0) = 0 = y'(L)$, the the eigenpair is

$$\phi_n(x) = \cos \lambda_n x, \quad \lambda_n = \frac{n\pi}{L}, \quad (n = 0, 1, 2, \cdots). \blacksquare$$

WWW `eigenpair.ma`, available in the Mathematica Notebook, can be used to obtain the eigenvalues and eigenfunctions for a given boundary value problem.

Tables 4.1 and 4.2 at the end of this chapter provide the data for the solution of the eigenvalue problem with three types of boundary conditions (4.32) in the Cartesian and the polar cylindrical coordinates, respectively. Note that the Bessel equation in the polar cylindrical coordinates

$$\frac{d^2y}{dr^2} + \frac{1}{r}\frac{dy}{dr} + \left(\lambda^2 - \frac{\nu^2}{r^2}\right) y = 0, \quad 0 \le r \le a, \quad \nu \ge -\frac{1}{2}, \tag{4.35}$$

is a Sturm–Liouville equation.

4.6. Bessel Functions

The Bessel functions of order ν are the solutions of the ordinary differential equation

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2)\, y = 0, \quad \nu \geq 0, \tag{4.36}$$

which is known as the Bessel equation of order ν. Its regular solutions for each $\nu \geq 0$ are the Bessel functions of the first kind

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+\nu)!} \left(\frac{x}{2}\right)^{2n+\nu}, \quad -\infty < x < \infty. \tag{4.37}$$

These power series solutions are obtained by the Frobenius method, details of which are available in any standard book on ordinary differential equation, e.g., Ross (1964), Boyce and DiPrima (1992). The infinite series (4.37) is uniformly convergent and can be differentiated or integrated term–by–term. The differentiation and integration formulas are as follows:

$$\begin{aligned}
&J_\nu(-x) = (-1)^\nu J_\nu(x),\\
&\frac{d}{dx}[x^\nu J_\nu(x)] = x^\nu J_{\nu-1}(x),\\
&\frac{d}{dx}[x^{-\nu} J_\nu(x)] = -x^{-\nu} J_{\nu+1}(x),\\
&x\, J_\nu'(x) = \nu\, J_\nu(x) - x\, J_{\nu+1}(x),\\
&x\, J_\nu'(x) = -\nu\, J_\nu(x) + x\, J_{\nu-1}(x),\\
&\int_0^x t^\nu J_{\nu-1}(t)\, dt = x^\nu J_\nu(x),
\end{aligned} \tag{4.38}$$

where $' \equiv d/dx$. In particular,

$$J_0'(x) = -J_1(x),$$

and

$$\int_0^x t\, J_0(t)\, dt = x J_1(x).$$

The integral representation for $J_\nu(x)$ is

$$J_\nu(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin\theta - \nu\theta)\, d\theta, \tag{4.39}$$

and, in particular,

$$J_0(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin\theta)\, d\theta = \frac{2}{\pi} \int_0^{\pi/2} \cos(x \cos\theta)\, d\theta. \tag{4.40}$$

Note that J_{2k} is an even function and J_{2k+1} an odd function. Moreover, $J_0(0) = 1$, but $J_\nu(0) = 0$ for $\nu \geq 1$. In fact, J_ν has a zero of multiplicity ν at $x = 0$. The integral representation (4.39) shows that $\|J_\nu(x)\| \leq 1$ for all real x and $\nu \geq 0$.

www Plots of $J_\nu(x)$ and $J_\nu'(x)$ for $\nu = 0, 1, 2, 3$ are available in `bessel.ma` in the Mathematica Notebook.

From the graphs of J_0, J_1, J_2, J_3 and their derivatives, it is found that each J_ν decays for large x, and their zeros are almost evenly spread and interlaced. In fact, for real ν the functions $J_\nu(x)$ and $J_\nu'(x)$ each have countably many real zeros, all of which are simple except $x = 0$. For nonnegative ν, let the n–th positive zero of these functions be denoted by $\alpha_{\nu,n}$ and $\alpha_{\nu,n}'$; then the zeros interlace according to the inequalities

$$\begin{gathered}
\alpha_{\nu,1} < \alpha_{\nu+1,1} < \alpha_{\nu,2} < \alpha_{\nu+1,2} < \alpha_{\nu,3} < \cdots,\\
\nu < \alpha_{\nu,1}' < \alpha_{\nu+1,1}' < \alpha_{\nu,2}' < \alpha_{\nu+1,2}' < \alpha_{\nu,3}' < \cdots,
\end{gathered}$$

i.e., each J_ν possesses an increasing unbounded sequence of positive zeros. In fact, all zeros of J_ν are real for $\nu \geq -1$. But for $\nu < -1$ and ν not an integer, the number of complex zeros of J_ν is twice the integer part of $(-\nu)$. If the integer part of $(-\nu)$ is odd, two of these zeros lie on the imaginary axis. If $\nu \geq 0$, all zeros of J_ν are real (Abramowitz and Stegun, 1965, p. 372).

By taking the limit of the integral representation (4.40) as $x \to \infty$, it can be shown that

$$\lim_{x\to\infty} J_\nu(x) = 0.$$

In fact, in view of the Riemann–Lebesgue lemma which states that $\lim_{\kappa\to\infty} \int_{-\pi}^{\pi} F(x) \cos\kappa x\, dx = \lim_{\kappa\to\infty} \int_{-\pi}^{\pi} F(x) \sin\kappa x\, dx = 0$, we get

$$\begin{aligned}
\lim_{x\to\infty} J_0(x) &= \lim_{x\to\infty} \frac{2}{\pi} \int_0^{\pi/2} \cos(x \cos\theta)\, d\theta\\
&= -\lim_{x\to\infty} \frac{2}{\pi} \int_0^1 \frac{\cos(xy)}{\sqrt{1-y^2}}\, dy = 0.
\end{aligned}$$

Similarly,

$$J_\nu(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta - \nu\theta)\,d\theta$$
$$= \frac{1}{\pi}\int_0^\pi [\cos\nu\theta\,\cos(x\sin\theta) + \sin\nu\theta\,\sin(x\sin\theta)]\,d\theta$$

tends to zero as $x \to \infty$.

EXAMPLE 4.14. Consider the polar cylindrical form of the Laplace operator

$$L \equiv \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}$$

in R^1. Then the corresponding eigenvalue problem is $L\phi + \lambda^2\phi = 0$. It can be shown that the radially symmetric eigenfunctions ϕ_k of the Laplace equation subject to the Dirichlet boundary condition $\phi(1) = 0$ on the unit disk U are

$$\phi_k = J_0(\lambda_k r), \quad (k = 1, 2, \ldots),$$

such that

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

are the positive zeros of J_0. These eigenfunctions form a complete orthogonal set, i.e., for $m \neq n$

$$\langle J_0(\lambda_m r), J_0(\lambda_n r)\rangle = \int_0^1 J_0(\lambda_m r)\,J_0(\lambda_n r)\,r\,dr = 0.$$

Then, for any function f with the norm

$$\|f\|^2 = \int_0^1 f(r)^2\,r\,dr < +\infty,$$

we have the *Fourier–Bessel expansion*

$$f(r) = \sum_{n=1}^\infty c_n\,J_0(\lambda_n r), \tag{4.41}$$

where

$$c_n = \frac{\langle f, J_0(\lambda_n r)\rangle}{\|J_0(\lambda_n r)\|^2},$$
$$\|J_0(\lambda_n r)\|^2 = \frac{J_1^2(\lambda_n r)}{2}. \blacksquare \tag{4.42}$$

In general, we have

THEOREM 4.4. *The eigenfunctions $\phi_n = J_\nu(\lambda_n r)$, $(n = 1, 2, \cdots)$, form a complete orthogonal set of radially symmetric square integrable functions such that for a square integrable function f the Fourier–Bessel expansion*

$$f(r) = \sum_{n=1}^\infty c_n\,J_\nu(\lambda_n r) \tag{4.43}$$

holds, where

$$c_n = \frac{\langle f, J_\nu(\lambda_n r)\rangle}{\|J_\nu(\lambda_n r)\|^2}, \tag{4.44}$$

and λ_n are the positive zeros of J_ν for $n = 1, 2, \cdots$, and

$$\begin{aligned}\|J_\nu(\lambda_n r)\|^2 &= \int_0^1 J_\nu^2(\lambda_n r)\,r\,dr\\ &= \frac{[J_\nu'(\lambda_n)]^2}{2}\\ &= \frac{J_{\nu+1}^2(\lambda_n)}{2}\\ &= \frac{J_{\nu-1}^2(\lambda_n)}{2}.\end{aligned} \tag{4.45}$$

THEOREM 4.5. *The eigenfunctions for the Laplace operator with zero Dirichlet condition on the unit disk $U = \{r \leq 1\}$ are*

$$J_0(\lambda_n r), \quad J_\nu(\lambda_n r)\cos\nu\theta, \quad J_\nu(\lambda_n r)\sin\nu\theta, \tag{4.46}$$

$n = 1, 2, \cdots$, and $\nu \geq 1$, with the eigenvalues λ_n which are the positive zeros of J_ν. These eigenfunctions form a complete orthogonal basis in the Hilbert space $L^2(U)$ of all square integrable functions on U.

Proofs of these theorems can be found in Watson (1944).

For the three types of boundary conditions the eigenpairs are defined in Table 4.2 at the end of this chapter. Hence, a function $f(r,\theta)$ defined on the set $U \times (0, 2\pi)$ has an eigenfunction expansion of the form

$$f(r,\theta) = \sum_{n=0}^\infty [f_n(r)\cos n\theta + g_n(r)\sin n\theta], \tag{4.47}$$

where each of the functions f_n and g_n has an expansion of the form (4.43).

4.7. Exercises

Show that each given set of functions is orthogonal on the given interval, and determine the corresponding orthonormal set of functions:

4.1. $\left\{\sin \frac{n\pi x}{l}\right\}, n=1,2,3,\cdots; \quad -l \le x \le l$

4.2. $\left\{\cos \frac{2n\pi x}{l}\right\}, n=0,1,2,\cdots; \quad -l \le x \le l$

4.3. $\{\sin 2nx\}, n=1,2,3,\cdots; \quad 0 \le x \le \pi$

4.4. $\{\cos 2nx\}, n=1,2,3,\cdots; \quad 0 \le x \le \pi$

4.5. $\{\sin 3nx\}, n=0,1,2,\cdots; \quad -\pi \le x \le \pi$

4.6. $\{\cos 3nx\}, n=0,1,2,3,\cdots; \quad -\pi \le x \le \pi$

4.7. $\{\sin 2nx, \cos 2nx\}, n=1,2,3,\cdots; \quad |x| \le \pi$

4.8. Find the Fourier series for the function $f(x)$ which is assumed to have the period 2π:

(a) $f(x) = \begin{cases} 1 & \text{if } -\pi/2 < x < \pi/2 \\ -1 & \text{if } \pi/2 < x < 3\pi/2. \end{cases}$

(b) $f(x) = x, \quad -\pi < x < \pi.$

(c) $f(x) = x^2, \quad -\pi < x < \pi.$

ANS. (a) $\frac{4}{\pi}\left(\cos x - \frac{1}{3}\cos 3x - + \frac{1}{5}\cos 5x + \cdots\right)$.
(b) $2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + - \cdots\right)$.
(c) $\frac{\pi^2}{3} - 4\left(\cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x - \frac{1}{16}\cos 4x + - \cdots\right)$.

4.9. Find the Fourier series of the period function $f(x)$ of period T:

(a) $f(x) = \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1, \quad T = 2. \end{cases}$

(b) $f(x) = 1 - x^2, \quad -1 < x < 1, \quad T = 2.$

ANS. (a) $\frac{4}{\pi}\left(\sin \pi x + \frac{1}{3}\sin 3\pi x + \frac{1}{5}\sin 5\pi x + \cdots\right)$.
(b) $\frac{2}{3} + \frac{4}{\pi^2}\left(\cos \pi x - \frac{1}{4}\cos 2\pi x + \frac{1}{9}\cos 3\pi x - + \cdots\right)$.

4.10. Find the trigonometric Fourier series of the function $f(x) = x$, $-4 \le x \le 4$.

ANS. Note that $f(x)$ is an odd function, which implies $a_n = 0$; and $L = 4$. Then

$$b_n = \frac{1}{2}\int_0^4 x \sin\frac{n\pi x}{4}\,dx = -\frac{8}{n\pi}\cos n\pi = -\frac{8(-1)^{n+1}}{n\pi}, \quad (n = 1, 2, \cdots),$$

and the series is

$$x = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{4}.$$

4.11. Find the trigonometric Fourier series of the function

$$f(x) = \begin{cases} \pi, & -\pi \le x < 0, \\ x, & 0 \le x \le \pi. \end{cases}$$

ANS. Note that $L = \pi$. The function f is neither odd nor even. Then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{3\pi}{2},$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\frac{\cos n\pi - 1}{n^2}\right] = \begin{cases} -\frac{2}{\pi n^2}, & \text{for odd } n, \\ 0, & \text{for even } n, \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = -\frac{1}{n}, \quad (n = 0, 1, 2, \cdots).$$

The series is

$$f(x) = \frac{3\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{\pi n^2} \cos nx - \frac{1}{n} \sin nx \right].$$

4.12. Find the trigonometric Fourier series of the function

$$g(x) = \begin{cases} \pi, & -\pi \le x < 0, \\ \dfrac{\pi}{2}, & x = 0, \\ x, & 0 \le x \le \pi. \end{cases}$$

HINT: The function $g(x)$ is the same as $f(x)$ in Exercise 4.11, except at $x = 0$. Since these two functions have the same values at all points except a finite number (only one in this case) in the same interval, the function $g(x)$ has the same Fourier series as that for $f(x)$ in Exercise 4.11.

4.13. Find the trigonometric Fourier series of the function $f(x) = a^2 - x^2$, $0 \le x \le a$.
ANS.

$$a^2 - x^2 = \frac{2}{3}a^2 - \frac{4a^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi x}{a}.$$

4.14. Show that

$$\int_0^{2\pi} \sin^{2n} \theta \, d\theta = \frac{2\pi (2n)!}{(n! 2^n)^2}.$$

HINT: Use induction, or the result

$$\int_0^{\pi/2} \cos^m \theta \sin^n \theta \, d\theta = \frac{\Gamma\left(\dfrac{m+1}{2}\right) \Gamma\left(\dfrac{n+1}{2}\right)}{2\Gamma\left(\dfrac{m+n}{2} + 1\right)}.$$

4.15. Show that

$$e^{x(t-1/t)/2} = \sum_{n=0}^{\infty} J_n(x) [t^n + (-1)^n t^{-n}].$$

HINT: Multiply the series expansions for $e^{xt/2}$ and $e^{-x/(2t)}$.

4.16. Show that

$$\cos x = J_0(x) + 2 \sum_{n=1}^{\infty} (-1)^n J_{2n}(x),$$
$$\sin x = 2 \sum_{n=1}^{\infty} (-1)^n J_{2n+1}(x).$$

HINT: set $t = i$ in Exercise 4.15.

4.17. Show that

$$\cos(x \sin\theta) = J_0(x) + 2 \sum_{n=1}^{\infty} J_{2n}(x) \cos 2n\theta,$$
$$\sin(x \sin\theta) = 2 \sum_{n=0}^{\infty} J_{2n+1}(x) \sin 2n\theta.$$

HINT: set $t = e^{i\theta}$ in Exercise 4.15.

4.18. From the results of Exercise 4.17, deduce that

$$J_{2n}(x) = \frac{1}{2\pi} \int_0^{\pi} \cos(x \sin\theta) \cos 2n\theta \, d\theta,$$
$$J_{2n+1}(x) = \frac{1}{2\pi} \int_0^{\pi} \sin(x \sin\theta) \sin 2n\theta \, d\theta.$$

4.19. Show that

$$\int_0^a J_\nu(y)^2 \, y \, dy = \frac{(a^2 - \nu^2) J_\nu(a)^2 + a^2 J'_\nu(a)^2}{2}.$$

HINT: Use the identity

$$x J'_\nu(x) \; (x J'_\nu(x))' = J'_\nu(x)[x^2 J''_\nu(x) + x J'_\nu(x)] = (\nu^2 - x^2) J_\nu(x) \, j'_\nu(x).$$

4.20. Show that if λ is a positive zero of $J_\nu(x)$, then (4.45) holds.
HINT: Set $x = ar$ and use Exercise 4.19.

Table 4.1: Eigenvalue Problem in Cartesian Coordinates

$\frac{d^2y}{dx^2} + \lambda^2 y = 0, 0 < x < L$, subject to the boundary conditions $a_1\, y(0) + b_1\, y'(0) = 0,\ a_2\, y(L) + b_2\, y'(L) = 0$.

Eigenfunction expansion: $f(x) = \sum_{n=1}^{\infty} c_n\, \phi_n(x)$, $c_n = \frac{1}{\|\phi_n\|^2} \int_0^L f(x)\phi_n(x)\, dx, \quad h_i = \frac{a_i}{b_i}, (i = 1, 2)$.

	Boundary conditions At $x = 0$	At $x = L$	ϕ_n	$\|\phi_n\|^2$	λ_n are the roots of
1.	Dirichlet $a_1 \neq 0, b_1 = 0$	Dirichlet $a_2 \neq 0, b_2 = 0$	$\sin \lambda_n x$	$\frac{L}{2}$	$\sin \lambda L = 0$, i.e., $\lambda_n = \frac{n\pi}{L}$, $(n = 0, 1, 2, \cdots)$
2.	Dirichlet $a_1 \neq 0, b_1 = 0$	Neumann $a_2 = 0, b_2 \neq 0$	$\sin \lambda_n x$	$\frac{L}{2}$	$\cos \lambda L = 0$, i.e., $\lambda_n = \frac{(2n-1)\pi}{2L}$, $(n = 1, 2, \cdots)$
3.	Dirichlet $a_1 \neq 0, b_1 = 0$	Robin $a_2 \neq 0, b_2 \neq 0$	$\sin \lambda_n x$	$\frac{\lambda_n L - \sin \lambda_n L \cos \lambda_n L}{2\lambda_n}$	$\lambda + h_2 \tan \lambda L = 0$ *
4.	Neumann $a_1 = 0, b_1 \neq 0$	Dirichlet $a_2 \neq 0, b_2 = 0$	$\cos \lambda_n x$	$\frac{L}{2}$	$\cos \lambda L = 0$, i.e., $\lambda_n = \frac{(2n-1)\pi}{L}$, $(n = 1, 2, \cdots)$
5.	Neumann $a_1 = 0, b_1 \neq 0$	Neumann $a_2 = 0, b_2 \neq 0$	$\cos \lambda_n x$	$\frac{L}{2}$ **	$\sin \lambda L = 0$, i.e., $\lambda_n = \frac{n\pi}{L}$, $(n = 0, 1, 2, \cdots)$
6.	Neumann $a_1 = 0, b_1 \neq 0$	Robin $a_2 \neq 0, b_2 \neq 0$	$\cos \lambda_n x$	$\frac{\lambda_n L + \sin \lambda_n L \cos \lambda_n L}{2\lambda_n}$	$\lambda \tan \lambda L = h_2$
7.	Robin $a_1 \neq 0, b_1 \neq 0$	Dirichlet $a_2 \neq 0, b_2 = 0$	$\sin \lambda_n(L - x)$	$\frac{\lambda_n L - \sin \lambda_n L \cos \lambda_n L}{2\lambda_n}$	$\lambda \cot \lambda L = h_1$
8.	Robin $a_1 \neq 0, b_1 \neq 0$	Neumann $a_2 = 0, b_2 \neq 0$	$\cos \lambda_n(L - x)$	$\frac{\lambda_n L + \sin \lambda_n L \cos \lambda_n L}{2\lambda_n}$	$\lambda \tan \lambda L = -h_1$
9.	Robin $a_1 \neq 0, b_1 \neq 0$	Robin $a_2 \neq 0, b_2 \neq 0$	$\lambda_n \cos \lambda_n x - h_1 \sin \lambda_n x$	$\frac{1}{2}\left[(\lambda_n^2 + h_1^2)\left(L + \frac{h_2}{\lambda_n^2 + h_2^2}\right) - h_1\right]$	$\tan \lambda L = -\frac{\lambda(h_1 - h_2)}{\lambda^2 + h_1 h_2}$ ***

* If $L = -\frac{b_2}{a_2} > 0$, then $\lambda_0 = 0$ is an eigenvalue with $\phi_0 = x$.

** Replace L by $2L$ for $n = 0$.

*** If $L = \frac{1}{h_1} - \frac{1}{h_2} > 0$, then $\lambda_0 = 0$ is an eigenvalue, with $\phi_0 = x - \frac{1}{h_1}$.

Table 4.2: Eigenvalue Problem in Polar Cylindrical Coordinates

$$\frac{d^2y}{dr^2}+\frac{1}{r}\frac{dy}{dr}+\left(\lambda^2-\frac{\nu^2}{r^2}\right)y=0,\quad 0<r\le a,$$ subject to the boundary conditions shown below, with ν and h real constants, $\nu \ge -\frac{1}{2}$.

	Boundary conditions at $r=a$	$\phi_n(\lambda_n r)$	$\Vert\phi_n\Vert^2$	λ_n are the roots of
1.	Dirichlet $[y(a)=0]$	$J_\nu(\lambda_n r)$	$\frac{a^2 J_\nu'^2(\lambda_n a)}{2}$	$J_\nu(\lambda a)=0$
2.	Neumann $\left[\frac{dy}{dx}(a)=0\right]$	$J_\nu(\lambda_n r)$	$\frac{(a^2\lambda_n^2-\nu^2)J_\nu^2(\lambda_n a)}{2\lambda_n^2}$	$J_\nu'(\lambda_n a)=0$ †
3.	Robin $\left[\frac{dy}{dx}(a)+hy(a)=0\right]$	$J_\nu(\lambda_n r)$	$\frac{[a^2(h^2+\lambda_n^2)-\nu^2)]J_\nu^2(\lambda_n a)}{2\lambda_n^2}$	$\lambda J_\nu'(\lambda a)+hJ_\nu(\lambda a)=0$

† $\lambda_0=0$ is also an eigenvalue for $\nu=0$; then $\phi_0=1$ and $\Vert\phi_0\Vert^2=\frac{a^2}{2}$.

5

Separation of Variables

The method of separation of variables is a well–established technique for solving ordinary differential equations. This method is easily adaptable to almost all linear homogeneous partial differential equations with constant coefficients in canonical form, and exhibits the power of the superposition principle to construct the general solution of such equations. Since linear first order partial differential equations can always be solved by the method of characteristics, the method of separation of variables is usually applied to solve higher order partial differential equations. The basic idea of this method is to transform a partial differential equation into as many ordinary differential equations as the number of independent variables in the partial differential equation by representing the solution as a product of functions of each independent variable. After these ordinary differential equations are solved, the method reduces to solving eigenvalue problems and constructing the general solution as an eigenfunction expansion where the coefficients are evaluated by using the boundary conditions and the initial conditions. In most cases the solution is written in terms of a series of orthogonal functions.

5.1. Introduction

Consider the partial differential equation

$$a\,u_{xx}+b\,u_{xy}+c\,u_{yy}+e\,u_x+f\,u_y+g\,u=0. \tag{5.1}$$

The first step in the general technique is to eliminate the term with mixed partial derivatives by introducing a new set of coordinates x', y' (called *characteristic coordinates*, see §2.7). Thus, we have

$$a_1\,u_{x'x'} + c_1\,u_{y'y'} + e_1\,u_{x'} + f_1\,u_{y'} + g_1\,u = 0. \tag{5.2}$$

We now assume a solution of the form $u = X(x')Y(y')$ in (5.2), and obtain

$$a_1X''Y + CXY'' + e_1X'Y + f_1XY' + g_1XY = 0,$$

or, formally,

$$\frac{L_1\,(D_x)\,X}{X} + \frac{L_2\,(D_y)\,Y}{Y} + g_1 = 0, \tag{5.3}$$

where $L_1\,(D_x)$ is a linear differential operator in x' and $L_2\,(D_y)$ is a linear differential operator in y'. Since the first term in (5.3) is a function of x' only and the second term is a function of y' only, while the third term is a constant, the only way Eq (5.3) can be solved is if each of the first two terms is also constant, thus

$$\frac{L_1X}{X} = \lambda, \quad \frac{L_2Y}{Y} = \mu, \quad \text{such that } \lambda + \mu + g_1 = 0.$$

We shall explain this method by some examples.

5.2. Hyperbolic Equations

EXAMPLE 5.1. The problem of a vibrating string is defined by the one–dimensional wave equation (§1.4). Consider the boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \tag{5.4}$$

$$u(0,t) = 0 = u(l,t), \quad t \ge 0, \tag{5.5}$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \le x \le l, \tag{5.6}$$

where $f \in C^1$ is a given function. We seek the solution of the form

$$u(x,t) = X(x)T(t), \tag{5.7}$$

where X is a function of x only and T a function of t only. We assume here that a solution of the form (5.7) exists. Sometimes this method

requires some modifications, as in Example 5.3. We shall, however, carry out the details to see if the method works for this problem. Note that

$$\frac{\partial^2 u}{\partial t^2} = X\,T'', \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = X''\,T,$$

where the primes denote the derivative with respect to its corresponding independent variable. The Eq (5.4) reduces to

$$X\,T'' = c^2X''\,T,$$

or, after separating the variables, it becomes

$$\frac{T''}{T} = c^2\frac{X''}{X}. \tag{5.8}$$

In Eq (5.8) we have been able to separate the variables. It is only at this stage in the development of this method that we may either continue or abandon the method depending on whether or not we are successful in separating the variables.

Let us now fix t, and let x vary over the interval $0 < x < l$. The only situation where $X(x) = T(t)$ for all x and t is when $X(x) = T(t) =$ const. Hence, from (5.8) we can write

$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} = k, \quad k = \text{const.} \tag{5.9}$$

Thus, the set of equations (5.9) is equivalent to two ordinary differential equations:

$$T'' - k\,c^2\,T = 0, \tag{5.10}$$

$$X'' - k\,X = 0. \tag{5.11}$$

Since the constant k is arbitrary, it is necessary for k to have the same value for Eqs (5.10) and (5.11) in order that Eq (5.9) be satisfied. The general solution of Eq (5.10) is

$$T(t) = \begin{cases} c_1e^{c\sqrt{k}t} + c_2e^{-c\sqrt{k}t} & \text{for } k > 0 \\ c_1t + c_2 & \text{for } k = 0 \\ c_1\cos c\sqrt{-k}t + c_2\sin c\sqrt{-k}t & \text{for } k < 0, \end{cases} \tag{5.12}$$

and of Eq (5.11) is

$$X(x) = \begin{cases} d_1e^{\sqrt{k}x} + d_2e^{-\sqrt{k}x} & \text{for } k > 0 \\ d_1x + d_2 & \text{for } k = 0 \\ d_1\cos\sqrt{-k}x + d_2\sin\sqrt{-k}x & \text{for } k < 0. \end{cases} \tag{5.13}$$

In view of the boundary conditions (5.5) we must have

$$X(0)T(t) = 0 = X(l)T(t) \quad \text{for all } t \geq 0. \tag{5.14}$$

Using these conditions in (5.13) for $k > 0$ we get the system of equations

$$\begin{aligned} X(0) &= d_1 + d_2 = 0, \\ X(l) &= d_1 e^{\sqrt{k}l=} + d_2 e^{-\sqrt{k}l} = 0. \end{aligned} \tag{5.15}$$

The system (5.15) is consistent, i.e., it has a nontrivial solution, iff the determinant of its coefficients vanishes. But since

$$\det \begin{vmatrix} 1 & 1 \\ e^{\sqrt{k}l} & e^{-\sqrt{k}l} \end{vmatrix} = e^{-\sqrt{k}l} - e^{\sqrt{k}l} \neq 0,$$

a nonzero solution for $X(x)$ in (5.13) for $k > 0$ is not possible. Next, for $k = 0$, the boundary conditions (5.14) imply that $d_1 = 0$ and $d_2 = 0$. Hence there is no nonzero solution for $k = 0$. Finally, for $k < 0$, let us set $k = -\lambda^2$. Then the general solution (5.13) in this case becomes

$$X(x) = d_1 \cos \lambda x + d_2 \sin \lambda x,$$

which under the boundary conditions (5.14) yields

$$X(0) = d_1 = 0, \quad \text{and} \quad X(l) = d_2 \sin \lambda l = 0.$$

In order to avoid a trivial solution in this case, we choose λ such that λl is a positive multiple of π, i.e., $\lambda l = n\pi$, or $\lambda = \dfrac{n\pi}{l}$. The positive values of λ are chosen because the negative multiples give the same eigenfunctions as the positive ones. This result leads to an infinite set of solutions which are denoted by

$$X_n(x) = d_{2,n} \sin \frac{n\pi x}{l},$$

where each solution corresponds to the eigenvalue

$$k = -\frac{n^2\pi^2}{l^2}. \tag{5.16}$$

The solutions for $T(t)$ for the choice of $k < 0$, as in (5.16), are then obtained from (5.12) as

$$T_n(t) = c_{1,n} \cos \frac{n\pi ct}{l} + c_{2,n} \sin \frac{n\pi ct}{l}.$$

In view of (5.7), then the infinite set of solutions is

$$u_n(x,t) = X_n(x)T_n(t) = \left[A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l}\right] \sin \frac{n\pi x}{l}, \tag{5.17}$$

where the constants A_n and B_n are determined from the boundary conditions. The eigenfunctions are contained in the solution (5.17), whereas the eigenvalues for this boundary value problem are given by (5.16).

The next step is to obtain the particular solution which satisfies the initial conditions (5.6). At this point it may so happen that no one solution in (5.17) will satisfy (5.6). In view of the superposition principle (see §1.5), any finite sum of the solutions (5.17) is also a solution of this boundary value problem. We should, therefore, find a linear combination of these solutions which also satisfies the initial conditions (5.6). Even if this technique fails, we can always try an infinite series of solutions (5.17), i.e.,

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) = \sum_{n=1}^{\infty} \left[A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l}\right] \sin \frac{n\pi x}{l}. \tag{5.18}$$

(For convergence of the series (5.18), see Chapter 4.) But we can take this series expansion formally and verify that the boundary conditions (5.5) are still satisfied. We shall now satisfy the initial conditions (5.6) and thereby obtain the solution of this problem. Using the first of the initial conditions (5.6) we get

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} = f(x). \tag{5.19}$$

Since $f \in C^1$, the infinite series (5.19) is a Fourier sine series. Hence $f(x)$ can be regarded as an odd function with period $2l$. Thus, we can expand this function $f(x)$ on the interval $0 \leq x \leq l$ such that $f(-x) = -f(x)$ on the interval $-l \leq x \leq 0$, and $f(x + 2l) = f(x)$ for all x, where $f(0) = 0$. Then the coefficients A_n for $n = 1, 2, \ldots$ are given by

$$A_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l}\, dx = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx, \tag{5.20}$$

where the last integral representation holds because both $f(x)$ and $\sin \dfrac{n\pi x}{l}$ are odd and their product is even. Then, taking the derivative of (5.18), we get

$$u_t = \frac{\pi c}{l} \sum_{n=1}^{\infty} n \left[B_n \cos \frac{n\pi ct}{l} - A_n \sin \frac{n\pi ct}{l} \right] \sin \frac{n\pi x}{l}, \tag{5.21}$$

which, in view of the second of the initial conditions (5.6), gives

$$u_t(x,0) = \frac{\pi c}{l} \sum_{n=1}^{\infty} n B_n \sin \frac{n\pi x}{l} = g(x),$$

where

$$B_n = \frac{2}{n\pi l} \int_0^l g(x) \sin \frac{n\pi x}{l}\, dx, \quad n = 1, 2, \dots . \tag{5.22}$$

Hence, the solution (5.18) is completely determined.

We shall now derive the d'Alembert solution for this problem. From (5.18) we have

$$\begin{aligned} u &= \frac{1}{2} \sum_{n=1}^{\infty} A_n \left\{ \sin \frac{n\pi(x+ct)}{l} + \sin \frac{n\pi(x-ct)}{l} \right\} \\ &\quad + \frac{1}{2} \sum_{n=1}^{\infty} B_n \left\{ \cos \frac{n\pi(x-ct)}{l} - \cos \frac{n\pi(x+ct)}{l} \right\} \\ &= \frac{1}{2} \big[f(x+ct) + f(x-ct) \big] + \frac{1}{2} \big[-G(x+ct) + G(x-ct) \big], \end{aligned} \tag{5.23}$$

where

$$f(z) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi z}{l},$$

as in (5.19), and

$$G(z) = \sum_{n=1}^{\infty} B_n \cos \frac{n\pi z}{l}.$$

Since

$$G'(z) = -\frac{1}{c} \frac{\pi c}{l} \sum_{n=1}^{\infty} n B_n \sin \frac{n\pi z}{l} = -\frac{1}{c} g(z),$$

we obtain from (5.23) the formal solution, known as the d'Alembert solution for this problem, as

$$u(x,t) = \phi(x+ct) + \psi(x-ct), \tag{5.24}$$

where c is the wave velocity, and

$$\begin{aligned} \phi(x+ct) &= \frac{1}{2} \big[F(x+ct) - G(x+ct) \big], \\ \psi(x-ct) &= \frac{1}{2} \big[F(x-ct) + G(x-ct) \big]. \end{aligned}$$

An interpretation of the solution of this problem is as follows: At each point x_0 of the string

$$u(x_0, t) = \sum_{n=1}^{\infty} n\alpha_n \cos \frac{n\pi c}{l}(t + \delta_n) \sin \frac{n\pi c x_0}{l}.$$

This equation describes a harmonic motion with amplitudes $\alpha_n \sin \frac{n\pi c x_0}{l}$. The points where $\sin \frac{n\pi c x_0}{l} = 0$, i.e., $x = \frac{ml}{n}$ $(m = 1, 2, \dots, n-1)$, remain fixed during the entire process; these points are called the *nodes* of the standing wave. But the points where $\sin \frac{n\pi c x_0}{l} = \pm 1$, i.e., $x = \frac{(2m+1)l}{2n}$, vibrate with the maximum amplitude α_n. These points are called the *maxima* of the standing wave. For any t the structure of the standing wave is described by

$$u(x,t) = \sum_{n=1}^{\infty} C_n(t) \sin \frac{n\pi c x_0}{l},$$

where

$$C_n(t) = \alpha_n \cos \omega_n (t + \delta_n), \quad \omega_n = \frac{n\pi c}{l}.$$

For those times t when $\cos \omega_n (t + \delta_n) = \pm 1$, the displacement reaches its maximum value where the velocity becomes zero. ∎

We shall illustrate an example by the following Mathematica session.

In[1]:=

```
L:= 1
c:= 1
```

```
f[x_]:= Sin[x]
g[x_]:= x^2+1
A[n_]:= A[n] = 2/(n Pi c) NIntegrate[g[x] Sin[n Pi x/L],
   {x, 0, L}]//N//Chop
B[n_]:= B[n] = 2/L NIntegrate[f[x] Sin[n Pi x/L],{x,0,L}]
   //Chop
Table[{n, A[n], B[n]}, {n,1,8}]//TableForm
```

Out[7]=

```
{ 1,  0.607927,   0.596094}
{ 2,  -0.0506606,  -0.27481}
{ 3,  0.0675475,   0.180599}
{ 4,  -0.0126651,  -0.134778}
{ 5,  0.0243171,   0.107575}
{ 6,  -0.00562895,  -0.0895348}
{ 7,  0.0124067,   0.0766867}
{ 8,  -0.00316629,  -0.0670683}
```

In[8]:=

```
u[x_,t_,n_] := (A[n] Sin[n Pi c t/L] + B[n] Cos[n Pi c t/L])
   Sin[n Pi x/L];

uapprox[x_,t_] := Sum[u[x,t,k],{k,8}]
```

In[10]:=

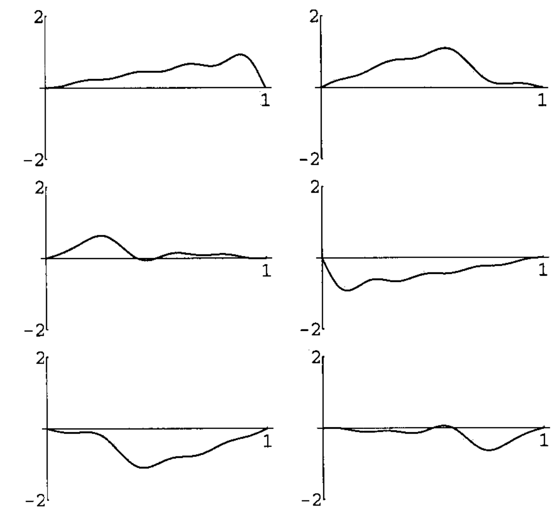
```
graphs =
Table[Plot[uapprox[x,t],{x,0,1},
PlotRange->{-2,2},
Ticks->{{0,1},{-2,2}},
DisplayFunction->Identity],
{t,0,2,1/3}];
```

In[11]:=

```
graphsarray = Partition[graphs,2];
```


In[12]:=

```
Show[GraphicsArray[graphsarray],
DisplayFunction->$DisplayFunction]
```




```
-GraphicsArray-
```

5.3. Parabolic Equations

EXAMPLE 5.2. Consider the one–dimensional heat conduction equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \tag{5.25}$$

subject to the boundary conditions

$$u(0,t) = 0 = u(l,t), \quad t \geq 0, \tag{5.26}$$

and the initial condition

$$u(x,0) = f(x), \quad 0 \leq x \leq l, \tag{5.27}$$

where $f \in C^1$ is a prescribed function. In physical terms, this problem represents the heat conduction in a rod when its ends are maintained at zero temperature while the initial temperature u at any point of the rod is prescribed as $f(x)$. Let us assume the solution in the form

$$u(x,t) = X(x)T(t),$$

which after substitution into Eq (5.25) yields the set of equations

$$\frac{1}{k}\frac{T'}{T} = \frac{X''}{X}. \tag{5.28}$$

As in Example 5.1, the only situation where these two expressions can be equal is for each of them to be constant, say each equal to c. Eq (5.28) then yields two ordinary differential equations

$$T' - ckT = 0, \tag{5.29}$$

$$X'' - cX = 0, \tag{5.30}$$

where the boundary conditions (5.26) reduce to

$$X(0)\,T(t) = 0 = X(l)\,T(t), \quad \text{or} \quad X(0) = 0 = X(l), \tag{5.31}$$

except for the case when the rod has zero initial temperature at every point. This situation, being uninteresting, can be neglected. As in the case of Example 5.1, we notice that for a nonzero solution of the problem (5.30)–(5.31) we must choose negative values of c. Hence we set $c = -\lambda^2$, and find that the eigenvalues $c = -n^2\pi^2/l^2$ have the corresponding eigenfunctions

$$X_n(x) = A_n \sin\frac{n\pi x}{l}.$$

Eq (5.29) then becomes

$$T' + \frac{kn^2\pi^2}{l^2}T = 0$$

whose general solution for each n is given by

$$T_n(t) = B_n e^{-kn^2\pi^2 t/l^2}.$$

Hence, we consider an infinite series of the form

$$u_n(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} e^{-kn^2\pi^2 t/l^2}. \tag{5.32}$$

Now, we use the initial condition (5.27) in (5.32) and obtain

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} = f(x), \tag{5.33}$$

which shows that $f(x)$ can be represented as a Fourier sine series, by extending f as an odd, piecewise continuous function of period $2l$ with piecewise continuous derivatives. Equation (5.33) gives the coefficients C_n as

$$C_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx, \; n = 1, 2, \dots. \tag{5.34}$$

Hence the solution (5.32) is completely determined for this problem. Note that the series in (5.33) converges since $u(x,0)$ does, and the exponential expression in (5.32) is less than 1 for each n and all $t > 0$ and approaches zero as $t \to \infty$. ∎

In[13]:=

```
L:= 1
k:= 1
f[x_]:= x^2
A[n_]:= A[n] = 2/(n Pi c) NIntegrate[g[x] Sin[n Pi x/L],
   {x, 0, L}]//N//Chop;

Table[{n, A[n]}, {n,1,8}]//TableForm
```

Out[17]=

```
{ 1,0.378607}
{ 2,-0.31831}
{ 3,0.202651}
{ 4,-0.159155}
{ 5,0.12526}
{ 6,-0.106103}
{ 7,0.0901935}
{ 8,-0.0795775}
```

In[18]:=

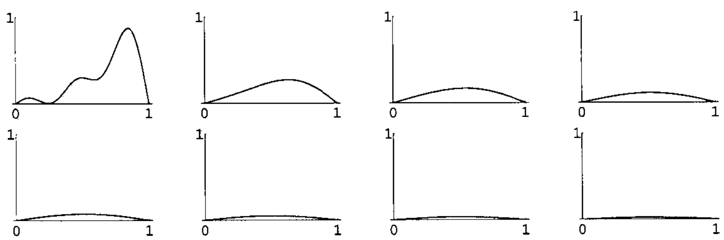
```
u[x_,t_,n_]:= A[n] Sin[n Pi x/L] Exp[-k t(n Pi/L)^2]

uapprox[x_,t_]:= Sum[u[x,t,j],{j,5}]
```

In[19]:=

```
graphs = Table[Plot[uapprox[x,t],{x,0,1},
PlotRange->{0,1},
Ticks->{{0,1},{0,1}},
DisplayFunction->Identity],
{t,0,1/3,1/24}];
```

In[20]:=

```
graphsarray = Partition[graphs,2];

Show[GraphicsArray[graphsarray],
DisplayFunction->$DisplayFunction]
```



```
-GraphicsArray-
```

We shall now present a Mathematica session with the nonhomogeneous initial condition

$$u(x,0) = \begin{cases} x^2 & 0 < x < 1/2 \\ x+1 & 1/2 < x < 1. \end{cases}$$

In[21]:=


```
L:= 1
k:= 1
f[x_]:= x^2
g[x_]:= x+1
```

In[25]:=

```
A[n_]:= A[n] = 2/(n Pi c) (NIntegrate[f[x] Sin[n Pi x/L],
{x, 0, 1/2}] +
NIntegrate[g[x] Sin[n Pi x/L],{x, 1/2, 1}])//N//Chop;

Table[{n, A[n], B[n]}, {n,1,8}]//TableForm
```

Out[26]=

```
{ 1,1.14423}
{ 2,-1.06676}
{ 3,0.419635}
{ 4,-0.119366}
{ 5,0.253616}
{ 6,-0.34603}
{ 7,0.181515}
{ 8,-0.0596831}
```

In[27]:=

```
u[x_,t_,n_] := A[n] Sin[n Pi x/L] Exp[-k t(n Pi/1)^2]

uapprox[x_,t_] := Sum[u[x,t,j],{j,5}]
```

In[29]:=

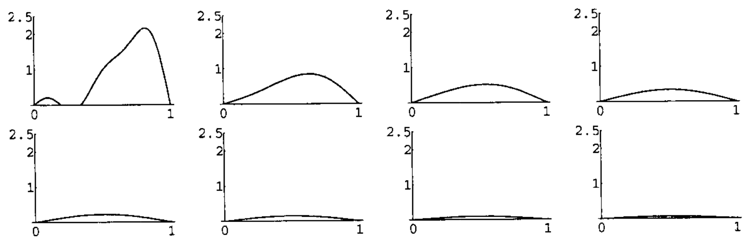
```
graphs =
Table[Plot[uapprox[x,t],{x,0,1},
PlotRange->{0,2.5},
Ticks->{{0,1},{0,1,2,2.5}},
DisplayFunction->Identity],
   {t,0,1/3,1/24}];
```

In[30]:=

```
graphsarray = Partition[graphs,4];
```

In[31]:=

```
Show[GraphicsArray[graphsarray],
DisplayFunction->$DisplayFunction]
```



```
-GraphicsArray-
```

An interesting situation arises if the function $f(x)$ is zero in the initial condition (5.27), but the boundary conditions are nonhomogeneous.

EXAMPLE 5.3. Consider the dimensionless partial differential equation governing the plane wall transient heat conduction

$$u_t = u_{xx}, \quad 0 < x < 1, \tag{5.35}$$

with the boundary conditions

$$u(0,t) = 1, \quad u(1,t) = 0, \quad t \geq 0, \tag{5.36}$$

and the initial condition

$$u(x,0) = 0, \quad 0 \leq x \leq 1. \tag{5.37}$$

Since the homogeneous initial condition (5.37) does not allow us compute the Fourier coefficients, as in (5.34), the standard technique used in the above examples is not directly applicable to this problem. Instead,

we proceed as follows: First, find a partial solution of the problem; although there is more than one way to determine the particular solution, we, e.g., take the steady state case, where the equation becomes $\tilde{u}_{xx} = 0$, which after integrating twice has the general solution

$$\tilde{u}(x) = c_1 x + c_2,$$

with the boundary conditions $\tilde{u}(0) = 1$, $\tilde{u}(1) = 0$. Thus, $c_1 = -1$, $c_2 = 1$, and the steady state solution is

$$\tilde{u}(x) = 1 - x.$$

Next, formulate a homogeneous problem by writing $u(x,t)$ as a sum of the steady state solution $\tilde{u}(x)$ and a transient term $v(x,t)$, i.e.,

$$u(x,t) = \tilde{u}(x) + v(x,t),$$

or

$$v(x,t) = u(x,t) - \tilde{u}(x). \tag{5.38}$$

Hence the problem reduces to finding $v(x,t)$. If we substitute v from (5.38) into (5.35), we get

$$v_t = v_{xx}, \tag{5.39}$$

where the boundary conditions (5.36) and the initial condition (5.37) reduce to

$$\begin{aligned} v(0,t) &= u(0,t) - \tilde{u}(0) = 0, \\ v(1,t) &= u(1,t) - \tilde{u}(1) = 0, \end{aligned} \tag{5.40}$$

and

$$v(x,0) = u(x,0) - \tilde{u}(x) = x - 1. \tag{5.41}$$

Notice that the problem (5.39)–(5.41) is the same as in Example 5.2 with $k = 1$, $l = 1$, $f(x) = x - 1$, and u replaced by v. Hence its general solution from (5.32) is given by

$$v(x,t) = \sum_{n=1}^{\infty} C_n \, e^{-n^2\pi^2 t} \sin n\pi x, \tag{5.42}$$

and the coefficients C_n are determined from (5.34) as

$$C_n = 2 \int_0^1 (x-1) \sin n\pi x \, dx = -\frac{2}{n\pi}.$$

Hence,

$$v(x,t) = -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \, e^{-n^2\pi^2 t} \sin n\pi x, \tag{5.43}$$

and finally from (5.38)

$$u(x,t) = 1 - x - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} e^{-n^2\pi^2 t} \sin n\pi x. \blacksquare \tag{5.44}$$

A Mathematica session for this example can be carried out as in the previous examples.

5.4. Elliptic Equations

EXAMPLE 5.4. We shall consider the potential problem for the rectangle $R: \{0 < x < a, 0 < y < b\}$:

$$u_{xx} + u_{yy} = 0, \quad x, y \in R, \tag{5.45}$$

subject to the Dirichlet boundary conditions

$$u(0,y) = 0 = u(a,y),\ u(x,0) = 0,\ u(x,b) = f(x). \tag{5.46}$$

Physically, this problem arises if three edges of a thin isotropic rectangular plate are insulated and maintained at zero temperature, while the fourth edge is subjected to a variable temperature $f(x)$ until the steady state conditions are attained throughout R. Then the steady state value of $u(x,y)$ represents the distribution of temperature in the interior of the plate. As before, we seek a solution of the form $u(x,y) = X(x)Y(y)$, which, after substitution into Eq (5.45) leads to the set of two ordinary differential equations :

$$X'' - cX = 0, \tag{5.47}$$

$$Y'' + cY = 0, \tag{5.48}$$

where c is a constant, as in Example 5.2. Since the first three boundary conditions in (5.46) are homogeneous, they become

$$X(0) = 0, \quad X(a) = 0, \quad Y(0) = 0, \tag{5.49}$$

but the fourth boundary condition which is nonhomogeneous must be used separately. Now, taking $c = -\lambda^2$, as before, the solution of (5.47)

subject to the first two boundary conditions in (5.49) leads to the eigenvalues and the corresponding eigenfunctions as

$$\lambda_n = \frac{n^2\pi^2}{a^2}, \quad X_n(x) = \sin\frac{n\pi x}{a}, \quad n = 1, 2, \dots,$$

while for these eigenvalues the solutions of (5.55) satisfying the third boundary condition in (5.49) are

$$Y_n(y) = \sinh\frac{n\pi y}{a}, \quad n = 1, 2, \dots. \tag{5.50}$$

Hence, for arbitrary constants C_n, $n = 1, 2, \dots$, we get

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}. \tag{5.51}$$

The coefficients C_n are then determined by using the fourth boundary condition in (5.46). Thus,

$$u(x,b) = f(x) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi b}{a}, \ 0 < x < a,$$

which, in view of the Fourier series expansion, yields

$$C_n \sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx, \quad n = 1, 2, \dots. \tag{5.52}$$

This solves the problem completely.

In particular, if $f(x) = f_0 = \text{const}$, then

$$C_n \sinh\frac{n\pi b}{a} = \frac{2f_0[1-(-1)^n]}{n\pi}.$$

Then from (5.51), we have

$$u(x,y) = \frac{2f_0}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \frac{\sin(n\pi x/a)\sinh(n\pi y/a)}{\sinh(n\pi b/a)}. \blacksquare \tag{5.53}$$

A Mathematica session for the following more general boundary value problem is presented below:

$$u_{xx} - u_{yy} = 0, \quad 0 < x < a, \quad o < y < b,$$

subject to the boundary conditions

$$u(x,0) = f_1(x), \quad u(x,b) = f_2(x) \quad \text{for } 0 < x < a,$$

and the initial conditions

$$u(0,y) = g_1(y), \quad u(a,y) = g_2(y) \quad \text{for } 0 < y < b.$$

The solution is $u(x,y) = u_1(x,y) + u_2(x,y)$, where

$$u_1(x,y) = \sum_{n=1}^{\infty} [A_n \cosh(\lambda_n y) + B_n \sinh(\lambda_n y)] \sin \lambda_n x,$$

$$A_n = \frac{2}{a}\int_0^a f_1(x) \sin \lambda_n x \, dx,$$
$$B_n = \frac{1}{\sinh \lambda_n b}\left[\frac{2}{a}\int_0^a f_2(x) \sin \lambda_n x \, dx - A_n \cosh \lambda_n b\right],$$

with $\lambda_n = \pm n\pi/a$, and

$$u_2(x,y) = \sum_{n=1}^{\infty} [a_n \cosh(\mu_n x) + b_n \sinh(\mu_n x)] \sin \mu_n y,$$

$$a_n = \frac{2}{b}\int_0^b g_1(y) \sin \mu_n y \, dy,$$
$$b_n = \frac{1}{\sinh \mu_n a}\left[\frac{2}{b}\int_0^b g_2(y) \sin \mu_n y \, dy - a_n \cosh \mu_n a\right],$$

with $\mu a_n = \pm n\pi/b$. We shall take $a = 1$, $b = 2$, $f_1(x) = x^2$, $f_2(x) = x + 2$, $g_1(y) = y$, and $g_2(y) = y + 1$.

In[32]:=

```
L:= 1
M:= 2
l[n_]:= n Pi/L//N
m[n_]:= n Pi/M//N
```


In[36]:=

```
f1[x_]:= x^2
f2[x_]:= x+2
g1[y_]:= y
g2[x_]:= y+1
```

In[40]:=

```
A[n_]:=
A[n] =2/L NIntegrate[f1[x] Sin[l[n] x], {x, 0, L}]//Chop;

B[n_]:=
   B[n] = 1/Sinh[l[n] M]
   (2/L NIntegrate[f2[x] Sin[l[n] x], {x, 0, L}]
   -A[n] Cosh[l[n] M])//Chop;

Table[{n, A[n],B[n]}, {n,1,8}]//ColumnForm
```

Out[43]=

```
{ 1,0.378607,-0.366722}
{ 2,-0.31831,0.318308}
{ 3,0.202651,-0.202651}
{ 4,-0.159155,0.159155}
{ 5,0.12526,-0.12526}
{ 6,-0.106103,0.106103}
{ 7,0.0901935,-0.0901935}
{ 8,-0.0795775,0.0795775}
```

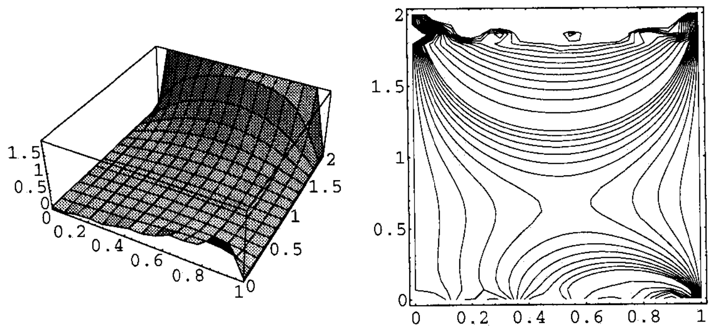
In[41]:=

```
u1[x_,y_,n_]:= (A[n] Cosh[l[n] y] + B[n] Sinh[l[n] y])
   Sin[l[n] x]
u1approx[x_,y_]:= Sum[u1[x,y,n],{n,8}]
```

In[43]:=

```
threeDplot1=Plot3D[u1approx[x,y],{x,0,L},{y,0,M},
   DisplayFunction->Identity];
```

```
Show[threeDplot1,DisplayFunction->$DisplayFunction]
```




```
-GraphicsArray-
```

In[45]:=

```
cvals1:= Table[i,{i,0,1/4,1/44}]
cvals2:= Table[i,{i,1/4,3/2,5/44}]
contourvals:= Union[cvals1,cvals2]
```

In[48]:=

```
u1[x_,y_,n_]:= (A[n] Cosh[l[n] y]+B[n] Sinh[l[n] y])
   Sin[l[n] x]
```

In[49]:=

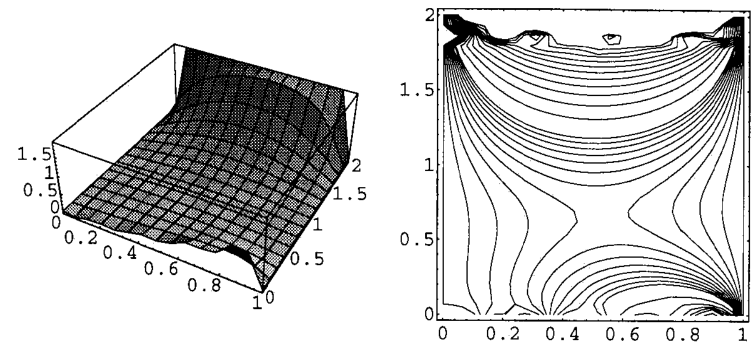
```
u1approx[x_,y_]:= Sum[u1[x,y,n],{n,8}]
```

In[50]:=

```
contourgraphs1= ContourPlot[u1approx[x,y],
   {x,0,1},{y,0,2},
   PlotPoints->40,
   Contours->contourvals,
   ContourShading -> False,
   DisplayFunction-> Identity];
```


In[51]:=

```
Show[GraphicsArray[{threeDplot1,contourgraphs1}],
DisplayFunction -> $DisplayFunction]
```




```
-GraphicsArray-
```

In[52]:=

```
a[n_]:= a[n] =
   2/M NIntegrate[g1[y] Sin[m[n] y], {y, 0, M}]//Chop
b[n_]:= b[n] = 1/Sinh[m[n] L]
   (2/M NIntegrate[g2[y] Sin[m[n] y], {y, 0, M}]-
   a[n] Cosh[m[n] L])//Chop
```

In[54]:=

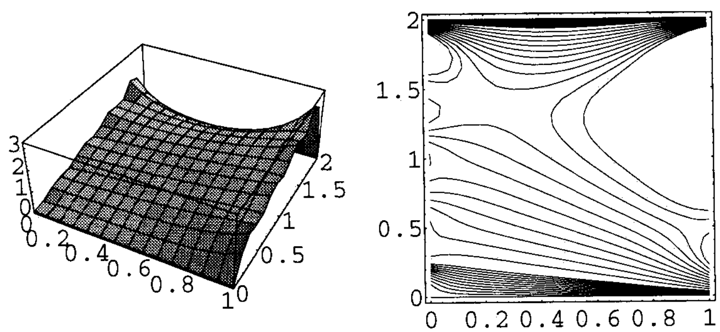
```
u2[x_,y_,n_]:= (a[n] Cosh[m[n] x]+
   b[n] Sinh[m[n] x])* Sin[m[n] y]


u2approx[x_,y_]:= Sum[u2[x,y,n],{n,1,8}]

threeDplot2= Plot3D[u2approx[x,y],{x,0,L},{y,0,M},
DisplayFunction->Identity];
```

In[56]:=

```
contourgraphs2=
ContourPlot[u2approx[x,y],
{x,0,1},{y,0,2},
PlotPoints->40,
Contours->contourvals,
ContourShading->False,
DisplayFunction->Identity];
```

In[57]:=

```
Show[GraphicsArray[{threeDplot2,contourgraphs2}],
  DisplayFunction->$DisplayFunction]
```




```
-GraphicsArray-
```

In[58]:=

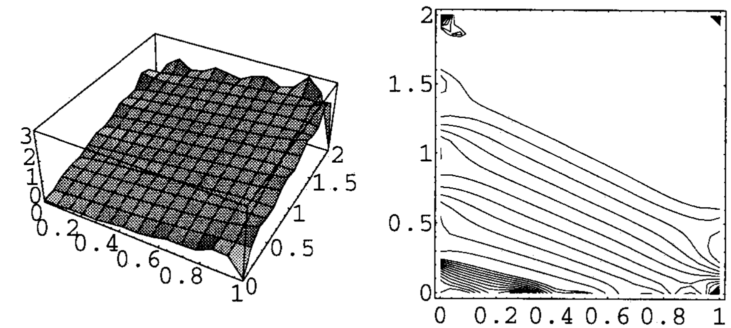
```
uapprox[x_,y_]:= u1approx[x,y] + u2approx[x,y];

threeDplot=
Plot3D[uapprox[x,y],{x,0,L},{y,0,M},
DisplayFunction->Identity];
```

In[60]:=

```
contourgraphsu=
ContourPlot[uapprox[x,y],{x,0,1},{y,0,2},
PlotPoints->40,Contours->contourvals,
```



```
ContourShading->False,
DisplayFunction->Identity];

Show[GraphicsArray[ {threeDplot,contourgraphsu}],
  DisplayFunction->$DisplayFunction]
```




```
-GraphicsArray-

(* Superpose the SurfaceGraphics plot over the
ContourGraphics plot *)
```

In[62]:=

```
myCP=
ContourPlot[uapprox[x,y],
{x,0,L},{y,0,M},
Contours->Union[Table[i,{i,0,1/4,1/44}],
Table[i,{i,1/4,3/2,1/8}]],
PlotPoints->30,
AspectRatio->Automatic,
PlotRange->All,
ContourLines->True,
ContourShading->True,
ContourStyle->
(Map[{Hue[#,1,Random[]],Thickness[.006]}&,
Range[0,1,1/12]]),
ColorFunction->Hue,
Ticks->{Range[0,1],Range[0,2],{-1.5,1.5}},
DisplayFunction->Identity]
```

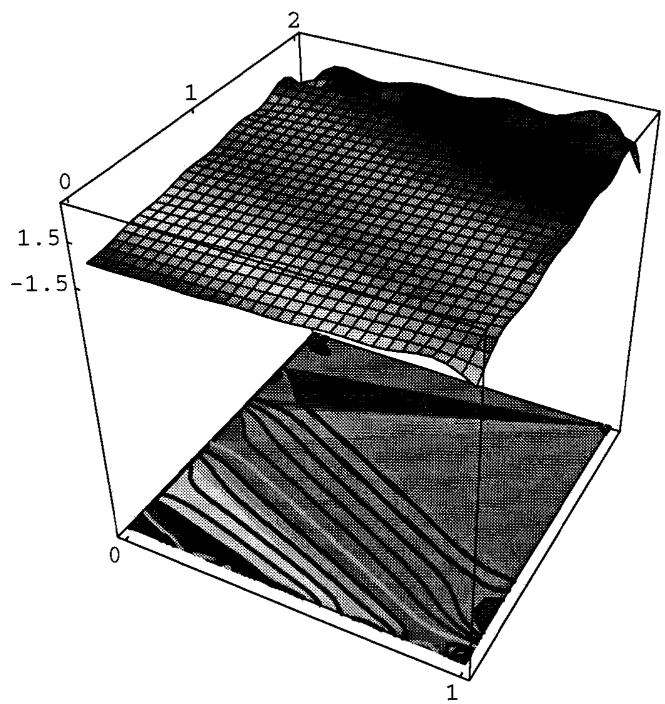
```
-GraphicsArray-
```

In[63]:=

```
myContourGP= First@Graphics@myCP;

myContourGP= N@myContourGP/.{x_AtomQ,y_AtomQ}->{x,y,-20};
```

In[65]:=

```
Show[
{SurfaceGraphics@myCP,Graphics3D@myContourGP},
Axes->True,
BoxRatios->{1,1,1},
DisplayFunction->$DisplayFunction]
```



Surface graph overlay of contour graph.

EXAMPLE 5.5. Consider the potential problem

$$u_{xx} + u_{yy} = 0, \quad 0 < x < \pi, \quad 0 < y < 1, \tag{5.54}$$

subject to the mixed boundary conditions

$$\begin{aligned} u(x,0) = u_0 \cos x, \quad u(x,1) = u_0 \sin^2 x, \\ u_x(0,y) = 0 = u_x(\pi, y). \end{aligned} \tag{5.55}$$

The separation of variables technique leads to the same set of ordinary differential equations as in (5.47)–(5.48), i.e.,

$$X'' + \lambda^2 X = 0, \quad X'(0) = 0 = X'(\pi), \tag{5.56}$$

and

$$Y'' - \lambda^2 Y = 0. \tag{5.57}$$

The eigenvalues and the corresponding eigenfunctions for (5.56) are

$$\begin{aligned} &\lambda_0 = 0, \quad X_0(x) = 1, \\ &\lambda_n = n^2, \quad X_n(x) = \cos nx, \quad n = 1, 2, \dots, \end{aligned}$$

and subsequently the solutions of (5.57) are

$$Y_n(y) = \begin{cases} A_0 + B_0 y, & n = 0, \\ A_n \cosh ny + B_n \sinh ny, & n = 1, 2, \dots. \end{cases} \tag{5.58}$$

Hence, using the superposition principle, we get

$$u(x,y) = A_0 + B_0 y + \sum_{n=1}^{\infty} \left[A_n \cosh ny + B_n \sinh ny\right] \cos nx. \tag{5.59}$$

Now, the first boundary condition in (5.55) leads to

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos nx = u_0 \cos x. \tag{5.60}$$

By matching the coefficients of similar terms on both sides of (5.60) we find that $A_0 = 0$, $A_1 = u_0$, and $A_n = 0$ for $n \geq 2$. Hence the solution becomes

$$u(x,y) = B_0 y + u_0 \cosh y \, \cos x + \sum_{n=1}^{\infty} B_n \sinh ny \, \cos nx. \tag{5.61}$$

Similarly, using the second boundary condition in (5.55) we find from (5.61) that

$$\begin{aligned} u(x,1) &= B_0 + u_0 \cosh 1 \cos x + \sum_{n=1}^{\infty} B_n \sinh n \cos nx \\ &= u_0 \sin^2 x = u_0 \frac{1-\cos 2x}{2}, \end{aligned}$$

from which, after comparing the coefficients of similar terms on both sides, we get

$$\begin{aligned} B_0 &= \frac{u_0}{2}, & B_1 &= -\frac{u_0 \cosh 1}{\sinh 1} \\ B_2 &= -\frac{u_0}{2\sinh 2}, & B_n &= 0 \quad \text{for } n \geq 3. \end{aligned}$$

Hence, from (5.61) the general solution is given by

$$\begin{aligned} u(x,y) &= \frac{u_0}{2} y + u_0 \left[\cosh y - \frac{\cosh 1 \sinh y}{\sinh 1}\right] \cos x - u_0 \frac{\sinh 2y}{2\sinh 2} \cos 2x \\ &= u_0 \left[\frac{1}{2} y + \frac{\sinh(1-y)}{\sinh 1} \cos x - \frac{\sinh 2y}{2 \sinh 2} \cos 2x\right]. \ \blacksquare \end{aligned} \tag{5.62}$$

5.5. Cylindrical Coordinates

The three–dimensional Laplacian in cylindrical coordinates is

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}. \tag{5.63}$$

EXAMPLE 5.6. (*Circular drum*) If a circular drum is struck in the center, its vibrations are radially symmetric. We shall solve the boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u \equiv \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}, \quad r < 1, \tag{5.64}$$

subject to the boundary conditions

$$\begin{aligned} &u(r,0) = f(r), \quad r < 1, \\ &u_t(r,0) = 0, \quad u(1,t) = 0, \quad t \geq 0. \end{aligned} \tag{5.65}$$

If we take $u(r,t) = T(t)R(r)$, then Eq (5.64) reduces to the system of ordinary differential equations

$$\frac{T''}{T} = \frac{R'' + \dfrac{R'}{R}}{R} = k. \tag{5.66}$$

Here, again, $k = -\lambda^2$ yields nontrivial solutions. Then the system (5.66) gives the uncoupled ordinary differential equation

$$\begin{aligned} &T'' + \lambda^2 T = 0, \\ &\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \lambda^2 R = 0, \end{aligned} \tag{5.67}$$

or

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + \lambda^2 r^2 R = 0, \tag{5.68}$$

which is the Bessel equation. The eigenvalues λ_n are the positive zeros of $J_0(\lambda)$, with the corresponding eigenfunctions $J_0(\lambda_n r)$. The solutions of the first equation in (5.67) are $T_n = \cos \lambda_n t$. Hence,the solution of the vibrating circular drum struck at the center is given by

$$u(x,t) = \sum_{n=1}^{\infty} C_n \cos \lambda_n t J_0(\lambda_n r), \tag{5.69}$$

where the coefficients C_n are (see Appendix B)

$$C_n = \frac{\int_0^1 f(r) J_0(\lambda_n r) r\, dr}{\int_0^1 [J_0(\lambda_n r)]^2 r\, dr}. \tag{5.70}$$

Marc Kac (1966) asked the question: "Can one hear the shape of a drum?" This means one should answer the question whether two drums of different shapes and struck in their centers have the same eigenvalues (Protter, 1987). This question has been resolved negatively by Gordon, Webb and Wolpert (1992) ■

WWW For an animation notebook, see `drum.ma`.

5.6. Spherical Coordinates

Using the transformation $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho \cos\phi$, where $\rho \geq 0$, $0 \leq \phi \leq \pi$, and $0 \leq \theta \leq 2\pi$, the Laplacian in the spherical coordinate system becomes

$$\nabla^2 \equiv \frac{\partial^2}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial^2}{\partial \theta^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\cot\phi}{\rho^2}\frac{\partial}{\partial \phi}.$$

EXAMPLE 5.7. (*Cooling ball*) Consider the boundary value problem

$$u_t = \nabla^2 u, \quad 0 \leq \rho < 1, \quad 0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi,$$

subject to the conditions $u(\rho, \phi, 0) = f(\rho, \phi)$, $u(1, \phi, t) = 0$. The problem describes the temperature distribution in the interior of the unit ball dropped in cold water. The first condition implies that the temperature u is not uniform but depends on ρ and ϕ but not on θ. Thus the solution can be assumed formally to be

$$u(\rho, \phi, t) = R(\rho)\,\Phi(\phi)\,T(t), \tag{5.71}$$

which after separating the variables gives

$$\frac{R'' + \frac{2}{\rho}R'}{R} + \frac{\Phi'' + \cot\phi\,\Phi'}{\rho^2\,\Phi} = -\alpha^2 = \frac{\dot T}{T} = \lambda. \tag{5.72a}$$

The left side of Eq (5.72a) can be expressed as

$$\left(\frac{R'' + \frac{2}{\rho}R'}{R} + \alpha^2\right)\rho^2 = -\frac{\Phi'' + \cot\phi\,\Phi'}{\Phi}.$$

In order that this equation be satisfied, the terms on each side must be constant. Thus,

$$\left(\frac{R'' + \frac{2}{\rho}R'}{R} + \alpha^2\right)\rho^2 = \mu = -\frac{\Phi'' + \cot\phi\,\Phi'}{\Phi}. \tag{5.72b}$$

It is known that $\mu = n(n+1)$ (Courant and Hilbert, 1963). Then Eq (5.72b) yields

$$\begin{aligned} &\Phi'' + \cot\phi\,\Phi' + n(n+1)\,\Phi = 0, \\ &\rho^2 R'' + 2\rho R' + \alpha^2\rho^2 R - n(n+1) - R = 0. \end{aligned} \tag{5.73}$$

If we set $x = \alpha\rho$ in the second equation, then it becomes

$$x^2 R'' + 2x\,R' + \left(x^2 - (n+1)\right) R = 0,$$

which under the transformation $w = \sqrt{x}\,R$ reduces to the Bessel equation

$$x^2 w'' + x\,w' + \left(x^2 - (n + \frac{1}{2})^2\right) w = 0, \tag{5.74}$$

and has a bounded solution $w = J_{n+1/2}(x)$. Hence

$$R(\rho) = \frac{J_{n+1/2}(\alpha\rho)}{\sqrt{\alpha\rho}}.$$

The eigenfunctions $\psi_{mn} = \dfrac{J_{n+1/2}(\alpha_m\rho)}{\sqrt{\alpha\rho}}\,P_n(\cos\phi)$ form an orthogonal basis in the L^2–space on the unit ball, independent of θ, where α_{mn} are positive zeros of $J_{n+1/2}(\alpha\rho)$. Hence the solution for the temperature distribution in the unit ball is given by

$$u(\rho, \phi, t) = \sum_{\substack{m=1\\ n=0}}^{\infty} C_{mn}\,\frac{e^{-\alpha_{mn}^2 t}\,J_{n+1/2}(\alpha_{mn}\rho)}{\sqrt{\alpha_{mn}\rho}}\,P_n(\cos\phi), \tag{5.75}$$

where, using $\displaystyle\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}$,

$$C_{mn} = \frac{(2n+1)\sqrt{\alpha_{mn}}}{\beta(\alpha_{mn})}\int_0^{\pi}\int_0^1 \rho^{3/2}\,J_{n+1/2}(\alpha_{mn}\rho)\,P_n(\cos\theta)\,f(\rho,\phi)\,\sin\phi\,d\rho\,d\phi,$$

and

$$\beta(\alpha_{mn}) = 2\int_0^1 \rho\,J_{n+1/2}(\alpha_{mn}\rho) = \left[J'_{n+1/2}(\alpha_{mn})\right]^2.$$

The factor $\sqrt{\alpha_{mn}}$ can be absorbed in C_{mn}. ■

5.7. Nonhomogeneous Problems

In the above examples we have seen that the method of separation of variables is applicable to steady state linear problems with homogeneous governing equations and three homogeneous and one nonhomogeneous boundary conditions. Nonhomogeneity, however, occurs from other conditions as well. For example, there may be more than one nonhomogeneous boundary conditions, or the governing equation may be nonhomogeneous. In order to use the method of separation of variables, a nonhomogeneous problem can be divided into finitely many simple problems with homogeneous equations and/or homogeneous boundary conditions. Then the solution of the given problem is obtained from the superposition of the solutions of all these simple problems.

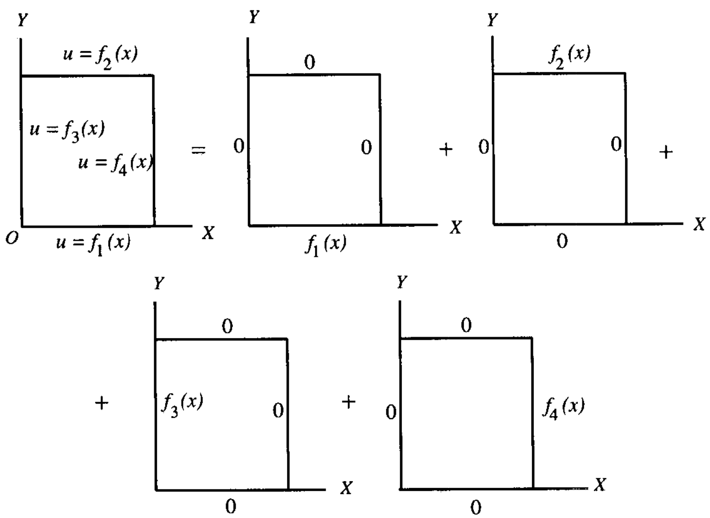


Fig. 5.1.

EXAMPLE 5.8. (*with four nonhomogeneous boundary conditions*) Consider the steady state temperature distributions governed by Eq (5.45) in the region R, with more than one nonhomogeneous boundary conditions, viz.,

$$\begin{aligned} u(x,0) &= f_1(x), \quad & u(x,b) &= f_2(x), \quad & 0 < x < a, \\ u(0,y) &= f_3(y), \quad & u(a,y) &= f_4(y), \quad & 0 < y < b. \end{aligned} \tag{5.76}$$

This problem can be resolved as a superposition of the four problems, shown in Fig. 5.1. Hence,

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y), \tag{5.77}$$

where the solution of each simple problem is obtained as in Example 5.5. ■

In certain cases with mixed boundary conditions, the method of separation of variables can be readily used by translating the function $u(x,y)$ which depends on the geometry and material symmetry of the problem.

EXAMPLE 5.9. Consider the Laplace equation (5.45) in a half-strip (see Fig. 5.2) subject to the boundary conditions

$$\begin{aligned} u(0,y) &= f(y), \quad & \lim_{x\to+\infty} u(x,y) &= u_\infty, \\ u_y(x,0) &= 0, \quad & u_y(x,b) + \beta[u(x,b) - u_\infty] &= 0, \end{aligned} \tag{5.78}$$

where β is known as the film coefficient.

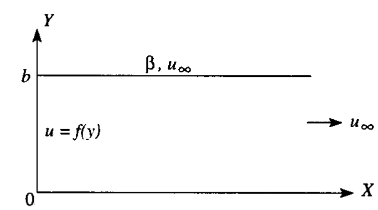


Fig. 5.2.

This problem has more than one nonhomogeneous boundary conditions. By using the translation $U(x,y) = u(x,y) - u_\infty$, the problem (5.45) and (5.78) reduces to

$$\begin{gathered} U_{xx} + U_{yy} = 0, \\ U(0,y) = f(y) - u_\infty = F(y), \quad \lim_{x\to+\infty} U(x,y) = 0, \\ U(y) = 0, \quad U_y(x,b) = \beta U(x,b). \end{gathered} \tag{5.79}$$

We can now assume that $U(x,y) = X(x)Y(y)$, which reduces the set of the two ordinary differential equations (with $c = \lambda^2$):

$$X'' - \lambda^2 X = 0, \quad Y'' + \lambda^2 Y = 0.$$

These two equations lead to the general solution

$$U(x,y) = \left(A_1\, e^{-\lambda x} + A_2\, e^{\lambda x}\right)\left(B_1 \cos \lambda y + B_2 \sin \lambda y\right). \tag{5.80}$$

Now, since $Y_y(0) = 0$ and $Y_y(b) + \beta Y(b) = 0$, we obtain the eigenfunctions as $\cos \lambda_n y$ with the corresponding eigenvalues λ_n, which are the positive roots of the equation

$$\lambda_n \tan \lambda_n b = \beta, \quad n = 1, 2, \dots. \tag{5.81}$$

Using the boundary condition $\lim_{x\to\infty} X(x) = 0$, we obtain

$$U(x,y) = \sum_{n=1}^{\infty} C_n\, e^{-\lambda_n x} \cos \lambda_n y + u_\infty. \tag{5.82}$$

Then, in view of the nonhomogeneous boundary condition $X(0) = F(y)$, we have

$$F(y) = f(y) - u_\infty = \sum_{n=1}^{\infty} C_n \cos \lambda_n y,$$

where the coefficients C_n are given by

$$C_n = \frac{2\lambda_n}{\lambda_n b + \sin \lambda_n b \cos \lambda_n b} \int_0^b \left[f(y) - u_\infty\right] \cos \lambda_n y\, dy.$$

(see the table in Chapter 4). Hence, the temperature distribution is

$$\begin{aligned} U(x,y) &= u(x,y) - u_\infty \\ &= 2\sum_{n=1}^{\infty} \frac{\lambda_n\, e^{-\lambda_n x} \cos \lambda_n y}{\lambda_n b + \sin \lambda_n b \cos \lambda_n b} \int_0^b \left[f(\eta) - u_\infty\right] \cos \lambda_n \eta\, d\eta. \end{aligned} \tag{5.83}$$

In particular, if $f(y) = u_0 = \text{const}$, the temperature distribution reduces to

$$\frac{u(x,y) - u_\infty}{u_0 - u_\infty} = 2\sum_{n=1}^{\infty} \frac{\sin \lambda_n b}{\lambda_n b + \sin \lambda_n b \cos \lambda_n b}\, e^{-\lambda_n x} \cos \lambda_n y\, dy. \tag{5.84}$$

In view of (5.81), the eigenvalues λ_n are the positive roots of $\tan \xi - \dfrac{\text{Bi}}{\xi} = 0$, where $\xi = \lambda_n b$, and $\text{Bi} = \beta b$ is the Biot number. Three of these roots, denoted by ξ_1, ξ_2, and ξ_3, are shown in Fig. 5.3. ■

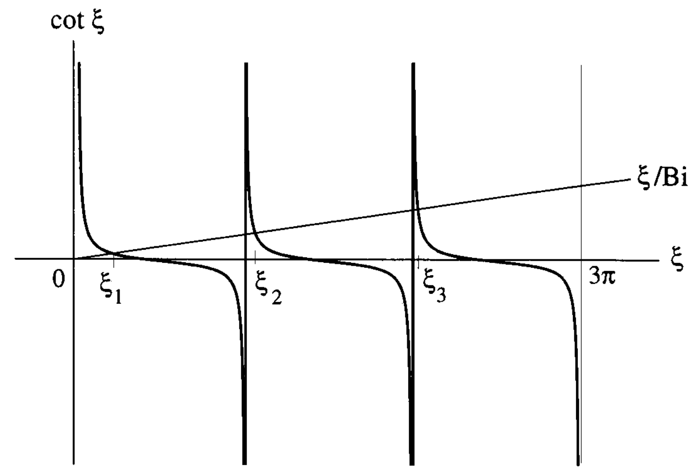


Fig. 5.3. Distribution of the eigenvalues ξ_n.

EXAMPLE 5.10. We shall consider a problem which is linear and nonhomogeneous in the governing equation, with linear and homogeneous boundary conditions. Assume that heat is generated in a rectangular bar at a constant rate q per unit volume, that there is no temperature gradient in the z-direction, and the thermal conductivity k of the bar is constant. Then the steady state temperature distribution is governed by the equation

$$k\,(u_{xx} + u_{yy}) + q = 0. \tag{5.85}$$

Let the linear and homogeneous boundary conditions be

$$\begin{aligned} u_x(0,y) &= 0, \qquad u(a,y) = 0, \\ u_y(x,0) &= 0, \qquad u(x,b) = 0. \end{aligned} \tag{5.86}$$

Equation (5.85), being nonhomogeneous, is not separable. But, if we assume the solution as

$$u(x,y) = V(x,y) + \phi(x), \tag{5.87}$$

then problem (5.85)–(5.86) reduces to the following two problems:

$$\frac{d^2\phi}{dx^2} + \frac{q}{k} = 0, \quad \frac{d\phi(0)}{dx} = 0, \quad \phi(a) = 0, \tag{5.88}$$

and

$$\begin{aligned} &V_{xx} + V_{yy} = 0, \\ &\quad V_x(0,y) = 0 = V(a,y), \quad V_y(x,0) = 0, \quad V(x,b) = -\phi(x). \end{aligned} \tag{5.89}$$

Solution of problem (5.88) is readily obtained as

$$\phi(x) = \frac{qa^2}{2k}\left(1 - \frac{x^2}{a^2}\right). \tag{5.90}$$

Problem (5.89) is separable and its solution is given by

$$V(x,y) = -\frac{2q}{ka}\sum_{n=0}^{\infty}\frac{(-1)^n}{\lambda_n^3}\frac{\cos\lambda_n x\cosh\lambda_n y}{\cosh\lambda_n b}, \tag{5.91}$$

where the eigenvalues $\lambda_n = \dfrac{(2n+1)\pi}{2a}$, $n = 0, 1, \dots$. The solution of this problem is then obtained by adding (5.91) and (5.90). Note that this problem can also be solved by taking

$$u(x,y) = V(x,y) + \psi(y). \tag{5.92}$$

In the case when the rate is variable, say, $q = q(x)$, we should use the substitution (5.87); if $q = q(y)$, then the substitution (5.92) will make the equation in V separable. ■

EXAMPLE 5.11. Consider the nonhomogeneous wave equation

$$u_{tt} = c^2 u_{xx} + f(x,t), \quad 0 < x < l, \tag{5.93}$$

with the homogeneous (Dirichlet) boundary conditions $u(0,t) = 0 = u(l,t)$, $t > 0$, and the initial conditions $u(x,0) = g(x)$, $u_t(x,0) = h(x)$, $0 \le x \le l$. Using the Fourier series method, which is the same as the method of separation of variables, we seek a solution in the form

$$u(x,t) = \sum_{n+1}^{\infty} u_n(t)\sin\frac{n\pi x}{l}, \tag{5.94}$$

where t is regarded as a parameter. The functions f, g, h are written as Fourier series

$$\begin{aligned} f(x,t) &= \sum_{n+1}^{\infty} f_n(t)\sin\frac{n\pi x}{l}, \quad f_n(t) = \frac{2}{l}\int_0^l f(\xi,t)\sin\frac{n\pi\xi}{l}\,d\xi, \\ g(x) &= \sum_{n+1}^{\infty} g_n\sin\frac{n\pi x}{l}, \quad g_n = \frac{2}{l}\int_0^l g(\xi)\sin\frac{n\pi\xi}{l}\,d\xi, \\ h(x) &= \sum_{n+1}^{\infty} h_n\sin\frac{n\pi x}{l}, \quad h_n = \frac{2}{l}\int_0^l h(\xi)\sin\frac{n\pi\xi}{l}\,d\xi. \end{aligned} \tag{5.95}$$

After substituting (5.95) into (5.93) we get

$$\sum_{n=1}^{\infty}\left\{\ddot u_n(t) + \frac{n^2\pi^2c^2}{l^2}u_n(t) - f_n(t)\right\}\sin\frac{n\pi x}{l} = 0,$$

where $\ddot u = d^2u/dt^2$. This relation is satisfied if all the coefficients of the series are zero, i.e., if

$$\ddot u_n(t) + \frac{n^2\pi^2c^2}{l^2}u_n(t) = f_n(t).$$

The solution $u_n(t)$ of the ordinary differential equation with constant coefficients can be easily obtained under the initial conditions

$$\begin{aligned} u(x,0) = g(x) &= \sum_{n=1}^{\infty} u_n(0)\sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} g_n\sin\frac{n\pi x}{l}, \\ u_t(x,0) = h(x) &= \sum_{n=1}^{\infty} \dot u_n(0)\sin\frac{n\pi x}{l} = \sum_{n=1}^{\infty} h_n\sin\frac{n\pi x}{l}. \end{aligned}$$

Thus, $u_n(0) = g_n$, and $u_t(0) = h_n$. Now, we define the solutions $u_n(t)$ in the form

$$u_n(t) = u_n^1(t) + u_n^2(t),$$

where

$$u_n^1(t) = \frac{1}{n\pi c}\int_0^t \sin\frac{n\pi c(t-\tau)}{l}\,f_n(t)\,d\tau$$

represents the solution of the nonhomogeneous equation with the homogeneous initial conditions, and

$$u_n^2(t) = g_n\cos\frac{n\pi ct}{l} + \frac{1}{n\pi c}h_n\sin\frac{n\pi ct}{l}$$

is the solution of the homogeneous equation with the prescribed initial conditions. Hence,

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty}\left[u_n^1(t) + u_n^2(t)\right] \\ &= \sum_{n=1}^{\infty}\frac{1}{n\pi c}\int_0^t \sin\frac{n\pi c(t-\tau)}{l}\sin\frac{n\pi x}{l}f_n(\tau)\,d\tau \\ &\quad + \sum_{n=1}^{\infty}\left(g_n\cos\frac{n\pi ct}{l} + \frac{1}{n\pi c}\sin\frac{n\pi ct)}{l}\right)\sin\frac{n\pi x}{l}. \end{aligned} \tag{5.96}$$

Note that the second term is the solution of the corresponding problem with $f = 0$ (representing a freely vibrating string with prescribed initial conditions; see Exercise 5.16). The first term represents the forced vibrations of the string under the influence of an external force. ∎

5.8. Exercises

5.1. Solve $u_{rr} + \frac{1}{r} u_r = u_{tt}$, subject to the conditions $u(r,0) = u_0 r$, $u_t(r,0) = 0$, and $u(a,t) = a u_0$, $\lim_{r\to 0} u(r,t) < +\infty$, where u_0 is a constant.

ANS. $u = \sum_{i=1}^{\infty} A_i\, J_0(\alpha_i r)\, \sin \alpha_i t$, where

$$A_i = u_0 \frac{\int_0^a r(r-a)\, J_0(\alpha_i r)\, dr}{\int_0^a r\, J_0^2(\alpha_i r)\, dr}.$$

5.2. Solve $x^2 u_{xy} + 3y^2 u = 0$, such that $u(x,0) = e^{1/x}$.

ANS. $u = e^{y^2 + 1/x}$.

5.3. Solve $u_{xx} - u_t = A\, e^{-\alpha x}$, $A \ge 0, \alpha > 0$, where $u(0,t) = 0 = u(L,t)$ for $t > 0$, and $u(x,0) = f(x)$ for $0 < x < L$.

ANS. $u = v - \frac{A}{\alpha^2} - \frac{A}{\alpha^2}\left(e^{-\alpha L} - 1\right) \frac{x}{L} + \frac{A}{\alpha^2} e^{-\alpha L}$, where

$$v = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}\, e^{-n^2\pi^2 t/L},$$

$$A_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L}\, dx.$$

5.4. Solve $u_t = a^2 u_{xx} + f(x,t)$, $0 < x < l$, with the boundary conditions $u(0,t) = 0 = u(l,t)$ for $0 \le x \le l$, and the initial condition $u(x,0) = 0$ for $t > 0$.

ANS. $u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{n\pi x}{l}$, where

$$u_n(t) = \int_0^t e^{n^2\pi^2 a^2 (t-\tau)/l^2} f_n(\tau)\, d\tau,$$

$$f_n(t) = \frac{2}{l} \int_0^l f(\xi,t) \sin \frac{n\pi\xi}{l}\, d\xi.$$

5.5. Solve $u_{tt} = c^2 u_{xx}$, $0 < x < L$, subject to the boundary conditions $u(0,t) = u_1$, $u(l,t) = u_2$ for $0 \le x \le l$, where u_1, u_2 are prescribed quantities, and the initial conditions $u(x,0) = g(x)$, $u_t(x,0) = h(x)$ for $t > 0$.

ANS. $u(x,t) = U(x) + v(x,t)$, where $U(x) = u_1 + (u_2 - u_1)\frac{x}{l}$, describes the steady state solution (static deflection), and $v(x)$ is the solution of the problem in Exercise 5.10.

5.6. Find the interior temperature of the cooling ball of Example 5.7, if $f(\rho,\phi) = \begin{cases} 1, & 0 \le \phi < \pi/2 \\ 0, & \pi/2 \le \phi < \pi. \end{cases}$

ANS.

$$C_{mn} = \frac{2\sqrt{\lambda_m} \int_0^1 J_{n+1/2}(\sqrt{\lambda_m}\rho)\, d\rho}{\pi \int_0^1 J_{n+1/2}(\lambda_m \rho)^2\, \rho\, d\rho}.$$

5.7. Determine the steady state temperature inside a solid hemisphere $0 \le \rho \le 1$, $0 \le \phi \le \pi/2$

(a) when the base $\phi = \pi/2$ is at 0° and the curved surface $\rho = 1$, $0 \le \phi < \pi/2$ is at 1°.

(b) when the base $\phi = \pi/2$ is insulated, but the temperature on the curved surface is $f(\phi)$. HINT: $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial \rho \cos\phi} = 0$ ON THE BASE.

ANS. (a) $u(x) = \frac{1}{2} \sum_{n=0}^{\infty} \rho^{2n+1} [P_{2n}(0) - P_{2n+2}(0)]\, P_{2n+1}(x)$.

(b) $u(x) = \sum_{n=0}^{\infty} c_n \rho^{2n}\, P_{2n}(\cos\phi)$, where

$c_n = (4n+1) \int_0^{\pi/2} f(\phi)\, P_{2n}(\cos\phi)\, \sin\phi\, d\phi$.

5.8. Solve $u_t = u_{xx}$, $-\pi < x < \pi$, subject to the conditions $u(x,0) = f(x)$, $u(-\pi,t) = u(\pi,t)$, and $u_x(-\pi,t) = u_t(\pi,t)$, where $f(x)$ is a periodic function of period 2π. This problem describes the heat flow inside a rod of length 2π which is shaped in the form of a closed circular ring. HINT: ASSUME $X(x) = A\cos\omega x + B \sin\omega x$.

ANS. $\omega_n = n$; $u(x,t) = \sum_{n=0}^{\infty} e^{-n^2 t} \left(a_n \cos nx + b_n \sin nx\right)$, where

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx,\ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx.$

5.9. Solve the problem $u_t = \nabla^2 u$, $r < 1$, $0 < z < 1$, such that $u(r,z,0) = 1$, $u(1,z,t) = 0$, and $u(r,0,t) = 0 = u(r,1,t)$. This problem describes the temperature distribution inside a homogeneous isotropic solid circular cylinder.
ANS.

$$u(r,z,t) = \sum_{m,n=1}^{\infty} C_{mn}\, e^{-(\lambda_m^2 + n^2\pi^2)t} J_0(\lambda_m r) \sin n\pi z,$$

where λ_m are the zeros of J_0, and

$$C_{mn} = \frac{4\,(1-(-1)^n)}{n\pi\lambda_m\, J_1(\lambda_m)}.$$

5.10. Find the steady state temperature in a solid circular cylinder of radius 1 and height 1 under the conditions that the flat faces are kept at 0° and the curved surface at 1°.
ANS.

$$u(r,z) = 4 \sum_{\substack{n=1 \\ n\,\text{odd}}}^{\infty} \frac{I_0(n\pi r)}{I_0(n\pi)}\, \frac{\sin n\pi z}{\sin n\pi}.$$

5.11. Solve the steady state problem of temperature distribution in a half-cylinder $0 \le r \le 1$, $0 \le \theta \le \pi$, $0 \le z \le 1$, where the flat faces are kept at 0° and the curved surface at 1°.
ANS.

$$u(r,\theta,z) = \frac{16}{\pi} \sum_{\substack{m,n=1 \\ m,n\,\text{odd}}}^{\infty} \frac{I_m(n\pi r)}{I_m(n\pi)}\, \frac{\sin n\pi z}{\sin n\pi}.$$

5.12. Solve $\dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(x\,\dfrac{\partial u}{\partial x}\right)$, $0 < x < 1$, $t > 0$, subject to the conditions $u(x,0) = f(x)$ and $u(1,t) = 0$. HINT: SET $4x = r^2$ AND SOLVE AS IN EXAMPLE 5.6.
ANS. $u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n^2 t/4} J_0(\lambda_n \sqrt{x})$, where λ_n are the zeros of J_0 and

$$C_n = \frac{\int_0^1 f(\sqrt{x})\, J_0(\lambda_n\sqrt{x})\sqrt{x}\, dx}{\int_0^1 [J_0(\lambda_n\sqrt{x})]^2 \sqrt{x}\, dx}.$$

5.13. Solve $u_{tt} = c^2\,(u_{xx} + u_{yy})$ in the rectangle $R = \{(x,y) : 0 < x < a,\ 0 < y < b\}$, subject to the condition $u = 0$ on the boundary of R for $t > 0$, and the initial conditions $u(x,y,0) = f(x,y)$, $u_t(x,y,0) = g(x,y)$. This problem describes a vibrating rectangular membrane. Interpret the solutions u_{11}, u_{12}, u_{21}, u_{22}, u_{13}, and u_{31} for a square membrane $a = b = 1$.
ANS.

$$u(x,y,t) = \sum_{\substack{m,n=1 \\ m,n\,\text{odd}}}^{\infty} (A_{mn} \cos \lambda_{mn} t + B_{mn} \sin \lambda_{mn}) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b},$$

where

$$A_{mn} = \frac{4}{ab} \int_0^b \int_0^a f(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dx\, dy,$$
$$B_{mn} = \frac{4}{ab\lambda_{mn}} \int_0^b \int_0^a g(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dx\, dy,$$

for $m, n = 1, 2, \dots$; the eigenvalues are

$$\lambda_{mn} = c\pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}.$$

The solutions u_{11}, u_{12}, u_{21}, u_{22}, u_{13}, and u_{31} are represented in Fig. 5.4.

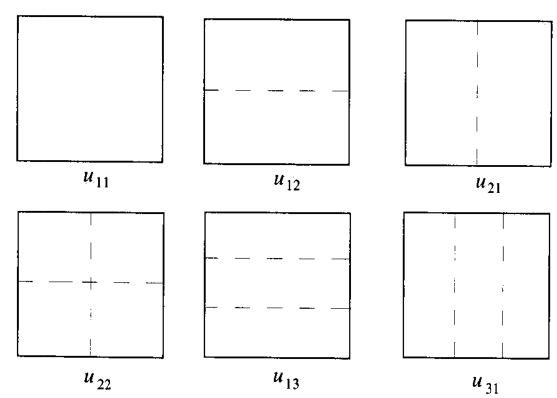


Fig. 5.4.

5.14. Solve $u_{xy} - 4xyu = 0$, such that $u(0,y) = e^{y^2}$.
ANS. $u = e^{x^2+y^2}$.

5.15. Solve $u_{xx} + u_{yy} = 0$, under the conditions $u(0, y) = 0 = u(\pi, y)$, $u(x, 0) = \sin x$, $\lim_{y\to\infty} u(x, y) < +\infty$.
ANS. $u = e^{-y} \sin x$.

5.16. Solve $u_{xx} - u_{tt} = e^{-a^2\pi^2 t} \sin a\pi x$, subject to the conditions $u(x, 0) = 0$, $u_t(x, 0) = 0$, and $u(0, t) = u(1, t) = 0$, where a is a constant.
ANS. $\dfrac{1}{a^2\pi^2(1 + a^2\pi^2)}[\cos a\pi t - e^{-a^2\pi^2 t} - a\pi \sin a\pi t] \sin a\pi x$.

5.17. Solve $r^2 u_{rr} + r u_r + u_{\theta\theta} = 0$, such that $u(b, \theta) = f(\theta)$, $u(r, \theta + 2\pi) = u(r, \theta)$, and $\lim_{r\to 0} u(r, \theta) < +\infty$ (circular disc problem).
HINT: Separate the variables and show that the only relevant part of the solution reduces to

$$u(r, \theta) = c_0 + \sum_\alpha r^\alpha (A(\alpha) \cos \alpha\theta + B(\alpha) \sin \alpha\theta).$$

Note that under the given conditions $u(r, \theta)$ must have a Fourier series representation in θ and therefore $\alpha = n$ is a positive integer.

5.18. Solve $u_{xx} + u_{yy} = 0$, under the conditions $u(x, 0) = 0 = u(x, \pi)$, $u(0, y) = 0$, and $u(\pi, y) = \cos^2 y$.
ANS.

$$u = \sum_{n=1}^{\infty} C_n \sinh nx \sin ny,$$

where

$$\begin{aligned} C_n &= \frac{2}{\pi \sinh n\pi} \int_0^\pi \cos^2 y \sin ny\,dy \\ &= \frac{2}{\pi \sinh n\pi} \left[\frac{1 - (-1)^n}{2n} - \frac{(-1)^n n}{n^2 - 4}\right], \quad n \neq 2, \end{aligned}$$

and $C_2 = 0$.

5.19. Solve $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$, subject to the conditions $u = 0$ for $\theta = 0$ or $\pi/2$, and $u_r = \sin\theta$ at $r = a$.
ANS.

$$u = \sum_{n=1}^{\infty} C_n r^{2n} \sin 2n\theta,$$

where

$$C_n = \frac{4n(-1)^{n+1}}{\pi(4n^2 - 1)a^{2n-1}}.$$

5.20. Solve the problem of transverse vibrations of a beam: $u_{tt} + a^2 u_{xxxx} = 0$, subject to the conditions $u(0, t) = u(L, t) = u_{xx}(0, t) = u_{xx}(L, t) = u_t(x, 0) = 0$, and $u(x, 0) = f(x)$.
ANS. Let $u = X(x)T(t)$, then we have $\dfrac{X^{(4)}}{X} = -\dfrac{T''}{a^2 T} = \lambda$, where λ is a parameter. By standard arguments it can be shown that the relevant values of λ are positive values. Let $\lambda = \alpha^4$. Then the solutions for X and T are given by

$$X = A \cos \alpha x + B \sin \alpha x + C \cosh \alpha x + D \sin \alpha x,$$

and

$$T = E \cos \alpha^2 t + F \sin \alpha^2 t.$$

$X(0) = 0$ means $A + C = 0$, and $X(L) = 0$ yields

$$A \cos \alpha L + B \sin \alpha L + C \cosh \alpha L + D \sin \alpha L = 0,$$

and $X_{xx}(0) = 0$ implies $2\alpha^2 A = 0$, which gives $A = C = 0$, and $X_{xx}(L) = 0$ which yields $\alpha^2(B \sinh \alpha L - D \sin \alpha L) = 0$. We thus have a pair of two homogeneous equations:

$$B \sinh \alpha L - D \sin \alpha L = 0, \quad B \sinh \alpha L + D \sin \alpha L = 0.$$

For B and D to have nontrivial values, we must have

$$\sinh \alpha L \, \sin \alpha L = 0,$$

i.e., $\alpha L = n\pi$, and $B = 0$ and $T(0) = 0$ are equivalent to $F = 0$. Absorbing E in D and using the initial condition we get

$$u = \sum_{n=1}^{\infty} D_n \sin \frac{n\pi x}{L} \cos \frac{n^2\pi^2 t}{L^2},$$

where $D_n = \dfrac{2}{L} \displaystyle\int_0^L f(x) \sin \frac{n\pi x}{L} dx$.

5.21. Solve $r u_{rr} + u_r + u_{zz} = 0$, $u(a, z) = u_0$, under the conditions $u(a, 0) = 0 = u(a, h)$, and $\lim_{r\to 0} u(r, z) < +\infty$ (steady state temperature in a finite cylinder).
ANS.

$$u = \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{J_0(n\pi r)}{J_0(n\pi a)} \sin \frac{n\pi z}{h}.$$

5.22. Solve $u_{xx} - u_{tt} = e^{-\pi^2 t} \sin \pi x$, such that $u(x,0) = 0 = u_t(x,0) = u(0,t) = u_x(1,t)$.

ANS. A particular solution is given by

$$u_p = -\frac{e^{-\pi^2 t} \sin \pi x}{\pi^2(1+\pi^2)}.$$

Now define $u = v + u_p$. Then the problem becomes

$$v_{xx} - v_{tt} = 0, \quad v(x,0) = \frac{\sin \pi x}{\pi^2(1+\pi^2)}, \quad v_t(x,0) = -\frac{\sin \pi x}{(1+\pi^2)},$$

$$v(0,t) = 0, \quad v_x(1,t) = -\frac{e^{-\pi^2 t}}{\pi(1+\pi^2)}.$$

Assume $v = X(x)T(t)$. Then we have

$$\frac{X''}{X} = \frac{T''}{T} = \text{const.}$$

When the constant is zero, the solution will not make any contribution to v. So we consider two cases: (i) when const $= \lambda^2$, and (ii) when const $= -\alpha^2$. In the first case the solution is

$$v_1 = \sum_\lambda e^{-\lambda t}(A \cosh \lambda x + B \sinh \lambda x),$$

and in the second case

$$v_2 = \sum_\alpha (A_1 \cos \alpha x \cos \alpha t + A_2 \cos \alpha x \sin \alpha t + A_3 \sin \alpha x \cos \alpha t + A_4 \sin \alpha x \sin \alpha t).$$

Applying the boundary conditions to $v + v_1 + v_2$ we get $A = 0$, $B = \dfrac{-1}{\pi^3(1+\pi^2)}$, $\lambda = \pi^2$, $A_1 = A_2 = 0$, $\alpha = (n + \frac{1}{2})\pi$, and A_3, A_4 are to be determined from the initial conditions. Thus we have

$$u = -\frac{e^{-\pi^2 t} \sin \pi x}{\pi^2(1+\pi^2)} - \frac{e^{-\pi^2 t} \sinh \pi^2 x}{\pi^3(1+\pi^2)} + \sum_\alpha (A_3 \sin \alpha x \cos \alpha t + A_4 \sin \alpha x \sin \alpha t).$$

We now apply the initial conditions

$$u(x,0) = -\frac{\sin \pi x}{\pi^2(1+\pi^2)} - \frac{\sinh \pi^2 x}{\pi^3(1+\pi^2)} + \sum_\alpha A_3 \sin \alpha x = 0,$$

$$\sum_\alpha A_3 \sin \alpha x = \frac{\sin \pi x}{\pi^2(1+\pi^2)} + \frac{\sinh \pi^2 x}{\pi^3(1+\pi^2)} = g_1(x)$$

and

$$u_t(x,0) = \frac{\sin \pi x}{(1+\pi^2)} + \frac{\sinh \pi^2 x}{\pi(1+\pi^2)} + \sum_\alpha \alpha A_4 \sin \alpha x = 0,$$

$$\sum_\alpha \alpha A_4 \sin \alpha x = -\frac{\sin \pi x}{(1+\pi^2)} - \frac{\sinh \pi^2 x}{\pi(1+\pi^2)} = g_2(x).$$

Define

$$I_1 = \int_0^1 \sin \pi x \sin \frac{(2n+1)\pi x}{2} dx = \frac{-4(-1)^n}{(2n+3)(2n-1)},$$

$$I_2 = \int_0^1 \sinh \pi^2 x \sin \frac{(2n+1)\pi x}{2} dx = \frac{4(-1)^n \pi^2 \cosh \pi^2}{2\pi^2 + (2n+1)^2}.$$

$$A_3 = 2\left[\frac{I_1}{\pi^2(1+\pi^2)} + \frac{I_4}{\pi^3(1+\pi^2)}\right],$$

$$A_4 = -\frac{4}{(2n+1)\pi}\left[\frac{I_1}{(1+\pi^2)} + \frac{I_4}{\pi(1+\pi^2)}\right].$$

5.23. Solve the Poisson equation $u_{xx} + u_{yy} = -1$, $0 < x, y < 1$, subject to the Dirichlet boundary conditions $u(0,y) = 0 = u(1,y) = u(x,0) = u(x,1)$.

ANS.

$$u(x,y) = \frac{16}{\pi^4} \sum_{\substack{j,k=1 \\ j,k \text{ odd}}}^{\infty} \frac{\sin j\pi x \sin k\pi y}{j^3k^2 + j^2k^3}.$$

6

Integral Transforms

The technique of integral transforms is a powerful tool for the solution of linear partial differential equations.

A function $F(x)$ may be transformed by the formula

$$F(s) = \int_a^b f(x)K(s,x)\,dx$$

provided $F(s)$ exists, where $K(s,x)$ is known as the *kernel* of the transform. A transform becomes useful if we can obtain $f(x)$ from $F(s)$ by some inversion formula. Some well–known integral transforms and their inversion formulas, known as the transform pairs, are given below.

1. The Fourier cosine transform $\tilde{f}_c(\alpha)$ of $f(x)$ is defined as

$$\mathcal{F}_c\{f(x)\} \equiv \tilde{f}_c(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(x\alpha)\,dx, \qquad (6.1a)$$

and its inverse is

$$\mathcal{F}^{-1}\{\tilde{f}_c(\alpha)\} \equiv f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \tilde{f}_c(\alpha)\cos(x\alpha)\,d\alpha. \qquad (6.1b)$$

2. The Fourier sine transform $\tilde{f}_s(\alpha)$ of $f(x)$ is defined as

$$\mathcal{F}_s\{f(x)\} \equiv \tilde{f}_s(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(x\alpha)\,dx, \qquad (6.2a)$$

and its inverse is

$$\mathcal{F}^{-1}\{\tilde{f}_s(\alpha)\} \equiv f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \tilde{f}_s(\alpha)\sin(x\alpha)\,d\alpha. \qquad (6.2b)$$

3. The Fourier complex transform $\mathcal{F}f(x) = \tilde{f}(\alpha)$ of $f(x)$ is defined as

$$\mathcal{F}\{f(x)\} = \tilde{f}(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x)e^{ix\alpha}\,dx, \qquad (6.3a)$$

and its inverse is

$$\mathcal{F}^{-1}\{\tilde{f}(\alpha)\} \equiv f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \tilde{f}(\alpha)e^{-ix\alpha}\,d\alpha. \qquad (6.3b)$$

4. The Laplace transform is defined as

$$\mathcal{L}\{f(t)\} \equiv F(s) = \bar{f}(s) = \int_0^\infty f(t)e^{-st}\,dt, \qquad (6.4a)$$

and its inverse is

$$\mathcal{L}^{-1}\{F(s)\} \equiv f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}\,ds. \qquad (6.4b)$$

5. The Mellin transform is defined as

$$\mathcal{M}\{f(x)\} \equiv F_M(s) = \int_0^\infty f(x)^x s - 1\,dx, \qquad (6.5a)$$

and its inverse is

$$\mathcal{M}^{-1}\{F_M(s)\} \equiv f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_M(s)x^{-s}\,ds. \qquad (6.5b)$$

6. The Hankel transform of order n is defined as

$$\mathcal{H}\{f(x)\} \equiv F_n(s) = \int_0^\infty xf(x)J_n(sx)\,dt, \qquad (6.6a)$$

where its inverse is

$$\mathcal{H}^{-1}\{F_n(s)\} \equiv f(x) = \int_0^\infty sF(s)J_n(sx)\,dt. \qquad (6.6b)$$

These definitions are not unique, particularly in the case of Fourier and Hankel transforms which are sometimes defined in a different manner. In fact, one can develop an infinity of transforms. However, the six transforms defined above are frequently used. Some of the other better–known transforms are Meijer, Kontorowich-Lebedev, Mehler-Foch, Hilbert, and Laguerre. We will, however, discuss only the Laplace and Fourier transforms. Once the use of one transform is completely understood, it is a simple matter to extend one's understanding to another transform.

In most cases a function has to satisfy Dirichlet's conditions in order to possess an integral transform. These conditions in the interval (a,b) are (i) a function has only a finite number of extremum points in (a,b), and (ii) a function has only a finite number of finite discontinuities in (a,b) and no infinite jumps. Unless otherwise stated, it will be assumed that all the functions in the sequel satisfy Dirichlet's conditions.

The student who lacks the knowledge of contour integration technique may omit all material in this chapter containing this technique.

Laplace Transforms

6.1. Notation

It is expected that the reader is familiar with the elementary theory of the Laplace transforms. The following notation is used:

$$\mathcal{L}\left\{f(t)\right\} = F(s) = \bar{f}(s) = \int_0^\infty f(t)e^{-st}\,dt,$$

and

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\bar{f}(s)\right\} = f(t),$$

where s is the variable of the transform which is, in general, a complex variable. Note that the Laplace transform $F(s)$ exists for $s > \alpha$, if the function $f(t)$ is piecewise continuous in every finite closed interval $0 \le t \le b$ $(b > 0)$, and $f(t)$ is of exponential order α, i.e., there exist α, M, and $t_0 > 0$ such that $e^{-\alpha t}\left|f(t)\right| < M$ for $t > t_0$.

We will now state some basic properties of the Laplace transforms.

(i) $\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a)$,
and $\mathcal{L}^{-1}\left\{F(s-a)\right\} = e^{at}f(t)$.

(ii) $\mathcal{L}\left\{H(t-a)f(t-a)\right\} = e^{-as}F(s)$,
and $\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\} = H(t-a)f(t-a)$.

(iii) Convolution Theorem:

$$\mathcal{L}^{-1}\left\{G(s)F(s)\right\} = \int_0^t f(t-u)g(u)du = \int_0^t f(u)g(t-u)du.$$

(iv) $\mathcal{L}\left\{\dfrac{d^n f(t)}{dt^n}\right\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$, and $\mathcal{L}^{-1}\left\{s^nF(s)\right\} = \dfrac{d^n}{dt^n}f(t)$.

(v) $\mathcal{L}^{-1}\left\{\dfrac{1}{s}F(s)\right\} = \displaystyle\int_0^t f(u)du.$

(vi) $\mathcal{L}\left\{t^nf(t)\right\} = (-1)^n\dfrac{d^nF}{ds^n}$, and $\mathcal{L}^{-1}\left\{(-1)^n\dfrac{d^nF}{ds^n}\right\} = t^nf(t)$.

(vii) If $\mathcal{L}\left\{f(x,t)\right\} = F(x,s)$, then

$$\mathcal{L}\left\{\frac{\partial f(x,t)}{\partial x}\right\} = \frac{\partial F(x,s)}{\partial x}, \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{\partial F(x,s)}{\partial x}\right\} = \frac{\partial f(x,t)}{\partial x}.$$

The last two results are based on the Leibniz rule and are extremely effective. The Leibniz rule states that if $g(x,t)$ is an integrable function of t for each value of x, and the partial derivative $\dfrac{\partial g(x,t)}{\partial x}$ exists and is continuous in the region under consideration, and if

$$f(x) = \int_a^b g(x,t)dt,$$

then

$$f'(x) = \int_a^b \frac{\partial g(x,t)}{\partial x}dt. \tag{6.7}$$

6.2. Basic Laplace Transforms

A table of basic Laplace transform pairs is given in Appendix B. We shall show the effectiveness of the above properties in the derivation of certain Laplace transforms.

We start with the easily established result that

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}. \tag{6.8}$$

Differentiating both sides with respect to a, we get

$$\mathcal{L}\left\{te^{at}\right\} = \frac{1}{(s-a)^2}, \tag{6.9}$$

and repeating this differentiation n times, we find that

$$\mathcal{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}}. \tag{6.10}$$

After replacing a by ib, choosing an appropriate n, and comparing the real and imaginary parts of both sides, we can get the Laplace transforms of functions $t^p \cos bt$ and $t^p \sin bt$, and then combining with Property (i), we can get the Laplace transforms of functions $t^p e^{at} \cos bt$ and $t^p e^{at} \sin bt$. For example, if we choose $n = 2$, then we have

$$\mathcal{L}\left\{t^2 e^{at}\right\} = \frac{2!}{(s-a)^3}. \tag{6.11}$$

Now letting $a = ib$, we get

$$\mathcal{L}\left\{t^2 e^{ibt}\right\} = \frac{2!}{(s-ib)^3}, \tag{6.12}$$

which yields

$$\mathcal{L}\left\{t^2(\cos bt + i \sin bt)\right\} = \frac{2(s+ib)^3}{(s^2+b^2)^3}. \tag{6.13}$$

Expanding the numerator on the right side of (6.13), we get

$$\mathcal{L}\left\{t^2(\cos bt + i \sin bt)\right\} = \frac{2(s^3 + 3is^2b - 3sb^2 - ib^3)}{(s^2+b^2)^3}. \tag{6.14}$$

Then equating the real and imaginary parts in (6.14), we obtain

$$\mathcal{L}\left\{t^2 \cos bt\right\} = \frac{2(s^3 - 3sb^2)}{(s^2+b^2)^3}, \tag{6.15}$$

and

$$\mathcal{L}\left\{t^2 \sin bt\right\} = \frac{2(3s^2b - b^3)}{(s^2+b^2)^3}. \tag{6.16}$$

In[1]:=

```
<<Calculus`LaplaceTransform`
```

In[2]:=

```
LaplaceTransform[t^n Exp[a t],t,s]
```

Out[2]=

```
(-a + s)^(-1 - n) Gamma[1 + n]
```

In[3]:=

```
LaplaceTransform[t^2 Exp[a t],t,s]
```

Out[3]=

$$\frac{2}{(-a + s)^3}$$

In[4]:=

```
LaplaceTransform[t^2 Exp[I b t],t,s]
```

Out[4]=

$$\frac{2}{(-I\ b + s)^3}$$

In[5]:=

```
LaplaceTransform[t^2 Cos[b t],t,s]//Simplify
```

Out[5]=

$$\frac{2\,s\,(-3\,b^2+s^2)}{(b^2+s^2)^3}$$

In[6]:=

```
LaplaceTransform[t^2 Sin[b t],t,s]//Simplify
```

Out[6]=

$$\frac{2\,b\,(-b^2+3\,s^2)}{(b^2+s^2)}$$

The Laplace transforms of $\mathcal{L}\left\{e^{at}t^2\cos bt\right\}$ and $\mathcal{L}e^{at}\left\{t^2\sin bt\right\}$ can now be easily obtained.

An important Laplace inverse is

$$\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\} = \operatorname{erfc}\frac{a}{2\sqrt{t}}, \tag{6.17}$$

where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du, \quad \operatorname{erfc}(x) = 1-\operatorname{erf}(x). \tag{6.18}$$

In[7]:=

```
(* An important formula *)

InverseLaplaceTransform[Exp[-a Sqrt[s]]/s,s,t]
```

Out[8]=

$$1 - \mathrm{Erf}\left[\frac{a}{2\,\mathrm{Sqrt}[t]}\right]$$

We can derive a large number of the Laplace inverses by using Properties (i)–(vii).

EXAMPLE 6.1. $\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{\sqrt{s}}\right\}$ can be obtained by differentiating formula (6.17) with respect to a. Thus,

$$\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi t}}e^{-a^2/4t}$$

is obtained after differentiating (6.17) with respect to a and canceling out the negative sign on both sides. Although the usual method of deriving the Laplace inverse of $\frac{e^{-a\sqrt{s}}}{\sqrt{s}}$ is by contour integration, or by using the Laplace inverse of $\frac{e^{-a\sqrt{s}}}{s}$, an interesting method is as follows (Churchill, 1972): Define $\frac{e^{-a\sqrt{s}}}{\sqrt{s}} = y$ and $e^{-a\sqrt{s}} = z$. Then

$$y' = -\frac{1}{2s^{3/2}}\,e^{-a\sqrt{s}} - \frac{a}{2s}\,e^{-a\sqrt{s}},$$

which yields

$$2sy' + y + az = 0.$$

Similarly, $z' = -\frac{a}{2\sqrt{s}}e^{-a\sqrt{s}}$ yields

$$2z' + ay = 0.$$

Taking the inverse transform of these equations, we get

$$aG - F - 2tF' = 0, \quad \text{and} \quad aF - 2tG = 0,$$

where $\mathcal{L}^{-1}\{y\} = F(t)$, and $\mathcal{L}^{-1}\{z\} = G(t)$. From these two equations in F and G we get

$$F' = \frac{1}{2t}\left(\frac{a^2F}{2t} - F\right), \tag{6.19}$$

The solution of (6.19) is

$$F = \frac{A}{\sqrt{t}} e^{-a^2/4t},$$

which gives

$$G = \frac{aA}{2\sqrt{t^3}} e^{-a^2/4t}.$$

Note that if $a = 0$, then $y = \frac{1}{\sqrt{s}}$, and $F(t) = \frac{1}{\sqrt{\pi t}}$ implies that $A = \frac{1}{\sqrt{\pi}}$. Hence

$$F(t) = \frac{1}{\sqrt{\pi t}} e^{-a^2/4t}, \quad G = \frac{a}{\sqrt{\pi t^3}} e^{-a^2/4t}. \tag{6.20}$$

One can then integrate $\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi t}} e^{-a^2/4t}$ with respect to a from 0 to a and obtain $\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\}$. In this problem we have assumed that $\mathcal{L}\frac{1}{\sqrt{t}} = \sqrt{\frac{\pi}{s}}$ (see Exercise 6.11). ■

In[8]:=

```
Needs["Calculus`LaplaceTransform`"]

(* An important Laplace Inverse *)
```

In[9]:=

```
f[s_]:= Exp[-a Sqrt[s]]/s
InverseLaplaceTransform[f[x],s,t] == Erfc[a/(2 Sqrt[t])]
```

Out[11]=

```
            a                   a
1 - Erf[----------] == Erfc[---------]
        2 Sqrt[t]           2 Sqrt[t]
```

In[11]:=


```
RHS:= Erfc[a/(2 Sqrt[t])]
```

In[12]:=

```
D[RHS,a]
```

Out[14]=

```
              1
-(----------------------------)
   a²/(4 t)
  E         Sqrt[Pi] Sqrt[t]
```

In[14]:=

```
InverseLaplaceTransform[-D[f[s],a],s,t] == -D[RHS,a]
```

Out[15]=

```
True
```

EXAMPLE 6.2. $\mathcal{L}^{-1}\left\{e^{-a\sqrt{s}}\right\} = \frac{a}{2\sqrt{\pi t^3}} e^{-a^2/4t}$ is obtained by differentiating the formula in the previous example and canceling out the negative sign. ■

In[1]:=

```
Needs["Calculus`LaplaceTransform`"];

(* f[s] is defined as in Example 6.1 *)

f[s] Sqrt[s]
```

Out[3]=

```
          1
-----------------------
  a Sqrt[s]
 E          Sqrt[s]
```

In[3]:=

```
D[f[s] Sqrt[s],a]
```

Out[4]=

```
-E^-(a Sqrt[s])
```

In[4]:=

```
InverseLaplaceTransform[-D[f[s] Sqrt[s],a],s,t]
```

Out[5]=

```
            a
------------------------
2 E^(a^2/(4 t)) Sqrt[Pi t^3]
```

EXAMPLE 6.3. If we integrate the formula $\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\} = \operatorname{erfc}\frac{a}{2\sqrt{t}}$ with respect to a from 0 to a, we get

$$\int_0^a \mathcal{L}^{-1}\left\{\frac{e^{-x\sqrt{s}}}{s}\right\} dx = \int_0^a \operatorname{erfc}\frac{x}{2\sqrt{t}} dx.$$

Now after changing the order of integration and the Laplace inversion and carrying out the integration on the left side, we get

$$\int_0^a \mathcal{L}^{-1}\left\{\frac{e^{-x\sqrt{s}}}{s}\right\} dx = \mathcal{L}^{-1}(s^{-3/2} - s^{-3/2}e^{-a\sqrt{s}}), \tag{6.21}$$

and the right side yields

$$\int_0^a \operatorname{erfc}\frac{x}{2\sqrt{t}}\, dx = \left[x \operatorname{erfc}\frac{x}{2\sqrt{t}}\right]_0^a + \frac{1}{\sqrt{\pi t}}\int_0^a x\, e^{-x^2/4t}\, dx$$
$$= a \operatorname{erfc}\frac{a}{2\sqrt{t}} - 2\sqrt{\frac{t}{\pi}}e^{-a^2/4t} + 2\sqrt{\frac{t}{\pi}}.$$

Since $\mathcal{L}^{-1}\left\{s^{-3/2}\right\} = 2\sqrt{\frac{t}{\pi}}$, we get

$$\mathcal{L}^{-1}\left\{s^{-3/2}e^{-a\sqrt{s}}\right\} = 2\sqrt{\frac{t}{\pi}}e^{-a^2/4t} - a \operatorname{erfc}\frac{a}{2\sqrt{t}}. \blacksquare \tag{6.22}$$

In[1]:=

```
Needs["Calculus`LaplaceTransform`"];

(* f[s] is defined as in Example 6.1 *)

f[s]/.a -> x Sqrt[s]
```

Out[2]=

```
       1
--------------
E^(Sqrt[s]x) s

(* Change the order of operations *)
```

In[3]:=

```
LHS = Integrate[ f[s]/.a -> x ,{x,0,a}]
```

Out[3]=

```
s^-(3/2) -          1
           -------------------
           E^(a Sqrt[s]) s^3/2
```

In[4]:=

```
InverseLaplaceTransform[f[s],s,t] == Erfc[a/(2 Sqrt[t])]
```

Out[4]=

```
1 - Erf[    a     ] == Erfc[    a    ]
        ----------            ---------
        2 Sqrt[t]             2 Sqrt[t]

(* Define a substitution *)
```

In[5]:=

```
trans =
InverseLaplaceTransform[f[s],s,t] -> Erfc[a/(2 Sqrt[t])]
```

Out[5]=

$$1 - \text{Erf}\left[\frac{a}{2\ \text{Sqrt}[t]}\right] \to \text{Erfc}\left[\frac{a}{2\,\text{Sqrt}[t]}\right]$$

In[6]:=

```
Integrate[Erfc[x/(2 Sqrt[t])],{x,0,a}]
```

Out[6]=

$$a + \frac{2\ \text{Sqrt}[t]}{\text{Sqrt}[\text{Pi}]} - \frac{2\ \text{Sqrt}[t]}{E^{a^2/(4t)}\ \text{Sqrt}[\text{Pi}]} - a\ \text{Erf}\left[\frac{a}{2\ \text{Sqrt}[t]}\right]$$

```
(*   Collect the terms involving a,
in order to apply the transformation *)
```

In[7]:=

```
RHS=
Collect[Integrate[Erfc[x/(2 Sqrt[t])],{x,0,a}],a]/.trans
```

Out[7]=

$$\frac{2\ \text{Sqrt}[t]}{\text{Sqrt}[\text{Pi}]} - \frac{2\ \text{Sqrt}[t]}{E^{a^2/(4t)}\ \text{Sqrt}[\text{Pi}]} + a\ \text{Erfc}\left[\frac{a}{2\ \text{Sqrt}[t]}\right]$$

```
(* Prevent evaluation of the InverseLaplaceTransform *)
```

In[8]:=

```
ILT[X_]:= Hold[InverseLaplaceTransform[X,s,t]]

sexpr:=
ILT[InverseLaplaceTransform[s^(-3/2) Exp[-a Sqrt[s]]]]
```


In[10]:=

```
Solve[
InverseLaplaceTransform[
s^(-3/2),s,t] - ILT[sexpr] == RHS,ILT[sexpr]]//Simplify
```

Out[8]=

$$\{\{\text{Hold}[\text{InverseLaplaceTransform}[\frac{1}{E^{a\ \text{Sqrt}[s]}\ s^{3/2}}, s, t]] \to \frac{2\ \text{Sqrt}[t]}{E^{a^2/(4t)}\ \text{Sqrt}[\text{Pi}]} - a\ \text{Erfc}[\frac{a}{2\ \text{Sqrt}[t]}]\}\}$$

EXAMPLE 6.4. Evaluate $\mathcal{L}^{-1}\left\{\dfrac{e^{-a\sqrt{s+c}}}{s}\right\}$. We know from (6.20) that

$$\mathcal{L}^{-1}\left\{e^{-a\sqrt{s}}\right\} = \frac{a}{2\sqrt{\pi t^3}}\, e^{-a^2/4t}. \tag{6.23}$$

Hence, using Property (i),

$$\mathcal{L}^{-1}\left\{e^{-a\sqrt{s+c}}\right\} = \frac{a}{2\sqrt{\pi t^3}} e^{-ct-a^2/4t}. \tag{6.24}$$

Using the convolution theorem with $F(s) = \dfrac{1}{s}$ and $G(s) = e^{-a\sqrt{s+c}}$, we get

$$\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s+c}}}{s}\right\} = \int_0^t \frac{a}{2\sqrt{\pi u^3}}\, e^{-cu-a^2/4u} du. \tag{6.25}$$

Note that

$$\frac{a}{2\sqrt{\pi u^3}} = \frac{a}{4\sqrt{\pi u^3}} + \frac{1}{2}\sqrt{\frac{c}{u}} + \frac{a}{4\sqrt{\pi u^3}} - \frac{1}{2}\sqrt{\frac{c}{u}},$$

and

$$cu + \frac{a^2}{4u} = (\sqrt{cu} + \frac{a}{2\sqrt{u}})^2 - a\sqrt{c} = (\sqrt{cu} - \frac{a}{2\sqrt{u}})^2 + a\sqrt{c}.$$

If we now substitute

$$x = \frac{a}{2\sqrt{u}} + \sqrt{cu}, \quad \text{and} \quad y = \frac{a}{2\sqrt{u}} - \sqrt{cu},$$

then the integral on the right side of (6.25) can be expressed as

$$\frac{1}{\sqrt{\pi}}\left[e^{a\sqrt{c}}\int_{x_1}^{\infty} e^{-x^2}dx + e^{-a\sqrt{c}}\int_{y_1}^{\infty} e^{-y^2}dy\right]$$
$$= \frac{1}{2}\left[e^{a\sqrt{c}}\operatorname{erfc}(\frac{a}{2\sqrt{t}} + \sqrt{ct}) + e^{-a\sqrt{c}}\operatorname{erfc}(\frac{a}{2\sqrt{t}} - \sqrt{ct})\right].$$

Hence

$$\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s+c}}}{s}\right\} = \frac{1}{2}\Big[e^{a\sqrt{c}}\operatorname{erfc}\left(\frac{a}{2\sqrt{t}} + \sqrt{ct}\right) + e^{-a\sqrt{c}}\operatorname{erfc}\left(\frac{a}{2\sqrt{t}} - \sqrt{ct}\right)\Big]. \blacksquare \tag{6.26}$$

We will state a very useful theorem without proof.

THEOREM 6.1. *If* $G(s) = \sum_{1}^{\infty} G_k(s)$ *is uniformly convergent, then*

$$\mathcal{L}^{-1}G(s) = g(t) = \sum_{1}^{\infty} g_k(t), \tag{6.27}$$

where $\mathcal{L}^{-1}G_k(s) = g_k(t)$.

EXAMPLE 6.5. Since

$$\mathcal{L}^{-1}\left\{s^{-3/2}e^{1/s}\right\}$$
$$= \mathcal{L}^{-1}\left\{\frac{1}{s^{3/2}}\left[1 - \frac{1}{s} + \frac{1}{2!s^2} - \frac{1}{3!s^3} + \cdots + (-1)^n\frac{1}{n!s^n} + \cdots\right]\right\}$$
$$= \mathcal{L}^{-1}\sum_{0}^{\infty}\frac{(-1)^n}{n!\,s^{n+3/2}} = \frac{1}{\sqrt{\pi}}\sum_{0}^{\infty}\frac{(-1)^n(2\sqrt{t})^{2n+1}}{(2n+1)!} = \frac{1}{\sqrt{\pi}}\sin(2\sqrt{t}), \tag{6.28}$$

we find that this result and Property (iv) give

$$\mathcal{L}^{-1}\left\{s^{-1/2}e^{1/s}\right\} = \frac{1}{\sqrt{\pi t}}\cos(2\sqrt{t}). \blacksquare \tag{6.29}$$


```
(* An interesting use of the package SymbolicSum *)

In[1]:=

Needs["Calculus`LaplaceTransform","Algebra`SymbolicSum`"];

(* Compute an infinite sum *)

In[2]:=

Sum[s^(3/2) (1/s)^k/k!,{k,0,Infinity}]

Out[2]=

 1/s  3/2
E    s

(* Compute an infinite sum *)

In[3]:=

Sum[(-1)^k/(s^(k+3/2)k!),{k,0,Infinity}]

Out[3]=

    1
----------
 1/s  3/2
E    s

(* Compute the InverseLaplaceTransform of a term *)

In[4]:=

InverseLaplaceTransform[(-1)^k/(s^(k+3/2)k!),s,t]

Out[4]=

     k  1/2 + k
 (-1)  t
----------------
          3
k! Gamma[- + k]
          2
```

```
(* Sum the infinite series *)
```

In[5]:=

```
1/Sqrt[Pi] Sum[((-1)^k ((2 Sqrt[t])^(2k+1)))/(2k+1)!,
   {k,0,Infinity}]
```

Out[5]=

```
Sin[2 Sqrt[t]]
--------------
  Sqrt[Pi]
```

```
(* Compute derivative and use Property 4 *)
```

In[6]:=

```
D[1/(Sqrt[Pi]) Sin[2 Sqrt[t]],t]//Simplify
```

Out[6]=

```
 Cos[2 Sqrt[t]]
----------------
Sqrt[Pi] Sqrt[t]
```

In[7]:=

```
Hold[InverseLaplaceTransform[Release[
   s^1 s^(-1/2)] Exp[1/s],s,t]] ==
   D[1/(Sqrt[Pi]) Sin[2 Sqrt[t]],t]//Simplify
```

Out[7]=

```
Hold[InverseLaplaceTransform[E^(1/s) Sqrt[s], s, t]]==
     Cos[2 Sqrt[t]]
   ----------------
   Sqrt[Pi] Sqrt[t]
```

```
(* Note Mma fails to compute the following transform *)
```

In[8]:=


```
InverseLaplaceTransform[s^(-1/2) Exp[1/s],s,t]
```

Out[8]=

```
$Failed
```

EXAMPLE 6.6. Consider a semi-infinite medium bounded by $0 \le x \le \infty$, $-\infty < y,\ z < \infty$, which is kept at an initial zero temperature, while its face $x = 0$ is maintained at a time-dependent temperature $f(t)$. The problem is to find the temperature for $t > 0$. By applying the Laplace transform to the heat conduction equation $kT_{xx} = T_t$, we get $\overline{T}_{xx} = \frac{s}{k}\overline{T}$, where $\overline{T} = \mathcal{L}\{T\}$. The solution of this equation is

$$\overline{T} = Ae^{mx} + Be^{-mx}, \tag{6.30}$$

where $m = \sqrt{\frac{s}{k}}$. Since $\overline{T}$ remains bounded as $x \to \infty$, we find that $A = 0$. The boundary condition at $x = 0$ in the transform domain yields $B = \bar{f}(s)$, where $\bar{f}(s)$ is the Laplace transform of $f(t)$. Thus, the solution in the transform domain is

$$\overline{T} = \bar{f}(s)\, e^{-mx}.$$

In order to carry out the inversion, we use the convolution property and Example 6.2 and get

$$T = \int_0^t \frac{x\, e^{-x^2/4k\tau}}{2\tau\sqrt{\pi k\tau}} f(t-\tau)\, d\tau.$$

If $\bar{f}(s) = 1$, then the solution for T reduces to

$$T = \int_0^t \frac{xe^{-x^2/4k\tau}}{2\tau\sqrt{\pi k\tau}}\, d\tau.$$

This solution is the fundamental solution for the heat conduction equation for the half-space. In the special case when $f(t) = 1$, the solution is given by

$$T = T_0 \operatorname{erfc}\left(\frac{x}{2\sqrt{kt}}\right). \blacksquare$$

In the above example, we have assumed a function whose Laplace transform is 1. The question arises: Is there such a function? We shall try to answer this question in a heuristic manner. Consider the step function $H(t)$ which is defined by

$$H(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases}.$$

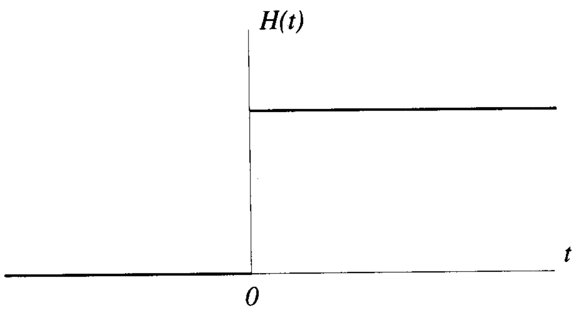


Fig. 6.1.

The Laplace transform of $H(t)$ is $\overline{H}(s) = \dfrac{1}{s}$. Then, by Property 4 of the Laplace transforms, $\mathcal{L}H'(t) = s\overline{H}(s) = 1$. Let us examine $H'(t)$ closely. Obviously, it vanishes for $|t| > 0$ and does not exist for $t = 0$. From the graph of $H(t)$, it is clear that there is a vertical jump at $t = 0$ (see Fig. 6.1). Therefore, it is reasonable to assume that $\lim_{t \to 0} H'(t) \to \infty$. But since $\dfrac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} H'(t)\, dt = 1$, it is obvious that a function like $H'(t)$ does not exist in the classical sense. Such a function is called generalized function or distribution. The function $H'(t)$ is a generally denoted by $\delta(t)$ and is known as the Dirac delta function. This function is defined such that

$$\delta(t) = \begin{cases} 0 & \text{for } |t| > 0 \\ \infty & \text{for } t = 0, \end{cases} \quad \text{and} \quad \int_{-\varepsilon}^{\varepsilon} \delta(t)\, dt = 1. \tag{6.31}$$

To make this function acceptable in the classical sense, we can modify the definition as follows:

$$\delta(t) = \begin{cases} 0 & \text{for } |t| > \varepsilon, \\ \dfrac{1}{2\varepsilon} & \text{for } |t| < \varepsilon \end{cases} \quad \text{as } \varepsilon \to 0.$$

This definition is consistent with the classical definition of a function and automatically satisfies Equation (6.31). An important consequence of Equation (6.31) is that if $f(t)$ is any continuous function, then

$$\int_{-\infty}^{\infty} \delta(t) f(t)\, dt = \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} \delta(t) f(t)\, dt = f(0).$$

To prove this assertion, we note that by definition

$$\int_{-\infty}^{\infty} \delta(t) f(t)\, dt = \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} \frac{1}{2\varepsilon} f(t)\, dt = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} 2\varepsilon f(t') = f(0),$$

where t' is a point at which $f(t)$ takes its average value in $(-\varepsilon, \varepsilon)$, such that $t' \in (-\varepsilon, \varepsilon)$, and, therefore, $t' \to 0$ as $\varepsilon \to 0$. For more details on the Dirac delta function, see §7.1.1.

EXAMPLE 6.7. Consider an infinite slab bounded by $0 \leq x \leq l$, $-\infty < y, z < \infty$, with initial zero temperature. The face $x = 0$ is maintained at a constant temperature T_0 and the face $x = l$ is maintained at zero temperature. The problem is to find the temperature in the slab for $t > 0$. Proceeding as in the above example, the solution in the transform domain is given by Equation (6.30). Applying the boundary conditions in the transform domain we get

$$A + B = \frac{T_0}{s},$$

and

$$A\, e^{ml} + B\, e^{-ml} = 0.$$

These two equations yield

$$A = \frac{-T_0 e^{ml}}{2s \sinh ml}, \quad \text{and} \quad B = \frac{T_0}{s} - A.$$

Using these values in $\overline{T}$ and simplifying, we find that

$$\overline{T} = \frac{T_0 \sinh(l - x)}{s \sinh l}.$$

Rewriting this solution as

$$\overline{T} = \frac{T_0}{s}\, e^{-ml} \left(e^{m(l-x)} - e^{-m(l-x)}\right) \left(1 - e^{-2ml}\right)^{-1},$$

and expanding the last factor by the binomial theorem, we get

$$\overline{T} = \frac{T_0}{s} e^{-ml} \left(e^{m(l-x)} - e^{-m(l-x)} \right) \sum_0^\infty e^{-2nml}$$
$$= \frac{T_0}{s} \sum_0^\infty \left(e^{-m(2nl+x)} - e^{-m[(2n+2)l-x]} \right),$$

which on inversion yields

$$T = T_0 \sum_0^\infty \left(\operatorname{erf} \frac{2(nl+1)-x}{2\sqrt{kt}} - \operatorname{erf} \frac{(2nl+x)}{2\sqrt{kt}} \right).$$

Alternately, we can use the Cauchy residue theorem and obtain a Fourier series type result. Thus

$$T = \sum \text{residues of } \frac{T_0 e^{st} \sinh(l-x)}{s \sinh l}$$
$$= T_0 \Big[1 - \frac{x}{l} - \sum_1^\infty \frac{2}{n\pi} e^{-n^2\pi^2 kt/l^2} \sin(n\pi x/l) \Big]. \blacksquare \tag{6.32}$$

EXAMPLE 6.8. Consider a solid sphere of radius a. Suppose its initial temperature is zero and its surface is maintained at a temperature T_0 for $t \geq 0$. The problem is to determine the temperature of the sphere at any subsequent time. The heat conduction equation in this case is

$$T_{rr} + \frac{2}{r} T_r = \frac{1}{k} T_t. \tag{6.33}$$

If we introduce a new independent variable u, such that $u = rT$, then the heat conduction equation reduces to

$$u_{rr} = \frac{1}{k} u_t, \tag{6.34}$$

which can be solved as in Example 6.2. ■

EXAMPLE 6.9. Solve the wave equation

$$u_{tt} = c^2 u_{xx}, \tag{6.35}$$

subject to the initial conditions

$$u = u_t = 0, \quad \text{for } t \leq 0,$$

and the boundary conditions

$$u = 0 \quad \text{at } x = 0, \quad \text{and} \quad u_x = T \quad \text{at } x = l.$$

If we apply the Laplace transform to the wave equation (6.35), we get

$$\bar{u}_{xx} = c^{-2} s^2 \bar{u}.$$

Its solution is

$$\bar{u} = A\, e^{-sx/c} + B\, e^{sx/c}.$$

Applying the boundary conditions in the transform domain, we get

$$A + B = 0, \quad \text{and} \quad -A\, e^{-sl/c} + B\, e^{sl/c} = \frac{cT}{s^2}.$$

Solving for A and B and substituting their values in the solution for $\bar{u}$ we get

$$\bar{u} = \frac{Tc}{s^2} \frac{\sinh \dfrac{sx}{c}}{\cosh \dfrac{sl}{c}}. \tag{6.36}$$

This equation can be expressed as

$$\bar{u} = \frac{Tc}{s^2} \left(\frac{e^{sy} - e^{-sy}}{e^{sL} + e^{-sL}} \right),$$

where $y = x/c$, and $L = l/c$. This, after some manipulation similar to that in Example 6.7, yields

$$\bar{u} = \frac{T}{s^2} \sum_0^\infty (-1)^n \left(e^{-s[(2n+1)L-y]} - e^{-s[(2n+1)L+y]} \right),$$

which, after inversion, gives

$$u = T \sum_0^\infty (-1)^n \big[(t-(2n+1)L+y) H(t-(2n+1)L+y) -$$
$$(t-(2n+1)L-y) H(t-(2n+1)L-y) \big]. \tag{6.37}$$

Alternately,

$$u = \sum \text{residues of } \{\bar{u}\, e^{st}\}$$
$$= T \Big[x - \frac{8l}{\pi^2} \sum_0^\infty \frac{(-1)^n}{(2n+1)^2} \sin \frac{(2n+1)\pi x}{2l} \cos \frac{(2n+1)\pi ct}{2l} \Big]. \blacksquare \tag{6.38}$$

EXAMPLE 6.10. The partial differential equation for propagation of sound waves produced by the motion of a sphere of radius a in an infinite expanse of fluid is given by

$$r^2 D_t^2 u = c^2 D_r(r^2 D_r)u, \tag{6.39}$$

with initial conditions

$$u(r,0) = D_t u(r,0) = 0, \tag{6.40}$$

and the boundary conditions

$$D_r u(a,t) = f(t), \quad \text{and} \quad u \to 0 \quad \text{as } r \to \infty. \tag{6.41}$$

We first introduce a new independent variable $v = ru$. This substitution reduces the partial differential equation to the standard wave equation

$$D_t^2 v = c^2 D_r^2 v. \tag{6.42}$$

Applying the Laplace transform and using the second boundary condition, the solution in the transform domain is given by

$$\bar{v} = A\, e^{-sr/c},$$

or

$$\bar{u} = \frac{A}{r} e^{-sr/c}.$$

Applying the first boundary condition in the transform domain, we get

$$\bar{u} = -\frac{ac}{s+\frac{c}{a}} \frac{F(s)}{r} e^{-s(r-a)/c}.$$

By the convolution property

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s+k}\right\} = \int_0^t e^{-k(t-x)} f(x)\, dx = \phi(t), \quad k = \frac{c}{a}. \tag{6.43}$$

Thus the solution is

$$u = \mathcal{L}^{-1}\left\{-\frac{ac}{s+\frac{c}{a}} \frac{F(s)}{r} e^{-s(r-a)/c}\right\}, \tag{6.44}$$

which by Property (ii) and Equation (6.44) yields

$$u = -\frac{ca}{r}\phi\left(t - \frac{r-a}{c}\right) H\left(t - \frac{r-a}{c}\right). \tag{6.45}$$

If $f(t) = \delta(t)$, then $\phi(t) = e^{-ct/a}$, and the solution becomes

$$u = -\frac{ac}{r} e^{-ct/a(t-(r-a)/c)} H\left(t - \frac{r-a}{c}\right). \tag{6.46}$$

We can derive solutions for other values of $f(t)$ by evaluating the appropriate ϕ function. ∎

6.3. Inversion Theorem

We will now establish the inversion theorem:

THEOREM 6.2. *If $F(s)$ is the Laplace transform of $f(t)$, then*

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\, e^{st}\, ds. \tag{6.47}$$

In order to prove this theorem, we first state and prove a lemma.

LEMMA 6.1. *If $f(z)$ is analytic and of order $O(z^{-k})$ in the half-plane $\Re z > \gamma$, where γ and k are real constants, then*

$$f(z_0) = \frac{1}{2\pi i}\lim_{\beta\to\infty}\int_{\gamma-i\beta}^{\gamma+i\beta} \frac{f(z)}{z_0 - z}\, dz, \quad \Re z_0 > \gamma. \tag{6.48}$$

PROOF. Consider the rectangle in Fig. 6.2. Choose $\beta > |\gamma|$ and such that z_0 lies in this rectangle. By the Cauchy integral formula

$$\int_\Gamma \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0), \tag{6.49}$$

where Γ is the contour ABCDA. Let S denote the contour ABCD, then

$$\int_\Gamma \frac{f(z)}{z-z_0}\, dz = \int_{DA} \frac{f(z)}{z-z_0}\, dz + \int_S \frac{f(z)}{z-z_0}\, dz.$$

Since

$$\int_{DA} \frac{f(z)}{z-z_0}\,dz = -\int_{AD} \frac{f(z)}{z-z_0}\,dz,$$

we have

$$-\int_{\gamma-i\beta}^{\gamma+i\beta} \frac{f(z)}{z-z_0}\,dz + \int_S \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0). \qquad (6.50)$$

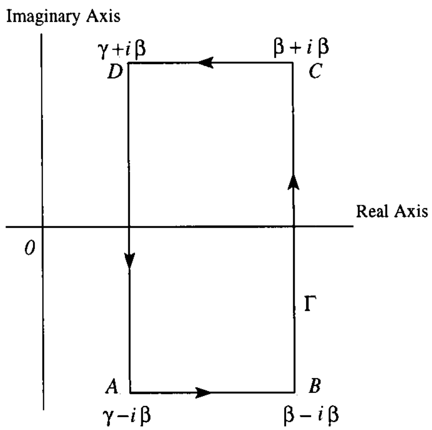


Fig. 6.2.

Now consider $\int_S \frac{f(z)}{z-z_0}\,dz$ as $\beta \to \infty$. Obviously, $\beta \to \infty$ implies that $|z| \to \infty$ on S. Thus $|z| \geq \beta$ for points on S. If we take β large enough so that $\beta > 2|z_0|$, then $|z_0| < \frac{1}{2}\beta \leq \frac{1}{2}|z|$, or $\left|\frac{z_0}{z}\right| < \frac{1}{2}$ implies that $\left|1-\frac{z_0}{z}\right| \geq 1 - \left|\frac{z_0}{z}\right| > \frac{1}{2}$. Noting that $|f(z)| < M|z|^{-k}$ for large z, we get

$$\left|\frac{f(z)}{z-z_0}\right| = \left|\frac{f(z)}{z}\frac{1}{\left(1-\frac{z_0}{z}\right)}\right| \leq \frac{M}{\left|z^{k+1}\left(1-\frac{z_0}{z}\right)\right|} \leq \frac{2M}{\beta^{k+1}}.$$

It now follows that

$$\begin{aligned}\left|\int_S \frac{f(z)}{z-z_0}\,dz\right| &< \frac{2M}{\beta^{k+1}}\int_S |dz| \\ &= \frac{2M}{\beta^{k+1}}\,(\text{length of } S) \\ &= \frac{2M}{\beta^k}\left(\frac{4\beta-2\gamma}{\beta}\right) = \frac{2M}{\beta^k}\left(4-\frac{2\gamma}{\beta}\right).\end{aligned}$$

Thus,

$$\int_S \frac{f(z)}{z-z_0}\,dz = 0.$$

Hence, from (6.50),

$$-\int_{\gamma-i\beta}^{\gamma+i\beta} \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0),$$

or

$$F(s) = \frac{1}{2\pi i}\int_{\gamma-i\beta}^{\gamma+i\beta} \frac{F(z)}{s-z}\,dz. \blacksquare \qquad (6.51)$$

The proof of Theorem 6.2 for the Laplace transform is now elementary. By taking the Laplace inverse of both sides of the above equation, we have

$$f(t) = L^{-1}F(s) = \frac{1}{2\pi i}\int_{\gamma-i\beta}^{\gamma+i\beta} L^{-1}\frac{F(z)}{s-z}\,dz = \frac{1}{2\pi i}\int_{\gamma-i\beta}^{\gamma+i\beta} F(z)e^{zt}\,dz. \blacksquare \qquad (6.52)$$

LEMMA 6.2. *If $f(z) < CR^{-k}$, $z = Re^{i\theta}, -\pi \leq \theta \leq \pi, R > R_0$, where R_0, C and k are constants, then $\int_\Gamma e^{zt} f(z)\,dz \to 0$ as $R \to \infty$, provided $t > 0$, where Γ is the arc $BB'C$ or $CA'A$, and R is the radius of the circular arc with chord AB (Fig. 6.3).*

PROOF. Consider the integral over the arc BB'. Since for BB', we have $\alpha = \cos^{-1}\frac{\gamma}{R}$, where α is the angle BOB', we get

$$\begin{aligned}\left|\int_{BB'} e^{zt} f(z)\,dz\right| &< \int_\alpha^{\pi/2} \left|CR^{-k}e^{Rte^{i\theta}}Rie^{i\theta}\right|\,d\theta \\ &= CR^{-k+1}\int_\alpha^{\pi/2} \left|e^{Rt\cos\theta}\right|\,d\theta \\ &\leq CR^{-k+1}\int_\alpha^{\pi/2} \left|e^{\gamma t}\right|\,d\theta \\ &= CR^{-k+1}(\pi/2-\alpha)e^{\gamma t} \\ &= CR^{-k+1}\,e^{\gamma t}\sin^{-1}\frac{\gamma}{R} \to 0 \quad \text{as } R \to \infty.\end{aligned}$$

Similarly, $\int_{A'A} e^{zt} f(z)\, dz \to 0$ as $R \to \infty$.

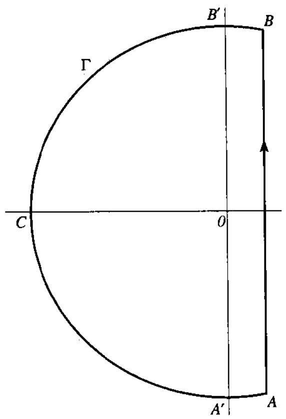


Fig. 6.3.

Let us now consider the integral over the arc $B'CA'$. By following the above procedure, we get

$$\begin{aligned}\left|\int_{BB'C} e^{zt} f(z)\, dz\right| &< CR^{-k+1}\int_{\pi/2}^{3\pi/2} \left|e^{Rt\cos\theta}\right| d\theta \\ &= CR^{-k+1}\int_0^{\pi} e^{-Rt\sin\phi} d\phi \quad \text{where } \theta = \pi/2+\phi \\ &= 2CR^{-k+1}\int_0^{\pi/2} e^{-Rt\sin\phi} d\phi \\ &\le 2CR^{-k+1}\int_0^{\pi/2} e^{-Rt\phi/\pi}\, d\phi \\ &= \frac{\pi C R^{k+1}}{t}(e^{-Rt}-1) \to 0 \quad \text{as } R\to\infty.\end{aligned}$$

Hence $\int_\Gamma e^{zt} f(z)\, dz \to 0$ as $R \to \infty$, provided $t > 0$. ■

This result enables us to convert the integral $\frac{1}{2\pi i}\int_{\gamma-i\beta}^{\gamma+i\beta} F(z)\, e^{zt}\, dz$ into an integral over the contour $-\Gamma$.

EXAMPLE 6.11. Evaluate $\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\}$ by contour integration. If $f(t) = \mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\}$, then, by the Laplace inversion theorem, we have

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{-a\sqrt{s}}}{s} e^{st}\, ds. \tag{6.53}$$

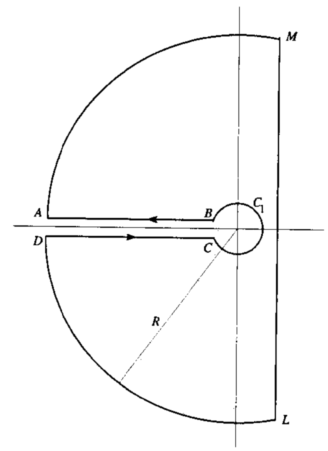


Fig. 6.4.

Consider the Bromwich contour $LABC_1CDL$ (Fig. 6.4). Then by Cauchy's theorem

$$\begin{aligned}I = \int_{c-i\infty}^{c+i\infty} \frac{e^{-a\sqrt{s}}}{s} e^{st}\, ds &= \int_{LD} F(s)\, ds + \int_{DC} F(s)\, ds + \\ &\int_{C_1} F(s)\, ds + \int_{BA} F(s)\, ds + \int_{AM} F(s) ds.\end{aligned}$$

Now, as established in Lemma 6.2,

$$\int_{LD} F(s)\, ds + \int_{AM} F(s)\, ds = 0,$$

where

$$F(s) = \frac{e^{-a\sqrt{s}}}{s} e^{st}.$$

The integral over the circle C_1 can be easily shown to be $2\pi i$. This can be done by taking the radius to be ε and substituting $s = \varepsilon e^{i\theta}$. On BA, $s = u\,e^{i\pi}$, and

$$\begin{aligned} I_{BA} &= \int_{\varepsilon\to 0}^{R\to\infty} \frac{1}{u\,e^{i\pi}} e^{-a\sqrt{u}e^{i\pi/2}+ute^{i\pi}} e^{i\pi}\,du \\ &= \int_0^\infty \frac{1}{u}\, e^{-ia\sqrt{u}-ut}\,du \\ &= \int_0^\infty \frac{1}{u} e^{-ut}(\cos a\sqrt{u} - i\sin a\sqrt{u})\,du \\ &= 2\int_0^\infty \frac{1}{v} e^{-v^2 t}(\cos av - i\sin av)\,dv, \end{aligned}$$

where $u = v^2$. Similarly

$$\int_{CD} = -2\int_0^\infty \frac{1}{v} e^{-v^2 t}(\cos av + i\sin av)\,dv.$$

Hence

$$\int_{CD} + \int_{BA} = -4i\int_0^\infty \frac{1}{v} e^{-v^2 t}\sin av\,dv.$$

In order to evaluate the integral $\int_0^\infty \frac{1}{v}\, e^{-v^2 t}\sin av\,dv$, we consider the integral $\int_0^\infty e^{-v^2 t}\cos av\,dv$. Then

$$\begin{aligned} \int_0^\infty e^{-v^2 t}\cos av\,dv &= \Re\int_0^\infty e^{-v^2 t + iav}\,dv \\ &= \Re\, e^{-a^2/4t}\int_0^\infty e^{-(v\sqrt{t}-ia/2\sqrt{t})^2}\,dv \\ &= \Re\, e^{-a^2/4t}\int_{-ia/2\sqrt{t}}^\infty e^{-u^2}\,du \quad \text{where } u = v\sqrt{t} - ia/2\sqrt{t} \\ &= \Re\frac{e^{-a^2/4t}}{\sqrt{t}}\left[\int_0^\infty e^{-u^2}\,du + \int_{-ia/2\sqrt{t}}^0 e^{-u^2}\,du\right]. \end{aligned}$$

Hence

$$\int_0^\infty e^{-v^2 t}\cos av\,dv = \frac{\sqrt{\pi}\,e^{-a^2/4t}}{2\sqrt{t}}. \tag{6.54}$$

Integrating both sides of this equation with respect to a from 0 to a, we get

$$\int_0^\infty \frac{1}{v}\, e^{-v^2 t}\sin av\,dv = \sqrt{\frac{\pi}{4t}}\int_0^a e^{-x^2/4t}\,dx = \frac{\pi}{2}\,\mathrm{erf}\,\frac{a}{2\sqrt{t}}.$$

Thus

$$\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\} = \frac{1}{2\pi i}\left[2\pi i - 4i\frac{\pi}{2}\,\mathrm{erf}\,\frac{a}{2\sqrt{t}}\right] = \mathrm{erfc}\,\frac{a}{2\sqrt{t}}. \blacksquare \tag{6.55}$$

6.4. Exercises

6.1. Using the techniques shown above and $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, derive the Laplace transform of $\sin at$, $\cos at$, $e^{bt}\sin at$, $e^{bt}\cos at$, $t^n e^{bt}$, and $\sinh bt$.

6.2. Show that $\mathcal{L}^{-1}\left\{e^{-a\sqrt{s}}\right\} = \frac{a}{2\sqrt{\pi t^3}}\,e^{-a^2/4t}$.

6.3. Using the Laplace transform method, solve the partial differential equation $u_t = u_{zz} - Mu$, given that $u(z,0) = 0$, and $u(0,t) = u_0$, $\lim_{x\to\infty} u(z,t) \to 0$ for $t > 0$. [This problem corresponds to the flow of a viscous fluid on an infinite moving plate under the influence of a constant magnetic field applied perpendicular to the plate.] Derive the solution when $M = 0$.
ANS.

$$u = \frac{1}{2}\left\{e^{z\sqrt{M}}\,\mathrm{erfc}\left(\frac{z}{2\sqrt{t}} + \sqrt{Mt}\right) + e^{-z\sqrt{M}}\,\mathrm{erfc}\left(\frac{z}{2\sqrt{t}} - \sqrt{Mt}\right)\right\}.$$

6.4. Using the Laplace transform method, solve the partial differential equation in the transform domain $u_t = u_{zz} + ku_{tzz}$, given that $u(z,0) = 0$, and $u(0,t) = u_0$, $\lim_{z\to\infty} u(z,t) \to 0$ for $t > 0$. Expand the solution in the transform domain in the form $\bar{u} = \frac{u_0}{s}\, e^{-z\sqrt{s}}[1 +$ powers of $k]$. Invert the first two terms of this expansion.
ANS. $\frac{u}{u_0} = \mathrm{erfc}\,\frac{z}{2\sqrt{t}} + \frac{kz}{4t\sqrt{\pi t}}\left(\frac{z^2}{2t} - 1\right)\, e^{-z^2/4t}$.
This problem corresponds to the flow of a viscoelastic fluid on an infinite moving plate. Obtain the exact solution in terms of definite integrals by using the contour integration.
ANS. $u(z,t) = 1 - \frac{1}{\pi}\int_0^\lambda \frac{1}{x} e^{-xt}\sin\sqrt{\frac{\lambda x}{\lambda - x}}\,dx$, where $\frac{1}{\lambda} = k$.

6.5. Using the Laplace transform method, solve the partial differential equation $u_t = u_{xx}$, with the initial condition $u(x,0) = 0$ and the boundary conditions $u_x(0,t) = 0$, and $u_x(1,t) = 1$.

ANS. The solution in the transform domain is $\bar{u} = \dfrac{\cosh x\sqrt{s}}{s^{3/2}\sinh\sqrt{s}}$. Find two different inverses of this solution, by expanding the solution in a series of the type shown in Example 6.7 and by the residue theorem. Thus,

$$u = \sum_{n=0}^{\infty}\Big\{2\sqrt{t/\pi} + \left(e^{-(2n+1-x)^2/4t} + e^{-(2n+1+x)^2/4t}\right) - (2n+1-x)\operatorname{erfc}\frac{2n+1-x}{2\sqrt{t}} - (2n+1+x)\operatorname{erfc}\frac{2n+1+x}{2\sqrt{t}}\Big\},$$

and

$$u = \frac{x^2}{2} + t - \frac{1}{6} - \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2\pi^2}e^{-n^2\pi^2 t}\cos n\pi x.$$

6.6. Using the Laplace transform method, solve the partial differential equation $u_{tt} = u_{xx}$, with the initial conditions $u(x,0) = -\dfrac{(1-x)^2}{2}$, $u_t(x,0) = 0$, and the boundary conditions $u_x(0,t) = 1$ and $u_x(1,t) = 0$.

ANS. $u = \dfrac{1}{2}t^2 - \dfrac{(1-x)^2}{2}$.

6.7. Using the Laplace transform method, solve the partial differential equation $u_t = u_{xx}$, with the initial condition $u(x,0) = 0$ and the boundary conditions $u_x(0,t) = 0$ and $u(1,t) = 1$.

ANS. The solution in the transform domain is $\bar{u} = \dfrac{\cosh x\sqrt{s}}{s\cosh\sqrt{s}}$. Find two different inverses of this solution, by expanding the solution in a series of the type shown in Example 6.7 and by the residue theorem.

ANS. $u = \displaystyle\sum_0^{\infty}(-1)^n\left[\operatorname{erfc}\frac{2n+1-x}{2\sqrt{t}} + \operatorname{erfc}\frac{2n+1+x}{2\sqrt{t}}\right]$, or

$$u = 1 - \sum_0^{\infty}(-1)^n\frac{4\cos(2n+1)\pi x/2}{(2n+1)\pi}e^{-(2n+1)^2\pi^2 t/4}.$$

6.8. Using the Laplace transform method, solve the partial differential equation $u_{tt} = u_{xx}$, with the initial condition $u(x,0) = -\dfrac{(1-x)^2}{2}$, $u_t(x,0) = 0$, and the boundary conditions $u(0,t) = 1$, $u_x(1,t) = 0$.

ANS.

$$\begin{aligned} u &= \sum_0^{\infty}(-1)^n\frac{3}{2}\{H(t-2n-2+x) + H(t-2n-x)\} \\ &+ \sum_0^{\infty}\frac{(-1)^n}{2}[(t-2n-2+x)^2H(t-2n-2+x) \\ &+ (t-2n-x)^2H(t-2n-x)] - \frac{1}{2}(1-x)^2H(t) - \frac{t^2}{2}H(t). \end{aligned}$$

6.9. Using the Laplace transform method, solve the partial differential equation

$$u_{xx} - u_{tt} = e^{-\pi^2 t}\sin\pi x,$$
$$u(x,0) = 0, \quad u_t(x,0) = 0, \quad u(0,t) = u_x(1,t) = 0.$$

ANS. Applying Laplace transform to the partial differential equation, we get $(D^2 - s^2)\bar{u} = \dfrac{1}{s+\pi^2}\sin\pi x$. Its solution is given by

$$\bar{u} = Ae^{sx} + Be^{-sx} - \frac{1}{(s+\pi^2)(s^2+\pi^2)}\sin\pi x.$$

Applying the boundary conditions we get $A + B = 0$, and

$$s(Ae^s - Be^{-s}) + \frac{\pi}{(s+\pi^2)(s^2+\pi^2)}\cos\pi = 0.$$

These two equations yield

$$A = -B = -\frac{\pi}{2s(s+\pi^2)(s^2+\pi^2)\sinh s}.$$

Thus,

$$\begin{aligned} \bar{u} &= -\frac{\pi\sinh sx}{s(s+\pi^2)(s^2+\pi^2)\sinh s} - \frac{1}{(s+\pi^2)(s^2+\pi^2)}\sin\pi x \\ &= \frac{1}{\pi}\left[\frac{1}{\pi^2(1+\pi^2)(s+\pi^2)} + \frac{1}{(1+\pi^2)(s^2+\pi^2)}(s+1) - \frac{1}{\pi^2 s}\right]\frac{\sinh sx}{\sinh s} \\ &\quad - \frac{1}{\pi^2(1+\pi^2)}\left[\frac{1}{s+\pi^2} - \frac{s}{s^2+\pi^2} + \frac{\pi^2}{s^2+\pi^2}\right]\sin\pi x \\ &= \frac{1}{\pi}\left[\frac{1}{\pi^2(1+\pi^2)(s+\pi^2)} + \frac{s+1}{(1+\pi^2)(s^2+\pi^2)} - \frac{1}{\pi^2 s}\right] \\ &\quad \left[\sum_0^{\infty}\left(e^{(x-2k-1)s} - e^{-(2k+1+x)s}\right)\right] - \frac{1}{\pi^2(1+\pi^2)}\left[\frac{1}{s+\pi^2} - \frac{s}{s^2+\pi^2} \right. \\ &\quad \left. + \frac{\pi^2}{s^2+\pi^2}\right]\sin\pi x. \end{aligned}$$

On inversion we find

$$
\begin{aligned}
u = \frac{1}{\pi}\sum_0^\infty \Big\{ & \frac{e^{-\pi^2(t+x-2k-1)}}{\pi^2(1+\pi^2)} \\
& + \frac{\cos(t+x-2k-1)+\sin(t+x-2k-1)}{(1+\pi^2)} - \frac{1}{\pi^2}\Big\} H(t+x-2k-1) \\
- \frac{1}{\pi}\sum_0^\infty \Big\{ & \frac{e^{-\pi^2(t-x-2k-1)}}{\pi^2(1+\pi^2)} \\
& + \frac{\cos(t-x-2k-1)+\sin(t-x-2k-1)}{(1+\pi^2)} - \frac{1}{\pi^2}\Big\} H(t-x-2k-1) \\
& - \frac{1}{\pi^2(1+\pi^2)}\left[e^{-\pi^2 t} - \cos\pi t + \pi\sin\pi t\right]\sin\pi x.
\end{aligned}
$$

6.10. Solve the diffusion equation

$$u_t = a\,u_{xx}, \quad 0 < x < \pi, \quad t > 0,$$

subject to the boundary conditions $u(0,t) = 1 - e^{-t}$ and $u(\pi,t) = 0$ for $t \ge 0$, and the initial condition $u(x,0) = 0$ for $0 < x < \pi$.
ANS. By the Laplace transform method, we get

$$\frac{d^2\bar u}{dx^2} = \frac{s}{a}\,\bar u,$$

with $\bar u(0,s) = \dfrac{1}{s(s+1)}$, and $\bar u(\pi,s) = 0$, which has the solution

$$\bar u(x,s) = \frac{1}{s(s+1)}\,\frac{\sinh\sqrt{s/a}(\pi - x)}{\sin\sqrt{s/a}\pi}.$$

Then the inversion formula gives

$$u(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}\sinh\sqrt{s/a}(\pi-x)}{s(s+1)\sinh\sqrt{s/a}\pi}\,ds,$$

where c is any positive constant. Assuming that a is not of the form n^{-2}, the integrand has simple poles at $s = 0, -1$, and $-an^2$, $n = 1, 2, \cdots$. The contour is completed by an infinite left side semicircle with $\Re s = c$ as diameter, which is defined as the limit of a sequence of semicircles Γ_n that cross the negative s–axis between the poles at

$-an^2$ and $-a(n+1)^2$. The limit of the integrand around Γ_n is zero as $n \to \infty$. The residue at the pole $s = 0$ is $(\pi - x)/\pi$, and at $s = -1$ it is

$$e^{-t}\,\frac{\sin[(\pi - x)/\sqrt{a}]}{\sin(\pi/\sqrt{a})}.$$

The residue at $s = -an^2$ is given by

$$\lim_{s\to an^2} \frac{s+n^2}{s(s+1)}\,e^{st}\,\frac{\sinh\sqrt{s/a}(\pi - x)}{\sinh\sqrt{s/a}\pi} = \frac{2\sin nx}{n\pi(an^2-1)}\,e^{-an^2 t}.$$

Hence

$$u(x,t) = \frac{\pi - x}{\pi}\,e^{-t}\,\frac{\sin[(\pi-x)/\sqrt{a}]}{\sin(\pi/\sqrt{a})} + \frac{2}{\pi}\sum_{n=1}^\infty \frac{\sin nx}{n(an^2-1)}\,e^{-an^2 t}.$$

Note that $u \to \dfrac{\pi - x}{\pi}$ as $t \to \infty$, which gives the steady state temperature in the interval $0 \le x \le \pi$.

6.11. Show that

$$\mathcal{L}\{t^p\} = \frac{\Gamma(p+1)}{s^{p+1}},$$

where $\Gamma(x)$ is the gamma function defined by $\Gamma(x) = \displaystyle\int_0^\infty x^p\,e^{-x}\,dx$.

ANS. In $\mathcal{L}\{t^p\} = \displaystyle\int_0^t t^p\,e^{-st}\,dt$, let $st = x$. Then

$$\int_0^\infty t^p\,e^{-st}\,dt = \frac{1}{s^{p+1}}\int_0^\infty x^{p+1}\,e^{-x}\,dx = \frac{\Gamma(p+1)}{s^{p+1}}.$$

6.12. Solve the nonhomogeneous Cauchy problem in $R^1 \times R^+$:

$$
\begin{aligned}
u_t - a u_{xx} &= f(x,t), \quad x \in R^1, \\
u(x,0) &= g(x), \quad t > 0,
\end{aligned}
$$

where $g(x)$ is prescribed.
SOLUTION. Using the Laplace transform we get

$$\frac{d^2\bar u}{dx^2} - s\bar u = -f(x),$$

with the boundary conditions $\bar{u}(0,s) = 0 = \bar{u}(l,s)$. The solution of the above homogeneous equation is given by

$$\bar{u}(x,s) = \frac{1}{\sqrt{s}\sinh(l\sqrt{s})}\Big[\sinh((l-x)\sqrt{s})\int_0^x \sinh(y\sqrt{s})f(y)\,dy + \sinh(x\sqrt{s})\int_x^l \sinh((l-y)\sqrt{s})f(y)\,dy\Big].$$

By inversion, the solution of this problem becomes

$$u(x,t) = \lim_{R\to\infty}\left[\frac{1}{2\pi i}\int_{u-iR}^{u+iR} e^{st}\bar{u}(x,s)\,ds\right].$$

The integrand in this solution has simple poles at $s = 0, -k_n^2$, where $k_n = n\pi/l, n = 1,2,\cdots$. We choose a contour of integration that avoids these poles and take $u > 0$. Then it can be shown that the residue at the pole $s = 0$ is zero, while at the poles $s = -k_n^2$ it is given by

$$\frac{2}{l}e^{-k_n^2 t}\sin k_n x \int_0^l f(y)\sin k_n y\,dy.$$

Hence the final formal solution of the problem is given by

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{l}\int_0^l f(y)\sin k_n y\,dy\right]e^{-k_n^2 t}\sin k_n x.$$

Alternately, if we use the series representation

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

for $|x| < 1$, then we can write

$$\sinh(l\sqrt{s}) = 2e^{-l\sqrt{s}}\sum_{n=0}^{\infty} e^{-2nl\sqrt{s}},$$

where $\Re s$ is chosen such that $e^{-2l\sqrt{s}} < 1$. Then expressing the hyperbolic functions in terms of exponentials, using the formula

$$\mathcal{L}^{-1}\left\{\frac{e^{-z\sqrt{s}}}{\sqrt{s}}\right\} = \frac{e^{-z^2/4t}}{\sqrt{\pi t}},$$

where z is independent of s and t (see §A.2), and interchanging the orders of summation and integration when needed, we obtain the solution for $|x| < 1$.

Fourier Transforms

We will not discuss the underlying theory of Fourier transforms. We will only define and discuss their properties and applications. From the definitions of the transform pairs (6.1a,b) and (6.2a,b) we note that the Fourier cosine and sine transforms and their inverses are symmetric. But the Fourier complex transform and its inverse are related in the following manner: If $\mathcal{F}f(x) = \tilde{f}(\alpha)$, then $\mathcal{F}\tilde{f}(x) = f(-\alpha)$. Various authors have defined the Fourier transform in different ways, but we shall follow the notation used by Sneddon (1957). A table of basic Fourier transform pairs is given in Appendix B.

6.5. Fourier Integral Theorems

THEOREM 6.3 (FOURIER INTEGRAL THEOREM). *If $f(x)$ satisfies the Dirichlet's conditions on the entire real line and is absolutely integrable on $(-\infty,\infty)$, then*

$$\frac{1}{2}[f(x+0)+f(x-0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\alpha x}\,d\alpha\int_{-\infty}^{\infty} f(u)e^{i\alpha u}\,du. \quad (6.56)$$

THEOREM 6.4 (FOURIER COSINE THEOREM). *If $f(x)$ satisfies the Dirichlet's conditions on the non-negative real line and is absolutely integrable on $(0,\infty)$, then*

$$\frac{1}{2}[f(x+0)+f(x-0)] = \frac{2}{\pi}\int_0^{\infty} d\alpha\int_0^{\infty} f(u)\cos(\alpha u)\cos(\alpha x)\,du. \quad (6.57)$$

THEOREM 6.5 (FOURIER SINE THEOREM). *If $f(x)$ satisfies the Dirichlet's conditions on the non-negative real line and is absolutely* *integrable on* $(0,\infty)$, *then*

$$\frac{1}{2}[f(x+0)+f(x-0)] = \frac{2}{\pi}\int_0^\infty d\alpha \int_0^\infty f(u)\sin(\alpha u)\sin(\alpha x)\,du. \quad (6.58)$$

If $f(x)$ is continuous then $\frac{1}{2}[f(x+0)+f(x-0)] = f(x)$. These three integrals form the basis of the Fourier transforms.*

6.6. Properties of Fourier Transforms

We will use the following notation: Let $\mathcal{F}f(x) = \tilde{f}(\alpha)$. Then

(1) $\mathcal{F}f(x-a) = e^{i\alpha a}\tilde{f}(\alpha)$.

(2) $\mathcal{F}f(ax) = \frac{1}{|a|}\tilde{f}(\alpha/a)$.

(3) $\mathcal{F}e^{iax}f(x) = \tilde{f}(\alpha + a)$.

(4) $\mathcal{F}\tilde{f}(x) = f(-\alpha)$.

(5) $\mathcal{F}x^n f(x) = (-1)^n \frac{d^n}{d\alpha^n}\tilde{f}(\alpha)$.

(6) $\mathcal{F}f(ax)e^{ibx} = \frac{1}{|a|}\tilde{f}\left(\frac{\alpha+b}{a}\right)$.

6.6.1. Fourier transforms of the derivatives of a function. Assuming that $f(x)$ is differentiable n times and the function and its derivatives approach zero as $|x| \to \infty$, then it can be easily established that

$$\tilde{f}^{(p)}(\alpha) = (-i\alpha)\tilde{f}^{(p-1)},$$

where $\tilde{f}^{(p)}$ is the Fourier transform of $f^{(p)}(x)$, which is the p-th derivative of $f(x)$ for $0 \le p \le n$.

*See Sneddon (1957) for proof.

If $\lim_{x\to\infty} f^{(p)}(x) = 0$, and $\lim_{x\to 0} f^{(p)}(x) = \sqrt{\frac{\pi}{2}}\,c_p$, then

$$\tilde{f}_c^{(p)} = -c_{p-1} + \alpha \tilde{f}_s^{(p-1)}, \quad (6.59)$$

and

$$\tilde{f}_s^{(p)} = -\alpha \tilde{f}_c^{(p-1)}. \quad (6.60)$$

6.6.2. Convolution theorems for Fourier transform. The convolution or Faltung of $f(t)$ and $g(t)$ over $(-\infty,\infty)$ is defined by

$$f\star g = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\eta)g(x-\eta)\,d\eta = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-\eta)g(\eta)\,d\eta. \quad (6.61)$$

THEOREM 6.6. *Let* $\tilde{f}(\alpha)$ *and* $\tilde{g}(\alpha)$ *be the Fourier transforms of* $f(x)$ *and* $g(x)$, *respectively. Then the inverse Fourier transform of* $\tilde{f}(\alpha)\,\tilde{g}(\alpha)$ *is*

$$\mathcal{F}^{-1}\left\{\tilde{f}(\alpha)\tilde{g}(\alpha)\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\eta)g(x-\eta)\,d\eta.$$

PROOF. Consider

$$\begin{aligned}\int_{-\infty}^{\infty} f(\eta)g(x-\eta)\,d\eta &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\eta)\,d\eta \int_{-\infty}^{\infty}\tilde{g}(\alpha)e^{-i\alpha(x-\eta)}d\alpha\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{g}(\alpha)e^{-i\alpha x}d\alpha\int_{-\infty}^{\infty} f(\eta)e^{i\alpha x\eta}\,d\eta\\ &= \int_{-\infty}^{\infty}\tilde{f}(\alpha)\tilde{g}(\alpha)e^{-i\alpha x}d\alpha,\end{aligned}$$

which proves the theorem. ∎

THEOREM 6.7. *Let* $\tilde{f}(\alpha)$ *and* $\tilde{g}(\alpha)$ *be the Fourier transforms of* $f(x)$ *and* $g(x)$, *respectively, then*

$$\int_{-\infty}^{\infty}\tilde{f}(\alpha)\tilde{g}(\alpha)d\alpha = \int_{-\infty}^{\infty} f(-\eta)g(\eta)\,d\eta. \quad (6.62)$$

PROOF. Consider

$$\begin{aligned}\int_{-\infty}^{\infty}\tilde{f}(\alpha)\tilde{g}(\alpha)\,d\alpha &= \int_{-\infty}^{\infty}\tilde{f}(\alpha)\,d\alpha\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\eta)e^{i\alpha\eta}d\eta\\ &= \int_{-\infty}^{\infty} g(\eta)\,d\eta\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{f}(\alpha)e^{i\alpha\eta}d\alpha\\ &= \int_{-\infty}^{\infty} g(\eta)f(-\eta)d\eta, \quad \text{by Property 4.}\ ∎\end{aligned}$$

6.6.3 Some Fourier transform formulas.

EXAMPLE 6.12. Find the Fourier transform of $f(x) = e^{-k|x|}, k > 0$.

$$\begin{aligned}\mathcal{F}f(x) = \tilde{f}(\alpha) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-k|x|}e^{ix\alpha}\,dx \\ &= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{kx}e^{ix\alpha}\,dx + \int_{0}^{\infty} e^{-kx}e^{ix\alpha}\,dx\right] \\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{k+i\alpha} - \frac{1}{-k+i\alpha}\right) = \frac{k\sqrt{2}}{\sqrt{\pi}(k^2+\alpha^2)}.\end{aligned}$$

Now by Property 4, the Fourier Transform of $f(x) = \dfrac{k\sqrt{2}}{\sqrt{\pi}(k^2+x^2)}$ should be $\tilde{f}(\alpha) = e^{-k|\alpha|}$. It is interesting as well as instructive to check if this is the case. Since $f(x) = f(-x)$, $\tilde{f}(\alpha) = \tilde{f}(-\alpha)$, we have

$$\begin{aligned}\mathcal{F}\frac{k\sqrt{2}}{\sqrt{\pi}(k^2+x^2)} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{k\sqrt{2}}{\sqrt{\pi}(k^2+x^2)}e^{ix\alpha}\,dx \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{ke^{ix\alpha}}{(k^2+x^2)}\,dx \\ &= \frac{1}{\pi}\left[\int_{-\infty}^{0}\frac{ke^{ix\alpha}}{(k^2+x^2)}\,dx + \int_{0}^{\infty}\frac{ke^{ix\alpha}}{(k^2+x^2)}\,dx\right] \\ &= \frac{1}{\pi}\left[\int_{0}^{\infty}\frac{ke^{-ix\alpha}}{(k^2+x^2)}\,dx + \int_{0}^{\infty}\frac{ke^{ix\alpha}}{(k^2+x^2)}\,dx\right] \\ &= \frac{2}{\pi}\int_{0}^{\infty}\frac{k\cos x\alpha}{(k^2+x^2)}\,dx \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{k\cos x\alpha}{(k^2+x^2)}\,dx.\end{aligned}$$

The standard method of evaluating this integral is by contour integration. The contour is the upper half–circle with radius R and center at the origin if $\alpha > 0$ and the lower half–circle if $\alpha < 0$. Its value is $e^{-k\alpha}$ if $\alpha > 0$, and $e^{k\alpha}$ if $\alpha < 0$. Thus the value can be expressed as $e^{-k|\alpha|}$.

A number of other Fourier transforms can be found by differentiating both sides of $\mathcal{F}e^{-k|x|} = \dfrac{k\sqrt{2}}{\sqrt{\pi}(k^2+\alpha^2)}$ with respect to k. For example, the Fourier transform of $|x|e^{-k|x|}$ is $\sqrt{\dfrac{2}{\pi}}\dfrac{k^2-\alpha^2}{(k^2+\alpha^2)^2}$. ■

```
(* Load the package *)

In[1]:=

Needs["Calculus`FourierTransform"];

In[2]:=

?$FourierOverallConstant

Out[2]=

$FourierOverallConstant is the default setting for the
option FourierOverallConstant (an option to FourierTrans-
form and related functions).

In[3]:=

$FourierOverallConstant

Out[3]=

1

(* Reset the constant *)

In[4]:=

$FourierOverallConstant = 1/Sqrt[2 Pi]

Out[4]=

      1
 ------------
 2 Sqrt[Pi]

In[5]:=
```

```
FourierTransform[Exp[-k Abs[x]],x,w]
```

Out[5]=

$$\frac{k\ \mathrm{Sqrt}[\frac{2}{Pi}]}{k^2+w^2}$$

```
(* Alternate calculation; takes time *)
```

In[6]:=

```
int1:= Integrate[Exp[k x] Exp[I x a],x,-Infinity,0]
int2:= Integrate[Exp[-k x] Exp[I x a],x,0,Infinity]
```

In[8]:=

```
result = 1/Sqrt[2 Pi] (int1 + int2)//Simplify
```

```
General::intinit:  Loading integration packages -- please
wait.
```

Out[7]=

$$\frac{k\ \mathrm{Sqrt}[\frac{2}{Pi}]}{k^2+a^2}$$

EXAMPLE 6.13. Find the Fourier transform of $f(x) = e^{-kx^2}$. Then

$$\begin{aligned}
\tilde{f}(\alpha) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-kx^2}e^{ix\alpha}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-k(x^2-ix\alpha/k-\alpha^2/4k^2+\alpha^2/4k^2)}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-k(x-i\alpha/k)^2-\alpha^2/4k}\,dx \\
&= \frac{1}{\sqrt{2\pi k}}e^{-\alpha^2/4k}\int_{-\infty}^{\infty} e^{-u^2}\,du \\
&= \frac{1}{\sqrt{2\pi k}}e^{-\alpha^2/4k}\sqrt{\pi} \\
&= \frac{1}{\sqrt{2k}}e^{-\alpha^2/4k}. \blacksquare
\end{aligned}$$

In[1]:=

```
Needs["Calculus`FourierTransform"];
```

In[2]:=

```
FourierTransform[Exp[-k t2^],t,w]
```

Out[2]=

$$\frac{1}{\mathrm{Sqrt}[2]\ E^{w^2/(4k)}\ \mathrm{Sqrt}[k]}$$

EXAMPLE 6.14. Find the Fourier transform of $f(x) = 0$ for $x < 0$ and $f(x) = xe^{-ax}$ for $x > 0$.

$$\begin{aligned}
\mathcal{F}f(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} xe^{-ax}e^{i\alpha x}\,dx \\
&= \frac{xe^{-ax+i\alpha x}}{\sqrt{2\pi}(i\alpha-a)}\Big|_0^{\infty} - \frac{1}{\sqrt{2\pi}(i\alpha-a)}\int_0^{\infty} e^{-ax}e^{i\alpha x}\,dx \\
&= -\frac{1}{\sqrt{2\pi}(i\alpha-a)}\int_0^{\infty} e^{-ax}e^{i\alpha x}\,dx \\
&= \frac{1}{\sqrt{2\pi}(i\alpha-a)^2}. \blacksquare
\end{aligned}$$

EXAMPLE 6.15. Find the Fourier transform of $f(x) = 0$ if $x < b$ and $f(x) = e^{-a^2x^2}$ if $0 < b < x$.

$$\begin{aligned}\tilde{f}(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_b^\infty e^{-a^2x^2} e^{ix\alpha}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_b^\infty e^{-a^2(x-i\alpha/2a^2)^2-\alpha^2/4a^2}\, dx \\ &= \frac{e^{-\alpha^2/4a^2}}{a\sqrt{2\pi}} \int_{(ab-i\alpha/2a)}^{\infty} e^{-u^2}\, du \\ &= \frac{1}{2a\sqrt{2}} e^{-\alpha^2/4a^2} \operatorname{erfc}\left(ab - \frac{i\alpha}{2a}\right). \ \blacksquare\end{aligned}$$

In[1]:=

```
<<Declare.m
```

Out[1]=

```
{Declare, NewDeclare, NonPositive, RealQ}
```

In[2]:=

```
Declare[a,Positive];
f[x_]:= Exp[-a^2 x^2]
int:= Integrate[f[x] Exp[I x alpha],{x,b,Infinity}]
sub[X_]:= 1 - Erf[X] -> Erfc[X//Together]
Simplify[1/Sqrt[2 Pi] int]/.sub[a*((-I/2*alpha)/a^2 + b)]
```

Out[6]=

```
       -I alpha + 2 a^2 b
  Erfc[------------------]
              2 a
  ------------------------
  2 Sqrt[2] a E^(alpha^2/(4 a^2))
```

EXAMPLE 6.16. Solve the partial differential equation $u_x + u_y + kyu = f(x)$, in the domain $|x| < 0, y > 0$, with the boundary conditions $u(x, 0) = 0$, $\lim_{x\to\pm\infty} u(x, y) = 0$, where $f(x)$ is a function such that $f(x) \to 0$ as $|x| \to \infty$.

The partial differential equation in the domain of the Fourier transform with respect to x is

$$\tilde{u}(\alpha, y)_y + (-i\alpha + ky)\,\tilde{u}(\alpha, y) = \tilde{f}(\alpha), \tag{6.63}$$

where $\tilde{u} = \mathcal{F}u$ and $\tilde{f}(\alpha) = \mathcal{F}f$. The differential equation in $\tilde{u}$ is a linear equation of first order and its solution subject to the given boundary conditions is

$$\tilde{u} = e^{i\alpha y - ky^2/2} \tilde{f}(\alpha) \int_0^y e^{-i\alpha t + kt^2/2}\, dt. \tag{6.64}$$

This solution on inversion yields

$$u = e^{-ky^2/2} \int_0^y f(x - y + t) e^{kt^2/2}\, dt. \tag{6.65}$$

Note that by using Property 1, $\mathcal{F}^{-1}\tilde{f}(\alpha)e^{-i\alpha(t-y)} = f(x - y + t)$. ■

EXAMPLE 6.17. Find the solution of the Laplace's equation $u_{xx} + u_{yy} = 0$ in the domain $|x| < \infty$ and $y \geq 0$, with the conditions that $u \to 0$ as $|x| \to \infty$ or as $y \to \infty$ and $u(x, 0) = \delta(x)$. After applying the Fourier transform to the partial differential equation with respect to x, we get

$$\tilde{u}_{yy} - \alpha^2 \tilde{u} = 0.$$

The appropriate solution is

$$\tilde{u} = Ae^{-|\alpha|y}.$$

Applying the boundary condition at $y = 0$ in the transform domain we get $\tilde{u}(\alpha, 0) = A = \frac{1}{\sqrt{2\pi}}$. Hence $\tilde{u} = \frac{1}{\sqrt{2\pi}} e^{-|\alpha|y}$. On inverting, we obtain

$$u(x, y) = \frac{1}{\pi} \frac{y}{x^2 + y^2}.$$

We can now use Duhamel's principle to obtain the solution to the problem with arbitrary condition $u(x, 0) = f(x)$. Then the solution is

$$u(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y f(\eta)}{(x - \eta)^2 + y^2}\, d\eta, \tag{6.66}$$

which is known as the Poisson integral representation for the Dirichlet problem in the half-plane. ■

EXAMPLE 6.18. Find the general solution of the diffusion equation $u_t = ku_{xx}$ for the homogeneous initial condition $u(x,0) = f(x)$, subject to the conditions that $\lim_{x\to\pm\infty} f(x), u(x,t) \to 0$. Applying Fourier transform to the partial differential equation we get

$$\tilde{u} + k\alpha^2\tilde{u} = 0,$$

with the initial condition $\tilde{u}(\alpha,0) = \tilde{f}(\alpha)$. Hence, the solution after applying the initial condition is

$$\tilde{u}(\alpha,t) = \tilde{f}(\alpha)e^{-k\alpha^2 t},$$

which on inversion yields

$$u(x,t) = \frac{1}{2\sqrt{\pi kt}}\int_{-\infty}^{\infty} f(\eta)e^{-(x-\eta)^2/4kt}\,d\eta,$$

where we have used the convolution property and Example 6.13. ∎

6.7. Fourier Sine and Cosine Transforms

We shall discuss Fourier sine and Fourier cosine transform simultaneously. This is because most of the time they are used simultaneously to solve any problem.

6.7.1. Properties of Fourier sine and cosine transforms.

$$\mathcal{F}_c f(x) = \tilde{f}_c(\alpha), \quad \mathcal{F}_s f(x) = \tilde{f}_s(\alpha), \tag{6.67}$$
$$\mathcal{F}_c \tilde{f}_c(x) = f(\alpha), \quad \mathcal{F}_s \tilde{f}_s(x) = f(\alpha), \tag{6.68}$$
$$\mathcal{F}_c f(kx) = \frac{1}{k}\tilde{f}_c(\frac{\alpha}{k}), \quad k>0, \tag{6.69}$$
$$\mathcal{F}_s f(kx) = \frac{1}{k}\tilde{f}_s(\frac{\alpha}{k}), \quad k>0, \tag{6.70}$$

$$\mathcal{F}_c f(kx)\cos bx = \frac{1}{2k}\left[\tilde{f}_c\left(\frac{\alpha+b}{k}\right) + \tilde{f}_c\left(\frac{\alpha-b}{k}\right)\right], \quad k>0, \tag{6.71}$$
$$\mathcal{F}_c f(kx)\sin bx = \frac{1}{2k}\left[\tilde{f}_s\left(\frac{\alpha+b}{k}\right) - \tilde{f}_s\left(\frac{\alpha-b}{k}\right)\right], \quad k>0, \tag{6.72}$$
$$\mathcal{F}_s f(kx)\cos bx = \frac{1}{2k}\left[\tilde{f}_s\left(\frac{\alpha+b}{k}\right) + \tilde{f}_s\left(\frac{\alpha-b}{k}\right)\right], \quad k>0, \tag{6.73}$$
$$\mathcal{F}_s f(kx)\sin bx = \frac{1}{2k}\left[\tilde{f}_c\left(\frac{\alpha-b}{k}\right) - \tilde{f}_c\left(\frac{\alpha+b}{k}\right)\right], \quad k>0, \tag{6.74}$$

$$\mathcal{F}_c x^{2n} f(x) = (-1)^n \frac{d^{2n}\tilde{f}_c(\alpha)}{d\alpha^{2n}}, \tag{6.75}$$
$$\mathcal{F}_c x^{2n+1} f(x) = (-1)^n \frac{d^{2n+1}\tilde{f}_s(\alpha)}{d\alpha^{2n}}, \tag{6.76}$$
$$\mathcal{F}_s x^{2n} f(x) = (-1)^n \frac{d^{2n}\tilde{f}_s(\alpha)}{d\alpha^{2n}}, \tag{6.77}$$
$$\mathcal{F}_s x^{2n+1} f(x) = (-1)^{n+1} \frac{d^{2n+1}\tilde{f}_c(\alpha)}{d\alpha^{2n}}. \tag{6.78}$$

6.7.2. Convolution theorems for Fourier sine and cosine transforms.

THEOREM 6.8. *Let $\tilde{f}_c(\alpha)$ and $\tilde{g}_c(\alpha)$ be the Fourier cosine transforms of $f(x)$ and $g(x)$, respectively, and let $\tilde{f}_s(\alpha)$ and $\tilde{g}_s(\alpha)$ be the Fourier sine transforms of $f(x)$ and $g(x)$, respectively. Then*

$$\tilde{f}_c^{-1}[\tilde{f}_c(\alpha)\tilde{g}_c(\alpha)] = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} g(\eta)[f(|x-\eta|) + f(x+\eta)]\,d\eta. \tag{6.79}$$

PROOF. We have

$$\begin{aligned}
\int_0^{\infty} \tilde{f}_c(\alpha)\,\tilde{g}_c(\alpha)\cos\alpha x d\alpha &= \frac{2}{\sqrt{\pi}}\int_0^{\infty} \tilde{f}_c(\alpha)\cos\alpha x d\alpha \int_0^{\infty} g(\eta)\cos\alpha\eta\,d\eta \\
&= \frac{2}{\sqrt{\pi}}\int_0^{\infty} g(\eta)\,d\eta \int_0^{\infty} \tilde{f}_c(\alpha)\cos\alpha x\cos\alpha\eta d\alpha \\
&= \frac{1}{2}\frac{2}{\sqrt{\pi}}\int_0^{\infty} g(\eta)\,d\eta \int_0^{\infty} \tilde{f}_c(\alpha)[\cos\alpha(|x-\eta|) + \cos\alpha(x+\eta)]\,d\alpha \\
&= \frac{1}{2}\int_0^{\infty} g(\eta)[f(|x-\eta|) + f(x+\eta)]\,d\eta. \blacksquare
\end{aligned}$$

THEOREM 6.9.

$$\int_0^\infty \tilde{f}_c(\alpha)\,\tilde{g}_s(\alpha)\sin\alpha x d\alpha = \frac{1}{2}\int_0^\infty g(\eta)[f(|x-\eta|) - f(x+\eta)]\,d\eta, \tag{6.80}$$

and

$$\int_0^\infty \tilde{f}_s(\alpha)\,\tilde{g}_c(\alpha)\sin\alpha x d\alpha = \frac{1}{2}\int_0^\infty f(\eta)[g(|x-\eta|) - g(x+\eta)]\,d\eta. \tag{6.81}$$

THEOREM 6.10.

$$\int_0^\infty \tilde{f}_c(\alpha)\,\tilde{g}_c(\alpha)\,d\alpha = \int_0^\infty f(\eta)g(\eta)d(\eta) = \int_0^\infty \tilde{f}_s(\alpha)\tilde{g}_s(\alpha)\,d\alpha. \tag{6.82}$$

The proofs for these theorems are left as exercises.

We will now derive Fourier sine and cosine transforms of some functions.

EXAMPLE 6.19. Define

$$I_1 = \int_0^\infty e^{-ax}\sin bx\,dx, \quad I_2 = \int_0^\infty e^{-ax}\cos bx\,dx.$$

Then $I_2 = \frac{1}{a} - \frac{b}{a}I_1$, and $I_1 = \frac{b}{a}I_2$. Solving for I_1 and I_2, we get

$$I_1 = \frac{b}{a^2+b^2}, \quad \text{and} \quad I_2 = \frac{a}{a^2+b^2}.$$

If $f(x) = e^{-bx}$, then

$$\tilde{f}_c(\alpha) = \sqrt{\frac{2}{\pi}}\frac{b}{\alpha^2+b^2},$$

and

$$\tilde{f}_s(\alpha) = \sqrt{\frac{2}{\pi}}\frac{\alpha}{\alpha^2+b^2}.$$

These two results yield on inversion

$$\int_0^\infty \frac{\cos\alpha x}{\alpha^2+b^2}\,d\alpha = \frac{\pi}{2b}e^{-bx},$$

and

$$\int_0^\infty \frac{\alpha\sin\alpha x}{\alpha^2+b^2}\,d\alpha = \frac{\pi}{2}e^{-bx}.$$

An interesting integral is obtained by defining

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < b, \\ 0 & \text{for } b < x. \end{cases}$$

Then $\tilde{f}_c(\alpha) = \sqrt{\frac{2}{\pi}}\frac{\sin\alpha b}{\alpha}$. Also, define $g(x) = e^{-ax}$. Then

$$\tilde{g}_c(\alpha) = \sqrt{\frac{2}{\pi}}\frac{a}{\alpha^2+a^2}.$$

Thus, we have

$$\begin{aligned} \int_0^\infty \tilde{f}_c(\alpha)\,\tilde{g}_c(\alpha)\,d\alpha &= \frac{2}{\pi}\int_0^\infty \frac{a}{\alpha}\frac{\sin\alpha b}{\alpha^2+a^2}\,d\alpha = \int_0^\infty f(\eta)g(\eta)\,d\eta \\ &= \int_0^b e^{-a\eta}\,d\eta = \frac{1-e^{-ab}}{a}. \blacksquare \end{aligned} \tag{6.83}$$

EXAMPLE 6.20. Show that

$$\int_0^\infty \frac{1}{\lambda^2}\sin\lambda a\sin\lambda b\,d\lambda = \frac{\pi}{2}\min(a,b).$$

In fact, if we define

$$f(x) = \begin{cases} 1 & \text{for } 0 \le x < a \\ 0 & \text{for } a < x \end{cases}, \qquad g(x) = \begin{cases} 1 & \text{for } 0 \le x < b \\ 0 & \text{for } b < x, \end{cases}$$

then

$$\int_0^\infty \tilde{f}_c(\lambda)\,\tilde{g}_c(\lambda)\,d\lambda = \int_0^\infty f(x)\,g(x)\,dx.$$

This, on using (6.83), yields

$$\frac{2}{\pi}\int_0^\infty \frac{1}{\lambda^2}\sin\lambda a\,\sin\lambda b\,d\lambda = \int_0^\infty f(x)g(x)\,dx = \min(a,b). \blacksquare$$

EXAMPLE 6.21. Find the Fourier cosine transform of $f(x)$, where

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1, \\ 2 - x & \text{for } 1 < x < 2, \\ 0 & \text{for } 2 < x < \infty. \end{cases}$$

Here,

$$\begin{aligned} \mathcal{F}_c[f(x)] &= \sqrt{\frac{2}{\pi}} \left[\int_0^1 x \cos x \, dx + \int_1^2 (2 - x) \cos x \, dx \right] \\ &= \sqrt{\frac{2}{\pi}} \left[\frac{x \sin \alpha x}{\alpha} \Big|_0^1 - \frac{1}{\alpha} \int_0^1 \sin \alpha x \, dx + \frac{(2 - x) \sin \alpha x}{\alpha} \Big|_1^2 + \right. \\ &\qquad \left. \frac{1}{\alpha} \int_1^2 \sin \alpha x \, dx \right] \\ &= \sqrt{\frac{2}{\pi}} \left[\frac{2 \cos \alpha - 1 - \cos 2\alpha}{\alpha^2} \right]. \blacksquare \end{aligned}$$

In[1]:=

```
Needs[
"Calculus`FourierTransform`","Algebra`Trigonometry`"];
```

In[2]:=

```
Clear[f];
f[x_]:= If[0< x <1, x ,If[1< x < 2, 2-x ,0]]
```

In[4]:=

```
FourierCosTransform[f[t],t,w]
```

```
On::none:  Message SeriesData::csa not found.
On::none:  Message SeriesData::csa not found.
On::none:  Message SeriesData::csa not found.
General::stop:  Further output of On::none
will be suppressed during this calculation.
```

Out[4]=


```
FourierCosTransform[f[t],t,w]
```

In[5]:=

```
int1:= Integrate[x Cos[alpha x],{x,0,1}]
int2:= Integrate[(2-x) Cos[alpha x],{x,1,2}]

result= Sqrt[2/Pi] (int1 + int2)//Simplify
```

Out[7]=

$$\frac{4\ \mathtt{Sqrt}[\frac{2}{\mathtt{Pi}}]\ \mathtt{Cos[alpha]}\ \mathtt{Sin}[\frac{\mathtt{alpha}}{2}]^2}{\mathtt{alpha}^2}$$

EXAMPLE 6.22. We shall now use the Fourier transform to solve Example 6.7, which was earlier solved by the Laplace transform. The partial differential equation along the boundary and initial conditions are

$$k\, u_{xx} = u_t,$$

$$u = 0, \quad u \to 0 \text{ as } x \to 0 \text{ for } t \le 0, \quad u = T_0 \quad \text{for } t > 0.$$

Applying Fourier sine transform to the partial differential equation, we get

$$\frac{\partial \tilde{u}_s}{\partial t} + k\alpha^2 \tilde{u}_s = k\, \alpha\, T_0,$$

where $\tilde{u}_s$ is the Fourier sine transform of u. Its solution is given by

$$\tilde{u}_s = A e^{k\alpha^2 t} + \sqrt{\frac{2}{\pi}} \frac{T_0}{\alpha}.$$

By applying the initial condition at $t = 0$, we get

$$A + \sqrt{\frac{2}{\pi}} \frac{T_0}{\alpha} = 0.$$

Hence

$$\tilde{u}_s = \sqrt{\frac{2}{\pi}} \frac{T_0}{\alpha} \left(1 - e^{k\alpha^2 t}\right).$$

Thus, $u(x,t)$ is given by

$$\begin{aligned} u(x,t) &= \frac{2}{\pi}T_0 \int_0^\infty \frac{1}{\alpha}\left(1-e^{-k\alpha^2 t}\right)\sin\alpha x\, d\alpha \\ &= \frac{2}{\pi}T_0\left[\int_0^\infty \frac{\sin\alpha x}{\alpha}\,d\alpha - \int_0^\infty \frac{\sin\alpha x}{\alpha}e^{-k\alpha^2 t}\,d\alpha\right] \\ &= \frac{2}{\pi}T_0\left[\frac{\pi}{2}-\frac{\pi}{2}\operatorname{erf}\frac{x}{2\sqrt{kt}}\right] \\ &= T_0\operatorname{erfc}\frac{x}{2\sqrt{kt}}. \blacksquare \end{aligned}$$

6.8. Finite Fourier Transforms

When the domain of the physical problem is finite, it is generally not convenient to use the transforms with an infinite range of integration. In many cases, finite Fourier transform can be used with advantage. We define

$$\tilde{f}_s(n) = \int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx \tag{6.84}$$

as the finite Fourier sine transform of $f(x)$. The function $f(x)$ is then given by

$$f(x) = \frac{2}{a}\sum_1^\infty \tilde{f}_s(n)\sin\left(\frac{n\pi x}{a}\right). \tag{6.85}$$

Similarly, the finite Fourier cosine transform is defined as

$$\tilde{f}_c(n) = \int_0^a f(x)\cos\left(\frac{n\pi x}{a}\right)dx, \tag{6.86}$$

and the inverse is

$$f(x) = \frac{\tilde{f}_c(0)}{a} + \frac{2}{a}\sum_1^\infty \tilde{f}_c(n)\cos\left(\frac{n\pi x}{a}\right). \tag{6.87}$$

EXAMPLE 6.23. Consider the Laplace equation for a rectangle

$$u_{xx}+u_{yy}=0, \tag{6.88}$$

with the boundary conditions

$$u(0,y)=u(a,y)=u(x,b)=0, \quad \text{and} \quad u(x,0)=f(x).$$

Applying the finite Fourier sine transform to $u(x,y)$ with respect to x from 0 to a, we have

$$\begin{aligned} (\tilde{u}_s)_{xx}(n) &= \int_0^a u_{xx}\sin\frac{n\pi x}{a}\,dx \\ &= \frac{n\pi}{a}\left[u(0,t)-(-1)^n u(a,t)\right] - \frac{n^2\pi^2}{a^2}\tilde{u}_s(n). \end{aligned} \tag{6.89}$$

Then Equation (6.88) becomes

$$\left[\frac{d^2}{dy^2}-\frac{n^2\pi^2}{a^2}\right]\tilde{u}_s(n,y)=0.$$

Solving for $\tilde{u}_s(n,y)$, we get

$$\tilde{u}_s(n,y) = Ae^{n\pi y/a}+Be^{-n\pi y/a}.$$

Since $\tilde{u}_s(n,b)=0$, we can express $\tilde{u}_s(n,y)$ as

$$\tilde{u}_s(n,y) = A_n(e^{n\pi(y-b)/a}-e^{-n\pi(y-b)/a}).$$

Applying the boundary condition at $y=0$, we get

$$A_n(e^{-n\pi b/a}-e^{n\pi b/a}) = \bar{f}_s(n),$$

which, after solving for A_n and substituting its value in $\tilde{u}_s(n,y)$, yields

$$\tilde{u}_s(n,y) = -\frac{\sinh[n\pi(y-b)/a]}{\sinh(n\pi b/a)}\bar{f}_s(n).$$

Hence,

$$u(x,y) = \frac{2}{a}\sum_1^\infty \frac{\sinh[n\pi(b-y)/a]}{\sinh(n\pi b/a)}\bar{f}_s(n)\sin(n\pi x/a),$$

where

$$\bar{f}_s(n) = \int_0^a f(\xi)\sin(n\pi\xi/a)d\xi. \blacksquare$$

EXAMPLE 6.24. Solve the wave equation

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < l,$$

subject to the conditions

$$u(0,t) = g(t) \quad \text{and} \quad u(l,t) = 0 \quad \text{for } 0 < x < l,$$
$$u(x,0) = 0 \quad \text{for } t > 0.$$

Taking the finite Fourier sine transform with the boundary conditions at $x = 0, l$, and using (6.89), we get

$$\frac{d^2\tilde{u}_s}{dt^2} + \frac{n^2\pi^2c^2}{l^2}\tilde{u}_s = \frac{n\pi c^2}{l} g(t).$$

The general solution of this equation is

$$\tilde{u}_s(n) = A \cos\frac{n\pi ct}{l} + B \sin\frac{n\pi ct}{l} + \tilde{u}_{s,P}(n),$$

where $\tilde{u}_{s,P}(n)$ is the particular solution which can be obtained by the variation of parameters as

$$\begin{aligned}\tilde{u}_{s,P}(n) &= c\int_0^t g(\tau)\sin\frac{n\pi c(t-\tau)}{l}\,dt \\ &= c\int_0^t g(t-\tau)\sin\frac{n\pi c\tau}{l}\,d\tau.\end{aligned}$$

With this choice of $\tilde{u}_{s,P}(n)$, the constants A and B become zero because of the initial condition $\tilde{u}_s(n,0) = 0$. Hence $\tilde{u}_s(n) = \tilde{u}_{s,P}(n)$. Thus, by (6.85)

$$u(x,t) = \frac{2c}{l}\sum_{n=1}^{\infty}\sin\frac{n\pi x}{l}\int_0^t g(t-\tau)\sin\frac{n\pi c\tau}{l}\,d\tau.$$

Alternately, by applying the Laplace transform, we have

$$\frac{d^2\bar{u}}{dx^2} - \frac{s^2}{c^2}\bar{u} = 0,$$

which, when solved with the boundary conditions $\bar{u}(0,s) = \bar{g}(s)$, $\bar{u}(l,s) = 0$, gives

$$\bar{u}(x,s) = \bar{g}(s)\,\frac{\sinh(s(l-x)/c)}{\sin(sl/c)}.$$

This can be inverted for different values of $g(t)$. Note that as $t \to \infty$, i.e., as $s \to 0$, we get

$$\bar{u}(x,s) = \bar{g}(s)\,e^{-sx/c},$$

which on inversion gives

$$u(x,t) = H(t - x/c)\,g(t - x/c).$$

This solution also follows from d'Alembert general solution (5.24) of the wave equation. ∎

Note that the finite cosine transforms for the derivatives of a function u can be obtained analogous to (6.89) (see Exercise 6.23).

6.9. Exercises

6.13. Find the complex Fourier transform of of the following functions:

(a) $f(x) = \begin{cases} 0 & \text{for } x < 0 \\ e^{-ax} & \text{for } x > 0. \end{cases}$

ANS. $\dfrac{1}{\sqrt{2\pi}(a - i\alpha)}$.

(b) $f(x) = \begin{cases} 0 & \text{for } x < 0, \\ \dfrac{1}{x}[e^{-ax} - e^{-bx}] & \text{for } x > 0 \text{ and } b > a > 0. \end{cases}$

HINT: Use part (a) and integrate with respect to a.

ANS. $\dfrac{1}{\sqrt{2\pi}}\ln\dfrac{b - i\alpha}{a - i\alpha}$.

(c) $f(x) = \begin{cases} 0 & \text{for } |x| > a, \\ 1 - \dfrac{|x|}{a} & \text{for } |x| < a. \end{cases}$

ANS. $\sqrt{\dfrac{2}{\pi}}\dfrac{1}{a\alpha^2}(1 - \cos\alpha a)$.

(d) $f(x) = \cos ax^2$ and $f(x) = \sin ax^2$.

HINT: Use Example 6.13 and define k as ia.

ANS. $\dfrac{1}{2\sqrt{a}}\left(\sin\dfrac{\alpha^2}{4a} + \cos\dfrac{\alpha^2}{4a}\right)$ and $\dfrac{1}{2\sqrt{a}}\left(\sin\dfrac{\alpha^2}{4a} - \cos\dfrac{\alpha^2}{4a}\right)$.

(e) $f(x) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1. \end{cases}$

ANS. $\dfrac{\sqrt{2}}{\alpha\sqrt{\pi}}\sin\alpha.$

(f) $f(x) = \begin{cases} \sin kx & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1. \end{cases}$

ANS. $\dfrac{i}{\sqrt{2\pi}}\left[\dfrac{\sin(k-\alpha)}{k-\alpha} - \dfrac{\sin(k+\alpha)}{k+\alpha}\right].$

(g) $f(x) = \begin{cases} \dfrac{1}{\sqrt{x}} & \text{for } x > 0 \\ = 0 & \text{for } x < 0 \end{cases}.$

HINT: Substitute $\alpha x = v^2 e^{i\pi/2}$.

ANS. $\dfrac{e^{i\pi/4}}{\sqrt{2\alpha}}, \quad \alpha > 0.$

(h) $f(x) = \begin{cases} \dfrac{1}{\sqrt{|x|}} & \text{for } x < 0 \\ 0 & \text{for } x > 0. \end{cases}$

ANS. $\dfrac{e^{-i\pi/4}}{\sqrt{2\alpha}}, \quad \alpha > 0.$

(i) $f(x) = \dfrac{1}{\sqrt{|x|}}.$

ANS. $\dfrac{1}{\sqrt{|\alpha|}}.$

6.14. Solve the partial differential equation $(a^2 D_x^2 - D_t^2)u = 0$, subject to the conditions $u(x,0) = f(x) + g(x)$ and $D_t u(x,0) = a(f'(x) - g'(x))$, where $u(x,t)$, $f(x)$, $g(x)$, $u'(x,t)$, $f'(x)$, and $g'(x)$ all go to zero as $|x| \to \infty$.
ANS. $u = f(x+at) + g(x-at)$.

6.15. Solve the following system: $D_t u - v = -D_x h$, $D_t v + u = 0$, $D_t h + c^2 D_x u = 0$, subject to the conditions $u(x,0) = v(x,0) = 0$ and $h(x,0) = k$ if $|x| < a$ and $h(x,0) = 0$ if $|x| > a$; in addition $u(x,t), v(x,t)$ and $h(x,t)$ approach 0 as $|x| \to \infty$. This problem is connected with shallow water waves.
ANS.

$$h(x,t) = \frac{2k}{\pi}\int_{-\infty}^{\infty} \frac{\sin(a\alpha)}{\alpha}\left[\frac{1+\alpha^2c^2\cos(t\sqrt{1+\alpha^2c^2})}{1+\alpha^2c^2}\right]\cos(\alpha x)\,dx.$$

6.16. Solve the partial differential equation $r^2 D_t^2 u = c^2 D_r(r^2 D_r)u$, with initial conditions $u(r,t) = D_t u(r,t) = 0$ for $t \le 0$ and the boundary condition $D_r u(a,t) = H(t)f(t)$, by using Fourier transform with respect to t.

ANS. See Example 5.10. HINT. $\mathcal{F}^{-1}\dfrac{1}{i\alpha - b} = -\sqrt{2\pi}H(t)e^{-bt}$ and $\mathcal{F}^{-1}\dfrac{e^{i\alpha k(r-a)}}{i\alpha - b} = -\sqrt{2\pi}H[t-k(r-a)]e^{-b[t-k(r-a)]}$. Use the convolution theorem.

6.17. Find the Fourier sine and cosine transforms of $f(x) = \dfrac{1}{\sqrt{x}}e^{-ax}$.

ANS. $\mathcal{F}_c f(x) = \dfrac{\sqrt{\sqrt{a^2+\alpha^2}+a}}{\sqrt{a^2+\alpha^2}}$, $\mathcal{F}_s f(x) = \dfrac{\sqrt{\sqrt{a^2+\alpha^2}-a}}{\sqrt{a^2+\alpha^2}}$.

6.18. Find the Fourier cosine transform of $\dfrac{1}{x}[e^{-ax} - e^{-bx}]$, $\Re a, \Re b > 0$.

ANS. $\dfrac{1}{2\pi}\ln\left(\dfrac{a^2+\alpha^2}{b^2+\alpha^2}\right).$

6.19. Find the Fourier sine transform of $\dfrac{1}{x}e^{-ax}$, $\Re a > 0$.

ANS. $\sqrt{\dfrac{2}{\pi}}\tan^{-1}\dfrac{\alpha}{a}.$

6.20. Derive additional formulas for Fourier sine and cosine transforms from Examples 5.5 and 5.6 by differentiating or integrating with respect to a.

6.21. Find the Fourier sine and cosine transforms of $f(x) = \dfrac{1}{\sqrt{x}}e^{-ax}$, $a > 0$.

ANS. $\tilde{f}_c(\alpha) = \dfrac{\sqrt{\sqrt{a^2+\alpha^2}+a}}{\sqrt{a^2+\alpha^2}}$, and $\tilde{f}_s(\alpha) = \dfrac{\sqrt{\sqrt{a^2+\alpha^2}-a}}{\sqrt{a^2+\alpha^2}}$.

6.22. Find Fourier sine and cosine transforms of $f(x) = \sqrt{x}e^{-ax}$, $a > 0$.

ANS. $\tilde{f}_c(\alpha) + i\tilde{f}_s(\alpha) = \dfrac{e^{3i\arctan(\alpha/a)}}{2\sqrt{2}(a^2+\alpha^2)^{3/4}}.$

HINT: $\tilde{f}_c(\alpha) + i\tilde{f}_s(\alpha) = \dfrac{1}{\sqrt{2(a-i\alpha)^3}}$; then express $(a - i\alpha)$ in polar form.

6.23. Derive the formulas for the finite sine and cosine transforms for the first, second, and third derivatives of a function $u(x)$ in the interval $[0, l]$.

ANS. For the finite sine transform

$$\left(\frac{du}{dx}\right)_s = -\frac{n\pi}{l}\,\tilde{u}_c(n),$$

$$\left(\frac{d^2u}{dx^2}\right)_s \text{ is given by (6.89)},$$

$$\left(\frac{d^3u}{dx^3}\right)_s = -\frac{n\pi}{l}\,[(-1)^n u'(l) - u'(0)] - \frac{n^3\pi^3}{l^3}\,\tilde{u}_c(n).$$

For the finite cosine transform

$$\left(\frac{du}{dx}\right)_c = [(-1)^n u(l) - u(0)] + \frac{n\pi}{l}\,\tilde{u}_s(n),$$

$$\left(\frac{d^2u}{dx^2}\right)_c = [(-1)^n u'(l) - u'(0)] - \frac{n^2\pi^2}{l^2}\,\tilde{u}_c(n),$$

$$\left(\frac{d^3u}{dx^3}\right)_c = [(-1)^n u''(l) - u''(0)] - \frac{n^2\pi^2}{l^2}\,[(-1)^n u(l) - u(0)] - \frac{n^3\pi^3}{l^3}\,\tilde{u}_s(n).$$

7

Green's Functions

The solution of a given linear partial differential equation due to a unit point source in the region under consideration subject to homogeneous boundary conditions is generally called a Green's function. This solution enables us to generate solutions for the partial differential equation subject to a range of boundary conditions and internal sources. This technique is of great importance in a variety of physical problems. For the derivation of the Green's functions one can assume the presence of an internal source or a certain boundary condition which results in the same effect as the point source. The point source, represented by the Dirac delta function, belongs to a class of functions known as generalized functions or distributions. With this in mind, we shall first study certain elementary aspects of the distribution theory. The definition and construction of Green's functions for different types of boundary value problems are carried out through various examples and exercises.

WWW Refer to the Mathematica Notebook `Greens.ma` for this chapter.

7.1. Definitions

Let R^n denote the Euclidean n–space, and R^+ the set of nonnegative real numbers. Then $|x - y|$ defines the Euclidean distance between points x and y in R^n. An open ball of radius r centered at a point $x_0 \in R^n$ is defined by $\{x : |x - x_0| < r\}$, and denoted by $B(x_0, r)$. The boundary (surface) of the open ball $B(x_0, r)$ will be denoted by $S(x_0, r) = \{x : |x - x_0| = r\}$, where $S_n(1) = \dfrac{2\pi^{n/2}}{\Gamma(n/2)}$ is the surface area of the unit ball in R^n. The ε–neighborhood of a set $A \subset R^n$ is $A_\varepsilon = \cup_{x\in A} B(x,\varepsilon)$. The complement of a set B with respect to a set A will be denoted by $A\backslash B$, the product of the sets A and B by $A \times B$, and the closure of a set A by $\overline{A}$. The characteristic (or indicator) function of the set A is defined by $\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$. A complex–valued function f is said to belong to the class $C^p(\Omega)$ if it is continuous together with the derivatives $D^k f(x)$, $|k| \le p$, $0 \le p < \infty$, in a domain Ω. The function f in the class $C^p(\Omega)$, for which all derivatives $D^k f(x)$, $|k| \le p$, admit continuous continuations in the closure $\overline{\Omega}$, form the class of functions $C^p(\overline{\Omega})$. The class $C^\infty(\Omega)$ consists of functions f which are infinitely differentiable on Ω, i.e., continuous partial derivatives of all orders exist. These classes are linear sets; thus, every linear combination $\lambda f + \mu g$, where λ and μ are arbitrary complex numbers, also belongs to the respective class.

A function defined on R^n is said to belong to the class $C_0^\infty(R^n)$ if it is infinitely differentiable on R^n and vanishes outside some bounded region.

The *support* of a continuous function f (written supp f) is the closure of the set $\{x \in R^n : f(x) \ne 0\}$. Then the class $C_0^p(R^n)$ denotes the set of functions in $C^p(R^n)$ that have compact support.*

The Hilbert space $L^2[a,b]$ is a complete inner product space with the norm defined by

$$\|f\| = \left(\int_a^b [f(t)]^2\, dt\right)^{1/2}, \quad t \in [a,b],$$

and is obtained from the inner product defined by

$$\langle f, g\rangle = \int_a^b f(t)\, g(t)\, dt.$$

Complex–valued functions which form a complex vector space become an inner product space if we define

$$\langle f, g\rangle = \int_a^b f(t)\, \overline{g(t)}\, dt,$$

*A complex–valued function f on R^n is said to have compact support if there exists a compact set K such that $f(\mathbf{x}) = 0$ for each $\mathbf{x}$ not in K.

where $t \in [a,b]$ is kept real, and the bar denotes the complex conjugate. The norm is then defined by

$$\|f\| = \left(\int_a^b |f(t)|^2\, dt\right)^{1/2},$$

since $f(t)\overline{f(t)} = |f(t)|^2$.

We shall now discuss some basic concepts and results from the theory functionals, generalized functions, and distributions. Proofs for most of the results can be easily found in the literature (see Friedlander (1982), Gelfand and Shilov (1964), Kythe (1995, 1996), Rubinstein (1969), and Stakgold (1979)).

Definition 7.1. Let a real number $\displaystyle\int_{R^n} f(\mathbf{x})\phi(\mathbf{x})\, d\mathbf{x} = \langle f, \phi\rangle$ be associated with each $\mathbf{x} \in R^n$ for every function $\phi \in C_0^\infty(R^n)$. Then f is said to be a *functional* on R^n. The function ϕ is known as the *test function.*

Thus, e.g., the Fourier series of $f \in C^1[0,\pi]$, defined by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad \text{where} \quad b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\, dx,$$

is a functional on R^1, with test functions in the set $\{\sin x, \sin 2x, \cdots\}$. Some useful properties of test functions are

(i) if $\phi_1(\mathbf{x})$ and $\phi_2(\mathbf{x})$ are test functions on R^n, so is their linear combination $c_1\phi_1(\mathbf{x}) + c_2\phi_2(\mathbf{x})$, where c_1 and c_2 are real numbers;

(ii) if $\phi(\mathbf{x}) \in C_0^\infty(R^n)$, so do all partial derivatives of $\phi(\mathbf{x})$ belong to the class $C_0^\infty(R^n)$;

(iii) if $\phi(\mathbf{x}) \in C_0^\infty(R^n)$ and $a(\mathbf{x})$ are infinitely differentiable, then the product $a(\mathbf{x})\phi(\mathbf{x})$ belongs to the class $C^\infty(R^n)$; and

(iv) if $\phi(x_1, \cdots, x_m) \in C_0^\infty(R^m)$ and $\psi(x_{m+1}, \cdots, x_n) \in C_0^\infty(R^{n-m})$, then $\phi(x_1, \cdots, x_m)\,\psi(x_{m+1}, \cdots, x_n) \in C_0^\infty(R^n)$.

Definition 7.2. A functional f on R^n is said to be *linear* if $\langle f, \lambda\phi_1 + \mu\phi_2\rangle = \lambda\langle f, \phi_1\rangle + \mu\langle f, \phi_2\rangle$ for all real numbers λ, μ and all $\phi_{1,2} \in C_0^\infty(R^n)$. Note that $\langle f, 0\rangle = 0$, and $\langle f, \sum_{n=1}^{m} a_n\phi_n\rangle = \sum_{n=1}^{m} a_n\langle f, \phi_n\rangle$.

DEFINITION 7.3. A linear functional f on $C_0^\infty(R^n)$ is said to be continuous if the numerical sequence $\langle f, \phi_m \rangle \to 0$ as $m \to \infty$, where $\{\phi_m(\mathbf{x})\}$ is a null sequence in $C^\infty(R^n)$, i.e., supp ϕ_m, $m = 1, 2, \cdots$, is contained in a sufficiently large ball, and $\lim_{m\to\infty} \max_{\mathbf{x}\in R^n} |D^k \phi_m(\mathbf{x})| = 0$ for every multi–index k, $|k| \le n$.

DEFINITION 7.4. A continuous linear functional f on $C_0^\infty(R^n)$ is said to be a *distribution*. The number $\langle f, \phi \rangle$ is called the value of f at ϕ, or the action of f on ϕ.

The space $\mathcal{D}'$ of all distributions on $C_0^\infty(R^n)$ is a linear space. A locally integrable function $f(\mathbf{x})$ in R^n generates an n–dimensional distribution f such that for all $\phi \in C_0^\infty(R^n)$

$$\begin{aligned}\langle f, \phi \rangle &= \int_{R^n} f(\mathbf{x})\phi(\mathbf{x})\, d\mathbf{x} \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \cdots, x_n)\, \phi(x_1, \cdots, x_n)\, dx_1 \cdots dx_n.\end{aligned} \tag{7.1}$$

Hence, every locally integrable function f can be regarded as a distribution. Let $f_1(\mathbf{x})$ and $f_2(\mathbf{x})$ be two different continuous functions. Then each generates a different distribution such that there exists a ϕ in $C_0^\infty(R^n)$ for which $\langle f_1, \phi \rangle \ne \langle f_2, \phi \rangle$, i.e., $\langle f_1 - f_2, \phi \rangle \ne 0$. Two functions f_1 and f_2 are said to be equal almost everywhere (a.e.) on a bounded domain Ω if $\int_\Omega |f_1 - f_2|\, d\mathbf{x} = 0$. Hence, two locally integrable functions that are equal a.e. generate the same distribution. A distribution of the form (7.1), where $f(\mathbf{x})$ is locally integrable, is said to be *regular*. All other distributions are called *singular*, although formula (7.1) can be used formally for such distributions.

EXAMPLE 7.1. The functional $\langle \chi_\Omega, \phi \rangle = \int_\Omega \phi(\mathbf{x})\, d\mathbf{x}$, $\Omega \in R^n$, where χ_Ω is the indicator function of the domain Ω, generates a linear, piecewise continuous and regular distribution. Note that $\chi_\Omega(x) = H(x)$ in R^1, where $H(x) = \begin{cases} 1, & x > 0 \\ 0, & x \le 0 \end{cases}$ is the Heaviside unit function with Ω as the interval $(0, \infty)$. ∎

7.1.1. Dirac distribution. Let $\mathbf{x}'$ be a fixed point in R^n. Consider the functional $\delta_{\mathbf{x}'}$ defined by $\langle \delta_{\mathbf{x}'}, \phi \rangle = \phi(\mathbf{x}')$, which assigns to each test function ϕ its value at $\mathbf{x}'$. The functional $\delta_{\mathbf{x}'}$ is linear and continuous on $C_0^\infty(R^n)$, and hence a distribution with pole $\mathbf{x}'$. We shall show that δ_0 (denoted simply by δ) is a singular distribution. The proof is by contradiction: Assume that δ is regular. Then there exists a locally integrable function $f(\mathbf{x})$ such that

$$\int_{R^n} f(\mathbf{x})\phi(\mathbf{x})\, d\mathbf{x} = \phi(0) \quad \text{for every } \phi \in C^\infty(R^n). \tag{7.2}$$

The functional $\phi(0)$ is taken as the definition of the density $\delta(\mathbf{x})$ which is known as the Dirac delta function. Hence,

$$\langle \delta(\mathbf{x}, \mathbf{x}'), \phi(\mathbf{x}) \rangle = \int_{R^n} \delta(\mathbf{x}, \mathbf{x}')\phi(\mathbf{x})\, d\mathbf{x} = \phi(\mathbf{x}'). \tag{7.3}$$

In the classical sense the Dirac delta function is defined as

$$\delta(x, x') \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if } x \ne x' \\ \infty, & \text{if } x = x'. \end{cases} \tag{7.4}$$

This function is a generalized function defined by

$$\delta(x, x') = \frac{1}{\pi} \lim_{n\to\infty} \frac{\sin(n(x - x'))}{x - x'}, \tag{7.5}$$

which, in the limit, is zero at every point $x \ne x'$ and infinite at $x = x'$. Thus, the Dirac delta function represents a point singularity at the source point x', i.e.,

$$\delta(x, x') \begin{cases} \to \infty & \text{as } x \to x' \\ = 0, & \text{otherwise}. \end{cases} \tag{7.6}$$

This function is used in defining a concentrated impulsive force in solid and fluid mechanics, a point mass in the theory of gravitational potential, a point charge in electronics, an impulsive force in acoustics and other similar situations in physics and mechanics.

A consequence of (7.3) defines a basic property of this function as

$$\iiint_\Omega \delta(\mathbf{x}, \mathbf{x}')\phi(\mathbf{x})\, dV = \begin{cases} \phi(\mathbf{x}'), & \text{if } \mathbf{x}' \in \Omega \\ 0, & \text{if } \mathbf{x}' \notin \Omega, \end{cases} \tag{7.7}$$

where $\mathbf{x}, \mathbf{x}' \in \Omega \subset R^3$, and $\phi \in C(\Omega)$. If $\phi(\mathbf{x}) = 1$ in (7.7), we get

$$\iiint_\Omega \delta(\mathbf{x}, \mathbf{x}')\, dV = \begin{cases} 1, & \text{if } \mathbf{x}' \in \Omega \\ 0, & \text{if } \mathbf{x}' \notin \Omega. \end{cases} \tag{7.8}$$

Also,

$$\iiint_\Omega \delta(k(\mathbf{x},\mathbf{x}'))\,d\mathbf{x} = \iiint_\Omega \frac{1}{k}\delta(\mathbf{x},\mathbf{x}')\,d\mathbf{x}. \tag{7.9}$$

For $(a,b) \in R^1$ the basic property (7.7) becomes

$$\int_a^b \delta(x,x')\phi(x)\,dx = \phi(x'), \tag{7.10}$$

where a,b can be $-\infty$ or $+\infty$ for unbounded intervals.

Note that $\delta(x,x')$ has the units $[L^{-1}]$, and $\delta(t,t')$ has the units $[T^{-1}]$ if t, and t' denote time, where L denotes the unit of length and T the unit of time. In general, $\delta(x,x')$ has the units such that $\iiint_\Omega \delta(x,x')\,dV = 1$. Also, we often write $\delta(x)$ for $\delta(x,0)$, and $\delta(x,t)$ for $\delta(x)\,\delta(t)$.

The Dirac delta function on a region Ω is defined in terms of the complete orthonormal set of eigenfunctions f_n for the region Ω, as

$$\delta_\Omega(\mathbf{x},\mathbf{x}') = \sum_{n=0}^{\infty} f_n(\mathbf{x})\bar{f}_n(\mathbf{x}'), \tag{7.11}$$

where $\mathbf{x}$ and $\mathbf{x}'$ are the field and the source point, respectively, in Ω.

Example 7.2. In view of Example 5.2, the spatial orthogonal eigenfunctions $f_n(x)$ for the one–dimensional problem $u_t = k\,u_{xx}$, $-a < x < a$, subject to the initial and boundary conditions $u(x,0) = F(x)$ for $-a < x < a$, and $u(-a,t) = 0 = u(a,t)$ for $t > 0$, are given by $\dfrac{1}{2a}\sin\dfrac{n\pi x}{a}$, which in complex form are

$$f_n = \frac{1}{2a}\,e^{in\pi x/a}. \tag{7.12}$$

Hence the Dirac delta function in the region $-a < x < a$ for the steady state (as $t \to \infty$) one–dimensional Laplace equation is represented by

$$\delta(x,x') = \begin{cases} \frac{1}{4a^2}\sum_{-\infty}^{\infty} e^{in\pi(x-x')/a}, & \text{for } |x| < a, \\ 0, & \text{for } |x| > a. \end{cases} \tag{7.13}$$

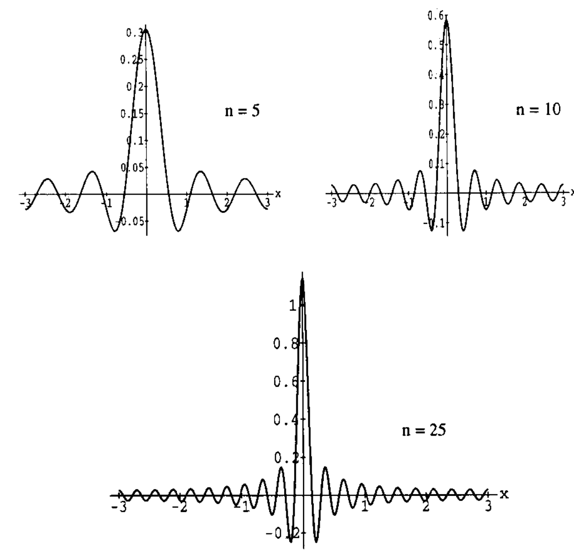


Fig. 7.1. Dirac delta function with 5, 10, and 25 terms, respectively.

The graphs of the real part of this function for the basic interval $-a < x < a$ are shown in Fig. 7.1, using 5, 10, and 25 terms in (7.13), with $a = 3$, and $x' = 0$. They show that the peak becomes infinitely higher and narrower as n increases. ■

Example 7.3. To determine the Fourier transform of the Dirac delta function $\delta(x)$, consider the function

$$f_\kappa(x) = \begin{cases} \dfrac{1}{2\kappa}, & |x| < \kappa, \\ 0, & |x| > \kappa. \end{cases}$$

Note that $\lim_{\kappa\to 0} f_\kappa(x) = \delta(x)$. Also, $e^{-ax} f_\kappa(x) \in L_1(-\infty,\infty)$ for all real a, which implies that $\mathcal{F} f_\kappa(x) = F_\kappa(\alpha)$ is analytic in the entire α–plane of the transform domain. Then

$$\begin{aligned} F_\kappa(\alpha) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_\kappa(x)\,e^{i\alpha x}\,dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\kappa}^{\kappa} \frac{1}{2\kappa}\,e^{i\alpha x}\,dx = \frac{1}{\sqrt{2\pi}}\,e^{i\alpha x_0}, \end{aligned}$$

by means of the mean–value theorem with $-\kappa < x_0 < \kappa$. Hence

$$\lim_{\kappa\to 0} F_\kappa(\alpha) = \frac{1}{\sqrt{2\pi}},$$

which implies that

$$\mathcal{F}\delta(x) = \frac{1}{\sqrt{2\pi}}, \quad \text{and} \quad \mathcal{F}\{1\} = \sqrt{2\pi}\,\delta(x). \blacksquare$$

7.1.2. Heaviside function. The Heaviside function $H(x)$ defines the distribution in $\langle H, \phi\rangle = \int_0^\infty \phi(x)\,dx$, and so, in the distributional sense (7.1), its derivative is defined by

$$H'(x) = \delta(x). \tag{7.14}$$

Note that since $H(x)$ is not differentiable at $x = 0$ in the classical sense, the distributional definition (7.14) for $H'(x)$ when formally integrated gives

$$H(x) = \int_0^x \delta(x)\,dx = \begin{cases} 0, & x < 0 \\ 1, & x > 0. \end{cases}$$

7.1.3. Harmonic functions. The functions whose Laplacian is zero are known as harmonic functions.

LEMMA 1. *The function*

$$\frac{1}{r} = \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} \tag{7.15}$$

is harmonic in a domain that does not contain the point $\mathbf{x}_0 = (x_0, y_0, z_0)$.

LEMMA 2. *Let* $u(\mathbf{x})$ *be in class* $C^2(\Omega)$, *and let its first derivatives be continuous up to the boundary* S *of the domain* Ω. *Let* $\mathbf{x}_0$ *be a fixed point in* Ω, *and* $\mathbf{x}$ *any other point in* Ω. *Then*

$$u(\mathbf{x}_0) = \frac{1}{4\pi} \iint_S \left(\frac{1}{r}\frac{\partial u}{\partial n} - u \frac{\partial(1/r)}{\partial n} \right) dS - \frac{1}{4\pi} \iiint_\Omega \frac{\nabla^2 u}{r}\,dV, \tag{7.16}$$

where r *is defined in* LEMMA 1, *and* $\mathbf{n}$ *is the outward normal vector to the surface* $S = \partial\Omega$.

PROOF. The proof will also establish a very basic technique in the case when $r = 0$, i.e., when $\mathbf{x} = \mathbf{x}_0$. Let $v = \dfrac{1}{r}$. Since this function is

undefined when $\mathbf{x} = \mathbf{x}_0$, we cannot apply Green's identity (A.7) to the entire domain Ω. So we indent the point $\mathbf{x}_0$ by a sphere Ω_ε centered at the point $\mathbf{x}_0$ and with a small enough radius ε. Let S_ε be the surface of Ω_ε, and $\Omega_1 = \Omega \backslash \Omega_\varepsilon$. Then both functions u and $v = \dfrac{1}{r}$ are in the class $C^2(\Omega_1)$, and Green's identity (A.7) is valid in Ω_1. Thus, from (A.7)

$$\iiint_{\Omega_1} \frac{\nabla^2 u}{r}\,dV = \iint_S \left(\frac{1}{r}\frac{\partial u}{\partial n} - u\frac{\partial(1/r)}{\partial n} \right) dS + \\ + \iint_{S_\varepsilon} \left(\frac{1}{r}\frac{\partial u}{\partial n} - u\frac{\partial(1/r)}{\partial n} \right) dS. \tag{7.17}$$

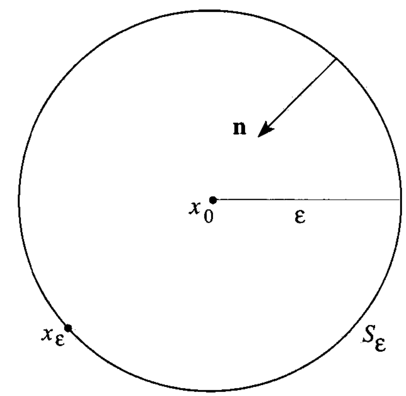


Fig. 7.2.

Let $\varepsilon \to 0$. The surface integral over S in (7.17) is independent of ε. So we consider the second integral over S_ε. Since $r = \varepsilon = \text{const}$, and the normal $\mathbf{n}$ is directed inward on the boundary S_ε (see Fig. 7.2), we get

$$\left.\frac{\partial\left(\frac{1}{r}\right)}{\partial n}\right|_{S_\varepsilon} = -\left.\frac{\partial(1/r)}{\partial r}\right|_{r=\varepsilon} = \frac{1}{\varepsilon^2},$$

which gives

$$\iint_{S_\varepsilon} u\frac{\partial(1/r)}{\partial n}\,dS = \frac{1}{\varepsilon^2}\iint_{S_\varepsilon} u\,dS = \frac{1}{\varepsilon^2} u(\mathbf{x}_\varepsilon)\,4\pi\varepsilon^2 \\ = 4\pi u(\mathbf{x}_\varepsilon) \to 4\pi u(\mathbf{x}_0) \quad \text{as } \varepsilon \to 0, \tag{7.18}$$

where $\mathbf{x}_\varepsilon$ is a point on S_ε, which satisfies the mean–value theorem for the surface integral over S_ε in (7.17) and approaches $\mathbf{x}_0$ as $\varepsilon \to 0$. Since

$u \in C^2(\Omega)$, the first derivatives of u are bounded in $\bar{\Omega} = \Omega \cup S$. Hence, there exists a positive number K such that $|\partial u/\partial n| < K$. Then

$$\left| \iint_{S_\varepsilon} \frac{1}{r} \frac{\partial u}{\partial n} \, dS \right| < \frac{K}{\varepsilon} \iint_{S_\varepsilon} dS = \frac{K}{\varepsilon} 4\pi\varepsilon^2 = 4\pi K\varepsilon \to 0 \quad \text{as } \varepsilon \to 0.$$

Thus, as $\varepsilon \to 0$, we get

$$\iiint_\Omega \frac{\nabla^2 u}{r} \, dV = \iint_S \left(\frac{1}{r} \frac{\partial u}{\partial n} - u \frac{\partial (1/r)}{\partial n} \right) dS - 4\pi u(\mathbf{x}_0),$$

which gives (7.16). ■

NOTE. In a finite domain Ω in R^2, with boundary Γ, the identities analogous to (A.7) and (7.16) are

$$\iint_R \left(u\nabla^2 v - v\nabla^2 u \right) dA = \int_\Gamma \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) ds. \tag{7.19}$$

and

$$u(\mathbf{x}_0) = \frac{1}{2\pi} \int_\Gamma \left(\ln \frac{1}{r} \frac{\partial u}{\partial n} - u \frac{\partial \ln(1/r)}{\partial n} \right) ds - \frac{1}{2\pi} \iint_R \ln \frac{1}{r} \nabla^2 u \, dA, \tag{7.20}$$

where $\mathbf{x}_0 = (x_0, y_0)$, $dA = dx\,dy$, and s is an arc–length along the curve Γ. The proof follows by using the technique of Lemma 2 with $v = \ln \frac{1}{r}$ and indenting the point $\mathbf{x}_0$ by a circle Γ_ε; then $\nabla^2 v = 0$, and

$$\left. \frac{\partial \ln(1/r)}{\partial n} \right|_{\Gamma_\varepsilon} = - \left. \frac{\partial \ln(1/r}{\partial r} \right|_{r=\varepsilon} = \varepsilon,$$

$$\int_{\Gamma_\varepsilon} u \frac{\partial \ln(1/r)}{\partial n} \, ds = \frac{1}{\varepsilon} \int_{\Gamma_\varepsilon} u \, ds = \frac{1}{\varepsilon} u(\mathbf{x}_\varepsilon) \, 2\pi\varepsilon$$
$$\to 2\pi u(\mathbf{x}_0) \quad \text{as } \varepsilon \to 0.$$

Thus,

$$\left| \int_{\Gamma_\varepsilon} \ln \frac{1}{r} \frac{\partial u}{\partial n} \, ds \right| < K \ln \frac{1}{\varepsilon} \int_{\Gamma_\varepsilon} ds = K \ln \frac{1}{\varepsilon} = 2\pi\varepsilon \ln \frac{1}{\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0,$$

where $\left| \dfrac{\partial u}{\partial n} \right| < K$. The result follows as $\varepsilon \to 0$. ■

Some important properties of harmonic functions are defined by the following theorems:

THEOREM 7.1. *Let the function $u(\mathbf{x}) \in C^2$ be harmonic in Ω. Then*

$$\iint_S u \frac{\partial u}{\partial n} \, dS \ge 0. \tag{7.21}$$

In fact, if we let $v = u$ in Green's first identity (A.5), we get

$$\iiint_\Omega \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial z} \right)^2 \right] dV = \iint_S u \frac{\partial u}{\partial n} \, dS. \tag{7.22}$$

Since the volume integral (left side) in (7.22) ≥ 0, we get (7.21). ■

THEOREM 7.2. *The integral of the normal derivative of a harmonic function over the boundary of the domain is zero, i.e.,*

$$\iint_S u \frac{\partial u}{\partial n} \, dS = 0. \tag{7.23}$$

The result follows by applying Green's identity (A.7) to $u = v = 1$. ■

THEOREM 7.3. *The value of a harmonic function at an interior point of a finite domain can be expressed in terms of its values and the values of its normal derivative on the surface of the domain, i.e.,*

$$u(\mathbf{x}_0) = \frac{1}{4\pi} \iint_S \left(\frac{1}{r} \frac{\partial u}{\partial n} - u \frac{\partial (1/r)}{\partial n} \right) dS. \tag{7.24}$$

This result is obtained by applying (7.16) to the harmonic function u. ■

NOTE that there are no second order derivatives in (7.21), (7.23), or (7.24). Thus, to ensure that these three results are valid we must assume that the harmonic function $u \in C^2(S)$. Consider a domain $\hat{\Omega} \subset \Omega$. Then we apply (7.16) to $\hat{\Omega}$, and by taking the limit process $\hat{\Omega} \to \Omega$ we find that the above analysis need not assume that the second derivatives of u are continuous up to the boundary S.

THEOREM 7.4. *A function $u(\mathbf{x})$ harmonic in the domain Ω has derivatives of all orders inside Ω.*

PROOF. Consider an arbitrary point $\mathbf{x}_0 \in \Omega$ and surround this point by a domain Ω' bounded by its surface S' such that $\Omega' \cup S' \subset \Omega$. Since u is harmonic in Ω, so it is harmonic in Ω' and $u \in C^2(\Omega')$. Then by (7.24)

$$u(\mathbf{x}_0) = \frac{1}{4\pi} \iint_{S'} \left(\frac{1}{r}\frac{\partial u}{\partial n} - u \frac{\partial (1/r)}{\partial n} \right) dS. \tag{7.25}$$

Since the point $\mathbf{x}_0$ does not lie on S', we find that

$$\frac{1}{r} = \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}}$$

is continuous and has continuous derivatives of all orders at the point $\mathbf{x}_0$. Hence the right side of (7.25) can be differentiated any number of times under the integral sign. ∎

THEOREM 7.5. *The value of a harmonic function at the center of a sphere is equal to the arithmetic mean of its value on the surface of this sphere.*

PROOF. Let $u(\mathbf{x}_0)$ be harmonic inside the sphere $|\mathbf{x} - \mathbf{x}_0| = R$. Then by (7.24)

$$u(\mathbf{x}_0) = \frac{1}{4\pi} \iint_{S_R} \left(\frac{1}{r}\frac{\partial u}{\partial n} - u \frac{\partial (1/r)}{\partial n} \right) dS. \tag{7.26}$$

Now, since $\left.\dfrac{\partial\left(\frac{1}{r}\right)}{\partial n}\right|_{S_R} = \left.\dfrac{\partial (1/r)}{\partial r}\right|_{r=R} = -\dfrac{1}{R^2}$, we get from (7.26)

$$u(\mathbf{x}_0) = \frac{1}{4\pi R^2} \iint_{S_R} u\, dS = \frac{1}{4\pi R^2} \int_0^{\pi} \int_0^{2\pi} u(R, \theta, \phi) \sin\theta \, d\theta \, d\phi. \;\blacksquare \tag{7.27}$$

THEOREM 7.6 (MAXIMUM PRINCIPLE). *A non-constant function which is harmonic inside a bounded domain Ω with boundary S and continuous in the closed domain $\bar{\Omega} = \Omega \cup S$ attains its maximum and minimum values only on the boundary of the domain.*

PROOF. Assume that $u(\mathbf{x})$ attains its maximum value at an interior point $\mathbf{x}_0 \in \Omega$. Draw a sphere $S_\rho : |\mathbf{x} - \mathbf{x}_0| = \rho$ such that $S_\rho \subset \Omega$. Then by Theorem 7.5,

$$u(\mathbf{x}_0) = \frac{1}{4\pi\rho^2} \iint_{S_\rho} u\, dS \le \frac{1}{4\pi\rho^2} \iint_{S_\rho} u_\rho^{\max}\, dS = u_\rho^{\max}, \tag{7.28}$$

where equality holds only if $u = u(\mathbf{x}_0) = \text{const}$ on S_ρ. Since $u(\mathbf{x})$ attains its maximum value in Ω, we must have $u(\mathbf{x}_0) \le u_\rho^{\max}$, which means that we must have the equality sign in (7.28). Thus, $u(\mathbf{x})$ becomes constant both inside and on the boundary of the sphere S_ρ. We shall now show that $u(\mathbf{x})$ is constant throughout Ω: Let $\tilde{\mathbf{x}}$ be any other interior point of Ω. Then we show that $u(\tilde{\mathbf{x}}) = u(\mathbf{x}_0)$. Join the points $\mathbf{x}_0$ and $\tilde{\mathbf{x}}$ by a line L which may be a polygonal (broken) line inside Ω, and let d be the shortest distance between L and S. Then by (7.28), $u(\mathbf{x})$ has the constant value $u(\mathbf{x}_0)$ in the sphere $S_0 : |\mathbf{x} - \mathbf{x}_0| = d^2/4$, where d is the diameter of the sphere. Suppose that $\mathbf{x}_1$ is the last point at the intersection of this sphere and the line L. Then we have $u(\mathbf{x}_1) = u(\mathbf{x}_0)$, and thus, we have $u(\mathbf{x}) = u(\mathbf{x}_0)$ on the sphere $S_1 : |\mathbf{x} - \mathbf{x}_1| = d^2/4$. Again, suppose that $\mathbf{x}_2$ is the last point at the intersection of the sphere S_1 and the line L. Then, $u(\mathbf{x}) = u(\mathbf{x}_0)$ on the sphere $S_2 : |\mathbf{x} - \mathbf{x}_2| = d^2/4$. Continuing this process of constructing spheres the entire line L can be covered by a finite number of these spheres, and the point $\tilde{\mathbf{x}}$ will lie inside one of these spheres, and therefore, $u(\tilde{\mathbf{x}}) = u(\mathbf{x}_0)$. An analogous argument can show that a harmonic function cannot attain its minimum value inside Ω. Hence, the function $u(\mathbf{x})$ attains its maximum and minimum values in a closed bounded domain Ω, and in particular, it attains them on the boundary of Ω, since a harmonic function cannot attain them inside Ω. ∎

COROLLARY 1. *If u and U are continuous in $\Omega \cup S$, and harmonic in Ω such that $u \le U$ on S, then $u \le U$ also at all points inside Ω.*

In fact, the function $U - u$ is continuous in Ω, and harmonic in Ω; hence $U - u \ge 0$ on S. Then, in view of the maximum principle (Theorem 7.6), we must have $U - u \ge 0$ at all points inside Ω. ∎

COROLLARY 2. *If u and U are continuous in $\Omega \cup S$ and harmonic in Ω for which $|u| \le U$ on S, then $|u| \le U$ also at all points inside Ω.*

In fact, the three harmonic functions $-U$, u, and U satisfy the relation $-U \le u \le U$ on S. If we apply Corollary 1 twice, we find that

$-U \le u \le U$ at all points inside Ω, or $|u| \le U$ inside Ω. ■

7.1.4. The concept of a Green's function. Consider the heat conduction problem in a region $\Omega \subset R^3$ with the boundary surface $S = \cup_{i=1}^{n} S_i$, defined by

$$\nabla^2 u(\mathbf{x}, t) + \frac{1}{k} f(\mathbf{x}, t) = \frac{1}{a}\frac{\partial u(\mathbf{x}, t)}{\partial t}, \quad \text{for } t > 0, \tag{7.29a}$$

$$k_i \frac{\partial u}{\partial n_i} + h_i u = g(\mathbf{x}, t) \quad \text{on } S_i,\ t > 0, \tag{7.29b}$$

$$u(\mathbf{x}, 0) = F(\mathbf{x}), \tag{7.29c}$$

where $\dfrac{\partial}{\partial n_i}$ denotes the derivative with respect to the outward normal $\mathbf{n}$ to the boundary surface S_i, the coefficients k_i and h_i are constant, and the function $g(\mathbf{x}, t)$ is the prescribed mixed boundary condition on each S_i. Note that with $k_i = 0$ the boundary condition reduces to the Dirichlet type, and with $h_i = 0$ to the Neumann type. In order to solve problem (7.29) we consider the auxiliary problem

$$\begin{aligned} &\nabla^2 G(\mathbf{x}, t; \mathbf{x}', t') + \frac{1}{k}\delta(\mathbf{x}, \mathbf{x}')\,\delta(t, t') = \frac{1}{a}\frac{\partial G}{\partial t} \quad \text{in } \Omega,\ t > 0, \\ &k_i \frac{\partial G}{\partial n_i} + h_i u = 0 \quad \text{on } S_i,\ t > 0, \end{aligned} \tag{7.30}$$

such that

$$G(\mathbf{x}, t; \mathbf{x}', t') = 0 \quad t' < t, \tag{7.31}$$

where $\delta(\mathbf{x}, \mathbf{x}') = \delta(x, x')\,\delta(y, y')\,\delta(z, z')$ is the Dirac delta function in R^3 for the space coordinates $\mathbf{x} = (x, y, z)$ and $\mathbf{x}' = (x', y', z')$, and $\delta(t, t')$ is the Dirac delta function for the time coordinate $t > t'$. Note that the function $G(\mathbf{x}, t; \mathbf{x}', t')$, known as *Green's function* for problem (7.29), satisfies the auxiliary problem (7.30) with homogeneous boundary conditions and zero initial condition (7.31), and has an impulsive heat source at the space–time source point $(\mathbf{x}', t')$.

DEFINITION 7.5. Green's function $G(\mathbf{x}, t; \mathbf{x}', t')$ in R^3, which is a solution of problem (7.30)–(7.31), represents the temperature distribution in the region Ω which initially at zero temperature is subjected to the homogeneous boundary conditions due to an impulsive heat source of unit strength situated at the space–time point $(\mathbf{x}', t')$.

The notation $G(\mathbf{x}, t; \mathbf{x}', t')$ is composed of two parts in its argument. The first part $\mathbf{x}, t$ denotes the field point where the *effect* of the impulsive heat source located at the source point signifies the temperature at the point $\mathbf{x}$ at time t. The second part $\mathbf{x}', t'$ denotes the *cause* which is the impulsive heat source situated at the point $\mathbf{x}'$ generating an instantaneous (impulsive) heat at an earlier time t'. The combined notation has the physical significance of an entire space–time process which can be visualized as $G(\textit{effect}; \textit{cause}) \equiv G(\mathbf{x}, t; \mathbf{x}', t')$.

DEFINITION 7.6. The *reciprocity relation* for Green's function $G(\mathbf{x}, t; \mathbf{x}', t')$ which satisfies the auxiliary problem (7.30)–(7.31) is defined by

$$G(\mathbf{x}, t; \mathbf{x}', t') = G(\mathbf{x}', -t'; \mathbf{x}, -t). \tag{7.32}$$

The physical significance of this relation is that the *effect* at $\mathbf{x}, t$ due to a *cause* at $\mathbf{x}', t'$ for $t' < t$ is the same as the *effect* at $\mathbf{x}', -t'$ due to a *cause* at $\mathbf{x}, -t$.

Green's function $G(\mathbf{x}, t; \mathbf{x}', t')$ is singular at the source point $\mathbf{x}' \in \Omega$, such that

$$LG(\mathbf{x}, t; \mathbf{x}', t') = \delta(\mathbf{x}, \mathbf{x}'), \tag{7.33}$$

where L is a differential operator.

7.2. Parabolic Equations

Before we determine Green's functions for parabolic equations, we will solve problem (7.29). In view of the reciprocity relation (7.32), Eq (7.30) is written for the function $G(\mathbf{x}', -t'; \mathbf{x}, -t)$ as

$$\nabla_0^2 G + \frac{1}{k}\delta(\mathbf{x}, \mathbf{x}')\,\delta(t, t') = -\frac{1}{a}\frac{\partial G}{\partial t'} \quad \text{in } \Omega, \tag{7.34}$$

where ∇_0^2 denotes the Laplacian in the variable $\mathbf{x}'$, i.e., $\nabla_0^2 \equiv \dfrac{\partial^2}{\partial x'^2} + \dfrac{\partial^2}{\partial y'^2} + \dfrac{\partial^2}{\partial z'^2}$, and the minus sign on the right side results because t has been replaced by $-t'$. Similarly, if we replace $\mathbf{x}$ by $\mathbf{x}'$ and t by t' in (7.29a), we get

$$\nabla_0^2 u + \frac{1}{k} f(\mathbf{x}', t') = \frac{1}{a}\frac{\partial u(\mathbf{x}', t')}{\partial t'} \quad \text{in } \Omega. \tag{7.35}$$

Then, if we multiply (7.34) by u and (7.35) by G, subtract, and integrate the resulting equation with respect to $\mathbf{x}'$ over the region Ω and with respect to t' from 0 to $t_0 = t+\varepsilon$, where $\varepsilon > 0$ is arbitrarily small, we find that

$$\int_0^{t_0} dt' \iiint_\Omega \left(G\nabla_0^2 u - u\nabla_0^2 G\right)\, d\Omega + \frac{1}{k}\int_0^{t_0} dt' \iiint_\Omega \Big[f(\mathbf{x}',t')G - \frac{1}{a}\delta(\mathbf{x}',\mathbf{x})\,\delta(t',t)\,u\Big]\, d\Omega = \frac{1}{a}\iiint_\Omega [Gu]_{t'=0}^{t_0}\, d\Omega. \quad (7.36)$$

In view of Green's identity (A.7), the first volume integral is changed to the surface integral; thus,

$$\iiint_\Omega \left(G\nabla_0^2 u - u\nabla_0^2 G\right)\, d\Omega = \sum_{i=1}^n \iint_{S_i} \left(F\frac{\partial u}{\partial n_i} - u\frac{\partial G}{\partial n_i}\right) dS_i.$$

Also, in view of the property (7.7) of the Dirac delta distribution, the second term in the second volume integral in (7.36) is

$$\int_0^{t_0} dt' \iiint_\Omega u\,\delta(\mathbf{x}',\mathbf{x})\,\delta(t',t_0)\, d\Omega = u(\mathbf{x},t_0);$$

and the integrand on the right side of (7.36) in the limit as $\varepsilon \to 0$ becomes

$$[Gu]_{t'=0}^{t_0} = G\Big|_{t'=t_0} u(t_0) - G\Big|_{t'=0} F(\mathbf{x}') = -G\Big|_{t'=0} F(\mathbf{x}'),$$

since $G\Big|_{t'=t_0} = G(\mathbf{x},t;\mathbf{x}',t_0) = G(\mathbf{x},t;\mathbf{x}',t+\varepsilon) = 0$ for $t < t_0$ because the time for the effect precedes the time for the cause (impulse). Hence, Eq (7.36) gives

$$u(\mathbf{x},t) = \iiint_\Omega G\Big|_{t'=0} F(\mathbf{x}')\, d\Omega + \frac{a}{k}\int_0^t dt' \iiint G\, f(\mathbf{x}',t')\, d\Omega + a\int_0^t dt' \sum_{i=1}^n \iint_{S_i} \left(F\frac{\partial u}{\partial n_i} - u\frac{\partial G}{\partial n_i}\right) dS_i, \quad (7.37)$$

where $G\Big|_{t'=0} = G(\mathbf{x},t;\mathbf{x}',0)$. If we multiply the boundary condition (7.29b) by G and (7.30) by u and subtract, we get

$$G\frac{\partial u}{\partial n_i} - u\frac{\partial G}{\partial n_i} = \frac{1}{k_i} G\Big|_{S_i} g_i(\mathbf{x},t), \quad (7.38)$$

where $G\Big|_{S_i} = G(\mathbf{x},t;\mathbf{x}_i{}',t')$ denotes Green's function evaluated on the boundary surfaces S_i, $i = 1,2,\cdots,n$. After substituting (7.38) into (7.37), we obtain the solution of the problem (7.29) in terms of Green's function $G(\mathbf{x},t;\mathbf{x}',t')$ as

$$u(\mathbf{x},t) = \iiint_\Omega G(\mathbf{x},t;\mathbf{x}',0)\, F(\mathbf{x}')\, d\Omega + \frac{a}{k}\int_0^t dt' \iiint G(\mathbf{x},t,\mathbf{x}';t')\, f(\mathbf{x}',t')\, d\Omega + a\int_0^t dt' \sum_{i=1}^n \frac{1}{k_i}\iint_{S_i} G(\mathbf{x},t;\mathbf{x}',t')\, f(\mathbf{x}',t')\, dS_i. \quad (7.39)$$

Now we will derive Green's functions for some parabolic equations. Without loss of generality, we will translate the source point to the origin, i.e., we will take $\mathbf{x} = \mathbf{0}$ and $t' = 0$, and denote $G(\mathbf{x},t;\mathbf{0},0)$ by $G(\mathbf{x},t)$. The results so obtained can then be translated again to the source point at $(\mathbf{x}',t')$ by replacing $\mathbf{x}$ by $\mathbf{x}-\mathbf{x}'$ and t by $t-t'$. This will, however, yield $G(\mathbf{x}-\mathbf{x}';t-t')$ which in our notation is $G(\mathbf{x},t;\mathbf{x}',t')$.

EXAMPLE 7.4. Green's function $G(x,t;x',t)$ for the homogeneous diffusion operator in R^1 satisfies the equation

$$\frac{\partial G}{\partial t} - a\frac{\partial^2 G}{\partial x^2} = \delta(x-x')\,\delta(t-t'). \quad (7.40)$$

Assuming that $G = 0$ for $t < t'$, we apply the Laplace transform to Eq (4.40) and get

$$s\overline{G} - a\frac{d^2\overline{G}}{dx^2} = e^{-st'}\,\delta(x-x').$$

The solution of this equation under the condition that $\overline{G} \to 0$ as $x \to \pm\infty$ is

$$\overline{G}(x,x';s,t') = \frac{e^{-st'}}{2}\frac{1}{\sqrt{as}}\, e^{-r\sqrt{as}}, \quad r = |x-x'|,$$

which on inversion gives

$$G(x,x';t,t') = \frac{H(t-t')}{2\sqrt{\pi a(t-t')}}\, e^{(x-x')^2/4a(t-t')}. \ \blacksquare$$

EXAMPLE 7.5. For the diffusion operator in R^n, Green's function $G(\mathbf{x}, t)$ satisfies the equation

$$\frac{\partial G(\mathbf{x}, t)}{\partial t} - a\nabla^2 G(\mathbf{x}, t) = \delta(\mathbf{x}, t) = \delta(\mathbf{x})\,\delta(t). \tag{7.41}$$

We apply the Fourier transform $\mathcal{F}_x$ to both sides of Eq (7.41). Then,

$$\mathcal{F}_x\left(\frac{\partial G}{\partial t}\right) - a\mathcal{F}_x[\nabla^2 G] = \mathcal{F}_x[\delta(x, t)].$$

Since

$$\mathcal{F}_x[\delta(x, t)] = \mathcal{F}_x[\delta(x) \cdot \delta(t)] = \mathcal{F}[\delta](\alpha) \cdot \delta(t) = \frac{\mathbf{1}(\alpha)}{(2\pi)^{n/2}} \cdot \delta(t),$$

where $\mathbf{1}$ is the identity function, and

$$\mathcal{F}_x\left[\frac{\partial G}{\partial t}\right] = \frac{\partial}{\partial t}\mathcal{F}_x[G],$$

$$\mathcal{F}_x[\nabla^2 G] = \mathcal{F}_x[\nabla^2 G] = -|\alpha|^2 \mathcal{F}_x[G],$$

Eq (7.41) is transformed into

$$\frac{\partial}{\partial t}\mathcal{F}_x G(\alpha, t) + a|\alpha|^2 \mathcal{F}_x G(\alpha, t) = \frac{\mathbf{1}(\alpha)}{(2\pi)^{n/2}} \cdot \delta(t),$$

which has the solution

$$\mathcal{F}_x G(\alpha, t) = \frac{H(t)}{(2\pi)^{n/2}}\, e^{-a|\alpha|^2 t}.$$

If we apply the inverse Fourier transform $\mathcal{F}_\alpha^{-1}$, we obtain

$$\begin{aligned} G(\mathbf{x}, t) &= \mathcal{F}_\alpha^{-1}[G(\alpha, t)] \\ &= \frac{H(t)}{(2\pi)^n}\int_{R^n} e^{-a|\alpha|^2 t - i(\alpha\cdot\mathbf{x})}\, d\alpha = \frac{H(t)}{(4\pi a t)^{n/2}} e^{-|\mathbf{x}|^2/4at}, \end{aligned} \tag{7.42}$$

which, on translating to the source point $(\mathbf{x}', t')$, yields

$$G(\mathbf{x}, t; \mathbf{x}', t') = \frac{H(t - t')}{[4\pi a(t - t')]^{n/2}} e^{-|\mathbf{x}-\mathbf{x}'|^2/4a(t-t')}. \tag{7.43}$$

The graphs for $G(\mathbf{x}, t)$ for $0 < t_1 < t_2 < t_3$ are shown in Fig. 7.3.

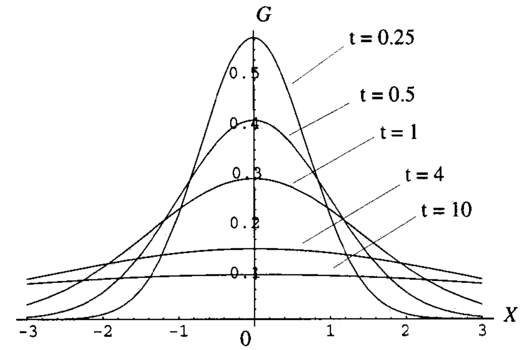


Fig. 7.3. Graphs of $G(\mathbf{x}, t)$ in R^2.

EXAMPLE 7.6. We will solve the Cauchy problem in R^1, i.e., solve the initial value problem

$$u_t = a\, u_{xx}, \tag{7.44a}$$

$$u(x, 0) = f(x), \quad -\infty < x < \infty, \quad t > 0, \tag{7.44b}$$

where $f(x)$ is a bounded and continuous function on the x–axis. Eq (7.44a) is also known as the transient Fourier equation in R^1. We shall prove that the function

$$u(x, t) = \frac{1}{\sqrt{4\pi a t}}\int_{-\infty}^{\infty} f(y)\, e^{-(y-x)^2/4at}\, dy, \tag{7.45}$$

which belongs to the class C^∞ with respect to x and t for $t > 0$, satisfies Eq (7.44a) such that $\lim_{t\to 0+} u(x, t) = f(x), \quad -\infty < x < \infty$.

METHOD 1: We use the method of separation of variables in the form $u(x, t) = X(x)T(t)$. Then we find that $T(t) = e^{-a\lambda^2 t}$ and $X(x) = A\cos\lambda x + B\sin\lambda x$, where λ, A, B are arbitrary. If we assume $A = A(\lambda)$, $B = B(\lambda)$, then

$$u_\lambda(x, t) = [A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]e^{-a\lambda^2 t}$$

is a solution of Eq (7.44a). Moreover,

$$\int_{-\infty}^{\infty} u_\lambda(x, t)\, d\lambda,$$

with proper choice of $A(\lambda)$ and $B(\lambda)$, is also a solution of Eq (7.44a). Now, in view of the initial condition (7.44b)

$$f(x) = \int_{-\infty}^{\infty} u_\lambda(x,0)\,d\lambda = \int_{-\infty}^{\infty} [A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]\,d\lambda. \quad (7.46)$$

The function $f(x)$, being continuous and bounded, has the Fourier integral representation

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} f(y)\cos\lambda(y-x)\,dy. \quad (7.47)$$

Comparing (7.46) and (7.47), we find that

$$A(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\cos\lambda y\,dy, \quad B(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\sin\lambda y\,dy.$$

Thus,

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} u_\lambda(x,t)\,d\lambda \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} f(y)\cos\lambda(y-x)\,e^{-a\lambda^2 t}\,dy \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)\,dy \int_{-\infty}^{\infty} \cos\lambda(y-x)\,e^{-a\lambda^2 t}\,d\lambda \\ &= \frac{1}{\sqrt{4\pi at}}\int_{-\infty}^{\infty} f(y)\,e^{-(y-x)^2/4at}\,dy. \end{aligned}$$

METHOD 2: We apply a Fourier transform in space (with α as the transform variable) and a Laplace transform in time (with s as the transform variable). Then Eq (7.44a) with the initial condition (7.44b) is transformed into

$$(s + a\alpha^2)\tilde{u}(\alpha,s) - \tilde{f}(\alpha) = 0, \quad (7.48)$$

where $\tilde{u} = \mathcal{F}[u]$, and $\tilde{f} = \mathcal{F}[f]$. Since $\tilde{u}(\alpha,s) = \dfrac{\tilde{f}(\alpha)}{s + a\alpha^2}$, the function $\tilde{u}(\alpha,s)$ has a simple pole at $s = -a\alpha^2$. Hence, by applying the inverse Fourier and Laplace transforms, we get the solution of the Cauchy problem (7.44) as

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left\{\lim_{R\to\infty}\left[\frac{1}{2\pi i}\int_{u-iR}^{u+iR}\frac{\tilde{f}(\alpha)}{s+a\alpha^2}e^{st}\,ds\right]\right\} e^{-i\alpha x}\,d\alpha.$$

A direct inversion of the Laplace transform yields

$$\tilde{u}(\alpha,t) = e^{-a\alpha^2 t}\tilde{f}(\alpha),$$

which, after the inversion of the Fourier transform, gives

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{u}(\alpha,t)e^{-i\alpha x}\,d\alpha \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\alpha)\,e^{-(i\alpha x + a\alpha^2 t)}\,d\alpha. \end{aligned}$$

If we identify $\tilde{g}(\alpha) = e^{-a\alpha^2 t}$, then for the convolution of the product $\tilde{f}(\alpha)\tilde{g}(\alpha)$, we have

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}(\alpha)\tilde{g}(\alpha)e^{-i\alpha x}\,d\alpha = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)g(x-y)\,dy. \quad (7.49)$$

Since $\tilde{f}(\alpha)$ is identified with the Fourier component of the Cauchy data $f(x)$ in (7.44b), we find that $f(y)$ in (7.49) corresponds to the initial condition and

$$g(x-y) = \frac{1}{\sqrt{4\pi at}}\,e^{-(x-y)^2/4at},$$

and hence by the convolution theorem we get the solution of the Cauchy problem from (7.49).

If $f(x) = \delta(x)$ in (7.44b), then the solution of the Cauchy problem (7.44) is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi at}}\,e^{-x^2/4at}, \quad t > 0. \blacksquare$$

EXAMPLE 7.7. (*Schrödinger equation*) If we consider the non–homogeneous case, then the Fourier heat equation

$$\left(\frac{1}{a}\frac{\partial}{\partial t} - \nabla^2\right)u(x,t) = f(x,t) \quad (7.50)$$

has two interpretations:
(i) If $a > 0$ is a real constant depending on specific heat and thermal conductivity of the medium, then $u(x,t)$ determines the temperature distribution. The function $f(x,t)$ on the right side describes local heat

production minus absorption.
(ii) The function $u(x,t)$ defines a particle density and a is the diffusion coefficient. If a is purely imaginary such that $a = \dfrac{i\hbar}{2m}$, where m is the mass of the quantum particle, and $\hbar = 1.054 \times 10^{-27}$ erg-sec is Planck's constant, then Eq (7.50) defines the Schrödinger equation

$$i\hbar \frac{\partial u(x,t)}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 u(x,t) = f(x,t). \tag{7.51}$$

In view of (7.42) we have $a = \hbar/2im$, and Green's function for Eq (7.50) in R^1 is given by

$$G(x,t) = H(t) \frac{1+i}{2} \sqrt{\frac{m}{\pi \hbar t}}\, e^{imx^2/2\hbar t}. \blacksquare \tag{7.52}$$

7.3. Elliptic Equations

We will consider the Laplace and the Helmholtz operators and derive Green's functions for them.

EXAMPLE 7.8. Assuming that the source point $G(\mathbf{x}')$ is at the origin, Green's function $G(\mathbf{x}) = G(\mathbf{x}, \mathbf{0})$ for the Laplacian ∇^2 in R^n satisfies the equation

$$\nabla^2 G(\mathbf{x}) = \delta(\mathbf{x}). \tag{7.53}$$

Then, in view of the property (7.11),

$$\iint_S \nabla^2 G(\mathbf{x})\, dS = \iint_S \delta(\mathbf{x})\, dS = 1. \tag{7.54}$$

We shall first consider the case $n = 3$. Since the operator ∇^2 is invariant under a rotation of coordinate axes, we shall seek a solution that depends only on $r = |x|$. For $r > 0$, $G(r)$ will satisfy the homogeneous equation $\nabla^2 G = 0$, i.e., in spherical coordinates

$$\frac{1}{r} \frac{\partial}{\partial r} \Big(r^2 \frac{\partial G}{\partial r} \Big) = 0,$$

which has a solution $G(r) = \dfrac{A}{r} + B$. If we require the potential to vanish at infinity, then $B = 0$, and $G(r) = A/r$. In order to determine A, we take into account the magnitude of the source at $x = 0$. Integrating (7.53) over a small sphere S_ε of radius ε and center at $x = 0$, we obtain

$$\iint_{S_\varepsilon} \nabla^2 G\, dx = 1,$$

which, by using (A.4), gives

$$\iint_{\partial S_\varepsilon} \frac{\partial G}{\partial r}\Big|_{r=\varepsilon} dS = 1, \tag{7.55}$$

where ∂S_ε is the surface of the sphere S_ε. Physically, Eq (7.55) expresses the conservation of charge, i.e., the flux of the electric field through the closed surface ∂S_ε (of area $4\pi\varepsilon^2$) is equal to the charge in the interior of S_ε. Now, substituting $G = A/r$ in (7.55), we find that $A = 1/(4\pi)$, and hence Green's function for the three-dimensional Laplace equation is

$$G = \frac{1}{4\pi r} = \frac{1}{4\pi |\mathbf{x} - \mathbf{x}'|}. \tag{7.56}$$

For $n = 2$, we have

$$\frac{1}{r} \frac{\partial}{\partial r} \Big(r \frac{\partial G}{\partial r} \Big) = 0, \quad r = |\mathbf{x}| = \sqrt{x^2 + y^2},$$

which has a solution $G(r) = C \ln r + D$. We set $D = 0$ to ensure a zero value at infinity, and use a result similar to (7.55) for the flux of an electric charge through the boundary ∂C_ε (of length $2\pi\varepsilon$) of a circle of radius ε. Then $C = -1/2\pi$, and

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{2\pi} \ln \frac{1}{r} = \frac{1}{2\pi} \ln \frac{1}{|\mathbf{x} - \mathbf{x}'|}. \tag{7.57}$$

In general, Green's function for the free-space Laplacian in R^n is given by

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{(n-2) S_n(1) r^{n-2}}, \quad n > 2, \tag{7.58}$$

where $S_n(1) = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of a sphere of radius unity, and $r = |\mathbf{x} - \mathbf{x}'|$. Note that the notation for this Green's function is not standard; it is sometimes defined with a minus sign, or without the factor $1/2\pi$. $\blacksquare$

EXAMPLE 7.9. (*Method of images*) The problem of finding Green's function $G(\mathbf{x}, \mathbf{x}') = G(x, y; x', y')$ inside some region Ω bounded by a closed curve Γ with the homogeneous boundary condition $G = 0$ on Γ amounts to that of finding the electrostatic potential due to a point charge at the point (x', y') inside a grounded conductor in the shape of the boundary Γ. Consider, for example, the region Ω as the half-plane $y > 0$. Then Green's function is given by the point charge of strength $1/2\pi$ at (x', y') together with an equal but opposite charge (of strength $-1/2\pi$) at the point $(x', -y')$ which is the image of the point (x', y') in the x-axis (Fig. 7.4). Thus, Green's function for the half-plane such that $G = 0$ on the x-axis and $G \to 0$ as $r \to \infty$, where $r = |\mathbf{x} - \mathbf{x}'| = \sqrt{(x - x')^2 + (y - y')^2}$, is given by

$$G(x, y; x', y') = \frac{1}{4\pi} \ln \frac{(x - x')^2 + (y - y')^2}{(x - x')^2 + (y + y')^2}. \tag{7.59}$$

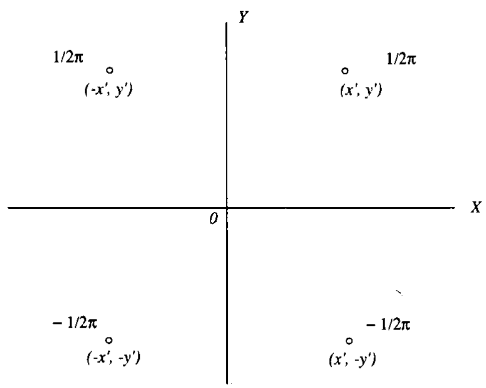


Fig. 7.4.

Since the algebraic sum of the charges over the entire (x, y)-plane is zero, the condition $G \to 0$ as $r \to \infty$ becomes possible because any nonzero residual charge will make Green's function behave like $\ln r$. This means that we cannot determine a Green's function for the half-plane $y > 0$ subject to the conditions $G_y = 0$ on the x-axis and $G \to 0$ as $r \to \infty$, because in this case that charge at each (x', y') and $(x', -y')$ are of the same sign and their algebraic sum is not zero. However, we can determine Green's function for the quarter-plane $x > 0$, $y > 0$, subject to the conditions $G = 0$ on $y = 0$ and $G_y = 0$ on $x = 0$. As seen in Fig. 7.4, we have the charge of strength $1/2\pi$ at each (x', y') and $(-x', y')$, and the charge of strength $-1/2\pi$ at each $(x', -y')$ and $(-x', y')$. Thus, Green's function in this case is

$$G(x, y; x', y') = \frac{1}{4\pi} \ln \frac{[(x - x')^2 + (y - y')^2][(x + x')^2 + (y - y')^2}{[(x - x')^2 + (y + y')^2][(x + x')^2 + (y + y')^2]}. \tag{7.60}$$

EXAMPLE 7.10. Green's function for the Helmholtz equation in R^n satisfies

$$(\nabla^2 + \mu) G(r) = \delta(r), \quad \text{or} \quad (\nabla^2 - h^2) G(r) = \delta(r), \quad r = |\mathbf{x} - \mathbf{x}'|, \tag{7.61}$$

where $h^2 = -\mu$, and $\sqrt{\mu}$ is defined such that it has a nonnegative imaginary part, i.e., $\sqrt{\mu} = \alpha + i\beta$, with $\beta \ge 0$, and $\beta = 0$ iff $\mu \in [0, \infty)$. We will therefore take $h = -i\sqrt{\mu}$, so that h is real positive when μ is real negative. We will assume that Green's function $G(r)$ is spherically symmetric. Then for $\mathbf{x} \ne \mathbf{0}$, the function $G(r)$ must satisfy in spherical coordinates

$$\frac{d}{dr}\left(r^{n-1} \frac{dG}{dr}\right) + \mu r^{n-1} G = 0. \tag{7.62}$$

If we substitute $G = w r^{1-(n/2)}$, then Eq (7.62) can be reduced to an equation of the Bessel type of order $(n/2) - 1$ with parameter μ, i.e.,

$$\frac{d}{dr}\left(r \frac{dw}{dr}\right) - \frac{w}{r}\left(1 - \frac{n}{2}\right)^2 + \mu r w = 0,$$

whose general solution can be written in terms of the Hankel functions as

$$w(r) = C_1 H^{(1)}_{(n/2)-1}(\sqrt{\mu}\, r) + C_2 H^{(2)}_{(n/2)-1}(\sqrt{\mu}\, r), \quad n \ge 2. \tag{7.63}$$

If $\mu \notin [0, \infty)$, then $\sqrt{\mu}$ has positive imaginary part and the Hankel function $H^{(2)}_{(n/2)-1}(\sqrt{\mu}\, r)$ becomes exponentially large as $r \to \infty$, but $H^{(1)}_{(n/2)-1}(\sqrt{\mu}, r)$ is exponentially small. Since G vanishes at $r = \infty$, we must have $C_2 = 0$, and then from (7.63) we get

$$G = C_1 H^{(1)}_{(n/2)-1}(\sqrt{\mu}\, r). \tag{7.64}$$

As in Example 7.8, we apply (A.3) to (7.61) and obtain

$$\int_{S_\varepsilon} \frac{\partial G}{\partial r}\, dS = 1,$$

or

$$\lim_{r\to 0} r^{n-1} S_n(1) \frac{\partial G}{\partial r} = 1, \tag{7.65}$$

where $S_n(1)$ is the surface area of a sphere of unit radius. For small r, we have the asymptotic expansion

$$H_n^{(1)}(r) \sim -\frac{i2^n(n-1)!}{\pi} r^{-n}.$$

Thus, substituting (7.64) into (7.65) and using (7.65) we find that

$$C_1 = \frac{i\pi 2^{-n/2}(\sqrt{\mu})^{(n/2)-1}}{[(n/2)-1]!S_n(1)} = \frac{i}{4}\left(\frac{\sqrt{\mu}}{2\pi}\right)^{(n/2)-1}.$$

Hence, for $n \ge 2$ and $\mu \notin [0,\infty)$, the required Green's function is given by

$$G(r,\mu) = \frac{i}{4}\left(\frac{\sqrt{\mu}}{2\pi}\right)^{(n/2)-1} H_{(n/2)-1}^{(1)}(\sqrt{\mu}\, r), \quad n \ge 2, \tag{7.66}$$

or, writing $h^2 = -\mu$, i.e., $h = -i\sqrt{\mu}$, or $\sqrt{\mu} = ih$, we have

$$H_{(n/2)-1}^{(1)}(\sqrt{\mu}\, r) = H_{(n/2)-1}^{(1)}(ihr) = \frac{2}{i\pi} K_{(n/2)-1}(hr),$$

where $K_{(n/2)-1}$ are the modified Bessel functions, thereby yielding

$$G(r,-h^2) = \frac{1}{2\pi}\left(\frac{h}{2\pi r}\right)^{(n/2)-1} K_{(n/2)-1}(hr), \quad n \ge 2, \tag{7.67}$$

which holds whenever $-h^2 \notin [0,\infty)$, i.e., for all h with $\Re h > 0$. For $n = 2$, Green's functions (7.66) and (7.67) become

$$G(r,\mu) = \frac{i}{4} H_0^{(1)}(\sqrt{\mu}\, r) = \frac{1}{2\pi} K_0(hr). \tag{7.68}$$

For $n = 3$, by using $H_{1/2}^{(1)}(z) = \frac{1}{i}\left(\frac{2}{\pi}\right)^{1/2} \frac{e^{iz}}{z^{1/2}}$, we get

$$G(r,\mu) = \frac{e^{i\sqrt{\mu}\, r}}{4\pi r} = \frac{e^{-hr}}{4\pi r}. \tag{7.69}$$

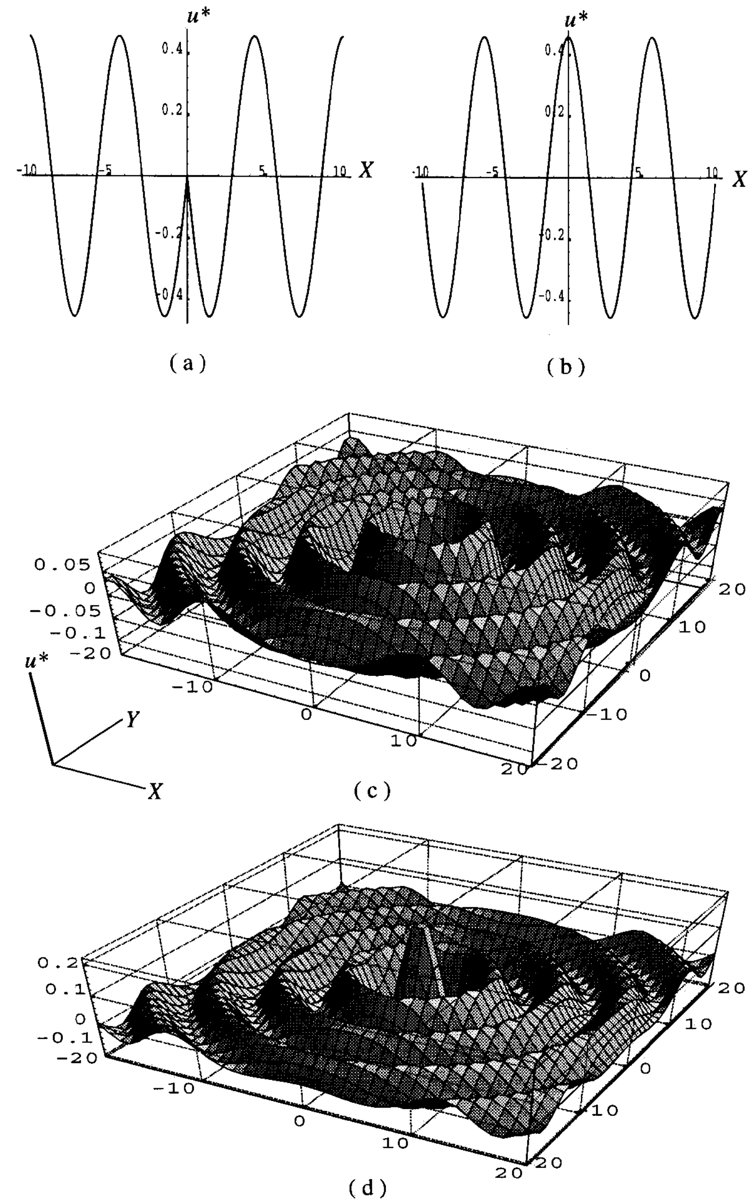


Fig. 7.5. Green's functions of the Helmholtz operator, with $\mu = 1.2$: (a), (b): Real and imaginary parts of one–dimensional solution; (c), (d): Real and imaginary parts of two–dimensional solution.

The one–dimensional Green's function is found directly, and it is

$$G(x,\mu) = \frac{i e^{i\sqrt{\mu}|x|}}{2\sqrt{\mu}} = \frac{e^{-h|x|}}{2h}. \tag{7.70}$$

Note that if the Helmholtz equation is taken as $(\nabla^2 + k^2)u = 0$, where the wave number $k > 0$ is real, then Green's function in R^2 is given by

$$G(r,k) = \begin{cases} -\frac{i}{4} H_0^{(2)}(\sqrt{\mu}\, r), & \text{in } R^2, \\ \frac{e^{-ikr}}{4\pi r} & \text{in } R^3. \end{cases} \tag{7.71}$$

It is obvious from Fig. 7.5 that the real and the imaginary parts of Green's functions of the Helmholtz equation in one– and two–dimensional cases exhibit wave structure. ∎

7.4. Hyperbolic Equations

We shall denote Green's function for the wave operator $\Box_c = \frac{\partial^2}{\partial t^2} - c^2\nabla^2$ in R^n by $G_n(\mathbf{x}, t)$. It satisfies the equation

$$\Box_c\, G_n(\mathbf{x}, t) = \delta(\mathbf{x}, t), \tag{7.72}$$

where $\mathbf{x} \in R^n$ and $t \in (0,t) \subset R^+$. We will use the Laplace transform method and derive Green's functions for $n = 1, 2$ and 3.

EXAMPLE 7.11. In R^1, Green's function $G_1(x, x'; t, t')$ satisfies the equation

$$\frac{\partial^2 G_1}{\partial t^2} - c^2 \frac{\partial^2 G_1}{\partial x^2} = \delta(x - x')\delta(t - t').$$

Taking the Laplace transform with zero initial data, we get

$$s^2 \overline{G}_1 - c^2 \frac{d^2 \overline{G}_1}{dx^2} = e^{-st'}\, \delta(x - x').$$

Assuming that $\overline{G}_1$ is finite at $x = \pm\infty$, the solution of the above equation is

$$\overline{G}_1(x, s; x', t') = \frac{e^{-st'}}{2cs}\, e^{-s|x-x'|/c},$$

which on inversion gives

$$\overline{G}_1(x,t;x',t') = \begin{cases} \dfrac{1}{2c} & \text{for } c(t-t') > |x - x'| \\ 0 & \text{for } c(t-t') < |x - x'| \end{cases} \tag{7.73}$$
$$= \frac{1}{2c} H(c(t-t') - |x - x'|). \;\blacksquare$$

EXAMPLE 7.12. In R^2, Green's function G_2 satisfies the equation

$$\frac{\partial^2 G_2}{\partial t^2} - c^2 \left(\frac{\partial^2 G_2}{\partial x^2} - \frac{\partial^2 G_2}{\partial y^2}\right) = \delta(x - x')\delta(y - y')\delta(t - t').$$

Applying the Laplace transform we get

$$s^2 \overline{G}_2 - c^2 \left(\frac{\partial^2 \overline{G}_2}{\partial x^2} + \frac{\partial^2 \overline{G}_2}{\partial y^2}\right) = e^{-st'}\delta(x - x')\delta(y - y'),$$

or using the axial symmetry with $r^2 = (x - x')^2 + (y - y')^2$,

$$s^2 \overline{G}_2 - c^2 \left(\frac{d^2 \overline{G}_2}{dr^2} + \frac{1}{r}\frac{\partial \overline{G}_2}{\partial r}\right) = e^{-st'}\delta(x - x')\delta(y - y').$$

The solution of this equation is

$$\overline{G}_2(r, s; t') = -\frac{e^{-st'}}{2\pi} K_0\left(\frac{c}{s} r\right),$$

where K_0 is the modified Bessel function of the third kind and zero order. Since

$$\mathcal{L}^{-1}\{K_0(\alpha s)\} = \frac{H(t-\alpha)}{\sqrt{t^2 - \alpha^2}}$$

(see Erdelyi et al., 1954, or Abramowitz and Stegun, 1965), on inversion we get

$$G(r,t;t') = \begin{cases} -\dfrac{1}{2\pi c\sqrt{c^2(t-t')^2 - r^2}} & \text{for } r < c(t-t') \\ 0 & \text{for } r > c(t-t') \end{cases} \tag{7.74}$$
$$= -\frac{H(c(t-t') - r)}{2\pi c\sqrt{c^2(t-t')^2 - r^2}}. \;\blacksquare$$

EXAMPLE 7.13. In R^3, Green's function G_3 satisfies the equation

$$\frac{\partial^2 G_3}{\partial t^2} - c^2\left(\frac{\partial^2 G_3}{\partial x^2} - \frac{\partial^2 G_3}{\partial y^2} + \frac{\partial^2 G_3}{\partial z^2}\right) = \delta(x-x')\delta(y-y')\delta(z-z')\delta(t-t').$$

Applying the Laplace transform we get

$$s^2\overline{G}_3 - c^2\left(\frac{\partial^2 \overline{G}_3}{\partial x^2} + \frac{\partial^2 \overline{G}_3}{\partial y^2} + \frac{\partial^2 \overline{G}_3}{\partial y^2}\right) = e^{-st'}\delta(x-x')\delta(z-z')\delta(y-y'),$$

or using the axial symmetry with $r^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$,

$$s^2\overline{G}_3 - c^2\left(\frac{d^2\overline{G}_3}{dr^2} + \frac{1}{r}\frac{\partial \overline{G}_3}{\partial r}\right) = e^{-st'}\delta(x-x')\delta(y-y')\delta(z-z'),$$

which has the solution (neglecting the other solution $e^{sr/c}$ since $G_3 \to 0$ as $r \to \infty$)

$$\overline{G}_3(r,s;t') = -\frac{e^{-st'}\,e^{-sr/c}}{4\pi r}, \quad r = |\mathbf{x} - \mathbf{x}'|.$$

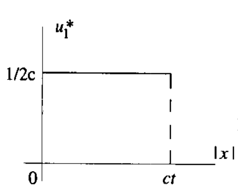
On inversion this gives

$$G_3(r,t;t') = \frac{1}{4\pi r}\,\delta\left(t - t' - \frac{r}{c}\right),$$

or

$$G_3(x,y,z,t;x',y',z',t') = \frac{\delta\left(t - t' - \frac{1}{c}\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}\right)}{4\pi r}. \blacksquare \quad (7.75)$$

The graphs of Green's functions G_1, G_2, and G_3 are presented in Figs. 7.6, 7.7, and 7.8.



Figs. 7.6.

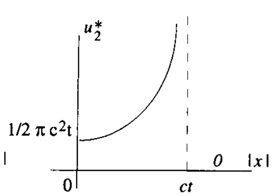


Fig. 7.7.

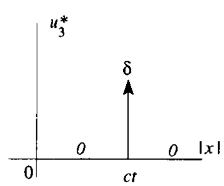
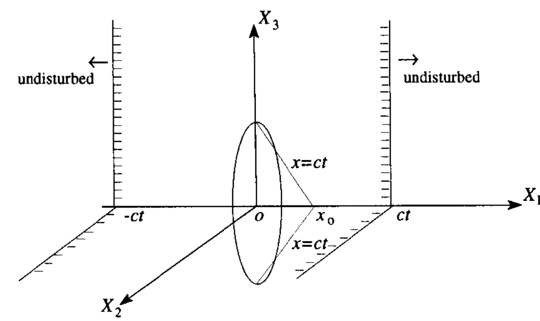


Fig. 7.8.

In the above solutions (7.73)–(7.75), we find that the impulsive point source at $\mathbf{x}'$ at time t' affects the position of the field point $\mathbf{x}$ at time t if $|\mathbf{x} - \mathbf{x}'| = c|t - t'|$, i.e., when the distance from the source to the field (or observation) point is c times the time. The source generates a wave propagating in all directions with velocity c. After a duration of time $|t - t'|$ the effect of the source is located at a distance $c|t - t'|$ away. The wave structure, and the non-occurrence of diffusion in wave propagation, which is known as *Huygens' principle*, in the three cases, represented by Green's functions (7.73)–(7.75) is as follows:



Figs. 7.9. Wave propagation in R^1.

In R^1, the solution (7.73) shows that the wave that originates instantaneously at a point source $\delta(x,t)$ at time $t > 0$ will cover the interval $-ct \le x \le ct$, where there exist two edges defined by $x = \pm ct$ that move forward with velocity c. This wave is observed behind the front edge and has amplitude $1/2c$. Hence, wave diffusion occurs in this case. A three–dimensional representation of Green's function in R^1 is shown in Fig. 7.9. It can be viewed as that of a wave starting at the point source and propagating as a plane wave $|x| \le ct$ whose front edge$|x| = ct$ moves with the velocity c perpendicular to the plane $x = 0$. There does not exist a rear edge of the wave in this case.

In R^2, Green's function defined by (7.74) shows that the disturbance originates instantaneously at the point source $\delta(x,t)$, and at time $t > 0$ it occupies the entire circle $|x| \le ct$ (see Fig. 7.10). The wavefront at $|x| = ct$ propagates throughout the plane with velocity c. But wave propagation exists behind the front edge at all subsequent times, and the wave has no rear edge. The wave diffusion occurs in this case, and Huygens' principle does not apply.

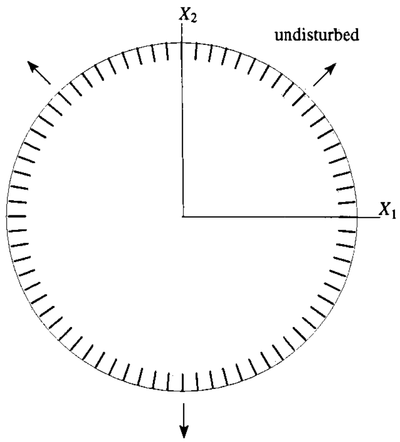
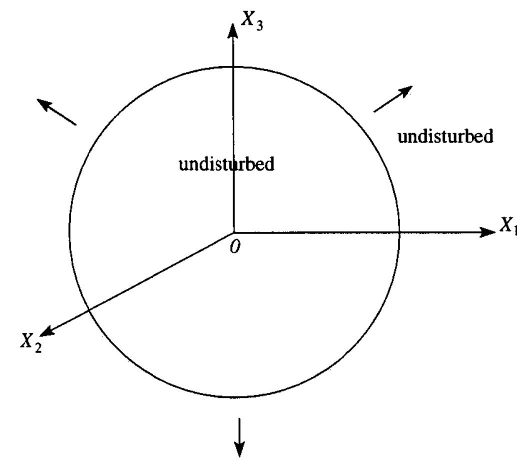


Fig. 7.10. Wave propagation in R^2.

In R^3, Green's function (7.75) implies that the disturbance that originates at a point source $\delta(x,t)$ at time $t > 0$ occupies a spherical surface of radius ct and center at the origin. The wave propagates as a spherical wave with wavefront at $|x| = ct$ and velocity c, and after the wave has passed there will be no disturbance (see Fig. 7.11). Huygens' principle is applied in this case. The amplitude of the wave decays like r^{-1} as the radius increases.

There is a significant difference between the two– and three–dimensional cases. If a stone is dropped in a calm shallow pond, the leading water wave spreads out in a circular form with its radius increasing uniformly with time, but the water contained by this wave continues to move after its passage. This is because of the Heaviside function in the solution (7.74) which leaves a wake behind it. On the other hand, in the three–dimensional case if a shot fired suddenly at time $t = t'$ in still air is heard only on expanding spherical surfaces with center at the firing gun and radius $c|t - t'|$, where c is the sound velocity. But the air does not continue to reverberate after the passage of this wave. This is because of the presence of the Dirac delta function in the solution (7.75), which represents a sharp bang and no tail effect.

Huygens' principle accounts for the simplicity of communications in our three–dimensional world. If it were two dimensional, the communications would have been impossible since utterances could be hardly distinguished from one another.



Figs. 7.11. Wave propagation in R^3.

7.5. Applications

An important application of Green's functions for a differential operator L with homogeneous boundary conditions is in finding the solution of nonhomogeneous boundary value problems. Since a Green's function $G(\mathbf{x}, \mathbf{x}'; t, t')$, or $G(\mathbf{x}, \mathbf{x}')$ in steady state problems, is required to satisfy the equation

$$L[G] = \delta(\mathbf{x} - \mathbf{x}')\,\delta(t - t'), \tag{7.76}$$

together with certain prescribed homogeneous boundary conditions, we can use the linear superposition principle to determine the solution of the nonhomogeneous equation

$$L[u(\mathbf{x}, t)] = F(\mathbf{x}, t), \tag{7.77}$$

which satisfies the same boundary conditions. The solution in a region $\Omega \subset R^n$ is given by

$$u(\mathbf{x}, t) = \int_\Omega G(\mathbf{x}, \mathbf{x}'; t, t')\, F(\mathbf{x}, t)\, d\mathbf{x}'. \tag{7.78}$$

We will consider only two–dimensional boundary value problems.

Consider the nonhomogeneous equation

$$\nabla^2 u(x, y) = F(x, y) \quad \text{in } \Omega \subset R^2, \tag{7.79}$$

which is subject to certain nonhomogeneous boundary conditions on the boundary Γ of the region Ω. Then, in view of (7.19) where we take $v = G$, we have

$$u(x',y') = \iint_\Omega G(x,y;x',y')\,F(x,y)\,dS + \int_\Gamma \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right)\,ds. \tag{7.80}$$

The boundary conditions on both u and $\dfrac{\partial u}{\partial n}$ are, in general, not prescribed at all points of Γ, but those on G are prescribed precisely, i.e., the boundary conditions on G are such that they annul whichever values of u or $\dfrac{\partial u}{\partial n}$ are unknown. For example, let u be known (but not $\dfrac{\partial u}{\partial n}$) on a portion Γ_1 of Γ; then, if $u = 0$ is prescribed on Γ_1, we must have $G = 0$ on Γ_1, because only then will the integrals in (7.80) vanish. We will consider the following three types of boundary conditions:
(a) (Dirichlet) u is prescribed on Γ;
(b) (Neumann) $\dfrac{\partial u}{\partial n}$ is prescribed on Γ;
(c) (Mixed) u is prescribed on Γ_1 and $\dfrac{\partial u}{\partial n}$ is prescribed on Γ_2, where $\Gamma_1 \cup \Gamma_2 = \Gamma$.
In each case Green's function G must satisfy the same homogeneous boundary condition as that satisfied by u.

In the case when $F(x,y) = 0$, Eq (7.80) with the Dirichlet boundary condition (a) becomes

$$u(x',y') = \int_\Gamma u\frac{\partial G}{\partial n}\,ds, \tag{7.81}$$

where $G = 0$ on Γ. But if $\dfrac{\partial u}{\partial n}$ is prescribed on Γ (Neumann condition (b)), Eq (7.80) becomes

$$u(x',y') = -\int_\Gamma G\frac{\partial u}{\partial n}\,ds, \tag{7.82}$$

where $\dfrac{\partial u}{\partial n} = 0$ on Γ.

EXAMPLE 7.14. Let Ω be the half–plane $y > 0$. Then Green's function associated with the Dirichlet boundary condition $u = 0$ on the

boundary $y = 0$ is given by (7.59). Hence, for $u(x,0) = f(x)$ we find from (7.81) that

$$\begin{aligned} u(x',y') &= -\frac{1}{4\pi}\int_{-\infty}^{\infty} f(x)\,\frac{\partial}{\partial y}\Big[\ln\frac{(x-x')^2+(y-y')^2}{(x-x')^2+(y+y')^2}\Big]_{y=0}\,dx \\ &= -\frac{1}{4\pi}\int_{-\infty}^{\infty} f(x)\,\frac{-4y'}{(x-x')^2+y'^2}\,dx \\ &= \frac{y'}{\pi}\int_{-\infty}^{\infty}\frac{f(x)}{(x-x')^2+y'^2}\,dx. \blacksquare \end{aligned}$$

EXAMPLE 7.15. Let Γ be the circle $r = a$, and let $u(a,\theta) = f(\theta)$. In this case Green's function is given in Exercise 7.15. Since $\dfrac{\partial}{\partial n} = \dfrac{\partial}{\partial r}$ on the circle Γ, we find from (7.81) that

$$\begin{aligned} u(r',\theta') &= \frac{1}{4\pi}\int_0^{2\pi} f(\theta)\,\frac{\partial}{\partial r}\Big[\ln\frac{a^2[r^2-2rr'\cos(\theta-\theta')+r'^2]}{r^2r'^2-2rr'a^2\cos(\theta-\theta')+a^4}\Big]_{r=a}\,a\,d\theta \\ &= \frac{1}{4\pi}\int_0^{2\pi} f(\theta)\,\Big[\frac{2r-2r'\cos(\theta-\theta')}{r^2-2rr'\cos(\theta-\theta')+r'^2} \\ &\qquad - \frac{2rr'^2-2a^2r'\cos(\theta-\theta')}{r^2r'^2-2rr'a^2\cos(\theta-\theta')+a^4}\Big]_{r=a}\,a\,d\theta \\ &= \frac{a^2-r'^2}{2\pi}\int_0^{2\pi}\frac{f(\theta)}{a^2[r^2-2rr'\cos(\theta-\theta')+r'^2}\,d\theta. \blacksquare \end{aligned}$$

EXAMPLE 7.16. To find the harmonic function ϕ in the quarter–plane $x > 0$, $y > 0$, subject to the boundary conditions

$$\phi(x,0) = f(x), \quad 0 < x < \infty, \quad \text{and} \quad \frac{\partial\phi}{\partial x}(0,y) = g(y), \quad 0 < y < \infty,$$

note that in view of (7.80), the solution is given by

$$\phi(x',y') = \int_\Gamma\left(\phi\frac{\partial G}{\partial n} - g\frac{\partial\phi}{\partial n}\right)\,ds,$$

where G is defined by Eq (7.60) (Example 7.8), and Γ is boundary of the quarter–plane $\{x > 0, y > 0\}$. Then

$$\begin{aligned} \phi(x',y') &= -\int_0^\infty f(x)\,\Big[\frac{\partial G}{\partial y}\Big]_{y=0}\,dx + \int_0^\infty [G]_{x=0}\,g(y)\,dy \\ &= \frac{y'}{\pi}\int_0^\infty f(x)\,\Big[\frac{1}{(x-x')^2+y'^2} + \frac{1}{(x+x')^2+y'^2}\Big]\,dx \\ &\quad + \frac{1}{2\pi}\int_0^\infty \ln\frac{x'^2+(y-y')^2}{x'^2+(y+y')^2}\,g(y)\,dy. \end{aligned}$$

Note that the signs in the two integrals above result from the fact that $\dfrac{\partial}{\partial n} = -\dfrac{\partial}{\partial y}$ and $\dfrac{\partial}{\partial n} = -\dfrac{\partial}{\partial x}$ for the quarter–region. ∎

7.6. Exercises

7.1. Find a harmonic function ϕ in the semicircular domain $r < a$, $0 < \theta < \pi$, such that $\phi(r,0) = 0$, $\phi(r,\pi) = k = \phi(a,\theta)$, where k is a constant.

ANS. $\phi(r,\theta) = A\,\theta + B + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n [c_n \sin n\theta + d_n \cos n\theta]$ is the general solution of the Laplace equation in polar cylindrical coordinates $\dfrac{\partial^2\phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}$. The boundary conditions on $\theta = 0$, π are satisfied if we take

$$\phi(r,\theta) = \frac{k\theta}{\pi} + \sum_{n=1}^{\infty} c_n \left(\frac{r}{a}\right)^n \sin n\theta.$$

From the last boundary condition on $r = a$ we get

$$k = \frac{k\theta}{\pi} + \sum_{n=1}^{\infty} c_n \sin n\theta,$$

which gives

$$c_n = \frac{2k}{\pi}\int_0^{\pi}\left(1 - \frac{\theta}{\pi}\right)\sin n\theta\, d\theta = \frac{2k}{n\pi}.$$

Thus, the required harmonic function is

$$\phi(r,\theta) = \frac{k\theta}{\pi} + \frac{2k}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \frac{\sin n\theta}{n}.$$

Note that this solution represents the stream function for the two–dimensional flow of a jet liquid from a Borda mouthpiece (Mackie, 1989).

7.2. Show that Green's function (7.42) satisfies the condition (7.41).

ANS. The function $G(x,t)$ is locally integrable in R^{n+1} and

$$\begin{aligned}\int_{R^n} G(\mathbf{x},t)\,d\mathbf{x} &= \frac{1}{(4\pi at)^{n/2}}\int_{R^n} e^{-|x|^2/4at}\,dx \\ &= \prod_{i=1}^{n}\frac{1}{\sqrt{\pi}}\int_{\infty}^{\infty} e^{-\alpha_i^2}\,d\alpha_i = 1.\end{aligned} \tag{7.83}$$

If $t > 0$, then $G \in C^\infty$. Thus,

$$\begin{aligned}\frac{\partial G}{\partial t} &= \left(\frac{|x|^2}{4at^2} - \frac{n}{2t}\right)G, \\ \frac{\partial G}{\partial x_i} &= -\frac{x_i}{2at}G, \\ \frac{\partial^2 G}{\partial {x_i}^2} &= \left(\frac{{x_i}^2}{4a^2t^2} - \frac{1}{2at}\right)G.\end{aligned} \tag{7.84}$$

Hence

$$\frac{\partial G}{\partial t} - a\nabla^2 G = \left(\frac{|x|^2}{4at^2} - \frac{n}{2t}\right)G - \left(\frac{|x|^2}{4at^2} - \frac{n}{2t}\right)G = 0.$$

Let $\phi \in \mathcal{D}(R^{n+1})$. Then, using (7.84), we have

$$\begin{aligned}\langle\frac{\partial G}{\partial t} - a\nabla^2 G, \phi\rangle &= -\langle G, \frac{\partial\phi}{\partial t} + a\nabla^2\phi\rangle \\ &= -\int_0^{\infty}\int_{R^n} G(\mathbf{x},t)\left(\frac{\partial\phi}{\partial t} + a\nabla^2\phi\right)dx\,dt \\ &= -\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty}\int_{R^n} G(\mathbf{x},t)\left(\frac{\partial\phi}{\partial t} + a\nabla^2\phi\right)dx\,dt \\ &= \lim_{\varepsilon\to 0}\left[\int_{R^n} G(\mathbf{x},\varepsilon)\phi(x,\varepsilon)\,dx + \right. \\ &\qquad \left. + \int_{\varepsilon}^{\infty}\int_{R^n}\left(\frac{\partial G}{\partial t} - a\nabla^2 G\right)\phi\,dx\,dt\right] \\ &= \lim_{\varepsilon\to 0}\int_{R^n} G(\mathbf{x},\varepsilon)\phi(x,\varepsilon)\,dx + \\ &\qquad + \lim_{\varepsilon\to 0}\int_{R^n} G(\mathbf{x},\varepsilon)[\phi(x,\varepsilon) - \phi(x,0)]\,dx \\ &= \lim_{\varepsilon\to 0}\int_{R^n} G(\mathbf{x},\varepsilon)\phi(x,0)\,dx,\end{aligned}$$

which, in view of (7.83), gives

$$\left| \int_{R^n} G(\mathbf{x},t)[\phi(x,\varepsilon) - \phi(x,0)]\, dx \right| \le K\varepsilon \int_{R^n} G(x,\varepsilon) dx = K\varepsilon.$$

We will now show that $G(\mathbf{x},t)$, defined by (7.42), converges to $\delta(x)$ as $t \to 0+$ in $\mathcal{D}'(R^n)$. Let $\phi(x) \in \mathcal{D}$. Since

$$\begin{aligned}\left| \int_{R^n} G(\mathbf{x},t)[\phi(x) - \phi(0)]\, dx \right| &\le \frac{K}{(4\pi at)^{n/2}} \int_{R^n} e^{-|x|^2/4at} |x|\, dx \\ &= \frac{K S_n(1)}{(4\pi at)^{n/2}} \int_0^\infty r^n e^{-r^2/4at}\, dr \\ &= \frac{2K S_n(1)\sqrt{at}}{\pi^{n/2}} \int_0^\infty u^n e^{-u^2}\, du = C\sqrt{t},\end{aligned}$$

then, as $t \to 0+$, we have

$$\begin{aligned}\langle G(\mathbf{x},t), \phi\rangle &= \int_{R^n} G(\mathbf{x},t)\phi(x) dx = \phi(0) \int_{R^n} G(\mathbf{x},t)\, dx + \\ &\quad + \int_{R^n} G(\mathbf{x},t)[\phi(x) - \phi(0)]\, dx \to \phi(0) = \langle \delta, \phi\rangle,\end{aligned}$$

i.e., $G(\mathbf{x},t) \to \delta(x)$ as $t \to 0+$ in $\mathcal{D}'(R^n)$.

7.3. Set $at = \varepsilon > 0$ in (7.42) for $n = 1$, and consider the sequence of functions $f_\varepsilon(x) = \dfrac{1}{2\sqrt{\pi\varepsilon}}\, e^{-x^2/4\varepsilon}$. Show that the sequence $\{f_\varepsilon(x)\}$ converges to $\delta(x)$ in R^1 as $\varepsilon \to 0$.
ANS. Notice from Fig. 7.2 that the peak profile for Green's function (7.42) smoothes out gradually as t increases. Hence, the functions $f_\varepsilon(x)$ vary appreciably over successively smaller intervals about the origin as $\varepsilon \to 0$. Thus, for any $\phi \in C^\infty(R^1)$,

$$\begin{aligned}\langle f_\varepsilon(x), \phi(x)\rangle &= \int_{-\infty}^\infty \phi(x) f_\varepsilon(x)\, dx = \frac{1}{2\sqrt{\pi\varepsilon}} \int_{-\infty}^\infty \phi(x) e^{-x^2/4\varepsilon}\, dx \\ &= \frac{1}{\sqrt{\pi}} \sum_{n=0}^\infty (4\varepsilon)^{n/2} \frac{\phi^{(n)}(0)}{n!} \int_{-\infty}^\infty z^n e^{-z^2}\, dz,\end{aligned} \tag{7.85}$$

where we have used the Maclaurin series expansion for $\phi(x)$ and then the substitution $x = 2\sqrt{\varepsilon}$. An evaluation of the integral on the right side of (7.85) shows that

$$\lim_{\varepsilon\to 0} \langle f_\varepsilon(x), \phi(x)\rangle = \phi(0).$$

Thus, the functions $f_\varepsilon(x)$ approach $\delta(x)$ as $\varepsilon \to 0$.

7.4. Show that $\displaystyle\int_{-\infty}^\infty G(\mathbf{x},t)\, dt = \frac{1}{(n-2)S_n(1)} |\mathbf{x}|^{2-n}$, $n \ge 3$, with $a = 1$.
ANS.

$$\begin{aligned}\int_{-\infty}^\infty G(\mathbf{x},t) dt &= \int_0^\infty \frac{1}{(2\sqrt{\pi t})^n} e^{-|x|^2/4t}\, dt \\ &= \frac{|x|^{2-n}}{4\pi^{n/2}} \int_0^\infty u^{n/2-2} e^{-u}\, du = \Gamma\left(\frac{n}{2} - 1\right) \frac{|x|^{2-n}}{4\pi^{n/2}} \\ &= \frac{1}{(n-2)S_n(1)} |x|^{2-n}, \quad n \ge 3.\end{aligned}$$

7.5. The complex conjugate of the Schrödinger equation in $R^1 \times R^+$ is

$$-i\hbar \frac{\partial u}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial u^2}{\partial x^2}.$$

Find its Green's function.
ANS. Set $a = \dfrac{i\hbar}{2m}$ in (7.42).

7.6. Find Green's function for the Fokker–Planck equation in R^1

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + x\right) u.$$

ANS. Green's function satisfies

$$\left[\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} + x\right)\right] G(x,t;x',t') = \delta(x,x')\delta(t,t').$$

Let $x = X\, e^{-t}$, $u = v\, e^t$. Then the Fokker–Plank equation becomes

$$\frac{\partial v}{\partial t} = e^{2t} \frac{\partial^2 v}{\partial X^2}.$$

If we set $2T = e^{2t}$, then the above equation reduces to

$$\frac{\partial v}{\partial T} = \frac{\partial^2 v}{\partial X^2},$$

which, in view of (7.45), has Green's function

$$G(x,t;x',t') = \frac{H(t-t')}{\sqrt{2\pi(1-e^{-2(t-t')})}} e^{-(x-x'e^{-(t-t')})^2/2(1-e^{-2(t-t')})}.$$

7.7. Find the axisymmetric solution of the equation

$$\nabla^2 G_\varepsilon(r) \equiv \frac{1}{r}\frac{d}{dr}\left(r\,\frac{dG_\varepsilon}{dr}\right) = \Delta_\varepsilon(r) \quad \text{in } R^2,$$

such that $G_\varepsilon(a) = 0$ and $G'_\varepsilon(r)$ is finite at $r = 0$, where $r > a > 0$, for the (a) $\Delta_\varepsilon(r) = \frac{1}{4\pi\varepsilon} e^{-r^2/4\varepsilon}$, and (b) $\Delta_\varepsilon(r) = \frac{1}{\pi\varepsilon^2} e^{-r^2/\varepsilon^2}$. Show that $G(r) = \lim_{\varepsilon\to 0} G_\varepsilon(r) = \frac{1}{2\pi}\ln r$ for any fixed $r = |\mathbf{x}| > 0$.
ANS. (a) Integrating the given equation with respect to r, and taking the constant of integration zero since $G'_\varepsilon(r)$ is finite at $r = 0$, we get

$$r\,\frac{dG_\varepsilon}{dr} = \frac{1}{2\pi}\left[1 - e^{-r^2/4\varepsilon}\right].$$

Hence

$$G_\varepsilon(r) = \frac{1}{2\pi}\ln\frac{r}{a} - \frac{1}{2\pi}\int_a^r \frac{e^{-z^2/4\varepsilon}}{z}\,dz, \quad r > a > 0,$$

where the integral on the right side tends to zero as $\varepsilon \to 0$ for a fixed $r > 0$. Hence

$$G(r) = \lim_{\varepsilon\to 0} G_\varepsilon(r) = \frac{1}{2\pi}\ln\frac{r}{a}.$$

Without loss of generality, we can take $a = 1$ since r extends to the entire R^2 plane, and the result follows. Part (b) can be solved similarly. Note that the function $\Delta_\varepsilon(r)$ reduces to the delta function as $\varepsilon \to 0$. Thus what we have shown is that $\nabla^2 G(r) = \delta(r)$, which is true in view of the property (7.8).

7.8. Find Green's function for the Laplacian in R^1.
ANS. For Green's function of the one-dimensional Laplace equation, we note that (7.53) becomes $-\frac{d^2G}{dx^2} = \delta(x,x')$, whose general solution for a fixed x' is

$$G(x) = -\frac{1}{2}|x - x'| + Ax + B.$$

If we require spherical symmetry, i.e., $G = G(|x - x'|)$, then $A = 0$, and we set $B = 0$. Then Green's function is given by $G(x) = -\frac{1}{2}|x - x'|$.

7.9. Prove that Green's function $G(\mathbf{x},\mathbf{x}')$ for the Laplacian, defined by (7.56), satisfies the following properties: (a) $G(\mathbf{x},\mathbf{x}') \ge 0$ throughout the domain $\Omega \subset R^3$, and (b) Green's function is symmetric, i.e., $G(\mathbf{x},\mathbf{x}') = G(\mathbf{x}',\mathbf{x})$.
ANS. (a) $G(\mathbf{x},\mathbf{x}') = 0$ on S and $G(\mathbf{x},\mathbf{x}') > 0$ on the surface of a small enough sphere centered at $\mathbf{x}'$ since $G(\mathbf{x},\mathbf{x}') \to +\infty$ as $\mathbf{x} \to \mathbf{x}'$. In view of Theorem 7.6 (maximum principle), it follows that $G(\mathbf{x},\mathbf{x}') > 0$ throughout Ω. Since $G(\mathbf{x},\mathbf{x}_0)\big|_S = -1/4\pi r$, we find that $G(\mathbf{x},\mathbf{x}') < 0$ in the closed domain $\bar{\Omega}$, which implies that

$$0 < G(\mathbf{x},\mathbf{x}') < \frac{1}{4\pi r}$$

inside Ω. (b) Apply Green's second identity (A.7) to $u = G(\mathbf{x},\mathbf{x}_1)$, $v = G(\mathbf{x},\mathbf{x}_2)$, where the integration domain Ω_2 is chosen as the domain Ω minus the two small spheres S_1 and S_2, centered, respectively, at $\mathbf{x}_1$ and $\mathbf{x}_2$, each with radius ε. Then

$$\iiint_{\Omega_2} \left[G(\mathbf{x},\mathbf{x}_1)\nabla^2 G(\mathbf{x},\mathbf{x}_2) - G(\mathbf{x},\mathbf{x}_2)\nabla^2 G(\mathbf{x},\mathbf{x}_1)\right] dV$$
$$= \iint_{S-S_1-S_2} \left(G(\mathbf{x},\mathbf{x}_1)\frac{\partial G(\mathbf{x},\mathbf{x}_2)}{\partial n} - G(\mathbf{x},\mathbf{x}_2)\frac{\partial G(\mathbf{x},\mathbf{x}_1)}{\partial n}\right) dS,$$

or

$$0 = \iint_{S_1} \left(G(\mathbf{x},\mathbf{x}_1)\frac{\partial G(\mathbf{x},\mathbf{x}_2)}{\partial n} - G(\mathbf{x},\mathbf{x}_2)\frac{\partial G(\mathbf{x},\mathbf{x}_1)}{\partial n}\right) dS +$$
$$+ \iint_{S_2} \left(G(\mathbf{x},\mathbf{x}_1)\frac{\partial G(\mathbf{x},\mathbf{x}_2)}{\partial n} - G(\mathbf{x},\mathbf{x}_2)\frac{\partial G(\mathbf{x},\mathbf{x}_1)}{\partial n}\right) dS.$$

As $\varepsilon \to 0$, the integral over S_1 (centered at $\mathbf{x}_1$) will tend to $G(\mathbf{x}_1,\mathbf{x}_2)$), and similarly the integral over S_2 (centered at $\mathbf{x}_2$) will tend to $G(\mathbf{x}_2,\mathbf{x}_1)$, which proves the symmetry of Green's function.

7.10. Find Green's function for the Laplace operator in a sphere of radius R.
ANS. In Fig. 7.12, we note that the point M with coordinates $\mathbf{x} = (x,y,z)$ is inside Ω, $\rho = |OM|$, $\rho_1 = |OM_1|$, where the point $M_1 = \mathbf{x}_1 = (x_1,y_1,z_1)$ is the inverse point to M such that

$$\rho\rho_1 = R^2.$$

Let $P = \boldsymbol{\xi} = (\xi, \eta, \zeta)$ be an arbitrary point on the surface S of the sphere. Then, since the triangles OMP and OM_1P are similar (having a common angle α at the origin), we find that $r/r_1 = \rho/R$, i.e.,

$$\frac{1}{r} = \frac{R}{\rho}\frac{1}{r_1}.$$

Hence, Green's function for the sphere is

$$G(\boldsymbol{\xi}, \mathbf{x}') = \frac{1}{4\pi r} = \frac{1}{4\pi}\frac{R}{\rho}\frac{1}{r_1}.$$

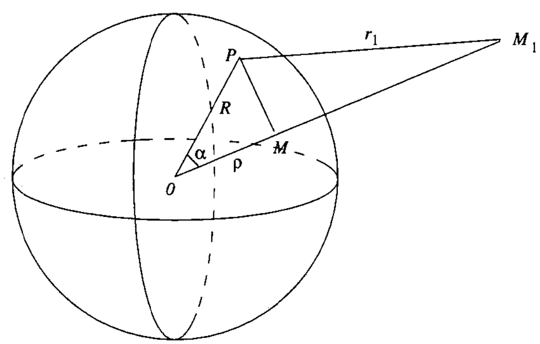


Fig. 7.12.

7.11. Find Green's function for the operator $\nabla^2 - k^2$, where $k > 0$ is real.
ANS. Follow the method of Example 7.10 and deduce that Green's function is

$$G(r) = \begin{cases} \dfrac{1}{2k}\sinh(kr) & \text{in } R^1, \\ \dfrac{1}{2\pi}K_0(kr) & \text{in } R^2, \\ \dfrac{1}{4\pi r}e^{ikr} & \text{in } R^3, \end{cases}$$

where $r = |\mathbf{x} - \mathbf{x}'|$. Green's functions are used in the problems of neutron diffusion.

7.12. Show that Green's function (7.71) satisfies Eq (7.61).
ANS. Since the functions $\cos k|x|$ and $\dfrac{\sin kx}{|k|}$ belong to the class

$C^\infty(R^n)$, and since, in view of Exercise 7.4,

$$\frac{\partial}{\partial x_i}\frac{1}{|x|} = -\frac{x_i}{|x|^3}, \quad \frac{\partial}{\partial x_i}e^{ik|x|} = \frac{ikx_i}{|x|}e^{ik|x|},$$
$$\nabla\frac{1}{|x|}\cdot\nabla e^{ik|x|} = \frac{-ik}{|x|^2}e^{ik|x|},$$
$$\nabla^2\frac{1}{|x|} = -4\pi\delta(x) \quad (\text{by } (2.13)),$$
$$\nabla^2 e^{ik|x|} = \left(\frac{2ik}{|x|} - k^2\right)e^{ik|x|},$$

we apply the Leibniz formula $\nabla^2(af) = f\nabla^2 a + 2\nabla a\cdot\nabla f + a\nabla^2 f$, and find that

$$\begin{aligned}(\nabla^2 + k^2)\frac{e^{ik|x|}}{|x|} &= e^{ik|x|}\nabla^2\frac{1}{|x|} + 2\left(\nabla\frac{1}{|x|}\right)\cdot\left(\nabla e^{ik|x|}\right) + \frac{1}{|x|}\nabla^2 e^{ik|x|} \\ &= -4\pi\delta(x) - \frac{2ik}{|x|^2}e^{ik|x|} + \frac{2ik}{|x|^2}e^{ik|x|} - \frac{k^2}{|x|}e^{ik|x|} + \frac{k^2}{|x|}e^{ik|x|} \\ &= -4\pi\delta(x).\end{aligned}$$

7.13. Show that Green's function for the Cauchy–Riemann operator $\dfrac{\partial}{\partial\bar z} = \dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}$, $z = x + iy$ is $G(z) = \dfrac{1}{\pi z}$.
ANS. We will show that $G(z)$ defined above satisfies the equation

$$\frac{\partial}{\partial\bar z}G(z) = \frac{\partial}{\partial\bar z}G(x,y) = \delta(x,y).$$

Let C be the boundary of a region $D \subset R^2$, and let $f \in C^1(\overline{D})$ be such that $f(z) \equiv f(x,y) = 0$ and $z \in R^2\backslash D$. Let us assume that the (closed) contour C is piecewise smooth and is traversed in the positive sense such that the region D remains to the left. Then, since

$$\frac{\partial f}{\partial\bar z} = \left[\frac{\partial f}{\partial\bar z}\right] - \frac{f}{2}\left[\cos nx + i\,\sin nx\right]\delta_c,$$

and

$$\begin{aligned}\left\langle\frac{\partial f}{\partial\bar z},\phi\right\rangle &= \int_D\left(f\frac{\partial\phi}{\partial\bar z} + \frac{\partial f}{\partial\bar z}\phi\right)dx\,dy \\ &= \frac{1}{2}\oint_C f\phi\left[\cos nx + i\,\cos ny\right]ds \\ &= \frac{1}{2}\oint_C f\phi(dy - i\,dx) = -\frac{i}{2}\oint_C f\phi\,dz,\end{aligned}$$

we have

$$\int_D \frac{\partial}{\partial \bar{z}}(f\phi)\,dx\,dy = -\frac{i}{2}\oint_C f\phi\,dz.$$

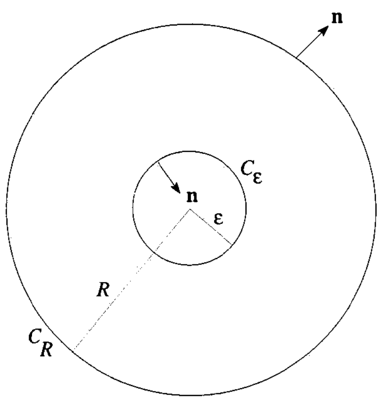


Fig. 7.13.

Since the function $1/z$ is locally integrable in R^2, we take $1/z$ and $D = \{z : \varepsilon < |z| < R\}$ (see Fig. 7.13, where C is the same as C_R). Then for all $\phi \in \mathcal{D}$ such that $\operatorname{supp}\phi \subset U_R$, we have, using the above result,

$$\begin{aligned}
\langle \frac{\partial}{\partial \bar{z}}\frac{1}{z}, \phi\rangle &= -\langle \frac{1}{z}, \frac{\partial}{\partial \bar{z}}\phi\rangle = -\int_{U_R} \frac{1}{z}\frac{\partial \phi}{\partial \bar{z}}\,dx\,dy \\
&= -\lim_{\varepsilon\to 0}\int_{\varepsilon<|z|<R} \frac{1}{z}\frac{\partial \phi}{\partial \bar{z}}\,dx\,dy \\
&= \lim_{\varepsilon\to 0}\Big[\int_{\varepsilon<|z|<R} \phi\frac{\partial}{\partial \bar{z}}\frac{1}{z}\,dx\,dy + \frac{i}{2}\Big(\int_{C_R} - \int_{C_\varepsilon}\Big)\frac{\phi}{z}\,dz\Big] \\
&= -\frac{i}{2}\lim_{\varepsilon\to 0}\int_{|z|=\varepsilon}\frac{\phi(z)}{z}\,dz = -\frac{i}{2}\lim_{\varepsilon\to 0} i\int_0^{2\pi}\phi(\varepsilon e^{i\vartheta})\,d\theta \\
&= \pi\phi(0) = <\pi\delta, \phi> .
\end{aligned}$$

7.14. Use the method of images to determine Green's function for a disk of radius a. HINT: The image of a charge at a point P' inside the disk has an equal but opposite charge at the inverse point Q' (Fig. 7.14). Note that if the coordinates of P' are (r', θ'), then those of Q' are $(a^2/r', \theta')$.

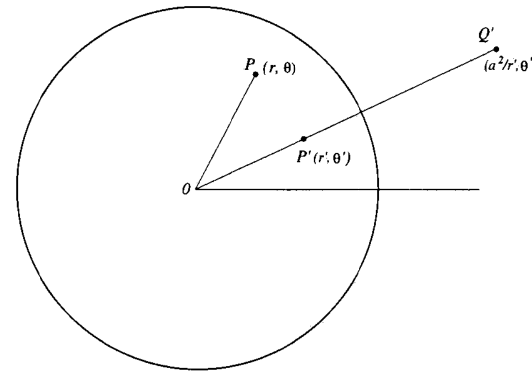


Fig. 7.14.

ANS. In polar coordinates

$$G(r,\theta;r',\theta') = \frac{1}{2\pi}\ln\frac{PP'}{PQ'} + C,$$

where C is a constant which is added to ensure that $G = 0$ on the boundary. Since $\dfrac{PP'}{PQ'} = \dfrac{r'}{a}$ when P is on the boundary, we have on the boundary $\dfrac{1}{2\pi}\ln\dfrac{r'}{a} + C = 0$, which gives $C = \dfrac{1}{2\pi}\ln\dfrac{a}{r'}$, and hence

$$\begin{aligned}
G(r,\theta;r',\theta') &= \frac{1}{4\pi}\ln\frac{a^2[r^2 - 2rr'\cos(\theta-\theta') + r'^2]}{r'^2\Big[r^2 - \dfrac{2ra^2}{r'}\cos(\theta-\theta') + \dfrac{a^4}{r'^2}\Big]} \\
&= \frac{1}{4\pi}\ln\frac{a^2[r^2 - 2rr'\cos(\theta-\theta') + r'^2]}{r^2r'^2 - 2rr'a^2\cos(\theta-\theta') + a^4}.
\end{aligned}$$

7.15. Solve the Laplace equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

subject to the conditions $u(r,0) = f(r)$ and $u(r,\pi) = \pi$ for $r > 0$.
ANS. This is the well-known problem of solving the Laplace equation in the half-plane $y > 0$. Taking the Fourier sine transform defined by

$$\tilde{u}(r,n) = \int_0^\pi u(r,\theta)\,\sin n\theta\,d\theta,$$

we get

$$r^2\frac{d^2\tilde{u}}{dr^2}+r\frac{d\tilde{u}}{dr}-n^2\,\tilde{u}=-nf(r).$$

This equation is solved such that the solution is finite at $r=0$ and tends to zero as $r\to\infty$. To do this, first we determine Green's function associated with the above problem. The solutions for the homogeneous equation are $r^{\pm n}$, and thus

$$G(r,r')=\begin{cases}A\left(\dfrac{r}{r'}\right)^n, & r\le r',\\ A\left(\dfrac{r'}{r}\right)^n, & r\ge r',\end{cases}$$

where A is such that $G(r.r')$ has a discontinuity of derivative of amount $1/r'^2$ at $r=r'$, i.e., $A=-1/2nr$. Then

$$\tilde{u}=-n\int_0^\infty G(r,r')\,f(r')\,dr',$$

and

$$\tilde{u}(r,n)=\frac{1}{2}\Big[\int_0^r\left(\frac{r'}{r}\right)^n\frac{f(r')}{r'}\,dr'+\int_r^\infty\left(\frac{r}{r'}\right)^n\frac{f(r')}{r'}\,dr'\Big],$$

which gives

$$\begin{aligned}u(r,\theta)&=\frac{1}{\pi}\Big[\int_0^r\frac{1}{r'}\sum_{n=1}^\infty\left(\frac{r'}{r}\right)^n\sin n\theta\,f(r')\,dr'\\&\quad+\int_r^\infty\frac{1}{r'}\sum_{n=1}^\infty\left(\frac{r}{r'}\right)^n\sin n\theta\,f(r')\,dr'\Big]\\&=\frac{1}{\pi}\Big[\int_0^r\frac{1}{r'}\Im\sum_{n=0}^\infty\left(\frac{r'}{r}e^{i\theta}\right)^n f(r')\,dr'\\&\quad+\int_r^\infty\frac{1}{r'}\Im\sum_{n=1}^\infty\left(\frac{r}{r'}e^{i\theta}\right)^n f(r')\,dr'\Big]\\&=\frac{1}{\pi}\Big[\int_0^r\frac{1}{r'}\Im\frac{r}{r-r'e^{i\theta}}\,f(r')\,dr'\\&\quad+\int_r^\infty\frac{1}{r'}\Im\frac{r'}{r'-re^{i\theta}}\,f(r')\,dr'\Big]\\&=\frac{1}{\pi}\Big[\int_0^r\frac{r\sin\theta}{r^2-2rr'\cos\theta+r'^2}\,f(r')\,dr'\end{aligned}$$

$$\begin{aligned}&\quad+\int_r^\infty\frac{r\sin\theta}{r'^2-2rr'\cos\theta+r^2}\,f(r')\,dr'\Big]\\&=\frac{1}{\pi}\int_0^\infty\frac{r\sin\theta\,f(r')}{r^2-2rr'\cos\theta+r'^2}\,dr'\\&=\frac{y}{\pi}\int_0^\infty\frac{f(r')}{(x-r')^2+y^2}\,dr'\end{aligned}$$

in Cartesian coordinates. Note that if we remove the restriction on u being zero on half of the x–axis, and take $u=f(x)$ on the entire x–axis, then we must add the solution for $x<0$, i.e., $\int_{-\infty}^0$ to the above solution, which will give the solution under this condition as

$$u(x,y)=\frac{y}{\pi}\int_{-\infty}^\infty\frac{f(r')}{(x-r')^2+y^2}\,dx'.$$

7.16. Let a function u_ε be defined by

$$u_\varepsilon(x,y)=\frac{1}{2\pi}\ln(r+\varepsilon),\quad r=\sqrt{x^2+y^2}.$$

Show that ∇^2u_ε gives a function which tends to the delta function $\delta(x)\delta(y)$ as $\varepsilon\to0$.

ANS. Since $\nabla^2u_\varepsilon=0$, and $u_\varepsilon\to\dfrac{1}{2\pi}\ln r$ as $\varepsilon\to0$, we find that u_ε behaves like Green's function for the Laplacian in R^2 in the limit as $\varepsilon\to0$. Using the requirement (7.53), then

$$\nabla^2u_\varepsilon(x,y)=\delta(x,y)=\delta(x)\delta(y).$$

7.17. Show that for the wave operator

$$\int_{-\infty}^\infty G_3(x,y,z,t;x',y',z',t')\,dz'=G_2(x,y,t;x',y',t').$$

ANS. Since $\int_{-\infty}^{\infty} \delta(z-z')\,dz' = 1$, we find from (7.75) that

$$\begin{aligned}
&\int_{-\infty}^{\infty} G_3(x,y,z,t;x',y',z',t')\,dz' \\
&= \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{\delta\left(t-t'-\frac{1}{c}\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}\right)}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\,dz' \\
&= -\frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{\delta\left(t-t'-\frac{1}{c}\sqrt{(x-x')^2+(y-y')^2+u^2}\right)}{\sqrt{(x-x')^2+(y-y')^2+u^2}}\,dz' \\
&\qquad \text{where } z-z'=u \\
&= -\frac{c}{2\pi} - \int_{r/c}^{\infty} \frac{\delta(t-t'-v)}{\sqrt{c^2v^2-r^2}}\,dv \\
&\qquad \text{where } r^2+u^2=v^2,\ r^2=(x-x')^2+(y-y')^2 \\
&= -\frac{c}{2\pi\sqrt{c^2v^2-r^2}} \quad \text{for } t-t'<r/c \\
&= G_2(x,y,t;x',y',t').
\end{aligned}$$

7.18. Show that for the wave operator

$$\int_{-\infty}^{\infty} G_2(x,y,t;x',y',t')\,dz' = G_1(x,t;x',t').$$

ANS. Follow the method in Exercise 7.19.

7.19. Use Fourier transform method to derive the Green's function $G_1(x,t;x',t')$, for the wave operator $\Box_c$.
ANS. Apply the Fourier transform $\mathcal{F}_x$ to the equation

$$\frac{\partial^2 G_1}{\partial t^2} - c^2\frac{\partial^2 G_1}{\partial x^2} = \delta(x-x')\delta(t-t').$$

For the sake of simplicity, let us translate the source point to the origin, take $t'=0$, and denote $\mathcal{F}_x[G_1(\mathbf{x},t)]$ by $\hat{G}_1(\alpha,t)$. Then after an application of this Fourier transform we get

$$\frac{d^2}{dt^2}\hat{G}_1(\alpha,t) + c^2|\alpha|^2\hat{G}_1(\alpha,t) = \mathbf{1}(\alpha)\cdot\delta(t),$$

which has the solution

$$\hat{G}_1(\alpha,t) = H(t)\,\frac{\sin c|\alpha|t}{c|\alpha|}.$$

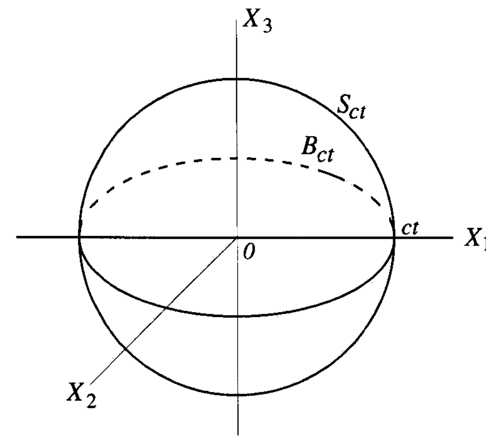


Fig. 7.15.

Since

$$\mathcal{F}^{-1}\left[\frac{\sin c|\alpha|t}{|\alpha|}\right] = \frac{1}{4\pi ct}\delta_{S_{ct}}(\mathbf{x})$$

(see Fig. 7.15), we find that

$$\begin{aligned}
G_1(x,t) &= \frac{1}{4\pi ct}\,\delta_{S_{ct}}(x) = \frac{1}{4\pi ct}H(ct-x)2\pi t \\
&= \frac{H(ct-x)}{2c}.
\end{aligned}$$

8

Weighted Residual Methods

Variational formulation of boundary value problems originates from the fact that weighted variational methods provide approximate solutions of such problems. Variational methods for solving boundary value problems are based on the techniques developed in the calculus of variations. They deal with the problem of minimizing a functional, and thus reducing the given problem to the solution of Euler–Lagrange differential equations. If the functional to be minimized has more than one independent variable, the Euler–Lagrange equations are partial differential equations. Conversely, a boundary value problem can be formulated as a minimizing problem. The functional which corresponds to the partial differential equation is generally known as the energy function. In the case when the solution is not available in a simple form, an approximate solution such that it minimizes the energy equation can be found. The approximating function is a linear combination of the form $\sum_{i=0}^{\infty} c_i\phi_i$, $c_0 = 1$, of specially chosen functions ϕ_i which are known as the test functions or interpolation functions. The function ϕ_0 satisfies the same boundary conditions as the original unknown function, while the remaining functions ϕ_i, $i \neq 0$, satisfy the homogeneous boundary conditions. The constants c_i, $i \neq 0$, are then determined by minimizing the energy function.

The weak variational formulation is defined in §8.4, and the problem of constructing an appropriate functional for a given partial differential equation is discussed there. The Galerkin and the Rayleigh–Ritz weighted residual methods are examined, with examples, in §8.5 and §8.6, and some less frequently used weighted residual methods, like the collocation method, the

least–square method, and the method of moments, are outlined in §8.9.

WWW Refer to the Mathematica Notebook `galerkin.ma` for this chapter.

8.1. Line Integrals

In many types of boundary value problems, the variational methods are used to provide precise formulations which can be applied in any prescribed system of coordinates. We shall derive the Euler equation which is the necessary condition for the solution of the following problem: Find a function $u(x)$ for which the integral

$$I(u) = \int_a^b F(x, u, u')\, dx \tag{8.1}$$

is a minimum, where F is twice–differentiable with respect to x, u, u'. Let two fixed points $A(a, c)$ and $B(b, d)$ in the xy–plane be joined by a curve $\Gamma : x \longmapsto u(x)$ (see Fig. 8.1(a)). Then $c = u(a)$, $d = u(b)$, and $u' = du/dx = u'(x)$ at each point Q of Γ. Thus, the curve Γ determines a value of the integral in (8.1). The value of the integral I will, however, change if we replace Γ by a new curve joining A and B. In order to investigate the variation of I with Γ, i.e., to determine as to for what curve Γ the integral I has a minimum (or a maximum) value, we shall confine to a set of curves Γ_α which are defined as follows: First we select any curve $u = \phi(x)$ such that

$$\phi(a) = 0 = \phi(b) \tag{8.2}$$

(see Fig. 8.1(b)). Then for any value of the parameter α, the curve Γ_α is, in view of (8.2), defined by

$$U = U(x) = u(x) + \alpha\phi(x), \quad U(a) = c, \quad U(b) = d. \tag{8.3}$$

Hence it passes through A and B. From (8.3) we find that on the curve Γ_α

$$U' = \frac{dU}{dx} = u'(x) + \alpha\phi'(x). \tag{8.4}$$

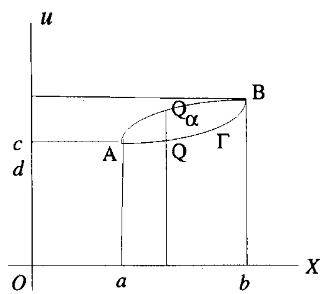


Fig. 8.1(a).

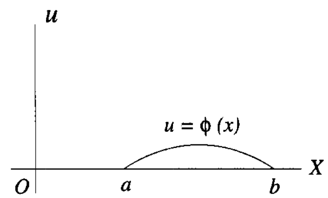


Fig. 8.1(b).

For any value of α, we thus have a curve Γ_α and may form the value of I along Γ_α by substituting the values from (8.3) and (8.4) as

$$I(u,\alpha)=\int_a^b F(x,U,U')\,dx. \tag{8.5}$$

Using the differentiation formula

$$\frac{\partial F}{\partial\alpha}=\frac{\partial F}{\partial U}\frac{\partial U}{\partial\alpha}+\frac{\partial F}{\partial U'}\frac{\partial U'}{\partial\alpha},$$

and noting from (8.3) and (8.4) that $\dfrac{\partial U}{\partial\alpha}=\phi(x)$, $\dfrac{\partial U'}{\partial\alpha}=\phi'(x)$, we find that

$$\frac{\partial F}{\partial\alpha}=\phi(x)\frac{\partial F}{\partial U}+\phi'(x)\frac{\partial F}{\partial U'}.$$

Hence,

$$\frac{\partial I}{\partial\alpha}=\int_a^b\left[\phi(x)\frac{\partial F}{\partial U}+\phi'(x)\frac{\partial F}{\partial U'}\right]dx. \tag{8.6}$$

If we integrate by parts the second term in the integrand in (8.6), and use (8.2), we get

$$\frac{\partial I}{\partial\alpha}=\int_a^b\phi(x)\left[\frac{\partial F}{\partial U}-\frac{d}{dx}\left(\frac{\partial F}{\partial U'}\right)\right]dx. \tag{8.7}$$

Let us assume that there is a twice–differentiable curve, say Γ, for which the value of I is a minimum. Then, the value of I on Γ will be less than the value of I on any other curve Γ_α. Thus, $I(\alpha)$ will assume a minimum value for $\alpha=0$, since $\partial I/\partial\alpha$ is continuous. But, then $U=u$ and $U'=u'$ when $\alpha=0$. Hence, taking $\alpha=0$ in (8.7), and $I'(0)=0$, we find that

$$\int_a^b\phi(x)\left[\frac{\partial F}{\partial u}-\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right]dx=0. \tag{8.8}$$

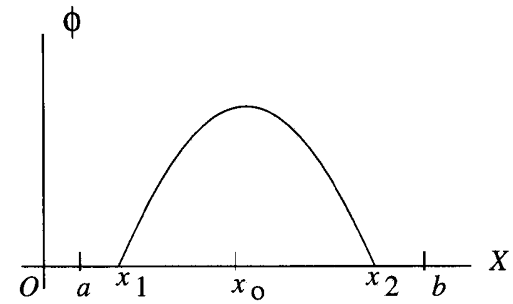


Fig. 8.2.

Since the solution $\phi(x)$ is arbitrary, except for the conditions (8.2), the factor in the square brackets in (8.8) is continuous, which implies that

$$\frac{\partial F}{\partial u}-\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)=0 \tag{8.9}$$

for all $x\in[a,b]$. In fact, if the expression within the square brackets in (8.8) were nonzero at any point, say x_0, there would be some small interval including this point, say $x_1<x_0<x_2$, in which it remains nonzero; then, by taking the function $\phi(x)$ in Fig. 8.2, the integrand in (8.9) would be zero except for the interval (x_1,x_2) where it is nonzero. Thus, we have proved: If the integral I, defined by (8.1), has a minimum (or a maximum) along any sufficiently smooth curve Γ joining A and B, then $u=u(x)$ will be a solution of the differential Eq (8.9). This is known as the *Euler–Lagrange equation*, and its solution $u(x)$ as the *extremal*. Note that the derivative d/dx in (8.9) is computed by recalling that $u=u(x)$ and $u'=u'(x)=du/dx$ are functions of x. After this differentiation is carried out, Eq (8.9) becomes

$$\frac{\partial F}{\partial u}-\frac{\partial^2F}{\partial x\partial u'}-\frac{\partial^2F}{\partial u\partial u'}\frac{du}{dx}-\frac{\partial^2F}{\partial u'^2}\frac{d^2u}{dx^2}=0, \tag{8.10}$$

which is a second order differential equation, and its solution contains two arbitrary constants which must be determined from the conditions that the curve passes through A and B.

Similarly, by following the above procedure and using integration by parts twice in the third term appearing in the integrand of

$$\frac{dI}{d\alpha}=\int_a^b\left[\phi(x)\frac{\partial F}{\partial U}+\phi'(x)\frac{\partial F}{\partial U'}+\phi''(x)\frac{\partial^2F}{\partial U''}\right]dx, \tag{8.11}$$

and taking the additional requirement that $\phi'(a) = 0 = \phi'(b)$, we find that the Euler equation which provides a necessary condition for the functional

$$I(u) = \int_a^b F(x, u, u', u'')\, dx \tag{8.12}$$

to be a minimum is

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\Big(\frac{\partial F}{\partial u'}\Big) + \frac{d^2}{dx^2}\Big(\frac{\partial F}{\partial u''}\Big) = 0 \tag{8.13}$$

(see Exercise 8.1.).

If the integral I depends on two functions u and v, i.e., if

$$I(u, v) = \int_a^b F(x, u, v, u', v')\, dx, \tag{8.14}$$

then there are two Euler equations:

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\Big(\frac{\partial F}{\partial u'}\Big) = 0, \quad \frac{\partial F}{\partial v} - \frac{d}{dx}\Big(\frac{\partial F}{\partial v'}\Big) = 0. \tag{8.15}$$

To derive (8.15), introduce two functions $\phi(x)$ and $\psi(x)$ and two parameters α and β such that $U = u + \alpha\phi(x)$, $V = v + \beta\psi(x)$. Then $\partial I/\partial\alpha = 0$ and $\partial I/\partial\beta = 0$.

8.2. Variational Notation

Eq (8.3) implies that the difference $U - u = \alpha\phi(x)$, or QQ_α in Fig. 8.1, is the change in U, starting from u and the value of $\alpha = 0$. Hence $\alpha\phi(x) = U - u$ is called the (first) *variation* in u and is denoted by δu. It may also be regarded as the differential dU, since $d\alpha = \alpha - 0 = \alpha$ at $\alpha = 0$; then $dU/d\alpha = \phi(x)$ so that

$$dU = \Big(\frac{dU}{d\alpha}\Big) d\alpha = \phi(x)\alpha = \delta u. \tag{8.16}$$

Also, Eq (8.4) may be rewritten as $U' = u' + \alpha\phi'(x)$, or

$$U' - u' = \alpha\phi'(x). \tag{8.17}$$

The term $\alpha\phi'(x) = U' - u'$ is the variation of u' and is denoted by $\delta u'$. It is also the differential

$$dU' = \Big(\frac{dU'}{d\alpha}\Big) d\alpha = \phi'(x)\alpha = \delta u'. \tag{8.18}$$

From (8.16) and (8.18) it follows that $\dfrac{d(\delta u)}{dx} = \delta\Big(\dfrac{du}{dx}\Big)$, or

$$(\delta u)' = \delta(u'), \tag{8.19}$$

i.e., the operator δ and d/dx are commutative. The variation of u'' and higher derivatives of u are defined analogously.

If u is changed to $u + \delta u = u + \alpha w$, then the corresponding change in a function $F = F(x, u, u')$ at u is defined by

$$\delta F = \frac{\partial F}{\partial u}\,\delta u + \frac{\partial F}{\partial u'}\,\delta u'. \tag{8.20}$$

Since δ behaves like a differential operator, it has the following properties:

(i) $\delta(f_1 \pm f_2) = \delta f_1 \pm \delta f_2$,
(ii) $\delta(f_1 f_2) = f_1\,\delta f_2 + f_2\,\delta f_1$,
(iii) $\delta(f_1/f_2) = (f_2\,\delta f_1 - f_1\,\delta f_2)/f_2^2$,
(iv) $\delta(f)^n = n(f)^{n-1}\,\delta f$,
(v) $D(\delta u) = \delta(Du)$,
(vi) $\delta\int_a^b u(x)\,dx = \int_a^b \delta u(x) dx$,

where $' \equiv D \equiv d/dx$, and f_1, f_2 are functions of x, u, u'. Note that (v) and (vi) are commutative properties under differential and integral operators. The above six properties (i)–(vi) can be easily proved by using (8.20).

Now, we find from (8.7) that

$$\delta I = \Big(\frac{dI}{d\alpha}\Big)_{\alpha=0} = \int_a^b \Big[\frac{\partial F}{\partial u} - \frac{d}{dx}\Big(\frac{\partial F}{\partial u'}\Big)\Big]\delta u\, dx, \tag{8.21}$$

where the variations at the end points are zero. Similarly, for I defined by (8.12) we have

$$\delta I = \int_a^b \Big[\frac{\partial F}{\partial u} - \frac{d}{dx}\Big(\frac{\partial F}{\partial u'}\Big) + \frac{d^2}{dx^2}\Big(\frac{\partial F}{\partial u''}\Big)\Big]\delta u\, dx, \tag{8.22}$$

where the variations are restricted by the conditions: $\phi'(a) = 0 = \phi'(b)$.

For the integral I in (8.14) whose integrand $F(x, u, u', v, v')$ contains two independent variables u and v, we define the variations as total differentials so that

$$\delta u = \alpha\phi(x),\ \delta u' = \alpha\phi'(x),\ \delta v = \alpha\psi(x),\ \delta v' = \alpha\psi'(x).$$

Then the definition in this case becomes

$$\delta F = \frac{\partial F}{\partial u}\,\delta u + \frac{\partial F}{\partial u'}\,\delta u' + \frac{\partial F}{\partial v}\,\delta v + \frac{\partial F}{\partial v'}\,\delta v'. \tag{8.23}$$

For the integral I in (8.14) we have

$$\delta I = \int_a^b \left[\frac{\partial F}{\partial u} - \frac{d}{dx}\Big(\frac{\partial F}{\partial u'}\Big)\right]\delta u\,dx + \int_a^b \left[\frac{\partial F}{\partial v} - \frac{d}{dx}\Big(\frac{\partial F}{\partial v'}\Big)\right]\delta v\,dx. \tag{8.24}$$

In the parametric case, we can introduce a new parameter t as the independent variable and regard x and u (or x, u and v) as the dependent variables. For example, let us consider the integrand $F(x, u, u')$. Using the hat to denote differentiation with respect to the parameter t, we have: $u' = \hat{u}/\hat{x}$, and $F(x, u, u')\,dx = F(x, u, \hat{u}/\hat{x})\hat{x}\,dt$. Let $x = a$ when $t = t_1$, and $x = b$ when $t = t_2$, and denote $F(x, u, \hat{u}/\hat{x})\hat{x}$ by $G(x, u, \hat{x}, \hat{u})$. Then, the integral (8.1) can be written as

$$I = \int_a^b F(x, u, u')\,dx = \int_{t_1}^{t_2} G(x, u, \hat{x}, \hat{u})\,dt,$$

and (8.21) becomes (using (8.24) with x, u, v replaced by t, x, u, respectively)

$$\delta I = \int_{t_1}^{t_2} \left[\frac{\partial G}{\partial x} - \frac{d}{dt}\Big(\frac{\partial G}{\partial \hat{x}}\Big)\right]\delta x\,dt + \int_{t_1}^{t_2} \left[\frac{\partial G}{\partial u} - \frac{d}{dt}\Big(\frac{\partial G}{\partial u}\Big)\right]\delta u\,dt, \tag{8.25}$$

where δx and δu are zero at t_1 and t_2.

EXAMPLE 8.1. Show that in R^2

$$\nabla(\delta u)\cdot\nabla u = \frac{1}{2}\delta|\nabla u|^2. \tag{8.26}$$

A direct computation shows that

$$\begin{aligned}\nabla(\delta u)\cdot\nabla u &= \Big(\mathbf{i}\frac{\partial}{\partial x}(\delta u) + \mathbf{j}\frac{\partial}{\partial y}(\delta u)\Big)\cdot(\mathbf{i}\,u_x + \mathbf{j}\,u_y)\\ &= (\mathbf{i}\,u_x + \mathbf{j}\,u_y)\cdot\delta\,(\mathbf{i}\,u_x + \mathbf{j}\,u_y)\\ &= \frac{1}{2}\delta\left[(\mathbf{i}\,u_x + \mathbf{j}\,u_y)\cdot(\mathbf{i}\,u_x + \mathbf{j}\,u_y)\right]\\ &= \frac{1}{2}\delta|\nabla u|^2. \blacksquare\end{aligned}$$

8.3. Multiple Integrals

Let $F(a, y, u, p, q)$ be a twice-differentiable function of five variables, where the dependent variable u is a function of x and y and $p = u_x = \partial u/\partial x$, $q = u_y = \partial u/\partial y$. We shall study the variation of the integral

$$I(u) = \iint_S F(x, y, u, p, q)\,dS \tag{8.27}$$

over a surface S which passes through a fixed boundary curve B and determine for which surface S the integral $I(u)$ is a minimum. Following the method of §8.1, we first choose any surface S defined by $u = \phi(x, y)$ such that $\phi(x, y) = 0$ on the curve Γ, where Γ is the projection of the boundary curve B in the xy-plane. Since $u(x, y)$ passes through Γ, we define a surface S_α for any α by

$$U(x, y) = u(x, y) + \alpha\phi(x, y).$$

The surface S_α also passes through Γ. On this surface

$$P \equiv U_x = p + \alpha\phi_x, \quad Q \equiv U_y = q + \alpha\phi_y,$$

so that the integral I is written as

$$I = \iint_{S_\alpha} F(x, y, U, P, Q)\,dx\,dy. \tag{8.28}$$

Hence

$$\frac{dI}{d\alpha} = \iint \left[\frac{\partial F}{\partial U}\phi + \frac{\partial F}{\partial P}\phi_x + \frac{\partial F}{\partial Q}\phi_y\right]dx\,dy. \tag{8.29}$$

Using (A.3) we find from (8.29) that

$$\frac{dI}{d\alpha}\Big|_{\alpha=0} = \iint \left[\frac{\partial F}{\partial U} - \frac{\partial}{\partial x}\Big(\frac{\partial F}{\partial P}\Big) - \frac{\partial}{\partial y}\frac{\partial F}{\partial Q}\right]\phi\,dx\,dy,$$

since $U = u$, $P = p$, $Q = q$ when $\alpha = 0$. If we multiply the value of $dI/d\alpha\big|_{\alpha=0}$ by $d\alpha = \alpha - 0 = \alpha$ and write δu for $\alpha\phi$, we find, as in (8.21), that

$$\delta I = \iint \left[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\Big(\frac{\partial F}{\partial p}\Big) - \frac{\partial}{\partial y}\Big(\frac{\partial F}{\partial q}\Big)\right]\delta u\,dx\,dy. \tag{8.30}$$

If $u(x,y)$ is the surface for which I is a minimum, then $\delta I = 0$ for arbitrary δu. Since the expression in the square brackets in (8.30) is continuous, it must be zero. This gives a necessary condition for I to be a minimum as

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial p}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial q}\right) = 0. \tag{8.31}$$

This is the Euler equation for (8.27), and any surface corresponding to a solution of this equation is an extremal.

If $F = F(x,y,u,p,q,r,s,t)$, where p and q are defined above and $r = u_{xx}$, $s = u_{xy}$, and $t = u_{yy}$, then the Euler equation to minimize the integral $I = \iint F(x,y,u,p,q,r,s,t)\,dx\,dy$ is given by

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial p}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial q}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial r}\right) + \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial F}{\partial s}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial t}\right) = 0. \tag{8.32}$$

EXAMPLE 8.2. The condition that the Dirichlet integral

$$I(u) = \iint \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dx\,dy$$

be a minimum is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

which is the Laplace equation. ■

8.4. Weak Variational Formulation

The weak variation formulation of boundary value problems is derived from the fact that variational methods for finding approximate solutions of boundary value problems, viz., Galerkin, Rayleigh–Ritz, collocation, or other weighted residual methods, are based on the weak variational statements of the boundary value problems. In fact, the

weak formulations are more general than the corresponding strong formulations, since even the irregular boundary conditions are easily managed in the weak formulation. We shall not discuss the evolution of the strong formulations, but rather explain the method of the weak variational formulation, which will in turn define the underlying concept.

We shall consider a general form of a second order mixed boundary value problem, defined by Eq (8.31), in a two-dimensional region Ω with the prescribed boundary conditions

$$u = u_0 \quad \text{on } \Gamma_1, \tag{8.33a}$$

and

$$\frac{\partial F}{\partial p} n_x + \frac{\partial F}{\partial q} n_y = q_0 \quad \text{on } \Gamma_2, \tag{8.33b}$$

where $F = F(x,y,u,p,q)$, and n_x, n_y are the direction cosines of the unit vector $\hat{\mathbf{n}}$ normal to the boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ of the region Ω such that $\Gamma_1 \cap \Gamma_2 = \emptyset$.

For example, a special case of (8.31) is when F is defined as

$$F = \frac{1}{2}\left[k_1\left(\frac{\partial u}{\partial x}\right)^2 + k_2\left(\frac{\partial u}{\partial y}\right)^2\right] - f\,u.$$

This equation arises in heat conduction problems in a two–dimensional region with k_1, k_2 as thermal conductivities in the x, y directions, and f being the heat source (sink). Here

$$\frac{\partial F}{\partial p} = k_1 \frac{\partial u}{\partial x}, \quad \frac{\partial F}{\partial q} = k_2 \frac{\partial u}{\partial y}, \quad \frac{\partial F}{\partial u} = -f,$$

and Eq (8.31) becomes

$$-\frac{\partial}{\partial x}\left(k_1 \frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(k_2 \frac{\partial u}{\partial y}\right) = f \quad \text{in} \quad \Omega.$$

If $k_1 = k_2 = 1$, then we get the Poisson equation $-\nabla^2 u = f$ with appropriate boundary conditions.

The weak variational formulation for Eq (8.31) can be obtained by the following three steps:

STEP 1: Multiply Eq (8.31) by a test function $w\,(\equiv \delta u)$ and integrate the product over the region Ω:

$$\iint_\Omega \left[\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial p}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial q}\right)\right] w\,dx\,dy = 0. \tag{8.34}$$

The test function w is arbitrary, but it must satisfy the homogeneous essential boundary conditions (8.33a) on u.

STEP 2: Use formula (A.3) componentwise to the second and third terms in (8.34), in order to transfer the differentiation from the dependent variable u to the test function w, and identify the type of the boundary conditions admissible by the variational form:

$$\iint_\Omega \left[w\frac{\partial F}{\partial u} + \frac{\partial w}{\partial x}\frac{\partial F}{\partial p} + \frac{\partial w}{\partial y}\frac{\partial F}{\partial q} \right] dx\,dy \\ - \int_{\Gamma=\Gamma_1\cup\Gamma_2} \left(\frac{\partial F}{\partial p} n_x + \frac{\partial F}{\partial q} n_y \right) w\,ds = 0. \quad (8.35)$$

Note that the formula (A.3) does not apply to the first term in the integrand in (8.34).

Notice that this step also yields boundary terms which determine the nature of the essential and natural boundary conditions for the problem. The general rule to identify the essential and natural boundary conditions for (8.31) is as follows: The essential boundary condition is prescribed on the dependent variable (u in this case), i.e.,

$$u = u_0 \quad \text{on} \quad \Gamma_1$$

is the essential boundary condition for (8.31). The test function w in the boundary integral (8.35) satisfies the homogeneous form of the same boundary condition as that prescribed on u. The natural boundary condition arises by specifying the coefficients of w and its derivatives in the boundary integral in (8.35). Thus,

$$\frac{\partial F}{\partial p} n_x + \frac{\partial F}{\partial q} n_y = q_0 \quad \text{on} \quad \Gamma_2$$

is the natural boundary condition in a Neumann boundary value problem. In one–dimensional problems, use integration by parts instead of the divergence formula (A.3).

In order to equalize the continuity requirements on u and w, the differentiation in the divergence formula (A.3) has been transferred from F to w. It imparts weaker continuity requirements on the solution u in the variational problem than in the original equation.

STEP 3: Simplify the boundary terms by using the prescribed boundary conditions. This will affect the boundary integral in (8.35) which

is split into two terms, one on Γ_1 and the other on Γ_2:

$$\iint_\Omega \left[w\frac{\partial F}{\partial u} + \frac{\partial w}{\partial x}\frac{\partial F}{\partial p} + \frac{\partial w}{\partial y}\frac{\partial F}{\partial q} \right] dx\,dy \\ - \int_{\Gamma_1\cup\Gamma_2} \left(\frac{\partial F}{\partial p} n_x + \frac{\partial F}{\partial u_y} n_y \right) w\,ds = 0. \quad (8.36)$$

The integral on Γ_1 vanishes since $w = \delta u = 0$ on Γ_1. The natural boundary condition is substituted in the integral on Γ_2. Then (8.36) reduces to

$$\iint_\Omega \left[w\frac{\partial F}{\partial u} + \frac{\partial w}{\partial x}\frac{\partial F}{\partial p} + \frac{\partial w}{\partial y}\frac{\partial F}{\partial q} \right] dx\,dy - \int_{\Gamma_2} w\,q_0\,ds = 0. \quad (8.37)$$

This is the weak variational form for the problem (8.31). We can write (8.37) in terms of the bilinear and linear differential forms as

$$b(w,u) = l(w), \quad (8.38)$$

where

$$\begin{aligned} b(w,u) &= \iint_\Omega \left[\frac{\partial w}{\partial x}\frac{\partial F}{\partial p} + \frac{\partial w}{\partial y}\frac{\partial F}{\partial q} \right] dx\,dy, \\ l(w) &= -\iint_\Omega w\frac{\partial F}{\partial u}\,dx\,dy + \int_{\Gamma_2} w q_0\,ds. \end{aligned} \quad (8.39)$$

Formula (8.38) defines the weak variational form for Eq (8.31) subject to the boundary conditions (8.33). The quadratic functional associated with this variational form is given by

$$I(u) = \frac{1}{2} b(u,u) - l(u). \quad (8.40)$$

EXAMPLE 8.3. Consider the system of Navier–Stokes equations for a two–dimensional flow of a viscous, incompressible fluid (pressure–velocity fields):

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right), \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \end{aligned}$$

in a region Ω, with boundary conditions $u = u_0$, $v = v_0$ on Γ_1, and

$$\nu\Big(\frac{\partial u}{\partial x}n_x + \frac{\partial u}{\partial y}n_y\Big) - \frac{1}{\rho}pn_x = \hat{t}_x,$$
$$\nu\Big(\frac{\partial v}{\partial x}n_x + \frac{\partial v}{\partial y}n_y\Big) - \frac{1}{\rho}pn_y = \hat{t}_y,$$

on Γ_2, where (u, v) denotes the velocity field, p the pressure, and $\hat{t}_x, \hat{t}_y$ the prescribed values of the secondary variables. Let w_1, w_2, w_3 be the test functions, one for each equation, such that w_1 and w_2 satisfy the essential boundary conditions on u and v, respectively, and w_3 does not satisfy any essential condition. Then

$$0 = \iint_\Omega \Big[w_1\Big(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\Big) - \frac{p}{\rho}\frac{\partial w_1}{\partial x} + \nu\Big(\frac{\partial w_1}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial w_1}{\partial y}\frac{\partial u}{\partial y}\Big)\Big]\,dx\,dy - \int_{\Gamma_2} w_1\hat{t}_x\,ds,$$

$$0 = \iint_\Omega \Big[w_2\Big(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\Big) - \frac{p}{\rho}\frac{\partial w_2}{\partial y} + \nu\Big(\frac{\partial w_2}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial w_2}{\partial y}\frac{\partial v}{\partial y}\Big)\Big]\,dx\,dy - \int_{\Gamma_1} w_2\hat{t}_y\,ds,$$

$$0 = \iint_\Omega w_3\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big)\,dx\,dy.$$

Then

$$\begin{aligned} b((w_1, w_2, w_3), (u, v)) &= \iint_\Omega \Big[w_1\Big(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\Big) + w_2\Big(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\Big) + w_3\Big(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y}\Big)\Big]\,dx\,dy \\ &\quad + \nu\iint_\Omega\Big(\frac{\partial w_1}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial w_1}{\partial y}\frac{\partial u}{\partial y} + \frac{\partial w_2}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial w_2}{\partial y}\frac{\partial v}{\partial y}\Big)\,dx\,dy \\ &\quad + \frac{p}{\rho}\iint_\Omega\Big(\frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y}\Big)\,dx\,dy, \end{aligned}$$

$$l(w_1, w_2, w_3) = \int_{\Gamma_2} (w_1\hat{t}_x + w_2\hat{t}_y)\,ds.$$

Note that the boundary integral in the linear form $l(w_1, w_2, w_3)$ has no term containing w_3. ■

We shall now discuss the Galerkin and the Rayleigh–Ritz methods, which are the two most widely used weighted residual methods for obtaining approximate numerical solutions of boundary value problems. It will be found that these two methods give the same results for homogeneous boundary value problems. In fact it can be proved that they are similar for homogeneous problems.

8.5. Galerkin Method

Consider the boundary value problem

$$L\,u = f \qquad \text{in } \Omega, \tag{8.41}$$

subject to the boundary conditions

$$u = g(s) \qquad \text{on } \Gamma_1, \tag{8.42}$$

$$\frac{\partial u}{\partial n} + k(s)\,u = h(s) \qquad \text{on } \Gamma_2, \tag{8.43}$$

where $\Gamma = \Gamma_1 \cup \Gamma_2$ is the boundary of the region Ω. Let us choose an approximate solution $\tilde{u}$ of the form

$$\tilde{u} = \sum_{i=1}^{N} c_i\phi_i. \tag{8.44}$$

An approximate solution does not, in general, satisfy the system (8.41)–(8.43). The residual (error) associated with an approximate solution is defined by

$$r(\tilde{u}) \equiv L\,\tilde{u} - f = L\Big(\sum_{i=1}^{N} c_i\phi_i\Big) - f. \tag{8.45}$$

Note that if u_0 is an exact solution of (8.41)–(8.43), then $r(u_0) = 0$. The Galerkin method requires that the residual be orthogonal with respect to the basis functions ϕ_i (also called the trial functions) used in (8.44), i.e.,

$$\langle r, \phi_i\rangle = 0. \tag{8.46}$$

Hence

$$\iint_\Omega \{L(\tilde{u}) - f\}\phi_i\,dx\,dy = 0, \qquad i = 1, \cdots, N, \tag{8.46a}$$

or

$$\sum_{j=1}^{n} c_j \iint_\Omega \phi_i\, L\phi_j\, dx\, dy = \iint_\Omega f\phi_i\, dx\, dy,$$

which in the matrix form is written as

$$[A]\{c\} = \{b\}, \tag{8.46b}$$

where

$$A_{ij} = \iint_\Omega \phi_i\, L\phi_j\, dx\, dy, \qquad b_i = \iint_\Omega f\phi_i\, dx\, dy. \tag{8.46c}$$

In the examples given below, we shall choose different values of N in (8.34) for the trial function $\tilde{u}$. There is some guidance from geometry for such choices; moreover, they should satisfy the essential conditions and exhibit the nature of the approximation solutions vis-a-vis the exact solutions (see §8.7 for some choices). However, the larger the N, the better the approximation becomes.

EXAMPLE 8.4. Consider the Poisson equation

$$-\nabla^2 u \equiv -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = c, \quad 0 < x < a, \quad 0 < y < b,$$

such that $u = 0$ at $x = 0, a$ and $y = 0, b$. First we choose the first order approximate solution as

$$\tilde{u}_1^{(1)} = \alpha xy(x-a)(y-b).$$

Note that this choice satisfies all four Dirichlet boundary conditions. The Galerkin equation (8.46a) gives

$$\int_0^b \int_0^a \left[-2\alpha(y^2 - by + x^2 - ax) - c\right] xy(x-a)(y-b)\, dx\, dy = 0$$

which simplifies to

$$\frac{\alpha}{90}\left[a^3b^3(a^2+b^2)\right] - \frac{a^3b^3c}{36} = 0.$$

Thus,

$$\alpha = \frac{5c}{2(a^2+b^2)}.$$

Hence

$$\tilde{u}_1^{(1)} = \frac{5c}{2(a^2+b^2)}\, xy(x-a)(y-b).$$

Alternately, we can solve this problem by choosing the first order approximate solution as

$$\tilde{u}_1^{(2)} = \sum_{j,k=1}^{N} \alpha_{jk} \sin\frac{j\pi x}{a} \sin\frac{k\pi y}{b},$$

which is an orthogonal trigonometric series with a finite number of terms. Note that $u_1^{(2)}$ satisfies the boundary conditions. Also note the orthogonality condition

$$\int_0^a \sin\frac{m\pi x}{a} \sin\frac{n\pi x}{a}\, dx = \begin{cases} 0, & m \neq n \\ a/2, & m = n. \end{cases}$$

The Galerkin equation (8.46a) in this case gives

$$\int_0^b \int_0^a \left[\alpha_{jk}\left(\frac{j^2\pi^2}{a^2} + \frac{k^2\pi^2}{b^2}\right) \sin\frac{j\pi x}{a} \sin\frac{k\pi y}{b} + c\right] \times \sin\frac{j\pi x}{a} \sin\frac{k\pi y}{b}\, dx\, dy = 0.$$

Hence

$$\alpha_{jk}\frac{\pi^2}{4}\left(\frac{j^2}{a^2} + \frac{k^2}{b^2}\right) = \frac{c}{jk\pi^2}(1-\cos j\pi)(1-\cos k\pi),$$

or

$$\alpha_{jk} = \frac{4c(1-\cos j\pi)(1-\cos k\pi)a^2b^2}{\pi^4 jk(b^2j^2 + a^2k^2)}.$$

Thus, this approximate solution is

$$\tilde{u}_1^{(2)} = \sum_{j,k=1}^{N} \frac{4a^2b^2c(1-\cos j\pi)(1-\cos k\pi)}{jk\pi^4(a^2k^2 + b^2j^2)} \sin\frac{j\pi x}{a} \sin\frac{k\pi y}{b}.$$

If the number of terms in each sum is infinite, then $\tilde{u}_1^{(2)}$ becomes the exact solution u_0.

At the center point (a/2,b/2), we have

$$\tilde{u}_{1,c}^{(2)} = \sum_{j,k=1}^{N} \frac{4a^2b^2c(1-\cos j\pi)(1-\cos k\pi)}{jk\pi^4(a^2k^2+b^2j^2)} \sin\frac{j\pi}{2} \sin\frac{k\pi}{2}.$$

If $a=b$, then at the center point $(a/2,a/2)$

$$\begin{aligned}\tilde{u}_{1,c}^{(2)} &= \sum_j \sum_k \frac{4a^2c(1-\cos j\pi)(1-\cos k\pi)}{jk\pi^4(j^2+k^2)} \sin\frac{j\pi}{2} \sin\frac{k\pi}{2} \\ &= \frac{a^2c}{\pi^4}\left[8+\frac{8}{15}+\frac{8}{15}+\frac{8}{81}+\cdots\right] \equiv u_0 \\ &\approx \frac{36.64}{\pi^4} c\left(\frac{a}{2}\right)^2.\end{aligned}$$

For the N–th approximation, the trial functions are chosen as $\phi_{jk}(x,y) = f_j(x)\, g_k(y)$, where

$$f_j(x) = x^j(x-a), \quad g_k(y) = y^k(y-b).$$

Then the N–th approximate solution is

$$\tilde{u}_N(x,y) = \sum_{j,k=1}^{N} \alpha_{jk}\, \phi_{jk}(x,y),$$

and the residual is

$$r = -c - \sum_{j,k=1}^{N} \left[f_j''(x)g_k(y) + f_j(x)g_k''(y)\right].$$

Hence, since the Galerkin method requires that $\langle \phi_{mn}, r\rangle = 0$ for $m,n = 1,2,\cdots,N$, we get

$$0 = \int_0^a \int_0^b \left\{-c - \sum_{j,k=1}^{N} \left[f_j''(x)g_k(y) + f_j(x)g_k''(y)\right]\right\} f_m(x)g_n(y)\,dx\,dy,$$

which after integration yields

$$\sum_{j,k=1}^{N} \alpha_{jk}\left[j\,p(j,m,a)\,q(k,n,b) + k\,p(k,n,b)\,q(j,m,a)\right] + ca^{m+2}\,b^{n+2}\,h(m,n) = 0, \quad (m,n = 1,2,\cdots,N),$$

where

$$\begin{aligned} p(j,m,a) &= a^{j+m+1}\left[\frac{j-1}{j+m-1} - \frac{2}{j+m} + \frac{j+1}{j+m+1}\right], \\ q(k,n,b) &= b^{k+n+3}\left[\frac{1}{k+n+1} - \frac{2}{k+n+2} + \frac{1}{k+n+3}\right], \\ h(m,n) &= \frac{1}{(m+1)(m+2)(n+1)(n+2)}. \end{aligned}$$

The coefficients α_{jk} for $j,k = 1,2,\cdots,N$ can be determined from the above system of equations. The result for $\tilde{u}_1$ found earlier follows from this general case.

The trial functions $\phi_{jk}(x,y) = \sin\frac{j\pi x}{a} \sin\frac{k\pi y}{b}$, used in the approximation $\tilde{u}_1^{(2)}$, belong to the set of orthogonal functions obtained by solving the given boundary value problem by the separation of variables method (see Table 4.1, Dirichlet–Dirichlet case, and Exercise 5.28). ■

8.6. Rayleigh–Ritz Method

Consider the Poisson equation $-\nabla^2 u = f$, with the homogeneous boundary conditions $u = 0$ on Γ_1 and $\partial u/\partial n = 0$ on Γ_2. Then, the weak variational formulation leads to

$$I(u) = \iint_\Omega \left\{\frac{1}{2}|\nabla u|^2 - fu\right\} dx\,dy = 0. \qquad (8.47)$$

A generalization of the result in (8.47) for the case of the system $u\,Lu = f$ with the above homogeneous boundary conditions, where L is a linear self–adjoint and positive definite operator, leads to the functional

$$I(u) = \frac{1}{2}\iint_\Omega \{uLu - 2fu\}\,dx\,dy. \qquad (8.48)$$

THEOREM 8.1. *If the operator L is self-adjoint and positive definite, then the unique solution of $Lu = f$ with homogeneous boundary conditions occurs at a minimum value of $I(u)$.*

An application of Theorem 8.1 is the Rayleigh–Ritz method, where we find the direct solution of the variational problem for the system $Lu = f$ by constructing minimizing sequences and securing the approximate solutions by a limiting process based on such sequences. Thus, we choose a complete set of linearly independent basis (test) functions ϕ_i, $i = 1, \cdots$, and then approximate the exact solution u_0 by taking the approximate solution $\tilde{u}$ in the form

$$\tilde{u} = \sum_{i=1}^{n} c_i \phi_i, \tag{8.49}$$

where the constants c_i are chosen such that the functional $I(\tilde{u})$ is minimized at each stage. If $\tilde{u} \to u_0$ as $n \to \infty$, then the method yields a convergent solution. At each stage the method reduces the problem to that of solving a set of linear algebraic equations. The details for the boundary value problem $-\nabla^2 u = f$ with homogeneous boundary conditions are as follows: Using (8.49) in the functional (8.48) we get

$$\begin{aligned} I(\tilde{u}) &= I(c_1, \cdots, c_n) \\ &= \iint_\Omega \left\{ \left(\frac{\partial \tilde{u}}{\partial x}\right)^2 + \left(\frac{\partial \tilde{u}}{\partial y}\right)^2 - 2\tilde{u} f \right\} dx\, dy \\ &= \iint_\Omega \left\{ \left(\sum c_i \frac{\partial \phi_i}{\partial x}\right)^2 + \left(\sum c_i \frac{\partial \phi_i}{\partial y}\right)^2 - 2f \sum c_i \phi_i \right\} dx\, dy, \end{aligned}$$

thus

$$\begin{aligned} I(c_i) &= c_i^2 \iint_\Omega \left\{ \left(\frac{\partial \phi_i}{\partial x}\right)^2 + \frac{\partial \phi_i}{\partial y}\right)^2 \right\} dx\, dy \\ &+ 2 \sum_{i \neq j} c_i c_j \iint_\Omega \left(\frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y}\frac{\partial \phi_j}{\partial y} \right) dx\, dy - 2c_i \iint_\Omega \phi_i f\, dx\, dy. \end{aligned}$$

Hence

$$\frac{\partial I}{\partial c_i} = 2A_{ii} c_i + 2 \sum_{i \neq j} A_{ij} c_j - 2h_i, \tag{8.50}$$

and

$$A_{ij} = \iint_\Omega \left(\frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y}\frac{\partial \phi_j}{\partial y} \right) dx\, dy, \tag{8.51}$$

$$h_i = \iint_\Omega \phi_i f\, dx\, dy. \tag{8.52}$$

Now, if we choose c_i such that $I(c_i)$ is a minimum (i.e., $\partial I/\partial c_i = 0$), then from (8.50) we get

$$\sum_{j=1}^{n} A_{ij} c_i = h_i, \quad i = 1, \cdots, n, \tag{8.53}$$

which in the matrix notation is

$$[A]\{c\} = \{h\}, \tag{8.54}$$

where the matrix $[A]$ has elements A_{ij} given by (8.51), $\{h\}$ has elements h_i given by (8.52), and $\{c\} = [c_1, \cdots, c_n]^T$. Note that (8.54) is a system of linear algebraic equations to be solved for the unknown parameter c_i, and $[A]$ is non-singular if L is positive definite.

The Rayleigh–Ritz method can be developed, alternately, by solving for u the equation (8.38), where we require that w satisfy the homogeneous essential conditions only. Then this problem is equivalent to minimizing the functional (8.40). In other words, we will find an approximate solution of (8.38) in the form

$$u_n = \sum_{j=1}^{n} c_j \phi_j + \phi_0, \tag{8.55}$$

where the coefficients c_j are chosen such that Eq (8.38) is true for $w = \phi_i$, $i = 1, \cdots, n$, i.e.,

$$b(\phi_i, u_n) = l(\phi_i), \quad i = 1, \cdots, n,$$

or

$$b\Big(\phi_i, \sum_{j=1}^{n} c_j \phi_j + \phi_0\Big) = l(\phi_i),$$

thus,

$$\sum_{j=1}^{n} c_j b(\phi_i, \phi_j) = l(\phi_i) - b(\phi_i, \phi_0). \tag{8.56}$$

This equation is a system of n linear algebraic equations in n unknowns c_j and has a unique solution if the coefficient matrix in (8.56) is non-singular and thus has an inverse.

The functions ϕ_i must satisfy the following requirements: (i) ϕ_i should be well-defined such that $b(\phi_i, \phi_j) \neq 0$, (ii) ϕ_i must satisfy at least the essential homogeneous boundary condition, (iii) the set $\{\phi_i\}_{i=1}^n$ must be linearly independent, and (iv) the set $\{\phi_i\}_{i=1}^n$ must be complete.

EXAMPLE 8.5. Consider the Bessel equation

$$x^2u'' + xu' + (x^2 - 1)u = 0, \quad u(1) = 1, u(2) = 2.$$

Put $u = v + x$. Then the given equation and the boundary conditions become

$$x^2v'' + xv' + (x^2 - 1)v + x^3 = 0, \quad v(1) = 0 = v(2).$$

In the self-adjoint form this equation is written as

$$xv'' + v' + \frac{x^2-1}{x}v + x^2 = 0.$$

For the 1st approximation, we take

$$v_1 = a_1\phi_1 = a_1(x-1)(x-2).$$

Then using (8.46) we get $\int_1^2 (Lv_1 - f)\phi_1\,dx = 0$, which gives

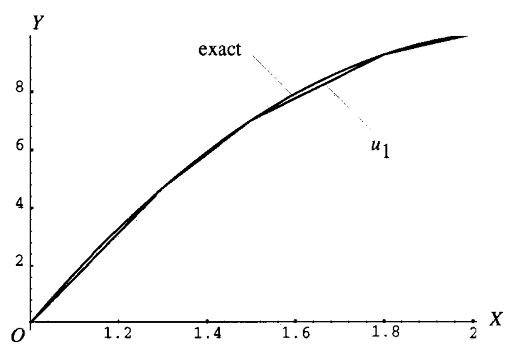
$$\int_1^2 \left[2a_1x - (3-2x)a_1 + \frac{x^2-1}{x}(x-1)(x-2)a_1 + x^2\right](x-1)(x-2)\,dx = 0,$$

which, on integration, yields $a_1 = -0.811$, and thus,

$$u_1 = v_1 + x = -0.811(x-1)(x-2) + x.$$

The exact solution is $u = c_1J_1(x) + c_2Y_1(x)$, where $c_1 = 3.60756$, $c_2 = 0.75229$. A comparison with the exact solution in the following table shows that u_1 is a good approximation:

x	u_1	u_{exact}
1.3	1.4703	1.4706
1.5	1.7027	1.7026
1.8	1.9297	1.9294



■

EXAMPLE 8.6. Consider the 4th–order equation

$$[(x+2l)u'']'' + bu - kx = 0, \quad 0 < x < l,$$

with the boundary conditions: $u(l) = 0 = u'(l)$, $(x+2l)u''(0) = 0$, $[(x+2l)u'']'(0) = 0$. We choose the test functions

$$\phi_1(x) = (x-l)^2(x^2 + 2lx + 3l^2)$$
$$\phi_2(x) = (x-l)^3(3x^2 + 4lx + 3l^2).$$

For the 1st approximation, we have $u_1 = a_1\phi_1(x)$. Then $\int_0^l (Lu_1 - f)\phi_1(x)\,dx = 0$, which gives

$$a_1(24l + 57.6 + 161bl^4/315 + 9ql^2/5) + kl/3 = 0.$$

If, e.g., we take $l = b = 1$, and $k = 3$, then $a_1 = 0.011917$, and

$$u_1 = 0.011917(x-1)^2(x^2 + 2x + 3).$$

For the 2nd approximation, we take $u_2 = a_1\phi_1(x) + a_2\phi_2(x)$. Then

$$\int_0^l (Lu_2 - f)\phi_1\,dx = 0, \quad \text{and} \quad \int_0^l (Lu_2 - f)\phi_2(x)\,dx = 0$$

which, with $l = b = 1$, $k = 3$, yield

$$83.911a_1 - 67.313a_2 = 1$$
$$67.213a_1 - 91.882a_2 = 0.7143.$$

Thus $a_1 = 0.013743$, $a_2 = 0.002279$, and

$$u_2 = 0.013743(x-1)^2(x^2 + 2x + 3) + 0.002279(x-1)^3(3x^2 + 4x + 3).$$

Instead of determining the exact solution, we can compare u_1 and u_2. Thus, e.g., $u_1(0.5) = 0.012662$, and $u_2(0.5) = 0.012964$, which give good results. ■

8.7. Choice of Test Functions

Note that a suitable choice of the test functions $\phi_i(x, y)$ can be made by taking linear combinations of polynomials, or trigonometric functions, such that they satisfy the boundary conditions. For example, we can choose a system of functions

$$\phi_0 = g, \quad \phi_1 = gx, \quad \phi_2 = gy, \quad \phi_3 = gx^2, \quad \phi_4 = gxy, \dots, \qquad (8.57)$$

where $g = g(x, y)$. It can be shown that the system (8.57) is complete.

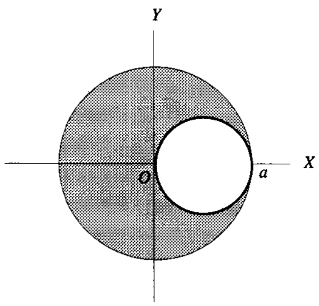


Fig. 8.3.

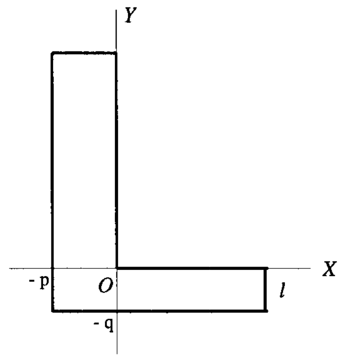


Fig. 8.4.

Some practical rules for constructing the functions $g(x, y)$ in (8.57) are as follows:
(i) For the rectangle $[-a, a; -b, b]$:

$$g(x, y) = (x^2 - a^2)(y^2 - b^2).$$

(ii) For a circle of radius r and center at origin:

$$g(x, y) = r^2 - x^2 - y^2.$$

(iii) If the boundary Γ of a region Ω is defined by $F(x, y) = 0$, where $F \in C^n$, then

$$g(x, y) = \pm F(x, y).$$

See (ii) above if Γ is a circle.
(iv) For the case of a convex polygon whose sides are defined by $a_1 x + b_1 y + c_1 = 0, \cdots, a_m x + b_m y + c_m = 0$, we have

$$g(x, y) = \pm(a_1 x + b_1 y + c_1) \cdots (a_m x + b_m y + c_m).$$

See (i) above for a rectangle.
(v) The choice in (iv) is also suitable in different types of regions bounded by curved lines; e.g., for a sector formed by the circles of radii r and $r/2$, as in Fig. 8.3, we have

$$g(x, y) = (r^2 - x^2 - y^2)(x^2 - rx + y^2).$$

(vi) For nonconvex polygons, the function $g(x, y)$ must be assigned piecewise in different parts of the region, and we must introduce moduli for any re–entrant angles. Thus, for the region in Fig. 8.4,

$$\begin{aligned} g(x, y) &= \big(|x| + |y| - x - y\big)(x + p)(y + q)(l - x)(h - y) \\ &= \begin{cases} -2y(x + p)(y + q)(l - x)(h - y), & \text{in } [0, l; -q, 0] \\ -2(x + y)(x + p)(y + q)(l - x)(h - y), & \text{in } [-p, 0; -q, 0] \\ -2x(x + p)(y + q)(l - x)(h - y) & \text{in } [-p, 0; 0, h]. \end{cases} \end{aligned}$$

We can also take (2nd choice)

$$g(x, y) = \big(x^2 + y^2 - x|x| - y|y|\big)(x + p)(y + q)(l - x)(h - y).$$

In this case $g \in C^1$. A third choice is to assign the functions $u_n(x, y)$ separately in the three parts of the corner region of Fig. 8.4:

$$u_n(x, y) = \begin{cases} (x + p)x(h - y)(a_1 + a_2 x + a_3 y + \cdots + a_n y^m) \\ \qquad \text{in } [-p, 0; 0, h] \\ (x + p)(y + q)(b_1 + b_2 x + b_3 y + \cdots + b_n y^m) \\ \qquad \text{in } [-p, 0; -q, 0] \\ (y + q)(l - x)(c_1 + c_2 x + c_3 y + \cdots + c_n y^m) \\ \qquad \text{in } [0, l; -q, 0], \end{cases}$$

where a_k, b_k, c_k $(k = 1, \cdots, n)$ are parameters which must be connected by the following condition on the axes $x = 0$ and $y = 0$:

$$\begin{aligned} (x + p)xh(a_1 + a_2 x + a_4 x^2 + \cdots) &= (x + p)q(b_1 + b_2 x + b_4 x^2 + \cdots) \\ p(y + q)(b_1 + b_2 y + \cdots + b_n y^m) &= (y + q)yl(c_1 + c_2 y + \cdots + c_n y^m). \end{aligned}$$

In view of the above considerations, the test functions $\phi_i(x, y)$ are also called the shape functions for the region Ω.

EXAMPLE 8.7. Torsion of a prismatic rod of rectangular cross–section of length $2a$ and width $2b$ is defined by

$$\nabla^2 u = 2, \quad u = 0 \quad \text{on } \Gamma.$$

For the rectangle shown in Fig. 8.5, we choose

$$\phi(x, y) = (a^2 - x^2)(b^2 - y^2),$$

and seek an approximate solution of the form

$$u_n(x,y) = (a^2-x^2)(b^2-y^2)(A_1 + A_2x^2 + A_3y^2 + \cdots + A_n x^{2i}y^{2j}).$$

First, for $n=1$, we use (8.46a) with $f=2$ and find that (with $\phi(x) = (a^2-x^2)(b^2-x^2)$)

$$\begin{aligned} 0 &= \iint_\Omega \Big(-\frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^2 u_1}{\partial y^2} - 2\Big)\phi\,dx\,dy \\ &= 2\int_{-a}^{a}\int_{-b}^{b} \big[1 - A_1(a^2-x^2) - A_1(b^2-y^2)\big](a^2-x^2)(b^2-y^2)\,dy\,dx \\ &= -\frac{128}{45}a^3b^3(a^2+b^2)A_1 + \frac{32}{9}a^2b^3. \end{aligned}$$

Thus, $A_1 = 5/4(a^2+b^2)$, and

$$u_1 = \frac{5}{4}\frac{(a^2-x^2)(b^2-y^2)}{a^2+b^2}.$$

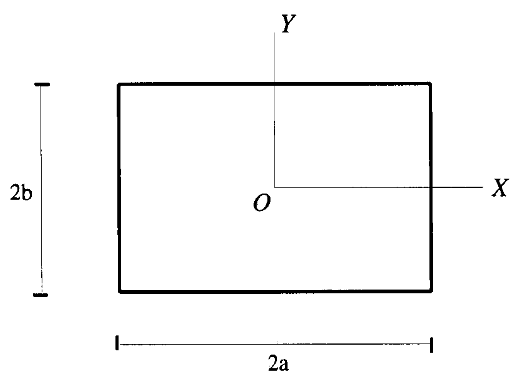


Fig. 8.5.

The torsional moment

$$M = 2G\theta \int_{-a}^{a}\int_{-b}^{b} u_1\,dy\,dx = \frac{40}{9}\frac{G\theta a^3b^3}{a^2+b^2},$$

where G is the shear modulus, and θ is the angle of twist per unit length. The tangential stresses τ_{zx} and τ_{zy} are given by

$$\tau_{zx} = G\theta\frac{\partial u_1}{\partial y}, \quad \tau_{zy} = G\theta\frac{\partial u_1}{\partial x}.$$

For $a=b$, we find that $M = 20G\theta a^4/9 \approx 0.1388(2a)^4G\theta$. The exact classical solution is given by

$$u = ax - x^2 - \frac{8a^2}{\pi^3}\sum_{n=1}^{\infty}\frac{\cosh\frac{(2n-1)\pi y}{2a}}{(2n-1)^3\cosh\frac{(2n-1)\pi b}{a}}\sin\frac{(2n-1)\pi x}{a},$$

which gives

$$M = 2G\theta\Big\{\frac{a^3b}{6} - \frac{32a^4}{\pi^5}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^5}\tanh\frac{(2n-1)\pi b}{2a}\Big\}.$$

For $a=b$, the exact value of M is $0.1406(2a)^4G\theta$, which compares very well with the approximate value obtained above by the Galerkin method. ■

EXAMPLE 8.8. Solve $\nabla^4 u = 0$ on the rectangle $[-a,a;-b,b]$ (Fig. 8.5) under the boundary conditions

$$\frac{\partial^2 u}{\partial x\partial y} = 0, \quad \frac{\partial^2 u}{\partial y^2} = c\Big(1 - \frac{y^2}{b^2}\Big) \quad \text{at } x = \pm a,$$

$$\frac{\partial^2 u}{\partial x\partial y} = 0, \quad \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{at } y = \pm b,$$

where c is a constant. This problem pertains to the expansion of a rectangular plate under tensile forces.

First, we will reduce the above boundary conditions to homogeneous boundary conditions: The function

$$u_0 = \frac{c}{2}y^2\Big(1 - \frac{y^2}{6b^2}\Big)$$

obviously satisfies the given boundary conditions (it follows by integrating each one of the above boundary conditions). Set $u = u_0 + \hat u$. Then $\nabla^4\hat u = 2c/b^2$, and the boundary conditions become

$$\frac{\partial \hat u}{\partial x\partial y} = 0, \quad \frac{\partial \hat u}{\partial y^2} = 0 \quad \text{at } x = \pm a,$$

$$\frac{\partial^2 \hat u}{\partial x\partial y} = 0, \quad \frac{\partial^2 \hat u}{\partial x^2} = 0 \quad \text{at } y = \pm b.$$

These boundary conditions will be satisfied if the following conditions are met:

$$\hat{u} = 0, \quad \frac{\partial \hat{u}}{\partial x} = 0 \quad \text{at } x = \pm a,$$
$$\hat{u} = 0, \quad \frac{\partial \hat{u}}{\partial y} = 0 \quad \text{at } y = \pm b.$$

Thus, the given problem reduces to that of minimizing the integral

$$I(u) = \iint_\Omega \left[\left(\nabla^2 \hat{u}\right)^2 - \frac{4c}{b^2} \hat{u} \right] dx\, dy.$$

Then, by Rayleigh–Ritz (or Galerkin) method

$$\iint_\Omega \left(\nabla^4 u_n - f\right) \phi_j \, dx\, dy = 0, \quad j = 1, \dots, n, \tag{8.58}$$

where u_n is the n–th approximate solution, which, in view of the geometric symmetry of the rectangle, is taken as

$$u_n = (x^2 - a^2)^2 (y^2 - b^2)^2 (a_1 + a_2 x^2 + a_3 y^2 + \cdots).$$

For $n = 1$, we find from (8.58) that

$$\int_{-a}^{a} \int_{-b}^{b} [24 a_1 (y^2 - b^2)^2 + 16 a_1 (3x^2 - a^2)(3y^2 - b^2)$$
$$+ 24 a_1 (x^2 - a^2)^2 - \frac{2c}{b}](x^2 - a^2)^2 (y^2 - b^2)^2 \, dy\, dx = 0,$$

or

$$\left(\frac{54}{7} + \frac{256}{49}\frac{b^2}{a^2} + \frac{64}{7}\frac{b^4}{a^4}\right) a_1 = \frac{c}{a^4 v^2},$$

which gives $a_1 = 0.043253 c / a^6$, and

$$u_1 = u_0 + \hat{u}_1 = \frac{c}{2} y^2 \left(1 - \frac{y^2}{6b^2}\right) + \frac{0.04253c}{a^6} (x^2 - a^2)^2 (y^2 - b^2)^2. \blacksquare$$

8.8. Transient Problems

For time-dependent problems the semi–discrete formulation is used to choose the basis functions. Thus, for one–dimensional problems the N–th approximate solution is taken as

$$\tilde{u}_N(x,t) = \phi_0 + \sum_{j=1}^{N} c_j(t)\, \phi_j(x), \tag{8.59}$$

where, as before, the functions ϕ_j satisfy the homogeneous boundary conditions and ϕ_0 is chosen as in (8.55). Then, using the Galerkin or Rayleigh–Ritz method such that the residual is orthogonal to the first N basis functions ϕ_i, $i = 1, 2, \cdots, N$, we obtain the N first order ordinary differential equations in t. For example, for the diffusion equation $u_t = \nabla^2 u$, this system is

$$\sum_{j=1}^{N} \dot{c}_j(t) \langle \phi_j, \phi_i \rangle = \sum_{j=1}^{N} c_j(t) \langle \phi_j, \phi_i \rangle + \langle \phi_j, \phi_0 \rangle,$$

where the dot denotes the time derivative. The initial conditions for this system are subject to another Galerkin approximation such that its residual $R = u(x,0) - \tilde{u}_N(x,0)$ is orthogonal to the first N basis functions ϕ_j. This yields the system of N algebraic equations

$$\sum_{j=1}^{N} c_j(0) \langle \phi_j, \phi_i \rangle = \langle \phi_j, u(x,0) - \phi_0(r) \rangle,$$

which is generally solved for the unknowns $c_i(0)$ by numerical methods.

EXAMPLE 8.9. Consider the heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{1}{r}\frac{\partial u}{\partial r}, \quad 0 < r < 1,$$

subject to the boundary conditions $u_r(0,t) = 0$, $u(1,t) = 0$, and the initial condition $u(r,0) = \ln r$. For the first order approximation, we can take the basis function as $\phi_1(r) = c_0 + c_1 r + c_2 r^2$. To determine the coefficients c_0, c_1, and c_2 we require that ϕ_1 satisfy the boundary conditions of the problem. Thus, $\phi_1(1) = c_0 + c_1 + c_2 = 0$, and $\dfrac{\partial \phi_1}{\partial r} = c_1 + 2c_2 a = 0$. By solving these two equations in terms of c_0, we find that $c_1 = 2ac_0/(1 - 2a)$, and $c_2 = -c_0/(1 - 2a)$. If we take $c_0 = 1 - 2a$, then $c_1 = 2a$, and $c_2 = -1$, and the basis function becomes $\phi_1(r) = 1 - 2a + 2ar - r^2$, or $\phi_1(r) = 1 - b + br - r^2$, with $b = 2a$. This suggests that for the N-th approximation we should choose the basis functions as

$$\phi_j(r) = 1 - b_j + b_j r - r^{j+1}, \quad b_j = (j+1)a^j, \quad j = 1, 2, \cdots, N,$$

with $\phi_0 = 0$. The N–th order approximate solution is then taken in the semi–discrete form as

$$\tilde{u}_N(r,t) = \sum_{j=1}^{N} c_j(t)\, \phi_j(r).$$

The residual is given by

$$\sum_{j=1}^{N}\left\{\dot c_j(t)\,\phi_j + c_j(t)\left[(j+1)^2 r^{j-1} - \frac{b_j}{r}\right]\right\}.$$

Then for the Galerkin method

$$\begin{aligned} 0 &= \sum_{j=1}^{N}\int_0^1 \left\{\dot c_j(t)\,\phi_j + c_j(t)\left[(j+1)^2 r^{j-1} - \frac{b_j}{r}\right]\right\} \\ &\qquad \cdot \left(1 - b_i + b_i r - r^{i+1}\right\}\, r\,dr \\ &= \sum_{j=1}^{N}\int_0^1 \left\{\dot c_j(t)\, f(i,j) + c_j(t)\, g(i,j)\right\}, \end{aligned}$$

for $i = 1, 2, \cdots, N$, where

$$\begin{aligned} f(i,j) &= \frac{(1-b_i)(1-b_j)}{2} + \frac{b_i + b_j - b_i b_j}{3} + \frac{b_i b_j}{4} - \frac{1-b_j}{i+3} \\ &\quad - \frac{1-b_i}{j+3} - \frac{b_j}{i+4} - \frac{b_i}{j+4} + \frac{1}{i+j+4}, \\ g(i,j) &= (j+1)^2\left[\frac{1-b_i}{j+1} + \frac{b_i}{j+2} - \frac{1}{i+j+2}\right] - b_j + \frac{b_i b_j}{2} + \frac{b_j}{i+2}. \end{aligned}$$

The initial condition is, in general, satisfied approximately. This is accomplished by requiring that the residual

$$R = \sum_{j=1}^{N} c_j(0)\,\phi_j(r) - \ln r$$

be orthogonal to the basis functions $\phi_i(r)$, i.e., $\langle \phi_i, R\rangle = 0$ for $i = 1, 2, \cdots, N$. This means that

$$\lim_{\varepsilon\to 0}\int_\varepsilon^1 R\,\phi_i(r)\,r\,dr = 0 \quad \text{for } i = 1, 2, \cdots, N,$$

since R has a logarithmic singularity at $r = 0$. After evaluating this improper integral, we obtain a system of N algebraic equations:

$$\sum_{j=1}^{N} c_j(0) f(i,j) = \frac{1}{4} + \frac{5}{36}\,b_j + \frac{1}{(i+3)^2}, \quad i = 1, 2, \cdots, N,$$

which can be solved for the unknowns $c_j(0)$. ■

8.9. Other Methods

There are other weighted residual methods, which we will mention in the sequel. These methods are are not frequently used, and so we will not present any examples. Interested readers will find detailed information on these methods in Connor and Brebbia (1973), Davies (1980), Kantorovitch and Krylov (1958), and Reddy (1984).

8.9.1. Collocation method. In this method, the trial function

$$u_n = \sum_i^n c_i \phi_i$$

is chosen to satisfy the boundary conditions, and the parameters c_i are determined by forcing u_n to satisfy the differential equation at a prescribed set of points, i.e., the residual r vanishes at these points.

8.9.2. Least–square method. This method is applied directly to the residual. The trial functions are chosen to satisfy the boundary conditions. The residual is minimized by choosing the parameters c_i such that the functional

$$I(u) = \iint_\Omega \{r(u)\}^2\,dx\,dy$$

is a minimum. Thus $\partial I/\partial c_i = 0$, $i = 1, \cdots, n$. Since

$$I(c_1, \cdots, c_n) = \iint_\Omega \left\{L\left(\sum c_i\phi_i\right) - f\right\}^2 dx\,dy$$

and L is linear, we get

$$\frac{\partial I}{\partial c_i} = 2\iint_\Omega \left\{L\left(\sum c_i\phi_i\right) - f\right\} L\phi_i\,dx\,dy = 0,$$

which implies

$$\sum_{i=1}^{n} c_i \iint_\Omega \left\{L\phi_i\, L\phi_i\right\} dx\,dy = \iint_\Omega f L\phi_i\,dx\,dy, \quad i = 1, \cdots, n.$$

8.9.3. Method of moments. In the equation $\langle r, w_i \rangle = 0$, where r is the residual and w_i are the weight functions, we can use any linearly independent and complete set of weight functions w_i. The simplest choice for 1–D problems is the set $\{1, x, x^2, x^3, \cdots\}$. Then the successive higher moments of the residual are forced to vanish, i.e.,

$$< r, x^i >= 0, \qquad i = 0, 1, 2, \cdots.$$

This scheme is called the method of moments.

8.10. Exercises

8.1. Fill in the details in the derivation of the Euler equation (8.13).
ANS. From (8.11), integrating by parts twice and using $\phi(x)\frac{\partial F}{\partial U'}\Big|_a^b = 0$, $\phi'(x)\frac{\partial F}{\partial U'} - \phi(x)\frac{d}{dx}\left(\frac{\partial F}{\partial U''}\right)\Big|_a^b = 0$, we get

$$0 = \frac{\partial I}{\partial \alpha} = \int_a^b \phi(x)\left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial u''}\right)\right] dx.$$

8.2. Fill in the details in the derivation of the Euler equation (8.15).
ANS. Introduce two functions $\phi(x)$ and $\psi(x)$, and two parameters α and β, respectively, such that $U = u + \alpha\phi(x)$, $V = v + \beta\psi(x)$. Then $\partial I/\partial\alpha = 0$ and $\partial I/\partial\beta = 0$ lead to (8.15).

8.3. Find the geodesics for the following problems:
(a) On the xy–plane, take $I = \int ds = \int \sqrt{1 + y'^2}\, dx$.
(b) On the xy–plane, take $I = \int ds = \int \sqrt{1 + r^2 (d\theta/dr)^2}\, dr$.
(c) On the cylinder $x^2 + y^2 = a^2$, $-\infty < z < \infty$, take $x = a\cos t$, $y = a \sin t$, and $I = \int ds = \int \sqrt{a^2 + (dz/dt)^2}\, dt$.
ANS. (a) Straight lines $y = c_1 x + c_2$; (b) Straight lines $r\cos(\theta - c_1) = c_2$; (c) $z = c_1 t + c_2$.

8.4. A ray of light moves between two fixed points in the xy–plane with variable velocity $v(x, y)$. By Fermat's law, its travel time is $\int \frac{ds}{v} = \int \frac{\sqrt{1 + y'^2}}{v}\, dx$. Show that the paths for a minimum travel time are given by

$$\frac{vy''}{\sqrt{1 + y'^2}} - \frac{\partial v}{\partial x} y' + \frac{\partial v}{\partial y} = 0.$$

8.5. Find the extremal when the following integral is minimized:
(a) $\int (y'^2 + y^2)\, dx$.
(b) $\int (y''^2 + y^2)\, dx$.
ANS. (a) $y = c_1 e^x + c_2 e^{-x}$; (b) $y = \left(c_1 e^{x/\sqrt{2}} + c_2 e^{-x/\sqrt{2}}\right)\cos(y/\sqrt{2}) + \left(c_3 e^{x/\sqrt{2}} + c_4 e^{-x/\sqrt{2}}\right)\sin(y/\sqrt{2})$.

8.6. If the end points are not fixed, derive

$$\int_a^b F(x, u, u')\, dx = \delta u \frac{\partial F}{\partial u'}\Big|_a^b + \int_a^b \left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right]\delta u\, dx.$$

8.7. Take $\delta u = \alpha\phi(x) + \beta\psi(x)$ in the integral (8.1), and let δI denote the total differential of I at $\alpha = 0 = \beta$, where $\delta\alpha = \alpha$, $\delta\beta = \beta$. Set $H \equiv \frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)$, and show that the variation δI in this case can be written as

$$\delta I = d\alpha \int_a^b H\phi\, dx + d\beta \int_a^b H\psi\, dx.$$

8.8. Find the extremal for the problem of determining a curve Γ of prescribed length l joining AB and maximizing the area $A = \int y\, dx$, bounded by Γ, x–axis and two fixed ordinates.
ANS. $(x - c_1)^2 + (y - c_2)^2 = k^2$, where c_1, c_2, and k make the arc Γ pass through A and B and have length l.

Derive the variational formulation for the boundary value problems **8.9–8.14** (here a, b, f, g are functions of x; u_0, h_0, m_0, q_0, T_∞, u_∞ are constants):

8.9. $-\frac{d}{dx}\left(a\frac{du}{dx}\right) - f = 0$, $u(0) = u_0$, $a\frac{du}{dx}(l) = q_0$, $0 < x < l$ (one–dimensional heat conduction).
ANS.

$$b(w, u) = \int_0^1 a \frac{dw}{dx}\frac{du}{dx}\, dx,$$

$$l(w) = \int_0^1 wf\, dx - w(0)\left[a\frac{du}{dx}\right]_{x=0} + q_0 w(l).$$

8.10. $-\frac{d}{dx}\left(a\frac{du}{dx}\right) - cu + x^2 = 0$; $u(0) = 0$, $a\frac{du}{dx}(1) = 1$, $0 < x < 1$ (one–dimensional deformation of a bar).

ANS.

$$b(w,u) = \int_0^1 \left(a\frac{dw}{dx}\frac{du}{dx} - cwu \right) dx, \quad l(w) = -\int_0^1 wx^2\,dx + w(1).$$

8.11. $-\dfrac{\partial}{\partial x}\Big(c_{11}\dfrac{\partial u}{\partial y} + c_{12}\dfrac{\partial u}{\partial x}\Big) - \dfrac{\partial}{\partial y}\Big(c_{21}\dfrac{\partial u}{\partial x} + c_{22}\dfrac{\partial u}{\partial y}\Big) + f = 0$ in Ω with boundary conditions $u = u_0$ on Γ_1, and $q_n \equiv \Big(c_{11}\dfrac{\partial u}{\partial x} + c_{12}\dfrac{\partial u}{\partial y}\Big)n_x + \Big(c_{21}\dfrac{\partial u}{\partial x} + c_{22}\dfrac{\partial u}{\partial y}\Big)n_y = q_0$, on Γ_2, where c_{ij}, u_0, and q_0 are prescribed.

ANS.

$$B(w,u) = \iint_\Omega \left[\frac{dw}{dx}\left(c_{11}\frac{du}{dx} + c_{12}\frac{du}{dy} \right) + \frac{dw}{dx}\left(c_{21}\frac{du}{dx} + c_{22}\frac{du}{dy} \right) + wf \right] dx\,dy,$$

$$l(w) = \int_\Gamma wq_n\,ds,$$

where $q_n = \left(c_{11}\dfrac{du}{dx} + c_{12}\dfrac{du}{dy} \right) n_x + \left(c_{21}\dfrac{du}{dx} + c_{22}\dfrac{du}{dy} \right) n_y$.

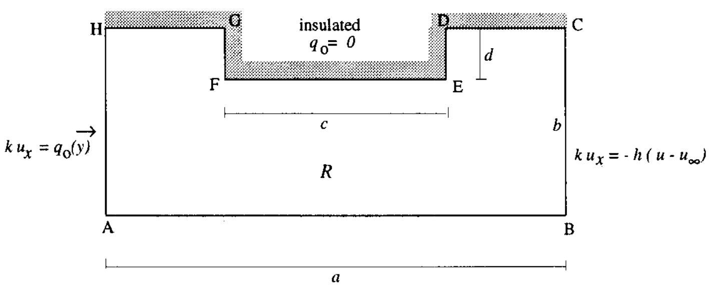


Fig. 8.6.

8.12. $-k\Big(\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2}\Big) = f$ in the region Ω with boundary conditions as shown in Fig. 8.6. The following boundary conditions are prescribed: $ku_x = q_0(y)$ on HA; $ku_x = -h(u - u_\infty)$ on BC; $u = u_0(x)$ on AB, and $\partial u/\partial n = q_0 = 0$ on $CDEFGH$ (insulated), where k is the thermal conductivity of the material of the region Ω, h and u_∞ are ambient quantities, and $\partial u/\partial n = -\partial u/\partial x = -u_x$ on HA (two–dimensional heat conduction).

ANS.

$$0 = -\iint_\Omega k\left(\frac{d^2T}{dx^2} + \frac{d^2T}{dy^2} \right) w\,dx\,dy,$$
$$= \iint_\Omega k\left(\frac{dw}{dx}\frac{dT}{dx} + \frac{dw}{dy}\frac{dT}{dy} \right) dx\,dy - \int_C kw\left(\frac{dT}{dx}n_x + \frac{dT}{dy}n_y \right) ds.$$

The boundary conditions on $C_1 = AB$ (prescribed temperature T_0): $n_x = 0$, $n_y = -1$; on $C_2 = BC$ (convective boundary, T_∞): $n_x = 1$, $n_y = 0$; on $C_3 = CDEFGH$ (insulated boundary): $q = \partial T/\partial n = 0$; and on $C_4 = HA$ (prescribed conduction $q_0(y)$): $n_x = -1$, $n_y = 0$. Thus,

$$b(w,u) = \iint_\Omega k\left(\frac{dw}{dx}\frac{dT}{dx} + \frac{dw}{dy}\frac{dT}{dy} \right) dx\,dy + h\int_0^b w(a,y)T(a,y)\,dy,$$
$$l(w) = -\int_0^b w(0,y)q_0(y)\,dy + hT_\infty \int_0^b w(a,y)\,dy.$$

8.13. $-\dfrac{d}{dx}\left\{ a\left[\dfrac{du}{dx} + \dfrac{1}{2}\Big(\dfrac{dv}{dx}\Big)^2 \right] \right\} + g = 0,$

$\dfrac{d^2}{dx^2}\Big(b\dfrac{d^2v}{dx^2}\Big) - \dfrac{d}{dx}\left\{ a\dfrac{dv}{dx}\left[\dfrac{du}{dx} + \dfrac{1}{2}\Big(\dfrac{dv}{dx}\Big)^2 \right] \right\} + f = 0;$

$u = v = 0$ at $x = 0, l$; $\dfrac{dv}{dx}\Big|_{x=0} = 0$, $\Big[b\dfrac{d^2v}{dx^2}\Big]_{x=l} = m_0$ (large-deflection bending of a beam).

ANS. Let w_1 and w_2 be the two test functions, one for each equation, such that they satisfy the essential boundary conditions on u and v. Then

$$0 = \int_0^l \left[a\frac{dw_1}{dx}\left\{ \frac{du}{dx} + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 \right\} + w_1 g \right] dx,$$

$$0 = \int_0^l \left[b\frac{d^2w_2}{dx^2}\frac{d^2v}{dx^2} + a\frac{dw_2}{dx}\frac{dv}{dx}\left\{ \frac{du}{dx} + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 \right\} + w_2 f \right] dx$$
$$- m_0\frac{dw_2}{dx}(l).$$

Thus,

$$b((w_1,w_2),(u,v)) = \int_0^l \left[a\frac{dw_1}{dx}\left\{ a\frac{du}{dx} + \frac{1}{2}\Big(\frac{dv}{dx}\Big)^2 \right\} + b\frac{d^2w_2}{dx^2}\frac{d^2v}{dx^2} + a\frac{dw_2}{dx}\frac{dv}{dx}\left\{ \frac{du}{dx} + \frac{1}{2}\Big(\frac{dv}{dx}\Big)^2 \right\} \right] dx,$$

$$l((w_1,w_2)) = -\int_0^l \Big(w_1 g + w_2 f \Big) dx + m_0\frac{dw_2}{dx}(l),$$

$$I[(u,v)] = \int_0^l \left[\frac{1}{2}\left\{a\left(\frac{du}{dx}\right)^2 + b\left(\frac{d^2v}{dx^2}\right)^2 + a\frac{du}{dx}\left(\frac{dv}{dx}\right)^2 + \frac{a}{4}\left(\frac{dv}{dx}\right)^4\right\} + \right.$$
$$\left. + w_1 g + w_2 f\right] dx - m_0 \frac{dw_2}{dx}(l).$$

8.14. Find the functional $I(u)$ for the transverse deflection u of a membrane stretched across a frame, in the shape of a curve C, subjected to a pressure loading $f(x,y)$ per unit area. Assume that the tension T in the membrane is constant. Note that u satisfies the equation

$$-\nabla^2 u = \frac{f}{T}.$$

ANS. The variation of the total work done by the force f/T is

$$\begin{aligned}
\delta \iint_\Omega \frac{fu}{T}\, dx\, dy &= \iint_\Omega \frac{f\delta u}{T}\, dx\, dy \\
&= -\iint_\Omega \nabla^2 u\, \delta u\, dx\, dy \\
&= -\iint_\Omega [\nabla\cdot(\nabla u \delta u) - \nabla(\delta u)\cdot\nabla u]\, dx\, dy \\
&= \frac{1}{2}\iint_\Omega \delta|\nabla u|^2\, dx\, dy - \int_C \frac{\partial u}{\partial n}\delta u\, ds,
\end{aligned}$$

which leads to

$$I[u] = \frac{1}{2}\iint_\Omega \{|\nabla u|^2 - 2fu\}\, dx\, dy.$$

Note that Example 8.1 is used. Thus,

$$\begin{aligned}
\nabla\cdot(\nabla u \delta u) &= \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y}\right)\cdot\left(\hat{\mathbf{i}}\frac{\partial u}{\partial x}\delta u + \hat{\mathbf{j}}\frac{\partial u}{\partial y}\delta u\right) \\
&= \nabla^2 u \delta u = -\frac{f}{T}\delta u.
\end{aligned}$$

8.15. Consider the Poisson boundary value problem: $-\nabla^2 u = f$ in Ω, with the boundary conditions $u = 0$ on C_1 and $\partial u/\partial n = 0$ on C_2. Show that

$$I(u) = \frac{1}{2}\iint_\Omega \left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 - 2fu\right\} dx\, dy.$$

ANS.

$$\begin{aligned}
b(w,u) &= \iint_\Omega \left(\frac{\partial w}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial u}{\partial y}\right) dx\, dy, \\
l(w) &= \iint_\Omega fw\, dx\, dy - \int_C w\frac{\partial u}{\partial n}\, ds.
\end{aligned}$$

8.16. Use the Galerkin method, and the Rayleigh–Ritz method, to solve: $\dfrac{d^2}{dx^2}\left(EI\dfrac{d^2u}{dx^2}\right) + f = 0$, $0 < x < l$, $EI > 0$, $f = \text{const}$, where EI is called the flexural rigidity of the beam, with

$$u(0) = 0 = \frac{du}{dx}(0), \quad EI\frac{d^2u}{dx^2}\bigg|_{x=l} = M_0, \quad \frac{d}{dx}\left(EI\frac{d^2u}{dx^2}\right)\bigg|_{x=l} = 0.$$

[Take $w = \phi_i = x^{i+1}$. The exact solution is

$$EI\, u = \frac{fl^4}{24} - \frac{fl^3}{6}x + \frac{M_0}{2}x^2 - \frac{f}{24}(l-x)^4.]$$

ANS.

$$\begin{aligned}
b(w,u) &= \int_0^l EI\frac{d^2w}{dx^2}\frac{d^2u}{dx^2}\, dx, \\
l(w) &= -\int_0^l wf\, dx + w(0)\left[\frac{d}{dx}\left(EI\frac{dw}{dx}\frac{du}{dx}\right)\right]_{x=0} - \\
&\quad - \left[\frac{dw}{dx}\right]_{x=0}\left[EI\frac{du}{dx}\right]_{x=0} - f_0 w(l) + m_0\left[\frac{dw}{dx}\right]_{x=l}.
\end{aligned}$$

Exact solution is obtained by direct integration.

8.17. $-\nabla^2 u = 1$ in $\Omega = \{(x,y): 0 < x, y < 1\}$ such that

$$u(1,y) = 0 = u(x,1), \quad \frac{\partial u}{\partial n}(0,y) = 0 = \frac{\partial u}{\partial n}(x,0).$$

ANS. If we take $w = \phi_i = (1-x^i)(1-y^i)$, $i = 1,\cdots,n$, then this choice satisfies the essential boundary conditions, but not the natural boundary conditions. Hence, we assume the first approximate solution as $u_1 = a\phi_1$, $\phi_1 = (1-x^2)(1-y^2)$. Alternatively, we can take $w = \phi_i = \cos\dfrac{(2i-1)\pi x}{2}\cos\dfrac{(2i-1)\pi y}{2}$, $i = 1,\cdots,n$. The exact solution is

$$\begin{aligned}
u(x,y) &= \frac{1}{2}\Big\{(1-y^2) \\
&\quad + \frac{32}{\pi^3}\sum_{k=1}^\infty \frac{(-1)^k \cos[(2k-1)\pi y/2]\cosh[(2k-1)\pi x/2]}{(2k-1)^3\cosh[(2k-1)\pi/2]}\Big\}.
\end{aligned}$$

8.18. Find the N–th approximate solution of Example 8.4 by taking the basis functions as $\phi_{jk}(x,y) = \sin\frac{j\pi x}{a}\,\sin\frac{k\pi y}{b}$.
ANS.

$$\sum_{j,k=1}^{N} \alpha_{jk}\,\frac{ab\pi^2}{4}\left(\frac{j^2}{a^2}+\frac{k^2}{b^2}\right) = \frac{4cab}{jk\pi^2}$$

for both j,k odd.

8.19. Find the approximate solution by the Galerkin method for the nonlinear problem $u_t = u_{xx} + \varepsilon u^2$ on $0 < x < 1$, subject to the boundary conditions $u(0,t) = 0 = u(1,t)4$ and the initial condition $u(x,0) = 1$.
HINT. Choose $\phi_j(x) = \sin j\pi x$.
ANS.

$$\sum_{j,k=1}^{N}\left\{\frac{1}{2}\dot{c}_j(t) + \frac{1}{2}j^2\pi^2 c_j(t) - \varepsilon\Big[\frac{2(1-\cos j\pi)}{3j\pi}\,c_j^2(t) + \frac{1}{2\pi}\sum_{m\neq n} c_m(t) f(m,n,j)\Big]\right\},\quad m,n = 1,2,\cdots,N,$$

where

$$f(m,n,j) = \frac{1-\cos(m-n+j)\pi}{m-n+j} - \frac{1-\cos(m-n-j)\pi}{m-n-j} - \frac{1-\cos(m+n+j)\pi}{m+n+j} + \frac{1-\cos(m+n-j)\pi}{m+n-j}.$$

To find $c_j(0)$, solve $\langle \phi_j, R\rangle = 0$, where $R = \sum_{j,k=1}^{N} c_j(0)\,\phi_j(x) - 1$.

8.20. Use the Galerkin method to solve the Poisson equation $\nabla^2 u = 2$ subject to the Dirichlet condition $u = 0$ along the boundary of the square $\{-a \le x, y \le a\}$ (Fig. 8.5).
HINT: Use the basis functions $\phi(x,y) = (a^2-x^2)(a^2-y^2)$, and consider the approximate solution

$$\tilde{u}_N(x,y) = (a^2-x^2)(a^2-y^2)(A_1 + A_2x^2 + A_3y^3 + \cdots + A_n x^{2i}y^{2j}).$$

ANS. For $N = 1$, we have

$$\int_{-a}^{a}\int_{-a}^{a}[-2(a^2-y^2)A_1 - 2(a^2-x^2)A_1 + 2](a^2-x^2)(a^2-y^2)\,dx\,dy = 0.$$

This yields

$$A_1 = \frac{5}{8a^2},\quad \tilde{u}_1 = \frac{5(a^2-x^2)(a^2-y^2)}{8a^2}.$$

We must have $A_2 = A_3$. Then for $N = 3$, take

$$\tilde{u}_2 = (a^2-x^2)(a^2-y^2)[A_1 + A_2(x^2+y^2)],$$

where

$$A_1 = \frac{1295}{1416a^2},\quad A_2 = \frac{525}{4432a^4},$$

and

$$\tilde{u}_2(x,y) = \frac{35}{4432a^2}(a^2-x^2)(a^2-y^2)\left[74 + \frac{15}{a^2}(x^2+y^2)\right].$$

Alternately, if we choose the basis functions as $\phi_{jk} = \cos\frac{j\pi x}{2a}\,\cos\frac{k\pi y}{a}$, j,k odd, then

$$\tilde{u}_N = \sum_{\substack{j,k=1\\ j,k\ \text{odd}}} \alpha_{jk}\cos\frac{j\pi x}{a}\,\cos\frac{k\pi y}{2a},$$

which leads to

$$\int_{-a}^{a}\int_{-a}^{a}\Big[\sum_{j,k}\alpha_{jk}\left(\frac{j^2\pi^2}{4a^2}+\frac{k^2\pi^2}{4a^2}\right)\cos\frac{j\pi x}{a}\,\cos\frac{k\pi xy}{2a}\Big] \times \cos\frac{m\pi x}{2a}\,\cos\frac{k\pi y}{2a}\,dx\,dy = 0.$$

Hence for $j = m$ and $k = n$

$$\alpha_{jk} = \frac{128a^2(-1)^{j+k-2)/2}}{jk(j^2+k^2)\pi^4}.$$

8.21. Use the Galerkin method to solve the eigenvalue problem $\nabla^2 u + \lambda u = 0$ in the cylindrical polar coordinates for $0 < rA$.
HINT: Solve $\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) + \lambda u = 0$, $0 < r < a$.
ANS. Take $\phi_j(r) = \cos\frac{j\pi r}{2a}$. For the first approximation, $\phi_1 = \cos\frac{\pi r}{2a}$, and $\tilde{u}_1 = \alpha_1\cos\frac{\pi r}{2a}$, which leads to

$$2\pi\int_0^a\left\{\frac{1}{r}\frac{d}{dr}\left[\frac{r\pi}{2a}\left(-\sin\frac{\pi r}{2a}\right)\right]\alpha_1 + \lambda\alpha_1\cos\frac{\pi r}{2a}\right\} r\,dr = 0.$$

This gives the equation for the eigenvalue λ as

$$\frac{\pi^2}{4}\left(\frac{1}{2}+\frac{2}{\pi^2}\right)-\lambda a^2\left(\frac{1}{2}-\frac{2}{\pi^2}\right)=0.$$

Hence,

$$\lambda_1=\frac{\pi^2(\pi^2+4)}{4a^2(\pi^2-4)}\approx\frac{5.8304}{a^2}.$$

The exact value is $\lambda_1=\dfrac{5.779}{a^2}$. For the second order approximation $\tilde{u}_2=\alpha_1\cos\dfrac{\pi r}{2a}+\alpha_2\cos\dfrac{3\pi r}{2a}$, which gives $\lambda_2=\dfrac{5.792}{a^2}$.

8.22. Use the Galerkin method to determine the lowest frequency (fundamental tone) of the vibration of a homogeneous circular plate Ω of radius a and center at at the origin of cylindrical coordinates, clamped at the entire edge, i.e., solve $\nabla^4 u=\lambda u$ subject to the conditions $u(a)=0=u_r(a)$.

HINT: Minimize the variational problem $I(u)=\displaystyle\iint_\Omega \nabla^4 u\,dx\,dy$, such that $\displaystyle\iint_\Omega u^2\,dx\,dy=1$, subject to the given conditions.

ANS. Solve

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}\right)=\lambda u.$$

Take $\tilde{u}_N=\displaystyle\sum_{j=1}^{N}\alpha_j\left(1-\frac{r^2}{a^2}\right)^{j+1}$. Then, e.g., for $\tilde{u}_2$, we have

$$\alpha_1\left(\frac{192}{9}-\frac{\lambda a^4}{5}\right)+\alpha_2\left(\frac{144}{9}-\frac{\lambda a^4}{6}\right)=0,$$
$$\alpha_1\left(\frac{144}{9}-\frac{\lambda a^4}{6}\right)+\alpha_2\left(\frac{96}{5}-\frac{\lambda a^4}{7}\right)=0,$$

and the equation for λ is

$$(\lambda a^4)^2-\frac{9792}{5}\lambda a^4+435456=0,$$

which has the smaller root as $\lambda=\dfrac{104.387654}{a^4}$. Using this value of λ in the above system of two equations, we find $\alpha_2=0.325\,\alpha_1$, and

$$\tilde{u}_2=\alpha_1\left[\left(1-\frac{r^2}{a^2}\right)^2+0.325\left(1-\frac{r^2}{a^2}\right)^2\right],$$

where α_1 can be found from the above system of two equations.

9

Perturbation Methods

The perturbation methods provide approximate solutions for boundary value and initial value problems. These methods are used when such problems contain a small parameter, say ε, and the solution for $\varepsilon=0$ is known. This parameter occurs, in general, in a partial differential equation of the form

$$L\,u+\varepsilon N\,u=0, \tag{9.1}$$

where L is a linear partial differential operator, and N is either a nonlinear or a linear differential operator which makes the solution of Eq (9.1) difficult. If $\varepsilon=0$ reduces Eq (9.1) to an ordinary differential equation, then the perturbation method will fail.

Another kind of perturbation problems arises by perturbing the boundary. In this case the parameter ε will appear in the boundary conditions. The two common perturbation methods discussed here are
(1) series expansion in powers of ε, and
(2) successive approximations.
These methods apply when the partial differential equation or the boundary is perturbed.

Quite often these two methods will yield the same approximate solution. Frequently the perturbation solution is equivalent to the iterative solution of the corresponding integral equation. For the perturbation method to be successful, it is assumed that u is a continuous function of the perturbation parameter and that the differences in the two problems (perturbed and unperturbed) are not singular in character. These methods are thus applied to problems in which the solution to an ideal simple problem is known and, for a more realistic situation, the differential equation or the boundary conditions or the region is perturbed.

There are other more complicated methods which deal with singular perturbation problems. We shall, however, not discuss such methods in this book. For more information about them the reader is referred to Kevorkian (1990).

WWW Refer to the Mathematica Notebook `perturbation.ma` for this chapter.

9.1. Taylor Series Expansions

The general scheme for this method is as follows: Consider Eq (9.1) subject to some prescribed boundary conditions and/or initial conditions. Assume that the solution of the homogeneous equation $Lu = 0$ subject to the same prescribed conditions is known. Then, in order to solve the given problem we shall assume that u possesses a series expansion in powers of ε of the form

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots = \sum_{n=0}^{\infty} \varepsilon^n u_n. \tag{9.2}$$

If we substitute this power series for u into Eq (9.1), we get

$$L\left(\sum_{n=0}^{\infty} \varepsilon^n u_n\right) + \varepsilon N\left(\sum_{n=0}^{\infty} \varepsilon^n u_n = 0\right). \tag{9.3}$$

Assuming that u_0 satisfies the prescribed conditions and that u_n, $n \neq 0$, satisfy homogeneous conditions, we can obtain a system of partial differential equations by comparing coefficients of various powers of ε on both sides of Eq (9.3). These partial differential equations are such that a partial differential equation in u_n will depend only on $u_0, u_1, \cdots, u_{n-1}$. The solutions to u_0, $u_1, \cdots, u_{n-1}$ are known successively, i.e., one solves first for u_0, which enables one to solve for u_1 and so on, and one can finally solve for u_n. We will demonstrate this method by some specific examples.

EXAMPLE 9.1. Consider

$$\nabla^2 u + \varepsilon f(u) = 0, \quad u(1, \theta) = g(\theta), \quad \lim_{r\to 0} u(r, \theta) \to 0. \tag{9.4}$$

We will consider two cases:
(a) For $f(u) = u$, let us assume the power series (9.2). The partial differential equation and the boundary conditions for each u_n, $n = 0, 1, 2, 3, \cdots$, are given by

$$\begin{aligned} \nabla^2 u_0 &= 0, \quad u_0(1,\theta) = g(\theta), \\ \nabla^2 u_1 + u_0 &= 0, \quad u_1(1,\theta) = 0, \\ \nabla^2 u_{n+1} + u_n &= 0, \quad u_{n+1}(1,\theta) = 0. \end{aligned} \tag{9.5}$$

The general solution for $\nabla^2 u_0 = 0$, by the method of separation of variables, is

$$u_0 = \sum_0^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta), \tag{9.6}$$

where

$$\sum_{n=0}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) = g(\theta). \tag{9.7}$$

If $g(\theta)$ is a periodic Dirichlet function, then A_n and B_n can be determined, and the subsequent equations for u_n can be solved. For example, let $g(\theta) = \cos\theta$. In this case $u_0 = r\cos\theta$, and the partial differential equation for u_1 becomes

$$\nabla^2 u_1 + r\cos\theta = 0. \tag{9.8}$$

Since the particular integral for a partial differential equation of the type

$$\nabla^2 v + c_1 r^n \cos p\theta + c_2 r^m \sin q\theta = 0$$

is of the form

$$v_p = c_{11} r^{n+2} \cos p\theta + c_{22} r^{m+2} \sin q\theta,$$

the solution for u_1 is given by

$$u_1 = \frac{1}{8} r(1 - r^2)\cos\theta. \tag{9.9}$$

Similarly, the solution for u_2 is given by

$$u_2 = \frac{1}{192} r(2 - 3r^2 + r^4)\cos\theta. \tag{9.10}$$

Other terms can be obtained similarly.

(b) For $f(u) = u^2$, we use the series (9.2) for u in powers of ε, where u_0 satisfies the boundary condition $u_0(1, \theta) = \cos\theta$, and u_p satisfies the homogeneous boundary condition. Thus, the partial differential equations for $u_0, u_1, u_2, \cdots$ are given by

$$\begin{aligned} u_{0rr} + \frac{1}{r}u_{0r} + \frac{1}{r^2}u_{0\theta\theta} &= 0, \\ u_{1rr} + \frac{1}{r}u_{1r} + \frac{1}{r^2}u_{1\theta\theta} + u_0^2 &= 0, \\ u_{2rr} + \frac{1}{r}u_{2r} + \frac{1}{r^2}u_{2\theta\theta} + 2u_0u_1 &= 0, \end{aligned} \tag{9.11}$$

and so on. The solution for u_0 is clearly $u_0 = r\cos\theta$, and the equation for u_1 becomes

$$u_{1rr} + \frac{1}{r}u_{1r} + \frac{1}{r^2}u_{1\theta\theta} + r^2\cos^2\theta = 0,$$

whose solution is then given by

$$u_1 = \frac{1}{32}(1 - r^4) + \frac{1}{24}(r^2 - r^4)\cos 2\theta. \tag{9.12}$$

We can continue the process to find $u_2, u_3, \cdots$. ■

EXAMPLE 9.2. To solve

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \varepsilon u u_\theta = 0, u(1, \theta) = \cos\theta, \tag{9.13}$$

up to the first three terms of the power series solution, we use the series (9.2) for u. Substituting this series into the partial differential equation and comparing coefficients of different powers of ε on both sides, we get

$$\begin{aligned} u_{0rr} + \frac{1}{r}u_{0r} + \frac{1}{r^2}u_{0\theta\theta} &= 0, \\ u_{1rr} + \frac{1}{r}u_{1r} + \frac{1}{r^2}u_{1\theta\theta} + u_0u_{0\theta} &= 0, \\ u_{2rr} + \frac{1}{r}u_{2r} + \frac{1}{r^2}u_{2\theta\theta} + u_1u_{0\theta} + u_0u_{1\theta} &= 0. \end{aligned} \tag{9.14}$$

The new boundary conditions are $u_0(1, \theta) = \cos\theta$, and $u_n(1, \theta) = 0$, $n \geq 1$. It is easy to see that $u_0(1, \theta) = r\cos\theta$, and the partial differential equations for u_1 is

$$u_{1rr} + \frac{1}{r}u_{1r} + \frac{1}{r^2}u_{1\theta\theta} = r^2\cos\theta\sin\theta = \frac{1}{2}r^2\sin 2\theta. \tag{9.15}$$

Its solution is $u_1 = \frac{1}{24}r^2(r^2 - 1)\sin 2\theta$. The partial differential equation for u_2 becomes

$$u_{2rr} + \frac{1}{r}u_{2r} + \frac{1}{r^2}u_{2\theta\theta} = \frac{1}{24}r^3(r^2 - 1)(\sin 2\theta\sin\theta - 2\cos 2\theta\cos\theta),$$

and its solution is given by

$$\begin{aligned} u_2 = \frac{1}{256}r^3(r^2 - 1)\cos 3\theta &+ \frac{1}{1152}r(r^4 - 1)\cos\theta \\ &- \frac{1}{640}r^3(r^4 - 1)\cos 3\theta - \frac{1}{2304}r(r^6 - 1)\cos\theta. \ ■ \end{aligned} \tag{9.16}$$

We can continue the process as long as we need to obtain the required degree of accuracy. There will, of course, be the question of convergence, but one observes that the coefficients are getting fairly small, so convergence for values of $\varepsilon < 1$ appears likely.

9.2. Successive Approximations

The general scheme for this method is as follows: We assume the first approximation to be u_0, which satisfies the given boundary and initial conditions and the homogeneous equation $Lu = 0$. The second approximation is then u_1, which satisfies the equation $Lu_1 + \varepsilon N u_0 = 0$ and the given boundary and initial conditions. The process is continued until the required degree of accuracy is achieved. It is obvious that the order of difficulty is directly proportional to the order of approximation. We will demonstrate this method by solving the above examples.

EXAMPLE 9.3. We now solve Example 9.1(b) for $f(u) = u^2$ by the method of successive approximations. The partial differential equation for u_0 is the same as in (9.11), i.e.,

$$u_{0rr} + \frac{1}{r}u_{0r} + \frac{1}{r^2}u_{0\theta\theta} = 0,$$

and its solution is $u_0 = r\cos\theta$. But the partial differential equation for u_1 is

$$u_{1rr} + \frac{1}{r}u_{1r} + \frac{1}{r^2}u_{1\theta\theta} + \varepsilon r^2\cos^2\theta = 0, u_1(1, \theta) = \cos\theta. \tag{9.17}$$

Its solution is

$$u_1 = r\cos\theta + \varepsilon\frac{1}{32}(1-r^4) + \frac{1}{24}(r^2-r^4)\cos 2\theta. \tag{9.18}$$

Notice that here u_1 is actually the sum $u_0 + \varepsilon u_1$ of Example 9.1. ∎

EXAMPLE 9.4. We now solve the Example 9.2 by the method of successive approximations. Obviously, the solution for u_0 is $u_0 = r\cos\theta$. Now the next approximation u_1 satisfies the given boundary condition and the partial differential equation

$$u_{1rr} + \frac{1}{r}u_{1r} + \frac{1}{r^2}u_{1\theta\theta} + \varepsilon u_0 u_{0\theta} = 0, \tag{9.19}$$

or

$$u_{1rr} + \frac{1}{r}u_{1r} + \frac{1}{r^2}u_{1\theta\theta} - \frac{\varepsilon}{2}r^2\sin 2\theta = 0. \tag{9.20}$$

The solution for u_1 is easily seen to be

$$u_1 = r\cos\theta + \frac{\varepsilon}{24}(r^4-r^2)\sin 2\theta. \tag{9.21}$$

The next approximation u_2 once again satisfies the same boundary conditions but the new partial differential equation is

$$u_{2rr} + \frac{1}{r}u_{2r} + \frac{1}{r^2}u_{2\theta\theta} + \varepsilon u_1 u_{1\theta} = 0. \tag{9.22}$$

Since

$$u_1 u_{1\theta} = -\frac{r^2}{2}\sin 2\theta + \frac{\varepsilon}{48}r^3(r^2-1)(\cos\theta + 3\cos\theta) + \frac{\varepsilon^2}{576}r^4(r^2-1)^2\sin 4\theta, \tag{9.23}$$

the equation for u_2 becomes

$$u_{2rr} + \frac{1}{r}u_{2r} + \frac{1}{r^2}u_{2\theta\theta} = \frac{r^2}{2}\sin 2\theta - \frac{\varepsilon}{48}r^3(r^2-1)(\cos\theta + 3\cos\theta) - \frac{\varepsilon^2}{576}r^4(r^2-1)^2\sin 4\theta. \tag{9.24}$$

Its solution is

$$\begin{aligned} u_2 &= r\cos\theta + \frac{\varepsilon}{24}r^2(r^2-1)\sin 2\theta + \frac{\varepsilon^2}{1152}r(r^4-1)\cos\theta \\ &- \frac{\varepsilon^2}{2304}r(r^6-1)\cos\theta + \frac{\varepsilon^2}{256}r^3(r^2-1)\cos 3\theta - \frac{\varepsilon^2}{640}r^3(r^4-1)\cos 3\theta \\ &- \varepsilon^3\left[\frac{1}{48384}r^4(r^6-1) - \frac{1}{13824}r^4(r^4-1) + \frac{1}{11520}r^4(r^2-1)\right]\sin 4\theta. \end{aligned} \tag{9.25}$$

It is clear that the solution (9.25) is almost similar to the one obtained in Example 9.2. These two solutions would be exactly the same if enough terms in the series solution and a large enough number of approximations are taken. ∎

9.3. Boundary Perturbations

For problems involving boundary perturbation the series (9.2) for u is used. The following examples illustrate this method.

EXAMPLE 9.5. Solve the harmonic (Laplace's) equation

$$u_{xx} + u_{yy} = 0,$$

such that

$$u(\varepsilon\sin\omega y, y) = 0, \quad u(\pi, y) = 0, \quad u(x, 0) = \sin x,$$

and $\lim_{y\to\infty} u(x,y)$ is bounded. Assuming the series (9.2) for u, we find that $\nabla u_n = 0$ for all n. Moreover,

$$u(\varepsilon\sin\omega y, y) = u(0,y) + \varepsilon\sin\omega y\, u_x(0,y) + \frac{1}{2}(\varepsilon\sin\omega y)^2 u_{xx}(0,y) + \cdots, \tag{9.26}$$

$$\begin{aligned} u(\varepsilon\sin\omega y, y) &= u_0(0,y) + \varepsilon u_1(0,y) + u_2(0,y) + \cdots \\ &+ \varepsilon\sin\omega y\,[u_{0,x}(0,y) + \varepsilon u_{1,x}(0,y) + \varepsilon^2 u_{2,x}(0,y) + \cdots] \\ &+ \frac{1}{2}(\varepsilon\sin\omega y)^2[u_{0,xx}(0,y) + \varepsilon u_{1,xx}(0,y) \\ &+ \varepsilon^2 u_{2,xx}(0,y) + \cdots] + \cdots. \end{aligned} \tag{9.27}$$

By comparing (9.26) and (9.27) we find that $u_0(x,y)$ must satisfy the conditions

$$u_0(0,y) = 0, \quad u_0(\pi,y) = 0, \quad \text{and} \quad u_0(x,0) = \sin x, \tag{9.28}$$

whereas $u_1(x,y)$ must satisfy the conditions

$$u_1(0,y) = -e^{-y}\sin\omega y, \quad \text{and} \quad u_1(\pi,y) = 0, u_1(x,0) = 0. \tag{9.29}$$

Note that $u_0(x,y) = e^{-y}\sin x$. Since the solution for u_1 is complicated, we will separate it into two parts: The first part will satisfy Laplace's equation and the boundary conditions at $x = 0, \pi$, while the second part, though satisfying Laplace's equation, will satisfy homogeneous boundary conditions at $x = 0, \pi$; then the sum of both parts will satisfy the conditions at $y = 0$. Thus, let $u_1 = v_1 + v_2$, where $\nabla v_1 = 0$, subject to the conditions $v_1(0,y) = -e^{-y}\sin\omega y, v_1(\pi, y) = 0$. Then

$$\begin{aligned} v_1 &= [f(x)\cos\omega y + g(x)\sin\omega y]\, e^{-y}, \\ f(0) &= f(\pi) = g(\pi) = 0, \quad g(0) = -1. \end{aligned} \tag{9.30}$$

Then substituting v_1 into Laplace's equation and comparing the coefficients of $\sin\omega x$ and $\cos\omega x$ on both sides, we get

$$\begin{aligned} f'' + (1-\omega^2)f - 2\omega g &= 0, \\ g'' + (1-\omega^2)g + 2\omega f &= 0. \end{aligned} \tag{9.31}$$

Let $z = f + ig$. Then, multiplying the second equation by i in (9.31) and adding it to the first we get

$$z'' + (1 + 2i\omega - \omega^2)z = 0,$$

or

$$z'' + (1 + i\omega)^2 z = 0, \tag{9.32}$$

with the boundary conditions $z(0) = -i, z(\pi) = 0$. Its solution can be expressed as

$$z = Ae^{(1+i\omega)x} + Be^{(1+i\omega)x}. \tag{9.33}$$

On applying the boundary conditions, we find that

$$A = \frac{-ie^{-i(1+i\omega)\pi}}{e^{-i(1+i\omega)\pi} - e^{i(1+i\omega)\pi}}, \quad B = \frac{ie^{i(1+i\omega)\pi}}{e^{-i(1+i\omega)\pi} - e^{i(1+i\omega)\pi}}. \tag{9.34}$$

Now solving for the real and imaginary parts of z, we find that

$$f(x) = \frac{\sin x\cosh\omega(\pi - x)}{\sinh\omega\pi}, \quad g(x) = -\frac{\cos x\sinh\omega(\pi - x)}{\sinh\omega\pi}. \tag{9.35}$$

Next, we determine v_2 by the separation of variables method with homogeneous boundary conditions with respect to x. The solution is of the form

$$v_2 = \sum_0^\infty A_n e^{-ny}\sin nx. \tag{9.36}$$

Then adding the solutions (9.30) and (9.36), and using (9.35), the complete solution for u_1 is given by

$$\begin{aligned} u_1 = \sum_0^\infty A_n e^{-ny}\sin nx + \big[&\cosh\omega(\pi - x)\sin x\cos\omega y \\ &- \sinh\omega(\pi - x)\cos x\sin\omega y\big]\frac{e^{-y}}{\sinh\omega\pi}. \end{aligned} \tag{9.37}$$

On applying the condition $u_1(x,0) = 0$, we get from (9.37)

$$A_n = -\left[\frac{1}{(n-1)^2 + \omega^2} - \frac{1}{(n+1)^2 + \omega^2}\right]\frac{\omega}{\sinh\omega\pi}.$$

Hence the solution for the problem is

$$\begin{aligned} u_1(x,y) = \big[&\cosh\omega(\pi - x)\sin x\cos\omega y \\ &- \sinh\omega(\pi - x)\cos x\sin\omega y\big]\frac{e^{-y}}{\sinh\omega\pi} \\ &- \sum_0^\infty\left[\frac{1}{(n-1)^2 + \omega^2} - \frac{1}{(n+1)^2 + \omega^2}\right]\frac{\omega\sin nx}{\sinh\omega\pi}e^{-ny}. \blacksquare \end{aligned} \tag{9.38}$$

EXAMPLE 9.6. Consider

$$\nabla^2 u = 0, u(1 + \varepsilon\sin\theta, \theta) = f(\theta). \tag{9.39}$$

By expanding $u(r,\theta), r = 1 + \varepsilon\sin\theta$ about $r = 1$ in a Taylor series, we have

$$u(1+\varepsilon\sin\theta,\theta) = f(\theta) = u(1,\theta) + \varepsilon\sin\theta u_r(1,\theta) + \frac{1}{2}\varepsilon^2\sin^2\theta u_{rr}(1,\theta) + \cdots. \tag{9.40}$$

We also assume that u has a series expansion in powers of ε of the form (9.2), i.e.,

$$u(r,\theta) = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots. \tag{9.41}$$

Combining (9.40) and (9.41), we get

$$\begin{aligned} u_0(1,\theta) &= f(\theta), \\ u_1(1,\theta) + \sin\theta u_{0r}(1,\theta) &= 0, \\ u_2(1,\theta) + \sin\theta u_{1r} + \frac{1}{2}\sin^2\theta u_{0rr}(1,\theta) &= 0, \end{aligned} \tag{9.42}$$

and so on. The partial differential equation to be satisfied by u_n for all n is $\nabla^2 u_n = 0$. Using the general solution and applying the boundary conditions for u_0, we get

$$u_0(1,\theta) = f(\theta) = \sum_0^\infty (A_n \cos n\theta + B_n \sin n\theta). \tag{9.43}$$

If $f(\theta)$ is a periodic Dirichlet function, then

$$\begin{aligned} A_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta, \quad A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta, \\ B_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta. \end{aligned} \tag{9.44}$$

Case 1. If $f(\theta) = \cos\theta$, then

$$\begin{aligned} u_0 &= r\cos\theta, u_1 = -\frac{1}{2}r^2 \sin 2\theta, \\ u_2 &= \frac{1}{2}(r^3\cos\theta - r\cos\theta). \end{aligned} \tag{9.45}$$

Case 2. If $f(\theta) = \sin\theta$, then

$$\begin{aligned} u_0 &= r\sin\theta, \\ u_1 &= \frac{1}{2}(r^2\cos\theta - 1), \\ u_2 &= \frac{1}{2}(r\sin\theta - r^3\sin 3\theta. \end{aligned} \tag{9.46}$$

For the solution of this example by the method of successive approximations see Exercise 9.4 below. ■

9.4. Exercises

9.1. Obtain a perturbation solution for the problem:

$$u_t = u_{yy} + k u_{yyt},$$

where $u(y,t) = 0$ for $0 \le t$ and $u(0,t) = 1$ for $t > 0$.
(a) By successive approximation (2 approximations).

(b) By expansion in powers of k (up to the first power of k).
HINT. Use Laplace transforms. [Compare with Exercise 6.4.]
ANS. $u = \operatorname{erfc}\left(\frac{y}{2\sqrt{t}}\right) + \frac{k}{4t\sqrt{\pi t}}\left(\frac{y^2}{t} - 1\right) e^{y^2/4\pi t}$.

9.2. Obtain a steady state perturbation solution for the problem:

$$\begin{gathered} u_t + k u_{yyt} = k u_{yyy} + u_{yy} + u_y, \quad u(0,t) = e^{i\omega t}, \\ \lim_{y\to\infty} u(y,t) = 0, \quad \lim_{k\to 0} u(y,t,k) = u(y,t,0). \end{gathered}$$

Steady state in this case implies that the initial conditions are to be ignored.
HINT: Assume a solution of the form $u(y,t) = F(y)e^{i\omega t}$. Instead of assuming a perturbation solution for $u(y,t)$, assume a perturbation solution for the roots of the characteristic equation.
ANS.

$$u(y,t) = \Re\, e^{-ay + i(\omega t + by)},$$

where

$$\begin{aligned} a &= \frac{1}{2}(1+\alpha_0) - \frac{Ak}{2\sqrt{1+16\omega^2}} - \frac{Ck^2}{\sqrt{1+16\omega^2}}, \\ b &= -\frac{1}{2}\beta_0 + \frac{Bk}{2\sqrt{1+16\omega^2}} + \frac{Dk^2}{\sqrt{1+16\omega^2}}, \\ \alpha_0 &= \sqrt{\frac{1}{2}\left[\sqrt{1+16\omega^2}+1\right]}, \\ \beta_0 &= \sqrt{\frac{1}{2}\left[\sqrt{1+16\omega^2}-1\right]}, \\ A &= \alpha_0 + \alpha_0^2 + \beta_0^2 - 2\omega^2\alpha_0 + 4\omega\beta_0, \\ B &= 4\omega\alpha_0 + 2\omega\alpha_0^2 + 2\omega^2\beta_0 + 2\omega\beta_0^2 - \beta_0, \\ C &= p\alpha_0 + q\beta_0, D = q\alpha_0 - p\beta_0, \\ p &= \frac{(2\omega^2\beta_0 - 3(1+\alpha_0))A + (8\omega + 2\omega\alpha_0 + 3\beta_0)B - \dfrac{(A^2 - B^2)}{\sqrt{1+16\omega^2}}}{4\sqrt{1+16\omega^2}}, \\ q &= \frac{(2\omega^2\beta_0 - 3(1+\alpha_0))B - (2\omega\alpha_0 - 3\beta_0 - 4\omega)A - \dfrac{2AB}{\sqrt{1+16\omega^2}}}{4\sqrt{1+16\omega^2}}. \end{aligned}$$

9.3. Solve $\nabla^2 u + \varepsilon f(u) = 0, u(1,\theta) = g(\theta)$, for $f(u) = u, u^2, u + u^2$, respectively, and $g(\theta) = \sin\theta, \sin^2\theta$, respectively. Find the first two

terms in each case.
ANS. For $f(\theta) = \sin\theta$:

$$u \approx r\,\sin\theta - \frac{\varepsilon}{8} r(1-r^2)\sin\theta,$$
$$u \approx r\,\sin\theta + \frac{\varepsilon}{8}\left\{\frac{1}{4}(1-r^4) - \frac{1}{3}r^2(1-r^2)\cos 2\theta\right\},$$
$$u \approx r\,\sin\theta + \frac{\varepsilon}{8}\left\{\frac{1}{4}(1-r^4) - r(1-r^2)\sin\theta - \frac{r^2}{3}(1-r^2)\cos 2\theta\right\}.$$

For $f(\theta) = \sin^2\theta$:

$$u \approx \frac{1}{2}(1-r^2\cos 2\theta) + \frac{\varepsilon}{8}\left\{1 - r^2 - \frac{r^2}{3}(1-r^2)\cos 2\theta\right\},$$
$$u \approx \frac{1}{2}(1-r^2\cos 2\theta) + \frac{\varepsilon}{8}\left\{\frac{1}{2}(1-r^2) - \frac{r^2}{3}(1-r^2)\cos 2\theta + \frac{1}{36}(1-r^4) + \frac{r^4}{20}(1-r^2)\cos 4\theta\right\},$$
$$u \approx \frac{1}{2}(1-r^2\cos 2\theta) + \frac{\varepsilon}{8}\left\{\frac{3}{2}(1-r^2) - \frac{2r^2}{3}(1-r^2)\cos 2\theta + \frac{1}{36}(1-r^4) + \frac{r^4}{20}(1-r^2)\cos 4\theta\right\}.$$

9.4. Solve $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \varepsilon u_r u_\theta = 0, u(1,\theta) = \cos\theta$.
SOLUTION. Substituting the series (9.2) into the partial differential equation and comparing coefficients of different powers of ε on both sides we get

$$u_{0rr} + \frac{1}{r}u_{0r} + \frac{1}{r^2}u_{0\theta\theta} = 0,$$
$$u_{1rr} + \frac{1}{r}u_{1r} + \frac{1}{r^2}u_{1\theta\theta} + u_{0r}u_{0\theta} = 0,$$
$$u_{2rr} + \frac{1}{r}u_{2r} + \frac{1}{r^2}u_{2\theta\theta} + u_{1r}u_{0\theta} + u_{0r}u_{1\theta} = 0.$$

The new boundary conditions are $u_0(1,\theta) = \cos\theta$, and $u_n(1,\theta) = 0$. It is easy to see that

$$u_0(1,\theta) = r\cos\theta,$$
$$u_1(r,\theta) = \frac{1}{10}(r^3 - r^2)\sin 2\theta,$$
$$u_2(r,\theta) = \frac{1}{480}(r^5 - r)\cos\theta + \left(\frac{1}{35}r^4 - \frac{1}{64}r^5 - \frac{29}{2240}r^3\right)\cos 3\theta.$$

Alternately, we will now solve this problem by the method of successive approximations: Note that the solution for u_0 is $r\cos\theta$. The next approximation u_1 satisfies the given boundary condition and the partial differential equation

$$u_{1rr} + \frac{1}{r}u_{1r} + \frac{1}{r^2}u_{1\theta\theta} + \varepsilon u_{0r}u_{0\theta} = 0.$$

The solution for u_1 is

$$u_1 = r\cos\theta + \varepsilon\frac{1}{10}(r^3 - r^2)\sin 2\theta.$$

The next approximation u_2 once again satisfies the same boundary conditions but the new partial differential equation is

$$u_{2rr} + \frac{1}{r}u_{2r} + \frac{1}{r^2}u_{2\theta\theta} + \varepsilon u_{1r}u_{1\theta} = 0,$$

whose solution is

$$u_2 = r\cos\theta + \varepsilon\frac{1}{10}(r^3 - r^2)\sin 2\theta + \varepsilon^2\left[\frac{1}{480}(r^5 - r)\cos\theta + \left(\frac{1}{35}r^4 - \frac{1}{64}r^5 - \frac{29}{2240}r^3\cos 3\theta\right)\right] - \frac{\varepsilon^3}{100}\left(\frac{r^7}{11} - \frac{r^6}{4} + \frac{2r^5}{9} - \frac{25r^4}{396}\right)\sin 4\theta.$$

9.5. Solve $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \varepsilon u_{r\theta} = 0, u(1,\theta) = \sin\theta$. Find the first three terms.
SOLUTION. Substituting the series (9.2) into the partial differential equation and comparing coefficients of different powers of ε on both sides we get

$$u_{0rr} + \frac{1}{r}u_{0r} + \frac{1}{r^2}u_{0\theta\theta} = 0,$$
$$u_{1rr} + \frac{1}{r}u_{1r} + \frac{1}{r^2}u_{1\theta\theta} + u_{0r\theta} = 0,$$
$$u_{2rr} + \frac{1}{r}u_{2r} + \frac{1}{r^2}u_{2\theta\theta} + u_{1r\theta} = 0.$$

The new boundary conditions are $u_0(1,\theta) = \sin\theta$, and $u_n(1,\theta) = 0, n \geq 1$. Clearly, $u_0(1,\theta) = r\sin\theta$, and the partial differential equations for u_1 is

$$u_{1rr} + \frac{1}{r}u_{1r} + \frac{1}{r^2}u_{1\theta\theta} = -\cos\theta.$$

Its solution is

$$u_1 = \frac{1}{3} r(1-r)\cos\theta.$$

The partial differential equation for u_2 becomes

$$u_{2rr} + \frac{1}{r}u_{2r} + \frac{1}{r^2}u_{2\theta\theta} = \frac{1}{3}(1-2r)\sin\theta,$$

whose solution is given by

$$u_2 = -\frac{1}{36} r(1-4r+3r^2)\sin\theta.$$

9.6. In Example 9.6, choose $f(\theta) = \sin^2\theta$, and solve the problem.
ANS.

$$u_0 = \frac{1}{2}(1 - r^2\cos 2\theta),$$
$$u_1 = \frac{1}{2}(r^3\cos\theta - r\cos\theta),$$
$$u_2 = \frac{1}{8} + \frac{1}{2} r\cos\theta - \frac{1}{4}r^2\cos 2\theta - \frac{3}{2}\cos 3\theta + \frac{1}{8}r^4\cos 4\theta.$$

9.7. Solve $\nabla^2 u = 0, u(1+\varepsilon\cos\theta,\theta) = f(\theta)$, for (a) $f(\theta) = \cos\theta$, and (b) $f(\theta) = \sin\theta$.
ANS.

$$\text{(a)}\quad \begin{cases} u_0 = r\cos\theta, \\ u_1 = -\frac{1}{2}(1 + r^2\cos 2\theta), \\ u_2 = \frac{1}{2}(r^3\cos 3\theta + r\cos\theta). \end{cases}$$

$$\text{(b)}\quad \begin{cases} u_0 = r\sin\theta, \\ u_1 = -\frac{1}{2} r^2\sin 2\theta, \\ u_2 = \frac{1}{2}(r^3\sin 3\theta - r\sin\theta). \end{cases}$$

9.8. Find the exact solution of $yu_x + (x + \varepsilon u)u_y = 0, u(x,1) = x$, by the methods of Chapter 2, and then find an approximate solution (up to three terms of a series solution or three approximations by the method of successive approximation) of this partial differential equation by perturbation methods. Make relevant comparisons.
ANS.

$$(1+2\varepsilon)u^2 - 2\varepsilon ux + y^2 - x^2 - 1 = 0,$$

$$u = \frac{\varepsilon x + \sqrt{[(1+x^2-y^2) + 2\varepsilon(1+x^2-y^2) + \varepsilon^2 x^2]}}{1+2\varepsilon}.$$

The series solution is

$$u = 1 + x^2 - y^2 + \left(x - \sqrt{1+x^2-y^2}\right)\varepsilon + \left(3x^2 - 2y^2 + 2 - 3x\sqrt{1+x^2-y^2}\right)\varepsilon^2 + \cdots.$$

9.9. Find the exact solution of

$$\frac{u_x}{1+y^2} + (x + \varepsilon u^2)u_y = 0, \quad u(x,0) = x,$$

by the methods of Chapter 2, and then find an approximate solution (up to three terms of a series solution or three approximations by the method of successive approximation) of the partial differential equation by perturbation techniques.
ANS.

$$u^2 + 2\varepsilon u^2(u - x) = x^2 - 2\tan^{-y},$$

$$u = \sqrt{x^2 - 2\tan^{-y}} + \varepsilon\sqrt{x^2 - 2\tan^{-y}}\left(x - \sqrt{x^2 - 2\tan^{-y}}\right) + \varepsilon^2\sqrt{x^2 - 2\tan^{-y}}\left(x - \sqrt{x^2 - 2\tan^{-y}}\right)^2 + \cdots.$$

9.10. Solve $\nabla^2 u = 0$, $0 < x < \pi$, $y > \varepsilon x$, $0 < \varepsilon \ll 1$, such that $u(0,y) = u(\pi,y) = 0$, and $u(x,\varepsilon x) = \sin x$.
SOLUTION. Using (9.2), we have

$$u(x,\varepsilon x) = \sum_0^\infty \varepsilon^n x^n \frac{\partial^n u}{\partial^n y}(x,0).$$

Hence

$$\sin x = \sum_0^\infty \varepsilon^n u_n(x,0) + \varepsilon x \sum_0^\infty \varepsilon^n \frac{\partial u_n}{\partial y}(x,0) \varepsilon^2 x^2 \sum_0^\infty \varepsilon^n \frac{\partial^2 u_n}{\partial y^2}(x,0) + \cdots.$$

Comparing powers of ε on both sides we get

$$u_0(x,0) = \sin x,$$
$$u_1(x,0) + x\frac{\partial u_0}{\partial y}(x,0) = 0,$$
$$u_2(x,0) + x\frac{\partial u_1}{\partial y}(x,0) + \frac{x^2}{2}\frac{\partial^2 u_0}{\partial y^2}(x,0) = 0,$$

and conditions on $u_n(x,0)$ can be derived similarly. All u_n satisfy Laplace's equation. The solution for u_0 is clearly given by $u_0 = e^{-y}\sin x$, and u_1 satisfies the following conditions:

$$u_1(0,y) = u_1(\pi,y) = 0, \quad u_1(x,0) = x\sin x.$$

Assuming a solution of the form

$$u_1 = \sum_{n=1}^{\infty} A_n e^{-ny}\sin x,$$

we find that

$$A_1 = \frac{\pi}{2}, \quad A_n = \frac{4n[1+(-1)^n]}{\pi(n^2-1)^2}, \quad n \neq 1.$$

Thus u_1 is completely determined. We can continue the process and determine u_n for higher values of n.

10

Finite Difference Methods

The development of high–speed digital and personal computers has made it possible to effectively use different numerical techniques for solving boundary and initial value problems involving partial differential equations. Among different methods available, the finite difference method is widely used. It has a straightforward structure which is derived from truncated Taylor's series, also known as Taylor's formula. We shall discuss difference schemes for first and second order partial derivatives, and then apply them to numerically solve boundary and initial value problems for second order partial differential equations.

In finite difference methods we replace the given differential equation and the boundary/initial conditions by a set of algebraic equations which are then solved by various well-known numerical techniques including sparse matrix methods and adaptive grid generation schemes. Although the finite difference method is just one of many numerical methods, it has advantages over other methods in its simplicity of analysis and computer codes in solving problems with complex geometries. It is used extensively in the area of computational fluid dynamics allowing the modeling of complex flows. It is also used along with finite element method in the solution of time–dependent problems.

10.1. Finite Difference Schemes

Consider a single–valued, finite function $u(x)$ which belongs to the class $C^\infty(R^1)$. Then by Taylor's theorem

$$u(x+h) = u(x) + h\,u'(x) + \frac{h^2}{2!}\,u''(x) + \frac{h^3}{3!}\,u'''(x) + \cdots, \tag{10.1}$$

$$u(x-h) = u(x) - h\,u'(x) + \frac{h^2}{2!}\,u''(x) - \frac{h^3}{3!}\,u'''(x) + \cdots. \tag{10.2}$$

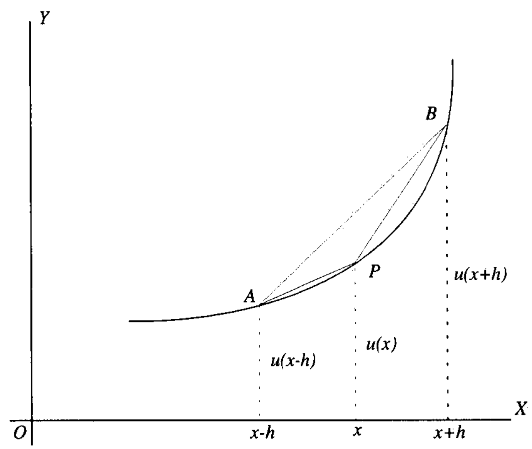


Fig. 10.1.

If we subtract (10.2) from (10.1), we get

$$u(x+h) - u(x-h) = 2h\,u'(x) + O(h^3).$$

Then

$$u'(x) = \frac{u(x+h) - u(x-h)}{2h}, \tag{10.3}$$

with a truncation error of $O(h^2)$. The approximation formula (10.3) is known as the first order *central difference* formula, and geometrically it represents the slope of the chord $\overline{AB}$ (Fig. 10.1). Similarly,

$$u'(x) = \frac{u(x) - u(x-h)}{2h}, \tag{10.4}$$

and

$$u'(x) = \frac{u(x+h) - u(x)}{2h}, \tag{10.5}$$

each with a truncation error of $O(h^2)$, are known as the backward and forward difference formulas, respectively, representing the slope of the chord $\overline{PA}$ and $\overline{BP}$, respectively (Fig. 10.1).

If we add (10.1) and (10.2), we get

$$u(x+h) + u(x-h) = 2\,u(x) + h^2\,u''(x) + O(h^4),$$

which yields the second order central difference formula

$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}, \tag{10.6}$$

with a truncation error of $O(h^4)$. In fact, based on the Taylor series expansion, the truncation error is approximately $h\,f'''(x)$ for both forward and backward schemes, and $\frac{1}{12}h^2 f^{(4)}(x)$ for the central scheme.

In the case of a function $u(x,t)$ of two independent variables x and t, we partition the x–axis into intervals of equal length h, and the t–axis into intervals of equal length k. The (x,t)–plane is divided into equal rectangles of area hk by the grid lines parallel to Ot, defined by $x_i = ih$, $i = 0, \pm1, \pm2, \ldots$, and by the grid lines parallel to Ox, defined by $y_j = ih$, $j = 0, \pm1, \pm2, \ldots$ (Fig. 10.2). We will use the following notation: Let $u_P = u(ih, jk) = u_{i,j}$ denote the value of the function $u(x,t)$ at a mesh point (node) $P\,(ih, jk)$.

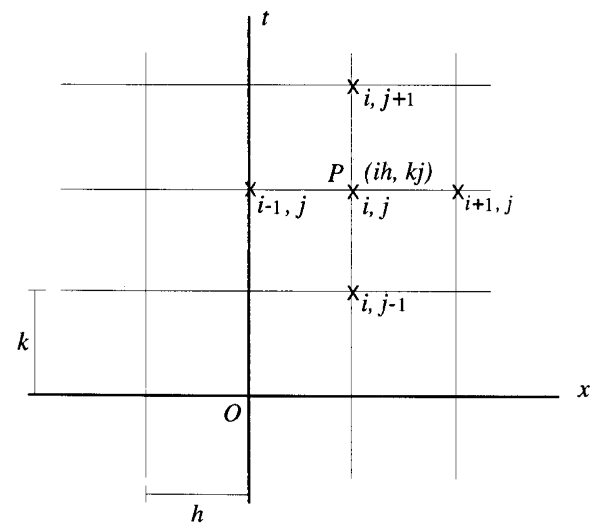


Fig. 10.2. Grid lines.

Then, in view of (10.6), we have the following three central difference schemes:

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}\bigg|_P = \frac{\partial^2 u}{\partial x^2}\bigg|_{i,j} &= \frac{u\left((i+1)h, jk\right) - 2u(ih, jk) + u\left((i-1)j, jk\right)}{h^2}\\ &= \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2}\\ &\equiv \frac{\delta_x^2\, U_{i,j}}{h^2}, \quad i = 1, 2, \dots, n-1,\end{aligned} \tag{10.7}$$

with a truncation error $-\dfrac{h^2}{12} u_{xxxx}(\bar{x}, t)$, where $x_{i-1} < \bar{x} < x_i$;

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2}\bigg|_P = \frac{\partial^2 u}{\partial t^2}\bigg|_{i,j} &= \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{k^2}\\ &\equiv \frac{\delta_t^2\, U_{i,j}}{k^2}, \quad j = 1, 2, \dots, m-1,\end{aligned} \tag{10.8}$$

with a truncation error $-\dfrac{k^2}{12} u_{tttt}(x, t')$, where $t_{j-1} < t' < t_j$; and

$$\frac{\partial^2 u}{\partial x \partial t}\bigg|_P = \frac{U_{i+1,j+1} - U_{i+1,j-1} - U_{i-1,j+1} + U_{i-1,j-1}}{4hk}, \tag{10.9}$$

with a truncation error $-\dfrac{h^2}{6} u_{xxxt}(\bar{x}, \bar{t}) - \dfrac{k^2}{6} u_{xttt}(x', t')$. Note that the difference operators δ_x^2 and δ_t^2 are the finite difference analogs of the partial differential operators $\dfrac{\partial^2}{\partial x^2}$ and $\dfrac{\partial^2}{\partial t^2}$, respectively. Moreover, from (10.3) we have the first order central difference formula

$$\begin{aligned}\frac{\partial u}{\partial x}\bigg|_P &= \frac{u\left((i+1)h, jk\right) - u\left((i-1)h, jk\right)}{h}\\ &= \frac{U_{i+1,j} - U_{i-1,j}}{h},\end{aligned} \tag{10.10}$$

with a truncation error $-\dfrac{h^2}{6} u_{xxx}(\bar{x}, \bar{t})$, and

$$\frac{\partial u}{\partial t}\bigg|_P = \frac{U_{i,j+1} - U_{i,j-1}}{k}, \tag{10.11}$$

with a truncation error $-\dfrac{k^2}{6} u_{ttt}(x, t')$. The first order backward and forward difference formulas for $u(x,t)$ can be similarly derived from (10.4) and (10.5).

The second order forward and backward difference schemes are respectively defined by

$$\frac{\partial^2 u}{\partial x^2}\bigg|_{i,j} = \frac{U_{i+2,j} - 2U_{i+1,j} + U_{i,j}}{h^2}, \quad i = 0, 1, 2, \dots, n-2, \tag{10.12}$$

and

$$\frac{\partial^2 u}{\partial x^2}\bigg|_{i,j} = \frac{U_{i,j} - 2U_{i-1,j} + U_{i-2,j}}{h^2}, \quad i = 0, 1, 2, \dots, n-2. \tag{10.13}$$

10.2. First Order Equations

Consider the first order quasi–linear partial differential equation of the form

$$a\, u_x + b\, u_y = c, \quad \text{or} \quad a\, p + b\, q = c, \tag{10.14}$$

where a, b, and c are functions of x, y, and u only (see §2.3). This equation yields the auxiliary system of ordinary differential equations

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}. \tag{10.15}$$

It was shown in Chapter 2 that at each point of the solution domain of Eq (10.14) there exists a direction, known as the characteristic, along which the solution of this partial differential equation coincides with the solutions of the ordinary differential equations (10.15). We shall denote the projection in the (x, y)–plane of this characteristic by C. Let us assume that the solution of u is known at every point of the characteristic C in the (x, y)–plane such that C is distinct from the initial curve Γ on which the initial value of u is prescribed.

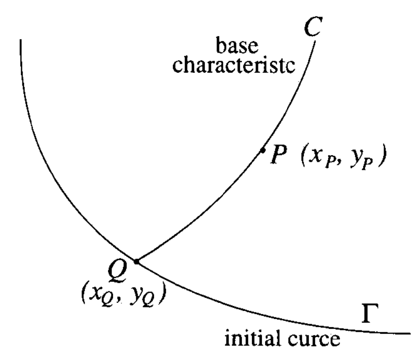


Fig. 10.3.

Let u be prescribed on the initial curve Γ which does not coincide with any characteristic C for the partial differential equation (10.14). Let $P\,(x_P, y_P)$ be a point on the characteristic C which passes through a point $Q\,(x_Q, y_Q)$ on Γ such that P and Q are close to each other, i.e., $|x_P - x_Q|$ is small (Fig. 10.3).

We shall denote the m–th approximation of u by $u^{(m)}$, and of y by $y^{(m)}$ for $m = 1, 2, \dots$. Let us assume that x_P is known. Then, in view of (10.3) and (10.4), the *first approximation* of $a\,dy = b\,dx$ is given by

$$a_Q\left(y_P^{(1)} - y_Q\right) = b_Q\,(x_P - x_Q), \tag{10.16}$$

which yields $y_P^{(1)}$, and from $a\,du = c\,dx$ we have

$$a_Q\left(u_P^{(1)} - u_Q\right) = c_Q\,(x_P - x_Q), \tag{10.17}$$

which yields $u_P^{(1)}$. For the *second approximation* we have, by using average values,

$$\frac{a_Q + a_P^{(1)}}{2}\left(y_P^{(2)} - y_Q\right) = \frac{b_Q + b_P^{(1)}}{2}\,(x_P - x_Q), \tag{10.18}$$

which yields $y_P^{(2)}$, and

$$\frac{a_Q + a_P^{(1)}}{2}\left(u_P^{(2)} - u_Q\right) = \frac{c_Q + c_P^{(1)}}{2}\,(x_P - x_Q), \tag{10.19}$$

which yields $u_P^{(2)}$. Subsequent (higher) approximations can be similarly obtained.

EXAMPLE 10.1. Consider the quasi–linear partial differential equation

$$x^{3/2}\,u_x - u\,u_y = u^2,$$

where $u = 1$ on the initial line $y = 0$, $0 < x < \infty$. The auxiliary system of equations is

$$\frac{dx}{x^{3/2}} = \frac{dy}{-u} = \frac{du}{u^2}.$$

Solving $\dfrac{dx}{x^{3/2}} = \dfrac{du}{u^2}$, we find that $\dfrac{2}{\sqrt{x}} = \dfrac{1}{u} - A$, where A is constant on a particular characteristic C. Thus, at a point $(x_Q, 0)$ on the initial curve Γ, we know that $u = 1$, and that gives $A = 1 - \dfrac{2}{\sqrt{x_Q}}$, which yields the solution along the base characteristic C_Q as

$$\frac{1}{u} = 1 + \frac{2}{\sqrt{x}} - \frac{2}{\sqrt{x_Q}}. \tag{10.20}$$

Similarly, solving $\dfrac{dy}{-u} = \dfrac{du}{u^2}$, we have $y = -\ln Bu$, and the solution on the characteristic C_Q is

$$u = e^{-y}. \tag{10.21}$$

Thus, the solution along a characteristic C_Q is given by (10.20) or (10.21). Eliminating u between Eqs (10.20) and (10.21), we find the equation of the characteristic C_Q as

$$y = \ln\left(1 + \frac{2}{\sqrt{x}} - \frac{2}{\sqrt{x_Q}}\right). \tag{10.22}$$

Now we will find the first and the second approximations at the point $P(1.1, y)$. Let $x_Q = 1$, $x = 1.1$. Then $dx = x_P - x_Q = 0.1$, $y_Q = 0$, and $u_Q = 1$.

First approximation: In view of (10.18) and (10.19), from $x^{3/2}\,dy = -u\,dx$ we have $x_Q^{3/2}\left(y_P^{(1)} - 0\right) = -u_Q\,dx$, which gives

$$y_P^{(1)} = -\frac{u_Q}{x_Q^{3/2}}\,dx = -0.1.$$

Also from $x_Q^{3/2}\,du = u_Q^2\,dx$, we have

$$x_Q^{3/2}\left(u_P^{(1)} - 1\right) = u_Q^2\,dx,$$

which yields

$$u_P^{(1)} = 1 + \frac{u_Q^2}{x_Q^{3/2}}\,dx = 1 + \frac{1}{(1)^{3/2}}\,(0.1) = 1.01.$$

Second approximation: The equation $u\,dx = -x^{3/2}\,dy$ gives

$$\frac{u_Q + u_P^{(1)}}{2}\,dx = -\frac{x_Q^{3/2} + x_P^{3/2}}{2}\left(y_P^{(2)} - y_Q\right),$$

$$\frac{1 + 1.01}{2}\,(0.1) = -\frac{(1)^{3/2} + (1.1)^{3/2}}{2}\,y_P^{(2)},$$

which yields $y_P^{(2)} = -0.097507$.
Also, from $x^{3/2}\,du = u^2\,dx$, we have

$$\frac{x_Q^{3/2} + x_P^{3/2}}{2}\left(u_P^{(2)} - u_Q\right) = \frac{u_Q^2 + u_P^2}{2}\,dx,$$
$$\frac{(1)^{3/2} + (1.1)^{3/2}}{2}\left(u_P^{(2)} - 1\right) = \frac{(1)^2 + (1.1)^2}{2}\,dx,$$

which yields $u_P^{(2)} = 1.10261$.
The exact values from (10.22) and (10.20) are

$$y_P = \ln\left(1 + \frac{2}{\sqrt{1.1}} - \frac{2}{1}\right) = -0.0976953,$$
$$u_P = e^{-y_P} = 0.91485.$$

Higher order approximations can be continued until the computed values differ from each other within the preassigned tolerance. ■

10.3. Second Order Equations

Let a region Ω in the (x,t)–plane be partitioned into a grid (x_i, t_j), $0 \le i \le n$, and $0 \le j \le m$, as in Fig. 10.2. By replacing all derivatives in a given partial differential equation

$$L\,u = f, \quad \text{for } x,\, t \in \Omega, \tag{10.23}$$

by their respective difference quotients, we obtain a finite difference equation of the form

$$D\,U_{i,j} = f_{i,j} \quad \text{for } (x_i, t_j) \in \Omega, \tag{10.24}$$

where D denotes the difference operator. Note that Eq (10.24) is the discretized form of the given equation (10.23) such that the solution $U_{i,j}$ approximates $u(x,t)$ at the grid nodes.

DEFINITION 10.1. The *local truncation error* $\varepsilon_{i,j}$ is the amount by which the solution $U_{i,j}$ fails to satisfy Eq (10.23), i.e.,

$$\varepsilon_{i,j} = D\,U_{i,j} - f_{i,j}. \tag{10.25}$$

DEFINITION 10.2. The difference equation (10.24) is said to be *consistent* with the given partial differential equation (10.23) if

$$\lim_{h,k\to 0} \varepsilon_{i,j} = 0. \tag{10.26}$$

DEFINITION 10.3. The *discretization error* $V_{i,j}$ is defined as $V_{i,j} = U_{i,j} - u_{i,j}$, where $U_{i,j}$ is the exact solution of Eq (10.24), and $u_{i,j}$ is the solution of Eq (10.23) evaluated at (x_i, t_j).

DEFINITION 10.4. The difference scheme, defined by Eq (10.24), is said to be *convergent* if

$$\lim_{h,k\to 0} \left|V_{i,j}\right| = \lim_{h,k\to 0} \left|U_{i,j} - u_{i,j}\right| = 0 \quad \text{for } (x_i, t_j) \in \Omega. \tag{10.27}$$

In some cases the difference method may not be convergent, although it may be consistent. There are examples discussing these issues in Abbott and Basco (1990). Also, an example suggested by Du Fort and Frankel is

$$\frac{U_{i,j+1} - U_{i,j-1}}{2k} = a^2\,\frac{U_{i+1,j} - U_{i,j+1} - U_i, j - 1}{}U_i - 1, jh^2,$$

which is always stable, where a is a constant, but is not consistent unless $k/h \to 0$ as k and $h \to 0$ (Carrier and Pearson, 1988, p. 263).

The concept of stability of a finite difference scheme is based on the propagation of the error $E_{i,0} = V_{i,0} - U_{i,0}$ with increasing j, where $U_{i,0}$ denote the initial values for the difference equation (10.24), and $V_{i,0}$ are the initial values obtained from the solution of a perturbed difference system. For a partial differential equation with a bounded solution, the difference scheme (10.24) is said to be *stable* if the errors $E_{i,j} = V_{i,j} - U_{i,j}$ are uniformly bounded in i as $j \to \infty$, i.e.,

$$\left|E_{i,j}\right| < M \quad \text{for } j > J,$$

where M is a positive constant and J a positive integer. A theorem, known as the Lax equivalence theorem, states that stability of a solution is both a necessary and sufficient condition for the convergence of a finite difference problem which is consistent with a well–posed initial and boundary value problem.

In order to illustrate the finite difference method for second order equations, we shall first consider a very simple example.

EXAMPLE 10.2. Let a one–dimensional steady state heat conduction problem be defined by

$$u'' = -x^2, \quad 0 < x < 1, \quad u(0) = 1, \quad u(1) = 2. \tag{10.28}$$

A partition of the interval $[0, 1]$ into equally spaced points is given by

$$0 = x_0 < x_1 < \cdots < x_n = 1,$$

with the step size $h = x_{i+1} - x_i = \frac{1}{n}$. Now, using the forward difference scheme (10.12) on the problem (10.28), we have

$$\frac{U_{i+2} - 2U_{i+1} + U_i}{h^2} = -x_i^2, \quad i = 0, 1, 2, \ldots, n-1.$$

In particular, say, for $n = 4$, we get the system of equations

$$\begin{aligned} 16\,(U_2 - 2U_1 + U_0) &= 0, \\ 16\,(U_3 - 2U_2 + U_1) &= -\frac{1}{16}, \\ 16\,(U_4 - 2U_3 + U_2) &= -\frac{1}{4}. \end{aligned}$$

In view of the boundary conditions, we have $U_0 = 1$, and $U_4 = 2$. Then the above system yields

$$\begin{aligned} 2U_1 - U_2 &= 1, \\ U_1 - 2U_2 + U_3 &= -\frac{1}{256}, \\ U_2 - 2U_3 &= -\frac{129}{4}, \end{aligned}$$

which, by using the Gauss elimination method, gives

$$U_1 = \frac{643}{512} = 1.25586, \quad U_2 = \frac{387}{256} = 1.51172, \quad U_3 = \frac{903}{512} = 1.7367.$$

Alternately, since only u'' occurs in problem (10.28), we can use the central difference scheme (10.7). Thus,

$$\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} = -x_i^2, \quad i = 1, 2, \ldots, n-1.$$

Then, with $n = 4$, we get the system of equations

$$\begin{aligned} 16\,(U_2 - 2U_1 + U_0) &= -\frac{1}{16}, \\ 16\,(U_3 - 2U_2 + U_1) &= -\frac{1}{4}, \\ 16\,(U_4 - 2U_3 + U_2) &= -\frac{9}{16}, \end{aligned}$$

or

$$\begin{aligned} 2U_1 - U_2 &= \frac{257}{256}, \\ U_1 - 2U_2 + U_3 &= -\frac{1}{64}, \\ U_2 - 2U_3 &= -\frac{521}{256}, \end{aligned}$$

which gives

$$U_1 = \frac{325}{256} = 1.29653, \quad U_2 = \frac{393}{256} = 1.53516, \quad U_3 = \frac{457}{256} = 1.7816.$$

The exact solution is

$$u(x) = 1 + \frac{13}{12}x - \frac{x^4}{12}.$$

A comparison with the exact solution shows that the central difference scheme gives a better approximation for problem (10.22). The results are shown in the following table.

x	Forward Difference	Central Difference	Exact
0.0	1.0	1.0	1.0
0.25	1.25586	1.26953	1.27051
0.5	1.51172	1.53516	1.53646
0.75	1.76367	1.78516	1.78613
1.0	2.0	2.0	2.0 ∎

Note that if the interval is $[a, b]$, then we use the transformation

$$\xi = \frac{x - a}{b - a},$$

to reduce this interval to $[0, 1]$.

10.3.1. Diffusion equations. Consider the one–dimensional diffusion equation

$$u_t = a^2\, u_{xx}, \tag{10.29}$$

where a^2 denotes the diffusivity. For the grid $(x_i, t_j) = (ih, jk)$, we shall discuss the following three finite difference schemes:

Forward Difference (Explicit Scheme):

$$\frac{U_{i,j+1} - U_{i,j}}{k} = a^2\, \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2},$$

or

$$U_{i,j+1} = \left(1 + r\,\delta_x^2\right)\, U_{i,j}, \tag{10.30}$$

where $r = a^2 k/h^2$.

Backward Difference (Implicit Scheme):

$$\frac{U_{i,j+1} - U_{i,j}}{k} = a^2\, \frac{U_{i+1,j+1} - 2U_{i,j+1} + U_{i-1,j+1}}{h^2},$$

or

$$\left(1 - r\,\delta_x^2\right)\, U_{i,j+1} = U_{i,j}. \tag{10.31}$$

Crank–Nicolson (Implicit Scheme):

$$\frac{U_{i,j+1} - U_{i,j}}{k} = a^2\, \frac{\delta_x^2 U_{i,j} + \delta_x^2 U_{i,j+1}}{h^2},$$

or

$$\left(1 - \frac{r}{2}\delta_x^2\right) U_{i,j+1} = \left(1 + \frac{r}{2}\delta_x^2\right) U_{i,j}. \tag{10.32}$$

Note that the Crank–Nicolson scheme is derived by averaging the finite differences at the points (i, j) and $(i, j+1)$ (see Exercise 10.2). Note that the above forward difference scheme (10.30) is conditionally stable iff $r < 1/2$, but the other two schemes (10.31) and (10.32) are always stable.

EXAMPLE 10.3. Consider the boundary value problem

$$\begin{aligned} &u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0,\\ &u(0,t) = 0 = u(1,t), \quad \text{for } t > 0,\\ &u(x,0) = f(x), \quad \text{for } 0 < x < 1. \end{aligned} \tag{10.33}$$

This problem has been studied in Example 5.2 by the separation of variables method. Since the space derivative is of second order, we shall use the central difference scheme (10.7), where $U_{i,j} = u(x_i, t_j)$ denotes the temperature at a point x_i at time t_j. For the time derivative we shall use the forward difference scheme

$$u_t = \frac{U_{i,j+1} - U_{i,j}}{k},$$

where $k = t_{j+1} - t_j$. Then Eq (10.33) is approximated by

$$\begin{aligned} &r\left[U_{i+1,j} - 2U_{i,j} + U_{i-1,j}\right] = U_{i,j+1} - U_{i,j},\\ &i = 1, 2, \dots, n-1, \quad \text{and} \quad j = 0, 1, 2, \dots, \end{aligned} \tag{10.34}$$

where $r = k/h^2$, and the boundary and initial conditions become

$$\begin{aligned} &U_{0,j} = 0 = U_{n,j}, \quad \text{for } j = 1, 2, \dots,\\ &U_{i,0} = f(x_i) = f_i, \quad \text{for } i = 0, 1, 2, \dots, n. \end{aligned}$$

After rearranging the terms in Eq (10.34), we get

$$U_{i,j+1} = r\, U_{i-1,j} + (1 - 2r)\, U_{i,j} + r\, U_{i+1,j}. \tag{10.35}$$

This difference equation allows us to compute $U_{i,j+1}$ from the values of U computed for earlier times. Note that the value $U_{i,0} = f_i$ is a prescribed value of u at time $t = 0$ and $x = x_i$.

We shall take $n = 4$, i.e., $h = 1/4$. The value of r in Eq (10.35) must be chosen properly so that the solution remains stable. It has been determined that for a stable solution the value of r must be such that the coefficients of u on the right side of Eq (10.35) remains non–negative (Smith, 1985, Ch. 3). Hence, we must have $0 < r \le \frac{1}{2}$. Then, in view of this restriction, we must have $k \le rh^2 = \frac{1}{32}$. In the marginal case when $r = 1/2$, we get $k = 1/32$. With these values of r and k, system (10.35) reduces to

$$U_{i,j+1} = \frac{1}{2}\left[U_{i-1,j} + U_{i+1,j}\right], \quad j = 0, 1, 2, \dots.$$

Since $U_{0,j} = U_{4,j} = 0$ for all j, we find
For j=0:

$$U_{i,1} = \frac{1}{2}\left[U_{i-1,0} + U_{i+1,0}\right],$$

or, successively,

$$U_{1,1} = \frac{1}{2} f_2, \quad U_{2,1} = \frac{1}{2}(f_1 + f_2), \quad U_{3,1} = \frac{1}{2} f_2;$$

For $j = 1$:

$$U_{1,2} = \frac{1}{2} U_{2,1}, \quad U_{2,2} = \frac{1}{2}(U_{1,1} + U_{3,1}), \quad U_{3,2} = \frac{1}{2} U_{2,1},$$

and so on. The values of $U_{i,j}$ for $f(x) = \cos \pi x$ are listed below in a tabular form for some successive values of t.

t	$x=0$	$x=0.25$	$x=0.5$	$x=0.75$	$x=1$
0	1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	−1
1/32	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0
1/16	0	0	0	0	0
3/32	0	0	0	0	0
1/8	0	0	0	0	0

Notice that the values average out in the outer columns. This will happen if $r = 1/2$ is chosen. The solution for problem (10.33) for different values of t with $r = 0.1$ is presented in the following table:

t	$x=0$	$x=0.25$	$x=0.5$	$x=0.75$	$x=1$
0	1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	−1
1/32	0	0.665685	0	−0.665685	0
1/16	0	0.532548	0	−0.532548	0
3/32	0	0.426039	0	−0.426039	0
1/8	0	0.340831	0	−0.340831	0 ■

10.3.2. Wave equation. To approximate the solution of the wave equation

$$u_{tt} = c^2 u_{xx}, \tag{10.36}$$

we can use the difference schemes (10.30)–(10.32) discussed earlier for the diffusion equation, but more frequently used schemes are the forward difference and the Crank-Nicolson. Let $(x_i, t_j) = (ih, jk)$,

$(i, j = 0, 1, \dots)$. Then
(a) *Explicit Scheme (Forward Difference)*:

$$\frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{k^2} = c^2 \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2},$$

or

$$\delta_t^2 U_{i,j} = c^2 \rho^2 \delta_x^2 U_{i,j}, \tag{10.37}$$

where $\rho = k/h$.

(b) *Implicit Scheme (Crank–Nicolson)*:

$$\frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{k^2} = \frac{c^2}{2} \Big[\frac{U_{i+1,j+1} - 2U_{i,j+1} + U_{i-1,j+1}}{h^2} + \frac{U_{i+1,j-1} - 2U_{i,j-1} + U_{i-1,j-1}}{h^2} \Big],$$

or

$$\delta_t^2 U_{i,j} = \frac{c^2 \rho^2}{2} \left[\delta_x^2 U_{i,j+1} + \delta_x^2 U_{i,j-1} \right]. \tag{10.38}$$

Note that the central difference schemes (10.7) and (10.8) are sometimes also used (see Example 10.5).

EXAMPLE 10.4. In order to use a finite difference scheme to solve the wave equation (10.36) with the Neumann initial conditions, i.e., for

$$u(x, 0) = f(x),$$
$$u_t(x, 0) = g(x),$$

we take $U_{i,0} = f(x_i) = f_i$. Now we will find the start–up value of $U_{i,1}$ as follows: Under the assumption that $f \in C^2$, we have by Taylor's theorem

$$\begin{aligned} u(x_i, t_0) &= u(x_i, 0) + k\, u_t(x_i, 0) + \frac{k^2}{2} u_{tt}(x_i, 0) + O(k^3) \\ &= u(x_i, 0) + k\, g(x_i) + \frac{k^2}{2} c^2 f''(x_i) + O(k^3) \\ &= u(x_i, 0) + k\, g_i + \frac{c^2 \rho^2}{2} \left[f_{i+1} - 2f_i + f_{i-1} \right] + O(k^2 h^2 + k^2), \end{aligned}$$

where $g_i = g(x_i)$, $f_i = f(x_i)$, and (10.6) is used to approximate $f''(x_i)$. The above expansion gives

$$g_i = \frac{U_{i,1} - U_{i,0}}{k} - \frac{1}{2} c^2 \rho^2 \left[f_{i+1} - 2f_i + f_{i-1} \right],$$

where $U_{i,0} = f_i$. Then the start–up value $U_{i,1}$ is given by

$$U_{i,1} = f_i + k\left[g_i + \frac{1}{2}c^2\rho^2\left(f_{i+1} - 2f_i + f_{i-1}\right)\right]. \blacksquare$$

EXAMPLE 10.5. Consider the wave equation $u_{tt} = u_{xx}$, $0 < x < 1$, $t > 0$, subject to the boundary conditions

$$u(0,t) = 0 = u(1,t) \quad \text{for } t > 0,$$

and the initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \quad \text{for } 0 < x < 1.$$

This problem is solved in Example 5.1 by the separation of variables method. We shall use the central difference schemes (10.7) and (10.8) for both u_{xx} and u_{tt}. Thus,

$$u_{xx} = \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2},$$
$$u_{tt} = \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{k^2}.$$

Then the wave equation, the boundary and the initial conditions reduce to

$$\begin{aligned} U_{i,j+1} &= \rho^2 U_{i-1,j} + 2(1-\rho^2)U_{i,j} + \rho^2 U_{i+1,j} - U_{i,j-1}, \\ &\quad i = 1, 2, \dots, n-1, \quad j = 0, 1, 2, \dots, \\ &\quad U_{0,j} = 0 = U_{n,j} \quad \text{for } j = 1, 2, \dots, \\ U_{i,0} &= f_i, \quad U_{i,1} - U_{i,0} = k\, g_i \quad \text{for } i = 0, 1, \dots, n, \end{aligned} \tag{10.39}$$

where $\rho = k/h$, and the forward difference scheme is used for the initial condition $u_t(x,0) = g(x)$.

Let $f(x) = \sin x$, and $g(x) = 1 - x$. Then, with $h = k = 1/4$, the initial conditions become

$$U_{i,0} = f_i = \sin x_i \quad \text{for } i = 0, 1, 2, 3, 4,$$
$$U_{i,1} - U_{i,0} = k\, g_i = \frac{i}{4}(1 - x_i) \quad \text{for } i = 0, 1, 2, 3, 4.$$

Then the first equation in (10.39) becomes

$$\begin{aligned} U_{1,j+1} &= U_{0,j} + U_{2,j} - U_{0,j-1}, \\ U_{2,j+1} &= U_{1,j} + U_{3,j} - U_{2,j-1}, \\ U_{3,j+1} &= U_{2,j} + U_{4,j} - U_{3,j-1}. \end{aligned}$$

Since the boundary conditions are $U_{0,j} = 0 = U_{4,j}$, we get the solution for successive values of $j = 1, 2, \dots$, which is presented in the following table.

t	$x = 0$	$x = 0.25$	$x = 0.5$	$x = 0.75$	$x = 1$
0	0	0	0	0	0
0.25	0.2474	0.4272	0.3421	0.5728	0.6579
0.5	0.4794	0.5895	0.6707	0.7526	0.9021
0.75	0.6816	0.7230	−0.0921	−0.0522	0.8447
1.0	0	0	0	0	0 ■

EXAMPLE 10.6. Solve $u_{tt} = 4\,u_{xx}$, $0 < x < 1$, $t > 0$, subject to the initial conditions $u(x,0) = \sin \pi x$, $u_t(x,0) = 0$ for $0 < x < 1$, and the boundary conditions $u(0,t) = 0 = u(1,t)$ for $t > 0$, by using (a) the explicit scheme (10.37), and (b) the implicit scheme (10.38). Note that the exact solution is

$$u(x,t) = \sin \pi x \, \cos 2\pi t,$$

and d'Alembert's solution is

$$u(x,t) = \frac{1}{2}\left[\sin(\pi x + 2\pi t) + \sin(\pi x - 2\pi t)\right].$$

The Mathematica program for (a) is given below, and the corresponding output is presented in a tabular form.

In[1]:=

```
Clear[u,n,m,i,j,r,k,h];
n:= 5
m:= 5
c:= 2
h:= 1/(n-1)
```

```
k:= 1/(m-1)
r:= c k/h;
u = Table[1,{n},{m}];
```

In[9]:=

```
f[i_]:= Sin[Pi 1/(n-1) (i-1)]
g[i_]:= 0
```

In[11]:=

```
Do[ u[[i,1]] = f[i] ;
u[[i,2]] = (1-r^2) f[i] + k g[i] +
   r^2/2 (f[i+1]+f[i-1]);,{i,1,n} ];
```

In[13]:=

```
Do[
u[[1,j]] = 0;
u[[n,j]] = 0;,
{j,1,n} ];
```

In[14]:=

```
Do[
u[[i,j]] = r^2 u[[i-1,j-1]] + 2(1-r^2) u[[i,j-1]] +
   r^2 u[[i+1,j-1]] - u[[i,j-2]], {i,2,n-1},
   {j,3,m}];

u//N//Chop//Transpose//TableForm
```

Out[15]=

0	0	0	0	0
0.707107	−0.12132	−0.665476	8.11418	−44.0196
1.	−0.171573	−0.941125	7.15642	−5.54069
0.707107	−0.12132	−0.665476	0.349676	27.1931
0	0	0	0	0


```
(* The exact values follow *)
```

In[15]:=

```
Table[Sin[Pi h (i-1)] Cos[2 Pi k (j-1)],
   {i,1,n},{j,1,m}]//N//TableForm
```

Out[20]=

0	0	0	0	0
0.707107	0	−0.707107	0	0.707107
1.	0	−1.	0	1.
0.707107	0	−0.707107	0	0.707107
0	0	0	0	0

```
(* Note the value of r and the instability *)
```

It is instructive to experiment with different values for h and k in order to study stability of solutions and find better approximations.

Part (b) can be similarly done by modifying the Mathematica code in (a) and using (10.38). ∎

10.3.3. Poisson's equation. Consider Poisson's equation in a rectangle $\Omega = \{0 < x < a,\ 0 < y < b\}$ with boundary Γ:

$$u_{xx} + u_{yy} = f(x, y), \tag{10.40}$$

subject to the Dirichlet boundary condition

$$u = g(x, y) \quad \text{on } \Gamma. \tag{10.41}$$

For $f = 0$, Eq (10.40) becomes Laplace's equation. We shall analyze the simple case when $a = b$, with the uniformly spaced grid lines of size $h = a/4$. The nodes are then given by $(x_m, y_n) = (mh, nk)$, where $m, n = 0, 1, 2, 3, 4$ (see Fig. 10.4). By using the central difference scheme (10.7), Eq (10.40) reduces to the finite difference equation

$$\delta_x^2 U_{m,n} + \delta_y^2 U_{m,n} = h^2 f_{m,n}, \tag{10.42}$$

where $f_{m,n} = f(x_m, y_n)$. The boundary condition (10.41) then becomes $U_{m,n} = g_{m,n} = g(x_m, y_n)$ for $m, n = 0, 1, 2, 3, 4$.

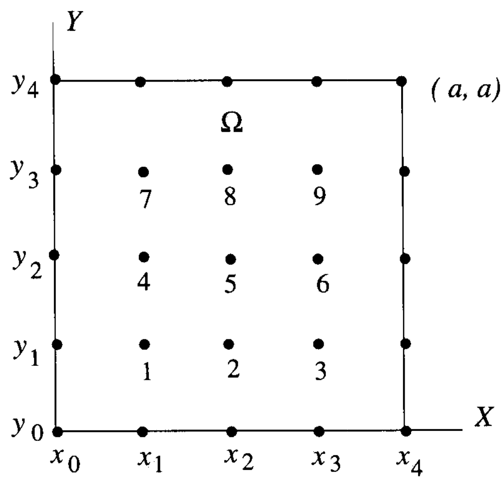


Fig. 10.4. Grid lines on the square Ω.

By reordering Eq (10.42), we get

$$4U_{m,n} - U_{m+1,n} - U_{m-1,n} - U_{m,n+1} - U_{m,n-1} = -h^2 f_{m,n}. \quad (10.43)$$

The unknown values of u are at the nodes $1, 2, \ldots, 9$ (Fig. 10.4). We shall use the notation:

$$U_{1,1} = U_1, \quad U_{2,1} = U_2, \quad U_{3,1} = U_3,$$
$$U_{1,2} = U_4, \quad U_{2,2} = U_5, \quad U_{3,2} = U_6,$$
$$U_{1,3} = U_7, \quad U_{2,3} = U_8, \quad U_{3,3} = U_9.$$

Then Eq (10.43) can be written in the form of a system of algebraic equation

$$[A]\{U\} = \{F\}, \quad (10.44)$$

where the matrix $[A]$ is a 9×9 symmetric matrix

$$[A] = \begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ & & 4 & -1 & 0 & -1 & 0 & 0 & 0 \\ & & & 4 & -1 & 0 & -1 & 0 & 0 \\ & & & & 4 & -1 & 0 & -1 & 0 \\ & & & & & 4 & 0 & 0 & -1 \\ & & & & & & 4 & -1 & 0 \\ & & & & & & & 4 & -1 \\ \text{sym} & & & & & & & & 4 \end{bmatrix}, \quad (10.45)$$

and

$$\{U\} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \\ U_8 \\ U_9 \end{Bmatrix}, \quad \{F\} = \begin{Bmatrix} -h^2 f_{1,1} + g_{0,1} + g_{1,0} \\ -h^2 f_{2,1} + g_{2,0} \\ -h^2 f_{3,1} + g_{3,0} \\ -h^2 f_{1,2} + g_{0,2} \\ -h^2 f_{2,2} \\ -h^2 f_{3,1} + g_{4,2} \\ -h^2 f_{1,3} + g_{0,3} + g_{1,4} \\ -h^2 f_{2,3} + g_{2,4} \\ -h^2 f_{2,3} + g_{4,3} + g_{3,4} \end{Bmatrix}. \quad (10.46)$$

The order of the matrix $[A]$ is $(l/h - 1) \times (l/h - 1)$. The dimensions of the vectors $\{U\}$ and $\{F\}$ are $(l/h - 1)$. The difference method explained above has a truncation error of $O(h^2)$. If the boundary value problem (10.40)–(10.41) has a unique solution, h is small and $[A]$ is nonsingular, then system (10.44) has a unique solution. The system (10.44) can be easily solved.

In the case of a curved boundary Γ, consider a node in Ω, which has at least one adjacent node outside Ω (see Fig. 10.5).

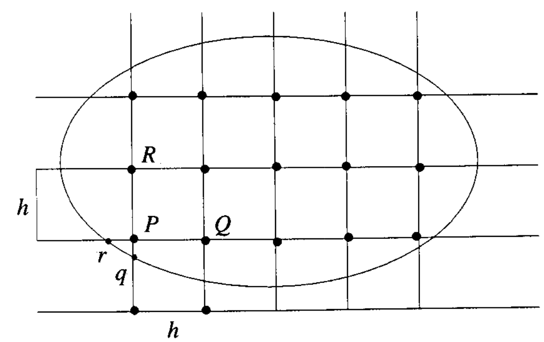


Fig. 10.5. Curved boundary Γ.

Let $P = (x_m, y_n)$ be a point inside Ω and near the boundary Γ. Then the coordinates of the adjacent points q and r on the boundary Γ are:

$$q = (x_{m+1}, y_n) = (x_m + \alpha h, y_n),$$
$$r = (x_m, y_{n+1}) = (x_m h, y_n + \beta h),$$

where $0 < \alpha,\ \beta < 1$. Moreover, the values of $u(q)$ and $u(r)$ are known, since

u is prescribed on the boundary Γ. By Taylor's theorem

$$u(q) = u(P) + \alpha\, h\, u_x(P) + \frac{\alpha^2 h^2}{2!}\, u_{xx}(P) + O(h^3),$$
$$u(Q) = u(P) - h\, u_x(P) + \frac{h^2}{2!}\, u_{xx}(P) + O(h^3).$$

After eliminating $u_x(P)$ from these two expansions, we obtain

$$u_{xx}(P) = \frac{2[u(q) - (1+\alpha)u(P) + \alpha\, u(Q)]}{\alpha(1+\alpha)} + O(h).$$

Similarly,

$$u_{yy}(P) = \frac{2[u(r) - (1+\beta)u(P) + \beta\, u(R)]}{\beta(1+\beta)} + O(h).$$

Hence the finite difference approximation for Poisson's equation (10.40) defined on a region with a curved boundary Γ is

$$\frac{U(Q)}{1+\alpha} + \frac{U(R)}{1+\beta} - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) U(P) + \frac{U(q)}{\alpha(1+\alpha)} + \frac{U(r)}{\beta(1+\beta)} = \frac{1}{2} h^2 f(P). \tag{10.47}$$

EXAMPLE 10.7. To find finite difference solutions of Laplace's equation $u_{xx} + u_{yy} = 0$ on the quarter–circular region $\Omega = \{x^2 + y^2 < 1, y > 0\}$, subject to the boundary conditions $u(x,y) = 10$ on $x^2 + y^2 = 1, y > 0\}$, and $u(x,y) = 0$ on $0 < x < 1, y = 0$ and $u_x = 0$ on $x = 0, 0 < y < 1$, we choose the grid with $h = 1/2$ (see Fig. 10.6).

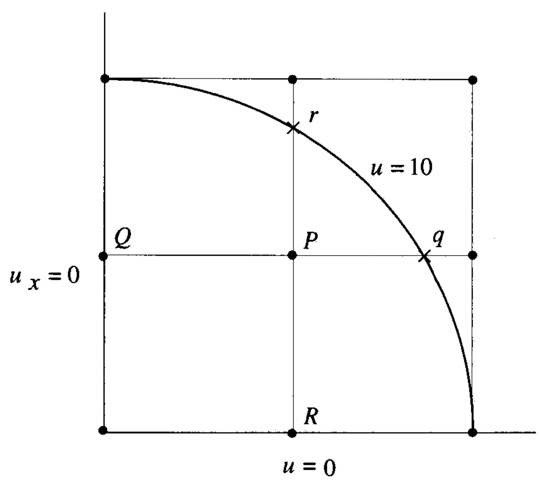


Fig. 10.6.

From the boundary conditions we have $U_{0,0} = 0 = U_{1,0}$ and $U_{2,0} = 10 = u(q) = u(r)$. The only unknown values of u are at the nodes $P = (x_1, y_1)$, and $Q = (x_0, y_1)$. From (10.43) the difference equation centered at Q is

$$4U_{0,1} - U_{1,1} - U_{-1,1} - U_{0,2} - U_{0,0} = 0. \tag{10.48}$$

Note that the coordinates of $q = (\sqrt{3}\,h, h)$, and of $r = (h, \sqrt{3}\,h)$. Hence $\alpha = \beta = \sqrt{3} - 1$. Also, the boundary conditions yields $U_{1,1} = 0$. Thus, from (10.48) we have

$$2U_{0,1} - U_{1,1} = 5. \tag{10.49}$$

Then the difference equation (10.47) gives

$$\frac{U(Q)}{\sqrt{3}} + \frac{U(R)}{\sqrt{3}} - \frac{2U(P)}{\sqrt{3}-1} + \frac{U(q)}{\sqrt{3}(\sqrt{3}-1)} = 0,$$

or

$$\frac{U_{0,1}}{\sqrt{3}} + \frac{U_{1,0}}{\sqrt{3}} - \frac{2U_{1,1}}{\sqrt{3}-1} + \frac{U(q)}{\sqrt{3}(\sqrt{3}-1)} = 0,$$

which yields

$$(1 - \sqrt{3})\, U_{0,1} + 2\sqrt{3}\, U_{1,1} = 20. \tag{10.50}$$

By solving (10.49) and (10.50), we get

$$U_{0,1} = \frac{20 + 10\sqrt{3}}{3\sqrt{3} + 1} = 6.02317, \quad U_{1,1} = \frac{35 + 5\sqrt{3}}{3\sqrt{3} + 1} = 7.046349. \blacksquare$$

WWW The Notebook `fd.ma` can be found on the CRC web server. This will help not only to solve problems but also generate Mathematica codes for difference schemes.

10.4. Exercises

10.1. Use the central difference scheme, with $n = 4$, to solve

$$u'' - u = -2, \quad 0 < x < 1, \quad u'(0) = 0, \quad u(1) = 1.$$

HINT: $16(U_{i+1} - 2U_i + U_{i-1}) - U_i = -2$ for $i = 1, 2, 3, 4$, $h = 1/4$, and the boundary conditions give $U_{i+1} = U_i$, and $U_4 = 1$.

10.2. Derive the Crank–Nicolson scheme (10.32).
ANS. The central difference at the point (i,j) is

$$u_{xx} = \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} = \frac{\delta_x^2 U_{i,j}}{h^2},$$

and at the point $(i, j+1)$ it is given by

$$u_{xx} = \frac{U_{i+1,j+1} - 2U_{i,j+1} + U_{i-1,j+1}}{h^2} = \frac{\delta_x^2 U_{i,j+1}}{h^2}.$$

The result follows by taking the average of these two differences.

10.3. (a) Show that the forward difference scheme (10.30) has a truncation error of $O(k+h^2)$; (b) Show that the local truncation error is of $O(k^2+h^4)$ if $r = 1/6$.
SOLUTION. (a) Note that for $(x_i, t_j) = (ih, jk)$, we have

$$\begin{aligned}(u_t - a^2 u_{xx})_{i,j} &= \frac{U_{i,j+1} - U_{i,j}}{k} - a^2 \frac{\delta_x^2 U_{i,j}}{h^2} \\ &\quad - \frac{k}{2} u_{tt}(x_i, \bar t_j) + \frac{a^2 h^2}{12} u_{xxxx}(\bar x_i, t_j),\end{aligned}$$

where $x_{i-1} < \bar x < x_i$ and $t_j < \bar t_j < t_{j+1}$. Then the truncation error is given by

$$\frac{k}{2} u_{tt}(x_i, \bar t) - \frac{a^2 h^2}{12} u_{xxxx}(x \bar x_i, t_j) = O(k + h^2),$$

provided u_{tt} and u_{xxxx} are bounded.
(b) If $r = 1/6$, then $(u_t - a^2 u_{xx})_{i,j} = 0$ leads to

$$\begin{aligned}\frac{U_{i,j+1} - U_{i,j}}{k} - a^2 \frac{\delta_x^2 U_{i,j}}{h^2} &= \left[\frac{k}{2} u_{tt}(x_i, \bar t) - \frac{a^2h^2}{12} u_{xxxx}(x\bar x_i, t_j)\right]_{i,j} \\ &= O(k^2) + O(h^4).\end{aligned}$$

10.4. Show that the forward scheme (10.30) is convergent for the problem $u_t = a^2 u_{xx}$, $0 < x < 1$, $t > 0$, subject to the Dirichlet boundary conditions $u(0,t) = f(t)$, $u(1,t) = g(t)$ for $t > 0$, and the initial condition $u(x, 0) = F(x)$ for $0 < x < 1$.
SOLUTION. With $(x_i, t_i) = (ih, jk)$ for $i = 0, 1, \dots, n$ and $j =$

$0, 1, \dots, m$, and with $nh = 1$, $jm = t_0$, where t_0 is a prescribed value, $0 \le t \le t_0$, we have

$$\begin{aligned}&U_{i,j+1} = U_{i,j} + r\,\delta_x^2 U_{i,j},\\ &U_{i,0} = F(x_i) = F_i, \quad U_{0,j} = f(t_j) = f_j, \quad U_{n,j} = g(t_j) = g_j.\end{aligned}$$

Let $V_{i,j} = U_{i,j} - u_{i,j}$ be the discretization error, as in Definition 10.3. Then

$$\begin{aligned}V_{i,j+1} = r\,V_{i-1,j} + (1-2r)V_{i,j} + r\,V_{i+1,j} &+ \frac{k^2}{2} u_{tt}(x_i, \bar t_j) \\ &+ \frac{a^2 k h^2}{12} u_{xxxx}(\bar x_i, t_j),\end{aligned}$$

where $x_{i-1} < \bar x_i < x_i$ and $t_j < \bar t_j < t_{j+1}$. Since $r \le 1/2$, we have

$$\begin{aligned}|V_{i,j+1}| &\le r\,|V_{i-1,j}| + (1-2r)|V_{i,j}| + Ak^2 + Bkh^2 \\ &\le \max_{0<i<n} |V_{i,j}| + Ak^2 + Bkh^2,\end{aligned}$$

where

$$A = \max\left|\frac{1}{2} u_{tt}(x,t)\right|, \quad B = \max\left|\frac{a^2}{12} u_{xxxx}(x,t)\right|,$$

since both u_{tt} and u_{xxxx} are assumed continuous. The above inequality gives

$$\|V_{j+1}\| \le \|V_j\| + Ak^2 + Bkh^2,$$

where $\|V_j\| = \max_{0<i<n} |V_{i,j}$, or, since $\|V_0\| = 0$,

$$\|V_j\| \le j\left(Ak^2 + Bkh^2\right) \le t_0\left(Ak + Bh^2\right),$$

i.e., $\|V_j\| \to 0$ uniformly in the domain of definition of the problem as $h, k \to 0$ for $0 \le t \le t_0$.

10.5. Derive the matrix form of the system of difference equations for the problem in Example 10.5 if the backward scheme (10.31) is used.
ANS. The matrix equation is $[A]\{U\} = \{B\}$, where

$$[A] = \begin{bmatrix} 1+2r & -2r & 0 & \cdots & 0 & 0 & 0 \\ -r & 1+2r & -r & \cdots & 0 & 0 & 0 \\ & -r & 1+2r & \cdots & 0 & 0 & 0 \\ & & & \cdots & & & \\ & & & \cdots & -r & 1+2r & -r \\ \text{sym} & & & \cdots & 0 & -2r & 1+2r \end{bmatrix},$$

$$\{U\} = \begin{Bmatrix} U_{0,j+1} \\ U_{1,j+1} \\ U_{2,j+1} \\ \cdots \\ U_{n-1,j+1} \\ U_{n,j+1} \end{Bmatrix}, \quad \{B\} = \begin{Bmatrix} U_{0,j} - 2hrf_{j+1} \\ U_{1,j} \\ U_{2,j} \\ \cdots \\ U_{n-1,j} \\ U_{n,j} + 2hrg_{j+1} \end{Bmatrix}.$$

10.6. Solve the heat equation $u_t = u_{xx}$, $0 < x < 1$, $t > 0$, with $n = 4$, $r = 1/2$, $k = 1/32$, subject to the boundary and initial conditions $u(0,t) = 0 = u(1,t)$ for $t > 0$, and $u(x,0) = x$.

ANS. Exact solution: $u = \sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi} \sin n\pi x \, e^{-n^2\pi^2 t}$.

t	$x=0$	$x=0.25$	$x=0.5$	$x=0.75$	$x=1$
0	0	1/4	1/2	3/4	1
1/32	0	0.25	0.5	0.75	0
1/16	0	0.25	0.5	0.25	0
3/32	0	0.25	0.25	0.25	0
1/8	0	0.125	0.25	0.125	0 ■

10.7. Solve $u_t = u_{xx} + 1$, $0 < x < 1$, $t > 0$, subject to the boundary and initial conditions $u(0,t) = 0 = u(1,t)$ for $t > 0$ and the initial condition $u(x,0) = 0$ for $0 < x < 1$, with $r = 1/2$, $k = 1/32$, and $n = 4$.

ANS.

t	$x=0$	$x=0.25$	$x=0.5$	$x=0.75$	$x=1$
0	0	0	0	0	0
1/32	0	0.03125	0.03125	0.03125	0
1/16	0	0.046875	0.0625	0.046875	0
3/32	0	0.0625	0.078125	0.0625	0
1/8	0	0.0703125	0.09375	0.0703125	0 ■

10.8. Solve the wave equation $u_{tt} = u_{xx}$, $0 < x < 1$, $t > 0$, with $r = 1/2$, $k = 1/32$, and $n = 4$, subject to the boundary and the initial conditions:
(a) $u(0,t) = 0 = u(1,t)$ for $t > 0$, and $u(x,0) = 0$, $u_t(x,0) = 1$ for $0 < x < 1$.
(b) $u(0,t) = 0 = u(1,t)$ for $t > 0$, and $u(x,0) = \sin \pi x$, $u_t(x,0) = 0$

for $0 < x < 1$.

ANS. (a)

t	$x=0$	$x=0.25$	$x=0.5$	$x=0.75$	$x=1$
0	0.0	0.0	0.0	0.0	0.0
0.25	0.0	0.25	0.25	0.25	0.0
0.5	0.0	0.25	0.5	0.25	0.0
0.75	0.0	0.25	0.25	0.25	0
1.0	0.0	0.0	0.0	0.0	0.0 ■

(b)

t	$x=0$	$x=0.25$	$x=0.5$	$x=0.75$	$x=1$
0	0.0	0.7071	1.0	0.7071	0.0
0.25		0.7071	1.	0.7071	0.0
0.5	0.0	0.2929	0.4142	0.2929	0.0
0.75	0.0	−0.2929	−0.4142	−0.2929	0.0
1.0	0.0	−0.7071	−1.0	−0.7071	0.0 ■

10.9. Find the system of equations $[A]\{U\} = \{F\}$ for the Neumann boundary value problem

$$u_{xx} + u_{yy} = f(x,y), \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} = g(x,y) \quad \text{on } \Gamma,$$

where Ω is the rectangle $\{0 < x < a, 0 < y < b\}$. Choose the nodes (mh, nh) for $m = 0,1,2,\dots,M$ and $n = 0,1,2,\dots,N$, such that $Nh = b$.

ANS. Eq (10.43) for $m = 0,1,2,\dots,M$ and $n = 0,1,2,\dots,N$, leads to

$$\begin{aligned} U_{M+1,N} - U_{M-1,N} &= 2h\, g_{m,n}, \quad n = 1,2,\dots,N-1, \\ U_{M,N+1} - U_{M,N-1} &= 2h\, g_{m,N}, \quad m = 1,2,\dots,M-1, \end{aligned}$$

and the boundary condition gives

$$\begin{aligned} U_{-1,n} - U_{1,n} &= 2h\, g_{0,n}, \quad n = 1,2,\dots,N-1, \\ U_{m,-1} - U_{m,1} &= 2h\, g_{m,0}, \quad m = 1,2,\dots,M-1. \end{aligned}$$

At a corner node where the outward normal $\mathbf{n}$ is undefined, we shall take the normal derivative as the average value of the two normal derivatives at the two adjacent boundary nodes. Thus, the boundary conditions reduce to

$$\begin{aligned} U_{-1,0} + U_{0,-1} &= U_{1,0} + U_{0,1} + 4h\, g_{0,0}, \\ U_{M,-1} + U_{M+1,0} &= U_{M,1} + U_{M-1,0} + 4h\, g_{M,0}, \\ U_{M+1,N} + U_{M,N=1} &= U_{M-1,N} + U_{M,N-1} + 4h\, g_{M,N}, \\ U_{0.N+1} + U_{-1,N} &= U_{0,N-1} + U_{1,N} + 4h\, g_{0,N}. \end{aligned}$$

10.10. In Exercise 10.9, take $M = N = 3$, and $g = 0$. Determine the matrix $[A]$ and the vectors $\{U\}$ and $\{F\}$.
ANS.

$$[A] = \begin{bmatrix} 4 & -2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ & 4 & -1 & 0 & -2 & 0 & 0 & 0 & 0 \\ & & 4 & 0 & 0 & -2 & 0 & 0 & 0 \\ & & & 4 & -2 & 0 & -1 & 0 & 0 \\ & & & & 4 & -1 & 0 & -1 & 0 \\ & & & & & 4 & 0 & 0 & -1 \\ & & & & & & 4 & -2 & 0 \\ & & & & & & & 4 & -1 \\ \text{sym} & & & & & & & & 4 \end{bmatrix},$$

and

$$\{U\} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \\ U_8 \\ U_9 \end{Bmatrix}, \qquad \{F\} = -h^2 \begin{Bmatrix} f_{1,1} \\ f_{2,1} \\ f_{3,1} \\ f_{1,2} \\ f_{2,2} \\ f_{3,1} \\ f_{1,3} \\ f_{2,3} \\ f_{2,3} \end{Bmatrix}.$$

10.11. Let Ω be the square region $\{0 < x, y < 1\}$ with boundary Γ. Find the finite difference equation for the boundary value problem

$$u_{xx} + u_{yy} + c\, u = f(x, y) \quad \text{in } \Omega,$$

subject to the Dirichlet boundary condition $u = g(x, y)$ on Γ.
ANS. $(c-4)U_{m,n} + U_{m,n-1} + U_{m-1,n} + U_{m,n-1} + U_{m+1,n} = f_{m,n}$.

10.12. Find the finite difference equation for the boundary value problem

$$(a\, u_x)_x + (b\, u_y)_y = 0, \quad \text{in } \Omega = \{0 < x, y < 1\},$$

where a and b are positive functions of x, y and u.
ANS. On the square grid $(x_m, y_n) = (mh, nk)$, the difference equation is given by

$$\begin{aligned} a_{m+1/2,n}\, U_{m+1,n} - \left(a_{m+1/2,n} + a_{m-1/2,n}\right) U_{m,n} + a_{m-1/2,n}\, U_{m-1,n} \\ + b_{m,n+1/2}\, U_{m,n+1} - \left(b_{m,n+1/2} + b_{m,n-1/2}\right) U_{m,n} + b_{m,n-1/2}\, U_{m,n-1} \\ = h^2 f_{m,n}. \end{aligned}$$

A

Green's Identities

A.1. Green's Identities

Let Ω be a finite domain in R^n bounded by a piecewise smooth orientable surface $\partial\Omega$, and let w and F be scalar functions and $\mathbf{G}$ a vector function in the class $C^0(\Omega)$. Then

$$\text{Gradient Theorem:} \quad \int_\Omega \nabla F\, d\Omega = \oint_{\partial\Omega} \mathbf{n} F\, dS,$$

$$\text{Divergence Theorem:} \quad \int_\Omega \nabla\cdot\mathbf{G}\, d\Omega = \oint_{\partial\Omega} \mathbf{n}\cdot\mathbf{G}\, dS,$$

$$\text{Stokes Theorem:} \quad \int_\Omega \nabla\times\mathbf{G}\, d\Omega = \oint_{\partial\Omega} \mathbf{G}\cdot\mathbf{t}\, dS,$$

where $\mathbf{n}$ is the outward normal to the surface $\partial\Omega$, $\mathbf{t}$ is the tangent vector at a point on $\partial\Omega$, $\oint$ denotes the surface or line integral, and dS denotes the surface or line element depending on the dimension of Ω. The divergence theorem in the above form is also known as the Gauss theorem. Stokes' theorem is a generalization of Green's theorem which in R^2 states that if $\mathbf{G} = (G_1, G_2)$ is a continuously differentiable vector field defined on a region containing $\Omega \cup \partial\Omega \subset R^2$ such that $\partial\Omega$ is a smooth closed contour, then

$$\int_\Omega \left(\frac{\partial G_2}{\partial x_1} - \frac{\partial G_1}{\partial x_2}\right) dx_1\, dx_2 = \oint_{\partial\Omega} G_1\, dx_1 + G_2\, dx_2.$$

The above theorems lead to the following two useful identities:

$$\int_\Omega (\nabla\mathbf{G}) w\, d\Omega = -\int_\Omega (\nabla w)\mathbf{G}\, d\Omega + \int_{\partial\Omega} \mathbf{n}\, w\, \mathbf{G}\, dS, \tag{A.1}$$

$$-\int_\Omega (\nabla^2\mathbf{G}) w\, d\Omega = \int_\Omega (\nabla w)\cdot(\nabla\mathbf{G})\, d\Omega - \oint_{\partial\Omega} \frac{\partial\mathbf{G}}{\partial n} w\, dS, \tag{A.2}$$

where $\dfrac{\partial}{\partial n} = \nabla\cdot\mathbf{n} = \sum n_{x_i}\dfrac{\partial}{\partial x_i}$, $i = 1, \cdots, n$, is the normal derivative operator. The i–th component of the formula (A.1) can be written in a useful form as

$$\int_\Omega w\frac{\partial\mathbf{G}}{\partial x_i}\, d\Omega = -\int_\Omega \frac{\partial w}{\partial x_i} + \oint_{\partial\Omega} n_{x_i}\, w\, \mathbf{G}\, dS. \tag{A.3}$$

In R^3 let the functions $M(x)$, $N(x)$, and $P(x)$, $x = (x_1, x_2, x_3) \equiv (x, y, z) \in \Omega$, be the components of the vector $\mathbf{G}$. Then, by the divergence theorem

$$\int_\Omega \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right) d\Omega = \oint_{\partial\Omega} [M\cos(\mathbf{n}, x) + N\cos(\mathbf{n}, y) + P\cos(\mathbf{n}, z)]\, dS, \tag{A.4}$$

with the direction cosines $\cos\mathbf{n}x$, $\cos\mathbf{n}y$ and $\cos\mathbf{n}z$. If we take $M = u\dfrac{\partial v}{\partial x}$, $N = u\dfrac{\partial v}{\partial y}$, and $P = u\dfrac{\partial v}{\partial z}$, then (A.4) yields

$$\int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\right) d\Omega = \int_{\partial\Omega} u\frac{\partial v}{\partial n}\, dS - \int_\Omega u\nabla^2 v\, d\Omega, \tag{A.5}$$

which is known as Green's first identity. Moreover, if we interchange u and v in (A.4), we get

$$\int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\right) d\Omega = \int_{\partial\Omega} v\frac{\partial u}{\partial n}\, dS - \int_\Omega v\nabla^2 u\, d\Omega. \tag{A.6}$$

If we subtract (A.5) from (A.6), we obtain Green's second identity:

$$\int_\Omega \left(u\nabla^2 v - v\nabla^2 u\right) d\Omega = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS, \tag{A.7}$$

which is also known as Green's reciprocity theorem. Note that Green's identities are valid even if the domain Ω is bounded by finitely many closed surfaces;

however, in that case the surface integrals must be evaluated over all surfaces that make the boundary of Ω. If we take $v = 1$ in (A.5), then

$$\int_{\partial\Omega} \frac{\partial u}{\partial n}\, dS = \int_{\Omega} \nabla^2 u\, d\Omega. \tag{A.8}$$

If we take $u = v$ in (A.5), then

$$\int_{\partial\Omega} \frac{\partial u}{\partial n}\, dS = \int_{\Omega} \left(u\nabla^2 u + |\nabla u|^2\right) d\Omega. \tag{A.9}$$

A.2. Exercises

A.1. Use Green's formulas to show that

$$\iint_R \nabla^2 u\, \nabla^2 w\, dx\, dy = \iint_R \nabla^4 u\, w\, dx\, dy + \int_C \nabla^2 u \frac{\partial w}{\partial n}\, ds - \int_C \frac{\partial}{\partial n} \nabla^2 u\, w\, ds.$$

ANS.

$$\begin{aligned}\iint_R \nabla^2 u\, \nabla^2 w\, dx\, dy &= \\ &= \iint_R \left[\frac{\partial}{\partial x}\left(\nabla^2 u \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nabla^2 u \frac{\partial w}{\partial y}\right)\right] dx\, dy \\ &- \iint_R \left[\frac{\partial}{\partial x}\left(w \frac{\partial}{\partial x}\nabla^2 u\right) + \frac{\partial}{\partial y}\left(w \frac{\partial}{\partial y}\nabla^2 u\right)\right] \\ &+ \iint_R w\left(\frac{\partial^2}{\partial x^2}\nabla^2 u + \frac{\partial^2}{\partial y^2}\nabla^2 u\right) dx\, dy \\ &= \iint_R (\nabla^4 u) w\, dx\, dy + \int_C \nabla^2 u\left(\frac{\partial w}{\partial x}\, dy - \frac{\partial w}{\partial y}\, dx\right) \\ &- \int_C w\left(\frac{\partial(\nabla^2 u)}{\partial x}\, dy - \frac{\partial(\nabla^2 u)}{\partial y}\, dx\right) \\ &= \iint_R \nabla^4 u\, w\, dx\, dy + \int_C \nabla^2 u \frac{\partial w}{\partial n}\, ds - \int_C \frac{\partial}{\partial n}\nabla^2 u\, w\, ds.\end{aligned}$$

A.2. Prove that $u\nabla \cdot (h\,\nabla v) = \nabla \cdot (u\,h\,\nabla v) - \nabla u \cdot h\,\nabla v$.
ANS. Using $\nabla \equiv \hat{\mathbf{i}}\,\partial/\partial x + \hat{\mathbf{j}}\,\partial/\partial y + \hat{\mathbf{k}}\,\partial/\partial z$,

the right side

$$\begin{aligned}&= u\left[\frac{\partial}{\partial x}\left(h\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(h\frac{\partial v}{\partial z}\right)\right] \\ &= u\left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) \cdot \left(\hat{\imath}h\frac{\partial v}{\partial x} + \hat{\jmath}h\frac{\partial v}{\partial y} + \hat{\mathbf{k}}h\frac{\partial v}{\partial z}\right) \\ &= u\nabla \cdot (h\nabla v) = \text{the left side.}\end{aligned}$$

A.3. Show that $u\nabla^2 v = \nabla \cdot (u\,\nabla v) - \nabla u \cdot \nabla v$.
ANS. Take $h = 1$ in **A.2**.

B

Tables of Transform Pairs

Some basic formulas for the pairs of the Laplace, complex (exponential) Fourier, Fourier sine, Fourier cosine, finite sine, and finite cosine transforms are provided below in tabular forms. Definitions of these transforms are given in §6.1 and §6.8.

B.1. Laplace Transform Pairs

	$f(t)$	$F(s) = \bar{f}(s)$
1.	1	$\frac{1}{s}, \quad s > 0$
2.	e^{at}	$\frac{1}{s-a}, \quad s > a$
3.	$\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
4.	$\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
5.	$\sinh at$	$\frac{a}{s^2-a^2}, \quad s > 0$
6.	$\cosh at$	$\frac{s}{s^2-a^2}, \quad s > 0$
7.	$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$

	$f(t)$	$F(s) = \bar{f}(s)$
8.	$e^{at} \cosh bt$	$\frac{s}{(s-a)^2+b^2}, \quad s > a$
9.	$t^n \quad (n = 1, 2, \cdots)$	$\frac{n!}{s^{n+1}}, \quad s > 0$
10.	$t^n e^{at} \quad (n = 1, 2, \cdots)$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
11.	$H(t-a)$	$\frac{e^{-as}}{s}, \quad s > 0$
12.	$H(t-a)\, f(t-a)$	$e^{-as} F(s) = e^{-as} \bar{f}(s)$
13.	$e^{at} f(t)$	$F(s-a) = \bar{f}(s-a)$
14.	$f(t) \star g(t)$	$F(s)\, G(s) = \bar{f}(s)\, \bar{g}(s)$
15.	$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{n-1}(0)$
16.	$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right), \quad a > 0$
17.	$\int_0^t f(t)\, dt$	$\frac{1}{s} F(s) = \frac{1}{s} \bar{f}(s)$
18.	$\delta(t-a)$	e^{-as}
19.	$t\, f(t)$	$-\frac{d}{ds} F(s)$
20	$\operatorname{erfc} \frac{a}{2\sqrt{t}}$	$\frac{e^{-a\sqrt{s}}}{s}$
21.	$f(t)$ with period $= T$ *	$\frac{\int_0^T e^{-st} f(t)\, dt}{1 - e^{-Ts}}$

The command `<<Calculus'LaplaceTransform'` loads the Mathematica package concerning Laplace transforms.

*$f(t)$ is continuous in $[0, T]$ and periodic with period T, $T > 0$.

B.2. Fourier Cosine Transform Pairs

	$f(x)$	$\mathcal{F}_c\{f(x)\} = \tilde{f}_c(\alpha)$
1.	$f(ax)$	$\frac{1}{a}\tilde{f}_c\left(\frac{\alpha}{a}\right)$
2.	e^{-ax}	$\sqrt{\frac{2}{\pi}}\frac{a}{\alpha^2+a^2}$
3.	$x^{-1/2}$	$\frac{1}{\sqrt{\alpha}}$
4.	e^{-ax^2}	$\frac{1}{\sqrt{2a}}e^{-\alpha^2/4a}$
5.	$\frac{a}{x^2+a^2}$	$\sqrt{\frac{\pi}{2}}e^{-a\alpha}$
6.	$x^2 f(x)$	$-\tilde{f}_c''(\alpha)$
7.	$\frac{\sin ax}{x}$	$\sqrt{\frac{\pi}{2}}H(a-\alpha)$
8.	$f''(x)$	$-\alpha^2\tilde{f}_c(\alpha)-\sqrt{\frac{2}{\pi}}f(0)$
9.	$\delta(x)$	$\sqrt{\frac{2}{\pi}}$
10.	$H(a-x)$	$\sqrt{\frac{2}{\pi}}\frac{\sin a\alpha}{\alpha}$
11.	$\left(\frac{\sqrt{x^2+a^2}+a}{x^2+a^2}\right)^{1/2}$	$\frac{\pi\sqrt{\alpha}}{2}e^{-a\alpha}$
12.	$\begin{cases}(x^2-a^2)^{1/2}, & x<a\\ 0, & x>a\end{cases}$	$\left(\frac{\pi}{2}\right)^{3/2}J_0(a\alpha)$
13.	$\begin{cases}(a^2-x^2)^{-1/2}, & x>a\\ 0, & x<a\end{cases}$	$-\left(\frac{\pi}{2}\right)^{3/2}Y_0(a\alpha)$
14.	$\sin(a^2x^2)$	$\frac{\pi}{4a}\left(\cos(\alpha^2/4a^2)-\sin(\alpha^2/4a^2)\right)$
15.	$\cos(a^2x^2)$	$\frac{\pi}{4a}\left(\cos(\alpha^2/4a^2)+\sin(\alpha^2/4a^2)\right)$

B.3. Fourier Sine Transform Pairs

	$f(x)$	$\mathcal{F}_s\{f(x)\} = \tilde{f}_s(\alpha)$
1.	$f(ax)$	$\frac{1}{a}\tilde{f}_s\left(\frac{\alpha}{a}\right)$
2.	e^{-ax}	$\sqrt{\frac{2}{\pi}}\frac{\alpha}{a^2+\alpha^2}$
3.	$x^{-1/2}$	$\frac{1}{\sqrt{\alpha}}$
4.	x^{-1}	$\sqrt{\frac{\pi}{2}}$
5.	$\frac{x}{x^2+a^2}$	$\sqrt{\frac{\pi}{2}}e^{-a\alpha}$
6.	$\arctan(a/x)$	$\sqrt{\frac{\pi}{2}}\frac{1-e^{-a\alpha}}{\alpha}$
7.	$x^2 f(x)$	$-\tilde{f}_s''(\alpha)$
8.	$\operatorname{erfc}\frac{x}{2\sqrt{a}}, \quad a>0$	$\sqrt{\frac{2}{\pi}}\frac{1-e^{-a\alpha^2}}{\alpha}$
9.	$f''(x)$	$-\alpha^2\tilde{f}_s(\alpha)+\sqrt{\frac{2}{\pi}}\alpha f(0)$
10.	$H(a-x)$	$\sqrt{\frac{2}{\pi}}\frac{1-\cos a\alpha}{\alpha}$
11.	$x\,e^{-a^2x^2}$	$\frac{\pi\alpha}{4\sqrt{2}\,a^3}e^{-\alpha^2/4a^2}$
12.	$\frac{1}{\sqrt{x}}e^{-a/x}$	$\frac{\pi}{2\sqrt{\alpha}}e^{-\sqrt{2a\alpha}}\left(\cos\sqrt{2a\alpha}+\sin\sqrt{2a\alpha}\right)$
13.	$\frac{x\cos ax}{b^2+x^2}$	$\begin{cases}-\left(\frac{\pi}{2}\right)^{3/2}e^{-ab}\sinh a\alpha, & \alpha<a\\ \left(\frac{\pi}{2}\right)^{3/2}e^{-ab}\cosh a\alpha, & \alpha>a\end{cases}$

B.4. Complex Fourier Transform Pairs

	$f(x)$	$\mathcal{F}\{f(x)\} = \tilde{f}(\alpha)$
1.	$f^{(n)}(x)$	$(i\alpha)^n \tilde{f}(\alpha)$
2.	$f(ax), \quad a > 0$	$\frac{1}{a}\tilde{f}\left(\frac{\alpha}{a}\right)$
3.	$f(x-a)$	$e^{-a\alpha}\tilde{f}(\alpha)$
4.	$\delta(x-a)$	$\frac{1}{\sqrt{2\pi}} e^{-ia\alpha}$
5.	$e^{-a\|x\|}$	$\sqrt{\frac{2}{\pi}} \frac{a}{a^2+\alpha^2}$
6.	$e^{-a^2x^2}$	$\frac{1}{a\sqrt{2}} e^{-\alpha^2/4a^2}$
7.	$\begin{cases} 1, & \|x\| < a \\ 0, & \|x\| > a \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin a\alpha}{\alpha}$
8.	$\begin{cases} 1, & \|x\| < 1 \\ 0, & \|x\| > 1 \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{i}{\alpha} \left(\cos\alpha - \frac{1}{\alpha}\sin\alpha\right)$
9.	$f(x) \star g(x)$	$\frac{1}{\sqrt{2\pi}} \tilde{f}(\alpha)\,\tilde{g}(\alpha)$
10.	$H(x+a) - H(x-a)$	$\sqrt{\frac{2}{\pi}} \frac{\sin a\alpha}{\alpha}$
11.	$x\,e^{-a\|x\|}, \quad a > 0$	$-\sqrt{\frac{2}{\pi}} \frac{2ia\alpha}{(\alpha^2+a^2)^2}$
12.	$\frac{a}{x^2+a^2}$	$\sqrt{\frac{\pi}{2}} e^{-a\|\alpha\|}$
13.	$\frac{ax}{(x^2+a^2)^2}$	$-\frac{i}{2}\sqrt{\frac{\pi}{2}}\,\alpha\, e^{-a\|\alpha\|}$
14.	$\cos ax$	$\sqrt{\frac{\pi}{2}} [\delta(\alpha+a) + \delta(\alpha-a)]$
15.	$\sin ax$	$i\sqrt{\frac{\pi}{2}} [\delta(\alpha+a) - \delta(\alpha-a)]$
16.	$\begin{cases} \cos ax, & \|x\| < \pi/2a \\ 0, & \|x\| > \pi/2a \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{a}{a^2-\alpha^2} \cos(\pi\alpha/2a)$
17.	$\frac{1}{1+x^2}$	$\sqrt{\frac{\pi}{2}} e^{-\|\alpha\|}$
18.	$\begin{cases} 1-\|x\|, & \|x\| < 1 \\ 0, & \|x\| > 1 \end{cases}$	$2\sqrt{\frac{2}{\pi}} \left(\frac{\sin(\alpha/2)}{\alpha}\right)^2$

B.5. Finite Sine Transform Pairs

The tables are for the interval $[0, \pi]$. If the interval is $[a, b]$, then it can be transformed into $[0, \pi]$ by

$$y = \frac{\pi(x-a)}{b-a}. \tag{B.1}$$

	$f(x)$	$\tilde{f}_s(n)$
1.	$\sin mx, \quad m = 1, 2, \cdots$	$\begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$
2.	$\sum_{n=1}^{\infty} a_n \sin nx$	a_n
3.	$\pi - x$	$\frac{2}{n}$
4.	x	$\frac{2}{n}(-1)^{n+1}$
5.	1	$\frac{2}{n\pi}[1-(-1)^n]$
6.	$\begin{cases} -x, & x \le a \\ \pi - x, & x > a \end{cases}$	$\frac{2}{n}\cos na, \quad 0 < a < \pi$
7.	$\begin{cases} x(\pi-a), & x \le a \\ a(\pi-x), & x > a \end{cases}$	$\frac{2}{n^2}\sin na, \quad 0 < a < \pi$
8.	e^{ax}	$\frac{2n}{\pi(a^2+n^2)}[1-(-1)^n e^{a\pi}]$
9.	$\frac{\sinh a(\pi-x)}{\sinh a\pi}$	$\frac{2n}{\pi(a^2+n^2)}$
10.	$f''(x)$	$-n^2 S_n + \frac{2n}{\pi}[f(0) - (-1)^n f(\pi)]$

B.6. Finite Cosine Transform Pairs

The tables are for the interval $[0, \pi]$. If the interval is $[a, b]$, then it can be transformed into $[0, \pi]$ by (B.1).

	$f(x)$	$\tilde{f}_c(n)$
1.	$\cos mx, \quad m = 1, 2, \cdots$	$\begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$
2.	$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \sin nx$	a_n
3.	$f(\pi - x)$	$(-1)^n \frac{2}{\pi} C_n$
4.	1	$\begin{cases} 2, & n = 0 \\ 0, & n = 1, 2, \cdots \end{cases}$
5.	x	$\begin{cases} \pi, & n = 0 \\ \frac{2}{\pi n^2} \left[(-1)^n - 1\right], & n = 1, 2, \cdots \end{cases}$
6.	x^2	$\begin{cases} 2\pi^2/3, & n = 0 \\ \frac{4}{n^2} (-1)^n, & n = 1, 2, \cdots \end{cases}$
7.	$\begin{cases} 1, & 0 < x < a \\ -1, & a < x < \pi \end{cases}$	$\begin{cases} \frac{2(2a - \pi)}{\pi}, & n = 0 \\ \frac{4}{n\pi} \sin na, & n = 1, 2, \cdots \end{cases}$
8.	$\frac{e^{ax}}{a}$	$\frac{2}{\pi} \frac{(-1)^n e^{a\pi} - 1}{a^2 + n^2}$
9.	$-\ln(2 \sin x/2)$	$\begin{cases} 0, & n = 0 \\ 1/n, & n = 1, 2, \cdots \end{cases}$
10.	$f''(x)$	$-n^2 S_n + \frac{2n}{\pi} \left[f(0) - (-1)^n f(\pi)\right]$

C

Glossary of Mathematica Functions

`All` All is a setting used for certain options.

`Apart` Apart[expr] rewrites a rational expression as a sum of terms with minimal denominators. Apart[expr, var] treats all variables other than var as constants.

`Append` Append[expr, elem] gives expr with elem appended.

`Apply` Apply[f, expr] or f @@ expr replaces the head of expr by f. Apply[f, expr, levelspec] replaces heads in parts of expr specified by levelspec.

`BesselI` BesselI[n, z] gives the modified Bessel function of the first kind I(n, z).

`BesselJ` BesselJ[n, z] gives the Bessel function of the first kind J(n, z).

`BesselK` BesselK[n, z] gives the modified Bessel function of the second kind K(n, z).

`Binomial` Binomial[n, m] gives the binomial coefficient.

`Blank` _ or Blank[ ] is a pattern object that can stand for any Mathematica expression. _h or Blank[h] can stand for any expression with head h.

`C` C[i] is the default form for the i-th constant of integration produced in solving a differential equation with DSolve.

`Cancel` Cancel[expr] cancels out common factors in the numerator and denominator of expr.

`CForm` CForm[expr] prints as a C language version of expr.

`CharacteristicPolynomial` CharacteristicPolynomial[m, x] gives the characteristic polynomial defined by the square matrix m and the variable x. The result is normally equivalent to Det[m - x IdentityMatrix[Length[m]]].

Chop Chop[expr] replaces approximate real numbers in expr that are close to zero by the exact integer 0. Chop[expr, tol] replaces approximate real numbers in expr that differ from zero by less than tol with 0.

Clear Clear[symbol1, symbol2, ...] clears values and definitions for the specified symbols. Clear[\pattern1\, \pattern2\, ...] clears values and definitions for all symbols whose names match any of the specified string patterns.

Collect Collect[expr, x] collects together terms involving the same power of x. Collect[expr, {x1, x2, ...}] collects together terms that involve the same powers of x1, x2, ...

ColumnForm ColumnForm[{e1, e2, ...}] prints as a column with e1 above e2, etc. ColumnForm[list, horiz] specifies the horizontal alignment of each element. ColumnForm[list, horiz, vert] also specifies the vertical alignment of the whole column.

ComplexExpand ComplexExpand[expr] expands expr assuming that all variables are real. ComplexExpand[expr, {x1, x2, ...}] expands expr assuming that variables matching any of the xi are complex.

Conjugate Conjugate[z] gives the complex conjugate of the complex number z.

Continuation Continuation[n] is output at the beginning of the nth line in a multiline printed expression.

ContourPlot ContourPlot[f,{x,xmin, xmax}, {y, ymin, ymax}] generates a contour plot of f as a function of x and y.

Cos Cos[z] gives the cosine of z.

D D[f, x] gives the partial derivative of f with respect to x. D[f, {x, n}] gives the nth partial derivative with respect to x. D[f, x1, x2, ...] gives a mixed derivative.

Denominator Denominator[expr] gives the denominator of expr.

DensityPlot DensityPlot[f, {x, xmin, xmax}, {y, ymin, ymax}] makes a density plot of f as a function of x and y.

Derivative f' represents the derivative of a function f of one argument. Derivative[n1, n2, ...][f] is the general form, representing a function obtained from f by differentiating n1 times with respect to the first argument, n2 times with respect to the second argument, and so on.

Det Det[m] gives the determinant of the square matrix m.

Dot a.b.c or Dot[a, b, c] gives products of vectors, matrices and tensors.

DSolve DSolve[eqn, y[x], x] solves a differential equation for the functions y[x], with independent variable x. DSolve[{eqn1, eqn2, ...}, {y1[x1, ...], ...}, {x1, ...}] solves a list of differential equations.

Dt Dt[f, x] gives the total derivative of f with respect to x. Dt[f] gives the total differential of f. Dt[f, {x, n}] gives the nth total derivative with respect to x. Dt[f, x1, x2, ...] gives a mixed total derivative.

E E is the exponential constant e (base of natural logarithms), with numerical value 2.71828....

Eigensystem Eigensystem[m] gives a list {values, vectors} of the eigenvalues and eigenvectors of the square matrix m.

Eigenvalues Eigenvalues[m] gives a list of the eigenvalues of the square matrix m.

Eigenvectors Eigenvectors[m] gives a list of the eigenvectors of the square matrix m.

Equal lhs == rhs returns True if lhs and rhs are identical.

Erf Erf[z] gives the error function erf(z). Erf[z0, z1] gives the generalized error function erf(z1) - erf(z0).

Erfc Erfc[z] gives the complementary error function erfc(z) == 1 - erf(z).

Exp Exp[z] is the exponential function.

Expand Expand[expr] expands out products and positive integer powers in expr. Expand[expr, patt] avoids expanding elements of expr which do not contain terms matching the pattern patt.

False False is the symbol for the Boolean value false.

FindRoot FindRoot[lhs == rhs, {x, x0}] searches for a numerical solution to the equation lhs == rhs, starting with x == x0.

Flatten Flatten[list] flattens out nested lists. Flatten[list, n] flattens to level n.

FortranForm FortranForm[expr] prints as a Fortran language version of expr.

Fourier Fourier[list] finds the discrete Fourier transform of a list of complex numbers.

Gamma Gamma[a] is the Euler gamma function Gamma(a). Gamma[a, z] is the incomplete gamma function Gamma(a, z). Gamma[a, z0, z1] is the generalized incomplete gamma function Gamma(a, z0) - Gamma(a, z1).

Gradient Gradient is an option for FindMinimum, which can be used to specify the gradient of the function whose minimum is being sought. With Gradient-> Automatic, the gradient is computed symbolically. A typical setting is Gradient -> {2 x, Sign[y]}.

Greater x > y yields True if x is determined to be greater than y. x1 > x2 > x3 yields True if the xi form a strictly decreasing sequence.

GreaterEqual x >= y yields True if x is determined to be greater than or equal to y. x1 >= x2 >= x3 yields True if the xi form a non-increasing sequence.

Hold Hold[expr] maintains expr in an unevaluated form.

`I` I represents the imaginary unit Sqrt[-1].

`Im` Im[z] gives the imaginary part of the complex number z.

`In` In[n] is a global object that is assigned to have a delayed value of the n-th input line.

`Infinity` Infinity is a symbol that represents a positive infinite quantity.

`Integrate` Integrate[f,x] gives the indefinite integral of f with respect to x. Integrate[f,{x,xmin,xmax}] gives the definite integral. Integrate[f,{x,xmin,xmax},{y,ymin,ymax}] gives a multiple integral.

`InverseFourier` InverseFourier[list] finds the discrete inverse Fourier transform of a list of complex numbers.

`Jacobian` Jacobian is an option for FindRoot. Jacobian -> Automatic attempts symbolic computation of the Jacobian of the system of functions whose root is being sought. A typical setting is Jacobian -> {{2 x, Sign[y]}, {y, x}}.

`Join` Join[list1, list2, ...] concatenates lists together. Join can be used on any set of expressions that have the same head.

`LegendreP` LegendreP[n, x] gives the n-th Legendre polynomial. LegendreP[n, m, x] gives the associated Legendre polynomial.

`Less` x < y yields True if x is determined to be less than y. x1 < x2 < x3 yields True if the xi form a strictly increasing sequence.

`LessEqual` x <= y yields True if x is determined to be less than or equal to y. x1 <= x2 <= x3 yields True if the xi form a non-decreasing sequence.

`Limit` Limit[expr, x->x0] finds the limiting value of expr when x approaches x0.

`ListPlot` ListPlot[{y1, y2, ...}] plots a list of values. The x coordinates for each point are taken to be 1, 2, ... ListPlot[{{x1, y1}, {x2, y2}, ...}] plots a list of values with specified x and y coordinates.

`ListPlot3D` ListPlot3D[array] generates a three-dimensional plot of a surface representing an array of height values. ListPlot3D[array, shades] generates a plot with each element of the surface shaded according to the specification in shades.

`Log` Log[z] gives the natural logarithm of z (logarithm to base E). Log[b, z] gives the logarithm to base b.

`Map` Map[f, expr] or f /@ expr applies f to each element on the first level in expr. Map[f, expr, levelspec] applies f to parts of expr specified by levelspec.

`MatrixForm` MatrixForm[list] prints with the elements of list arranged in a regular array.

`Max` Max[x1, x2, ...] yields the numerically largest of the xi. Max[{x1, x2, ...}, {y1, ...}, ...] yields the largest element of any of the lists.

`Min` Min[x1, x2, ...] yields the numerically smallest of the xi. Min[{x1, x2, ...}, {y1, ...}, ...] yields the smallest element of any of the lists.

`N` N[expr] gives the numerical value of expr. N[expr, n] does computations to n-digit precision.

`NDSolve` NDSolve[eqns, y, {x, xmin, xmax}] finds a numerical solution to the differential equations eqns for the function y with the independent variable x in the range xmin to xmax. NDSolve[eqns, {y1, y2, ...}, {x, xmin, xmax}] finds numerical solutions for the functions yi. NDSolve[eqns, y, {x, x1, x2, ...}] forces a function evaluation at each of x1, x2, ... The range of numerical integration is from Min[x1, x2, ...] to Max[x1, x2, ...].

`Needs` Needs[\context`\, \file\] loads file if the specified context is not already in $Packages. Needs[\context`\] loads the file specified by ContextToFilename[\context`\] if the specified context is not already in $Packages.

`Negative` Negative[x] gives True if x is a negative number.

`NIntegrate` NIntegrate[f, {x, xmin, xmax}] gives a numerical approximation to the integral of f with respect to x over the interval xmin to xmax.

`NonNegative` NonNegative[x] gives True if x is a non-negative number.

`Not` !expr is the logical NOT function. It gives False if expr is True, and True if it is False.

`NSolve` NSolve[eqns, vars] attempts to solve numerically an equation or set of equations for the variables vars. Any variable in eqns but not vars is regarded as a parameter. NSolve[eqns] treats all variables encountered as vars above. NSolve[eqns, vars, prec] attempts to solve numerically the equations for vars using prec digits precision.

`NullSpace` NullSpace[m] gives a list of vectors that forms a basis for the null space of the matrix m.

`Number` Number represents an exact integer or an approximate real number in Read.

`Off` Off[symbol tag] switches off a message, so that it is no longer printed. Off[s] switches off tracing messages associated with the symbol s. Off[m1, m2, ...] switches off several messages. Off[ ] switches off all tracing messages.

`On` On[symbol tag] switches on a message, so that it can be printed. On[s] switches on tracing for the symbol s. On[m1, m2, ...] switches on several messages. On[ ] switches on tracing for all symbols.

`Out` %n or Out[n] is a global object that is assigned to be the value produced on the n-th output line. % gives the last result generated. %% gives the result before last. %%...% (k times) gives the k-th previous result.

`ParametricPlot` ParametricPlot[{fx, fy}, {t, tmin, tmax}] produces a parametric plot with x and y coordinates fx and fy generated as a function of t. ParametricPlot[{{fx, fy}, {gx, gy}, ...}, {t, tmin, tmax}] plots several parametric curves.

`ParametricPlot3D` ParametricPlot3D[{fx, fy, fz}, {t, tmin, tmax}] produces a three-dimensional space curve parameterized by a variable t which runs from tmin to tmax. ParametricPlot3D[{fx, fy, fz}, {t, tmin, tmax}, {u, umin, umax}] produces a three-dimensional surface parametrized by t and u. ParametricPlot3D[{fx, fy, fz, s}, ...] shades the plot according to the color specification s. ParametricPlot3D[{{fx, fy, fz}, {gx, gy, gz}, ...}, ...] plots several objects together.

`Part` expr[[i]] or Part[expr, i] gives the i-th part of expr. expr[[-i]] counts from the end. expr[[0]] gives the head of expr. expr[[i, j, ...]] or Part[expr, i, j, ...] is equivalent to expr[[i]] [[j]] .... expr[[{i1, i2, ...}]] gives a list of the parts i1, i2, ... of expr.

`Partition` Partition[list, n] partitions list into non-overlapping sublists of length n. Partition[list, n, d] generates sublists with offset d. Partition[list, {n1, n2, ...}, {d1, d2, ...}] partitions successive levels in list into length ni sublists with offsets di.

`Pi` Pi is pi, with numerical value 3.14159....

`Plot` Plot[f, {x, xmin, xmax}] generates a plot of f as a function of x from xmin to xmax. Plot[{f1, f2, ...}, {x, xmin, xmax}] plots several functions fi.

`Plot3D` Plot3D[f, {x, xmin, xmax}, {y, ymin, ymax}] generates a three-dimensional plot of f as a function of x and y. Plot3D[{f, s}, {x, xmin, xmax}, {y, ymin, ymax}] generates a three-dimensional plot in which the height of the surface is specified by f, and the shading is specified by s.

`Print` Print[expr1, expr2, ...] prints the expri, followed by a new line (line feed).

`Quit` Quit[ ] terminates a Mathematica session.

`Range` Range[imax] generates the list {1, 2, ..., imax}. Range[imin, imax] generates the list {imin, ..., imax}.

`Re` Re[z] gives the real part of the complex number z.

`Remove` Remove[symbol1, ...] removes symbols completely, so that their names are no longer recognized by Mathematica. Remove[\form1\, \form2\, ...] removes all symbols whose names match any of the string patterns formi.

`Replace` Replace[expr, rules] applies a rule or list of rules in an attempt to transform the entire expression expr.

`Rest` Rest[expr] gives expr with the first element removed.

`Rule` lhs -> rhs represents a rule that transforms lhs to rhs.

`SameQ` lhs === rhs yields True if the expression lhs is identical to rhs, and yields False otherwise.

`Series` Series[f, {x, x0, n}] generates a power series expansion for f about the point x = x0 to order (x - x0)^n. Series[f, {x, x0, nx}, {y, y0, ny}] successively finds series expansions with respect to y, then x.

`Set` lhs = rhs evaluates rhs and assigns the result to be the value of lhs. From then on, lhs is replaced by rhs whenever it appears. {l1, l2, ...} = {r1, r2, ...} evaluates the ri, and assigns the results to be the values of the corresponding li.

`Show` Show[graphics, options] displays two- and three-dimensional graphics, using the options specified. Show[g1, g2, ...] shows several plots combined. Show can also be used to play Sound objects.

`Simplify` Simplify[expr] performs a sequence of transformations on expr, and returns the simplest form it finds.

`Sin` Sin[z] gives the sine of z.

`Solve` Solve[eqns, vars] attempts to solve an equation or set of equations for the variables vars. Any variable in eqns but not vars is regarded as a parameter. Solve[eqns] treats all variables encountered as vars above. Solve[eqns, vars, elims] attempts to solve the equations for vars, eliminating the variables elims.

`Sqrt` Sqrt[z] gives the square root of z.

`Table` Table[expr, {imax}] generates a list of imax copies of expr. Table[expr, {i, imax}] generates a list of the values of expr when i runs from 1 to imax. Table[expr, {i, imin, imax}] starts with i = imin. Table[expr, {i, imin, imax, di}] uses steps di. Table[expr, {i, imin, imax}, {j, jmin, jmax}, ...] gives a nested list. The list associated with i is outermost.

`Tan` Tan[z] gives the tangent of z.

`TeXForm` TeXForm[expr] prints as a TeX language version of expr.

`Times` x*y*z or x y z represents a product of terms.

`Together` Together[expr] puts terms in a sum over a common denominator, and cancels factors in the result.

`Transpose` Transpose[list] transposes the first two levels in list. Transpose[list, {n1, n2, ...}] transposes list so that the nk-th level in list is the k-th level in the result.

`True` True is the symbol for the Boolean value true.

`VectorQ` VectorQ[expr] gives True if expr is a list, none of whose elements are themselves lists, and gives False otherwise. VectorQ[expr, test] gives True only if test yields True when applied to each of the elements in expr.

D

Mathematica Packages and Notebooks

A list of Mathematica packages and notebooks available on the CRC web server is given below. The numbers refer to the pages in the text of this book.

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