Solution for Chapter 9

(compiled by Xinkai Wu)

1. Ex. 9.2 The holographically reconstructed wave [by Kip Thorne, Jaemo Park, and Alexei Dvoretskii/99]

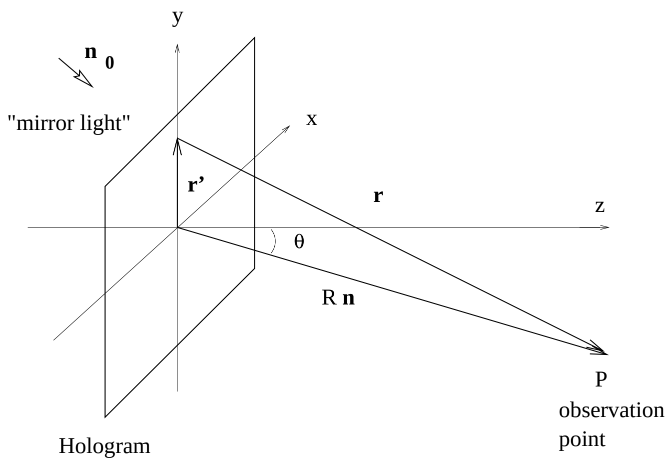

Figure 1: holographically reconstructed wave

(a) See Fig. 1, in which $r = |\mathbf{r}| = |R\mathbf{n} - \mathbf{r}'| \approx R - \mathbf{n}\cdot\mathbf{r}'$. Using the Helmholtz-Kirchhoff formula, we get

$$\psi_P = \frac{ik}{2\pi}\int d\mathbf{S}' \left(\frac{\mathbf{n}+\mathbf{n}_0}{2}\right)\frac{e^{i(kR-k\mathbf{r}'\cdot\mathbf{n})}}{r}\psi_0(x',y')$$

where

$$\begin{aligned}
\psi_0(x',y') &= T(x',y')e^{ik\mathbf{r}'\cdot\mathbf{n}_0} \\
&\sim \left(|O|^2 + M^2 + MOe^{-ik\mathbf{r}'\cdot\mathbf{n}_0} + MO^* e^{ik\mathbf{r}'\cdot\mathbf{n}_0}\right)e^{ik\mathbf{r}'\cdot\mathbf{n}_0} \\
&\sim (|O|^2 + M^2)e^{ik\mathbf{r}'\cdot\mathbf{n}_0} + MO + MO^* e^{2ik\mathbf{r}'\cdot\mathbf{n}_0}
\end{aligned}$$

and thus

$$\begin{aligned}
\psi_P &= \psi_P^{(1)} + \psi_P^{(2)} + \psi_P^{(3)} \\
\psi_P^{(1)} &= \frac{ik}{2\pi}\int d\mathbf{S}'\left(\frac{\mathbf{n}+\mathbf{n}_0}{2}\right)(|O|^2+M^2)\frac{e^{ikR}}{r}e^{ik\mathbf{r}'\cdot(\mathbf{n}_0-\mathbf{n})} \\
\psi_P^{(2)} &= \frac{ik}{2\pi}\int d\mathbf{S}'\left(\frac{\mathbf{n}+\mathbf{n}_0}{2}\right)MO\frac{e^{ikR}}{r}e^{-ik\mathbf{r}'\cdot\mathbf{n}} \\
\psi_P^{(3)} &= \frac{ik}{2\pi}\int d\mathbf{S}'\left(\frac{\mathbf{n}+\mathbf{n}_0}{2}\right)MO^*\frac{e^{ikR}}{r}e^{ik\mathbf{r}'\cdot(2\mathbf{n}_0-\mathbf{n})}
\end{aligned}$$

If the mirror wave had been absent and the photographic plate replaced by a window, then by using Helmholtz-Kirchhoff with $\tilde{\psi}_0 = O$ we would have gotten $\psi_P = \frac{ik}{2\pi} \int d\mathbf{S}' \left(\frac{\mathbf{n}+\mathbf{n}_0}{2}\right) O \frac{e^{ikR - ik\mathbf{r}'\cdot\mathbf{n}}}{r}$ which is the same as $\psi_P^{(2)}$ (to within a multiplicative constant).

The direction of propagation for different terms can be obtained by finding the stationary phase points for rapidly oscillating terms (note the last factor in the expressions for $\psi_P^{(1)}, \psi_P^{(2)}, \psi_P^{(3)}$ are rapidly oscillating, except in the direction of output wave). We find

for $\psi_P^{(1)}$, it's $\mathbf{n} = \mathbf{n}_0$, i.e. the wave propagates along the mirror wave direction.

for $\psi_P^{(2)}$, $\mathbf{n} = \mathbf{e}_z$, i.e. the wave propagates along $\mathbf{e}_z$, the direction perpendicular to the hologram.

for $\psi_P^{(3)}$, $2sin\theta_0 = sin\theta$. (Here we assume that both $\mathbf{n}$ and $\mathbf{n}_0$ are in the y-z plane, and θ and θ_0 are the angles between them and $\mathbf{e}_z$ respectively. We'll make the same assumption in part (b) and (c).) Note that if $\theta_0 < \pi/6$, then there exists a solution to this equation and the secondary image actually exists.

(b) Now if all angles are small (paraxial optics), then

$$\psi_P^{(3)} = \frac{ik}{2\pi} \int d\mathbf{S}' M O^* \frac{e^{ikz}}{z} e^{iky'(2\theta_0 - \theta)}, \quad \theta_0 \approx 0$$

The field at the object can be obtained by propagating back to point $(0, 0, -z)$

$$\psi_i = \frac{ik}{2\pi} \int d\mathbf{S}' O \frac{e^{-ikz}}{z} e^{iky'\theta_i}, \quad \theta_s \approx \theta_i$$

i.e. $\psi_P(z) \propto \psi_i^*(-z)$, we see that indeed the secondary image resides in front of the hologram and is turned inside out. See Fig. 2.

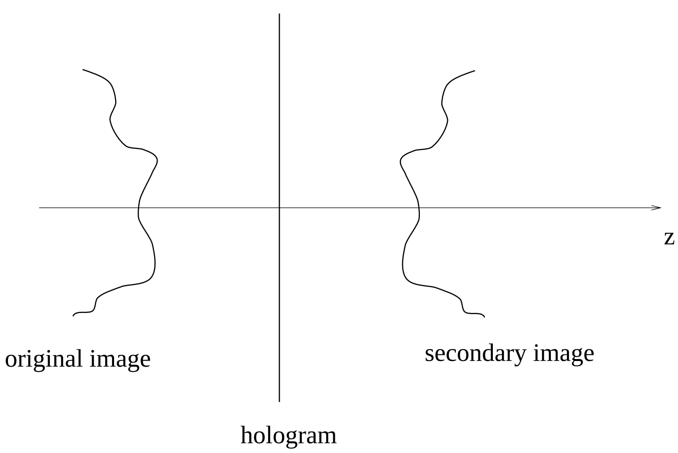

Figure 2: the secondary image

The angles in the wave scattered from the object and in the secondary wave are related by

$$-sin\theta_i = sin2\theta_0 - sin\theta_s$$

and for $\theta_i \approx 0$, if θ_0 is not small, then

$$\Delta\theta_i = cos\theta_s \Delta\theta_s \Rightarrow \Delta\theta_s = \frac{\Delta\theta_i}{cos2\theta_0} > \Delta\theta_i$$

i.e. the image is stretched.

(c) If light with wavenumber k is shined at the original angle θ_0 for reconstruction, then

$$\begin{aligned}
\psi_0 &\sim MOe^{-ik_0y'sin\theta_0}e^{iky'sin\theta_0} \\
\psi_P^{(2)} &\sim \int d\mathbf{S}' \left(\frac{\mathbf{n}+\mathbf{n}_0}{2}\right) MO\frac{e^{ikR}}{r}e^{iy'(-k_0sin\theta_0+ksin\theta_0-ksin\theta)}
\end{aligned}$$

So again by looking for a stationary phase condition we see that the direction of propagation will be given by $\sin\theta = sin\theta_0\left(1-\frac{k_0}{k}\right)$. (the sign of θ is indicated in Fig. 3)

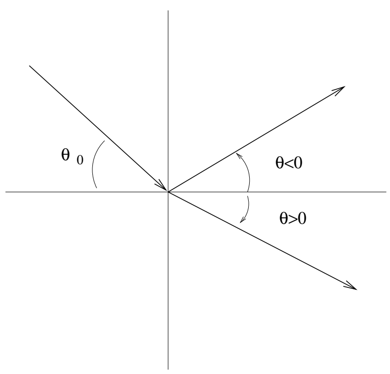

Figure 3: plane-parallel white light

$\theta_0 = \pi/4$, $k_0 = k_{green}$, and $sin\theta = \sin\theta_0\left(1-\frac{\lambda}{\lambda_{green}}\right)$.

(1) $\lambda = \lambda_{green}$, $sin\theta = 0$, $\theta = 0$

(2) $\lambda = \lambda_{red}$, $sin\theta = \frac{1}{\sqrt{2}}\left(1-\frac{\lambda_{red}}{\lambda_{green}}\right) \approx \frac{1}{\sqrt{2}}\left(1-\frac{700nm}{500nm}\right) \approx -0.28$. Thus $\theta \approx -16.4°$.

2. Ex 9.3 Compact disks [by Alexei Dvoretskii/99]

To record information on a CD, pits are burned in it by using a laser with $\lambda \approx 0.5\mu m$. Because of diffraction it's not possible to focus the light much better than that.

So in the best case 1 bit can be written per $0.25\mu m^2$.

Estimate: Disc area: $\pi\left(r_{out}^2 - r_{inner}^2\right) \approx 3[(6cm)^2 - (2cm)^2] \approx 100cm^2$. and thus in principle the total amount of information that can be stored is $I = \frac{100cm^2}{0.25\mu m^2} = 4 \times 10^{10} bit$. Taking 5 per cent of this, we get $I = 2 \times 10^9 bits$.

Now for Encyclopedia Britannica:

1 page $\sim$ 1kbyte $\sim 10^4$bit

1 volume $\sim 10^3$pages $\sim 10^7 bit$.

i.e. approximately 200 volumes could be stored on a CD.

3. Ex. 9.4 Nonlinear susceptibility for an isotropic medium [by Alexei Dvoretskii/99]

For an isotropic medium its susceptibility tensors should be symmetric in all indices. The only isotropic symmetric tensor available to us is the metric tensor g_{ij}, the Levi-Civita tensor being antisymmetric.

Therefore, all susceptibility tensors with an odd number of indices are zero ($\chi_{ijk} = 0$, $\chi_{ijklm} = 0$, ...) and the susceptibility tensors with an even number of indices must be proportional to symmetrized sums of products of metric tensors.

Thus $\chi_{ij} = \chi_0 g_{ij}$

For χ_{ijkl}, it should be a linear combination of $g_{ij}g_{kl}$; $g_{ik}g_{jl}$; $g_{il}g_{jk}$. And since these tensors can be obtained from each other by permutations of indices their only combination symmetric w.r.t. all permutations is their sum. Hence

$$\chi_{ijkl} = \frac{\chi_4}{3}(g_{ij}g_{kl} + g_{ik}g_{jl} + g_{il}g_{jk})$$

In general, a $(2n)$th rank susceptibility tensor will decompose for an isotropic medium into a symmetric sum of products of n metric tensors with $\frac{(2n)!}{2^n n!}$ terms.

$$\begin{aligned}\chi_{ijklmn} = \; & \frac{\chi_6}{15} \cdot \\ & (g_{ij}g_{kl}g_{mn} + g_{ik}g_{jl}g_{mn} + g_{il}g_{jk}g_{mn} \\ & +g_{ij}g_{km}g_{ln} + g_{ij}g_{kn}g_{lm} + g_{ik}g_{jm}g_{ln} \\ & +g_{ik}g_{jn}g_{lm} + g_{il}g_{jm}g_{kn} + g_{il}g_{jn}g_{km} \\ & +g_{im}g_{jk}g_{ln} + g_{im}g_{jl}g_{kn} + g_{im}g_{jn}g_{kl} \\ & +g_{in}g_{jk}g_{lm} + g_{in}g_{jl}g_{km} + g_{in}g_{jm}g_{lk})\end{aligned}$$

4. Ex. 9.6 Dispersion relation for an anisotropic medium [by Xinkai Wu/02]

(a) It's known that $P_i = \epsilon_0 \chi_{ij} E_j$ when nonlinear effects are unimportant. Hence we have

$$D_i = \epsilon_0 E_i + P_i = \epsilon_0 E_i + \epsilon_0 \chi_{ij} E_j = \epsilon_0 \epsilon_{ij} E_j$$

if we define $\epsilon_{ij} \equiv \delta_{ij} + \chi_{ij}$.

(b) This is very straightforward (recall that by definition $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$).

$$\begin{aligned}
&\nabla \cdot \mathbf{D} = \epsilon_0 \nabla \cdot \mathbf{E} + \nabla \cdot \mathbf{P} \\
&\text{using the first identity in eq. (9.15)} \\
&\quad = \epsilon_0 \left(- \frac{\nabla \cdot \mathbf{P}}{\epsilon_0} \right) + \nabla \cdot \mathbf{P} \\
&\quad = 0
\end{aligned}$$

Also the last identity in eq. (9.15) reads

$$\begin{aligned}
\nabla \times \mathbf{B} &= \mu_0 \left(\frac{\partial \mathbf{P}}{\partial t} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \\
&= \mu_0 \frac{\partial}{\partial t} \left(\mathbf{P} + \epsilon_0 \mathbf{E} \right) \\
&= \mu_0 \frac{\partial \mathbf{D}}{\partial t}
\end{aligned}$$

The other two sourceless equations remain the same as in eq. (9.15)
(c) Taking the curl of the third identity in eq. (9.37), we get

$$\begin{aligned}
\nabla \times (\nabla \times \mathbf{E}) &= \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \\
&= \nabla \times \left(- \frac{\partial \mathbf{B}}{\partial t} \right) = - \frac{\partial}{\partial t} (\nabla \times \mathbf{B}) \\
&\quad \text{using the fourth identity in eq. (9.37)} \\
&= - \mu_0 \frac{\partial^2 \mathbf{D}}{\partial t^2} \\
&\quad \text{using the fact that } \mathbf{D} = \epsilon_0 \epsilon \cdot \mathbf{E} \\
&= - \mu_0 \epsilon_0 \epsilon \cdot \frac{\partial^2 \mathbf{E}}{\partial t^2} \\
&= - \epsilon \cdot \frac{\partial^2 \mathbf{E}}{c^2 \partial t^2}
\end{aligned}$$

(d) Consider the plane wave $E_j(\mathbf{x}, t) = E_j e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}$, where E_j is a constant amplitude. Plugging this plane waveform into eq. (9.38), one readily get

$$L_{ij} E_j = 0, \quad \text{where } L_{ij} = k_i k_j - k^2 \delta_{ij} + \frac{\omega^2}{c^2} \epsilon_{ij}$$

(e) Consider waves propagating in the x-z plane, $\mathbf{k} = k(sin\theta \mathbf{e}_x + cos\theta \mathbf{e}_z)$. Then it's straightforward (and we omit the details here) to find that $||L_{ij}|| = k^2 \cdot$[the matrix given in eq. (9.41)], where $k = n\frac{\omega}{c}$. Computing the determinant of this matrix and pulling out some factors, one find that the dispersion relation (9.40) reduces to

$$\left(\frac{1}{n^2} - \frac{1}{n_o^2} \right) \left(\frac{1}{n^2} - \frac{cos^2\theta}{n_o^2} - \frac{sin^2\theta}{n_e^2} \right) = 0$$

(f) Solving the dispersion relation we found above, we find two solutions

(i) $n = n_o$.

In this case, solving for the eigenvectors, one finds there's no constraint on E_y, while E_x and E_z must satisfy

$$sin^2\theta E_x + sin\theta cos\theta E_z = 0$$
$$sin\theta cos\theta E_x + \left[\left(\frac{n_e}{n_o}\right)^2 - sin^2\theta\right] E_z = 0$$

the determinant of the coefficient matrix for the above 2 equations is $sin^2\theta\left[\left(\frac{n_e}{n_o}\right)^2 - 1\right]$, which is nonzero if we consider $sin\theta \neq 0$ and $n_e \neq n_o$. Hence $E_x = 0, E_z = 0$. And $\mathbf{E}$ is along the y-direction.

This is the polarization described in eq. (9.30) and associated text.

(ii) $\frac{1}{n} = \sqrt{\frac{cos^2\theta}{n_o^2} + \frac{sin^2\theta}{n_e^2}}$.

Solving for the eigenvectors, one finds that (again assuming $sin\theta \neq 0$ and $n_e \neq n_o$) $E_y = 0$, and

$$\left(\frac{n_o}{n_e}\right)^2 sin\theta E_x + cos\theta E_z = 0$$

i.e.

$$\frac{E_x}{E_z} = -\left(\frac{n_e}{n_o}\right)^2 cot\theta$$

This is the polarization described in eq. (9.31) and associated text.