Solution for Chapter 8

(compiled by Xinkai Wu)

A.

1. Exercise 8.2 Lateral coherence of solar radiation [by Yu Cao]

$$
\begin{aligned}
& \text{Maximum separation between slits} \\
\sim\ & \text{lateral coherence length } l_\perp \\
\sim\ & \lambda/\delta\alpha \\
\sim\ & \frac{5\times 10^{-7}m}{0.5^\circ \times \pi/180^\circ} \\
\sim\ & 50\mu m
\end{aligned}
$$

$$
\begin{aligned}
& \text{number of visible fringes} \\
\sim\ & \text{number of fringes within central peak of diffraction envelope} \\
\sim\ & \frac{2\lambda}{10\mu m}\Big/\frac{\lambda}{50\mu m} \\
\sim\ & 10
\end{aligned}
$$

where in the above equations, $2\lambda/10\mu m$ is the angular diameter of the central peak of the diffraction pattern created by a single slit of width $10\mu m$.

2. Exercise 8.4 Longitudinal coherence of radio waves

For a broad band source like this, the coherence length is $\sim$ the longest wavelength that contains significant power, not the bandwidth which is based on the shortes wavelength at significant power. Here we take the lowest frequency (corresponding to the longest wavelength) to be the lower end of the audio range, $f \sim 100Hz$, which gives us a coherence length of $c/f \sim 3\times10^8/100 \sim 3\times10^3 km$.

B. Exercise 8.7 Complex random processes [by Xinkai Wu/99]

(a)

$$C_\Psi(\xi) = \overline{\Psi(\mathbf{x})\Psi^*(\mathbf{x}+\xi)} = lim_{L\to\infty}\frac{1}{L^n}\int \Psi_L(\mathbf{x})\Psi_L^*(\mathbf{x}+\xi)d^n x$$

When using Parseval's theorem, let $A(\mathbf{x}) = \Psi_L(\mathbf{x})$, and $B(\mathbf{x}) = \Psi_L(\mathbf{x}+\xi)$. Then we have $\tilde{B}(\mathbf{k}) = e^{i\mathbf{k}\cdot\xi}\tilde{\Psi}_L(\mathbf{k})$. By Parseval's theorem, we get

$$\int \Psi_L(\mathbf{x})\Psi_L^*(\mathbf{x}+\xi)d^n x = \frac{1}{(2\pi)^n}\int \tilde{\Psi}_L(\mathbf{k})e^{-i\mathbf{k}\cdot\xi}\tilde{\Psi}_L^*(\mathbf{k})d^n k$$

Using this, we find

$$\begin{aligned}
\int C_\Psi(\xi) e^{i\mathbf{k}\cdot\xi} d^n\xi &= \int d^n\xi \; e^{i\mathbf{k}\cdot\xi}\left[lim_{L\to\infty}\frac{1}{L^n}\frac{1}{(2\pi)^n}\int \tilde{\Psi}_L(\mathbf{k}')e^{-i\mathbf{k}'\cdot\xi}\tilde{\Psi}^*_L(\mathbf{k}')d^n k'\right] \\
&= lim_{L\to\infty}\frac{1}{L^n}\int d^n k' \tilde{\Psi}_L(\mathbf{k}')\tilde{\Psi}^*_L(\mathbf{k}')\frac{1}{(2\pi)^n}\int d^n\xi \; e^{i(\mathbf{k}-\mathbf{k}')\cdot\xi} \\
&= lim_{L\to\infty}\frac{1}{L^n}\tilde{\Psi}_L(\mathbf{k})\tilde{\Psi}^*_L(\mathbf{k}) \\
&= S_\Psi(\mathbf{k})
\end{aligned}$$

which is the desired eq. (8.34).

Doing the inverse Fourier transform, one gets eq. (8.35)

$$C_\Psi(\xi) = \frac{1}{(2\pi)^n}\int S_\Psi(\mathbf{k})e^{-i\mathbf{k}\cdot\xi}d^n k$$

Now let's prove that $C_\Psi(-\xi) = C^*_\Psi(\xi)$.

$$\begin{aligned}
C_\Psi(-\xi) &= \overline{\Psi(\mathbf{x})\Psi^*(\mathbf{x}-\xi)} \\
&= \lim_{L\to\infty}\frac{1}{L^n}\int \Psi_L(\mathbf{x})\Psi^*_L(\mathbf{x}-\xi)d^n x \\
\text{let} \quad & \mathbf{x}' = \mathbf{x}-\xi \\
&= \lim_{L\to\infty}\frac{1}{L^n}\int \Psi_L(\mathbf{x}'+\xi)\Psi^*_L(\mathbf{x}')d^n x' \\
&= \left[\lim_{L\to\infty}\frac{1}{L^n}\int \Psi^*_L(\mathbf{x}'+\xi)\Psi_L(\mathbf{x}')d^n x'\right]^* \\
&= C^*_\Psi(\xi)
\end{aligned}$$

(b)

$$\begin{aligned}
C_\Psi(\mathbf{a},\tau) &= \overline{\Psi(\mathbf{x},t)\Psi^*(\mathbf{x}+\mathbf{a},t+\tau)} \\
&= \overline{\left[\frac{\psi(\mathbf{x},t)}{\left(\overline{|\psi(\mathbf{x},t)|^2}\right)^{1/2}}\cdot\frac{\psi^*(\mathbf{x}+\mathbf{a},t+\tau)}{\left(\overline{|\psi(\mathbf{x}+\mathbf{a},t+\tau)|^2}\right)^{1/2}}\right]} \\
&= \frac{\overline{\psi(\mathbf{x},t)\psi^*(\mathbf{x}+\mathbf{a},t+\tau)}}{\left[\overline{|\psi(\mathbf{x},t)|^2}\;\overline{|\psi(\mathbf{x}+\mathbf{a},t+\tau)|^2}\right]^{1/2}} \\
&= \gamma_{12}(k\mathbf{a},\tau)
\end{aligned}$$

$$\begin{aligned}
S_\Psi(-\alpha k,-\omega) &\propto \left|\tilde{\Psi}_L(-\alpha k,-\omega)\right|^2 \\
&\propto \left|\int d^n x dt \; \psi_L(\mathbf{x},t)e^{i\alpha k\cdot\mathbf{x}+i\omega t}\right|^2 \\
&\propto I_\omega(\alpha,\omega)
\end{aligned}$$

Thus we can write S_Ψ as

$$S_\Psi(-\alpha k, -\omega) = const \times \frac{I_\omega(\alpha,\omega)}{F_s}$$
$$\text{where } F_s = \int I_\omega(\alpha,\omega) d\Omega_\alpha d\omega$$

To find the constant, we use the fact that $C_\Psi(\mathbf{0},0) = 1$, then

$$1 = C_\Psi(\mathbf{0},0) = \frac{1}{(2\pi)^3}\int S_\Psi(\mathbf{k}) d^3k = \frac{1}{(2\pi)^3} const \cdot \int \frac{I_\omega(\alpha,\omega)d\Omega_\alpha d\omega}{F_s}$$
$$= \frac{1}{(2\pi)^3} const$$
$$\Rightarrow const = (2\pi)^3$$

So we get

$$S_\Psi(-\alpha k, -\omega) = (2\pi)^3 \frac{I_\omega(\alpha,\omega)}{F_s}$$

Combining eq. (8.36) and eq. (8.35), we have

$$\begin{aligned}
\gamma_{12}(k\mathbf{a},\tau) &= C_\Psi(\mathbf{a},\tau) = \frac{1}{(2\pi)^3}\int S_\Psi(\alpha k,\omega) e^{-ik\alpha\cdot\mathbf{a} - i\omega\tau} d\Omega_\alpha d\omega \\
&= \int \frac{I_\omega(-\alpha,-\omega)}{F_s} e^{-ik\alpha\cdot\mathbf{a} - i\omega\tau} d\Omega_\alpha d\omega \\
&= \frac{\int d\Omega_\alpha d\omega I_\omega(\alpha,\omega) e^{i(k\mathbf{a}\cdot\alpha + \omega\tau)}}{F_s}
\end{aligned}$$

which is eq. (8.28)

Combining eq. (8.36) and eq. (8.34), we have

$$\begin{aligned}
I_\omega(\alpha,\omega) &= F_s \frac{1}{(2\pi)^3} S_\Psi(-\alpha k, -\omega) \\
&= F_s \frac{1}{(2\pi)^3}\int \gamma_{12}(k\mathbf{a},\tau) e^{-ik\alpha\cdot\mathbf{a} - i\omega\tau} d\tau d^2ka \\
&= F_s \int \frac{d\tau d^2 ka}{(2\pi)^3} \gamma_{12}(k\mathbf{a},\tau) e^{-i(k\mathbf{a}\cdot\alpha + \omega\tau)}
\end{aligned}$$

which is eq. (8.29).

C.

1. Exercise 8.5 Microwave background radiation [by Xinkai Wu/99]

(a)

$$\begin{aligned}
\gamma_{\|}(\tau) &\propto \int_{-\infty}^{+\infty} F_\omega(\omega) e^{i\omega\tau} d\omega \\
&\quad [\text{note } F_\omega(-\omega) = F_\omega(\omega)] \\
&= 2Re\left[\int_0^{+\infty} F_\omega(\omega) e^{i\omega\tau} d\omega\right]
\end{aligned}$$

For a Wien spectrum,

$$\begin{aligned}\int_0^{+\infty} F_\omega(\omega) e^{i\omega\tau} d\omega &\propto \int_0^{+\infty} \omega^3 e^{-\hbar\omega/kT} e^{i\omega\tau} d\omega \\ &= \int_0^{+\infty} \omega^3 e^{\omega\left(i\tau - \frac{\hbar}{kT}\right)} d\omega \\ &\quad \text{integrate by part several times} \\ &\propto \frac{1}{\left(i\tau - \frac{\hbar}{kT}\right)^4}\end{aligned}$$

Thus

$$\begin{aligned}\gamma_{\|}(\tau) &\propto Re\left[\frac{1}{\left(i\tau - \frac{\hbar}{kT}\right)^4}\right] \\ &\propto \frac{\left(\frac{\hbar}{kT}\right)^4 - 6\left(\frac{\hbar}{kT}\right)^2 \tau^2 + \tau^4}{\left[\left(\frac{\hbar}{kT}\right)^2 + \tau^2\right]^4} \\ &\propto \frac{s^4 - 6s_0^2 s^2 + s_0^4}{(s^2 + s_0^2)^4}\end{aligned}$$

where $s_0 \equiv \frac{\hbar c}{kT} = 0.83mm$, $s \equiv c\tau$. Thus we have

$$V = |\gamma_{\|}| \propto \frac{|s^4 - 6s_0^2 s^2 + s_0^4|}{(s^2 + s_0^2)^4}$$

(b) Using Mathematica, we find that for a Planck spectrum:

$$\gamma_{\|}(\tau) \propto \frac{-3 + \left(\pi\frac{\tau}{\tau_0}\right)^4 \left[2 + cosh\left(2\pi\frac{\tau}{\tau_0}\right)\right] csch^4\left(\pi\frac{\tau}{\tau_0}\right)}{(\tau/\tau_0)^4}$$

where $\tau_0 \equiv \frac{\hbar}{kT}$.

The plots of the visibility V for Wien and Planck spectrum are given in Fig. 1. As can be seen, they are quite similar to each other.

2. Exercise 8.8 Interferometry from space [by Xinkai Wu/02]

(a) Now that we only consider a single polarization, the specific intensity for a black body is half that given in chapter 2:

$$[I_\nu]_b = \frac{(h/c^2)\nu^3}{e^{h\nu/k_B T_b} - 1} \approx \frac{\nu^2 k_B T_b}{c^2}$$

where we only consider the low-frequency part $h\nu << k_B T_b$.

Equating the above expression to $2\pi I_\omega$ with I_ω being the specific intensity of the astronomical source, we find the brightness temperature for this source

$$T_b = \frac{(2\pi)^3 c^2 I_\omega}{k_B \omega^2} = \frac{(2\pi)^3 c^2 F_\omega}{k_B \omega^2 \Delta\Omega}$$

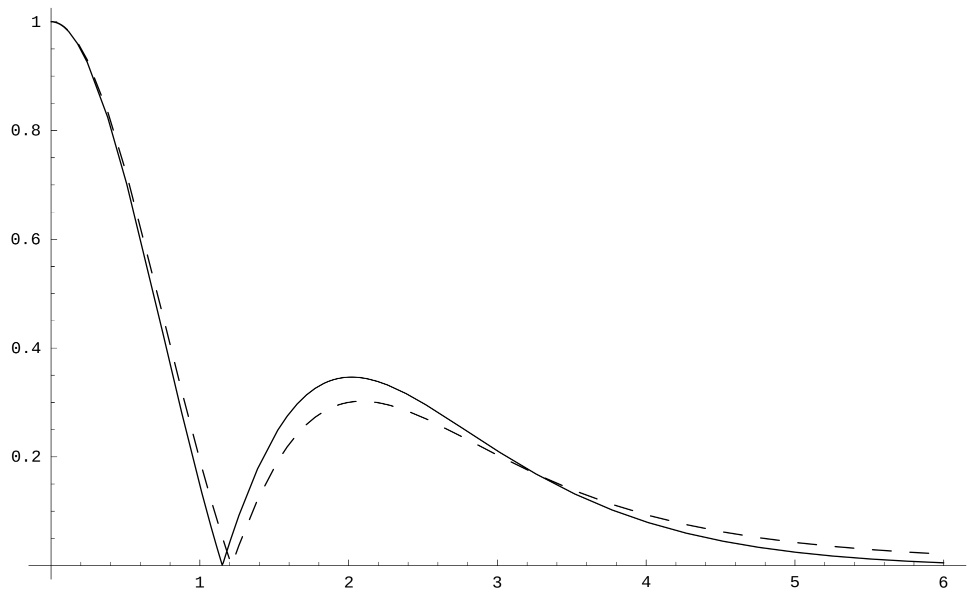

Figure 1: Microwave background radiation: x-axis is τ in unit of $10^{-12} sec$, y-axis is $|\gamma_{||}(\tau)|$; the solid line is for Wien spectrum, and the dashed line is for Plank spectrum

(b) The smallest source angular size α_r the telescope can resolve is given by the first zero of the *sinc* function,

$$ka\alpha_r = \pi \Rightarrow \alpha_r = 0.5\frac{\lambda}{a}$$

The corresponding solid angle subtended by the source is

$$\Delta\Omega = \pi\alpha_r^2$$

Plugging the above expressions into the formula for the brightness temperature we found in part (a), we get the maximum brightness temperature measurable

$$T_b = \frac{2F_\omega a^2}{(0.5)^2 k_B}$$

with a being the spatial separation of the telescopes. This maximum temperature has no ω dependence.

Taking $F_\omega = 10^{-25} Wm^{-2}Hz^{-1}$, $a = 6R_e = 6 \times 6378km$, we get

$$T_b = 8.5 \times 10^{13} K$$

D.

1. Exercise 8.10 Antireflection coating
(a) [by Xinkai Wu/99]
See Fig 2.

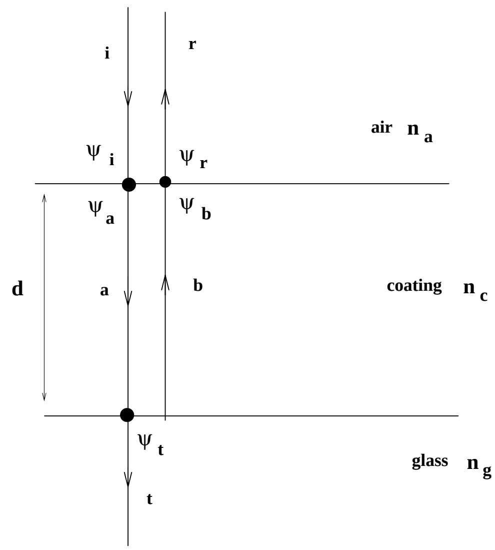

Figure 2: Anti-reflection coating

Notation of the amplitude reflection and transmission coefficients:
n_a, n_c, n_g: refraction indices of the air, coating, and glass, respectively.
r_1, t_1: air to coating
r_1', t_1': coating to air
r_2, t_2: coating to glass
Then similar to the derivation in the text, we have the following equations:

$$\psi_r = r_1\psi_i + t_1'\psi_b$$
$$\psi_a = t_1\psi_i + r_1'\psi_b$$
$$\psi_b e^{-ikd} = r_2\psi_a e^{ikd}$$
$$\psi_t = t_2\psi_a e^{ikd}$$

where $k = \frac{n_c\omega}{c} = \frac{2\pi}{\lambda_c} = \frac{2\pi n_c}{\lambda_{vac}}$ is the wavenumber in the coating (λ_c is the wavelength in the coating, and λ_{vac} is the wavelength in the vacuum).

Solving these equations, we get

$$\psi_r = \frac{r_1 + r_2(t_1t_1' - r_1r_1')e^{2ikd}}{1 - r_1'r_2e^{2ikd}}\psi_i$$
$$\psi_t = \frac{t_1t_2e^{ikd}}{1 - r_1'r_2e^{2ikd}}\psi_i$$

All the amplitude reflection and transmission coefficients here are real because both interfaces are perfectly sharp, and we have eq. (8.43)

$$t_1 t_1' - r_1 r_1' = 1$$
$$r_1' = -r_1$$

Using the above reciprocity relations, one finds

$$\frac{\psi_r}{\psi_i} = \frac{r_1 + r_2 e^{2ikd}}{1 + r_1 r_2 e^{2ikd}}, \quad \frac{\psi_t}{\psi_i} = \frac{t_1 t_2 e^{ikd}}{1 + r_1 r_2 e^{2ikd}} \tag{1}$$

Now for a given wavelength λ_c, if we take $d = \frac{\lambda_c}{4}$, then $2kd = \pi$, $e^{2ikd} = -1$.

We also have $n_c = \sqrt{n_a n_g}$, which implies $r_1 = r_2$ as one can explicitly verify using eq. (8.54):

$$r_1 = \frac{n_a - n_c}{n_a + n_c} = \frac{\sqrt{n_a} - \sqrt{n_g}}{\sqrt{n_a} + \sqrt{n_g}}$$
$$r_2 = \frac{n_c - n_g}{n_c + n_g} = \frac{\sqrt{n_a} - \sqrt{n_g}}{\sqrt{n_a} + \sqrt{n_g}}$$

So in this case we have

$$\frac{\psi_r}{\psi_i} = \frac{r_1 - r_2}{1 - r_1 r_2} = 0$$

namely there's no reflection, and thus we have perfect transmission by energy conservation.

(b) [by unknown author]

For blue light, $\lambda_{vac} \approx 400nm$.

$n_a \approx 1.0$, $n_g \approx 1.6$, thus $n_c = \sqrt{n_a n_g} \approx 1.3$.

So to avoid reflection of blue light, the thickness of the coating should be $\approx \frac{\lambda_{vac}}{4n_c} \sim 75nm$. Using the amplitude reflection coefficient in eq. (1)

$$R = \frac{|\psi_r|^2}{|\psi_i|^2} = \frac{(r_1 + r_2 cos\phi)^2 + r_2^2 sin^2\phi}{(1 + r_1 r_2 cos\phi)^2 + r_1^2 r_2^2 sin^2\phi}$$

where $\phi = 2kd$.

For red light, $\lambda_{vac} \sim 600nm$,

$$\phi = \pi \frac{400nm}{600nm} = \frac{2}{3}\pi$$

Also,

$$r_1 = r_2 = \frac{n_a - n_c}{n_a + n_c} \approx \frac{1.0 - 1.3}{1.0 + 1.3} \approx -0.13$$

where we neglect the variation of the refraction index w.r.t. to frequency.

$r_1, r_2 << 1$, so

$$R \approx (r_1 + r_2 cos\phi)^2 + r_2^2 sin^2\phi = 2r_1^2(1 + cos\phi) \approx 0.017$$

2. Exercise 8.13 Electron intensity interferometer [by Xinkai Wu/02]

To make a intensity interferometer for electron, just replace the photodetectors in Fig. 8.12 of the text with Ampere meters.

Photons are bosons so they tend to exhibit a positive intensity correlation; electrons are fermions and thus are expected to exhibit a negative intensity correlation. Now let's show that this is indeed the case.

Setting the multiplicative constant K to unity, then $I = |\psi|^2$. The intensity correlation is derived as follows:

Write

$$\psi(t) = \sum_j \psi_j e^{i\omega_j t}$$

where ψ_j can be thought of as the field (wave function) for an individual particle with energy $\hbar\omega_j$ and the sum is over all the particles.

Then

$$\begin{aligned} \overline{I(t)I(t+\tau)} &= \overline{\psi^*(t)\psi(t)\psi^*(t+\tau)\psi(t+\tau)} \\ &= \sum_{jkmn} \psi_j^*\psi_k\psi_m^*\psi_n e^{-i\omega_m\tau+i\omega_n\tau}\overline{e^{-i\omega_j t+i\omega_k t-i\omega_m t+i\omega_n t}} \\ &= \sum_{jm} \psi_j^*\psi_j\psi_m^*\psi_m + \sum_{jk} \psi_j^*\psi_k\psi_k^*\psi_j e^{-i\omega_k\tau+i\omega_j\tau} \end{aligned} \tag{2}$$

where in the last line, the first term comes from the case $j = k, m = n$ and is equal to

$$\sum_j \psi_j^*\psi_j \sum_m \psi_m^*\psi_m = \overline{I}^2$$

while the second term comes from the case $j = n, k = m$. Uptill now, we have been general, not specifying whether the particle is boson or fermion.

Now if the particle is boson, then the ψ_j's commute and the second term is equal to

$$\begin{aligned} &\sum_j \psi_j^*\psi_j e^{i\omega_j\tau} \sum_k \psi_k^*\psi_k e^{-i\omega_k\tau} \\ &= \overline{\psi^*(t)\psi(t+\tau)} \cdot \overline{\psi^*(t+\tau)\psi(t)} \\ &= \left|\overline{\psi^*(t+\tau)\psi(t)}\right|^2 \\ &= \overline{I}^2 \left|\gamma_{||}(\tau)\right|^2 \end{aligned}$$

Thus we find that for bosons,

$$\overline{I(t)I(t+\tau)} = \overline{I}^2 \left[1 + \left|\gamma_{\|}(\tau)\right|^2\right]$$

which is, of course, eq. (8.61) in the text.

Setting $\delta I(t) = I(t) - \overline{I}$, we see that

$$\frac{\overline{\delta I(t)\delta I(t+\tau)}}{\overline{I(t)}^2} = \left|\gamma_{\|}(\tau)\right|^2$$

which is eq. (8.62) in the text.

If the particle is fermion, then the ψ_j's (being regarded as wave functions for individual particles) anticommute, thus the second term in the last line of eq.(2) becomes

$$\begin{aligned}
&\sum_j \psi_j^* \psi_j e^{i\omega_j \tau} \cdot (-1) \sum_k \psi_k^* \psi_k e^{-i\omega_k \tau} \\
&= \overline{\psi^*(t)\psi(t+\tau)} \cdot (-1) \overline{\psi^*(t+\tau)\psi(t)} \\
&= (-1) \left|\overline{\psi^*(t+\tau)\psi(t)}\right|^2 \\
&= (-1)\overline{I}^2 \left|\gamma_{\|}(\tau)\right|^2
\end{aligned}$$

so we conclude that for fermions,

$$\overline{I(t)I(t+\tau)} = \overline{I}^2 \left[1 - \left|\gamma_{\|}(\tau)\right|^2\right]$$

and correspondingly

$$\frac{\overline{\delta I(t)\delta I(t+\tau)}}{\overline{I(t)}^2} = -\left|\gamma_{\|}(\tau)\right|^2$$

which exhibits the fermions' avoidance of each other (refusal to be in the same state), by contrast with the bosons' desire to be in the same state as exhibited by their positive correlation function.