The Cambridge Handbook of Physics Formulas

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Chapter 3 Dynamics and mechanics

3.1 Introduction

Unusually in physics, there is no pithy phrase that sums up the study of *dynamics* (the way in which forces produce motion), *kinematics* (the motion of matter), *mechanics* (the study of the forces and the motion they produce), and *statics* (the way forces combine to produce equilibrium). We will take the phrase *dynamics and mechanics* to encompass all the above, although it clearly does not!

To some extent this is because the equations governing the motion of matter include some of our oldest insights into the physical world and are consequentially steeped in tradition. One of the more delightful, or for some annoying, facets of this is the occasional use of arcane vocabulary in the description of motion. The epitome must be what Goldstein[1] calls "the jabberwockian sounding statement" *the polhode rolls without slipping on the herpolhode lying in the invariable plane*, describing "Poinsot's construction" – a method of visualising the free motion of a spinning rigid body. Despite this, dynamics and mechanics, including fluid mechanics, is arguably the most practically applicable of all the branches of physics.

Moreover, and in common with electromagnetism, the study of dynamics and mechanics has spawned a good deal of mathematical apparatus that has found uses in other fields. Most notably, the ideas behind the generalised dynamics of Lagrange and Hamilton lie behind much of quantum mechanics.

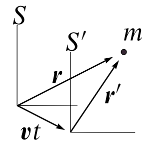
[1] H. Goldstein, *Classical Mechanics*, 2nd ed., 1980, Addison-Wesley.

3.2 Frames of reference

Galilean transformations

Time and position[a]	$\boldsymbol{r}=\boldsymbol{r}'+\boldsymbol{v}t$	(3.1)
	$t=t'$	(3.2)
Velocity	$\boldsymbol{u}=\boldsymbol{u}'+\boldsymbol{v}$	(3.3)
Momentum	$\boldsymbol{p}=\boldsymbol{p}'+m\boldsymbol{v}$	(3.4)
Angular momentum	$\boldsymbol{J}=\boldsymbol{J}'+m\boldsymbol{r}'\times\boldsymbol{v}+\boldsymbol{v}\times\boldsymbol{p}'t$	(3.5)
Kinetic energy	$T=T'+m\boldsymbol{u}'\cdot\boldsymbol{v}+\frac{1}{2}mv^2$	(3.6)

$\boldsymbol{r},\boldsymbol{r}'$	position in frames S and S'
$\boldsymbol{v}$	velocity of S' in S
t,t'	time in S and S'
$\boldsymbol{u},\boldsymbol{u}'$	velocity in frames S and S'
$\boldsymbol{p},\boldsymbol{p}'$	particle momentum in frames S and S'
m	particle mass
$\boldsymbol{J},\boldsymbol{J}'$	angular momentum in frames S and S'
T,T'	kinetic energy in frames S and S'

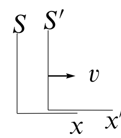


[a]Frames coincide at $t=0$.

Lorentz (spacetime) transformations[a]

Lorentz factor	$\gamma=\left(1-\frac{v^2}{c^2}\right)^{-1/2}$	(3.7)
Time and position	$x=\gamma(x'+vt')$; $\quad x'=\gamma(x-vt)$	(3.8)
	$y=y'$; $\quad y'=y$	(3.9)
	$z=z'$; $\quad z'=z$	(3.10)
	$t=\gamma\left(t'+\frac{v}{c^2}x'\right)$; $\quad t'=\gamma\left(t-\frac{v}{c^2}x\right)$	(3.11)
Differential four-vector[b]	$\mathrm{d}\boldsymbol{X}=(c\,\mathrm{d}t,-\mathrm{d}x,-\mathrm{d}y,-\mathrm{d}z)$	(3.12)

γ	Lorentz factor
v	velocity of S' in S
c	speed of light
x,x'	x-position in frames S and S' (similarly for y and z)
t,t'	time in frames S and S'
$\boldsymbol{X}$	spacetime four-vector

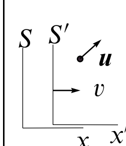


[a]For frames S and S' coincident at $t=0$ in relative motion along x. See page 141 for the transformations of electromagnetic quantities.
[b]Covariant components, using the (1,−1,−1,−1) signature.

Velocity transformations[a]

Velocity	$u_x=\frac{u'_x+v}{1+u'_xv/c^2}$; $\quad u'_x=\frac{u_x-v}{1-u_xv/c^2}$	(3.13)
	$u_y=\frac{u'_y}{\gamma(1+u'_xv/c^2)}$; $\quad u'_y=\frac{u_y}{\gamma(1-u_xv/c^2)}$	(3.14)
	$u_z=\frac{u'_z}{\gamma(1+u'_xv/c^2)}$; $\quad u'_z=\frac{u_z}{\gamma(1-u_xv/c^2)}$	(3.15)

γ	Lorentz factor $=[1-(v/c)^2]^{-1/2}$
v	velocity of S' in S
c	speed of light
u_i,u'_i	particle velocity components in frames S and S'

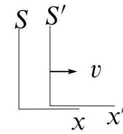


[a]For frames S and S' coincident at $t=0$ in relative motion along x.

Momentum and energy transformations[a]

Momentum and energy	$p_x=\gamma(p'_x+vE'/c^2)$;	$p'_x=\gamma(p_x-vE/c^2)$	(3.16)
	$p_y=p'_y$;	$p'_y=p_y$	(3.17)
	$p_z=p'_z$;	$p'_z=p_z$	(3.18)
	$E=\gamma(E'+vp'_x)$;	$E'=\gamma(E-vp_x)$	(3.19)
	$E^2-p^2c^2=E'^2-p'^2c^2=m_0^2c^4$		(3.20)
Four-vector[b]	$\boldsymbol{P}=(E/c,-p_x,-p_y,-p_z)$		(3.21)

γ	Lorentz factor $=[1-(v/c)^2]^{-1/2}$
v	velocity of S' in S
c	speed of light
p_x,p'_x	x components of momentum in S and S' (sim. for y and z)
E,E'	energy in S and S'
m_0	(rest) mass
p	total momentum in S
$\boldsymbol{P}$	momentum four-vector

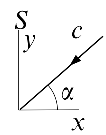


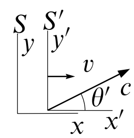
[a]For frames S and S' coincident at $t=0$ in relative motion along x.
[b]Covariant components, using the $(1,-1,-1,-1)$ signature.

Propagation of light[a]

Doppler effect	$\dfrac{\nu'}{\nu}=\gamma\left(1+\dfrac{v}{c}\cos\alpha\right)$	(3.22)
Aberration[b]	$\cos\theta=\dfrac{\cos\theta'+v/c}{1+(v/c)\cos\theta'}$	(3.23)
	$\cos\theta'=\dfrac{\cos\theta-v/c}{1-(v/c)\cos\theta}$	(3.24)
Relativistic beaming[c]	$P(\theta)=\dfrac{\sin\theta}{2\gamma^2[1-(v/c)\cos\theta]^2}$	(3.25)

ν	frequency received in S
ν'	frequency emitted in S'
α	arrival angle in S
γ	Lorentz factor $=[1-(v/c)^2]^{-1/2}$
v	velocity of S' in S
c	speed of light
θ,θ'	emission angle of light in S and S'
$P(\theta)$	angular distribution of photons in S





[a]For frames S and S' coincident at $t=0$ in relative motion along x.
[b]Light travelling in the opposite sense has a propagation angle of $\pi+\theta$ radians.
[c]Angular distribution of photons from a source, isotropic and stationary in S'. $\int_0^\pi P(\theta)\,\mathrm{d}\theta=1$.

Four-vectors[a]

Covariant and contravariant components	$x_0=x^0$ $\quad x_1=-x^1$		
	$x_2=-x^2$ $\quad x_3=-x^3$		(3.26)
Scalar product	$x^iy_i=x^0y_0+x^1y_1+x^2y_2+x^3y_3$		(3.27)
Lorentz transformations	$x^0=\gamma[x'^0+(v/c)x'^1]$;	$x'^0=\gamma[x^0-(v/c)x^1]$	(3.28)
	$x^1=\gamma[x'^1+(v/c)x'^0]$;	$x'^1=\gamma[x^1-(v/c)x^0]$	(3.29)
	$x^2=x'^2$;	$x'^3=x^3$	(3.30)

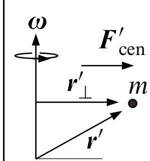
x_i	covariant vector components
x^i	contravariant components
x^i,x'^i	four-vector components in frames S and S'
γ	Lorentz factor $=[1-(v/c)^2]^{-1/2}$
v	velocity of S' in S
c	speed of light

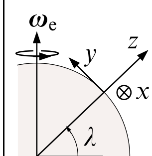
[a]For frames S and S', coincident at $t=0$ in relative motion along the (1) direction. Note that the $(1,-1,-1,-1)$ signature used here is common in special relativity, whereas $(-1,1,1,1)$ is often used in connection with general relativity (page 67).

Rotating frames

Vector transformation	$\left[\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}\right]_S = \left[\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}\right]_{S'} + \boldsymbol{\omega}\times\boldsymbol{A}$	(3.31)
Acceleration	$\dot{\boldsymbol{v}} = \dot{\boldsymbol{v}}' + 2\boldsymbol{\omega}\times\boldsymbol{v}' + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}')$	(3.32)
Coriolis force	$\boldsymbol{F}'_{\text{cor}} = -2m\boldsymbol{\omega}\times\boldsymbol{v}'$	(3.33)
Centrifugal force	$\boldsymbol{F}'_{\text{cen}} = -m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}')$	(3.34)
	$= +m\omega^2\boldsymbol{r}'_{\perp}$	(3.35)
Motion relative to Earth	$m\ddot{x} = F_x + 2m\omega_{\text{e}}(\dot{y}\sin\lambda - \dot{z}\cos\lambda)$	(3.36)
	$m\ddot{y} = F_y - 2m\omega_{\text{e}}\dot{x}\sin\lambda$	(3.37)
	$m\ddot{z} = F_z - mg + 2m\omega_{\text{e}}\dot{x}\cos\lambda$	(3.38)
Foucault's pendulum[a]	$\Omega_{\text{f}} = -\omega_{\text{e}}\sin\lambda$	(3.39)

$\boldsymbol{A}$	any vector
S	stationary frame
S'	rotating frame
$\boldsymbol{\omega}$	angular velocity of S' in S
$\dot{\boldsymbol{v}}, \dot{\boldsymbol{v}}'$	accelerations in S and S'
$\boldsymbol{v}'$	velocity in S'
$\boldsymbol{r}'$	position in S'
$\boldsymbol{F}'_{\text{cor}}$	coriolis force
m	particle mass
$\boldsymbol{F}'_{\text{cen}}$	centrifugal force
$\boldsymbol{r}'_{\perp}$	perpendicular to particle from rotation axis
F_i	nongravitational force
λ	latitude
z	local vertical axis
y	northerly axis
x	easterly axis
Ω_{f}	pendulum's rate of turn
ω_{e}	Earth's spin rate





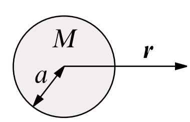
[a]The sign is such as to make the rotation clockwise in the northern hemisphere.

3.3 Gravitation

Newtonian gravitation

Newton's law of gravitation	$\boldsymbol{F}_1 = \frac{Gm_1m_2}{r_{12}^2}\hat{\boldsymbol{r}}_{12}$	(3.40)
Newtonian field equations[a]	$\boldsymbol{g} = -\nabla\phi$	(3.41)
	$\nabla^2\phi = -\nabla\cdot\boldsymbol{g} = 4\pi G\rho$	(3.42)
Fields from an isolated uniform sphere, mass M, $\boldsymbol{r}$ from the centre	$\boldsymbol{g}(\boldsymbol{r}) = \begin{cases} -\frac{GM}{r^2}\hat{\boldsymbol{r}} & (r > a) \\ -\frac{GMr}{a^3}\hat{\boldsymbol{r}} & (r < a) \end{cases}$	(3.43)
	$\phi(\boldsymbol{r}) = \begin{cases} -\frac{GM}{r} & (r > a) \\ \frac{GM}{2a^3}(r^2 - 3a^2) & (r < a) \end{cases}$	(3.44)

$m_{1,2}$	masses
$\boldsymbol{F}_1$	force on m_1 ($= -\boldsymbol{F}_2$)
$\boldsymbol{r}_{12}$	vector from m_1 to m_2
$\hat{}$	unit vector
G	constant of gravitation
$\boldsymbol{g}$	gravitational field strength
ϕ	gravitational potential
ρ	mass density
$\boldsymbol{r}$	vector from sphere centre
M	mass of sphere
a	radius of sphere



[a]The gravitational force on a mass m is $m\boldsymbol{g}$.

General relativity[a]

Line element	$ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu = -d\tau^2$	(3.45)	ds $d\tau$ $g_{\mu\nu}$	invariant interval proper time interval metric tensor
Christoffel symbols and covariant differentiation	$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}(g_{\delta\beta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta})$	(3.46)	dx^μ $\Gamma^\alpha{}_{\beta\gamma}$	differential of x^μ Christoffel symbols
	$\phi_{;\gamma} = \phi_{,\gamma} \equiv \partial\phi/\partial x^\gamma$	(3.47)	$,\alpha$ $;\alpha$	partial diff. w.r.t. x^α covariant diff. w.r.t. x^α
	$A^\alpha_{;\gamma} = A^\alpha_{,\gamma} + \Gamma^\alpha{}_{\beta\gamma} A^\beta$	(3.48)	ϕ A^α	scalar contravariant vector
	$B_{\alpha;\gamma} = B_{\alpha,\gamma} - \Gamma^\beta{}_{\alpha\gamma} B_\beta$	(3.49)	B_α	covariant vector
Riemann tensor	$R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha{}_{\mu\gamma}\Gamma^\mu{}_{\beta\delta} - \Gamma^\alpha{}_{\mu\delta}\Gamma^\mu{}_{\beta\gamma} + \Gamma^\alpha{}_{\beta\delta,\gamma} - \Gamma^\alpha{}_{\beta\gamma,\delta}$	(3.50)	$R^\alpha{}_{\beta\gamma\delta}$	Riemann tensor
	$B_{\mu;\alpha;\beta} - B_{\mu;\beta;\alpha} = R^\gamma{}_{\mu\alpha\beta} B_\gamma$	(3.51)		
	$R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma}\,;\quad R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\gamma\delta}$	(3.52)		
	$R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0$	(3.53)		
Geodesic equation	$\frac{Dv^\mu}{D\lambda} = 0$	(3.54)	v^μ	tangent vector ($= dx^\mu/d\lambda$)
	where $\frac{DA^\mu}{D\lambda} \equiv \frac{dA^\mu}{d\lambda} + \Gamma^\mu{}_{\alpha\beta} A^\alpha v^\beta$	(3.55)	λ	affine parameter (e.g., τ for material particles)
Geodesic deviation	$\frac{D^2\xi^\mu}{D\lambda^2} = -R^\mu{}_{\alpha\beta\gamma} v^\alpha \xi^\beta v^\gamma$	(3.56)	ξ^μ	geodesic deviation
Ricci tensor	$R_{\alpha\beta} \equiv R^\sigma{}_{\alpha\sigma\beta} = g^{\sigma\delta} R_{\delta\alpha\sigma\beta} = R_{\beta\alpha}$	(3.57)	$R_{\alpha\beta}$	Ricci tensor
Einstein tensor	$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$	(3.58)	$G^{\mu\nu}$ R	Einstein tensor Ricci scalar ($= g^{\mu\nu} R_{\mu\nu}$)
Einstein's field equations	$G^{\mu\nu} = 8\pi T^{\mu\nu}$	(3.59)	$T^{\mu\nu}$ p	stress-energy tensor pressure (in rest frame)
Perfect fluid	$T^{\mu\nu} = (p+\rho)u^\mu u^\nu + p g^{\mu\nu}$	(3.60)	ρ u^ν	density (in rest frame) fluid four-velocity
Schwarzschild solution (exterior)	$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$	(3.61)	M (r,θ,ϕ) t	spherically symmetric mass (see Section 9.5) spherical polar coords. time
Kerr solution (outside a spinning black hole)	$ds^2 = -\frac{\Delta - a^2\sin^2\theta}{\varrho^2} dt^2 - 2a\frac{2Mr\sin^2\theta}{\varrho^2} dt\, d\phi + \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\varrho^2}\sin^2\theta\, d\phi^2 + \frac{\varrho^2}{\Delta} dr^2 + \varrho^2 d\theta^2$	(3.62)	J a Δ ϱ^2	angular momentum (along z) $\equiv J/M$ $\equiv r^2 - 2Mr + a^2$ $\equiv r^2 + a^2\cos^2\theta$

[a]General relativity conventionally uses "geometrized units" in which $G=1$ and $c=1$. Thus, $1\,\mathrm{kg} = 7.425\times10^{-28}\,\mathrm{m}$ etc. Contravariant indices are written as superscripts and covariant indices as subscripts. Note also that ds^2 means $(ds)^2$ etc.

3.4 Particle motion

Dynamics definitions[a]

Newtonian force	$\boldsymbol{F}=m\ddot{\boldsymbol{r}}=\dot{\boldsymbol{p}}$	(3.63)	$\boldsymbol{F}$ m $\boldsymbol{r}$	force mass of particle particle position vector
Momentum	$\boldsymbol{p}=m\dot{\boldsymbol{r}}$	(3.64)	$\boldsymbol{p}$	momentum
Kinetic energy	$T=\frac{1}{2}mv^2$	(3.65)	T v	kinetic energy particle velocity
Angular momentum	$\boldsymbol{J}=\boldsymbol{r}\times\boldsymbol{p}$	(3.66)	$\boldsymbol{J}$	angular momentum
Couple (or torque)	$\boldsymbol{G}=\boldsymbol{r}\times\boldsymbol{F}$	(3.67)	$\boldsymbol{G}$	couple
Centre of mass (ensemble of N particles)	$\boldsymbol{R}_0=\frac{\sum_{i=1}^{N}m_i\boldsymbol{r}_i}{\sum_{i=1}^{N}m_i}$	(3.68)	$\boldsymbol{R}_0$ m_i $\boldsymbol{r}_i$	position vector of centre of mass mass of ith particle position vector of ith particle

[a]In the Newtonian limit, $v \ll c$, assuming m is constant.

Relativistic dynamics[a]

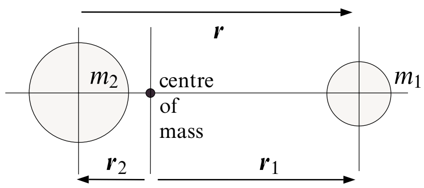
Lorentz factor	$\gamma=\left(1-\frac{v^2}{c^2}\right)^{-1/2}$	(3.69)	γ $\boldsymbol{v}$ c	Lorentz factor particle velocity speed of light
Momentum	$\boldsymbol{p}=\gamma m_0\boldsymbol{v}$	(3.70)	$\boldsymbol{p}$ m_0	relativistic momentum particle (rest) mass
Force	$\boldsymbol{F}=\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}$	(3.71)	$\boldsymbol{F}$ t	force on particle time
Rest energy	$E_{\mathrm{r}}=m_0c^2$	(3.72)	E_{r}	particle rest energy
Kinetic energy	$T=m_0c^2(\gamma-1)$	(3.73)	T	relativistic kinetic energy
Total energy	$E=\gamma m_0c^2$ $=(p^2c^2+m_0^2c^4)^{1/2}$	(3.74) (3.75)	E	total energy $(=E_{\mathrm{r}}+T)$

[a]It is now common to regard mass as a Lorentz invariant property and to drop the term "rest mass." The symbol m_0 is used here to avoid confusion with the idea of "relativistic mass" $(=\gamma m_0)$ used by some authors.

Constant acceleration

$v=u+at$	(3.76)	u	initial velocity
$v^2=u^2+2as$	(3.77)	v	final velocity
$s=ut+\frac{1}{2}at^2$	(3.78)	t	time
$s=\frac{u+v}{2}t$	(3.79)	s	distance travelled
		a	acceleration

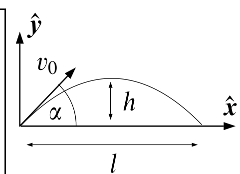
Reduced mass (of two interacting bodies)



Reduced mass	$\mu = \frac{m_1 m_2}{m_1 + m_2}$	(3.80)	μ reduced mass m_i interacting masses
Distances from centre of mass	$\boldsymbol{r}_1 = \frac{m_2}{m_1 + m_2}\boldsymbol{r}$	(3.81)	$\boldsymbol{r}_i$ position vectors from centre of mass $\boldsymbol{r}$ $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$
	$\boldsymbol{r}_2 = \frac{-m_1}{m_1 + m_2}\boldsymbol{r}$	(3.82)	$\lvert\boldsymbol{r}\rvert$ distance between masses
Moment of inertia	$I = \mu \lvert\boldsymbol{r}\rvert^2$	(3.83)	I moment of inertia
Total angular momentum	$\boldsymbol{J} = \mu \boldsymbol{r} \times \dot{\boldsymbol{r}}$	(3.84)	$\boldsymbol{J}$ angular momentum
Lagrangian	$L = \frac{1}{2}\mu \lvert\dot{\boldsymbol{r}}\rvert^2 - U(\lvert\boldsymbol{r}\rvert)$	(3.85)	L Lagrangian U potential energy of interaction

Ballistics[a]

Velocity	$\boldsymbol{v} = v_0 \cos\alpha\, \hat{\boldsymbol{x}} + (v_0 \sin\alpha - gt)\hat{\boldsymbol{y}}$	(3.86)	v_0 initial velocity $\boldsymbol{v}$ velocity at t α elevation angle
	$v^2 = v_0^2 - 2gy$	(3.87)	g gravitational acceleration
Trajectory	$y = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha}$	(3.88)	$\hat{}$ unit vector t time
Maximum height	$h = \frac{v_0^2}{2g}\sin^2\alpha$	(3.89)	h maximum height
Horizontal range	$l = \frac{v_0^2}{g}\sin 2\alpha$	(3.90)	l range

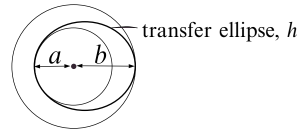


[a]Ignoring the curvature and rotation of the Earth and frictional losses. g is assumed constant.

Rocketry

Escape velocity[a]	$v_{\rm esc}=\left(\frac{2GM}{r}\right)^{1/2}$	(3.91)
Specific impulse	$I_{\rm sp}=\frac{u}{g}$	(3.92)
Exhaust velocity (into a vacuum)	$u=\left[\frac{2\gamma RT_{\rm c}}{(\gamma-1)\mu}\right]^{1/2}$	(3.93)
Rocket equation ($g=0$)	$\Delta v=u\ln\left(\frac{M_{\rm i}}{M_{\rm f}}\right)\equiv u\ln\mathscr{M}$	(3.94)
Multistage rocket	$\Delta v=\sum_{i=1}^{N}u_i\ln\mathscr{M}_i$	(3.95)
In a constant gravitational field	$\Delta v=u\ln\mathscr{M}-gt\cos\theta$	(3.96)
Hohmann cotangential transfer[b]	$\Delta v_{ah}=\left(\frac{GM}{r_a}\right)^{1/2}\left[\left(\frac{2r_b}{r_a+r_b}\right)^{1/2}-1\right]$	(3.97)
	$\Delta v_{hb}=\left(\frac{GM}{r_b}\right)^{1/2}\left[1-\left(\frac{2r_a}{r_a+r_b}\right)^{1/2}\right]$	(3.98)

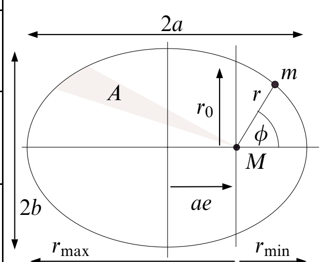
$v_{\rm esc}$	escape velocity
G	constant of gravitation
M	mass of central body
r	central body radius
$I_{\rm sp}$	specific impulse
u	effective exhaust velocity
g	acceleration due to gravity
R	molar gas constant
γ	ratio of heat capacities
$T_{\rm c}$	combustion temperature
μ	effective molecular mass of exhaust gas
Δv	rocket velocity increment
$M_{\rm i}$	pre-burn rocket mass
$M_{\rm f}$	post-burn rocket mass
$\mathscr{M}$	mass ratio
N	number of stages
$\mathscr{M}_i$	mass ratio for ith burn
u_i	exhaust velocity of ith burn
t	burn time
θ	rocket zenith angle
Δv_{ah}	velocity increment, a to h
Δv_{hb}	velocity increment, h to b
r_a	radius of inner orbit
r_b	radius of outer orbit



[a]From the surface of a spherically symmetric, nonrotating body, mass M.
[b]Transfer between coplanar, circular orbits a and b, via ellipse h with a minimal expenditure of energy.

Gravitationally bound orbital motion[a]

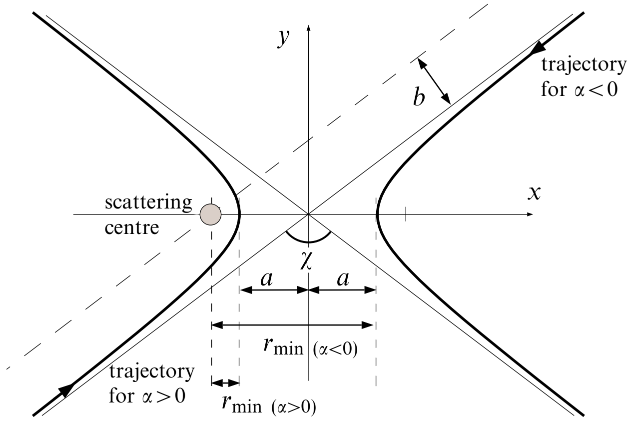
Potential energy of interaction	$U(r) = -\frac{GMm}{r} \equiv -\frac{\alpha}{r}$	(3.99)	$U(r)$ potential energy; G constant of gravitation; M central mass; m orbiting mass ($\ll M$); α positive constant
Total energy	$E = -\frac{\alpha}{r} + \frac{J^2}{2mr^2} = -\frac{\alpha}{2a}$	(3.100)	E total energy (constant); J total angular momentum (constant)
Virial theorem ($1/r$ potential)	$E = \langle U \rangle /2 = -\langle T \rangle$	(3.101)	T kinetic energy
	$\langle U \rangle = -2\langle T \rangle$	(3.102)	$\langle\cdot\rangle$ mean value
Orbital equation (Kepler's 1st law)	$\frac{r_0}{r} = 1 + e\cos\phi$, or	(3.103)	r_0 semi-latus-rectum; r distance of m from M
	$r = \frac{a(1-e^2)}{1+e\cos\phi}$	(3.104)	e eccentricity
Rate of sweeping area (Kepler's 2nd law)	$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{J}{2m} = \text{constant}$	(3.105)	A area swept out by radius vector (total area $= \pi ab$)
Semi-major axis	$a = \frac{r_0}{1-e^2} = \frac{\alpha}{2\lvert E \rvert}$	(3.106)	a semi-major axis; b semi-minor axis
Semi-minor axis	$b = \frac{r_0}{(1-e^2)^{1/2}} = \frac{J}{(2m\lvert E \rvert)^{1/2}}$	(3.107)	
Eccentricity[b]	$e = \left(1 + \frac{2EJ^2}{m\alpha^2}\right)^{1/2} = \left(1 - \frac{b^2}{a^2}\right)^{1/2}$	(3.108)	
Semi-latus-rectum	$r_0 = \frac{J^2}{m\alpha} = \frac{b^2}{a} = a(1-e^2)$	(3.109)	
Pericentre	$r_{\min} = \frac{r_0}{1+e} = a(1-e)$	(3.110)	$r_{\min}$ pericentre distance
Apocentre	$r_{\max} = \frac{r_0}{1-e} = a(1+e)$	(3.111)	$r_{\max}$ apocentre distance
Phase	$\cos\phi = \frac{(J/r) - (m\alpha/J)}{(2mE + m^2\alpha^2/J^2)^{1/2}}$	(3.112)	ϕ orbital phase
Period (Kepler's 3rd law)	$P = \pi\alpha \left(\frac{m}{2\lvert E \rvert^3}\right)^{1/2} = 2\pi a^{3/2}\left(\frac{m}{\alpha}\right)^{1/2}$	(3.113)	P orbital period



[a]For an inverse-square law of attraction between two isolated bodies in the nonrelativistic limit. If m is not $\ll M$, all explicit references to m in Equations (3.100) to (3.113) should be replaced by the reduced mass, $\mu = Mm/(M+m)$, and r taken as the body separation. The distance of mass m from the centre of mass is then $r\mu/m$ (see earlier table on *Reduced mass*). Other orbital dimensions scale similarly.

[b]Note that if the total energy, E, is <0 then $e<1$ and the orbit is an ellipse (a circle if $e=0$). If $E=0$, then $e=1$ and the orbit is a parabola. If $E>0$ then $e>1$ and the orbit becomes a hyperbola (see *Rutherford scattering* on next page).

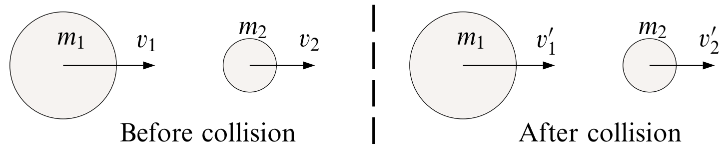
Rutherford scattering[a]



Scattering potential energy	$U(r) = -\dfrac{\alpha}{r}$	(3.114)	$U(r)$ potential energy r particle separation α constant
	$\alpha \begin{cases} <0 & \text{repulsive} \\ >0 & \text{attractive} \end{cases}$	(3.115)	
Scattering angle	$\tan\dfrac{\chi}{2} = \dfrac{\lvert\alpha\rvert}{2Eb}$	(3.116)	χ scattering angle E total energy (>0) b impact parameter
Closest approach	$r_{\min} = \dfrac{\lvert\alpha\rvert}{2E}\left(\csc\dfrac{\chi}{2} - \dfrac{\alpha}{\lvert\alpha\rvert}\right)$	(3.117)	$r_{\min}$ closest approach a hyperbola semi-axis e eccentricity
	$= a(e \pm 1)$	(3.118)	
Semi-axis	$a = \dfrac{\lvert\alpha\rvert}{2E}$	(3.119)	
Eccentricity	$e = \left(\dfrac{4E^2b^2}{\alpha^2} + 1\right)^{1/2} = \csc\dfrac{\chi}{2}$	(3.120)	
Motion trajectory[b]	$\dfrac{4E^2}{\alpha^2}x^2 - \dfrac{y^2}{b^2} = 1$	(3.121)	x, y position with respect to hyperbola centre
Scattering centre[c]	$x = \pm\left(\dfrac{\alpha^2}{4E^2} + b^2\right)^{1/2}$	(3.122)	
Rutherford scattering formula[d]	$\dfrac{d\sigma}{d\Omega} = \dfrac{1}{n}\dfrac{dN}{d\Omega}$	(3.123)	$\frac{d\sigma}{d\Omega}$ differential scattering cross section n beam flux density dN number of particles scattered into $d\Omega$ Ω solid angle
	$= \left(\dfrac{\alpha}{4E}\right)^2 \csc^4\dfrac{\chi}{2}$	(3.124)	

[a]Nonrelativistic treatment for an inverse-square force law and a fixed scattering centre. Similar scattering results from either an attractive or repulsive force. See also *Conic sections* on page 38.
[b]The correct branch can be chosen by inspection.
[c]Also the focal points of the hyperbola.
[d]n is the number of particles per second passing through unit area perpendicular to the beam.

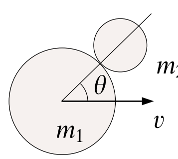
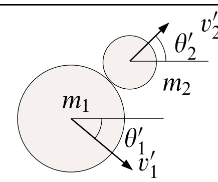
Inelastic collisions[a]



Coefficient of restitution	$v_2' - v_1' = \epsilon(v_1 - v_2)$	(3.125)	ϵ	coefficient of restitution
	$\epsilon = 1$ if perfectly elastic	(3.126)	v_i	pre-collision velocities
	$\epsilon = 0$ if perfectly inelastic	(3.127)	v_i'	post-collision velocities
Loss of kinetic energy[b]	$\dfrac{T - T'}{T} = 1 - \epsilon^2$	(3.128)	T, T'	total KE in zero momentum frame before and after collision
Final velocities	$v_1' = \dfrac{m_1 - \epsilon m_2}{m_1 + m_2} v_1 + \dfrac{(1+\epsilon) m_2}{m_1 + m_2} v_2$	(3.129)	m_i	particle masses
	$v_2' = \dfrac{m_2 - \epsilon m_1}{m_1 + m_2} v_2 + \dfrac{(1+\epsilon) m_1}{m_1 + m_2} v_1$	(3.130)		

[a]Along the line of centres, $v_1, v_2 \ll c$.
[b]In zero momentum frame.

Oblique elastic collisions[a]

Before collision		After collision		
Directions of motion	$\tan\theta_1' = \dfrac{m_2 \sin 2\theta}{m_1 - m_2 \cos 2\theta}$	(3.131)	θ	angle between centre line and incident velocity
	$\theta_2' = \theta$	(3.132)	θ_i'	final trajectories
			m_i	sphere masses
Relative separation angle	$\theta_1' + \theta_2' \begin{cases} > \pi/2 & \text{if } m_1 < m_2 \\ = \pi/2 & \text{if } m_1 = m_2 \\ < \pi/2 & \text{if } m_1 > m_2 \end{cases}$	(3.133)		
Final velocities	$v_1' = \dfrac{(m_1^2 + m_2^2 - 2 m_1 m_2 \cos 2\theta)^{1/2}}{m_1 + m_2} v$	(3.134)	v	incident velocity of m_1
	$v_2' = \dfrac{2 m_1 v}{m_1 + m_2} \cos\theta$	(3.135)	v_i'	final velocities

[a]Collision between two perfectly elastic spheres: m_2 initially at rest, velocities $\ll c$.

3.5 Rigid body dynamics

Moment of inertia tensor

Moment of inertia tensor[a]	$I_{ij}=\int (r^2\delta_{ij}-x_ix_j)\,\mathrm{d}m$	(3.136)
	$\mathsf{I}=\begin{pmatrix}\int(y^2+z^2)\,\mathrm{d}m & -\int xy\,\mathrm{d}m & -\int xz\,\mathrm{d}m\\ -\int xy\,\mathrm{d}m & \int(x^2+z^2)\,\mathrm{d}m & -\int yz\,\mathrm{d}m\\ -\int xz\,\mathrm{d}m & -\int yz\,\mathrm{d}m & \int(x^2+y^2)\,\mathrm{d}m\end{pmatrix}$	(3.137)
Parallel axis theorem	$I_{12}=I^\star_{12}-ma_1a_2$	(3.138)
	$I_{11}=I^\star_{11}+m(a_2^2+a_3^2)$	(3.139)
	$I_{ij}=I^\star_{ij}+m(\lvert\boldsymbol{a}\rvert^2\delta_{ij}-a_ia_j)$	(3.140)
Angular momentum	$\boldsymbol{J}=\mathsf{I}\boldsymbol{\omega}$	(3.141)
Rotational kinetic energy	$T=\frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{J}=\frac{1}{2}I_{ij}\omega_i\omega_j$	(3.142)

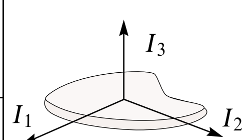
r	$r^2=x^2+y^2+z^2$
δ_{ij}	Kronecker delta
I	moment of inertia tensor
$\mathrm{d}m$	mass element
x_i	position vector of $\mathrm{d}m$
I_{ij}	components of I
$I^\star_{ij}$	tensor with respect to centre of mass
$a_i,\boldsymbol{a}$	position vector of centre of mass
m	mass of body
$\boldsymbol{J}$	angular momentum
$\boldsymbol{\omega}$	angular velocity
T	kinetic energy

[a] I_{ii} are the moments of inertia of the body. I_{ij} $(i\neq j)$ are its products of inertia. The integrals are over the body volume.

Principal axes

Principal moment of inertia tensor	$\mathsf{I}'=\begin{pmatrix}I_1 & 0 & 0\\ 0 & I_2 & 0\\ 0 & 0 & I_3\end{pmatrix}$	(3.143)
Angular momentum	$\boldsymbol{J}=(I_1\omega_1,I_2\omega_2,I_3\omega_3)$	(3.144)
Rotational kinetic energy	$T=\frac{1}{2}(I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2)$	(3.145)
Moment of inertia ellipsoid[a]	$T=T(\omega_1,\omega_2,\omega_3)$	(3.146)
	$J_i=\frac{\partial T}{\partial\omega_i}$ ($\boldsymbol{J}$ is $\perp$ ellipsoid surface)	(3.147)
Perpendicular axis theorem	$I_1+I_2\begin{cases}\geq I_3 & \text{generally}\\ =I_3 & \text{flat lamina} \perp \text{to 3-axis}\end{cases}$	(3.148)
Symmetries	$I_1\neq I_2\neq I_3$ asymmetric top	(3.149)
	$I_1=I_2\neq I_3$ symmetric top	
	$I_1=I_2=I_3$ spherical top	

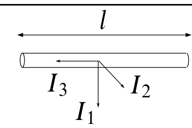
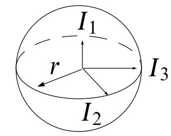
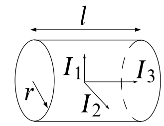
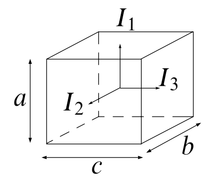
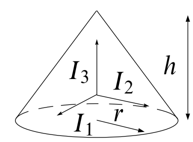
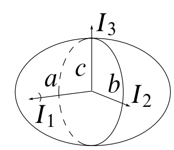
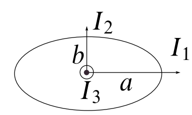
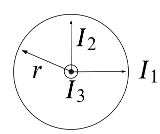
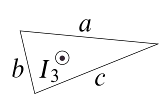
I'	principal moment of inertia tensor
I_i	principal moments of inertia
$\boldsymbol{J}$	angular momentum
ω_i	components of $\boldsymbol{\omega}$ along principal axes
T	kinetic energy



lamina

[a] The ellipsoid is defined by the surface of constant T.

Moments of inertia[a]

Thin rod, length l	$I_1 = I_2 = \dfrac{ml^2}{12}$ $I_3 \simeq 0$	(3.150) (3.151)	
Solid sphere, radius r	$I_1 = I_2 = I_3 = \dfrac{2}{5}mr^2$	(3.152)	
Spherical shell, radius r	$I_1 = I_2 = I_3 = \dfrac{2}{3}mr^2$	(3.153)	
Solid cylinder, radius r, length l	$I_1 = I_2 = \dfrac{m}{4}\left(r^2 + \dfrac{l^2}{3}\right)$ $I_3 = \dfrac{1}{2}mr^2$	(3.154) (3.155)	
Solid cuboid, sides a,b,c	$I_1 = m(b^2 + c^2)/12$ $I_2 = m(c^2 + a^2)/12$ $I_3 = m(a^2 + b^2)/12$	(3.156) (3.157) (3.158)	
Solid circular cone, base radius r, height h[b]	$I_1 = I_2 = \dfrac{3}{20}m\left(r^2 + \dfrac{h^2}{4}\right)$ $I_3 = \dfrac{3}{10}mr^2$	(3.159) (3.160)	
Solid ellipsoid, semi-axes a,b,c	$I_1 = m(b^2 + c^2)/5$ $I_2 = m(c^2 + a^2)/5$ $I_3 = m(a^2 + b^2)/5$	(3.161) (3.162) (3.163)	
Elliptical lamina, semi-axes a,b	$I_1 = mb^2/4$ $I_2 = ma^2/4$ $I_3 = m(a^2 + b^2)/4$	(3.164) (3.165) (3.166)	
Disk, radius r	$I_1 = I_2 = mr^2/4$ $I_3 = mr^2/2$	(3.167) (3.168)	
Triangular plate[c]	$I_3 = \dfrac{m}{36}(a^2 + b^2 + c^2)$	(3.169)	

[a]With respect to principal axes for bodies of mass m and uniform density. The radius of gyration is defined as $k = (I/m)^{1/2}$.
[b]Origin of axes is at the centre of mass ($h/4$ above the base).
[c]Around an axis through the centre of mass and perpendicular to the plane of the plate.

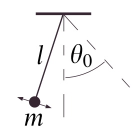
Centres of mass

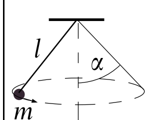
Solid hemisphere, radius r	$d=3r/8$ from sphere centre	(3.170)
Hemispherical shell, radius r	$d=r/2$ from sphere centre	(3.171)
Sector of disk, radius r, angle 2θ	$d=\frac{2}{3}r\frac{\sin\theta}{\theta}$ from disk centre	(3.172)
Arc of circle, radius r, angle 2θ	$d=r\frac{\sin\theta}{\theta}$ from circle centre	(3.173)
Arbitrary triangular lamina, height h[a]	$d=h/3$ perpendicular from base	(3.174)
Solid cone or pyramid, height h	$d=h/4$ perpendicular from base	(3.175)
Spherical cap, height h, sphere radius r	solid: $d=\frac{3}{4}\frac{(2r-h)^2}{3r-h}$ from sphere centre	(3.176)
	shell: $d=r-h/2$ from sphere centre	(3.177)
Semi-elliptical lamina, height h	$d=\frac{4h}{3\pi}$ from base	(3.178)

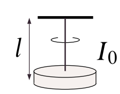
[a] h is the perpendicular distance between the base and apex of the triangle.

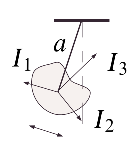
Pendulums

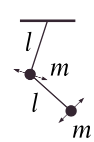
Simple pendulum	$P=2\pi\sqrt{\frac{l}{g}}\left(1+\frac{\theta_0^2}{16}+\cdots\right)$	(3.179)	P period; g gravitational acceleration; l length; θ_0 maximum angular displacement
Conical pendulum	$P=2\pi\left(\frac{l\cos\alpha}{g}\right)^{1/2}$	(3.180)	α cone half-angle
Torsional pendulum[a]	$P=2\pi\left(\frac{lI_0}{C}\right)^{1/2}$	(3.181)	I_0 moment of inertia of bob; C torsional rigidity of wire (see page 81)
Compound pendulum[b]	$P\simeq 2\pi\left[\frac{1}{mga}(ma^2+I_1\cos^2\gamma_1+I_2\cos^2\gamma_2+I_3\cos^2\gamma_3)\right]^{1/2}$	(3.182)	a distance of rotation axis from centre of mass; m mass of body; I_i principal moments of inertia; γ_i angles between rotation axis and principal axes
Equal double pendulum[c]	$P\simeq 2\pi\left[\frac{l}{(2\pm\sqrt{2})g}\right]^{1/2}$	(3.183)	





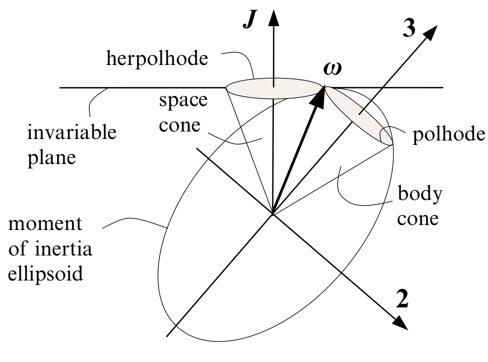




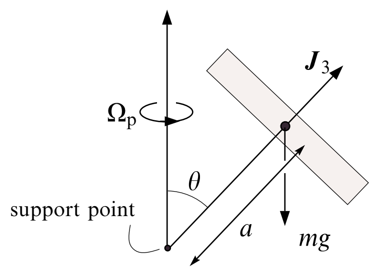


[a] Assuming the bob is supported parallel to a principal rotation axis.
[b] I.e., an arbitrary triaxial rigid body.
[c] For very small oscillations (two eigenmodes).

Tops and gyroscopes



prolate symmetric top



gyroscope

Euler's equations[a]	$G_1 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3$	(3.184)	G_i external couple (=0 for free rotation)
	$G_2 = I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1$	(3.185)	I_i principal moments of inertia
	$G_3 = I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2$	(3.186)	ω_i angular velocity of rotation
Free symmetric top[b] ($I_3 < I_2 = I_1$)	$\Omega_\mathrm{b} = \dfrac{I_1 - I_3}{I_1}\omega_3$	(3.187)	Ω_b body frequency; Ω_s space frequency
	$\Omega_\mathrm{s} = \dfrac{J}{I_1}$	(3.188)	J total angular momentum
Free asymmetric top[c]	$\Omega_\mathrm{b}^2 = \dfrac{(I_1 - I_3)(I_2 - I_3)}{I_1 I_2}\omega_3^2$	(3.189)	
Steady gyroscopic precession	$\Omega_\mathrm{p}^2 I_1' \cos\theta - \Omega_\mathrm{p} J_3 + mga = 0$	(3.190)	Ω_p precession angular velocity; θ angle from vertical; J_3 angular momentum around symmetry axis
	$\Omega_\mathrm{p} \simeq \begin{cases} Mga/J_3 & \text{(slow)} \\ J_3/(I_1'\cos\theta) & \text{(fast)} \end{cases}$	(3.191)	m mass; g gravitational acceleration
Gyroscopic stability	$J_3^2 \geq 4I_1' mga\cos\theta$	(3.192)	a distance of centre of mass from support point; I_1' moment of inertia about support point
Gyroscopic limit ("sleeping top")	$J_3^2 \gg I_1' mga$	(3.193)	
Nutation rate	$\Omega_\mathrm{n} = J_3/I_1'$	(3.194)	Ω_n nutation angular velocity
Gyroscope released from rest	$\Omega_\mathrm{p} = \dfrac{mga}{J_3}(1 - \cos\Omega_\mathrm{n} t)$	(3.195)	t time

[a] Components are with respect to the principal axes, rotating with the body.

[b] The body frequency is the angular velocity (with respect to principal axes) of **ω** around the **3**-axis. The space frequency is the angular velocity of the **3**-axis around **J**, i.e., the angular velocity at which the body cone moves around the space cone.

[c] **J** close to **3**-axis. If $\Omega_\mathrm{b}^2 < 0$, the body tumbles.

3.6 Oscillating systems

Free oscillations

Differential equation	$$\frac{d^2x}{dt^2}+2\gamma\frac{dx}{dt}+\omega_0^2x=0$$	(3.196)	x oscillating variable t time γ damping factor (per unit mass) ω_0 undamped angular frequency
Underdamped solution ($\gamma<\omega_0$)	$x=Ae^{-\gamma t}\cos(\omega t+\phi)$ where $\omega=(\omega_0^2-\gamma^2)^{1/2}$	(3.197) (3.198)	A amplitude constant ϕ phase constant ω angular eigenfrequency
Critically damped solution ($\gamma=\omega_0$)	$x=e^{-\gamma t}(A_1+A_2t)$	(3.199)	A_i amplitude constants
Overdamped solution ($\gamma>\omega_0$)	$x=e^{-\gamma t}(A_1e^{qt}+A_2e^{-qt})$ where $q=(\gamma^2-\omega_0^2)^{1/2}$	(3.200) (3.201)	
Logarithmic decrement[a]	$$\Delta=\ln\frac{a_n}{a_{n+1}}=\frac{2\pi\gamma}{\omega}$$	(3.202)	Δ logarithmic decrement a_n nth displacement maximum
Quality factor	$$Q=\frac{\omega_0}{2\gamma}\quad\left[\simeq\frac{\pi}{\Delta}\quad\text{if}\quad Q\gg 1\right]$$	(3.203)	Q quality factor

[a]The *decrement* is usually the ratio of successive displacement *maxima* but is sometimes taken as the ratio of successive displacement *extrema*, reducing Δ by a factor of 2. Logarithms are sometimes taken to base 10, introducing a further factor of $\log_{10}e$.

Forced oscillations

Differential equation	$$\frac{d^2x}{dt^2}+2\gamma\frac{dx}{dt}+\omega_0^2x=F_0e^{\mathbf{i}\omega_f t}$$	(3.204)	x oscillating variable t time γ damping factor (per unit mass)
Steady-state solution[a]	$x=Ae^{\mathbf{i}(\omega_f t-\phi)}$, where $A=F_0[(\omega_0^2-\omega_f^2)^2+(2\gamma\omega_f)^2]^{-1/2}$ $\simeq\dfrac{F_0/(2\omega_0)}{[(\omega_0-\omega_f)^2+\gamma^2]^{1/2}}\quad(\gamma\ll\omega_f)$ $\tan\phi=\dfrac{2\gamma\omega_f}{\omega_0^2-\omega_f^2}$	(3.205) (3.206) (3.207) (3.208)	ω_0 undamped angular frequency F_0 force amplitude (per unit mass) ω_f forcing angular frequency A amplitude ϕ phase lag of response behind driving force
Amplitude resonance[b]	$\omega_{ar}^2=\omega_0^2-2\gamma^2$	(3.209)	ω_{ar} amplitude resonant forcing angular frequency
Velocity resonance[c]	$\omega_{vr}=\omega_0$	(3.210)	ω_{vr} velocity resonant forcing angular frequency
Quality factor	$$Q=\frac{\omega_0}{2\gamma}$$	(3.211)	Q quality factor
Impedance	$$Z=2\gamma+\mathbf{i}\frac{\omega_f^2-\omega_0^2}{\omega_f}$$	(3.212)	Z impedance (per unit mass)

[a]Excluding the free oscillation terms.
[b]Forcing frequency for maximum displacement.
[c]Forcing frequency for maximum velocity. Note $\phi=\pi/2$ at this frequency.