MATHEMATICAL TRACTS FOR PHYSICISTS

INTRODUCTION
TO THE
CALCULUS OF VARIATIONS

BY

WILLIAM ELWOOD BYERLY
PERKINS PROFESSOR OF MATHEMATICS EMERITUS
HARVARD UNIVERSITY

CAMBRIDGE
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CHAPTER I

INTRODUCTION

1. The Calculus of Variations owed its origin to the attempt to solve a very interesting and rather narrow class of problems in Maxima and Minima, in which it is required to find the form of a function such that the definite integral of an expression involving that function and its derivative shall be a maximum or a minimum.

Let us consider three simple examples: The Shortest Line, The Curve of Quickest Descent, and The Minimum Surface of Revolution.

(*a*) *The Shortest Line.* Let it be required to find the equation of the shortest plane curve joining two given points.

We shall use rectangular coördinates in the plane in question taking one of the points as the origin. Call the coördinates of the second point x_1, y_1.

If $y = f(x)$ is a curve through $(0, 0)$ and (x_1, y_1) and I is the length of the arc between the points, obviously

$$I = \int_0^{x_1} \sqrt{dx^2 + dy^2}$$

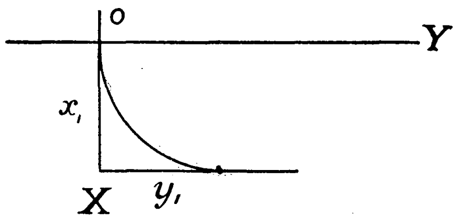
or

$$I = \int_0^{x_1} \sqrt{1 + y'^2}\,dx, \tag{1}$$

and we wish to determine the form of the function f so that this integral shall be a minimum.

(*b*) *The Curve of Quickest Descent.* Let it be required to find the form of a smooth curve lying in a vertical plane and joining two given points, down which a particle starting from rest will slide under gravity from the first point to the second in the least possible time.

We shall use rectangular axes in the vertical plane taking the higher point as the origin and taking the axis of X downward. Call the coördinates of the second point x_1, y_1.



Let $y = f(x)$ be a curve through $(0, 0)$ and (x_1, y_1) and use the well-known fact that $\frac{ds}{dt}$, the velocity of the moving particle at any time, is $\sqrt{2gx}$.

We have $$\frac{ds}{dt} = \sqrt{2gx},$$

whence $$dt = \frac{ds}{\sqrt{2gx}} = \frac{\sqrt{1 + y'^2}}{\sqrt{2gx}}\,dx$$

and $$t = \int_0^{x_1} \frac{1}{\sqrt{2gx}} \sqrt{1 + y'^2}\,dx.$$

Let $$I = \int_0^{x_1} \frac{1}{\sqrt{x}} \sqrt{1 + y'^2}\,dx, \tag{2}$$

and the form of the function f is to be determined so that this integral shall be a minimum.

(*c*) *The Minimum Surface of Revolution.* Given two points and a line which are co-planar, let it be required to find the form of a curve terminated by the two points and lying in the plane which, by its revolution about the given line, shall generate a surface of the least possible area.

Take the line as the axis of X and use an axis of Y through one of the points. Call the coördinates of the points 0, y_0, and x_1, y_1. Let $y = f(x)$ be a curve through $(0, y_0)$ and (x_1, y_1).

If S is the area of the surface of revolution generated by the curve,

$$S = 2\pi \int_0^{x_1} y ds = 2\pi \int_0^{x_1} y \sqrt{1 + y'^2} dx.$$

Let $$I = \int_0^{x_1} y \sqrt{1 + y'^2} dx, \tag{3}$$

and we wish to determine the form of the function f so that I shall be a minimum.

2. The three problems just considered are special cases of what we shall call our *fundamental* problem which is, to determine the form of the function f so that if $y = f(x)$, $\int_{x_0}^{x_1} \phi\, (x, y, y')\, dx$ shall be a maximum or a minimum; ϕ being a given function and x_0 and x_1 being given constants, as are y_0 and y_1, the corresponding values of y.

In ordinary problems in maxima and minima $y = f(x)$ is a given function and we wish to find a value, x_0, of x for which y is greater, if we seek a maximum, less, if we seek a minimum, than for *neighboring* values of x; that is, for values of x differing from x_0 by a sufficiently small amount whether that amount is positive or negative.

In our new problems, to speak in geometrical language, we have to find the *form* of a curve for which our integral, I, is greater or less than for any *neighboring* curve having the same end-points.

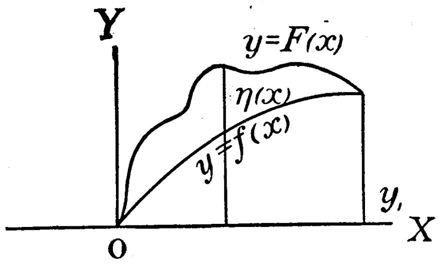
3. Let us now attack our first problem, that of the *shortest line.* We have to find the form of the function f so that if

$$I = \int_0^{x_1} \sqrt{1 + y'^2}\, dx$$

I shall be a minimum when $y = f(x)$. *v.* Art. 1 (*a*).

Let $y = F(x)$ be any other continuous curve joining the given points, and let $\eta(x) = F(x) - f(x)$.

Then $y = f(x) + \eta(x)$ is our curve $y = F(x)$. Consider the curve $y = f(x) + \alpha\eta(x)$ where α is a parameter independent of x.



For any particular value of α the curve $y = f(x) + \alpha\eta(x)$ is one of a family of curves including $y = f(x)$ for $\alpha = 0$ and $y = F(x)$ for $\alpha = 1$.

By taking a sufficiently small value for α we can make $\alpha\eta(x)$ less in absolute value for that and all less values of α, and for all values of x between 0 and x_1, than any previously chosen quantity ξ; and for such values of α the curve $y = f(x) + \alpha\eta(x)$ is said to be a curve in the neighborhood of $y = f(x)$.

If $y = f(x)$ and $y = F(x)$ are given, $I(\alpha)$, the I for any one of our curves $y = f(x) + \alpha\eta(x)$, is $\int_0^{x_1} \sqrt{1 + (y' + \alpha\eta'(x))^2}\, dx$, and $I(\alpha)$ is a function of α only.

A necessary condition that $I(\alpha)$ should be a minimum when $\alpha = 0$ is well known to be that $\frac{d}{d\alpha} I(\alpha)$ should be zero when $\alpha = 0$. This condition we shall express as $I'(0) = 0$.

Since the limits 0 and x_1 are constants

$$I'(\alpha) = \int_0^{x_1} \frac{d}{d\alpha} \sqrt{1 + [y' + \alpha\eta'(x)]^2} dx$$

$$= \int_0^{x_1} \frac{y' + \alpha\eta'(x)}{\sqrt{1 + [y' + \alpha\eta'(x)]^2}} \eta'(x) dx,$$

and

$$I'(0) = \int_0^{x_1} \frac{y'}{\sqrt{1 + y'^2}} \eta'(x) dx.$$

Integrating by parts

$$I'(0) = \left[\frac{y'}{\sqrt{1+y'^2}}\eta(x)\right]_0^{x_1} - \int_0^{x_1}\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}}\eta(x)dx$$

$$= -\int_0^{x_1}\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}}\eta(x)dx,$$

since $\eta(x)$ vanishes when $x = 0$ and when $x = x_1$.

A necessary condition that $I(0)$ shall be less than $I(\alpha)$ for some value of α and for all less values of α no matter what $F(x)$ may be, in which case the length of the curve $y = f(x)$ is less than that of any neighboring curve, is that $I'(0) = 0$ independently of $\eta(x)$, i. e. that $\int_0^{x_1}\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}}\eta(x)dx = 0$ no matter what the form of the arbitrary function $\eta(x)$.

This condition will be satisfied if and only if $\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}} = 0$; and we thus are led to a differential equation of the second order between y and x.

Its solution will express y as a function of x involving two arbitrary constants.

$$\frac{y'}{\sqrt{1+y'^2}} = C,$$

whence $y' = K,$ a constant;

$$y = Kx + L.$$

The required curve is to pass through $(0, 0)$ and (x_1, y_1) and thus we are able to determine K and L.

$$L = 0,\ K = \frac{y_1}{x_1}.$$

Hence $y = \frac{y_1}{x_1}x$; and our curve is a straight line through the given points.

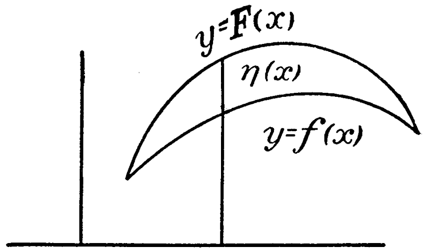
If certain other conditions, depending on the fact that when $I(\alpha)$ is a minimum $I''(\alpha)$ must be positive, are satisfied, $y = \frac{y_1}{x_1}x$ must be the required shortest line.

As our necessary conditions gave us but a single solution it is clear that if there is any shortest line it must be our line $y = \frac{y_1}{x_1}x$.

We may note in passing that in simplifying $I'(\alpha)$ by integration by parts we tacitly assumed that $f(x)$ and $F(x)$ were continuous and had continuous first derivatives over the range of integration.

4. We can deal with the general problem formulated in Art. 2 precisely as we have dealt with the *shortest line* problem.

Let it be required to determine the form of the function f so that $y = f(x)$ shall make $\int_{x_0}^{x_1} \phi\,(x, y, y')\,dx$ a maximum or a minimum; given that $y = y_0$ when $x = x_0$ and $y = y_1$ when $x = x_1$.



As in the last article let $\eta(x) = F(x) - f(x)$, and consider the family of curves

$$y = f(x) + a\eta(x).$$

Let
$$I(a) = \int_{x_0}^{x_1} \phi(x, y + a\eta(x), y' + a\eta'(x))dx.$$

$$I'(a) = \int_{x_0}^{x_1} \frac{d}{da}\phi(x, y + a\eta(x), y' + a\eta'(x))dx.$$

Let
$$u = y + a\eta(x), \quad v = y' + a\eta'(x).$$

$$\frac{du}{da} = \eta(x), \quad \frac{dv}{da} = \eta'(x).$$

$$I'(a) = \int_{x_0}^{x_1} \frac{d}{da}\phi(x, u, v)\,dx$$

$$= \int_{x_0}^{x_1} \left[\frac{\partial\phi}{\partial u}\eta(x) + \frac{\partial\phi}{\partial v}\eta'(x)\right]dx.$$

$$I'(0) = \int_{x_0}^{x_1} \left[\frac{\partial \phi}{\partial y} \eta(x) + \frac{\partial \phi}{\partial y'} \eta'(x) \right] dx \tag{1}$$

and $I'(0)$ must be made equal to zero.

Integrating by parts,

$$I'(0) = \int_{x_0}^{x_1} \frac{\partial \phi}{\partial y} \eta(x) dx + \left[\frac{\partial \phi}{\partial y'} \eta(x) \right]_{x_0}^{x_1} - \int_{x_0}^{x_1} \frac{d}{dx} \frac{\partial \phi}{\partial y'} \eta(x) dx,$$

$$I'(0) = \int_{x_0}^{x_1} \left[\frac{\partial \phi}{\partial y} - \frac{d}{dx} \frac{\partial \phi}{\partial y'} \right] \eta(x) dx$$

since $\eta(x)$ vanishes when $x = x_0$ and when $x = x_1$.

$I'(0)$ must be zero independently of the form of $\eta(x)$ therefore

$$\frac{\partial \phi}{\partial y} - \frac{d}{dx} \frac{\partial \phi}{\partial y'} = 0, \tag{2}$$

and as before we are led to a differential equation of the second order which if solved gives y as a function of x involving two arbitrary constants which must be determined from the facts that $y = y_0$ when $x = x_0$ and $y = y_1$ when $x = x_1$. This differential equation is known as *Lagrange's Equation* and it is a necessary condition that I should be a maximum or a minimum.

Any particular solution of Lagrange's Equation is called an *extremal,* and if the given problem has a solution it is that extremal which passes through the given end-points.

If ϕ is a function of y' only Lagrange's Equation becomes

$$\frac{\partial \phi}{\partial y'} = \psi(y') = C.$$

Whence $$y' = k,$$

a constant, and the extremals are straight lines; and therefore the required solution is the straight line through the given end-points as in Art. 3.

The problems of Art. 1 (*b*) and (*c*) can be solved by substituting in Lagrange's Equation the appropriate value of ϕ and then solving the resulting equation.

For *the curve of quickest descent*

$$\phi = \frac{1}{\sqrt{x}} \sqrt{1 + y'^2} \qquad v. \text{ Art. 1 } (b)\ (2).$$

$$\frac{\partial \phi}{\partial y} = 0,$$

$$\frac{\partial \phi}{\partial y'} = \frac{1}{\sqrt{x}} \frac{y'}{\sqrt{1 + y'^2}}.$$

Lagrange's Equation becomes

$$\frac{d}{dx} \frac{1}{\sqrt{x}} \frac{y'}{\sqrt{1 + y'^2}} = 0. \tag{3}$$

For *the minimum surface*

$$\phi = y \sqrt{1 + y'^2} \qquad v. \text{ Art. 1 } (c)\ (3).$$

$$\frac{\partial \phi}{\partial y} = \sqrt{1 + y'^2},$$

$$\frac{\partial \phi}{\partial y'} = \frac{yy'}{\sqrt{1 + y'^2}}.$$

Lagrange's Equation becomes

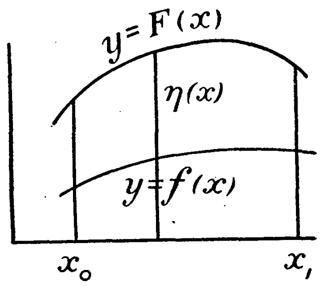
$$\sqrt{1 + y'^2} - \frac{d}{dx} \frac{yy'}{\sqrt{1 + y'^2}} = 0. \tag{4}$$

We shall reserve the solving of (3) and (4) for a later article.

CHAPTER II

VARIATIONS. NOTATION AND NOMENCLATURE. ILLUSTRATIVE PROBLEMS

5. If $y = f(x)$ and $y = F(x)$ are the equations of two curves and $\eta(x) = F(x) - f(x)$, $\eta(x)$ can be regarded as the increment produced in y by changing $f(x)$ into $F(x)$ without changing x. We shall call it the *variation* of y and represent it by δy. It is a function of x and usually a wholly arbitrary function of x.



$\eta'(x) = \dfrac{d}{dx}\,\delta y$ and is easily shown to be the increment produced in y' by changing $f(x)$ into $F(x)$, x being held fast, since when y becomes $y + \eta(x)$, y' becomes $y' + \eta'(x)$ and is changed by $\eta'(x)$. We shall call this increment the *variation* of y' and represent it by $\delta y'$. Like δy it is a function of x. As we have seen

$$\delta y' = \frac{d}{dx}\,\delta y. \tag{1}$$

If y is increased by δy,

$$\frac{d}{dy}\,\phi(y)\,\delta y$$

is an approximation to the increment produced in $\phi(y)$, the closeness of the approximation depending on the magnitude of δy. If δy is infinitesimal this approximation differs from the actual increment by an infinitesimal of higher order than δy.

We shall call this approximate increment the *variation* of $\phi(y)$ and represent it by $\delta\phi(y)$; so that

$$\delta\phi(y) = \frac{d}{dy}\phi(y)\delta y. \tag{2}$$

In like manner

$$\frac{\partial\phi(y, y')}{\partial y}\delta y + \frac{\partial\phi(y, y')}{\partial y'}\delta y'$$

is ~~in like manner~~ an approximation to the increment produced in $\phi(y, y')$ by giving y the increment δy and y' the increment $\delta y'$; the closeness of the approximation depending on the magnitudes of δy and $\delta y'$. We shall call it the *variation* of $\phi(y, y')$ and represent it by $\delta\phi(y, y')$, so that

$$\delta\phi(y, y') = \frac{\partial\phi(y, y')}{\partial y}\delta y + \frac{\partial\phi(y, y')}{\partial y'}\delta y'. \tag{3}$$

As all our variations are supposed to be caused by changing the form of the function f in $y = f(x)$ without changing x,

$$\delta\phi(x, y, y') = \frac{\partial\phi(x, y, y')}{\partial y}\delta y + \frac{\partial\phi(x, y, y')}{\partial y'}\delta y'. \tag{4}$$

If in (2) and (3) we replace the symbol δ by the symbol d we obtain the familiar formulas for $d\phi$. Hence variations are calculated by the use of the familiar formulas and processes of the Differential Calculus. Indeed if y is a function, ϕ of x, $d\phi$ and $\delta\phi$ are approximate increments of precisely the same type but differently caused, $d\phi$, by changing x without changing the form of the function and $\delta\phi$, by changing the form of the function without changing x.

With our new notation our necessary condition that I, the $\int_{x_0}^{x_1}\phi(x, y, y')\,dx$, shall be a maximum or a minimum can be written

$$\int_{x_0}^{x_1}\delta\phi(x, y, y')dx = 0, \tag{5}$$

v. Art. 4 (1).

If we define the variation of a definite integral as the integral of the variation of the integrand, so that

$$\delta\int_{x_0}^{x_1}\phi.dx = \int_{x_0}^{x_1}\delta\phi.dx, \text{ our condition (5)}$$

may be abridged into

$$\delta I = 0. \tag{6}$$

6. To illustrate the ordinary technique of the Calculus of Variations we shall now work out at length the problem of Art. 1 (*b*); *the brachistochrone* or *curve of quickest descent.*

We wish to make I a minimum where (*v.* Art. 1 (*b*))

$$I = \int_0^{x_1}\frac{\sqrt{1+y'^2}}{\sqrt{x}}\,dx.$$

We must make $\delta I = 0$.

$$\delta I = \int_0^{x_1}\delta\left(\frac{1}{\sqrt{x}}\sqrt{1+y'^2}\right)dx$$

$$= \int_0^{x_1}\frac{1}{\sqrt{x}}\frac{y'\delta y'}{\sqrt{1+y'^2}}\,dx$$

$$= \int_0^{x_1}\frac{1}{\sqrt{x}}\frac{y'}{\sqrt{1+y'^2}}\frac{d}{dx}\,\delta y.dx.$$

Integrating by parts,

$$\delta I = \left[\frac{1}{\sqrt{x}}\frac{y'}{\sqrt{1+y'^2}}\delta y\right]_0^{x_1} - \int_0^{x_1}\frac{d}{dx}\left[\frac{1}{\sqrt{x}}\frac{y'}{\sqrt{1+y'^2}}\right]\delta y dx.$$

Since our end points are fixed $\delta y = 0$ when $x = 0$ and when $x = x_1$. Therefore

$$\delta I = -\int_0^{x_1}\frac{d}{dx}\left[\frac{1}{\sqrt{x}}\frac{y'}{\sqrt{1+y'^2}}\right]\delta y dx.$$

As δI must be equal to zero no matter what the value of our arbitrary function δy we must have

$$\frac{d}{dx}\left[\frac{1}{\sqrt{x}}\frac{y'}{\sqrt{1+y'^2}}\right] = 0, \tag{1}$$

and (1) is our differential equation for y, i. e. the differential equation of our required curve. *v.* Art. 4 (3).

(1) is easily integrated. We have

$$\frac{1}{\sqrt{x}}\frac{y'}{\sqrt{1+y'^2}} = C;$$

whence
$$y' = \frac{C\sqrt{x}}{\sqrt{1-C^2x}} = \frac{x}{\sqrt{\frac{1}{C^2}x - x^2}}.$$

If we let
$$2a = \frac{1}{C^2},$$

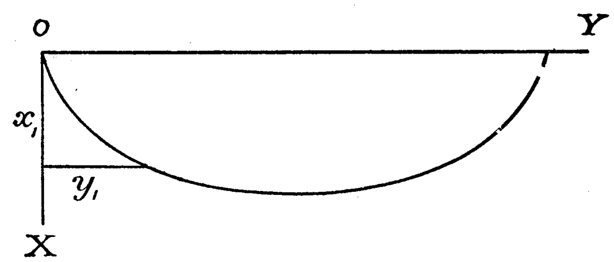
$$y = \int \frac{xdx}{\sqrt{2ax - x^2}} = -\sqrt{2ax - x^2} + a \operatorname{vers}^{-1}\frac{x}{a} + C.$$

Since $y = 0$ when $x = 0$, $C = 0$;

and
$$y = -\sqrt{2ax - x^2} + a \operatorname{vers}^{-1}\frac{x}{a}, \tag{2}$$

where a is to be determined so that the curve shall pass through the point (x_1, y_1), is our required curve of quickest descent.

(2) is the equation of a cycloid having its base in the axis of Y, its cusp at the origin, and lying on the positive side of the axis of Y.



Our solution, then, is an inverted cycloid with a horizontal base and it has a cusp at the higher of the given points.

Our treatment has assumed that in the required curve y is a single-valued function of x. Our solution is, therefore, apparently valid only if a cycloid suitably situated can be drawn so that (x_1, y_1) shall lie between the origin and the vertex of the curve. As a matter of fact this limitation is unnecessary as may be shown by interchanging the axes of X and Y and working the problem anew.

If this is done $I = \int_0^{x_1} \frac{\sqrt{1 + y'^2}}{\sqrt{y}} dx$ and it is not difficult to make $\delta I = 0$. The resulting differential equation

$$2yy'' + 1 + y'^2 = 0$$

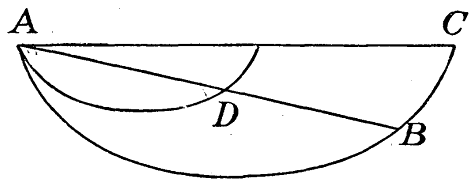
can be solved by the method employed in the following section and we find as the equation of our required curve

$$x = -\sqrt{2ay - y^2} + a \operatorname{vers}^{-1}\frac{y}{a}. \qquad (3)$$

Throughout the whole arch of the cycloid (3) y is a single valued function of x, and (x_1, y_1) is, therefore, not restricted to lying in the first half of the arch.

In any concrete problem the numerical determination of a is a matter of some difficulty. This, however, may be avoided by Newton's famous solution for the curve of quickest descent from A to B.

"Through A on the horizontal line AC as a base draw any cycloid cutting the straight line AB in D. Through A on the line AC draw



a second cycloid whose base and altitude are to the base and altitude of the first as AB is to AD; it will pass through B and will be the curve required."

7. As another example we shall take the problem of Art. 1 (*c*), *the minimum surface of revolution.* Here (*v.* Art. 1 (*c*), (3)),

$$I = \int_0^{x_1} y\sqrt{1 + y'^2}dx.$$

$$\delta I = \int_0^{x_1} \delta[y\sqrt{1 + y'^2}]dx$$

$$= \int_0^{x_1} \left[\sqrt{1 + y'^2}\delta y + \frac{yy'}{\sqrt{1 + y'^2}} \delta y' \right] dx$$

$$= \int_0^{x_1} \left[\sqrt{1 + y'^2}\delta y + \frac{yy'}{\sqrt{1 + y'^2}} \frac{d}{dx} \delta y \right] dx.$$

Integrating by parts,

$$\delta I = \left[\frac{yy'}{\sqrt{1 + y'^2}} \delta y \right]_0^{x_1} + \int_0^{x_1} \left[\sqrt{1 + y'^2} - \frac{d}{dx} \frac{yy'}{\sqrt{1 + y'^2}} \right] \delta y dx$$

$$= \int_0^{x_1} \left[\sqrt{1 + y'^2} - \frac{d}{dx} \frac{yy'}{\sqrt{1 + y'^2}} \right] \delta y dx.$$

To make $\delta I = 0$ we must make

$$\sqrt{1 + y'^2} - \frac{d}{dx} \frac{yy'}{\sqrt{1 + y'^2}} = 0, \tag{1}$$

and this is the differential equation of our required curve. *v.* Art. 4 (4).

Performing the differentiation indicated in (1) and reducing, the equation becomes

$$yy'' - 1 - y'^2 = 0. \tag{2}$$

Equation (2) can be integrated by a regulation device as follows:

Let $z = y'$.

$$y'' = \frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx} = y' \frac{dz}{dy} = z \frac{dz}{dy}.$$

(2) becomes

$$yz \frac{dz}{dy} = 1 + z^2.$$

$$\frac{z}{1 + z^2} dz = \frac{dy}{y}.$$

$$\tfrac{1}{2} \log (1 + z^2) - \log y = C.$$

$$\frac{\sqrt{1 + z^2}}{y} = K = \frac{1}{a},$$

where a is an arbitrary constant.

$$z = y' = \sqrt{\frac{y^2}{a^2} - 1},$$

$$\frac{dy}{\sqrt{y^2 - a^2}} = \frac{dx}{a},$$

$$\cosh^{-1}\frac{y}{a} = \frac{x}{a} + b,$$

$$y = a \cosh\left(\frac{x}{a} + b\right). \qquad (3)$$

(3) is the equation of a *catenary* having the axis of x as its directrix. The constants a and b must be determined so that (3) shall pass through the two given points $(0, y_0)$ and (x_1, y_1). To calculate their numerical values in a concrete case is not easy but the curve required may be obtained mechanically by the following process.

Insert smooth pegs, at the two given points, in the vertical plane through these points. Throw over the pegs a uniform flexible string so long that one end will hang below the given axis; and then pull down on the other end until the free end is drawn up to the axis. Then the portion of the string between the pegs will be the required catenary.

For it is well known that if one end of a uniform flexible string is fastened and the other end is thrown over a smooth horizontal peg, the curved portion of the string when in equilibrium will be a catenary whose directrix is a horizontal line through the hanging end of the string.

8. Since any function of a single variable can be geometrically represented by a graph by treating the independent variable as an abscissa and the corresponding value of the function as an ordinate, our theory covers the case where we are to make the definite integral of a given function of the independent variable, and of the dependent variable and its derivative a maximum or a minimum, no matter what the ordinary interpretation of the variables in question.

For example let us work the *shortest line* problem (Art. 1 (a)) using polar coördinates.

Let $r = f(\phi)$ be the required shortest plane curve connecting the points (r_0, ϕ_0) and (r_1, ϕ_1).

Since
$$ds = \sqrt{dr^2 + r^2 d\phi^2}$$
$$I = \int_{\phi_0}^{\phi_1} \sqrt{r^2 + r'^2}\, d\phi,$$
and we must make
$$\delta I = 0.$$
$$\delta I = \int_{\phi_0}^{\phi_1} \delta\sqrt{r^2 + r'^2}\, d\phi$$
$$= \int_{\phi_0}^{\phi_1} \frac{r\delta r + r'\delta r'}{\sqrt{r^2 + r'^2}}\, d\phi$$
$$= \int_{\phi_0}^{\phi_1} \frac{r\delta r}{\sqrt{r^2 + r'^2}}\, d\phi + \int_{\phi_0}^{\phi_1} \frac{r'\dfrac{d}{d\phi}\delta r}{\sqrt{r^2 + r'^2}}\, d\phi.$$

Integrating by parts,
$$\int_{\phi_0}^{\phi_1} \frac{r'}{\sqrt{r^2 + r'^2}} \frac{d}{d\phi}\delta r . d\phi = \left[\frac{r'}{\sqrt{r^2 + r'^2}}\delta r\right]_{\phi_0}^{\phi_1} - \int_{\phi_0}^{\phi_1} \frac{d}{d\phi} \frac{r'}{\sqrt{r^2 + r'^2}}\delta r d\phi$$

δr vanishes when $\phi = \phi_0$ and when $\phi = \phi_1$.

Therefore
$$\delta I = \int_{\phi_0}^{\phi_1} \left[\frac{r}{\sqrt{r^2 + r'^2}} - \frac{d}{d\phi} \frac{r'}{\sqrt{r^2 + r'^2}}\right] \delta r d\phi.$$

To make $\delta I = 0$ we must make
$$\frac{r}{\sqrt{r^2 + r'^2}} - \frac{d}{d\phi} \frac{r'}{\sqrt{r^2 + r'^2}} = 0; \qquad (1)$$
and this is the differential equation of our required line.

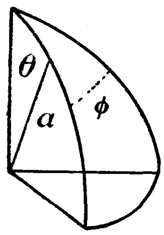
If we perform the indicated differentiation and reduce, (1) assumes the comparatively simple form
$$\frac{rr'' - 2r'^2 - r^2}{(r^2 + r'^2)^{\frac{3}{2}}} = 0. \qquad (2)$$

We can clear of fractions and solve without difficulty, but if we happen to remember the Calculus formula for the curvature of a curve in polar coördinates

$$\kappa = -\frac{r^2 - r\dfrac{d^2r}{d\phi^2} + 2\left(\dfrac{dr}{d\phi}\right)^2}{\left[r^2 + \left(\dfrac{dr}{d\phi}\right)^2\right]^{\frac{3}{2}}},$$

we see that our differential equation (2) represents a curve of zero curvature, i. e. a straight line.

9. As another example let us find a geodesic line joining two given points on the surface of a sphere.



The element of arc of a curve on the spherical surface is

$$ds = \sqrt{a^2 d\theta^2 + a^2 \sin^2\theta d\phi^2}$$

$$= a\sqrt{1 + \sin^2\theta\phi'^2}d\theta;$$

and we may take $I = \int_{\theta_0}^{\theta_1} \sqrt{1 + \sin^2\theta\phi'^2}\, d\theta.$

$$\delta I = \int_{\theta_0}^{\theta_1} \frac{\sin^2\theta\phi'\delta\phi'}{\sqrt{1 + \sin^2\theta\phi'^2}} d\theta$$

$$= \int_{\theta_0}^{\theta_1} \frac{\sin^2\theta\phi'}{\sqrt{1 + \sin^2\theta\phi'^2}} \frac{d}{d\theta}\delta\phi.d\theta.$$

Integrating by parts and remembering that $\delta\phi$ vanishes at both limits θ_0 and θ_1, we have

$$\delta I = -\int_{\theta_0}^{\theta_1} \frac{d}{d\theta} \frac{\sin^2\theta\phi'}{\sqrt{1 + \sin^2\theta\phi'^2}} \delta\phi d\theta.$$

Since $\delta I = 0$ whatever the form of $\delta\phi$

$$\frac{d}{d\theta}\frac{\sin^2\theta\phi'}{\sqrt{1+\sin^2\theta\phi'^2}} = 0. \qquad (1)$$

Integrating,

$$\sin^2\theta\phi' = C\sqrt{1+\sin^2\theta\phi'^2},$$

$$\phi' = \frac{C\csc\theta}{\sqrt{\sin^2\theta - C^2}} = \frac{C\csc^2\theta}{\sqrt{1-C^2\csc^2\theta}} = \frac{C\csc^2\theta}{\sqrt{1-C^2-C^2\operatorname{ctn}^2\theta}}.$$

If we let $z = \operatorname{ctn}\theta$ and $K^2 = \dfrac{C^2}{1-C^2}$

$$\phi = -\int\frac{dz}{\sqrt{\frac{1}{K^2}-z^2}} = -\sin^{-1}Kz + \alpha = -\sin^{-1}(K\operatorname{ctn}\theta) + \alpha$$

where α is an arbitrary constant.

$$\sin(\alpha - \phi) = K\operatorname{ctn}\theta,$$

$$\sin\alpha\sin\theta\cos\phi - \cos\alpha\sin\theta\sin\phi = K\cos\theta. \qquad (2)$$

If we multiply by a and change to rectangular coördinates we get

$$y\sin\alpha - z\cos\alpha = Kx,$$

a plane through the centre of the sphere. The constants must be determined so that this plane shall pass through the given points and then the great circle in which it cuts the sphere is the required geodesic.

Examples

(1) Work the *brachistochrone* problem taking the axis of Y vertically downward. *v.* Art. 6, page 12.

(2) Integrate $rr'' - 2r'^2 - r^2 = 0$, the polar differential equation of the *shortest line*. *v.* Art. 8 (2).

Ans. $r\cos(\phi - \alpha) = p$; a straight line.

(3) Find the shortest line that can be drawn on the surface of a cylinder of revolution and joining two given points on the surface; a *geodesic*. *Suggestion.* Use cylindrical coördinates a, ϕ, z, taking z as the independent variable.

Then $$I = \int_{z_0}^{z_1} \sqrt{1 + a^2\phi'^2}dz.$$

Ans. The geodesic is the line of intersection of the cylinder with the helicoid $\phi = bz + c$; and is a helix.

(4) Find a *geodesic* on a cone of revolution.

Suggestion. Use spherical coördinates r, α, ϕ, taking r as the independent variable.

Then $$I = \int_{r_0}^{r_1} \sqrt{1 + r^2 \sin^2 \alpha\phi'^2}dr.$$

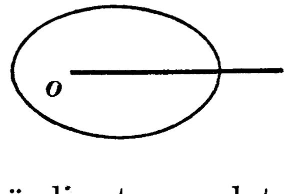
Ans. The geodesic is the intersection of the cone with the surface $r \cos [(\phi - \beta) \sin \alpha] = c$.

(5) A mountain is in the shape of a hemisphere and the velocity of a man walking upon it varies as his height above its base; show that his path of least time between two given points upon its surface lies in the vertical plane through the points.

10. *Isoperimetrical Problems.*

There is an important class of problems in which it is required to make I a maximum or a minimum while keeping constant the integral of a second given function of x, y, and y' taken between the same limits.

(*a*) For example take the problem to find the closed plane curve of given perimeter and maximum area.



We shall use polar coördinates and take the origin within the required curve.

If $r = f(\phi)$ is our curve, its area, A, and its perimeter, S, are equal

respectively to $\int_0^{2\pi} \frac{1}{2} r^2 d\phi$ and $\int_0^{2\pi} \sqrt{r^2 + r'^2}d\phi$.

We are to make A a maximum while S is kept equal to a given constant l.

If λ is a constant multiplier at present undetermined $A + \lambda S$ will be a maximum if our problem is correctly solved.

Let, then,
$$I = \int_0^{2\pi} [\tfrac{1}{2}r^2 + \lambda\sqrt{r^2 + r'^2}]d\phi,$$

and proceed as usual.

$$\delta I = \int_0^{2\pi} \left[r\delta r + \lambda \frac{r\delta r + r'\delta r'}{\sqrt{r^2 + r'^2}} \right] d\phi$$

$$= \int_0^{2\pi} \left[\left(1 + \frac{\lambda}{\sqrt{r^2 + r'^2}}\right) r\delta r + \frac{\lambda r'}{\sqrt{r^2 + r'^2}} \frac{d}{d\phi} \delta r \right] d\phi$$

$$= \int_0^{2\pi} \left[\left(1 + \frac{\lambda}{\sqrt{r^2 + r'^2}}\right) r - \frac{d}{d\phi} \frac{\lambda r'}{\sqrt{r^2 + r'^2}} \right] \delta r \, d\phi.$$

Since $\delta I = 0$,
$$r\left(\frac{1}{\lambda} + \frac{1}{\sqrt{r^2 + r'^2}}\right) - \frac{d}{d\phi} \frac{r'}{\sqrt{r^2 + r'^2}} = 0.$$

Performing the indicated differentiation and reducing we get
$$\frac{rr'' - 2r'^2 - r^2}{(r^2 + r'^2)^{\frac{3}{2}}} = \frac{1}{\lambda};$$

as the differential equation of our required curve.

The first member is the curvature of the curve, and as the curvature is $\frac{1}{\lambda}$, a constant, the required curve is a circle. *v.* Art. 8, page 17.

As λ is the radius of the circle its given perimeter l is $2\pi\lambda$ and our λ proves to be $\frac{l}{2\pi}$.

(*b*) The ends of a uniform string are fastened at given points; find the equation of the curve in which it must hang that its centre of gravity may be as low as possible.

Let (x_0, y_0) and (x_1, y_1) be the end points, and let l be the length of the string.

$$\int_{x_0}^{x_1} \sqrt{1 + y'^2}\,dx = l.$$

If $\bar{y}$ is the ordinate of the centre of gravity

$$\bar{y} = \frac{1}{l}\int_{x_0}^{x_1} y\sqrt{1 + y'^2}\,dx.$$

We must make

$$\bar{y} + \lambda l, \text{ i. e. } \frac{1}{l}\int_{x_0}^{x_1} (y + \lambda l)\sqrt{1 + y'^2}\,dx,$$

a minimum.

Let $\kappa = \lambda l$ and let $I = \int_{x_0}^{x_1} (y + \kappa)\sqrt{1 + y'^2}\,dx.$

Turning to Art. 7 we see that we have only to replace y by $y + \kappa$ in the solution there given to get the solution of the present problem, and that our required curve is $y + \kappa = a \cosh\left(\frac{x}{a} + b\right)$, a catenary. Unlike the catenary in Art. 7 it does not generally have the axis of X for its directrix.

In a numerical problem a and b would have to be determined so that the catenary would pass through (x_0, y_0) and (x_1, y_1) and κ would have to be found by the condition that

$$\int_{x_0}^{x_1} \sqrt{1 + y'^2}\,dx = l.$$

Problems like (*a*) and (*b*) are known as *isoperimetrical* problems and can all be handled by the device we have illustrated.

Examples

(1) Find the form of a curve if the area bounded by an arc of given length, the ordinates of its given end points, and the axis of X is to be a maximum.

Ans. A circle.

(2) Find the form of a curve if the sectorial area bounded by an arc of given length and the radii vectores of its end points is to be a maximum.

Ans. A circle.

CHAPTER III

PROBLEMS INVOLVING SEVERAL DEPENDENT VARIABLES

11. It is easy to extend our theory to problems where instead of a single dependent variable y and its derivative there are several variables to be determined as functions of the same independent variable. That is, when $I = \int_{x_0}^{x_1} \phi(x, y, z, \cdots, y', z', \cdots,)dx$ and I is to be made a maximum or a minimum.

Let $$\delta\phi = \frac{\partial\phi}{\partial y}\delta y + \frac{\partial\phi}{\partial z}\delta z + \cdots + \frac{\partial\phi}{\partial y'}\delta y' + \frac{\partial\phi}{\partial z'}\delta z' + \cdots,$$

and by the method of Art. 4 it can be shown that we must make $\delta I = 0$.

Note that $$\delta y' = \frac{d}{dx}\delta y, \quad \delta z' = \frac{d}{dx}\delta z, \cdots.$$

As an example let it be required to find the shortest curve not necessarily plane joining (0, 0, 0) and (x_1, y_1, z_1).

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{1 + y'^2 + z'^2}dx.$$

$$I = \int_0^{x_1} \sqrt{1 + y'^2 + z'^2}dx.$$

$$\delta I = \int_0^{x_1} \frac{y'\delta y' + z'\delta z'}{\sqrt{1 + y'^2 + z'^2}}dx.$$

By *integration by parts*

$$\delta I = -\int_0^{x_1}\left[\frac{d}{dx}\frac{y'}{\sqrt{1 + y'^2 + z'^2}}\delta y + \frac{d}{dx}\frac{z'}{\sqrt{1 + y'^2 + z'^2}}\delta z\right]dx.$$

To make $\delta I = 0$ we must make

$$\frac{d}{dx}\frac{y'}{\sqrt{1 + y'^2 + z'^2}}\delta y + \frac{d}{dx}\frac{z'}{\sqrt{1 + y'^2 + z'^2}}\delta z$$

vanish when δy and δz are independent arbitrary functions of x. Hence

$$\frac{d}{dx}\frac{y'}{\sqrt{1 + y'^2 + z'^2}} = 0$$

and

$$\frac{d}{dx}\frac{z'}{\sqrt{1 + y'^2 + z'^2}} = 0.$$

These are a pair of simultaneous differential equations of the second order connecting y and z with x; and can be solved without difficulty. We get $\frac{x}{A} = \frac{y}{B} = \frac{z}{C}$, a straight line.

12. *Hamilton's Principle,* called by Professor E. B. Wilson "the most fundamental and important single theorem in mathematical physics," and closely allied to the so-called principle of *Least Action* is that *in the actual motion of any conservative system the time-integral of the sum of the Energy and the force function is a minimum.*

Formulating, $\delta\int_{t_0}^{t_1}[T + U]\,dt = 0$, if U is the *force function* and T the *kinetic energy.*

We give a proof of Hamilton's Principle for a single moving particle. It can be extended without difficulty to a moving system.

Let a particle of mass m moving under forces X, Y, Z, actually move from its initial to its final position in the time t_1. Suppose that by the introduction of additional forces it had been made to move from its initial to its final position in the same time t_1, but by a slightly different path. Let (x, y, z) be its position at any time t in the actual motion and $(x + \delta x, y + \delta y, z + \delta z)$ its position in the hypothetical motion after the lapse of the same time. If t is our independent variable we have from Mechanics $mx'' = X$, $my'' = Y$, $mz'' = Z$; and if the forces are conservative and U is the *force function* $X\delta x + Y\delta y + Z\delta z = \delta U.$

$$X\delta x = mx''\delta x = m\left[\frac{d}{dt}(x'\delta x) - x'\delta x'\right] = m\frac{d}{dt}(x'\delta x) - \delta\frac{mx'^2}{2}.$$

$$\delta U = m\frac{d}{dt}[x'\delta x + y'\delta y + z'\delta z] - \delta T,$$

where $T = \frac{m}{2}[x'^2 + y'^2 + z'^2]$ and is the *Kinetic Energy.*

$$\int_0^{t_1} \delta(T + U)dt = m[x'\delta x + y'\delta y + z'\delta z]_{t=0}^{t=t_1} = 0,$$

since the initial and final positions of the particle are the same in the actual motion and the hypothetical motion.

As an illustration of the use of Hamilton's Principle we shall obtain the differential equations for planetary motion by its aid. If we use polar coördinates and take the sun as origin T, the *energy* of the planet, is $\frac{m}{2}(r'^2 + r^2\phi'^2)$. The force, F, exerted by the sun is an attractive force directed toward the sun, proportional to the product of the masses and inversely proportional to the square of the distance.

$$F = -\frac{\mu Mm}{r^2}.$$

The work done in any motion of the planet is $\int F dr$

and
$$\int F dr = -\mu Mm \int \frac{dr}{r^2} = \frac{\mu Mm}{r} + C.$$

The simplest *force function, U,* is equal to $\frac{\mu Mm}{r}$.

By Hamilton's Principle

$$\int_0^{t_1} \delta\left[\frac{m}{2}(r'^2 + r^2\phi'^2) + \frac{\mu Mm}{r}\right]dt = 0.$$

$$\int_0^{t_1} \left[r'\delta r' + r^2\phi'\delta\phi' + r\phi'^2\delta r - \frac{\mu M}{r^2}\delta r\right]dt = 0,$$

or
$$\int_0^{t_1} \left[r'\frac{d}{dt}\delta r + r^2\phi'\frac{d}{dt}\delta\phi + \left(r\phi'^2 - \frac{\mu M}{r^2}\right)\delta r\right]dt = 0.$$

Integrating by parts,

$$\int_0^{t_1}\left[-\frac{d}{dt}r'\delta r - \frac{d}{dt}(r^2\phi')\delta\phi + \left(r\phi'^2 - \frac{\mu M}{r^2}\right)\delta r\right]dt = 0.$$

Whence $$r'' - r\phi'^2 = -\frac{\mu M}{r^2}$$

and $$r^2\phi' = k.$$

Example

A heavy cube containing a smooth spherical cavity rests on a smooth horizontal plane, and a heavy particle lies at the bottom of the cavity. The cube is given a horizontal velocity. Find the equations for the subsequent motion of the system.

Suggestion. Let M be the mass of the cube and m the mass of the particle, and use as coördinates x, the distance the cube has moved, and θ, the angle a radius of the cavity drawn through the particle makes with the vertical. The horizontal velocity of the particle is easily seen to be $x' - a\cos\theta\theta'$, its vertical velocity is $a\sin\theta\theta'$.

$$T = \tfrac{1}{2}(M + m)x'^2 + ma^2\theta'^2 - 2a\cos\theta x'\theta'.$$
$$U = mga\cos\theta.$$

The equations required are

$$\frac{d}{dt}[(M + m)x' - ma\cos\theta\theta'] = 0$$

$$\frac{d}{dt}(a\theta' - \cos\theta x') - \sin\theta\,(x'\theta' - g) = 0,$$

or $$(M + m)x' - ma\cos\theta\theta' = C$$

$$a\theta'' - \cos\theta x'' + g\sin\theta = 0.$$

13. If in the proof of Hamilton's Principle in the last article we do not replace $X\delta x + Y\delta y + Z\delta z$ by its equal δU we get

$$\int_0^{t_1}[\delta T + X\delta x + Y\delta y + Z\delta z]dt = 0,$$

and this equation is valid even if the forces are not conservative. It is to be noted that $X\delta x + Y\delta y + Z\delta z$ is the work that would be done by the actual forces if the particle were displaced from its actual position at any time to its corresponding position in the hypothetical path.

The differential equations in polar coördinates for the motion of a particle in a plane are easily obtained by the aid of our new equation. Let the component of the force along the radius vector be R and perpendicular to the radius vector be Φ. If (r, ϕ) is the actual position of the particle at the time t the corresponding position in the hypothetical path is $(r + \delta r, \quad \phi + \delta\phi)$.

$$X\delta x + Y\delta y + Z\delta z$$

becomes $\quad R\delta r + \Phi r\delta\phi. \qquad T = \frac{m}{2}[r'^2 + r^2\phi'^2].$

$$\int_0^{t_1}\left[\delta\frac{m}{2}(r'^2 + r^2\phi'^2) + R\delta r + \Phi r\delta\phi\right]dt = 0,$$

$$\int_0^{t_1}[m(r'\delta r' + r^2\phi'\delta\phi' + r\phi'^2\delta r) + R\delta r + \Phi r\delta\phi]dt = 0,$$

$$\int_0^{t_1}\left[(-mr'' + r\phi'^2 + R)\delta r - \left(\frac{d}{dt}(r^2\phi') - \Phi r\right)\delta\phi\right]dt = 0.$$

$$m(r'' - r\phi'^2) = R,$$

$$\frac{1}{r}\frac{d}{dt}(r^2\phi') = \Phi.$$

Examples

(1) Find the differential equations for the motion of a spherical pendulum.

$$Ans. \quad \theta'' - \sin\theta\cos\theta\phi'^2 + \frac{g}{a}\sin\theta = 0$$

$$\sin^2\theta\phi' = C.$$

(2) Obtain the differential equations for the motion of a particle in space, using cylindrical coördinates.

$$Ans. \quad m(r'' - r\phi'^2) = R, \quad \frac{m}{r}\frac{d}{dt}(r^2\phi') = \Phi, \quad mz'' = Z.$$

(3) Obtain the differential equations for the motion of a particle in space, using spherical coördinates.

Ans.

$$m[r'' - r\,(\theta'^2 + \sin^2\theta\phi'^2)] = R$$

$$\frac{m}{r}\left[\frac{d}{dt}(r^2\theta') - r^2 \sin\theta \cos\theta\phi'^2\right] = \Theta$$

$$\frac{m}{r \sin\theta}\frac{d}{dt}(r^2 \sin^2\theta\phi') = \Phi.$$

CHAPTER IV

MULTIPLE INTEGRALS

14. If I is a multiple integral no serious additional difficulty is presented. We have still to make $\delta I = 0$.

If
$$I = \iint \phi(x, y, z, p, q)dxdy,$$

where
$$p = \frac{\partial z}{\partial x}$$

and
$$q = \frac{\partial z}{\partial y},$$

and the integration is over a given area in the XY plane, let it be required to make I a maximum or a minimum by suitably determining z as a function, f, of x and y, z being given for all values of x and y corresponding to points in the boundary of the area.

Here δz is the increment produced in z by changing the form of the function f, x and y being held fast, and is a function of x and y. If we let $\delta p = \frac{\partial}{\partial x}\delta z$, and $\delta q = \frac{\partial}{\partial y}\delta z$ then δp and δq are the increments produced in p and q.

$z = f(x, y) + \alpha\delta z$ is any one of a family of surfaces of which $z = f(x, y)$ is one.

$$I(\alpha) = \iint \phi(x, y, z + \alpha\delta z,\ p + \alpha\delta p,\ q + \alpha\delta q)dxdy.$$

$$I'(0) = \iint \left[\frac{\partial \phi}{\partial z}\delta z + \frac{\partial \phi}{\partial p}\delta p + \frac{\partial \phi}{\partial q}\delta q\right]dxdy$$

$$= \iint \delta\phi\, dxdy = \delta \iint \phi dxdy = \delta I,$$

and, as before, the necessary condition that I shall be a maximum or a minimum is $\delta I = 0$.

15. *Minimal Surface.* As an example let us find the form of the surface of least area that can be bounded by a given closed curve not necessarily plane.

$I = \iint \sqrt{1 + p^2 + q^2}\, dxdy$, the integral being taken over the area bounded by the projection of the given curve on the XY plane.

$$\delta I = \iint \frac{p\delta p + q\delta q}{\sqrt{1 + p^2 + q^2}} dxdy$$

$$= \iint \frac{p \dfrac{\partial}{\partial x}\delta z + q \dfrac{\partial}{\partial y}\delta z}{\sqrt{1 + p^2 + q^2}} dxdy$$

$$= \iint \frac{p \dfrac{\partial}{\partial x}\delta z}{\sqrt{1 + p^2 + q^2}} dxdy + \iint \frac{q \dfrac{\partial}{\partial y}\delta z}{\sqrt{1 + p^2 + q^2}} dydx.$$

Integrating by parts,

$$\int \frac{p \dfrac{\partial}{\partial x}\delta z}{\sqrt{1 + p^2 + q^2}} dx = \frac{p\delta z}{\sqrt{1 + p^2 + q^2}} - \int \frac{\partial}{\partial x} \frac{p}{\sqrt{1 + p^2 + q^2}} \delta z dx.$$

Since in our x-integration y is held fast our limits are the abscissas of the points on the projection of the given boundary which correspond to the value of y in question, and for these points $\delta z = 0$. Therefore the term outside of the sign of integration vanishes and

$$\int \frac{p \dfrac{\partial}{\partial x}\delta z}{\sqrt{1 + p^2 + q^2}} dx = -\int \frac{\partial}{\partial x} \frac{p}{\sqrt{1 + p^2 + q^2}} \delta z dx.$$

In like manner it may be shown that

$$\int \frac{q \dfrac{\partial}{\partial y}\partial z}{\sqrt{1 + p^2 + q^2}} dy = -\int \frac{\partial}{\partial y} \frac{q}{\sqrt{1 + p^2 + q^2}} \delta z dy.$$

Hence

$$\delta I = -\iint\left[\frac{\partial}{\partial x}\frac{p}{\sqrt{1+p^2+q^2}} + \frac{\partial}{\partial y}\frac{q}{\sqrt{1+p^2+q^2}}\right]\delta z dx dy$$

$$= -\iint\frac{(1+q^2)\frac{\partial p}{\partial x} - pq\left(\frac{\partial q}{\partial x}+\frac{\partial p}{\partial y}\right) + (1+p^2)\frac{\partial q}{\partial y}}{(1+p^2+q^2)^{\frac{3}{2}}}\delta z dx dy.$$

$\delta I = 0$ when and only when

$$\frac{\left[1+\left(\frac{\partial z}{\partial y}\right)^2\right]\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial x \partial y} + \left[1+\left(\frac{\partial z}{\partial x}\right)^2\right]\frac{\partial^2 z}{\partial y^2}}{\left[1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]^{\frac{3}{2}}} = 0, \qquad (1)$$

and (1) is the differential equation of the required *minimal surface.*

It is shown in works on *Differential Geometry* that the first member of (1) is the sum of the curvatures of any two mutually perpendicular normal sections of the surface, and its half is called the *mean curvature* of the surface at any point. Therefore a minimal surface is a surface of zero mean curvature.

Examples

(1) Show that the mean curvature of a helicoid $z = k \tan^{-1}\frac{y}{x}$ is zero.

(2) A cylindrical cup with a plane bottom of any shape and a given upper rim of any shape is to be capped so that the closed cup shall have a given volume and a minimum upper surface.

Show that the cap must be a surface of constant mean curvature.

16. The differential equation for small transverse oscillations of a stretched elastic string can now be obtained from Hamilton's Principle.

Let l be the length and p the tension of the string in its position of equilibrium, and let the motion be so small that higher powers than the first of the slope of the string at any time may be neglected. Let us assume also that longitudinal motions and the change in tension due to the stretching of the string during its oscillation are negligible.

If m is the mass of a unit length of the string the kinetic energy T is equal to $\int_0^l \frac{m}{2}\left(\frac{\partial y}{\partial t}\right)^2 dx$. The force function U is the negative of the potential energy, V, of the string, and V is equal to the work that would be done by the tension, p, in restoring the string to its length when in equilibrium: i. e. $V = p\,(s - l)$,

$$\text{or}\quad V = p\left[\int_0^l \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}\, dx - l\right] = p\left[\int_0^l \left(1+\frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 + \cdots\right)dx - l\right],$$

so that approximately $\quad V = \int_0^l \frac{p}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx.$

By Hamilton's Principle $\quad \delta I = 0,$

where $$I = \int_0^t (T - V)\, dt.$$

$$I = \int_0^t\int_0^l \left[\frac{m}{2}\left(\frac{\partial y}{\partial t}\right)^2 - \frac{p}{2}\left(\frac{\partial y}{\partial x}\right)^2\right] dxdt.$$

$$\delta I = \int_0^t\int_0^l \left[m\frac{\partial y}{\partial t}\,\delta\frac{\partial y}{\partial t} - p\frac{\partial y}{\partial x}\,\delta\frac{\partial y}{\partial x}\right] dxdt$$

$$= \int_0^t\int_0^l \left[m\frac{\partial y}{\partial t}\frac{\partial}{\partial t}\delta y - p\frac{\partial y}{\partial x}\frac{\partial}{\partial x}\delta y\right] dxdt$$

$$= \int_0^l\int_0^t m\frac{\partial y}{\partial t}\frac{\partial}{\partial t}\delta y\, dtdx - \int_0^t\int_0^l p\frac{\partial y}{\partial x}\frac{\partial}{\partial x}\delta y\, dxdt.$$

$$\int_0^t m\frac{\partial y}{\partial t}\frac{\partial}{\partial t}\delta y dt = -\int_0^t m\frac{\partial^2 y}{\partial t^2}\delta y dt,$$

since $\delta y = 0$ when $t = 0$ and when $t = t$. *v.* Art. 12.

$$\int_0^l p\frac{\partial y}{\partial x}\frac{\partial}{\partial x}\delta y dx = -\int_0^l p\frac{\partial^2 y}{\partial x^2}\delta y dx.$$

$$\delta I = -\int_0^t\int_0^l \left[m\frac{\partial^2 y}{\partial t^2} - p\frac{\partial^2 y}{\partial x^2}\right]\delta y dx dt = 0.$$

Hence
$$\frac{\partial^2 y}{\partial t^2} = \frac{p}{m}\frac{\partial^2 y}{\partial x^2}.$$

Example

Obtain the differential equation for small transverse oscillations of a stretched elastic membrane.

Suggestion. $V = p\iint\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dxdy - A$

where $A = \iint dxdy$ is the area of the membrane at rest.

Ans. $\frac{\partial^2 z}{\partial t^2} = \frac{p}{m}\left[\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right].$

CHAPTER V

VARIATION OF THE LIMITS. PRINCIPLE OF LEAST ACTION

17. In our fundamental problem it is required to determine y as a function of x so that $\int_{x_0}^{x_1} \phi(x, y, y')dx$ shall be a maximum or a minimum, x_0, x_1, and the corresponding values y_0 and y_1 being given, so that $\delta y = 0$ when $x = x_0$ and when $x = x_1$.

If while keeping the range of integration fixed we remove the restriction that y_0 and y_1 are given the old reasoning by which we established the necessary condition

$$\delta I = \int_{x_0}^{x_1} \left[\frac{\partial \phi}{\partial y}\delta y + \frac{\partial \phi}{\partial y'}\delta y'\right]dx = 0,$$

still holds good; but when we integrate by parts and get

$$\delta I = \int_{x_0}^{x_1} \left[\frac{\partial \phi}{\partial y} - \frac{d}{dx}\frac{\partial \phi}{\partial y'}\right]\delta y dx + \left[\frac{\partial \phi}{\partial y'}\delta y\right]_{x_0}^{x_1} = 0$$

the last term no longer disappears, but becomes

$$\left[\frac{\partial \phi}{\partial y'}\right]_{x = x_1}\delta y_1 - \left[\frac{\partial \phi}{\partial y'}\right]_{x = x_0}\delta y_0$$

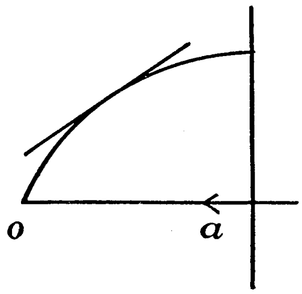
where δy_0 and δy_1 are entirely arbitrary; so that $\delta I = 0$ when and only when

$$\frac{\partial \phi}{\partial y} - \frac{d}{dx}\frac{\partial \phi}{\partial y'} = 0, \quad \left[\frac{\partial \phi}{\partial y'}\right]_{x = x_0} = 0, \quad \text{and} \quad \left[\frac{\partial \phi}{\partial y'}\right]_{x = x_1} = 0,$$

and we must determine the two arbitrary constants in the solution of the first equation by the aid of the second and third equations.

The problem of sailing to windward is an interesting illustration. The speed of a sailing vessel is a function, more or less complicated, of the angle her course makes with the direction of the wind. It is required to find how she must sail to get a given distance, a, to windward in the least possible time.

Let the starting point be taken as origin and an axis of X be drawn directly to windward. If $y = f(x)$ is the required path y' is



the tangent of the angle the course makes at any time with the wind's direction, and the velocity can be expressed in terms of y'.

$$\frac{ds}{dt} = F(y').$$

$$dt = \frac{ds}{F(y')} = \frac{\sqrt{1+y'^2}}{F(y')}dx;$$

and
$$I = \int_0^a \frac{\sqrt{1+y'^2}}{F(y')}dx.$$

$$\delta I = \int_0^a \frac{\partial}{\partial y'}\frac{\sqrt{1+y'^2}}{F(y')}\delta y' dx$$

$$= -\int_0^a \frac{d}{dx}\frac{\partial}{\partial y'}\frac{\sqrt{1+y'^2}}{F(y')}\delta y dx + \left[\frac{\partial}{\partial y'}\frac{\sqrt{1+y'^2}}{F(y')}\delta y\right]_{x=a}.$$

Since we must make $\delta I = 0$

$$\frac{\partial}{\partial y'}\frac{\sqrt{1+y'^2}}{F(y')} = C \tag{1}$$

and
$$\left[\frac{\partial}{\partial y'}\frac{\sqrt{1+y'^2}}{F(y')}\delta y\right]_{x=a} = 0. \tag{2}$$

(1) gives us $y' = c$, and the required path is a straight line. (v. Art. 3.)

(2) reduces to

$$y'F(y') - (1 + y'^2)F'(y') = 0,$$

or since $y' = c$ to $cF(c) - (1 + c^2)F'(c) = 0,$

which determines c, the constant slope of the required path.

As a rough approximation of the speed law for a rather sluggish boat let

$$v = F(y') = k\,(\tan^{-1} y' - \alpha).$$

$$F'(y') = \frac{k}{1 + y'^2}$$

and

$$ck\,(\tan^{-1}c - \alpha) = k.$$

$$(\theta - \alpha)\tan\theta = 1,$$

then, gives us the angle, θ, the course (known to sailors as *full and by*) should make with the direction of the wind; and as θ does not depend upon k this course is independent of the strength of the wind.

If the point aimed at lies within the angle formed by lines through the starting point making angles θ and $-\theta$ with the wind's direction it is reached by tacking, both tacks being steered full and by. If it lies outside of this angle it should be steered for directly.

EXAMPLES

(1) If $\alpha = \frac{\pi}{2} - 1 = 33° -$, $\theta = 62\frac{1}{2}°$.

(2) Find the *curve of quickest descent* from a given point to a given vertical line. (v. Art. 9, Ex. 1.)

Ans. An inverted cycloid with a horizontal base, a cusp at the given point, and the vertex in the given line.

(3) Given two mutually perpendicular lines and a point in their plane; find the curve terminated by the point and one of the lines which by its revolution about the other line shall generate a surface of minimum area. (v. Art. 7.)

Ans. A catenary having the axis of revolution as its directrix and having its vertex on the other given line.

18. If $I = \int_{x_0}^{x_1} \phi\,(x, y, y')\,dx$ and we vary x_0 and x_1 as well as y_0 and y_1, so that the range of integration is no longer fixed, the problem is more complicated.

For the sake of simplicity we shall suppose that x_0 and y_0 are given but not x_1 or y_1, and that we are to determine the curve $y = f(x)$ so that I shall be less if we seek a minimum, greater if we seek a maximum than when $y = f(x)$ is replaced by any *neighboring* curve and x_1 is replaced by any value, $x_1 + \delta x_1$, differing from x_1 by a sufficiently small amount.

With our ordinary notation $y = f(x) + \alpha\delta y$ is a neighboring curve if α is sufficiently small (*v.* Art. 3); and $I(\alpha)$, where

$$I(\alpha) = \int_{x_0}^{x_1+\alpha\delta x_1} \phi(x, y + \alpha\delta y,\ y' + \alpha\delta y')\,dx,$$

must be a maximum or a minimum when $\alpha = 0$.

Let $u = y + \alpha\delta y$ and $v = y' + \alpha\delta y'$,

then $$I(\alpha) = \int_{x_0}^{x_1+\alpha\delta x_1} \phi(x, u, v)\,dx.$$

$$I'(\alpha) = \int_{x_0}^{x_1+\alpha\delta x_1} \left(\frac{\partial\phi}{\partial u}\,\delta y + \frac{\partial\phi}{\partial v}\,\delta y'\right)dx + [\phi(x, u, v)]_{x = x_1+\alpha\delta x_1}\,\delta x_1$$

and $$I'(0) = \int_{x_0}^{x_1} \left(\frac{\partial\phi}{\partial y}\,\delta y + \frac{\partial\phi}{\partial y'}\,\delta y'\right)dx + [\phi(x, y, y')]_{x = x_1}\,\delta x_1$$

$$= \int_{x_0}^{x_1} \delta\phi\,dx + [\phi(x, y, y')]_{x = x_1}\,\delta x_1$$

$$= \int_{x_0}^{x_1} \left[\frac{\partial\phi}{\partial y} - \frac{d}{dx}\frac{\partial\phi}{\partial y'}\right]\delta y dx + \left[\frac{\partial\phi}{\partial y'}\,\delta y\right]_{x = x_1} + [\phi(x, y, y')]_{x = x_1}\,\delta x_1.$$

To make $I'(0) = 0$ we have as usual Lagrange's Equation

$$\frac{\partial \phi}{\partial y} - \frac{d}{dx}\frac{\partial \phi}{\partial y'} = 0; \tag{1}$$

and we have the conditions

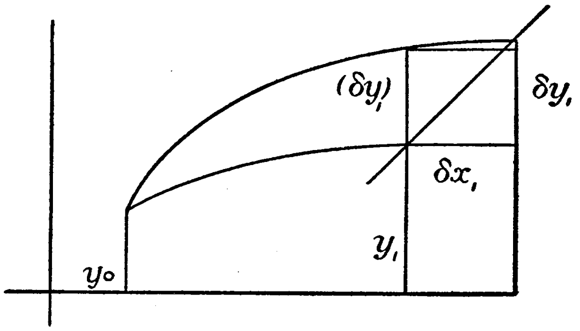
$$y = y_0 \text{ when } x = x_0,$$

and
$$\left[\frac{\partial \phi}{\partial y'}\,\delta y\right]_{x = x_1} + [\phi(x, y, y')]\delta x_1{}_{x = x_1} = 0, \tag{2}$$

to determine the two arbitrary constants in the solution of Lagrange's Equation.

If $(\delta y)_1$, the value δy assumes when $x = x_1$, and δx_1 are independent the solution is generally impossible, but in many interesting problems $(\delta y)_1$ depends upon δx_1.

For instance suppose that the required curve $y = f(x)$ which makes I a maximum or a minimum is to be terminated by a given



point (x_0, y_0) and a given curve $y = \psi(x)$, so that $y_1 = \psi(x_1)$. Let the second end point in the varied curve be $(x_1 + \delta x_1, y_1 + \delta y_1)$.

Then
$$\delta y_1 = \psi'(x_1)\delta x_1 + \epsilon$$
and
$$\begin{aligned}(\delta y)_1 &= \delta y_1 - y'_1\,\delta x_1 - \eta \\ &= (\psi'(x_1) - y'_1)\delta x_1 + \epsilon - \eta,\end{aligned}$$

where ϵ and η depend upon δx_1, and $\frac{\epsilon}{\delta x_1}$ and $\frac{\eta}{\delta x_1}$ can be made as small as we choose by taking a sufficiently small value of δx_1. The substitution of this value for $(\partial y)_1$ in (2) gives

$$\left[\frac{\partial \phi}{\partial y'}\left(\psi'(x) - y' + \frac{\epsilon - \eta}{\delta x_1}\right) + \phi\,(x, y, y')\right]_{x = x_1}\delta x_1 = 0, \tag{3}$$

and as (3) is to hold independently of the value of δx_1,

$$\left[\frac{\partial \phi}{\partial y'}(\psi'(x) - y') + \phi(x, y, y')\right]_{x = x_1} = 0. \tag{4}$$

Comparing (4) with (2) we see that it is (2) with δy replaced by $d\psi(x) - dy$, and δx_1 by dx.

If x_0 is varied as well as x we have in addition to (2) the condition

$$\left[\frac{\partial \phi}{\partial y'}\delta y + \phi(x, y, y')\delta x_0\right]_{x = x_0} = 0$$

and if $y = f(x)$ is to start at a point on the curve $y = \chi(x)$ we must replace δy and δx_0 by $d\chi(x) - dy$ and dx respectively.

19. (*a*) Suppose that in our Minimum Surface problem of Art. 7 the required generating curve is to be terminated by the point $(0, y_0)$ and the given curve $y = \psi(x)$.

We get as in Art. 7

$$\sqrt{1 + y'^2} - \frac{d}{dx}\frac{yy'}{\sqrt{1 + y'^2}} = 0$$

as the differential equation of the curve, and the condition

$$\left[\frac{yy'}{\sqrt{1 + y'^2}}\delta y + y\sqrt{1 + y'^2}\,\delta x_1\right]_{x = x_1} = 0,$$

which by Art. 18 gives us

$$\left[\frac{yy'}{\sqrt{1 + y'^2}}(\psi'(x) - y') + y\sqrt{1 + y'^2}\right]_{x = x_1} = 0$$

or

$$[y'\psi'(x)]_{x = x_1} + 1 = 0.$$

Therefore the required curve meets the given curve at right angles.

It has been shown in Art. 7 that it is a catenary having the axis of revolution as its directrix.

(*b*) As a second example let it be required to find the curve of given length and terminated by the axes, which with the axes encloses the maximum area. Here

$$I = \int_0^{x_1} [y + \lambda\sqrt{1 + y'^2}]\,dx.$$

$$\delta I = \int_0^{x_1}\left[\delta y + \lambda \frac{y'}{\sqrt{1+y'^2}}\delta y'\right]dx + \Big[y + \lambda\sqrt{1+y'^2}\Big]_{x=x_1}\delta x,$$

$$= \int_0^{x_1}\left[1 - \frac{d}{dx}\frac{\lambda y'}{\sqrt{1+y'^2}}\right]\delta y\,dx + \left[\frac{\lambda y'}{\sqrt{1+y'^2}}\delta y\right]_{x=0}^{x=x_1} + \Big[y + \lambda\sqrt{1+y'^2}\Big]_{x=x_1}\delta x_1 = 0.$$

$$1 - \frac{d}{dx}\frac{\lambda y'}{\sqrt{1+y'^2}} = 0 \qquad (1)$$

$$\left[\frac{\lambda y'}{\sqrt{1+y'^2}}\right]_{x=0} = 0 \qquad (2)$$

$$\left[\frac{\lambda y'}{\sqrt{1+y'^2}}(0 - y') + y + \lambda\sqrt{1+y'^2}\right]_{x=x_1} = 0. \qquad (3)$$

(1) integrates into

$$(x + C)^2 + (y - K)^2 = \lambda^2,$$

a circle.

(2) gives $[y']_{x=0} = 0$, and the circle crosses the axis of Y at right angles.

(3) gives $\left[\frac{\lambda}{\sqrt{1+y'^2}}\right]_{x=x_1} = 0$ since $y = 0$ when $x = x_1$, and the circle crosses the axis of X at right angles. Therefore the area in question is the quadrant of a circle.

20. The Principle of Least Action.

If T is the *kinetic energy* of a moving system the time-integral of $2T$ is called the *action* of the system. If V is the *potential energy* of the system and the forces are conservative it is well known that the so-called *equation of energy* $T + V = h$, where h is a constant, holds good during the motion.

The *Principle of Least Action* is that in the actual motion of the system the *action* is less than in any hypothetical motion, between the same initial and final configurations, in which the equation of energy holds good.

In Hamilton's Principle the actual motion is compared with a hypothetical motion in which initial and final configurations and *time of transit* are the same as in the actual motion, and to bring about the hypothetical motion additional forces would have to be imposed on the system, and these forces would do work.

In the Principle of Least Action the actual motion is compared with a hypothetical motion in which initial and final configurations are the same as in the actual motion, but as the equation of energy holds good, the time of transit is different in the two motions; and the hypothetical motion could be brought about by imposing constraints that would do no work.

The formulation of the Principle of Least Action is

$$\delta \int_0^{t_1} 2T dt = 0, \qquad T + V = h.$$

As in the case of Hamilton's Principle we shall prove the Principle of Least Action for a single moving particle, and we shall use the notation and as far as possible the results of Art. 12.

Keeping in mind that the potential energy is the negative of the force function, i. e. $V = -U$, and that t_1, the time of transit, is varied.

$$\delta \int_0^{t_1} 2T dt = \delta \int_0^{t_1} (T + h - V) dt = \int_0^{t_1} (\delta T - \delta V) dt + [T + h - V]_{t = t_1} \delta t_1.$$

$$\int_0^{t_1} \delta (T - V) dt = m[x'\delta x + y'\delta y + z'\delta z]_{t=0}^{t=t_1},$$

as in Art. 12.

Since the initial positions are the same in the two motions, $\delta x = \delta y = \delta z = 0$ when $t = 0$. Since the final positions are the same but are reached in the time t_1 in the actual motion and in the time $t_1 + \delta t_1$ in the hypothetical motion $\delta x = \delta y = \delta z = 0$ when $t = t_1 + \delta t_1$, and are approximately $-x' \delta t_1$, $-y' \delta t_1$, and $-z' \delta t_1$ when $t = t_1$. Therefore (*v.* Art. 18)

$$\int_0^{t_1} \delta (T - V) dt = -m[x'^2 + y'^2 + z'^2]_{t = t_1} \delta t_1 = -[2T]_{t = t_1} \delta t_1$$

approximately, and

$$\delta\int_0^{t_1} 2T dt = -[T + V - h]_{t=t_1}\delta t_1 \quad \text{approximately.}$$
$$= 0.$$

In any concrete problem of course the Principle of Least Action leads to the same differential equations of motion as Hamilton's Principle.

As an example we shall take the problem of planetary motion (*v*. Art. 12).

$$T = \frac{m}{2}(r'^2 + r^2\phi'^2), \quad V = -\frac{\mu M m}{r}, \quad \text{as in Art. 12.}$$

By the Principle of Least Action

$$\delta I = \delta\int_0^{t_1} m(r'^2 + r^2\phi'^2)dt = 0$$

and
$$\frac{m}{2}(r'^2 + r^2\phi'^2) - \frac{\mu M m}{r} = h.$$

$$\delta I = \delta\int_0^{t_1}\left[\frac{m}{2}(r'^2 + r^2\phi'^2) + h + \frac{\mu M m}{r}\right]dt$$

$$= \int_0^{t_1} m\left[r'\delta r' + r^2\phi'\delta\phi' + r\phi'^2\delta r - \frac{\mu M}{r^2}\delta r\right]dt$$
$$+ \left[\frac{m}{2}(r'^2 + r^2\phi'^2) + h + \frac{\mu M m}{r}\right]_{t=t_1}\delta t_1.$$

Integrating by parts

$$\delta I = m\int_0^{t_1}\left[-\frac{d}{dt}r'\delta r - \frac{d}{dt}(r^2\phi')\delta\phi + \left(r\phi'^2 - \frac{\mu M}{r^2}\right)\delta r\right]dt$$
$$+ m\Big[r'\delta r + r^2\phi'\delta\phi\Big]_0^{t_1} + \left[\frac{m}{2}(r'^2 + r^2\phi'^2) + h + \frac{\mu M m}{r}\right]_{t=t_1}\delta t_1.$$

The terms outside the integral sign reduce to

$$\left[-\frac{m}{2}(r'^2 + r^2\phi'^2) + h + \frac{\mu M m}{r}\right]_{t=t_1}\delta t_1,$$

and are equal to zero. Therefore

$$\delta I = m\int_0^{t_1}\left[-\frac{d}{dt}r'\delta r - \frac{d}{dt}(r^2\phi')\delta\phi + \left(r\phi'^2 - \frac{\mu M}{r^2}\right)\delta r\right]dt = 0$$

as in Art. 12.

And we get $$r'' - r\phi'^2 = -\frac{\mu M}{r^2}$$

and $$r^2\phi' = k,$$ as in Art. 12.

As Hamilton's Principle is more general than the older Principle of Least Action and is decidedly easier to use it is much more commonly employed in practice than the latter principle.

21. The whole subject of Variation of the Limits can be treated from a different point of view, that of parametric representation.

Instead of trying to determine y as a function of x so that $I = \int_{x_0}^{x_1}\phi\,(x, y, y')dx$ shall be a maximum or a minimum let us start to determine x and y as functions of a parameter τ.

We shall find it convenient to represent $\frac{dx}{d\tau}$ by $\dot{x}$ and $\frac{dy}{d\tau}$ by $\dot{y}$, in which case $y' = \frac{\dot{y}}{\dot{x}}$, and $dx = \dot{x}d\tau$. τ of course will be our independent variable, and we shall suppose that τ_0 and τ_1 are fixed while the corresponding values x_0 and x_1 may be varied.

$$I = \int_{x_0}^{x_1}\phi(x, y, y')\,dx = \int_{\tau_0}^{\tau_1}\phi(x, y, y')\dot{x}d\tau.$$

$$\delta I = \int_{t_0}^{t_1}\left[\phi\delta\dot{x} + \dot{x}\left(\frac{\partial\phi}{\partial x}\delta x + \frac{\partial\phi}{\partial y}\delta y + \frac{\partial\phi}{\partial y'}\frac{\dot{x}\delta\dot{y} - \dot{y}\delta\dot{x}}{\dot{x}^2}\right)\right]d\tau$$

$$= \int_{t_0}^{t_1}\left[\dot{x}\frac{\partial\phi}{\partial x}\delta x + \left(\phi - \frac{\dot{y}}{\dot{x}}\frac{\partial\phi}{\partial y'}\right)\delta\dot{x} + \dot{x}\frac{\partial\phi}{\partial y}\delta y + \frac{\partial\phi}{\partial y'}\delta\dot{y}\right]d\tau.$$

Integrating by parts

$$\delta I = \int_{\tau_0}^{\tau_1} \left\{ \left[\dot{x}\frac{\partial \phi}{\partial x} - \frac{d}{d\tau}\left(\phi - \frac{\dot{y}}{\dot{x}}\frac{\partial \phi}{\partial y'}\right) \right] \delta x + \left(\dot{x}\frac{\partial \phi}{\partial y} - \frac{d}{d\tau}\frac{\partial \phi}{\partial y'} \right) \delta y \right\} d\tau$$
$$+ \left[\left(\phi - \frac{\dot{y}}{\dot{x}}\frac{\partial \phi}{\partial y'} \right) \delta x + \frac{\partial \phi}{\partial y'}\delta y \right]_{\tau_0}^{\tau_1}.$$

To make $\delta I = 0$ we must make

$$\dot{x}\frac{\partial \phi}{\partial y} - \frac{d}{d\tau}\frac{\partial \phi}{\partial y'} = 0 \tag{1}$$

$$\dot{x}\frac{\partial \phi}{\partial x} - \frac{d}{d\tau}\left(\phi - \frac{\dot{y}}{\dot{x}}\frac{\partial \phi}{\partial y'}\right) = 0 \tag{2}$$

$$\left(\phi - \frac{\dot{y}}{\dot{x}}\frac{\partial \phi}{\partial y'}\right)\delta x_0 + \frac{\partial \phi}{\partial y'}\delta y_0 = 0 \tag{3}$$

when $\tau = \tau_0$,

and
$$\left(\phi - \frac{\dot{y}}{\dot{x}}\frac{\partial \phi}{\partial y'}\right)\delta x_1 + \frac{\partial \phi}{\partial y'}\delta y_1 = 0 \tag{4}$$

when $\tau = \tau_1$.

Dividing through by $\dot{x}$ (1) and (2) become

$$\frac{\partial \phi}{\partial y} - \frac{d}{dx}\frac{\partial \phi}{\partial y'} = 0 \tag{5}$$

$$\frac{\partial \phi}{\partial x} - \frac{d}{dx}\left(\phi - y'\frac{\partial \phi}{\partial y'}\right) = 0. \tag{6}$$

(3) and (4) reduce to

$$\left(\phi - y'\frac{\partial \phi}{\partial y'}\right)\delta x_0 + \frac{\partial \phi}{\partial y'}\delta y_0 = 0 \tag{7}$$

when $x = x_0$,

$$\left(\phi - y'\frac{\partial \phi}{\partial y'}\right)\delta x_1 + \frac{\partial \phi}{\partial y'}\delta y_1 = 0 \tag{8}$$

when $x = x_1$.

(5) is Lagrange's Equation. (6) which can be shown to have the same solution as (5) we can disregard. (7) and (8), if we have a relation between δx_0 and δy_0, and between δx_1 and δy_1, serve to determine the two arbitrary constants in the solution of (5).

For instance if the required curve is to be drawn from a given curve $y = \chi(x)$ to a given curve $y = \psi(x)$ as in Art. 18, $\delta y_0 = \chi'(x)\,\delta x_0$ and $\delta y_1 = \psi'(x)\,\delta x_1$, approximately, as in Art. 18; and (7) and (8) reduce to

$$\phi + (\chi'(x) - y')\frac{\partial\phi}{\partial y'} = 0 \qquad \text{when } x = x_0$$

$$\text{and } \phi + (\psi'(x) - y')\frac{\partial\phi}{\partial y'} = 0 \qquad \text{when } x = x_1 \quad v. \text{ Art. 18.}$$

22. To illustrate the practical working of this method we shall apply it to the second example in Art. 19, to the proof of the Principle of Least Action (*v.* Art. 20) and to the problem solved in Art. 20.

(*a*) (*v.* Art. 19). $\qquad I = \int_0^{x_1}[y + \lambda\sqrt{1 + y'^2}]\,dx.$

If x and y are functions of a parameter τ, $\delta x = 0$ and δy is arbitrary at the start, i. e. when $\tau = \tau_0$; and δx is arbitrary and $\delta y = 0$ at the finish, i. e. when $\tau = \tau_1$.

$$I = \int_{\tau_0}^{\tau_1}\left[y + \lambda\sqrt{1 + \frac{\dot y^2}{\dot x^2}}\right]\dot x\,d\tau = \int_{\tau_0}^{\tau_1}[\dot x y + \lambda\sqrt{\dot x^2 + \dot y^2}]\,d\tau.$$

$$\delta I = \int_{\tau_0}^{\tau_1}\left[\dot x\delta y + y\delta\dot x + \frac{\lambda(\dot x\delta\dot x + \dot y\delta\dot y)}{\sqrt{\dot x^2 + \dot y^2}}\right]d\tau$$

$$= \int_{\tau_0}^{\tau_1}\left\{\left[\dot x - \frac{d}{d\tau}\frac{\lambda\dot y}{\sqrt{\dot x^2 + \dot y^2}}\right]\delta y - \left[\dot y + \frac{d}{d\tau}\frac{\lambda\dot x}{\sqrt{\dot x^2 + \dot y^2}}\right]\delta x\right\}d\tau$$

$$+ \left[\frac{\lambda\dot y}{\sqrt{\dot x^2 + \dot y^2}}\delta y + \left(y + \frac{\lambda\dot x}{\sqrt{\dot x^2 + \dot y^2}}\right)\delta x\right]_{\tau_0}^{\tau_1}.$$

Since $\delta I = 0$

$$\dot x - \frac{d}{d\tau}\frac{\lambda\dot y}{\sqrt{\dot x^2 + \dot y^2}} = 0$$

$$\dot y + \frac{d}{d\tau}\frac{\lambda\dot x}{\sqrt{\dot x^2 + \dot y^2}} = 0;$$

$$\frac{\lambda \dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = 0 \qquad \text{when } \tau = \tau_0$$

$$y + \frac{\lambda \dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = 0 \qquad \text{when } \tau = \tau_1.$$

These equations reduce to

$$1 - \frac{d}{dx}\frac{\lambda y'}{\sqrt{1 + y'^2}} = 0, \tag{1}$$

$$y' + \frac{d}{dx}\frac{\lambda}{\sqrt{1 + y'^2}} = 0; \tag{2}$$

$$\frac{y'}{\sqrt{1 + y'^2}} = 0 \qquad \text{when } x = 0$$

$$\frac{1}{\sqrt{1 + y'^2}} = 0 \qquad \text{when } x = x_1.$$

(1) is (1) Art. 18 and integrates into $(x + C)^2 + (y - K)^2 = \lambda^2$, as does (2). The curve is a circle; and as $y' = 0$ when $x = 0$ and $y' = \infty$ when $y = 0$, the circle crosses both axes at right angles and the centre must be at the origin.

(*b*) (*v.* Art. 20).

We shall suppose that the time, t, and therefore the coördinates x and y, depends upon an independent variable τ in such a way that when $\tau = \tau_0$, $t = 0$ in the actual and in the hypothetical motion, and when $\tau = \tau_1$, $t = t_1$ in the actual motion and $t = t_1 + \delta t_1$ in the hypothetical motion. Note that δx and δy are zero when $\tau = \tau_0$ and when $\tau = \tau_1$, and that $\delta t = 0$ when $\tau = \tau_0$.

$$\int_0^{t_1} 2T dt = \int_0^{t_1} (T + h - V) dt = \int_{\tau_0}^{\tau_1} [T + h - V] \dot{t} d\tau.$$

$$\delta \int_0^{t_1} 2T dt = \int_{\tau_0}^{\tau_1} [(\delta T - \delta V)\dot{t} + (T + h - V)\delta\dot{t}] d\tau$$

$$= \int_{\tau_0}^{\tau_1} [(\delta T - \delta V)\dot{t} + 2T\delta\dot{t}] d\tau.$$

$$- \delta V = m(x''\delta x + y''\delta y + z''\delta z).$$

$$x''\delta x = \frac{d}{dt}(x'\delta x) - x'\frac{d}{dt}\delta x$$

$$= \frac{1}{\dot t}\frac{d}{d\tau}\left(\frac{\dot x}{\dot t}\delta x\right) - \frac{\dot x}{\dot t}\frac{\partial \dot x}{\dot t}.$$

$$-\delta V = \frac{m}{\dot t}\frac{d}{d\tau}\frac{\dot x\delta x + \dot y\delta y + \dot z\delta z}{\dot t} - \frac{1}{\dot t^2}\delta(\dot t^2 T)$$

$$= \frac{m}{\dot t}\frac{d}{d\tau}\frac{\dot x\delta x + \dot y\delta y + \dot z\delta z}{\dot t} - \delta T - 2T\frac{\delta\dot t}{\dot t}.$$

$$[\delta T - \delta V]\dot t + 2T\delta\dot t = m\frac{d}{d\tau}\frac{\dot x\delta x + \dot y\delta y + \dot z\delta z}{\dot t}.$$

$$\delta\int_0^{t_1} 2T\delta t = m[x'\delta x + y'\delta y + z'\delta z]_{\tau=\tau_0}^{\tau=\tau_1} = 0$$

since $\delta x = \delta y = \delta z = 0$ when $\tau = \tau_0$ and when $\tau = \tau_1$.

(*c*) Planetary motion (*v.* Art. 20).

We are to make

$$I = \int_0^{t_1}\left[\frac{m}{2}(r'^2 + r^2\phi'^2) + h + \frac{\mu M m}{r}\right]dt$$

a minimum, while keeping

$$\frac{m}{2}(r'^2 + r^2\phi'^2) - \frac{\mu M m}{r} = h.$$

$$I = \int_{\tau_0}^{\tau_1}\left[\frac{m}{2}\left(\frac{\dot r^2}{\dot t^2} + \frac{r^2\dot\phi^2}{\dot t^2}\right) + h + \frac{\mu M m}{r}\right]\dot t\,d\tau$$

$$= m\int_{\tau_0}^{\tau_1}\left[\frac{\dot r^2 + r^2\dot\phi^2}{2\dot t} + \left(\frac{h}{m} + \frac{\mu M}{r}\right)\dot t\right]d\tau.$$

$$\delta I = m\int_{\tau_0}^{\tau_1}\left[\frac{\dot t(\dot r\delta\dot r + r^2\dot\phi\delta\dot\phi + r\dot\phi^2\delta r) - \frac{1}{2}(\dot r^2 + r^2\dot\phi^2)\delta\dot t}{\dot t^2}\right.$$

$$\left. + \left(\frac{h}{m} + \frac{\mu M}{r}\right)\delta\dot t - \frac{\mu M\dot t}{r^2}\delta r\right]d\tau$$

$$= m\int_{\tau_0}^{\tau_1}\left[\left(\frac{r\dot\phi^2}{\dot t} - \frac{\mu M\dot t}{r^2}\right)\delta r + \frac{\dot r}{\dot t}\delta\dot r + \frac{r^2\dot\phi}{\dot t}\delta\dot\phi + \left(\frac{h}{m} + \frac{\mu M}{r} - \frac{\dot r^2 + r^2\dot\phi^2}{2\dot t^2}\right)\delta\dot t\right]d\tau.$$

$$\delta I = m\int_{\tau_0}^{\tau_1}\left[\left(\frac{r\dot\phi^2}{\dot t}-\frac{\mu M\dot t}{r^2}-\frac{d}{d\tau}\frac{\dot r}{\dot t}\right)\delta r-\frac{d}{d\tau}\frac{r^2\dot\phi}{\dot t}\,\delta\phi-\frac{d}{d\tau}\left(\frac{\mu M}{r}-\frac{\dot r^2+r^2\dot\phi^2}{2\dot t^2}\right)\delta t\right]d\tau$$

$$+\ m\left[\frac{\dot r}{\dot t}\,\delta r+\frac{r^2\dot\phi}{\dot t}\,\delta\phi+\left(\frac{h}{m}+\frac{\mu M}{r}-\frac{\dot r^2+r^2\dot\phi^2}{2\dot t^2}\right)\delta t\right]_{\tau=\tau_0}^{\tau=\tau_1}.$$

To make $\delta I = 0$ we must make

$$\frac{r\dot\phi^2}{\dot t}-\frac{\mu M\dot t}{r^2}-\frac{d}{d\tau}\frac{\dot r}{\dot t}=0,$$

$$\frac{d}{d\tau}\frac{r^2\dot\phi}{\dot t}=0,$$

$$\frac{d}{d\tau}\left(\frac{\mu M}{r}-\frac{\dot r^2+r^2\dot\phi^2}{2\dot t^2}\right)=0;$$

and $$\frac{h}{m}+\frac{\mu M}{r}-\frac{\dot r^2+r^2\dot\phi^2}{2\dot t^2}=0 \quad \text{when } \tau=\tau_1;$$

or $$r''-r\phi'^2+\frac{\mu M}{r^2}=0 \qquad (1)$$

$$r^2\phi'=k \qquad (2)$$

$$\frac{d}{dt}\left[\frac{\mu Mm}{r}-\frac{m}{2}(r'^2+r^2\phi'^2)\right]=0; \qquad (3)$$

and $$h+\frac{\mu Mm}{r}-\frac{m}{2}(r'^2+r^2\phi'^2)=0 \qquad (4)$$

when $t = t_1$.

(1) and (2) are the required equations of motion and (3) and (4) follow from the Equation of Energy.

Examples

(1) Work Art. 19 (*a*) by the method of Art. 21.

(2) Work examples 2 and 3 of Art. 17 by the method of Art. 21.

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