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CALC FOR THE CLUELESS

"Bob Miller saved my grade. Not only did I pass, but I got an A!"—**Sandy Smith, University of Pittsburgh**

Chapter 1—
Logarithms

Most of you, at this point in your mathematical journey, have not seen logs for at least a year, maybe longer. The normal high school course emphasizes the wrong areas. You spend most of the time doing endless calculations, none of which you need here. By the year 2000, students will do almost no log calculations due to calculators. In case you feel tortured, just remember you only spent weeks on log calculations. I spent months!!!

The Basic Laws of Logs

1. Defined, $\log_b x = y$ (log of x to the base b is y) if $b^y = x$; $\log_5 25 = 2$ because $5^2 = 25$.

2. What can the base b be? It can't be negative, such as -2, since $(-2)^{1/2}$ is imaginary. It can't be 0, since 0^n is either equal to 0 if n is positive or undefined if n is 0 or negative. b also can't be 1 since 1^n always $= 1$. Therefore b can be any positive number except 1.

Note

The base can be $2^{1/2}$, but it won't do you any good because there are no $2^{1/2}$ tables. The two most common bases are 10, because we have 10 fingers, and e, a number that occurs a lot in mathematics starting now.

A. e equals approximately 2.7.

B. What is e more exactly? On a calculator press 1, inv, ln.

C. $\log = \log_{10}$

D. $\ln = \log_e$ (In is the natural logarithm).

3. A log y is an exponent, and exponents can be positive, negative, and zero. The range is all real numbers.

4. Since the base is positive, whether the exponent is positive, zero, or negative, the answer is positive. The domain, therefore, is positive numbers.

Note

In order to avoid getting too technical, most books write log |x|, thereby excluding only x=0.

5. $\log_b x + \log_b y = \log_b xy$; $\log 2 + \log 3 = \log 6$.

6. $\log_b x - \log_b y = \log_b (x/y)$; $\log 7 - \log 3 = \log (7/3)$.

7. $\log_b x^p = p \log_b x$; $\ln 6^7 = 7 \ln 6$ is OK.

> **Note**
> *Laws 5, 6, and 7 are most important. If you can simplify using these laws, about half the battle (the easy half) is done.*

Example 1—

Write the following as simpler logs with no exponents:

$$\ln \left(\frac{a^4 b^5}{c^6 \sqrt{d}} \right)$$

4 ln a+ 5 ln b- 6 ln c- ½ ln d

8. $\log_b b = 1$ since $b^1 = b$. $\log_7 7 = 1$. In $e = 1$. $\log 10 = 1$.

9. $\log_b 1 = 0$ since $b^0 = 1$. $\log_8 1 = \log 1 = \ln 1 = 0$.

10. Log is a 1:1 function. This means if $\log c = \log d$, $c = d$.

Note

Not everything is 1:1. If $x^2 = y2$, $x = \pm y$.

11. Log is an increasing function. If $m < n$, then $\log m < \log n$.

12. $\lim\limits_{x \to \infty} (1 + 1/x)^x = e$

13. $b^{\log_b x}$ is a weird way of writing x. $e^{\ln x} = x$.

14. $\log_b b^x = x$; $\ln e^x = x$.

15. $\log_d c = \dfrac{\log_b c}{\log_b d}$; $\log_d c = \dfrac{\ln c}{\ln d}$; $\log_2 5 = \dfrac{\ln 5}{\ln 2}$.

You should now be able to solve the following kinds of log equations:

Example 2—

Solve for x: $4 \cdot 3^{x+2} = 28$.

Divide by 4; isolate exponent.

$3^{x+2} = 7$

Take logs. It now becomes an elementary algebra equation, which we solve for x, using the same technique as in the implicit differentiation section of *Calc 1*.

$(x + 2) \log 3 = \log 7$

$x = \dfrac{\log 7 - 2 \log 3}{\log 3}$

16. $a^x = e^{x \ln a}$. Also $x^x = e^{x \ln x}$ and $x^{\sin x} = e^{\sin x \ln x}$.

Example 3—

$4^{2x+5} = 9^{3x-7}$

$(2x + 5) \log 4 = (3x - 7) \log 9$

Using the same algebraic tricks, we get

Eliminate excess minus signs.

$x = \dfrac{-7 \log 9 - 5 \log 4}{2 \log 4 - 3 \log 9}$ or $\dfrac{7 \log 9 + 5 \log 4}{3 \log 9 - 2 \log 4}$

All this should be known about logs before the calculus. Now we are ready to get serious.

17. Major theorem: given f(x) = ln x; then f'(x) = 1/x. Proof (worth looking at):

Definition of derivative

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{\ln(x + \Delta x) - \ln x}{\Delta x}$$

Rule 6

$$= \lim_{\Delta x \to 0} \frac{\ln\left(\frac{x + \Delta x}{x}\right)}{\Delta x}$$

Normal trick I—multiply by I = x/x

$$= \lim_{\Delta x \to 0} \frac{x \ln\left(\frac{x + \Delta x}{x}\right)}{x \Delta x}$$

Algebra

$$= \lim_{\Delta x \to 0} \frac{\frac{x}{\Delta x} \ln\left(1 + \frac{\Delta x}{x}\right)}{x}$$

Rule 7

$$= \lim_{\Delta x \to 0} \frac{\ln\left(1 + \frac{\Delta x}{x}\right)^{x/\Delta x}}{x}$$

Use trick 2, $\frac{x}{\Delta x} = w$ As $\Delta x \to 0, w \to \infty$

$$= \lim_{w \to \infty} \frac{\ln(1 + 1/w)^w}{x}$$

Rule 12

$$= \frac{\ln e}{x}$$

Rule 8

$$= 1/x$$

This theorem is important, since it has a lot of log rules together with two normal math tricks. The theorem gives us the following result:

$$\int \frac{1}{x}\, dx = \ln |x| + C$$

Chapter 2—
Derivatives of Ex, Ax, Logs., Trig Functions, Etc., Etc.

We will now take derivatives involving ln x, e^x, a^x, $f(x)^{8(x)}$, trig functions, and inverse trig functions.

Example 1—

$$y = \ln (x^2 + 5x + 7)$$

Let $u = x^2 + 5X + 7$. Then $y = \ln u$. So $dy/dx = (dy/du)(du/dx) = (1/u)(2x + 5) = (2x + 5)/(x^2 + 5x + 7)$.

Notice, taking derivatives of logs is not difficult. However, you do not want to substitute $u = x^2 + 5x + 7$. You must do that in your head. If $y = \ln u$, do $y' = (1/u)(du/dx)$ in your head!

Example 2—

$$y = \ln \frac{(x^2 + 7)^9(x + 3)}{x^6}$$

The simplest way to do this is to use laws 3, 4, and 5 of the preceding chapter and simplify the expression before we take the derivative. So $y = 9 \ln (x^2 + 7) + \ln (x + 3) - 6 \ln x$. Therefore

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Remember to simplify by multiplying
$9(2x) = 18x$.

$$y' = \frac{9(2x)}{x^2 + 7} + \frac{1}{x + 3} - \frac{6}{x}$$

Example 3—

$$y = \log_2 x$$

Using law 15, $y = \ln x/\ln 2$, where $\ln 2$ is a number (a constant). Therefore

$$y' = (1/\ln 2)(\ln x)' = 1/(x \ln 2)$$

Law 18

$y = e^u$. $y' = e^u(du/dx)$. If $y = e$ to the power u, where $u = $ a function of x, the derivative is the original function untouched times the derivative of the exponent.

Example 4—

$$y = e^{\sin x}$$

$$y' = e^{\sin x} (\sin x)' = e^{\sin x} (\cos x)$$

Law 19

$y = a^u$. $y' = a^u \ln a(du/dx)$. If $y = a^u$, the derivative is a^u (the original function untouched) times the log of the base times the derivative of the exponent.

Example 5—

$y = 7^{x^2}$

$y' = (7^{x^2})(\ln 7)(2x)$

= (whole function) (ln of base) (derivative of exponent)

Let us, for completeness, recall the trig derivatives and do one longish chain rule.

Law 20

A. $y = \sin x$, $y' = \cos x$

B. $y = \cos x$, $y' = -\sin x$

C. $y = \tan x$, $y' = \sec^2 x$

D. $y = \cot x$, $y' = -\csc^2 x$

E. $y = \sec x$, $y' = \tan x \sec x$

F. $y = \csc x$, $y' = -\cot x \csc x$

Example 6—

$y = \tan^6 (4x^2 + 3x + 7)$

Since this is a function of a function, we must use the extended chain rule.

Let $u = \tan (4x^2 + 3x + 7)$. $y = u^6$ and $dy/du = 6u^5$. Let $v = 4x^2 + 3x + 7$. $u = \tan v$. $du/dv = \sec^2 v$ and $dv/dx = 8x + 3$. So

dy/dx =	(dy/du) times	(du/dv) times	(dv/dx)
=	$6u^5$ times	$\sec^2 v$ times	$(8x + 3)$
=	$[6 \tan^s (4x^2 + 3x + 7)]$	$[\sec^2 (4x^2 + 3x + 7)]$	$(8x + 3)$
	Power rule—leave trig function and crazy angle untouched	Derivative of trig function—leave crazy angle untouched	Derivative of crazy angle

You should be able to do this without substituting for u and v. It really is not that difficult with a little practice.

Law 21

A. $y = \sin^{-1} u$. $y' = \dfrac{1}{(1 - u^2)^{1/2}} \, du/dx$ $|u| < 1$.

B. $y = \tan^{-1} u$. $y' = \dfrac{1}{1 + u^2} \, du/dx$.

C. $y = \sec^{-1} u$. $y' = \dfrac{1}{u(u^2 - 1)^{1/2}} \, du/dx \quad |u| > 1$.

Example 7—

$y = \tan^{-1}(e^{5x})$

$u = e^{5x}$. $u' = e^{5x} 5$.

$$y' = \frac{1u'}{1 + u^2} = \frac{5e^{5x}}{1 + (e^{5x})^2} = \frac{5e^{5x}}{1 + e^{10x}}$$

Example 8—

$y = (\tan^{-1} x)/(1 + x^2)$

Use the quotient rule.

$$y' = \frac{(1 + x^2)[1/(1 + x^2)] - (\tan^{-1} x)(2x)}{(1 + x^2)^2} = \frac{1 - 2x \tan^{-1} x}{(1 + x^2)^2}$$

I like this one. I don't know why, but I do.

Example 9—

$y = \sin^{-1}(\cos^2 x)$

$u = \cos^2 x$. $u' = 2(\cos x)(-\sin x)$.

$$y' = \frac{u'}{(1 - u^2)^{1/2}} = \frac{-2 \cos x \sin x}{(1 - \cos^4 x)^{1/2}}$$

Example 10—

$y = \sec^{-1}(x^2 + 1)$

$u = x^2 + 1$. $u' = 2x$.

$$y' = \frac{u'}{u(u^2 - 1)^{1/2}} = \frac{2x}{(x^2 + 1)[(x^2 + 1)^2 - 1]^{1/2}}$$

If $y = f(x)^{g(x)}$, we take logs of both sides and differentiate implicitly (if you've forgotten implicit differentiation, see my *Calc I*).

Example 11—

$y = x^{\sin x}$

$\ln y = \sin x \ln x$

$(1/y)(dy/dx) = (\sin x)(1/x) + (\ln x)(\cos x)$

$\qquad dy/dx = y[(\sin x)/x + (\ln x)(\cos x)]$

$\qquad\qquad = x^{\sin x}[(\sin x)/x + (\ln x)(\cos x)]$

Example 11, Alternative Method—

$y = x^{\sin x} = e^{\sin x \ln x}$

$y' = e^{\sin x \ln x}(\sin x \ln x)'$

$\qquad = e^{\sin x \ln x}[(\sin x)/x + (\ln x)(\cos x)]$

$\qquad = x^{\sin x}[(\sin x)/x + (\ln x)(\cos x)]$

Example 12—

$$y = \frac{(x^3 + 1)^7(\sin^6 x)(x^2 + 4)^9}{e^{7x}x^8}$$

To take derivatives without logs is long and leads to errors. However, taking logs first and differentiating implicitly makes things much shorter and easier.

$\ln y = 7 \ln (x^3 + 1) + 6 \ln \sin x + 9 \ln (x^2 + 4) - 7x - 8 \ln x$

$(1/y)\, dy/dx = \dfrac{7(3x^2)}{x^3 + 1} + \dfrac{6(\cos x)}{\sin x} + \dfrac{9(2x)}{x^2 + 4} - 7 - \dfrac{8}{x}$

$\dfrac{dy}{dx} = y\left[\dfrac{21x^2}{x^3 + 1} + 6 \cot x + \dfrac{18x}{x^2 + 4} - 7 - \dfrac{8}{x}\right]$

This looks neat. But remember what y really is!! This method is still pretty short!

Chapter 3—
Shorter Integrals

In most schools, the largest part of the second semester of a three-term calc sequence involves integrals. It usually covers more than 50 percent of this course. It is essential to learn these shorter ones as perfectly as possible so that Chap. 6 will not be overwhelming. Also, it is impossible to put every pertinent example in without making the book too long. The purpose of this book is to give you enough examples so that you can do the rest by yourself. If you think an example should be added, write me.

Rule 1

$$\int \frac{f'(x)\, dx}{f(x)} = \ln\ |f(x)| + C$$

One of the first new things we look for is that the numerator is the derivative of the denominator. This gives us a ln for an answer.

Example 1—

Let $u = 5x^2 - 7$ and $du = 10x\ dx$.

$$\int \frac{x\, dx}{5x^2 - 7}$$

$$= (1/10) \int \frac{10x\, dx}{5x^2 - 7} = (1/10) \int \frac{du}{u}$$

$$= (1/10) \ln\ |u| + C = (1/10) \ln\ |5x^2 - 7| + C$$

Example 2—

$u = 1 + \sin x$ and $du = \cos x\ dx$

$$\int \frac{\cos x\, dx}{1 + \sin x}$$

Exclude $x = 3\pi/2$ and so on. Then $\sin x > -1$, so the absolute value is not needed in the answer.

$$= \int \frac{du}{u} = \ln\ |u| + C = \ln\ (1 + \sin x) + C$$

Example 3—

This one looks kind of weird. Sometimes we just have to try something. Let $u = x^{1/2} + 3$. (Note that $u = x^{1/2}$ will also work.) $du = \frac{1}{2} x^{-1/2}\ dx$, so $dx = 2x^{1/2}\ du$.

$$\int \frac{6\, dx}{x^{1/2}(x^{1/2} + 3)}$$

$$= \int \frac{6(2x^{1/2}\, du)}{x^{1/2} u} = \int \frac{12\, du}{u}$$

$$= 12 \ln u + C = 12 \ln (x^{1/2} + 3) + C$$

Let's try a definite integral.

Example 4—

u = ln x; du = (1/x) dx

$$\int_{e}^{e^e} \frac{dx}{x \ln x}$$

$$= \int (1/u)\, du = \ln u = \ln(\ln x) \Big[_{e}^{e^e}$$

$$= \ln(\ln e^e) - \ln(\ln e) = \ln(e) - \ln 1 = 1 - 0 = 1$$

Example 5—

We need to divide the long way since the degree of the top is greater than that of the bottom:

$$\begin{array}{r} x - 2 + -1/(x-2) \\ x - 2 \overline{)x^2 - 4x + 3} \\ \underline{x^2 - 2x} \\ -2x + 3 \\ \underline{-2x + 4} \\ -1 \end{array}$$

$$\int \left(\frac{x^2 - 4x + 3}{x - 2} \right) dx$$

$$= \int \left[x - 2 + \frac{-1}{x - 2} \right] dx$$

$$= \frac{x^2}{2} - 2x - \ln |x - 2| + C$$

Example 6—

u = x^2 + 3; du = 2x dx. But be careful! This is *not a* logarithm!! The exponent on the bottom is 2!!! It must be a 1 to be a log!!!

$$\int \frac{2x\, dx}{(x^2 + 3)^2}$$

$$= \int \frac{du}{u^2} = \int u^{-2}\, du = \frac{-1}{u} + C = \frac{-1}{x^2 + 3} + C$$

Trig Integrals

Rule 2

A. $\int \sin x\, dx = -\cos x + C$

B. $\int \cos x\, dx = \sin x + C$

C. $\int \sec^2 x \, dx = \tan x + C$

D. $\int \tan x \, dx = -\ln |\cos x| + C \quad$ or $\ln |\sec x| + C$

E. $\int \csc^2 x \, dx = -\cot x + C$

F. $\int \cot x \, dx = \ln |\sin x| + C$

G. $\int \tan x \sec x \, dx = \sec x + C$

H. $\int \sec x \, dx = \ln |\tan x + \sec x| + C$

I. $\int \csc x \cot x \, dx = -\csc x + C$

J. $\int \csc x \, dx = -\ln |\cot x + \csc x| + C$ or
$\ln |\csc x - \cot x| + C$

You must know these integrals perfectly!!

Example 7—

$u = 4x; \, dx = \frac{1}{4} \, du.$

$\int \cot 4x \, dx$

$= \frac{1}{4} \int \cot u \, du = \frac{1}{4} \ln |\sin u| + C = \frac{1}{4} \ln |\sin (4x)| + C$

Note

Whenever you have the integral of one of these trig functions and there is a constant multiplying the angle, you must, by sight, integrate this without letting u equal the angle. Otherwise, the integrals in Chap. 6 will take forever.

Example 8—

This is the crazy angle substitution:
$u = $ crazy angle $= 1 - 3x^3; \, du = -9x^2 \, dx.$

$\int x^2 \cos (1 - 3x^3) \, dx$

$= -(1/9) \int (-9x^2) \cos (1 - 3x^3) \, dx$

$= (-1/9) \int \cos u \, du = (-1/9) \sin u + C$

$= (-1/9) \sin (1 - 3x^3) + C$

Example 9—

$$\int \tan 2x \ \sec^2 2x \ dx$$

$$= \frac{1}{2} \int u \ du = \frac{1}{2}u^2/2 + C = \frac{1}{4} \tan^2 2x + C$$

Example 10—

This one requires splitting the integrand into two fractions and uses identities.

$$\int \frac{1 + \sin x \ dx}{\cos^2 x}$$

$$= \int \left(\frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} \right) dx$$

$$= \int (\sec^2 x + \tan x \sec x) \ dx = \tan x + \sec x + C$$

It's an easy one if (a *big* if) you know your identities and trig integrals.

Exponential Integrals

Rule 3

$$\int e^{bx} \ dx = \frac{e^{bx}}{b} + C$$. Know this perfectly by sight!

Example 11—

$$\int e^{5x} \ dx = \frac{e^{5x}}{5} + C$$

Rule 4

$$\int a^{bx} \ dx = \frac{a^{bx}}{b \ln a} + C$$. Know this one perfectly also!

Example 12—

$$\int 7^{5x} \ dx = \frac{7^{5x}}{5 \ln 7} + C$$

Example 13—

Crazy exponent substitutions: u = 7/x; du = -7/x² dx.

$$\int \frac{e^{7/x}}{x^2} \, dx$$

$$= \int -\frac{e^u}{7} \, du = -\frac{e^u}{7} + C = -\frac{e^{7/x}}{7} + C$$

Example 14—

Crazy exponent: u - In x; du = (1/x) dx.

$$\int \frac{7^{\ln x}}{x} \, dx$$

$$= \int 7^u \, du = \frac{7^u}{\ln 7} + C = \frac{7^{\ln x}}{\ln 7} + C$$

Example 15—

Crazy exponent (only real choice) plus trig identity: u = sin x; du = cos x dx = dx/sec x.

$$\int \frac{e^{\sin x} \, dx}{\sec x}$$

$$= \int e^u \, du = e^u + C = e^{\sin x} + C$$

Example 16—

One of my favorites. This one looks exactly like the one in Example 16A but is really different: $u = e^{4x} + 1$; $du = 4e^{4x}$ dx.

$$\int \frac{e^{4x} \, dx}{(e^{4x} + 1)^2}$$

$$= \frac{1}{4} \int \frac{4e^{4x} \, dx}{(e^{4x} + 1)^2} = \frac{1}{4} \int \frac{du}{u^2} = \frac{-1}{4u} + C = \frac{-1}{4(e^{4x} + 1)} + C$$

Inverse Trig Functions

This part is the last of the basic integrals that you must know by sight. In some schools, all six inverse trig functions must be known; in some, you learn three; and in some, like in my school, you learn two. We will do three—arc sin, arc tan, arc sec.

Rule 5

$$\int \frac{1 \, dx}{(1 - x^2)^{1/2}} = \sin^{-1} x + C \qquad\qquad \int \frac{1 \, dx}{(a^2 - x^2)^{1/2}} = \sin^{-1} (x/a) + C$$

$$\int \frac{1 \, dx}{1 + x^2} = \tan^{-1} x + C \qquad\qquad \int \frac{1 \, dx}{a^2 + x^2} = (1/a) \tan^{-1} (x/a) + C$$

$$\int \frac{1 \, dx}{x(x^2 - 1)^{1/2}} = \sec^{-1} x + C \qquad\qquad \int \frac{1 \, dx}{x(x^2 - a^2)^{1/2}} = (1/a) \sec^{-1} (x/a) + C$$

Memorize these also!

Example 16a (The Next One 1 Left Out)—

This is different.

$$\int \frac{(e^{4x} + 1)^2}{e^{4x}} \, dx$$

$$= \int \frac{e^{8x} + 2e^{4x} + 1}{e^{4x}} \, dx = \int e^{4x} + 2 + e^{-4x} \, dx$$

$$= \frac{e^{4x}}{4} + 2x - \frac{e^{-4x}}{4} + C$$

These integrals are not long, but you must study them because there are a lot of differences.

You will be able to identify these integrals by sight with practice. As for me, I know arc sin and arc tan very well, because I've practiced them. However, in all the years I've taught, no one has ever required arc sec, so I have to struggle, since I need practice also. Practice is what is needed!

Example 17—

You must see this is an arc sin. $u = e^{3x}$; $u^2 = (e^{3x})^2 = e^{6x}$; $du = 3e^{3x} \, dx$; $du/3 = e^{3x} \, dx$.

$$\int \frac{e^{3x} \, dx}{(1 - e^{6x})^{1/2}}$$

$$= \int \frac{(1/3) \, du}{(1 - u^2)^{1/2}} = (1/3) \arcsin u + C = (1/3) \arcsin (e^{3x}) + C$$

Example 18—

You must see this is an arc tan. u: X^6; $u^2 = (x^6)^2 = x^{12}$; $du = 6x^5 \, dx$; $a = 7^{1/2}$ since $a^2 = 7$.

$$\int \frac{x^5 \, dx}{7 + x^{12}}$$

$$= (1/6) \int \frac{du}{7 + u^2} = \frac{1}{6 \cdot 7^{1/2}} \tan^{-1} (x^6/7^{1/2}) + C$$

Example 19—

This is harder to tell. It is an arc sec with $u = x^2$, $du = 2x\, dx$. Multiply top and bottom by 2x. a = $11^{1/2}$.

$$\int \frac{dx}{x(x^4 - 11)^{1/2}}$$

$$= \frac{1}{2} \int \frac{2x\, dx}{x^2(x^4 - 11)^{1/2}} = \frac{1}{2} \int \frac{du}{u(u^2 - 11)^{1/2}}$$

$$= \frac{1}{2} (1/11^{1/2})\, \text{arc sec}\, (u/11^{1/2}) + C = \frac{\text{arc sec}\, (x^2/11^{1/2})}{2 \cdot 11^{1/2}} + C$$

You can all do it if you concentrate and practice a little.

Warning!! Beware! Danger! Now that you know these three, be careful of those that look similar but are not arcs.

Example 20—

This looks like an arc tan, but a u-substitution will give us a log. $u = x^2 + 4$; $du = 2x\, dx$; $du/2 = x\, dx$.

$$\int \frac{x\, dx}{x^2 + 4}$$

$$= \tfrac{1}{2} \int du/u = \tfrac{1}{2} \ln u + C = \tfrac{1}{2} \ln (x^2 + 4) + C$$

Example 21—

This is not an arc sin. $u = (1 - x^2)$; $du = -2x\, dx$.

$$\int \frac{x\, dx}{(1 - x^2)^{1/2}}$$

$$= -\tfrac{1}{2} \int u^{-1/2}\, du = -u^{1/2} + C = -(1 - x^2)^{1/2} + C$$

Chapter 4—
Exponential Growth and Decay

In every book on calculus, there is a little on differential equations, which are equations with derivatives. Usually, one chapter is devoted to this topic, which is almost never used. Parts of one or two other chapters may have differential equations in them. This topic is almost universally covered by all courses.

Example 1—

The rate of change of marlenium is proportional to the amount.

Ten pounds of marlenium become 90 pounds in 4 hours.

A. Write the equation.

B. How many pounds of marlenium will there be in 10 hours?

C. When will there be 500 pounds of marlenium?

1A. The differential equation to solve is $dM/dt = kM$ where k is a konstant.

We solve this by separation of variables.

Integrate.

$$\frac{dM}{M} = k\, dt$$

$$\ln M = kt + C$$

We need a trick. Let $C = \ln M_o$

$$\ln M = kt + \ln M_o$$

where M_o = the amount of marlenium at $t = 0$.

$$\ln M - \ln M_o = kt$$

By law 6 of logs,

$$\ln \frac{M}{M_o} = kt$$

By the definition of logs,

$$\frac{M}{M_o} = e^{kt} \text{ or } M = M_o\, e^{kt}$$

$$M_o = 10 \quad M = 90 \quad t = 4 \qquad M = M_o\, e^{kt}$$

Divide by 10

$$90 = 10\, e^{4k}$$

Take Ins

$$9 = e^{4k}$$

$$\ln 9 = 4k \ln e \quad \text{or} \quad k = \ln 9/4$$

But $e^{\ln 9} = 9$

$M = 10 \left(e^{(\ln 9/4)t} \right)$

Whew!!!!

sooo $\quad M = 10 \left(9^{t/4} \right)$

1B. $\quad t = 10$

$\quad M = 10 \left(9^{10/4} \right) = 10 \left(9^{5/2} \right) = 10 \left(3^5 \right) = 2430$ pounds.

1C. $\quad M = 500$ pounds. Solve for t. $500 = 10 \left(9^{t/4} \right)$

$$50 = 9^{t/4}$$

In $50 = (t/4)$ In 9. So $t = 4$ In $50/\ln 9 = 7.12$ hours, by calculator.

The simpler way to get part 1A. If you notice the numbers, you will see that 9 comes from 90/10. Although the time, 4 hours, originally is in the numerator, after the derivation, the 4 turns up in the bottom. Sooo, by observation

$M = $ Original times (Future amount/original)$^{t/time}$

$M = 10 \left(90/10 \right)^{t/4}; M = 10 \left(9^{t/4} \right)$

Let's try another one. Suppose 76 exponentially decays to 31 in 5 days.

The equation is $N = 76 \left(31/76 \right)^{t/5}$. Simple, isn't it?

Example 2—

Radioactive strontium 90 exponentially decays. Its half-life is 28 years. After an atomic attack, strontium 90 enters all higher life and is not safe until it decreases by a factor of 1000. How many years will it take strontium 90 to decay to safe levels after an atomic attack?

The equation, the short way, is $S = So \left(\frac{1}{2} \right)^{t/28}$. The $\frac{1}{2}$ is for the half-life, or half the amount of radioactivity.

We can let $So = 1000$ and $S = 1$ for a reduction factor of 1000.

$\quad 1 = 1000 \ \frac{1}{2}^{\ t/28}$

$.001 = (.5)^{t/28}$

ln $(.001) = (t/28)$ In $(.5)$. $t = 28$ In $(.001)/\ln (.5) = 279$ years to be safe.

We must truly be careful not to unleash nuclear bombs!!

Interest is also an exponential function. Simple interest = principle times rate times time. If $t = 1$ yr, $i = pr$ and the total amount $A = p + pr = p(1 + r)$. In other words, after 1 year, the principle is multiplied by $1 = pr$. After 2 years? $A = p (1 + r)^2$. After t years? $A = p (1 + r)^t$.

Suppose we have compounding interest twice a year, or half the interest rate $(r/2)$ but twice as many periods $(2t)$. $A = p (1 + r/2)^{2t}$. Compounded n times a year, the formula is $A = p (1 + r/n)^{nt}$.

Finally, if the interest is compounded continuously, $n \to \infty$, and $A = p \ e^{rt}$.

Note 1

If you use a bank with simple interest, go to another bank.

Note 2

For all intents and purposes, daily compounding is continuous compounding unless you have 10 billion dollars.

Note 3

For continuous compounding formula verification, look at L'Hopital's rule.

Let's try a problem.

Example 3—

Suppose we have $100,000 invested at 10 percent.

  A. How much would you have after 10 years, compounding yearly? continuously?

  B. When would you double your money, compounded yearly? continuously?

3A. $p = 100,000, r = .10, n = 1.$

    $A = p(1 + r)^n = 100,000(1 + .1)^{10} = \$259,374.25$

    $A = pe^{rt} = (100,000)e^{.1(10)} \qquad = \$271,828.18$

That's a difference of $12,453.93. Now all we need is the $100,000.

3B. $A = 200,000; p = 100,000; r = .10$

    $A = p(1 + r)^t; 200,000 = 100,000(1.1)^t;$
    $t = \ln 2/\ln (1.1) = 7.27$ years

    $A = pe^{rt}; 200,000 = 100,000\ e^{.1t}; t = \ln 2/.1 = 6.93$
    years

Continuous compounding does pay.

Chapter 5—
What You Should Know from Before To Do the Next

We have now come to the part of the book that requires you to work harder than perhaps at any time in the entire calculus sequence. We are about to embark on learning new, long integration techniques. Since the product and quotient rules do not hold for integrals, we are forced to learn many techniques, most of which are long.

In order to make these integrals shorter, we are listing some crucial facts from previous chapters. If you have properly learned them, this chapter will be much easier.

1. The definition of the six trig functions

2. The values of the six trig functions for multiples of 30, 45, 60, and 90 unless your instructor allows you to cheat and use calculators

3. The derivatives of the six trig functions

4. For the last time, the following identities:

A. $\sin x \csc x = 1$

B. $\cos x \sec x = 1$

C. $\tan x \cot x = 1$

D. $\tan x = \sin x/\cos x$

E. $\cot x = \cos x/\sin x$

F. $\sin^2 x + \cos^2 x = 1$

G. $1 + \tan^2 x = \sec^2 x$

H. $1 + \cot^2 x = \csc^2 x$

I. $\sin 2x = 2 \sin x \cos x$

Note

It is of interest to note that you really don't need to know cos 2x, as we will see shortly.

J. $\sin^2 x = \dfrac{1 - \cos 2x}{2}$

K. $\cos^2 x = \dfrac{1 + \cos 2x}{2}$

5. The beginning integrals

A. Integral of $x^n \quad n \neq -1$

B. Multiplying out

C. Dividing out

D. u-substitution in a parenthesis

E. u-substitution for a crazy angle

F. u-substitution for a crazy exponent

6. Trig integrals

A. $\int \sin ax\, dx = (1/a)(-\cos ax) + C$

B. $\int \cos ax\, dx = (1/a)(\sin ax) + C$

C. $\int \sec^2 ax\, dx = (1/a)(\tan ax) + C$

D. $\int \csc^2 ax\, dx = (1/a)(-\cot ax) + C$

E. $\int \tan ax \sec ax\, dx = (1/a) \sec ax + C$

F. $\int \cot ax \csc ax\, dx = -(1/a) \csc ax + C$

G. $\int \tan ax\, dx = -(1/a) \ln |\cos ax| + C$

$\quad\quad\quad$ or $(1/a) \ln |\sec ax| + C$

H. $\int \cot ax\, dx = (1/a) \ln |\sin ax| + C$

I. $\int \sec ax\, dx = (1/a) \ln |\sec ax + \tan ax| + C$

J. $\int \csc ax\, dx = -(1/a) \ln |\csc ax + \cot ax| + C$ or

$\quad\quad\quad = (1/a) \ln |\csc ax - \cot ax| + C$

7. *Definition*: certain values involving multiples of 30, 45, 60, and 90°, and derivatives of arc sin, arc tan, and arc sec

8. Inverse trig integrals

A. $\int \dfrac{dx}{(a^2 - x^2)^{1/2}} = \text{arc sin } (x/a) + C$

B. $\int \dfrac{dx}{a^2 + x^2} = (1/a) \text{ arc tan } (x/a) + C$

C. $\int \dfrac{dx}{x(x^2 - a^2)^{1/2}} = (1/a) \text{ arc sec } (x/a) + C$

9. Other integrals you should know:

A. $\int 1/x \, dx = \ln \, |x| + C$

B. $\int f'(x)/f(x) \, dx = \ln \, |f(x)| + C$

C. $\int \sinh ax \, dx = (1/a) \cosh ax + C$

D. $\int \cosh ax \, dx = (1/a) \sinh ax + C$

E. $\int e^{ax} \, dx = (1/a)e^{ax} + C$

F. $\int b^{ax} \, dx = \dfrac{b^{ax}}{a \ln b} + C$

It is quite a list, but, as you will see, all are needed.

Chapter 6—
Integration By Parts

As you will see, there is very little theory in this chapter—only *hard* work.

Integration by parts comes from the product rule for differentials, which is the same as the product rule for derivatives.

Let u and v be functions of x.

$$d(uv) = u\,dv + v\,du \quad \text{or} \quad u\,dv = d(uv) - v\,du$$

Integrating, we get

$$\int u\,dv = \int d(uv) - \int v\,du \quad \text{or}$$

$$\int u\,dv = uv - \int v\,du$$

What have we done? In the first integral, we have the function u and the differential of v. In the last integral, we have the differential of u and the function v. By reversing the roles of u and v, we hope to either have a very easy second integral or be allowed to proceed more easily to an answer.

Example 1—

$$\int xe^{3x}\,dx$$

If a polynomial multiplies e^{ax}, sin ax, and cos ax, we always let u = polynomial and $dv = e^{ax}\,dx$, sin ax dx, or cos ax dx. In this example,

$$u = x \quad dv = e^{3x}\,dx \quad v = \frac{e^{3x}}{3} \quad du = 1\,dx$$

$$\begin{array}{cccc} u & dv & u & v & v & du \end{array}$$

$$\int x\,e^{3x}\,dx = x\,\frac{e^{3x}}{3} - \int \frac{e^{3x}}{3}\,dx$$

$$= \frac{xe^{3x}}{3} - \frac{e^{3x}}{9} + C$$

Example 2—

$$\int x^4 e^{3x}\,dx$$

We must let u be a polynomial and $dv = e^{3x}\,dx$ 4 times!! However, if you observe the pattern, in time you may be able to do this in your head. Yes, I mean you. Signs alternate, polynomials get the derivative taken, a 3 is multiplied on the bottom each time, and e^{3x} multiplies each term.

The answer will be

$$\left(\frac{x^4}{3} - \frac{4x^3}{9} + \frac{12x^2}{27} - \frac{24x}{81} + \frac{24}{243} \right)e^{3x} + C.$$

Note

If we have $\int f(x) \cdot g(x)\, dx$ and f(x) is a polynomial and g(x) is e^{kx}, sin kx, or cos kx, we let u be a polynomial and dv = g(x), and we integrate by parts, of course.

Next we will consider integrating the arc sin, arc tan, and ln. If you had never seen them before, you probably would never guess that all are done by integration by parts, since there appears to be only one function.

However, mathematicians, being clever little devils, invented a second function so that all three of these integrals are rather easily done.

Example 3—

$$\int \sin^{-1} dx$$

Let $u = \sin^{-1} x$ $du = \dfrac{1\, dx}{(1-x^2)^{1/2}}$

Let $dv = 1\, dx$!! $v = x$

$$\begin{array}{cccccc} u & dv & v & u & v & du \end{array}$$

$$\int \sin^{-1} x\, dx = \int \sin^{-1} x\, dx = x \sin^{-1} x - \int x\, \dfrac{1\, dx}{(1-x^2)^{1/2}}$$

$$= x \sin^{-1} x + \int \dfrac{-x\, dx}{(1-x^2)^{1/2}}$$

$$w = 1 - x^2;\ dw = -2x\, dx$$

$$= x \sin^{-1} x + \dfrac{1}{2} \int \dfrac{-2x\, dx}{(1-x^2)^{1/2}}$$

$$= x \sin^{-1} x + \dfrac{1}{2} \int w^{-1/2}\, dw$$

$$= x \sin^{-1} x + \dfrac{1}{2} \dfrac{w^{1/2}}{1/2} + C$$

$$= x \sin^{-1} x + w^{1/2} + C$$

$$= x \sin^{-1} x + (1-x^2)^{1/2} + C$$

Note 1

If we have $\int f(x) \cdot g(x)\, dx$, f(x) = a polynomial or is not there (= 1), and g(x) = ln x or $\sin^{-1}$ x or $\tan^{-1}$ x or $\sec^{-1}$ x, we let dv be a polynomial or i and u = g(x). Integrate by parts.

Note 2

Although Example 3 is relatively short, some of these are verrrry long and use techniques we will learn later in this chapter.

We will now do a more complicated problem, e^{5x} cos 3x dx. Based on what we did before, we can take either function as u and the rest as dv. It turns out both will work. However, the problem is not quite so easy, as we will see. Being a glutton for punishment, I will show that the problem can be done two ways.

Example 4—

$$\int e^{5x} \cos 3x \, dx; \quad u = e^{5x}; \quad du = 5e^{5x} \, dx; \quad dv = \cos 3x;$$

$$v = (\sin 3x)/3.$$

$$\int \overset{u}{e^{5x}} \overset{dv}{\cos 3x} \, dx = \frac{\overset{u}{e^{5x}} \overset{v}{\sin 3x}}{3} - \int \frac{\overset{v}{\sin 3x}}{3} \overset{du}{(5e^{5x})} \, dx$$

$$= \frac{e^{5x} \sin 3x}{3} - (5/3) \int e^{5x} \sin 3x \, dx$$

At this point you might say, "This doesn't do anything for us." You'd be right. Let's do it again. We let $U = e^{5x}$ because, if we reversed, we would wind up with the original integral and would have accomplished nothing. dV $= \sin 3x \, dx$. V $= (-\cos 3x)/3$. (Note that, in the third line, the product of three minus signs is a minus.)

$$\int e^{5x} \cos 3x \, dx = \frac{e^{5x} \sin 3x}{3} - (5/3) \int \overset{U}{e^{5x}} \overset{dV}{\sin 3x} \, dx$$

$$= \frac{e^{5x} \sin 3x}{3} - (5/3) \overset{U}{e^{5x}} \frac{\overset{V}{(-\cos 3x)}}{3}$$

$$- \frac{5}{3} \int \overset{dU}{5e^{5x}} \frac{\overset{V}{\cos 3x}}{3} \, dx$$

$$\int e^{5x} \cos 3x \, dx = \frac{e^{5x} \sin 3x}{3} + \frac{5e^{5x} \cos 3x}{9}$$

$$- (25/9) \int e^{5x} \cos 3x \, dx$$

It looks like we will be going forever. However, notice that the original integral and the last integral are the same except for a constant. Call the original integral I (for integral, of course). The last line becomes

$$I = \frac{e^{5x} \sin 3x}{3} + \frac{5e^{5x} \cos 3x}{9} - (25/9)I$$

now $I = (9/9)I$

$$(34/9)I = \frac{e^{5x} \sin 3x}{3} + \frac{5e^{5x} \cos 3x}{9}$$

so

$$I = \int e^{5x} \cos 3x \, dx = (9/34) \left(\frac{e^{5x} \sin 3x}{3} + \frac{5e^{5x} \cos 3x}{9} \right) + C$$

$$= \frac{3e^{5x} \sin 3x}{34} + \frac{5e^{5x} \cos 3x}{34} + C$$

Note

You do not have to multiply out the last step, but I wanted to show you that doing the problem two ways gives the same answer. Also note that you do not have to do the problem two ways, and I am a little crazy to try.

$\int e^{5x} \cos 3x \, dx; \; dv = e^{5x}; \; v = e^{5x}/5;$

$u = \cos 3x; \; du = -3 \sin 3x \, dx$

$$\overset{u \quad dv}{\int \cos 3x \, e^{5x} \, dx} = \overset{u \quad v}{\frac{\cos 3x e^{5x}}{5}} - \int \overset{v}{\frac{e^{5x}}{5}} \overset{du}{(-3 \sin 3x) \, dx}$$

$$\overset{dv \quad u}{\int e^{5x} \cos 3x \, dx} = \frac{\cos 3x e^{5x}}{5} + (3/5) \int \overset{dV \quad U}{e^{5x} \sin 3x \, dx}$$

$$dV = e^{5x}; \; V = e^{5x}/5; \; U = \sin 3x; \; dU = 3 \cos 3x \, dx$$

$$\int e^{5x} \cos 3x \, dx = \frac{e^{5x} \cos 3x}{5} + \frac{(3/5)e^{5x} \sin 3x}{5} - (3/5) \int \overset{V \quad dU}{\frac{e^{5x} \, 3 \cos 3x \, dx}{5}}$$

$$= \frac{e^{5x} \cos 3x}{5} + \frac{3e^{5x} \sin 3x}{25} - (9/25) \int e^{5x} \cos 3x \, dx$$

$$I = \frac{e^{5x} \cos 3x}{5} + \frac{3e^{5x} \sin 3x}{25} - (9/25)I \qquad I = (25/25)I$$

$$(34/25)I = \frac{e^{5x} \cos 3x}{5} + \frac{3e^{5x} \sin 3x}{25}$$

$$I = \int e^{5x} \cos 3x \, dx = (25/34)\left(\frac{e^{5x} \cos 3x}{5} + \frac{3e^{5x} \sin 3x}{25}\right) + C$$

$$= \frac{5e^{5x} \cos 3x}{34} + \frac{3e^{5x} \sin 3x}{34} + C$$

Our two answers check. Now that I've done it two ways to show you that both ways give the same answer, I will never, never do it twice again!!!!!!!!

The last integration by parts, unless I think of another, is the integral of $\sec^3 x$. I think it more properly belongs later (Example 10).

The next section involves integrals of trig functions. It is absolutely essential that you know the trig identities and integrals we listed before.

Let's consider integrals of the form $\sin^m x \cos^n x$.

Example 5—

$\int \sin^6 x \cos^3 x \, dx$ when m or n is odd.

The technique is to break off the trig function that is to an odd power (if both are, break off the one that is to the lower degree), and write all the others using the identity $\sin^2 x + \cos^2 x = 1$.

$$\int \sin^6 x \cos^3 x \, dx = \int (\sin^6 x)(1 - \sin^2 x)(\cos x) \, dx$$

$$u = \sin x; \; du = \cos x \, dx$$

$$= \int u^6(1 - u^2) \, du = \int (u^6 - u^8) \, du$$

$$= \frac{u^7}{7} - \frac{u^9}{9} + C$$

so $\int \sin^6 x \cos^3 x \, dx = \dfrac{\sin^7 x}{7} - \dfrac{\sin^9 x}{9} + C$

Pretty simple, eh? However, when m and n are to an even power, the integrals are usually much longer.

Example 6—

$\int \sin^6 x \, dx; \; m = 6$ and $n = 0$ with both exponents even.

In doing these problems, we will use the half-angle formulas:

$$\sin^2 x = (1 - \cos 2x)/2 \quad \text{and} \quad \cos^2 x = (1 + \cos 2x)/2$$

$$\int \sin^6 x \, dx = \int (\sin^2 x)(\sin^2 x)(\sin^2 x) \, dx$$

$$= \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 - \cos 2x}{2}\right) dx$$

$$= \frac{1}{8} \int dx - \frac{3}{8} \int \cos 2x \, dx + \frac{3}{8} \int \cos^2 2x \, dx$$

$$- \frac{1}{8} \int \cos^3 2x \, dx$$

$$= \frac{1}{8} x - \frac{3 \sin 2x}{16} + A + B$$

Integral
$$A = \frac{3}{8} \int \cos^2 2x \, dx = \frac{3}{8} \int \frac{1 + \cos 4x}{2} \, dx$$

$$= \frac{3}{16} \int 1 \cdot dx + \frac{3}{16} \int \cos 4x \, dx$$

$$= \frac{3}{16} x + \frac{3 \sin 4x}{64}$$

Integral
$$B = -\frac{1}{8} \int \cos^3 2x \, dx$$

$$= -\frac{1}{8} \int (1 - \sin^2 2x)(\cos 2x) \, dx$$

$$u = \sin 2x; \; du = 2 \cos 2x \, dx$$

$$= -\frac{1}{16} \int (1 - \sin^2 2x)(2 \cos 2x) \, dx$$

$$= -\frac{1}{16} \int (1 - u^2) \, du = -\frac{1}{16} \left(u - \frac{u^3}{3} \right)$$

$$= -\frac{1}{16} \left(\sin 2x - \frac{\sin^3 2x}{3} \right)$$

$$= -\frac{\sin 2x}{16} + \frac{\sin^3 2x}{48}$$

Combining all parts of the integral, we get

$$\int \sin^6 x \, dx = \frac{x}{8} - \frac{3 \sin 2x}{16} + \frac{3x}{16} + \frac{3 \sin 4x}{64}$$

$$- \frac{\sin 2x}{16} + \frac{\sin^3 2x}{48} + C$$

$$= \frac{5x}{16} - \frac{\sin 2x}{4} + \frac{\sin^3 2x}{48} + \frac{3 \sin 4x}{64} + C$$

Quite a problem. It certainly is much nicer if the sin or cos has an odd exponent.

We now examine the integrals involving $\tan^m x \sec^n x$. Before we start, we will make two notes: (1) whatever we say for tan-sec goes for cot-csc, and (2) tan-sec and cot-csc are grouped together whether for trig identities or trig integrals.

Example 7—

$$\int \tan^5 x \sec^5 x \, dx$$

m and n are odd; u = sec x; du = tan x sec x dx. Break off one tan and one sec. Write each remaining $\tan^2 x$ as $\sec^2 x - 1$.

$$\int \tan^5 x \sec^5 x \, dx = \int (\sec^2 x - 1)(\sec^2 x - 1)(\sec^4 x)(\tan x \sec x) \, dx$$

$$= \int (u^2 - 1)(u^2 - 1)(u^4) \, du$$

$$= \int (u^8 - 2u^6 + u^4) \, du = \frac{u^9}{9} - \frac{2u^7}{7} + \frac{u^5}{5} + C$$

$$= \frac{\sec^9 x}{9} - \frac{2 \sec^7 x}{7} + \frac{\sec^5 x}{5} + C$$

Example 8—

$$\int \sec^4 x \, dx$$

where m and n are even. This is one of my favorites. If m and n are even, and, in this problem, even though there is no tangent, we let u = tan x, du = $\sec^2 x$ dx, break off two of the secants, and write the remaining $\sec^2 x$ as $\tan^2 x + 1$.

$$\int \sec^4 x \, dx = \int (1 + \tan^2 x)(\sec^2 x) \, dx$$

$$u = \tan x; \, du = \sec^2 x \, dx$$

$$= \int (1 + u^2) \, du = u + u^3/3 + C$$

$$= \tan x + (\tan^3 x)/3 + C$$

Example 9—

$$\int \tan^3 x \sec^4 x \, dx$$

where m (power of tan) is odd and n (power of sec) is even. We can let u = tan x or v = sec x. We will do it both ways. Neither is too bad.

Example 9A—

$$\int \tan^3 x \sec^4 x \, dx \qquad u = \tan x, \, du = \sec^2 x \, dx$$

$$= \int (\tan^3 x)(\tan^2 x + 1) \sec^2 x \, dx$$

$$= \int u^3(u^2 + 1) \, du = \int (u^5 + u^3) \, du = \frac{u^6}{6} + \frac{u^4}{4} + C$$

$$= \frac{\tan^6 x}{6} + \frac{\tan^4 x}{4} + C$$

Example 9B—

$$\int \tan^3 x \sec^4 x \, dx \qquad v = \sec x; \, dv = \tan x \sec x \, dx$$

$$= \int (\sec^2 x - 1)(\sec^3 x)(\tan x \sec x) \, dx$$

$$= \int (v^2 - 1)(v^3) \, dv = \int (v^5 - v^3) \, dv = \frac{v^6}{6} - \frac{v^4}{4} + C$$

$$= \frac{\sec^6 x}{6} - \frac{\sec^4 x}{4} + C$$

You might try to show that the answers to 9A and 9B are the same using the identity $\sec^2 x = \tan^2 x + 1$.

Example 10—

$$\int \sec^3 x \, dx$$

This is the worst case: m, the power of tan x, is even— m = 0—and n, the power of the sec x, is odd—n = 3. All cases where m is even and n is odd are done by integrating by parts. They get long fast as the powers of m and n increase, and all involve the same tricks.

$$\int \sec^3 x \, dx = \int \overset{u}{\sec x} \, \overset{dv}{\sec^2 x \, dx} = \overset{u}{\sec x} \, \overset{v}{\tan x}$$

$$- \int \overset{v}{(\tan x)} \, \overset{du}{\tan x \sec x \, dx}$$

$$= \sec x \tan x - \int [(\sec^2 x - 1) \sec x] \, dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \ln |\sec x + \tan x|$$

Solving for the unknown integral, we get

$$\int \sec^3 x \, dx = \frac{\sec x \tan x + \ln |\sec x + \tan x|}{2} + C$$

Example 11—

$$\int \tan^2 x \csc x \, dx$$

Anytime you have a mixed integral, that is, where the tan is not with the sec, you will have to use trig identities and usually tricks and sometimes long problems involving techniques that may not have been done here yet. The one I've given is a rather mild one.

$$\int \tan^2 x \csc x \, dx = \int \frac{\sin^2 x}{\cos^2 x} \frac{1}{\sin x} \, dx$$

$$u = \cos x; \, du = -\sin x \, dx$$

$$= \int \frac{-du}{u^2} = \int -u^{-2} \, du$$

$$= 1/u = 1/\cos x + C = \sec x + C$$

We now have integrals involving square roots. Our goal is to get rid of the radicals. The first area here is trig substitutions.

Type 1

$(a^2 - x^2)^{1/2}$. We use $x = a \sin u$ ($dx = a \cos u \, du$).

$$(a^2 - x^2)^{1/2} = (a^2 - a^2 \sin^2 u)^{1/2} = [a^2(1 - \sin^2 u)]^{1/2} = a \cos u$$

and the square root is gone. Here are the other two cases.

Type 2

$(a^2 + x^2)^{1/2}$; we use $x = a \tan u$ ($dx = a \sec^2 u \, du$)

$$(a^2 + x^2)^{1/2} = (a^2 + a^2 \tan^2 u)^{1/2} = [a^2(1 + \tan^2 a)]^{1/2} = a \sec u$$

Type 3

$(x^2 - a^2)^{1/2}$; we use $x = a \sec u$ ($dx = a \tan u \sec u \, du$)

$$(x^2 - a^2)^{1/2} = (a^2 \sec^2 u - a^2)^{1/2} = [a^2(\sec^2 u - 1)]^{1/2} = a \tan u$$

I have demonstrated each of the three types. However, it is essential that you know by sight what the answer is without substituting. Otherwise, the problems will take forever. If I wake you up in the middle of the night and ask you, "What do you get if you have $(7 - x^2)^{1/2}$?" You should instantly say, "Square root of 7 cosine u—now let me go back to sleep!!!!"

Example 12—
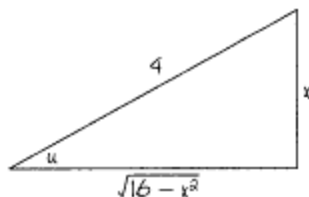
$$\int \frac{dx}{x^2(16 - x^2)^{1/2}} \qquad x = 4 \sin u; \, dx = 4 \cos u \, du$$

$$= \int \frac{4 \cos u \, du}{(16 \sin^2 u)(4 \cos u)}$$

$$= \int \frac{du}{16 \sin^2 u} = \int \frac{\csc^2 u \, du}{16} = \frac{-\cot u}{16} + C$$

We didn't start with u; we started with x. We must draw a triangle with $x = 4 \sin u$. $\sin u = x/4$. Note the missing side will be what the square root is.

$$\frac{-\cot u}{16} + C = -\frac{1}{16} \left(\frac{\sqrt{16 - x^2}}{x} \right) + C$$

Example 13—

$$\int \frac{x^2\,dx}{(16-x^2)^{1/2}}$$

This problem appears to be exactly the same as the last except the x^2 is on top instead of the bottom. This problem is given to show that the techniques are different, even in problems that look the same—some longer, some shorter, some easier, some harder. The kind of problem is known only after lots of study. Do them and hope they are short and easy.

Again let x = 4 sin u. dx = 4 cos u du.

$$\int \frac{x^2\,dx}{(16-x^2)^{1/2}}$$

$$= \int \frac{(16\sin^2 u)(4\cos u)\,du}{4\cos u} = \int 16\sin^2 u\,du$$

$$= \int \frac{16(1-\cos 2u)\,du}{2}$$

$$= \int 8\,du - \int 8\cos 2u\,du$$

$$= 8u - 4\sin 2u$$

$$= 8u - 8\sin u\cos u$$

$$= 8\ \text{arc}\ \sin(x/4) - 8\left[\frac{x}{4}\ \frac{(16-x^2)^{1/2}}{4}\right] + C$$

Notes

$x = 4\sin u, \qquad x/4 = \sin u, \qquad \sin^{-1}(x/4) = u$

$-4\sin 2u = -4(2\sin u\cos u) = -8\sin u\cos u$

$\dfrac{x}{4} = \sin u \qquad$ so $\qquad \dfrac{\sqrt{16-x^2}}{4} = \cos u$

As you can see, these two problems are quite different, although they look basically the same.

We will finish by showing that the area of a circle really is πr^2. We will find one-quarter the area of the circle $x^2 + y^2 = r^2$ and then multiply it by 4.

Example 14—

---

x = r sin u; dx = r cos u du.

$$4\int_0^r (r^2-x^2)^{1/2}\,dx$$

---

x = r, r = r sin u, 1 = sin u, u = π/2; x = 0, 0 = r sin u, 0 = sin u, u = 0.

$$= 4 \int_0^{\pi/2} (r \cos u)(r \cos u)\, du$$
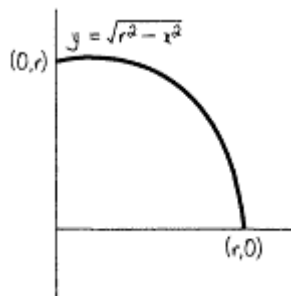
$$= 4r^2 \int_0^{\pi/2} \cos^2 u\, du$$

$$= 4r^2 \int_0^{\pi/2} \frac{(1 + \cos 2u)\, du}{2}$$

$$= 2r^2 \int_0^{\pi/2} (1 + \cos 2u)\, du$$

$$= 2r^2(u + \sin 2u)\Big[\,_0^{\pi/2} = 2r^2\left[\left(\frac{\pi}{2} + \sin \pi\right) - (0 - \sin 0)\right]$$

$$= 2r^2(\pi/2) = \pi r^2$$

The area of a circle really *is* πr^2, and you haven't been lied to all these years. It's nice to know.

Example 15—

$$x = 5 \tan u; \quad dx = 5 \sec^2 u\, du.$$
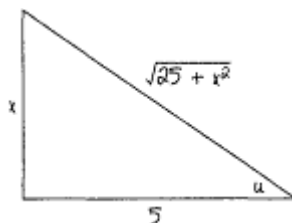
$$\int \frac{dx}{(25 + x^2)^{1/2}}$$

$$= \int \frac{5 \sec^2 u\, du}{5 \sec u}$$

$$= \int \sec u\, du$$

$$= \ln |\sec u + \tan u| + C$$

$$= \ln \left| \frac{\sqrt{25 + x^2}}{5} + \frac{x}{5} \right| + C$$

Example 16—

$$x = 7^{1/2} \sec u; \quad dx = 7^{1/2} \tan u \sec u\, du.$$
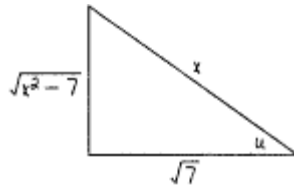
$$\int \frac{dx}{x^2(x^2 - 7)^{1/2}}$$

$$= \int \frac{7^{1/2} \tan u \sec u \, du}{(7 \sec^2 u)(7^{1/2} \tan u)}$$

$$= \int \frac{du}{7 \sec u} = \int \frac{\cos u \, du}{7}$$

$$= \frac{1}{7} \sin u = \frac{(x^2 - 7)^{1/2}}{7x} + C$$

Example 17—

Sometimes you use a trig substitution even though there is no square root; x = 3 tan u. dx = 3 sec^2 u du.
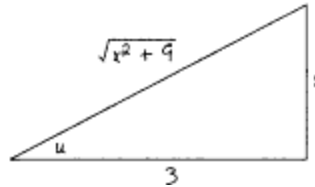
$$\int \frac{dx}{(x^2 + 9)^2}$$

$$= \int \frac{3 \sec^2 u \, du}{81 \sec^4 u} = \int \frac{\cos^2 u \, du}{27} = \int \frac{(1 + \cos 2u) \, du}{54}$$

$$= (1/54)u + (\sin 2u)/108$$

$$= (1/54)u + (\sin u \cos u)/54 + C$$

$$= \frac{1}{54} \tan^{-1} \frac{x}{3} + \frac{1}{54} \left(\frac{x}{\sqrt{x^2 + 9}} \cdot \frac{3}{\sqrt{x^2 + 9}} \right) + C$$

$$= \frac{1}{54} \tan^{-1} \frac{x}{3} + \frac{x}{18(x^2 + 9)} + C$$

One last example, but be warned! A trig substitution may work, but another method may be a lot easier.

Example 18—

---

x = 3 tan u will work. But you may not finish for two years. The best? u = x^2 + 9; du = 2x dx.

$$\int \frac{2x \, dx}{(x^2 + 9)^{100}}$$

$$= \int u^{-100} \, du = \frac{u^{-99}}{-99} + C = \frac{-1}{99(x^2 + 9)^{99}} + C$$

Another group of integrals are related to the last group. They can be very involved, but we will do two moderate ones.

Example 19—

	We do this by completing the square: $3x^2 - 18x + 75 = 3(x^2 - 6x + 25) = 3(x^2 - 6x + 9 + 16) = 3[(x - 3)^2 + 16]$.

$$\int \frac{(2x - 3) \, dx}{3x^2 - 18x + 75}$$

	$u = x - 3$; $x = u + 3$; $2x - 3 = 2(u + 3) - 3 = 2u + 3$; $du = dx$.

$$= \int \frac{(2x - 3) \, dx}{3[(x - 3)^2 + 16]}$$

	Now split the integral.

$$= \frac{1}{3} \int \frac{(2u + 3) \, du}{u^2 + 16}$$

	Both of these integrals should be known by sight!

$$= \frac{1}{3} \int \frac{2u \, du}{u^2 + 16} + \int \frac{1 \, du}{u^2 + 16}$$

$$= \frac{1}{3} \ln (u^2 + 16) + \frac{1}{4} \tan^{-1} (u/4) + C$$

$$= \frac{1}{3} \ln (x^2 - 6x + 25) + \frac{1}{4} \tan^{-1} [(x - 3)/4] + C$$

since $u = x - 3$ and $u^2 + 16 = (x - 3)^2 + 16 = x^2 - 6x + 9 + 16 = x^2 - 6x + 25$.

Example 20—

	Again we complete the square: $15 + 2x - x^2 = -1(x^2 - 2x) + 15 = -1(x^2 - 2x + 1) + 16 = 16 - (x - 1)^2$.

$$\int \frac{dx}{(15 + 2x - x^2)^{1/2}}$$

	Again you should be able to tell this is an arc sin; that is, $\int dx/(a^2 - u^2)^{1/2}$ where $a = 4$ and $u = x - 1$.

$$= \int \frac{dx}{[16 - (x - 1)^2]^{1/2}}$$

$$= \text{arc sin } [(x - 1)/4] + C$$

We now do the section I like the least. It is uninteresting, unimaginative, frequently overly long, and ... necessary. Unless we have only linear factors, it is best to avoid this technique if possible.

We wish to do the integrals by partial fractions. Suppose we have $R(x) = P(x)/Q(x)$, where the degree of $P(x)$ is less than the degree of $Q(x)$. We wish to break up $R(x)$ into simpler rational fractions; each piece is called a *partial fraction*. There will be one or more pieces for each linear factor $x + a$ or quadratic factor $x^2 + b^2$ of $Q(x)$. Here's how it looks in a particular case:

$$\frac{P(x)}{x^2(x-3)^3(x^2+4)^2} = \underbrace{\frac{A}{x} + \frac{B}{x^2}}_{2 \text{ from } x} + \underbrace{\frac{C}{x-3} + \frac{D}{(x-3)^2} + \frac{E}{(x-3)^3}}_{3 \text{ from } (x-3)}$$

$$+ \underbrace{\frac{Fx+G}{x^2+4} + \frac{Hx+I}{(x^2+4)^2}}_{2 \text{ from } (x^2+4)}$$

Notice that each linear factor gives pieces with constants on top, and each quadratic factor gives pieces with first-degree polynomials on top. The bottoms of the partial fractions are powers of the factors running from 1 to the power that occurs in $Q(x)$. The constants A, B, C, D, E, F, G, H, and I have to be solved for, which I hope you never have to do. If you added all the fractions on the right, you would get the left fraction.

One more thing. Suppose $Ax^3 + Bx^2 + Cx + D = 4x^3 - 7x - 1$. Two polynomials are equal if their coefficients match. So, $A = 4$, $B = 0$, $C = -7$, $D = -1$.

There are a number of techniques that will allow you to solve for A, B, C, and so on. Two of them (combinations of) will serve us best.

Example 21—

> Since the degree of the top is greater than or equal to the degree of the bottom, long-divide the bottom into the top until the degree of the top is less than the degree of the bottom.

$$\int \frac{x^3 - 7x + 18}{x^2 - 9} \, dx$$

> Look at the fractional part only.

$$= \int \left(x + \frac{2x + 18}{x^2 - 9} \right) dx$$

> We will solve for A and B in two different ways. We now add the fractions and equate the tops since the bottoms are the same.

$$\frac{2x + 18}{x^2 - 9} = \frac{A}{x - 3} + \frac{B}{x + 3}$$

$$2x + 18 = A(x + 3) + B(x - 3)$$

Method 1

> Multiply out left side and group terms.

$$A(x + 3) + B(x - 3) = 2x + 18$$

$$(A + B)x + (3A - 3B) = 2x + 18$$

$$A + B = 2$$

$$3A - 3B = 18$$

$$3A + 3B = 6$$

$$3A - 3B = 18$$

$$6A = 24$$

$$A = 4$$

$$B = -2$$

Method 2

$$A(x + 3) + B(x - 3) = 2x + 18$$

If x = 3, A(3 + 3) + B(3 - 3) = 2(3) + 18; 6A = 24; A = 4. If x = -3, A(-3 + 3) + B(-3 - 3)= 2(-3) + 18; -6B = 12; B = -2.

This way is so much easier—why don't we always use it? It is only perfect if we have all linear factors to the first power. Otherwise, it will not totally work. If there are no linear factors, you can't use this method. That is why both methods are needed. Let us finally finish the problem!

$$\int \frac{x^3 - 7x + 18 \, dx}{x^2 - 9} = \int x \, dx + \int \frac{4 \, dx}{x - 3} + \int \frac{-2 \, dx}{x + 3}$$

$$= \frac{x^2}{2} + 4 \ln |x - 3| - 2 \ln |x + 3| + C$$

Note how easy the calculus part is. The algebra can be overwhelming.

Example 22—

$$\int \frac{9x^2 - 5x + 19 \, dx}{(x^2 + 5)(x - 2)}$$

Notice the degree of the top (2) is less than the degree of the bottom (3), so long division is not necessary. The bottom is already factored. There is one quadratic factor and one linear factor. The form is

$$\frac{Ax + B}{x^2 + 5} + \frac{C}{x - 2}$$

$$\frac{9x^2 - 5x + 19}{(x^2 + 5)(x - 2)} = \frac{Ax + B}{x^2 + 5} + \frac{C}{x - 2}$$

$$= \frac{(Ax + B)(x - 2) + C(x^2 + 5)}{(x^2 + 5)(x - 2)}$$

We now multiply out the top on the right and set the left numerator equal to the right numerator.

$$9x^2 - 5x + 19 = (A + C)x^2 + (-2A + B)x + (-2B + 5C)$$

$$A + C = 9 \tag{1}$$

$$-2A + B = -5 \tag{2}$$

$$-2B + 5C = 19 \tag{3}$$

These three equations in three unknowns are not bad but not particularly nice to solve. So we can use the other technique. Going back to the original top on the right, we have $(Ax + B)(x - 2) + C(x^2 + 5) = 9x^2 - 5x + 19$. There is only one linear factor, $x - 2$, but it is enough. Substituting $x = 2$ in this equation, we get $[A(2) + B](2 - 2) + C(2^2 + 5) = 9(2)^2 - 5(2) + 19$. From this we get $9C = 45$ or $C = 5$. Substituting $C = 5$ into Eq. (1), we get $A = 4$. Substituting $A = 4$ into Eq. (2), we get $B = 3$.

$$\int \frac{9x^2 - 5x + 19}{(x^2 + 5)(x - 2)}\, dx = \int \left(\frac{4x + 3}{x^2 + 5} + \frac{5}{x - 2} \right) dx$$

Splitting the first fraction on the right,

$$\int \frac{9x^2 - 5x + 19}{(x^2 + 5)(x - 2)}\, dx = 2\int \frac{2x\, dx}{x^2 + 5} + \int \frac{3\, dx}{x^2 + 5} + \int \frac{5\, dx}{x - 2}$$

$$= 2 \ln (x^2 + 5) + (3/5^{1/2})\, \text{arc tan}\, (x/5^{1/2})$$

$$+ 5 \ln |x - 2| + C$$

Example 23—

$$\int \frac{(6x^2 + 17x - 5)\, dx}{(x - 3)(x + 2)^2}$$

We have two linear factors, and 1 is to the second power—soooo .....

$$\frac{6x^2 + 17x - 5}{(x - 3)(x + 2)^2} = \frac{A}{x - 3} + \frac{B}{x + 2} + \frac{C}{(x + 2)^2}$$

$$= \frac{A(x + 2)^2 + B(x + 2)(x - 3) + C(x - 3)}{(x - 3)(x + 2)^2}$$

$$6x^2 + 17x - 5 = A(x + 2)^2 + B(x + 2)(x - 3) + C(x - 3) \tag{1}$$

Multiply and group; we get:

$$= A(x^2 + 4x + 4) + B(x^2 - x - 6) + C(x - 3) \quad (2)$$

$$A + B = 6 \qquad\qquad\qquad\qquad\qquad\qquad\qquad (3)$$

$$4A - B + C = 17 \qquad\qquad\qquad\qquad\qquad\qquad (4)$$

$$4A - 6B - 3C = -5 \qquad\qquad\qquad\qquad\qquad (5)$$

Now there are two good numbers, -2 and 3, but, as we will see, 3 is enough. Putting x = 3 into both sides of Eq. (1), we get A = 4. Putting A = 4 into Eq. (3), we get B = 2. Putting B = 2 and A = 4 into Eqs. (4) or (5), we get C = 3.

$$\int \frac{6x^2 + 17x - 5\ dx}{(x+2)^2(x-3)} = \int \frac{4\ dx}{x-3} + \int \frac{2\ dx}{x+2} + \int \frac{3\ dx}{(x+2)^2}$$

$$= 4\ \ln\ |x - 3| + 2\ \ln\ |x + 2| - 3/(x + 2) + C$$

The last part of this long integral chapter is called—

Miscellaneous

Miscellaneous means anything that doesn't fit into any other part. So all the extra goes into this part, which makes it more miserable for you.

Example 24—

$$\int x(x + 1)^{80}\ dx$$

Sometimes the simplest substitutions work. We let u = x + 1. du = dx. This transfers the power to the monomial and allows us to multiply out the expression (x = u - 1).

$$\int x(x + 1)^{80}\ dx = \int (u - 1)u^{80}\ du = \int (u^{81} - u^{80})\ du$$

$$= (u^{82})/82 - (u^{81})/81 + C$$

$$= \frac{(x + 1)^{82}}{82} - \frac{(x + 1)^{81}}{81} + C$$

You sharp-eyed readers will note that there are at least two other ways to do this problem. The first is to multiply out the original. This is dismissed on grounds of sanity. The second is by parts.

Example 25—

$$\int \frac{dx}{x^{1/2} - x^{1/3}}$$

This one is perhaps the easiest to identify. LCD for 1/2, 1/3 is 6. So, we let $u = x^{1/6}$; $u^2 = (x^{1/6})^2 = x^{1/3}$; $u^3 = (x^{1/6})^3 = x^{1/2}$; and $u^6 = x$. So $6u^5\ du = dx$. Substituting, we get ...

$$\int \frac{dx}{x^{1/2} - x^{1/3}} = \int \frac{6u^5\ du}{u^3 - u^2} = 6 \int \frac{u^5\ du}{u^2(u - 1)}$$

$$\begin{array}{r}
u^2 + u + 1 + 1/(u-1) \\[2pt]
\hline
u - 1 \,)\overline{u^3} \\
\underline{u^3 - u^2} \\
u^2 \\
\underline{u^2 - u} \\
u \\
\underline{u - 1} \\
1
\end{array}$$

$$= 6 \int \frac{u^3\,du}{u-1}$$

$$= 6 \int [u^2 + u + 1 + 1/(u-1)]\,du$$

$$= 6(u^3/3 + u^2/2 + u + \ln|u - 1|) + C$$

$$= 2(x^{1/6})^3 + 3(x^{1/6})^2 + 6x^{1/6} + \ln|x^{1/6} - 1| + C$$

$$= 2x^{1/2} + 3x^{1/3} + 6x^{1/6} + \ln|x^{1/6} - 1| + C$$

Note the nice pattern of the coefficients and the exponents of this answer. Well, *I* like it!

Example 26—

$$\int \frac{x^3\,dx}{(x^2+4)^{1/2}}$$

This integral can be done in two new ways, both of which are useful.

Method A

Let u = the whole radical.

$u = (x^2 + 4)^{1/2}$; $u^2 = u^2 + 4$; $x^2 = u^2 - 4$; $2x\,dx = 2u\,du$; or $x\,dx = u\,du$. Substituting,

$$\int \frac{x^3\,dx}{(x^2+4)^{1/2}} = \int \frac{x^2x\,dx}{(x^2+4)^{1/2}} = \int \frac{(u^2-4)u\,du}{u}$$

$$= \int (u^2 - 4)\,du = u^3/3 - 4u + C$$

$$= \frac{(x^2+4)^{3/2}}{3} - 4(x^2+4)^{1/2} + C$$

Method B

Let v = what's under the radical sign.

$v = x^2 + 4$; $x^2 = v - 4$; and $2x\,dx = dv$. Substituting,

$$\int \frac{x^3\,dx}{(x^2+4)^{1/2}} = \frac{1}{2}\int \frac{x^2 2x\,dx}{(x^2+4)^{1/2}} = \frac{1}{2}\int \frac{(v-4)\,dv}{v^{1/2}}$$

$$= \frac{1}{2}\int (v^{1/2}-4v^{-1/2})\,dv = \frac{1}{2}[(2/3)v^{3/2}-8v^{1/2}]+C$$

$$= (1/3)(x^2+4)^{3/2}-4(x^2+4)^{1/2}+C$$

The sharp-eyed reader will discover there are many, many other ways to do this problem. When a publisher becomes smart enough to publish this book, this problem will become a contest.
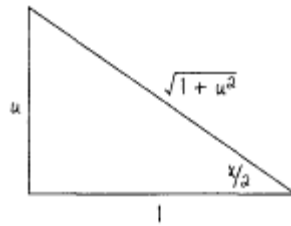
Here is the weirdest miscellaneous item of all! If we have sin x or cos x in the denominator, and if they are added or subtracted to each other (with one being multiplied by a number or being added or subtracted by itself from a number), then the substitution is u = tan (x/2)!!!!!!!! I don't know who discovered it or why, but it works.

Let us derive all the parts. Otherwise you would never believe it. Let tan (x/2) = u = u/1. Draw the triangle for x/2. We get sin (x/2) = u/(1 + u²)^{1/2} and cos (x/2)= 1/(1 + u²)^{1/2}.
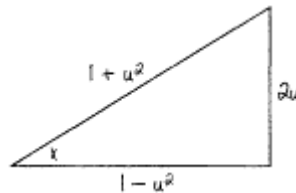
$\sin x = 2 \sin (x/2) \cos (x/2)$

$$= 2\frac{u}{(1+u^2)^{1/2}}\frac{1}{(1+u^2)^{1/2}} = \frac{2u}{1+u^2}$$

Draw triangle for angle x.

$$\cos x = \frac{1-u^2}{1+u^2}$$

Finally if u = tan (x/2), then tan⁻¹ u = x/2. Taking differentials, we get

$$\frac{2\,du}{1+u^2} = dx$$

In summary,

$$u = \tan (x/2) \qquad \frac{2\,du}{(1+u^2)} = dx \qquad \sin x = \frac{2u}{1+u^2}$$

$$\cos x = \frac{1-u^2}{1+u^2}$$

Example 27—

$$\int \frac{1 + \sin x}{1 + \cos x} \, dx$$

$$= \int \frac{1 + \dfrac{2u}{1 + u^2}}{1 + \dfrac{1 - u^2}{1 + u^2}} \cdot \frac{2 \, du}{1 + u^2}$$

$$= \int \frac{\dfrac{u^2 + 2u + 1}{1 + u^2}}{\dfrac{1 + u^2 + 1 - u^2}{1 + u^2}} \cdot \frac{2 \, du}{1 + u^2}$$

$$= \int \frac{u^2 + 2u + 1}{2} \cdot \frac{2 \, du}{1 + u^2}$$

$$= \int \frac{u^2 + 2u + 1}{u^2 + 1} \, du$$

$$= \int \frac{(u^2 + 1) \, du}{u^2 + 1} + \int \frac{2u \, du}{1 + u^2} = \int 1 \, du + \int \frac{2u \, du}{1 + u^2}$$

$$= u + \ln(u^2 + 1) + C = \tan(x/2) + \ln[\tan^2(x/2) + 1] + C$$

or, if you want to be fancy,

$$= \tan(x/2) + \ln[\sec^2(x/2)] + C$$

I recently found a new miscellaneous example. It's not too bad if you know your trig.

Example 28—

$$\int \sin(2 \tan^{-1} x) \, dx$$

Draw the triangle
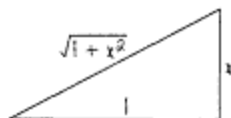
$$= \int \frac{2x}{1 + x^2} \, dx$$

$$\sin 2u = 2 \sin u \cos u =$$

$$2 \frac{x}{(1 + x^2)^{1/2}} \cdot \frac{1}{(1 + x^2)^{1/2}} =$$

$$\frac{2x}{1 + x^2}$$

$$= \ln(x^2 + 1) + C$$

Pretty cute, eh??!!

Chapter 7—
L'hopital's Rule

There was a little on L'Hopital's rule in the volume before. Let us give a complete discussion since we now know logs.

L'hopital's Rule

1. $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = 0/0$ or ∞/∞

2. $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)} = L$

then

$$\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = L$$

The value a is any number or plus or minus infinity, and L could be any number or infinity.

This rule states that, if the original limit is zero over zero or infinity over infinity or can be made into that form, then by taking the derivative of the top and the derivative of the bottom instead of the quotient rule, we can find the limit of the original.

Example 1—

$$\lim\limits_{x \to 0} \dfrac{x}{x^2 + 1} = \dfrac{0}{1} = 0$$

No L'Hopital's rule.

Example 2—

$$\lim\limits_{x \to \infty} \dfrac{x^2 + 1}{1/x} = \dfrac{\infty}{0}$$

which means the limit does not exist. Also no L'Hopital's rule.

Example 3—

$$\lim\limits_{x \to 0} \dfrac{\sin x}{x} = \dfrac{0}{0}$$

Yay! L'Hopital's rule can be used.

$$\lim\limits_{x \to 0} \dfrac{(\sin x)'}{(x)'} = \lim\limits_{x \to 0} \dfrac{\cos x}{1} = 1/1 = 1$$

so

$$\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$$

Example 4—

Also 0/0.

$$\lim_{x \to 1} \frac{\ln x}{x - 1} = \lim_{x \to 1} \frac{(\ln x)'}{(x - 1)'} = \lim_{x \to 1} \frac{(1/x)}{1} = 1$$

so

$$\lim_{x \to 1} \frac{\ln x}{x - 1} = 1$$

Example 5—

∞/∞

Here we need to use L'Hopital's rule twice.

$$\lim_{x \to \infty} \frac{2x^2 - 3x + 1}{5 - 7x^2} = \lim_{x \to \infty} \frac{4x - 3}{-14x} = \frac{4}{-14} = -2/7$$

so

$$\lim_{x \to \infty} \frac{2x^2 - 3x + 1}{5 - 7x^2} = -2/7$$

Example 6—

This turns out to be zero times infinity (minus infinity). We must rearrange it so it is either 0/0 or ∞/∞. We use a little trick to get it ∞/∞.

$$\lim_{x \to 0} (x \ln x)$$

$$\lim_{x \to 0} \frac{\ln x}{1/x} = \lim_{x \to 0} \frac{(\ln x)'}{(1/x)'} = \lim_{x \to 0} \frac{1/x}{-1/x^2} = \lim_{x \to 0} (-x) = 0$$

so

$$\lim_{x \to 0} (x \ln x) = 0$$

Example 7—

Trig identity. Now it's 0/0.

$$0 \times \infty; \lim_{x \to 0} (x \cot x) = \lim_{x \to 0} (x/\tan x)$$

$$\lim_{x \to 0} (x)'/(\tan x)' = 1/\sec^2 x = 1/1 = 1 \qquad \lim_{x \to 0} (x \cot x) = 1$$

Example 8—

Trick—add the fractions, get 0/0.

$$\lim_{x \to 0^+} (1/\sin x - 1/x)$$

$$\lim_{x \to 0^+} \frac{x - \sin x}{x \sin x} = \lim_{x \to 0^+} \frac{(1 - \cos x)}{x \cos x + \sin x}$$

$$= \lim_{x \to 0^+} \frac{\sin x}{2 \cos x - (\sin x)x} = 0/2 = 0$$

Using L'Hopital's rule twice, we find the original limit to be 0.

If you exclude the log examples, all of the preceding could have been done in the first semester. However, the following examples require logs. Those requiring logs are of the form 0^0, ∞^0, and 1^∞.

Example 9—

0^0; $\displaystyle\lim_{x \to 0^+} x^{4x}$; $y = x^{4x}$. $\ln y = 4x \ln x \to 0$ by Exmaple 6. So,

$y \to e^0 = 1$.

Example 10—

∞^0; $\displaystyle\lim_{x \to \infty} x^{1/x}$; $y = x^{1/x}$; $\ln y = (1/x) \ln x$; ∞/∞

$$\lim_{x \to \infty} \frac{(\ln x)'}{x'} = \lim_{x \to \infty} \frac{(1/x)}{1} = 0/1 = 0$$

Since $\ln y \to 0$, $y \to e^0 = 1$.

Example 11—

1^∞; $\displaystyle\lim_{x \to \infty} [1 + (1/x)]^x$

$$y = \left(x + \frac{1}{x}\right)^x \ln y = x \ln [1 + (1/x)] = \frac{\ln [1 + (1/x)]}{1/x} = 0/0$$

Taking derivatives top and bottom, we get

$$\frac{\dfrac{-1/x^2}{1 + (1/x)}}{-1/x^2} = \frac{1}{1 + (1/x)} \to 1$$

Thus, $\ln y \to 1$, so $y \to e^1 = e$.

Note

$$\lim_{x \to \infty} (1 + [(a/bx)])^{cx} = e^{ac/b}$$

Example 12—

0^∞. No L'Hopital's rule, because $0^k = 0$ for all positive k.

Note

We use L'Hopital's rule if we have 0/0, ∞/∞, $0 \cdot \infty$, $\infty - \infty$, 0^0, ∞^0, 1^∞. It's a no-no for $0/\infty$, $\infty/0$, $\infty \cdot \infty$, 0^∞, ∞^∞.

Chapter 8—
Improper Integrals

In discussing an improper integral, it would seem to be a good idea to recall what a proper integral is. In *Calc I*, we defined the integral of f(x) from a to b this way: break up the interval (a,b) into n parts. Let w_i be any point in the interval Δx_i. Form the sum $f(w_1)\Delta x_1 + f(w_2)\Delta x_2 + f(w_3)\Delta x_3 + ... + f(w_n).\Delta x_n$. Form the sum $\Sigma_{i=1}^{n} f(w_i)\Delta x_i$. If the limit exists as n goes to infinity and all the deltas go to zero, we have

$$\lim_{\substack{n \to \infty \\ \text{all } \Delta x's \to 0}} \sum_{i=1}^{n} f(w_i)\Delta x_i = \int_a^b f(x)\, dx$$

At the start, we usually take f(x) to be continuous, although that can be weakened. However, implied in the definition is that everything is *finite*; that is, both a and b are finite and f(x) is always finite. What happens if we have an infinity? In effect we close our eyes and pretend the infinity is not there. We then take the limit as we go to that infinity. If the limit gives us a single finite number, we will say the integral *converges* to that number. Otherwise, the integral *diverges*. Let us be more formal.

Example 1—

$$\int_{-1}^{\infty} \frac{1\, dx}{1 + x^2}$$

We rewrite this as

$$\lim_{a \to \infty} \int_{-1}^{a} \frac{1\, dx}{1 + x^2} = \lim_{a \to \infty} \tan^{-1} x \Big|_{-1}^{a} = \lim_{a \to \infty} [\tan^{-1} a - \tan^{-1}(-1)]$$

$$= \pi/2 - (-\pi/4) = \frac{3\pi}{4}$$

You might ask, "Are they all this easy?" In most books, the vast majority of the improper integrals are relatively easy in order to make sure that you understand what an improper integral is without worrying about a complicated integral.

In summary, this integral *converges* to the value $3\pi/4$.

Example 2—

$$\int_4^{\infty} 1/x^{1/2}\, dx$$

We write

$$\lim_{a \to \infty} \int_4^a 1/x^{1/2}\, dx = \lim_{a \to \infty} \int_4^a x^{-1/2}\, dx$$

$$= \lim_{a \to \infty} 2x^{1/2} \Big|_4^a = \lim_{a \to \infty} 2a^{1/2} - 2(4)^{1/2}$$

But $a^{1/2}$ goes to infinity as $a \to \infty$. Therefore, this integral *diverges*.

Note

In the kind of integral of Example 2, if the exponent in the denominator is less than or equal to 1, the integral diverges. If the exponent is greater than 1, the integral converges.

Example 3—

$$\int_0^\infty \cos x \, dx = \lim_{a \to \infty} \int_0^a \cos x \, dx = \lim_{a \to \infty} \sin x \Big|_0^a$$

$$= \lim_{a \to \infty} \sin a - \sin 0$$

This integral does not go to infinity. Yet it still diverges since, as a goes to infinity, sin (a) takes on every value between -1 and 1. The integral diverges because it doesn't go to one finite value.

If we have $\int_{-\infty}^{\infty} f(x) \, dx$, we break it up into two pieces, $\int_{-\infty}^{c} f(x) \, dx + \int_{c}^{\infty} f(x) \, dx$, where, for convenience, c is often 0 but certainly does not have to be.

Example 4—

$$\int_{-\infty}^{\infty} e^{-x} \, dx$$

Whenever you have infinity at both ends, you should try to do the piece that diverges first. If you choose the piece that diverges, you do not have to do the other piece, since the integral diverges (the whole integral diverges). If the first piece converges, then you still must do the other piece. You sharp-eyed readers have probably spotted the fact that the negative infinity piece diverges since, roughly speaking, $e^{-(-\infty)}$ goes to infinity. Note $e^{-(\infty)}$ is 0.

Let us take a look at the other infinity kind of improper integral.

Example 5—

The improper part, f(x) = (x - 1)$^{-1/2}$, is infinite at x= 1.

$$\int_1^5 \frac{1 \, dx}{(x-1)^{1/2}}$$

$$= \lim_{a \to 1^+} \int_a^5 (x-1)^{-1/2} \, dx = \lim_{a \to 1^+} 2(x-1)^{1/2} \Big|_a^5$$

$$= \lim_{a \to 1^+} 2(5-1)^{1/2} - 2(a-1)^{1/2}$$

$$= 4 - 0 = 4$$

This integral *converges* to 4.

Note

For this kind of improper integral, if the exponent in the denominator is less than 1, the integral converges; if the exponent is greater than or equal to 1, it diverges.

Example 6—

First, note that this *is* an improper integral since f(3) is undefined. Second, most of the time, if one piece diverges, both diverge, so that it is not important which piece to choose first.

$$\int_2^5 \frac{1 \, dx}{(x-3)^3}$$

$$= \lim_{c \to 3^-} \int_2^c (x - 3)^{-3}\, dx + \lim_{d \to 3^+} + \int_d^5 (x - 3)^{-3}\, dx$$

$$= \lim_{c \to 3^-} - \frac{1}{2(x - 3)^2}\bigg[_2^c - \lim_{d \to 3^+} \frac{1}{2(x - 3)^2}\bigg[_d^5$$

$$= \lim_{c \to 3^-} \left[- \frac{1}{2(c - 3)^2} + \frac{1}{2(2 - 3)^2} \right]$$

$$+ \lim_{d \to 3^+} \left[- \frac{1}{2(5 - 3)^2} + \frac{1}{2(d - 3)^2} \right]$$

Each bracketed piece goes to infinity. The integral *diverges*. However, if you did the problem, you should calculate only one piece. Once it diverges, you need not do the other piece.

The next two are a couple of my most favorite examples. It makes you believe that mathematicians can do *anything*. This is not true, of course, but the examples are extraordinary.

Example 7—

We will take an infinite area, rotate the region, and get a finite volume!!!!!!!!! Here's our infinite area:
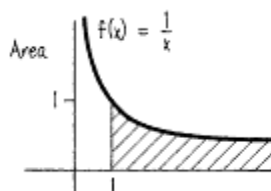
$$\int_1^\infty \frac{1}{x}\, dx = \lim_{a \to \infty} \int_1^a \frac{1}{x}\, dx$$

$$= \lim_{a \to \infty} \ln x \bigg[_1^a = \lim_{a \to \infty} \ln a - \ln 1 \to \infty$$

Volumes are determined by sections. Much more on this is in Chap. 11 in this book.
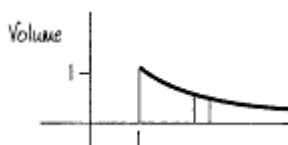
$$\pi \int_1^\infty \left(\frac{1}{x}\right)^2 dx = \lim_{a \to \infty} \pi \int_1^a x^{-2}\, dx = \lim_{a \to \infty} \pi \left[-\frac{1}{x} \right]_1^a$$

$$= \lim_{a \to \infty} \pi \left[-\frac{1}{a} + \frac{1}{1} \right] = \pi$$

Amazing!!!!!! Infinite area rotated gives finite volume.

This one will totally blow your mind. We will now take a finite region, rotate it, and get an infinite volume, which would seem impossible after the last example. It is *not!!!!*

Example 8—

Area, $R(x) = \frac{1}{x^{2/3}}$
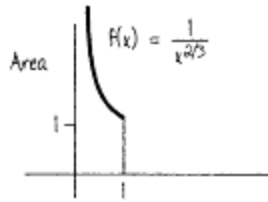
$$\int_0^1 \frac{1}{x^{2/3}}\, dx = \lim_{a\to 0^+} \int_a^1 x^{-2/3}\, dx = \lim_{a\to 0^+} 3x^{1/3} \Big[_a^1$$

$$= \lim_{a\to 0^+} (3 - 3a^{1/3}) = 3$$

Integral converges. Area is 3.

Volume
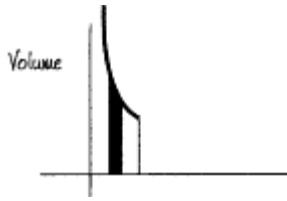
$$\pi \int_0^1 \left(\frac{1}{x^{2/3}}\right)^2 dx = \lim_{b\to 0^+} \pi \int_b^1 x^{-4/3}\, dx = \lim_{b\to 0^+} \pi\left(-\frac{3}{x^{1/3}}\right)\Big[_b^1$$

$$= \lim_{b\to 0^+} \pi\left(-3 + \frac{3}{b^{1/3}}\right) = \infty$$

Diverges!!!! Amazing!!!!!

Here is one more example I like a lot because it is slightly nasty, slightly tricky, and verrry instructive.

Example 9—

Factor—improper part is at "infinity."

$$\int_0^\infty \frac{1}{x^2 + 3x + 2}\, dx$$

Break up using partial fractions!

$$= \lim_{a\to\infty} \int_0^a \frac{1}{(x+1)(x+2)}\, dx$$

$$= \lim_{a\to\infty} \int_0^a \frac{1}{x+1} - \frac{1}{x+2}\, dx$$

$$= \lim_{a\to\infty} \ln(x+1) - \ln(x+2) \Big[_0^a$$

Law 6 of logs!!!

$$= \lim_{a\to\infty} \ln(a+1) - \ln(a+2) - (\ln 1 - \ln 2)$$

Now $\lim_{a\to\infty} \frac{a+1}{a+2} = 1$

$$= \lim_{a \to \infty} \ln \frac{a+1}{a+2} + \ln 2$$

$$= \ln 1 + \ln 2 = \ln 2.$$

Nice example!!!

Chapter 9—
Parameters, Polar Coordinates Arc Length, Surface of Revolution

If the cartesian coordinate system were superior to all others, this chapter would not be necessary. However, it is not. *Parameters* are variables that are introduced to make life easier.

Example 1—

$x = 2t;\ y = t^2/4 - 1;\ -2 \leq t \leq 4$

We need to make a table with three columns' t, x, and y. We will use only the x and y on the graph.

t	x	y
-2	-4	0
-1	-2	-3/4
0	0	-1
1	2	-3/4
2	4	0
3	6	5/4
4	8	3
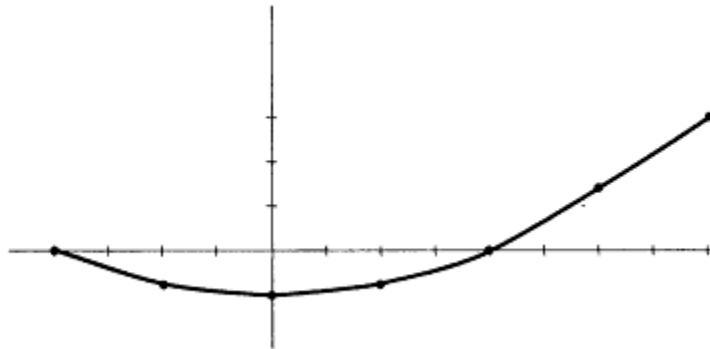
We can eliminate t: $x = 2t;\ t = x/2;\ y = (x/2)^2/4 - 1 = x^2/16 - 1$. It is clear we did not really need the parame ter t in this case. However, we did not want to take an example that was too difficult in order to show the use of a parameter.

Example 2—

Another common example is $x = \cos t,\ y = \sin t$. Again, we don't need a parameter, since $x^2 + y^2 = 1$.

Example 3—

Take a circle tangent to the x-axis at the origin. Put a dot on that point. Roll the circle on the x-axis. If we trace out the path of that dot, we will get a curve called a *cycloid*. Since this curve is derived in most books, we will put the equations down and then draw the picture.

$$x = a(t - \sin t)$$

$$y = a(1 - \cos t)$$
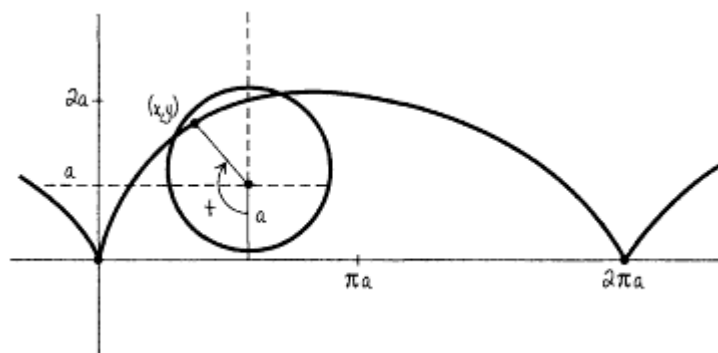
In order to eliminate t, we solve for t in the equation for y: $t = \cos^{-1}(1 - y/a)$. Therefore, $x = a\{\cos^{-1}(1 - y/a) - \sin[\cos^{-1}(1 - y/a)]\}$. Pretty awful, isn't it?! For all practical purposes, this form is impossible to use. We really need parameters here!!!!

We wish to take derivatives using parameters. The first derivative will give us the slope, and the second will tell its upness and downness.

Example 4—

$$x = t^4 + 1, \ y = t^8 - 5$$

$$dy/dx = \frac{dy/xt}{dx/dt}$$

$$dy/dt = 8t^7$$

$$dx/dt = 4t^3 \qquad \text{so} \qquad dy/dx = 8t^7/4t^3 = 2t^4$$

You might think we came up with d^2y/dx^2 by taking dy/dx and taking the derivative of the top over the derivative of the bottom. You'd be wrong. You say, "Of course. You must use the quotient rule!" Again you'd be wrong. The second derivative is the derivative of the first derivative. So...

$$\frac{d^2y}{dx^2} = \frac{d(y')}{dx} = \frac{d(y')/dt}{dx/dt}$$

$$= \frac{d(2t^4)/dt}{d(t^4 + 1)/dt} = \frac{8t^3}{4t^3} = 2$$

Again, in this particular example, you could eliminate the t, but in the cycloid, you really could not.

Example 5—

The a's cancel.

Cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$

$dx/dt = a(1 - \cos t)$ $dy/dt = a \sin t$

$dy/dx = (dy/dt)/(dx/dt) = (\sin t)/(1 - \cos t)$

$$dy'/dt = \frac{(1 - \cos t)(\cos t) - (\sin t)(\sin t)}{(1 - \cos t)^2}$$

$$= \frac{\cos t - \cos^2 t - \sin^2 t}{(1 - \cos t)^2}$$

$$= \frac{\cos t - 1}{(1 - \cos t)^2} = \frac{-1}{1 - \cos t}$$

$$\frac{dy'}{dx} = \frac{dy'}{dt} \bigg/ \frac{dx}{dt} = \frac{-1}{1 - \cos t} \bigg/ a(1 - \cos t) = \frac{-1}{a(1 - \cos t)^2}$$

If you look at the picture of the cycloid, the second derivative shows the curve is always down, since the second derivative is always negative a is positive, except at multiples of 2π, where the curve comes to a point. The parameter is extremely useful here, as it always is when it is used.

Polar Coordinates

In the past, you should have had a teeny, tiny bit of experience with polar coordinates, namely how to graph a point, say $(4, \pi/6)$. The following is a summary of all the things you should know on this topic.

Rule 1

Relationship between (x, y) and (r, θ):

A. $r = [x^2 + y^2]^{1/2}$

B. θ = angle measured from positive x-axis, positive in a counterclockwise direction

C. $x = r \cos \theta$

D. $y = r \sin \theta$

Rule 2

Every point has an infinite number of representations.

n integer

$(3, \pi/6) = (3, \pi/6 \pm 2n\pi) = (-3, 7\pi/6 \pm 2n\pi)$

Rule 3

A. $r = a$ is a circle center at the origin, radius a.

B. $\theta = c$ is a line through the origin.
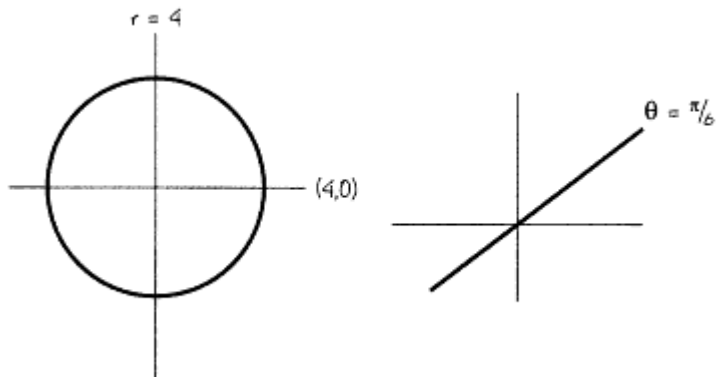
Rule 4

Symmetries to aid polar curve sketching:*

A. *x-axis sym.* If replacing (r,θ) by (r,θ) or $(-r,\pi - \theta)$ gives the same or equivalent equation.

B. *y-axis sym.* If replacing (r,θ) by $(-r,-\theta)$ or $(r,\pi - \theta)$ gives the same or equivalent equation.

C. *Origin sym.* If replacing (r,θ) by $(-r,\theta)$ or $(r,\pi + \theta)$ gives the same or equivalent equation.

There are several curves that you should know by looking at the equation. It is always easier to sketch a curve if you know what it looks like before you start.

Rule 5

r = a sin θ

r = a cos θ
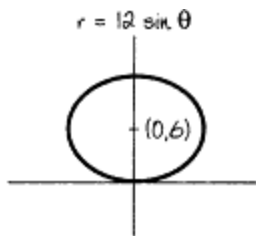
Circles tangent to origin, radius |a|/2, x- or y-axis is the axis of symmetry.

Rule 6

Cardioid:

r = a(1 ± cos θ)

r = a(1 ± sin θ)

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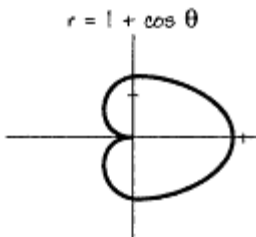
* Two of the symmetries mean all three hold.

Rule 7

Rose-n positive odd:

r = cos 3θ, 3 petals

r = a sin nθ or a cos nθ
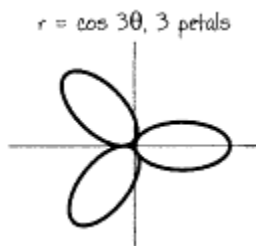
n petals

Rule 8

Rose-n positive even:

r = sin 2θ, 4 petals

r = a sin nθ or a cos nθ
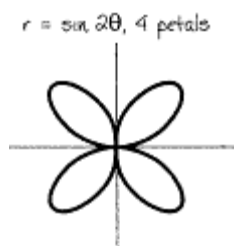
2n petals

Rule 9

Lemniscate:

$r^2 = \pm a \cos 2\theta$

$r^2 = \pm a \sin 2\theta$

$r^2 = \sin 2\theta$ $r^2 = \cos 2\theta$
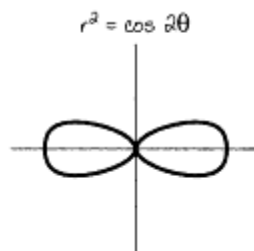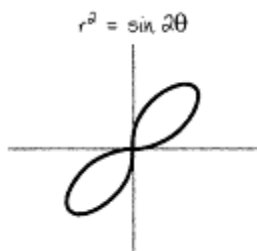
Rule 10

Spirals:

Spiral of Archimedes Logarithmic spiral		

r = aθ

$r = ae^{b\theta}$

$$r = \theta$$
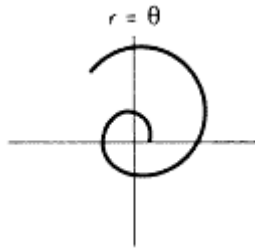
There are a few others, but these are the main ones. Let us graph a couple of these to show the technique.

Example 6—

Graph r = 4(1 - cos θ).

θ	r
0	4-4=0
π/6	$4 - 2(3)^{1/2} = .6$
π/4	$4 - 2(2)^{1/2} = 1.2$
π/3	4 - 2 = 2
π/2	4 - 0 =4
2π/3	4 + 2 = 6
3π/4	$4 + 2(2)^{1/2} = 6.8$
5π/6	$4 + 2(3)^{1/2} = 7.4$
π	4+4=8
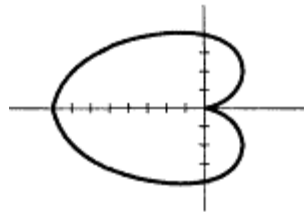
Note

We have symmetry with respect to the x-axis. So we need only values of θ between 0 and π to reflect the image in the x-axis. Note the chart. If you study the patterns, you should be able to do much of the table by sight. Also, you know the approximate value of $2^{1/2}$ (1.4) and $3^{1/2}$ (1.7).

Note

There is a short way to draw the previous graph (and other graphs) if you know what the graph is. Find the intercepts 0, π/2, 0, 3π/2, and sketch the graph around them (which is exactly how I did the graph).

Example 7—

$r^2 = 9 \cos 2\theta$

Symmetry is about x-axis, y-axis, and the origin. So we only need to graph the first quadrant and then repeat the curve perfectly in the other three. However, since we have 2θ, we need 3 columns—2θ, θ, and r—with only the last two on the graph.
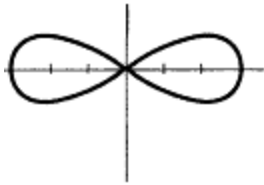
2θ	θ	r
0	0	3
$\pi/6$	$\pi/12$	2.8
$\pi/4$	$\pi/8$	2.5
$\pi/3$	$\pi/6$	2.1
$\pi/2$	$\pi/4$	0

Although this problem can be done without a calculator if you can approximate well, it is convenient to use a calculator for this one.

We would now like to find the area in polar coordinates. In almost any calc book, you will find the area is

$$\frac{1}{2} \int_{\theta = \theta_1}^{\theta_2} r^2 \, d\theta.$$

Example 8—

Find the area of the cardioid $r = 2 + 2 \cos \theta$.

As we have seen before, this curve is symmetric with respect to the x-axis. We can integrate from 0 to 2π or double the integral from 0 to π.

$$2(\tfrac{1}{2}) \int_0^\pi (2 + 2 \cos \theta)^2 \, d\theta = \int_0^\pi (4 + 8 \cos \theta + 4 \cos^2 \theta) \, d\theta$$

$$= \int_0^\pi [4 + 8 \cos \theta + 4(1 + \cos 2\theta)/2] \, d\theta$$

$$= \int_0^\pi (6 + 8 \cos \theta + 2 \cos 2\theta) \, d\theta$$

$$= 6\theta + 8 \sin \theta + \sin 2\theta \Big[_0^\pi = 6\pi$$

Example 9—

Find area of lemniscate $r^2 = 9 \cos 2\theta$.

From Example 7 we have four-quadrant symmetry. So we will find the area in the first quadrant and multiply by 4.

$$\frac{1}{2} \int r^2 \, d\theta = 4\left(\frac{1}{2}\right) \int_0^{\pi/4} 9 \cos 2\theta \, d\theta = (2)(9) \left. \frac{\sin 2\theta}{2} \right|_0^{\pi/4} = 9$$

Here's a little trick about finding the area of a curve with petals.

Example 10—

Find the area of $r = 4 \sin 6\theta$.

Just like the previous example, the graph as 12 petals, but all cosines are symmetric with respect to the x-axis. $\cos 0 = 1$. We find the smallest positive θ that makes $\cos 6\theta = 0$. $6\theta = \pi/2$. So $\theta = \pi/12$. From 0 to $\pi/12$ represents one-half petal. So we have 24 ½ petals to get the area. The integral is

$$(24)\tfrac{1}{2} \int_0^{\pi/12} (4 \cos 6\theta)^2 \, d\theta, \text{ etc.}$$

Here's another trick with this problem.

This has 12 petals. We will find the area of one of them and multiply by 12. To find the area of one petal, we find the first two values; $\sin 6\theta = 0$. $6\theta = 0$ or $\theta = 0$. $6\theta = \pi$. $\theta = \pi/6$. So the integral is

$$(12)\tfrac{1}{2} \int_0^{\pi/6} (4 \sin 6\theta)^2 \, d\theta, \text{ and so on.}$$

Note

I know this last trick is one almost no one will use. But I've left it in because my original editor, David Beckwith, was so great. This is one of his favorite tricks. Thanks, David.

Example 11—

Find the area of $r = 4 \cos 6\theta$.

We know we can slide the curve $y = f(x) + a$ units to the right by replacing x by x - a. In the same way, we can rotate $r = f(\theta)$ through a counterclockwise angle $+\alpha$ by replacing θ by $\theta - \alpha$. Thus, by rotating our curve by $15° = \pi/12$ radians, $r = 4 \cos 6(\theta - \pi/12) = 4 \cos(6\theta - \pi/2) = 4 \sin 6\theta$, which is exactly the curve in Example 10!!!!!! Rotating a curve doesn't affect its area. So... the answer must be the same as in Example 10!!

Just like with x-y coordinates, we would like to find the area between two curves. In the case of x - y, it was the top curve minus the bottom curve or the right curve minus the left curve. In polar coordinates, it is the outside curve minus the inside one. The formula is

$$\tfrac{1}{2} \int_{\theta_1}^{\theta_2} (r_{outside}^2 - r_{inside}^2) \, d\theta$$

where θ_1 and θ_2 are the intersection of the two curves.

Example 12—
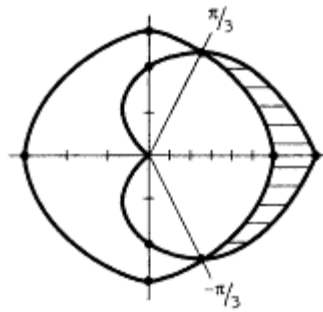
Find the area of the region bounded by the cardioid r = 2 + 2 cos θ (inside this curve) and the circle r = 3 (outside this curve).

See the figure.

Since the area is inside the cardioid, the cardioid is the *outside* curve. Outside r = 3 makes the circle the *inside* curve. In order to find the limits of the integral, we set the r's equal to each other; 3 = 2 + 2 cos θ; cos θ=½; θ=±π/3. Also, there is symmetry about the x-axis. So we can double the integral from 0 to π/3.

$$2(½) \int_{θ=0}^{π/3} (r^2_{outside} - r^2_{inside}) \, dθ = \int_0^{π/3} [(2 + 2 \cos θ)^2 - 3^2] \, dθ$$

$$= \int_0^{π/3} [8 \cos θ + 4 \cos^2 θ - 5] \, dθ$$

$$= \int_0^{π/3} [8 \cos θ + (2 + 2 \cos 2θ) - 5] \, dθ$$

$$= 8 \sin θ + \sin 2θ - 3 = 8(3)^{1/2}/2 + 3^{1/2}/2 - 3(π/3)$$

$$= \frac{9\sqrt{3}}{2} - π$$

There are two more problems that we can illustrate with the same diagram.

Example 13—
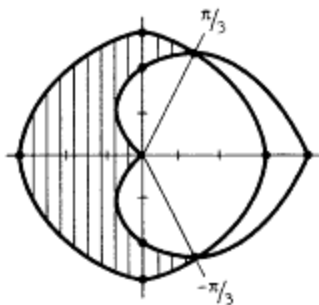
Find the area outside the cardioid and inside the circle.

If it's inside the circle r = 3, r = 3 becomes the outside r. "Outside the cardioid" means r = 2 + 2 cos θ becomes the inside curve. Again, we have x-axis symmetry. Again, setting the curves equal, we get ±π/3. The picture tells us twice the integral from π/3 to π. Since we did the last problem, we need not integrate twice, since it is the same integral. We need only change all the signs.

$$2(½) \int_{\pi/3}^{\pi} [3^2 - (2 + 2 \cos \theta)^2] \, d\theta = 3\theta - 8 \sin \theta - \sin 2\theta \Big[_{\pi/3}^{\pi}$$

$$= [3\pi - 8 \sin \pi - \sin 2\pi] - [\pi - 8 \sin \pi/3 - \sin 2\pi/3]$$

$$= 2\pi + \frac{9\sqrt{3}}{2}$$

Example 14—
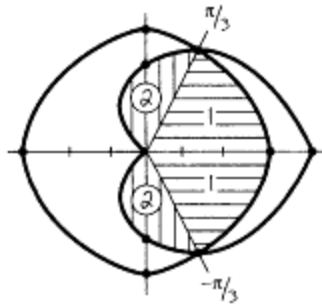
Find the area inside both the circle and the cardioid.

Now we know your instructor would never give such a problem, but .... This is a split region at the rays $0 = \pm\pi/3$. Region I is a sector of the circle. Region 2 is the "top" of the heart, the cardioid. So we get

$$2(½) \int_0^{\pi/3} 3^2 \, d\theta + 2(½) \int_{\pi/3}^{\pi} (2 + 2 \cos \theta)^2 \, d\theta$$

$$= \int_0^{\pi/3} 9 \, d\theta + \int_{\pi/3}^{\pi} (6 + 8 \cos \theta + 2 \cos 2\theta) \, d\theta$$

$$= 9\theta \Big[_0^{\pi/3} + 6\theta + 8 \sin \theta + \sin 2\theta \Big[_{\pi/3}^{\pi}$$

$$= 3\pi + 4\pi - \frac{9\sqrt{3}}{2} = 7\pi - \frac{9\sqrt{3}}{2}$$

Please note that polar coordinates are really not hard. It is just that many instructors rush through this section. This makes this section appear difficult. It really is not.

Arc Length and Surface Area of Revolution

The derivations are in most books. All that remains is memorizing the formulas and applying them.

The arc length is:

$$\int_a^b [1 + f'(x)^2]^{1/2} \, dx \qquad \text{or} \qquad \int_c^d [1 + f'(y)^2]^{1/2} \, dy$$

or, in parameters:

$$\int_{t=t_1}^{t=t_2} [(dx/dt)^2 + (dy/dt)^2]^{1/2} dt \quad \text{or}$$

$$\int_{\theta=\theta_1}^{\theta=\theta_2} [f(\theta)^2 + f'(\theta)^2]^{1/2} d\theta \text{ with a curve of } r = f(\theta)$$

The surface of revolution about the x-axis (distance is y) is:

$$\int_a^b 2\pi y[1 + (y')^2]^{1/2} dx \quad \text{or} \quad \int_c^d 2\pi y[1 + (x')^2]^{1/2} dy$$

Parametric:

$$\int_{t=t_1}^{t=t_2} 2\pi y(t)[(dx/dt)^2 + (dy/dt)^2]^{1/2} dt$$

Polar form:

$$\int_{\theta=\theta_1}^{\theta=\theta_2} 2\pi f(\theta) \sin\theta \, [f(\theta)^2 + f'(\theta)^2]^{1/2} d\theta$$

For the y-axis, x replaces y and $\theta \sin\theta$ is replaced by $\theta \cos\theta$.

Let us do one simple example, one clever example, and one known result.

Example 15—

$x = f(y) = (2/3)y^{3/2}$, arc length $0 \le y \le 1$

$f'(y) = y^{1/2}$ so $[1 + f'(y)^2]^{1/2} = (1 + y)^{1/2}$

$$\text{Arc length} = \int_0^1 (1 + y)^{1/2} dy = (2/3)(1 + y)^{3/2} \Big[_0^1$$

$$= (2/3)(2^{3/2} - 1)$$

Example 16 (Clever, Very Clever)—

$y = f(x) = (1/3)x^3 + 1/(4x)$, arc length $1 \le x \le 2$.

$f'(x) = x^2 - (1/4x^2)$ $f'(x)^2 = x^4 - \frac{1}{2} + 1/16x^4 = (x^2 - 1/4x^2)^2$

$$+1 = +1$$

$$1 + f'(x)^2 = x^4 + \frac{1}{2} + 1/16x^4 = (x^2 + 1/4x^2)^2$$

$$[1 + f'(x)^2]^{1/2} = x^2 + 1/4x^2$$

Pretty clever ???? Not when you have to see this for yourself, it isn't.

$$\text{Arc length} = \int_1^2 (x^2 + 1/4x^2) dx = x^3/3 - 1/4x \Big[_1^2$$

$$= (8/3 - \frac{1}{2}) - (1/3 - \frac{1}{4}) = 25/12$$

Exaple 17—

Find the surface area of a sphere of radius a.

A sphere is a semicircle rotated about the x-axis (or y-axis): $x = a \cos t$; $dx/dt = -a \sin t$; $y = a \sin t$; $dy/dt = a \cos t$.

$$SA = \int_0^\pi 2\pi y[(dx/dt)^2 + (dy/dt)^2]^{1/2} \, dt$$

$$= \int_0^\pi 2\pi a \sin t[(-a \sin t)^2 + (a \cos t)^2]^{1/2} \, dt$$

The math under the square root sign simplifies to a^2. So we get...

$$SA = 2\pi a^2 \int_0^\pi \sin t \, dt = 2\pi a^2(-\cos t) \Big[\,_0^\pi = 2\pi a^2[-(-1) + 1]$$

Again we haven't been lied to all these years. The surface area of a sphere really is $4\pi a^2$.

Chapter 10—
Work, Work, Work

This topic is usually presented in a physics book and scares everyone to death. If done my way, I'm pretty sure it won't bother you again. Work is defined as force times distance. However if the force is a function of distance, theory tells us that work is the integral of F(x) dx. We give the usual examples: springs and a couple on pumping water over the top.

Example 1—

It takes a force of 20 pounds to stretch a 7-foot spring to 11 feet. How much work to pull out the spring from 13 feet to 27 feet?

For a spring force F = kx. F = 20, and x = 11 - 7 = 4, the distance the spring is stretched from its natural length. So k = 5 and F = 5x.

Lower limit = 13 - 7 = 6, upper limit 27 - 7 = 20.

$$\text{Work} = \int kx \, dx$$

$$\text{Work} = \int_6^{20} 5x \, dx = (5x^2/2) \Big|_6^{20} = 5(20)^2/2 - 5(6^2)/2$$

$$= 1000 - 90 = 910 \text{ foot-pounds}$$

Example 2—
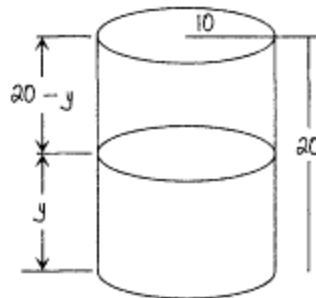
How much work is done to pump the water out of a full cylindrical can radius 10 feet, height 20 feet—if the water is to be pumped over the top?

W = density (weight/volume) x volume x height over which the water is pumped. Each section of water pumped is a thin cylinder V = π(10)² dy. Height pumped (see the figure) is 20 - y. The density for water is approximately 62.5 pounds per cubic foot.

$$\int_0^{20} 62.5\pi(10)^2(20 - y) \, dy = 6250\pi(20y - y^2/2) \Big|_0^{20}$$

$$= 1{,}250{,}000\pi \text{ foot-pounds}$$

Note 1

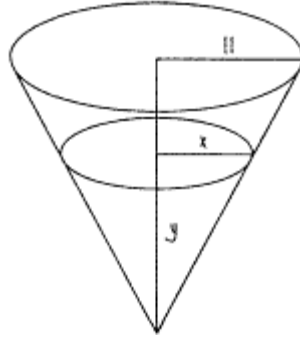If the outlet were 17 feet over the can, we would have 37 - y.

Note 2

If the can were three-fourths full, integral limits would be 0 to 15.

Note 3

If we did the same problem with a box, the cross section would be a thin sheet with length and width constant and height dy.

Example 3—

How much work is done to pump water out of a cone, diameter 22 feet, height 10 feet, if the outlet is 7 feet over the top of the cone and the cone is filled 2 feet deep?

First, a slight trick. The diameter is 22, so the radius is 11. Next, note the cross section is again a thin cylinder, but this time the radius changes: $V = \pi r^2 h = \pi x^2 \, dy$. We must see a similar triangle $x/y = 11/10$ or $x = 11y/10$. The height of the pipe makes the pumping distance 17 - y.

We leave the integration to you.

$$\int_0^2 62.5\pi[(11/10)y]^2(17-y) \, dy$$

Note
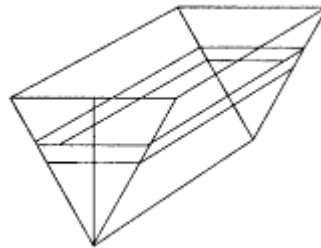
A trough is a similar problem except the horizontal cross section is a rectangular sheet whose width changes but whose length (length of the trough) remains the same.

Also note similar triangles, just like in the cone.

Chapter 11—
Volumes By Rotation and Section

This chapter is the first of two that are based on material from my first book, *Calc Helper*, the beginnings of this course. The material in this chapter is in many Calc I courses. However, at my school, it is in Calc II. The material in the next chapter, "Conic Sections—Circle, Ellipse, Parabola, Hyperbola," is taught in either Calc I or Calc II, depending on the book and the course. In my school, it is in the non-math-major Calc I and math-major Calc II. Therefore, this subject deserves to be treated in both books.

The first topic is to find volumes of rotations. This is very visual. If you see the picture, the volume is easy. If not, this topic is very hard.

Imagine a perfectly formed apple with a line through the middle from top to bottom. We can find the volume two different ways. One way is by making slices perpendicular to the line (axis). (We will do the other way later with an onion.) Each slice is a disc, a thin cylinder. Its volume is $\pi r^2 h$, where h is very small. If we add up all the discs, taking the limits properly, we get the volume.

We will take the same region in six different problems, rotating this region differently six times and getting six different volumes.
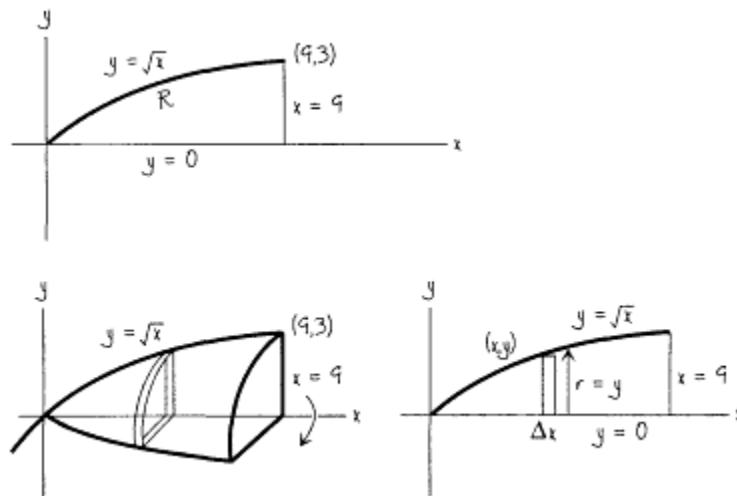
Example 5—
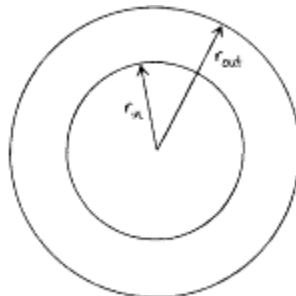
Find the volume if the region R is rotated about the x-axis.

The volume of each disc is $\pi r^2 h$: $h = \Delta x$; $r = y$. So $r^2 = y^2 = x$, and x goes from 0 to 9.

$$V = \int_0^9 \pi r^2 \, dh = \pi \int_{x=0}^9 x \, dx = \frac{\pi x^2}{2} \Big|_0^9 = \frac{81\pi}{2}$$

The integrals are almost always easy. Once you understand the picture, all will be easy. But it takes most people time, studying the pictures.

Let us get back to our apple. Suppose we core it. When we take slices perpendicular to the axis, we get a ring. The area of a ring is the area of the outside minus the area of the inside.

The volume of each disc is $(\pi r^2_{outside} - \pi r^2_{inside})h$. Again, Again, h is small.

Example 6—

Find the volume if our region is rotated about the y-axis.
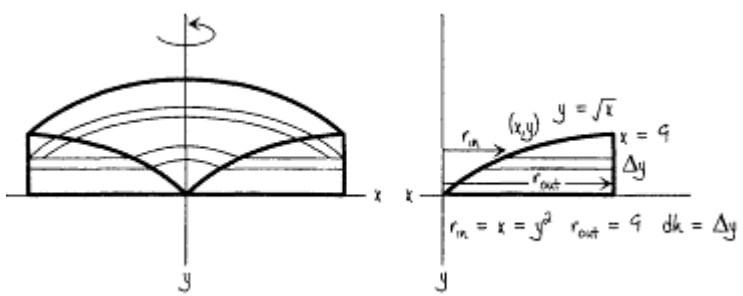
As you rotate this region, there is a hole. Outside radius is always 9 and the inside radius is always the x value. But $x = y^2$. $r^2 = x^2 = y^4$.

$$V = \int_{y=0}^{3} (\pi r^2_{out} - \pi r^2_{in}) \, dh = \pi \int_{0}^{3} (9^2 - y^4) \, dy$$

$$= \pi \left[81y - \frac{y^5}{5} \right]_{0}^{3} = \pi \left(81(3) - \frac{3^5}{5} \right) = \frac{972}{5} \pi$$

Example 7—

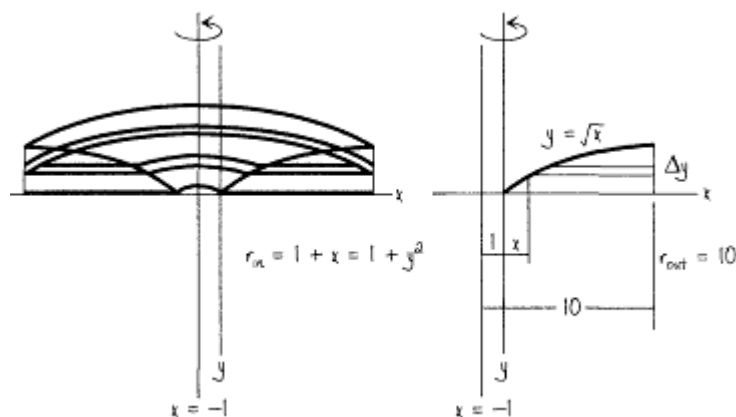Find the volume of our glorious region if it is rotated about the line x = 9.
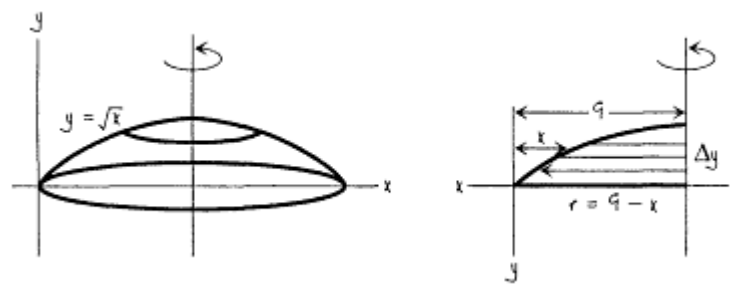
Notice that when we rotate the region about x = 9, there is no hole. $V_{sect} = \pi r^2 h$. $r = 9 - x = 9 - y^2$. $r^2 = 81 - 18y^2 + y^4$. $h = \Delta y$.

$$V = \pi \int_0^3 (81 - 18y^2 + y^4)\, dy = 81y - 6y^3 + \frac{y^5}{5} \Big|_0^3 \; \pi$$

$$= \pi\left[81(3) - 6(3)^3 + \frac{3^5}{5} - 0\right] = \frac{648\pi}{5}$$

Example 8—

Find the volume if the same region R is rotated about the line x =-1.

$V_{section} = \pi(r_{out}^2 - r_{in}^2)h$; $r_{out} = 9 + 1 = 10$; $r_{in} = 1 + x = 1 + y^2$; $r^2 = 1 + 2y^2 + y^4$; $h = \Delta y$.
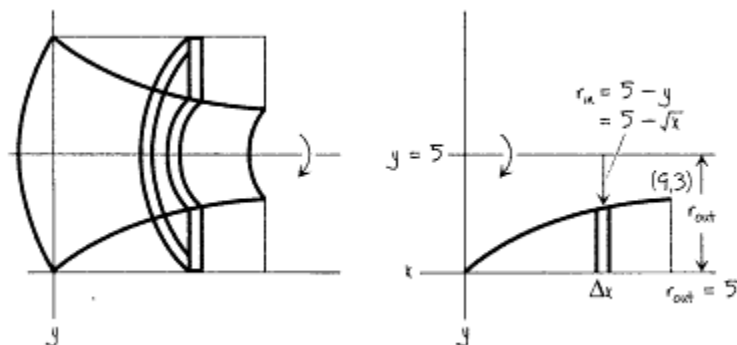
$$V = \int_0^3 (\pi r_{out}^2 - \pi r_{in}^2)\, dy = \pi \int_0^3 [10^2 - (1 + 2y^2 + y^4)]\, dy$$

$$= \pi \int_0^3 (99 - 2y^2 - y^4)\, dy = 99y - \frac{2y^3}{3} - \frac{y^5}{5} \Big|_0^3 \; \pi$$

$$= \pi\left[99(3) - \frac{2(3)}{3} - \frac{3^5}{5} - 0\right] = \frac{1152\pi}{5}$$

Example 9—

Find the volume if our beloved region is rotated about y=5.

$V_{section} = \pi(r_{out}^2 - r_{in}^2)h; \; r_{out} = 5; \; r_{in} = 5 - y = 5 - x^{1/2};$

$r_{in}^2 = 25 - 10x^{1/2} + x; \; h = \Delta x.$

$$V = \int_0^9 (\pi r_{out}^2 - \pi r_{in}^2) \, dx = \pi \int_0^9 [25 - (25 - 10x^{1/2} + x)] \, dx$$

$$= \pi \int_0^9 (10x^{1/2} - x) \, dx = \left. \frac{20}{3} x^{3/2} - \frac{x^2}{2} \right|_0^9 \pi$$

$$= \pi\left(\frac{20}{3} \, 9^{3/2} - \frac{9^2}{2} - 0 \right) = 139.5\pi$$

Example 10—

And for our final attraction, we will take the same region and rotate it about the line y = -2.

$V_{sect} = (r_{out}^2 - r_{in}^2)h; \; r_{in} = 2, \; r_{out} = 2 + y = 2 + x^{1/2}.$

$r_{out}^2 = 4 + 4x^{1/2} + x.$

$$V = \int_0^9 (\pi r_{out}^2 - \pi r_{in}^2) \, dx = \pi \int_0^9 [(4 + 4x^{1/2} + x) - 4] \, dx$$

$$= \pi \int_0^9 (4x^{1/2} + x) \, dx = \left. \frac{8}{3} x^{3/2} + \frac{x^2}{2} \right|_0^9 \pi$$

$$= \pi\left(\frac{8}{3} \cdot 9^{3/2} + \frac{9^2}{2} \right) = \frac{225\pi}{2}$$

The next kind of volumes we will consider are rotations again, but we will do it a different way. Think about an onion with each layer a whole piece. We will add up layer by layer until we get a volume. We will add up cylindrical shells, tall cylindrical shells. We will see one of these shells, a general illustration, and two examples.

$y = -2$

$r_{in} = 2$

$r_{out} = 2 + y = 2 + \sqrt{x}$

Δx

Top
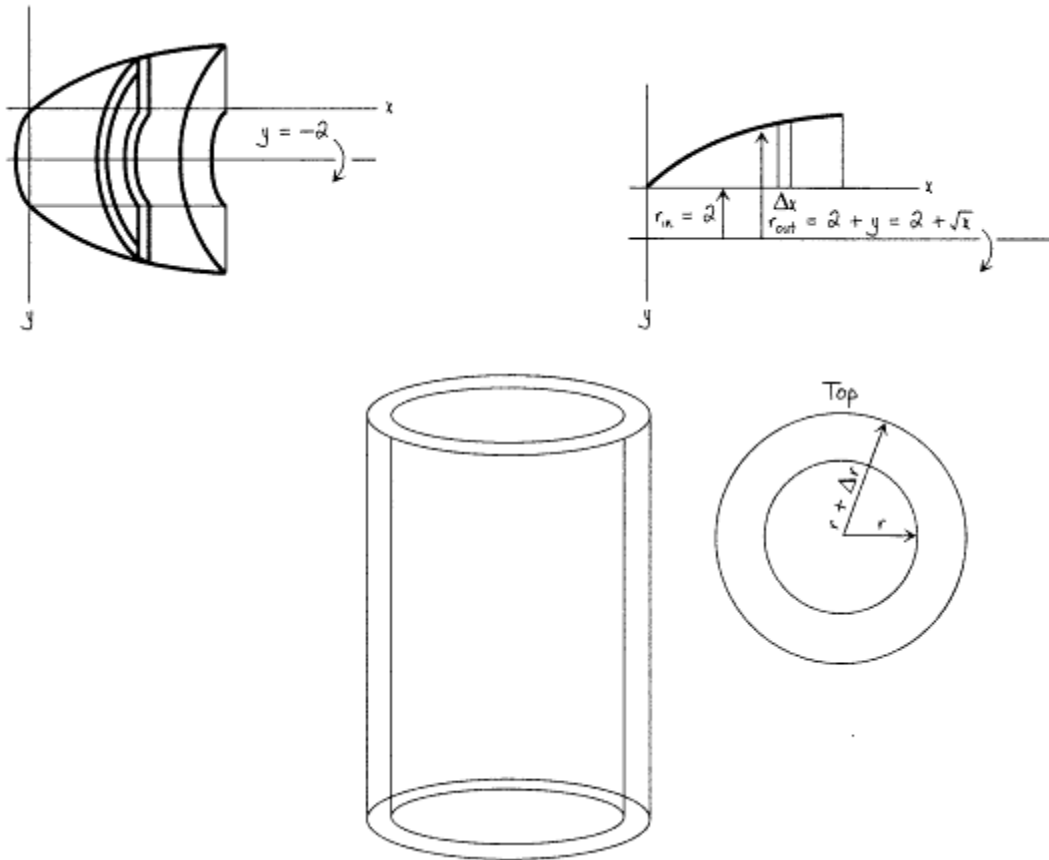
$V_{shell} = \pi r_{out}^2 h - \pi r_{in}^2 h = \pi h(r_{out}^2 - r_{in}^2)$

$$= \pi h(r_{out} + r_{in})(r_{out} - r_{in}) = 2\,\pi h\left(\frac{r_{out} + r_{in}}{2}\right) \Delta r$$

$= 2\pi \cdot \text{height} \cdot \text{average radius} \cdot \text{thickness}$

This is an example of a rotation about the y-axis of the region bounded by $y = f(x)$, $y = 0$, $x = a$, $x = b$. Notice the axis of the cylinder is the axis which the curve is rotated around.
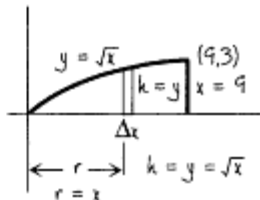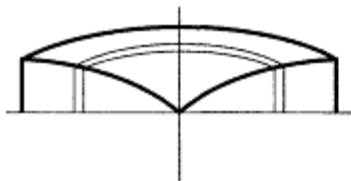

$y = f(x)$

$x = a$

$y = b$

$y = 0$
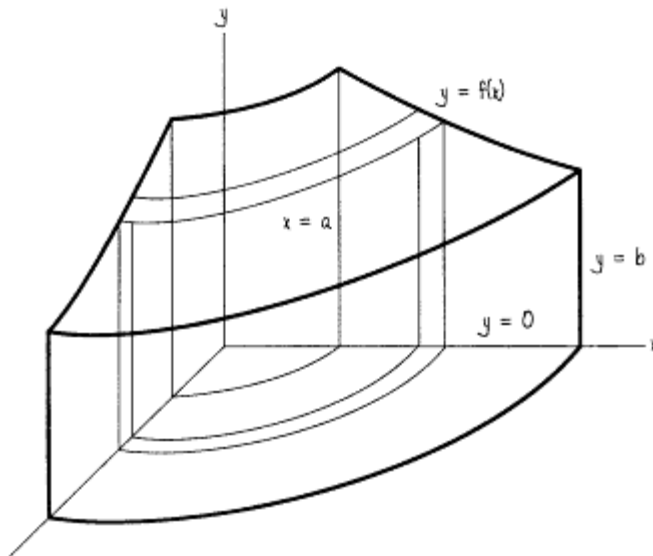
Example 11—

Let us try to do the same region R we had before and rotate it about the y-axis. We take the same region for two reasons: (1) you do not have to worry about different curves, and (2) we would like to show the answers. are the same.

$$V = \int 2\pi \cdot \text{average radius} \cdot \text{average height} \cdot \text{thickness}$$
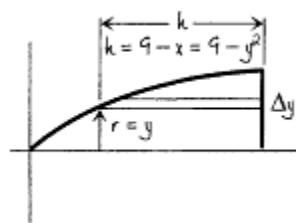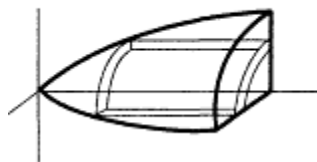
$$= 2\pi \int_0^9 x \cdot x^{1/2} \, dx = 2\pi \int_0^9 x^{3/2} \, dx = \frac{4}{5} x^{5/2} \Bigg[_0^9 \pi$$

$$= \frac{4\pi}{5} (9^{5/2} - 0) = \frac{972\pi}{5}$$

Example 12—

We will do our region one last time about the x-axis.

$$V = 2\pi \int_{y=0}^3 \text{average radius} \cdot \text{average height} \cdot \text{thickness}$$

$$= 2\pi \int_0^3 y(9 - y^2) \, dy$$

$$= 2\pi \int_0^3 (9y - y^3) \, dy = \frac{9y^2}{2} - \frac{y^4}{4} \Bigg[_0^3 \, 2\pi$$

$$= 2\pi \left[\frac{9(3)^2}{2} - \frac{3^4}{4} \right] = \frac{81\pi}{2}$$

The last part of this chapter is volume by sections. The sections are not circles but other shapes. Examples are given.

Example 13—

Find the volume of the following figure: base bounded by $y = 1/4x^2$ and $y = 4$ in the x-y plane; sections perpendicular to the y-axis are rectangles with height 1/3 the base.
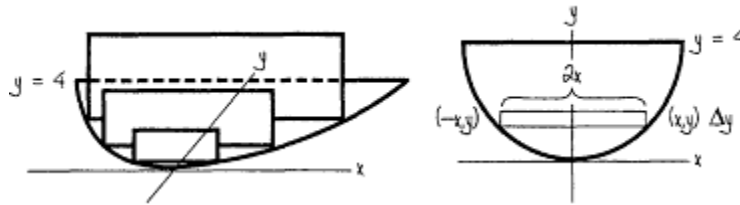
$A = bh = b(1/3b) = b^2/3$ $b = x - (-x) = 2x$

$y = \frac{1}{4}x^2$ $x^2 = 4y$ $x = 2y^{1/2}$ $2x = 4y^{1/2} = b$
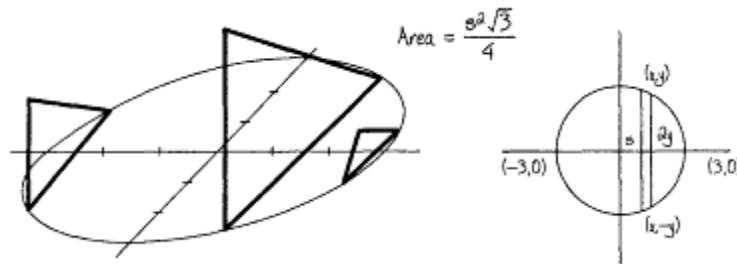
$A = (4y^{1/2})^2/3 = 16y/3$

The volume of each section $= 16y/3 \; \Delta y$.

$$V = \int_0^4 \frac{16y}{3} \, dy = \frac{8y^2}{3} \Big[_0^4 = \frac{8(4)^2}{3} - \frac{8(0)^2}{3} = \frac{128}{3}$$

Example 14—

Base $x^2 + y^2 = 9$ sections perpendicular to the x-axis are equilateral triangles. Find the volume.
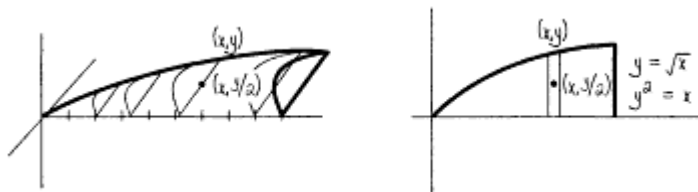

$$\int_{-3}^3 A(x) \, dx \qquad A = \frac{s^2\sqrt{3}}{4} = \frac{(2y)^2\sqrt{3}}{4} = \sqrt{3}y^2 = \sqrt{3}[9 - x^2]$$

$$\sqrt{3} \int_{-3}^3 (9 - x^2) \, dx = \sqrt{3}\left[9x - \frac{x^3}{3} \right]_{-3}^3 = \sqrt{3}[18 - (-18)]$$

$$= 36\sqrt{3}$$

Even though the base is a circle, we are adding triangular slices. Since the area of a triangle does not involve π, neither does our volume.

Example 15—

Base bounded by $y = x^{1/2}$, $x = 9$, x-axis. Sections perpendicular to the x-axis are semicircles. Find the volume.


$$\int_0^9 A(x)\,dx \qquad A = \frac{1}{2}\,\pi r^2 = \frac{1}{2}\,\pi(y/2)^2 = \frac{\pi}{8}\,y^2 = \frac{\pi x}{8}$$

$$\frac{\pi}{8}\int_0^9 x\,dx = \frac{\pi x^2}{16}\bigg|_0^9 = \frac{81\pi}{16}$$

Since we are adding circular slices (well, to be absolutely accurate, semicircular slices), the volume does have a π in it.

Chapter 12—
Conic Sections—Circle, Ellipse, Parabola, Hyperbola

Most books call the circle, parabola, ellipse, and hyperbola *conic sections* without explaining why. These curves are found by passing a plane through the truncated (cut-off) right circular cone pictured here. They are formed as follows:
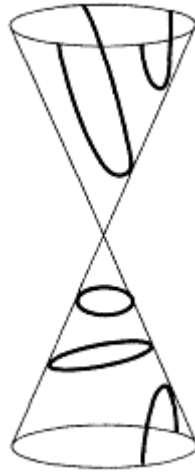
Circle Plane parallel to the top or bottom

Ellipse Plane on the top or bottom not parallel to the top or bottom but hitting all parts of the outside.

Parabola A plane parallel to an edge of the cone

Hyperbola Figure formed by a plane intersecting the top and bottom

Definition

Circle—The set of points that are equidistant from a point, the center (h, k). That distance is r, the radius.

The distance formula:

$$r = [(x - h)^2 + (y - k)^2]^{1/2}$$

Squaring, we get $(x - h)^2 + (y - k)^2 = r^2$.

Example 1—
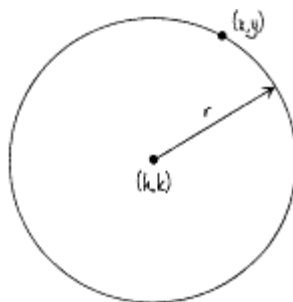
Find the radius and center if $(x - 4)^2 + (y + 1/2)^2 = 11$. $r = 11\ 1/2$, center $(4, -1/2)$.

Find the center and radius of the circle $2x^2 + 2y^2 + 8x - 16y + 6 = 0$. In order to do this, we have to complete the square, something we have not done since the derivation of the quadratic formula.

Example 2—

$2x^2 + 2y^2 + 8x - 16y + 6 = 0$

Divide by the coefficient of x^2

$x^2 + y^2 + 4x - 8y + 3 = 0$

Group the x terms and y terms, and get the constant to the other side.

$x^2 + 4x + y^2 - 8y = -3$

Take half of 4, square it, add it to both sides; take half of -8, square it, add it to both sides.

$x^2 + 4x + 4 + y^2 - 8y + 16 = -3 + 4 + 16$

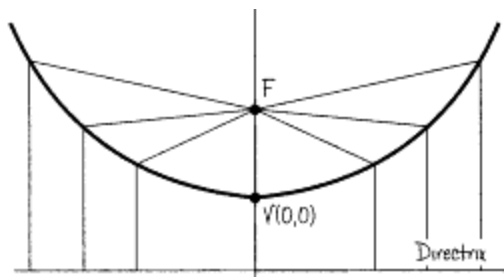| Factor into perfect squares (that was the idea) and add the terms on the right. |
| --- |

$(x + 2)^2 + (y - 4)^2 = 17$

The center is (-2,4); r = $17^{1/2}$.

For the parabola, ellipse, and hyperbola, it is essential to relate the equation to the picture. If you do, these curves are very simple.

Definition

Parabola—the set of all points that are equidistant from a point, called a *focus*, and a line called a *directrix*.
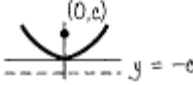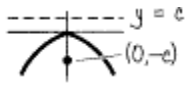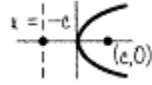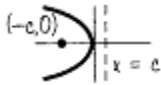
Point V is the vertex, equidistant from the focus and directrix and the closest to the directrix and to the focus.
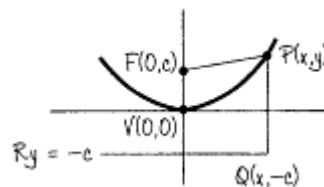
Let us do this development algebraically. Let the vertex be at (0,0). The focus is (0,c). The directrix is y = -c. Let (x,y) be any point on the parabola. The definition of a parabola says FP = PQ. Just like before, everything on PQ has the same x value, and everything on RQ has the same y value. The coordinates of Q are (x,-c). Since the x values are the same, the length of PQ = y - (-c). Using the distance formula to get FP and setting it equal to FP, we get $[(x - 0)^2 + (y - c)^2]^{1/2} = y + c$. Squaring, we get $x^2 + y^2 - 2cy + c^2 = y^2 + 2cy + c^2$. Simplifying, we get $x^2 = 4cy$.

We will make a small chart relating the vertex, focus, directrix, equation, and graph.

Vertex	Focus	Directrix	Equation	Graph	Comment
(0,0)	(0,c)	$y = -c$	$x^2 = 4cy$		The original derivation
(0,0)	(0,-c)	$y = c$	$x^2 = -4cy$		y replaced by -y
(0,0)	(c,0)	$x = -c$	$y2 = 4cx$		x,y interchange in 1
(0,0)	(-c,0)	$x = c$	$y^2 - 4cx$		x replaced by -x in 3

If you relate the picture to the original equation, the sketching will be easy.

Example 3—
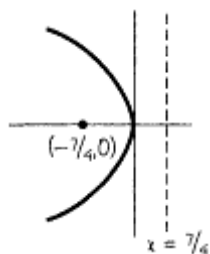
Given $y^2 = -7x$, sketch. Label the vertex, focus, and directrix.

From the chart, we know the sketch is picture 4. Now let $4c = 7$ (ignore the minus sign); $c = 7/4$. The vertex is (0,0). The focus is (-7/4,0), because it is on x-axis to the left of the origin. The directrix is $y = 7/4$; y, a vertical line, $= +7/4$, because it is to the right of the origin.

Example 4—
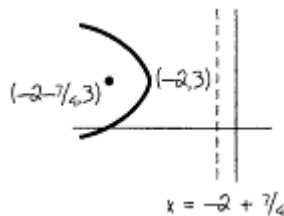
Sketch $(y- 3)^2 = -7(x + 2)$.

To understand the following, we need only note the difference between $x^2 + y^2 = 25$ and $(x- 3)^2 + (y + 6)^2 = 25$. Has the shape changed? No. Has the radius changed? No. What has changed? The center. Instead of being at the point (0,0), the center is at the point (3,-6).

In the case of the parabola, what has changed is the vertex. Instead of being at the point (0,0), the vertex is at the point (-2,3). The shape is the same. The value for 4c is still 7. So c = 7/4. The focus now becomes (-2 - 7/4,3), 7/4 to the left of the vertex (-7/4 from the x coordinate). The directrix is x = -2 + 7/4.

Example 5—

Sketch the parabola $2X^2 + 8X + 6y + 10 = 0$:

Original

$2x^2 + 8x + 6y + 10 = 0$

Divide through by the coefficient of the squared variable.

$x^2 + 4x + 3y + 5 = 0$

On one side, get all the terms that have the squared letter; everything else to the other side.

$x^2 + 4x = -3y - 5$

Complete the square; add to both sides.

$x^2 + 4x + 4 = -3y - 5 + 4$

Factor and simplify.

$(x + 2)^2 = -3y - 1$

Weird thing. No matter what the coefficient on the right side is, factor the whole coefficient out, even if there is a fraction in the parentheses.

$(x + 2)^2 = -3(y + 1/3)$

Sketch v(-2,-1/3), shape 2, ∩; 4c = 3, c = 3/4; F(-2, -1/3, -3/4); directrix y =-1/3 + 3/4.
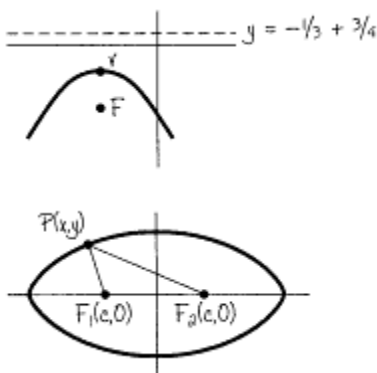
We will now look at the ellipse. Algebraically, the *ellipse* is defined as $PF_1 + PF_2 = 2a$, where $2a > 2c$, the distance between F_1 and F_2. In words, given two points, F_1 and F_2, you get two foci. If we find all points P, such that if we go from F_1 to P and then from P to F_2, add those two distances together, and always get the same number, 2a, where a will be determined later, we will get an ellipse.

I know you would desperately like to know how to draw an ellipse. This is how. Take anonelastic string. Attach both ends with thumbtacks to the table. Take the pencil point and stretch the string as far as it will go. Go 360º. You will trace out an ellipse.

Some of you have seen the equation for an ellipse, but few of you have seen the derivation. It is an excellent algebraic exercise for you to try. You will see there is a lot that goes into a rather simple equation.

$$PF_1 + PF_2 = 2a$$

$$\sqrt{[x - (-c)]^2 + (y - 0)^2} + \sqrt{(x - c)^2 + (y - 0)^2} = 2a$$

$$[\sqrt{(x + c)^2 + y^2}]^2 = [2a - \sqrt{(x - c)^2 + y^2}]^2$$

$$x^2 + 2cx + c^2 + y^2$$

$$= 4a^2 + x^2 - 2cx + c^2 + y^2 - 4a\sqrt{(x - c)^2 + y^2}$$

Combine like terms; isolate the radical.

$$4cx - 4a^2 = -4a\sqrt{(x - c)^2 + y^2}$$

$$(cx - a^2)^2 = [-a\sqrt{(x - c)^2 + y^2}]^2$$

$$a^4 - 2a^2cx + c^2x^2 = a^2(x^2 - 2cx + c^2 + y^2)$$

$$a^4 - a^2c^2 = a^2x^2 - c^2x^2 + a^2y^2$$

Reverse sides; take out common factors. Divide on both sides by $a^2(a^2 - c^2)$.

$$\frac{(a^2 - c^2)x^2 + a^2y^2}{(a^2 - c^2)a^2} = \frac{a^2(a^2 - c^2)}{a^2(a^2 - c^2)} \frac{a^2(a^2 - c^2)}{a^2(a^2 - c^2)}$$

Let $a^2 - c^2 = b^2$.

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

Finally we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Whew!

We are still not finished. What is a and what is b? Let's investigate.

Since T is any point on the ellipse, $F_1T + TF_2 = 2a$. By symmetry, $F_1T = TF_2$. So $F_1T = a$. Since $a^2 - c^2 = b^2$, $GT = b$. The coordinates of T are $(0, b)$, and the coordinates of T' are $(0, -b)$.
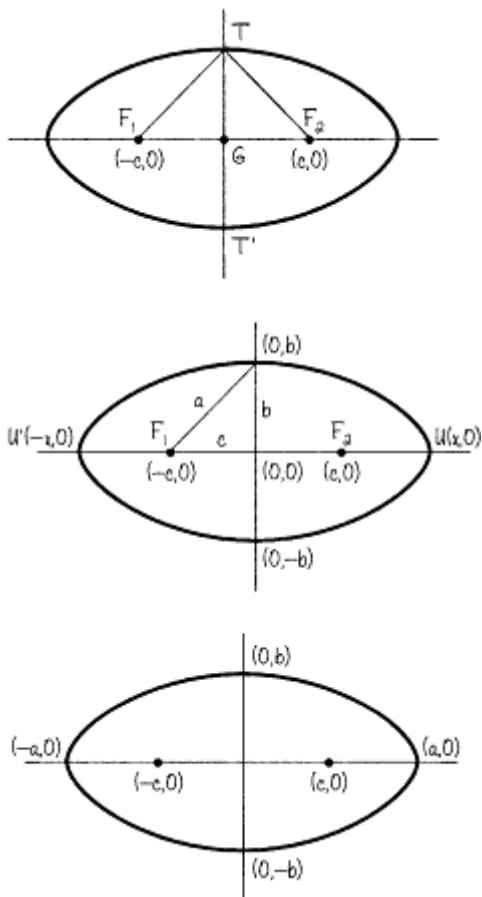
We would like to find the coordinates of U, and we have used up the letters a, b, and c. Oh, well, let's see what happens. $F_2U + UF_1 = 2a$; $F_2U = x - c$; $UF_1 = x + c$; $x + c + x - c = 2a$. So $x = a$. The coordinates of U are (a,0). The coordinates of U' are (-a,0).

Now, c = half the distance between the foci; b = length of the *semiminor* axis, (*semi* means half, *minor* means smaller, *axis* means line); a = length of the *semimajor axis* = *distance* from a focus to a minor vertex; (±a,O) are the *major vertices*; (0,±b) are the *minor vertices* or *co-vertices*. (±c,0) are the foci.

Although the derivation is very long, sketching should be short.

Example 6—

Sketch $x^2/11 + y^2/8 = 1$.

In the case of an ellipse, the longer axis is indicated by which number is larger under x^2 or y^2. That term is a^2. (Try not to remember a or b—remember the figure.) This ellipse is longer in the x direction.

Letting $y = 0$, we get the major vertices $(\pm\sqrt{11},0)$. Letting $x = 0$, we get the minor vertices $(0,\pm\sqrt{8})$. $c = \sqrt{11 - 8}$. The foci are $(\pm\sqrt{3},0)$. See the figure.
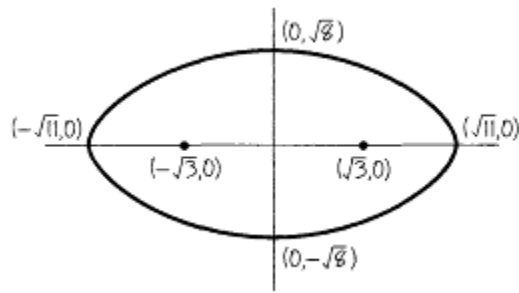
Example 7—

Sketch $x^2/5 + y^2/26 = 1$.

Major vertices $(0, \pm\sqrt{26})$; minor vertices $(\pm\sqrt{5}, 0)$; foci $(0, \pm\sqrt{21})$—foci are always on the longer axis.
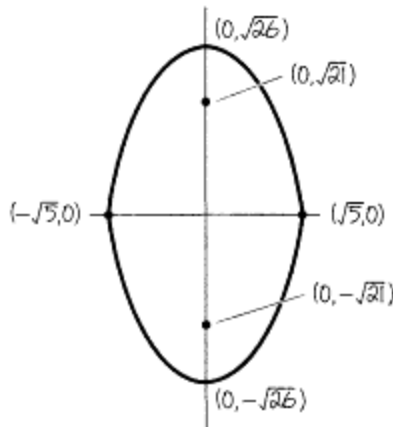
Example 8—

$$\frac{(x-6)^2}{11} + \frac{(y+4)^2}{8} = 1$$

This is the same as Example 12 except the center is no longer at the point (0,0). It is moved to the point (6,-4). Now the major vertices are $(6 \pm \sqrt{11}, -4)$. The minor vertices are $(6, -4 \pm \sqrt{8})$. The foci are $(6 \pm \sqrt{3}, -4)$.
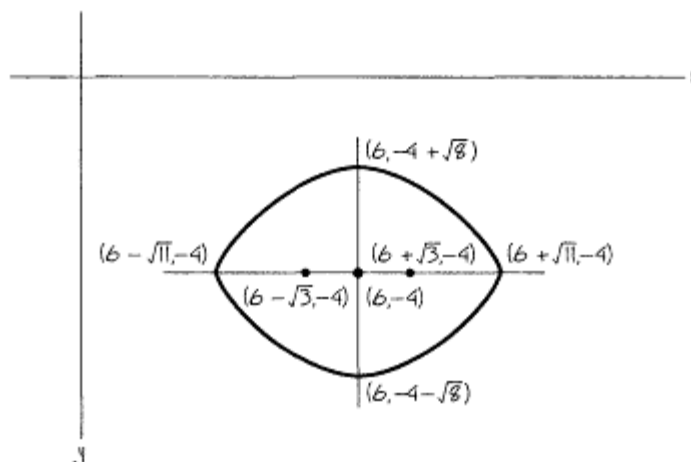
Weird numbers are intentionally chosen so that you know exactly where they come from.

Just a few months after the first edition of this book went to press, I realized a second way to find the vertices that made it much clearer to you, but my editor wouldn't change the book. So now I finally have a chance to show you. The center is again at the point (6,-4). The vertices are directly east-west and north-south of the point (6,-4). All points east-west of (6,-4) must have the same y value, y = -4. Soooo, letting y =-4, we get

$$\frac{(x-6)^2}{11} + \frac{(-4+4)^2}{8} = 1$$

$$\frac{(x-6)^2}{11} + 0 = 1$$

$$(x-6)^2 = 11; \quad x - 6 = \pm 11^{1/2}$$

Therefore, $x = 6 \pm 11^{1/2}$ and the major vertices are $(6 \pm 11^{1/2}, -4)$. Points north-south of (6,-4) have the same x value, $x = 6$.

$$(6-6)^2/11 + (y+4)^2/8 = 1$$

$$0 \quad + (y+4)^2/8 = 1$$

$$(y+4)^2 = 8. \quad y + 4 = \pm 8^{1/2}$$

Therefore, $y = -4 \pm 8^{1/2}$, and the minor vertices are $(6, -4 \pm 8^{1/2})$. For the same reasons the foci, always on the major axis, are $(6 \pm 3^{1/2}, -4)$. The sketch is, of course, the same.

Example 9—

Sketch and discuss $4x^2 + 5y^2 + 30y - 40x + 45 = 0$. Here we must complete the square in a slightly different manner.

$$4x^2 + 5y^2 + 30y - 40x + 45 = 0$$

$$4x^2 - 40x + 5y^2 + 30y = -45$$

$$4[x^2 - 10x + (-10/2)^2] + 5[y^2 + 6y + (6/2)^2]$$

$$= -45 + 4(-10/2)^2 + 5(6/2)^2$$

$$\frac{4(x-5)^2}{100} + \frac{5(y+3)^2}{100} = \frac{100}{100}$$

$$\frac{(x-5)^2}{25} + \frac{(y+3)^2}{20} = 1$$

Center (5,-3); vertices $(5 \pm \sqrt{25}, -3)$; $(5, -3 \pm \sqrt{20})$; $c = \sqrt{25-20} = \sqrt{5}$; foci $(5 \pm \sqrt{5}, -3)$. Of course, you should put 5 instead of $\sqrt{25}$. I leave the $\sqrt{25}$ for the sake of clarity.
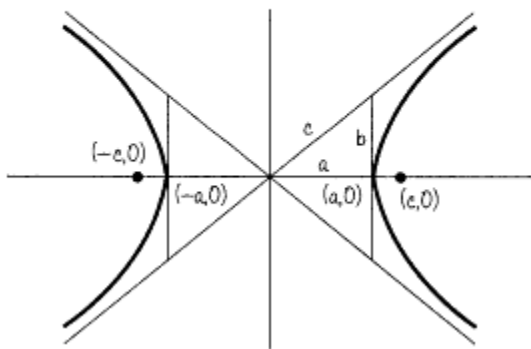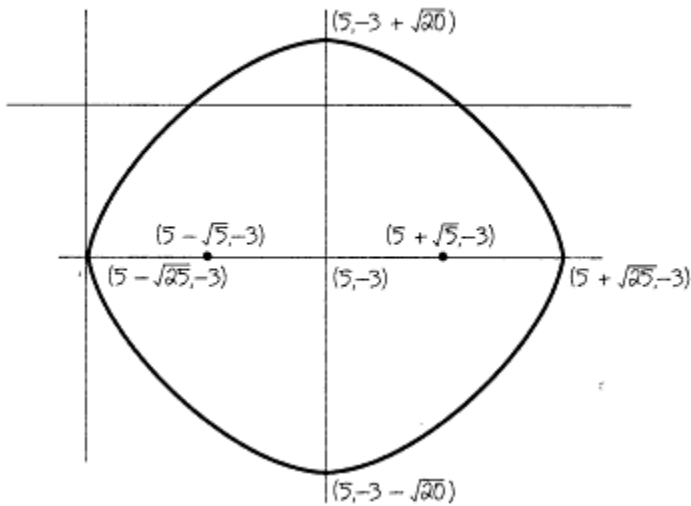
The definition of the *hyperbola* is $F_1P - PF_2 = 2a$ where the foci are $(\pm c, 0)$. The derivation is exactly the same as for an ellipse. Once is enough!! The equation we get is $x^2/a^2 - y^2/b^2 = 1$, where $a^2 + b^2 = c^2$. The coordinates $(\pm a, 0)$ are called the *transverse vertices*. The hyperbola has asymptotes $y = \pm(b/a)x$.

Note 1

The shape of a hyperbola is determined by the location of the minus sign, not which number is larger under the x^2 or y^2.

Note 2

In the case of the asymptote, the slope of the line b/a is the square root of the number under the y^2 divided by the square root of the term under the x^2 term.

Example 10—

Sketch and label $x^2/7 - y^2/11 = 1$.

Transverse vertices, $y = 0$, $x = \pm\sqrt{7}(\pm\sqrt{7}, 0)$,. Note that if $x = 0$, $y = \pm\sqrt{-11}$, which is imaginary. The curve does not hit the y-axis; $c = \sqrt{7 + 11}$; foci are $(\pm\sqrt{18}, 0)$; and $y = \pm(\sqrt{11}/\sqrt{7})x$.

Labels on first figure: $(\sqrt{7},0)$, $(-\sqrt{7},0)$, $(\sqrt{18},0)$, $(-\sqrt{18},0)$, $\sqrt{11}$, $\sqrt{7}$
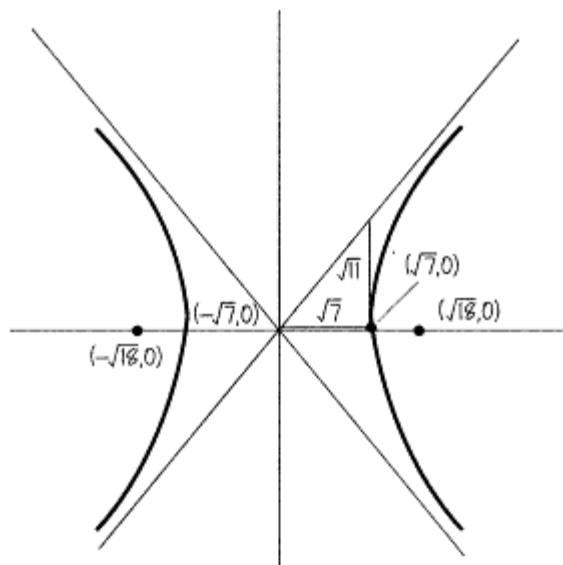
Example 11—

Sketch and discuss $y^2/5 - x^2/17 = 1$.

Transverse vertices $(0,\pm\sqrt{5})$; foci $(0,\pm\sqrt{22})$; asymptotes $y = \pm(\sqrt{5}/\sqrt{17})x$.

Labels on second figure: $(0,\sqrt{22})$, $(0,\sqrt{5})$, $\sqrt{17}$, $\sqrt{5}$, $(0,-\sqrt{5})$, $(0,-\sqrt{22})$
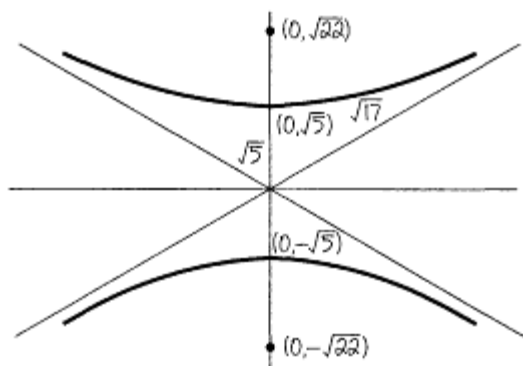
Example 12—

The same form $\dfrac{(y-6)^2}{5} - \dfrac{(x+7)^2}{17} = 1$.

This is the same as the previous sketch except the "center of the hyperbola," the place where the asymptotes cross, is no longer (0,0). It is now (-7,6). Transverse vertices $(-7,6 \pm \sqrt{5})$, foci $(-7,6 \pm \sqrt{22})$, asymptotes $(y-6) = \pm(\sqrt{5}/\sqrt{17})(x+7)$.
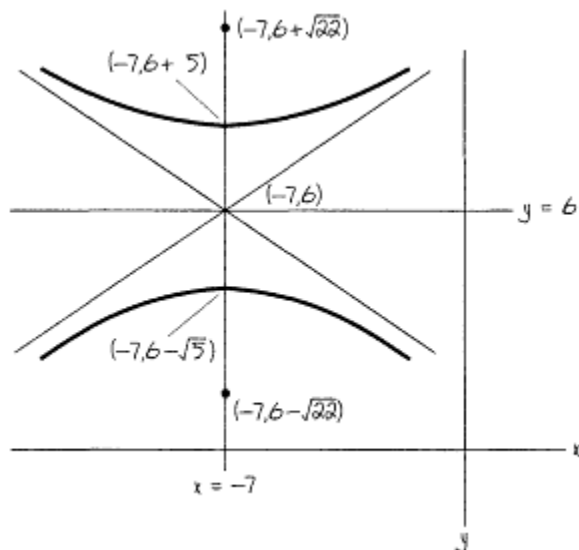
Just like with the ellipse, let's clarify the location of the vertices. In this case, the vertices occur north-south of the center (-7,6). So the x value of the vertices is the same as the center, x = -7. Substituting, we get

$(y - 6)^2/5 - (-7 + 7)^2/17 = 1$

$(y - 6)^2 = 5; y = 6 \pm 5^{1/2}$

The vertices are (-7,6 $\pm$ 5$^{1/2}$).

For the same reason, the foci, always on the axis through the vertices, are (-7,6 $\pm$ 22$^{1/2}$). The asymptotes and the sketch are the same!!

Example 13—

Sketch and discuss $25X^2 - 4y^2 + 50x - 12y + 116 = 0$.

For the last time, we will complete the square, again a little bit differently.

$25x^2 - 4y^2 + 50x - 12y + 116 = 0$

$25x^2 + 50x - 4y^2 - 12y = -116$

$25[x^2 + 2x + (2/2)^2] - 4[y^2 + 3y + (3/2)^2]$

$= -116 + 25(2/2)^2 - 4(3/2)^2$

$\dfrac{25(x + 1)^2}{-100} - \dfrac{4(y + 3/2)^2}{-100} = \dfrac{-100}{-100}$

$\dfrac{(y + 3/2)^2}{25} - \dfrac{(x + 1)^2}{4} = 1$

Center (-1,-3/2); vertices $(-1,-3/2 \pm \sqrt{25})$; foci $(-1,-3/2 \pm \sqrt{29})$; asymptotes $y + 3/2 = \pm(\sqrt{25}/\sqrt{4}) \cdot (x + 1)$.

Sometimes we have a puzzle. Given some information, can we find the equation? You must *always draw the picture* and relate the picture to its equation.

$\left(-1, \frac{-3}{2} + \sqrt{29}\right)$

$\left(-1, \frac{-3}{2} + \sqrt{25}\right)$

$\left(-1, \frac{-3}{2}\right)$

$y = \frac{-3}{2}$

$\left(-1, \frac{-3}{2} - \sqrt{25}\right)$

$x = -1$

$\left(-1, \frac{-3}{2} - \sqrt{29}\right)$
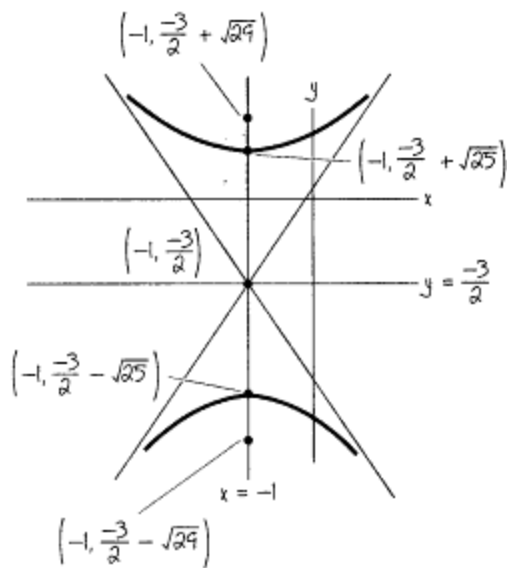
Example 14—

Find the equation of the parabola with focus (1,3), directrix x = 11.

Draw F and the directrix. Your sketch should match the figure. Vertex is halfway between the x numbers. So x = (11 + 1)/2 = 6; V = (6,3); c = the distance between V and F = 5. The equation is (y - 3)² = -4c(x - 6) = -20(x - 6). Remember, the minus sign comes from the shape, and c is always positive for these problems.

V(6,3)

(1,3)

x = 11
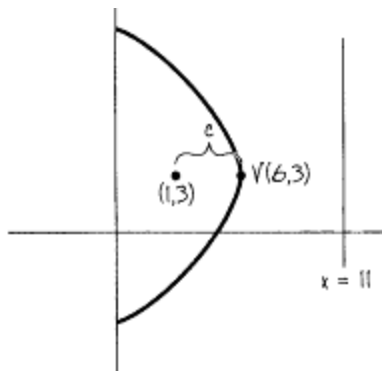
Example 15—

Given vertices (2,3) and (12,3) and one focus (11,3), find the equation of the ellipse.

Two vertices give the center; $(12 + 2)/(2,3) = (7,3)$. F(11,3). $(x - 7)^2/a^2 + (y - 3)^2/b^2 = 1$. $a = 12 - 7 = 5$. $c = 11 - 7 = 4$. $a^2 - b^2 = c^2$. $5^2 - b^2$. $b^2 = 9$ (no need for b). $(x - 7)^2/25 + (y - 3)^2/9 = 1$.
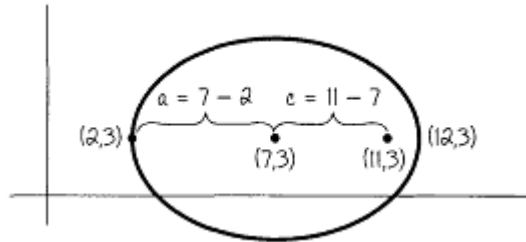
Example 16—

Find the equation of the hyperbola with vertices $(0,\pm6)$ and asymptotes $y = \pm (3/2)x$.
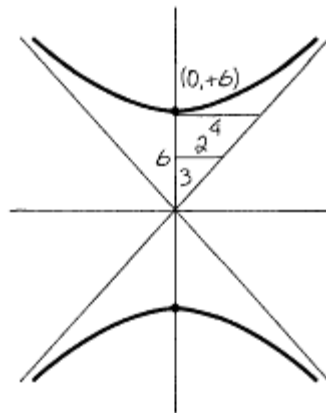
$V(0,\pm6)$ says the center is (0,0) and its shape is $y^2/36 - x^2/a^2 = 1$. The slope of the asymptotes is $3/2 = $ square root of the number under y^2 over the square root of the number under the x^2 term. So $3/2 = 6/\_$. So $\_= 4$. So $a^2 = 4^2 = 16$. The equation is $y^2/36 - x^2/16 = 1$.

This kind of question is shorter in length, but it does take practice. So practice!!

Chapter 13—
Odds and Ends

Approximations, Approximations

In an age where computers and calculators—especially those fun graphing calculators—do so many things, some things cannot be done exactly.

We know that all quadratics can be solved using the quadratic formula. Similarly, there is a cubic formula and a quartic (4th-degree formula) that can solve all cubics or quartics (although they are truly ugly and messy). However, in higher mathematics we can prove that most general 5th-degree equations cannot be solved. More simply, an equation like $2x = \cos x$ cannot be solved exactly. However, we can approximate a solution verrry closely.

Newton's Method

Suppose we have an equation $y = f(x)$. Let us say it crosses the x axis at $x = r$. That is the root $f(r) = 0$, but it is not exact. If we can find $f(a) < 0$ and $f(b) > 0$, then if $f(x)$ is continuous, there is a point r such that $f(r) = 0$
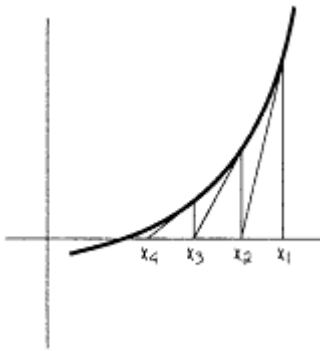
and r is between a and b. We can use *Newton's method.*

1. Let x_1 be the first approximation. Draw a tangent line at the point $[x_1, f(x_1)]$ until it hits the x-axis at x_2, which is usually closer to r than x_1. Continue.

2. Let us give a formula using point slope, $y - f(x_1) = f'(x_1)(x - x_1)$. If $f'(x_1) \neq 0$, the line is not parallel to the x-axis, and the line hits it at, let's say, the point $(x_2, 0)$. Substitute this point in the equation given.

3. $0 - f(x_1) = f'(x_1)(x_2 - x_1)$. We solve for x_2.

4. $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$. Repeating, we get the general formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Let us do an example.

Example 1—
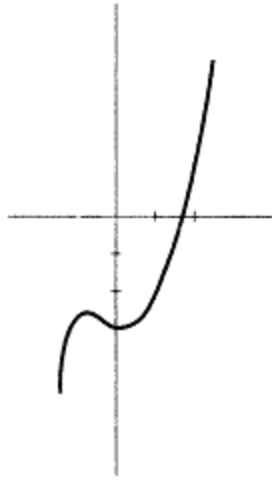
Find the root of $f(x) = x^3 - X - 3$ using Newton's method. The graph in the figure was made using the fun TI 82 calculator, $f(x) = x^3 - x - 3$. Therefore, $f'(x) = 3x^2 - 1$.

Newton's formula becomes

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^3 - x_n - 3}{3x_n^2 - 1}$$

Simplifying.......we get

$$x_{n+1} = \frac{2x_n^3 + 3}{3x_n^2 - 1}$$

$f(1) = -3$ and $f(2) = 3$.

Let us take $x_1 = 1.5$ for the first approximation. Probably $x_1 = 1$ or $x_1 = 2$ would work just fine.

$$x_2 = \frac{2(1.5)^3 + 3}{3(1.5)^2 - 1} = 1.695652174$$

$$x_3 = \frac{2(1.695652174)^3 + 3}{3(1.695652174)^2 - 1} = 1.672080792$$

Continuuuuing, we would get

$x_4 = 1.671700271$ $x_5 = 1.671699882$

$x_6 = 1.671699882$

We have reached the accuracy of our calculator: nine decimal places, which is probably more accurate than we would ever need.

Note 1

Sometimes the method doesn't work. A full study is left to other courses.

Note 2

When it works, it usually works very quickly and with great accuracy.

Note 3

This procedure, as well as the other two approximations, are suitable for many academic levels. Most appropriately, they occur in either Calc I or II.

The Trapezoidal Method
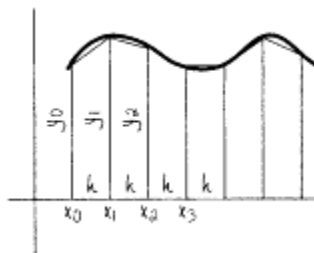
This method, the only one of the three that does not actually require calculus, approximates the area under the curve by trapezoids by approximating the top of the region by a line. Divide the region into n equal parts. See the trapezoids. They are standing on their heights. The area of a trapezoid is $\frac{1}{2}h(b_1 + b_2)$. All of the h's are the same. For the first trapezoid, $b_1 = y_1$
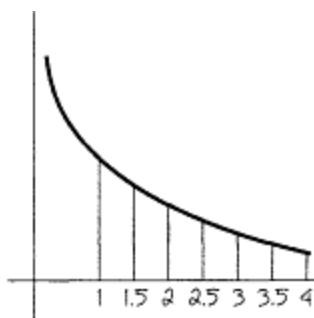
and $b_2 = y_2$; $A = \frac{1}{2}h(y_1 + y_2)$. For the second trapezoid, $b_1 = y_2$ and $b_2 = y_3$; $A = \frac{1}{2}h(y_2 + y_3)$. Notice the lower base of the first trapezoid is the upper base of the second trapezoid. Every base is doubled except the first upper base and the last lower base. The formula is $A = \frac{1}{2}h(y_0 + 2y_1 + 2y_2 + ... + 2y_{n-1} + y_n)$.

Example 2—

Approximate $\int_1^4 \frac{1}{x} \, dx$ using six equal subdivisions.

The interval is of length 4 - 1 = 3. Six equal parts? Each h = 3/6 = ½.

$x_0 = 1$ $y_0 = 1/1 = 1$

$x_1 = 3/2$ $y_1 = 1(3/2) = 2/3$

$x_2 = 2$ $y_2 = 1/2$, $x_3 = 5/2$ $y_3 = 2/5$, $y_4 = 1/3$,

$y_5 = 2/7$, $y_6 = 1/4$

$A = \frac{1}{2}h(y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5 + y_6)$

    $= \frac{1}{4}[1 + 2(2/3) + 2(\frac{1}{2}) + 2(2/5) + 2(1/3) + 2(2/7) + \frac{1}{4}]$

    $= 1.405357143$

Parabolic Method
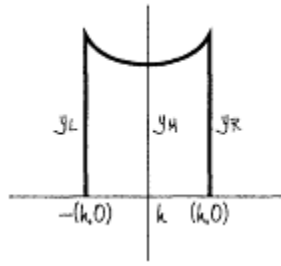
Another method is to approximate the region using parabolas on the top. We will isolate one of these regions. As you will see, this method works only if n is an even number of intervals.

Let the parabola be given by $y = f(x) = ax^2 + bx + c$: y at the left end is $y_L = f(-h) = ah^2 - bh + c$; y in the middle is $y_M = f(0) = c$; and y at the right is $y_R = f(h) = ah^2 + bh + c$.

The area of this region is

$$\int_{-h}^{h} (ax^2 + bx + c)\, dx$$

$$= ax^3/3 + bx^2/2 + cx \Big[_{-h}^{h}$$

$$= ah^3/3 + bh^2/2 + ch - (-ah^3/3 + bh^2/2 - ch)$$

$$= 2ah^3/3 + 2ch$$

Do something weird. Factor out h/3 because it works!!

$$= h/3(2ah^2 + 6ch)$$

$$= h/3(2ah^2 + 2ch + 4ch)$$

$$= h/3(y_L + y_R + 4y_M)$$

Just like before, the lower base of the first region is the upper base of the second region. Four times the middle never changes. The formula issss ... $A = h/3(y_0 + 4y_1 + 2y_2 + 4y_3 + ... + 4y_{n-1} + y_n)$, n being even.

Example 3—

Let's do the same example. Approximate $\int_1^4 \frac{1}{x}\, dx$

$$A = h/3(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6)$$

$$A = 1/6[1 + 4(2/3) + 2(\tfrac{1}{2}) + 4(2/5) + 2(1/3) + 4(2/7) + \tfrac{1}{4}]$$

$$= 1.387698413$$

As some of you know and the rest will find out soon, the approximate answer is that In 4 = 1.386294361.

Example 4—

A. Graph $y = xe^{-x}$. B. Find the area $x \geq 0$. C. Find the volume if the region is rotated about the x-axis.

Graphing exponentials is new, but we'll review graphing in general, areas, volumes of revolution, improper integrals, L'Hopital's rule, and integration by parts.

A. Graph $y = xe^{-x} = x/e^x$; $y' = (1 - x)e^{-x} = (1 - x)/e^x$; $y'' = (x-2)e^{-x} = (x-2)/e^x$:

1. x intercept(s) $y = 0$; top of fraction $= 0$; $x = 0$; point $(0,0)$.

2. $(0,0)$ is also the y intercept.

3. No vertical asymptotes, since the bottom of y is never 0.

4. Horizontal asymptote. This is different.

Asymptote is y = 0, but

$$\lim_{x \to \infty} \frac{x}{e^x} \overset{\text{\textcircled{L}}}{=} \frac{1}{e^x} = 0$$

---
since x is negative and e⁻ˣ gets very big as x goes to $-\infty$

$$\lim_{x \to -\infty} xe^{-x} \to -\infty$$

We have a one-sided asymptote. Many exponentials have a one-sided asymptote.
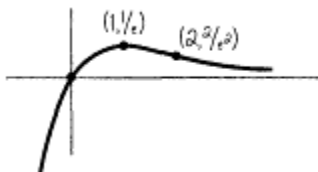
5. Possible max, min $y' = 0$; top $= 0$; $1 - x = 0$; $x = 1$; $y = 1/e^1$ $(1,1/e)$ $y''(1) = (1 - 2)/e$ is negative, so $(1,1/e)$ is a max.

6. Possible inflection points $y'' = 0$. Again the top $= 0$; $(x - 2) = 0$; $x = 2$; $y = 2/e^2$. The point is $(2,2/e^2)$. It is an inflection point, since $y''(2^+)$, $y''(2^-)$ are different signs.

Note

\textcircled{L}, of course, means equal by L'Hopital's rule. If you need to review curve sketching, look at *Calc I Helper*. Now let's find the area of the right part.

B. Find the area of $y = x e^{-x}$ where $x \geq 0$:

The improper part is infinity.

$$\int_0^{\infty} x e^{-x} \, dx$$

Integrate by parts: u = x and dv = e⁻ˣ.

$$= \lim_{a \to \infty} \int_0^a x e^{-x} \, dx$$

$$= \lim_{a \to \infty} \left(\frac{x\,e^{-x}}{-1} \Big[^a_0 + \int_0^a e^{-x}\,dx \right)$$

$$= \lim_{a \to \infty} \left(\frac{xe^{-x}}{-1} \Big[^a_0 - e^{-x} \Big[^a_0 \right)$$

$$= \lim_{a \to \infty} (-a/e^a + 0/e^0 - 1/e^a + 1/e^0)$$

$$= \qquad -0 \;+\; 0 \;-\; 0 \;+\; 1 \;=\; 1 \text{ square unit.}$$

C. Find the volume if the region $y = x\,e^{-x}$ is rotated about the x-axis, where $x \geq 0$:

Improper part is again infinity.

$$\pi \int_0^\infty r^2\,dx \qquad r = y = xe^{-x};\; r^2 = y^2 = x^2 e^{-2x}$$

Again, integration by parts: $u = x^2$, dv $= e^{-2x}$.

$$= \lim_{a \to \infty} \pi \int_0^a x^2\,e^{-2x}\,dx$$

$u = x,\; dv = e^{-2x}$.

$$= \lim_{a \to \infty} \left(\frac{\pi x^2\,e^{-2x}}{-2} \Big[^a_0 + \pi \int_0^a \frac{2x\,e^{-2x}}{2}\,dx \right)$$

$$= \lim_{a \to \infty} \left(\frac{\pi x^2\,e^{-2x}}{-2} \Big[^a_0 + \frac{\pi x\,e^{-2x}}{-2} \Big[^a_0 + \pi \int_0^a \frac{e^{-2x}}{2}\,dx \right)$$

$$= \lim_{a \to \infty} \left(\frac{\pi}{-2} x^2\,e^{-2x} \Big[^a_0 - \frac{\pi}{2} xe^{-2x} \Big[^a_0 - \frac{\pi}{4} e^{-2x} \Big[^a_0 \right.$$

$$= \lim_{a \to \infty} - \frac{\pi\,a^2}{2\,e^{4a}} + \frac{\pi\,0^2}{2\,e^0} - \frac{\pi\,a}{2\,e^{4a}} + \frac{\pi\,0}{2\,e^0} - \frac{\pi\,1}{4\,e^{4a}} + \frac{\pi\,1}{4\,e^0}$$

$$= \qquad -0 \;+\; 0 \;-\; 0 \;+\; 0 \;-\; 0 \;+\; \frac{\pi}{4}$$

The volume is $\pi/4$ cubic units.

Chapter 14—
Infinite Sequences

This topic brings some controversy. Some people think it is very difficult. Some think it is very easy. I believe if you understand the beginning, the rest of the chapter is not too bad.

Definition

Sequence—A sequence of terms, technically, is a function for which the domain is the positive integers. Nontechnically, there is a first term called a_1 (read "a sub-one," where the "one" is a subscript, not an exponent) denoting the first term, a2 ("a sub-two") denoting the second term, and so on. The notation for an infinite sequence is $\{a_n\}$.

Let us give some examples. We will list some sequences, write the first four terms, and then term number 100 by substituting 1, 2, 3, 4,...,100 for n in a_n.

Example 1—

$\{a_n\}$	1st	2nd	3rd	4th	100th
$\left\{\dfrac{n}{n+1}\right\}$	$\dfrac{1}{2}$	$\dfrac{2}{3}$	$\dfrac{3}{4}$	$\dfrac{4}{5}$	$\dfrac{100}{101}$
$\left\{\dfrac{(-1)^{n+1}(4n+1)}{n^2+1}\right\}$	$\dfrac{5}{2}$	$\dfrac{-9}{5}$	$\dfrac{13}{10}$	$\dfrac{-17}{17}$	$\dfrac{-401}{10001}$
$\{6\}$	6	6	6	6	6

Definition (Nontechnical)

We write $\lim_{n \to \infty} a_n = L$ if, the larger n gets, the closer an gets to L.

In this case, we say that $\{a_n\}$ converges to L (or has the limit L). If an goes to plus or minus infinity or does not go to a single number, then $\{a_n\}$ diverges (or has *no* limit).

Example 2—

Find the limit of $\{(n + 9)/n^2\}$.

$a_n = (n + 9)/n^2 = 1/n + 9/n^2$. As n goes to infinity, both terms go to 0. Therefore, the sequence converges to 0.

Example 3—

Find the limit of $\{(2n^2 + 3n + 2)/(5 - 7n^2)\}$.

Divide top and bottom of an by n^2. We get $[2 + (3/n) + (2/n^2)]/[(5/n^2 -7)]$. As n goes to infinity, an goes to 2/(-7). The sequence has the limit-2/7.

Note

This should look very familiar. This is how we found horizontal asymptotes. Also note that we can use L'Hopital's rule.

Example 4—

Find the limit of $\{a_n\} = \{\ln (n + 1) - \ln n\}$.

$$\lim_{n \to \infty} \ln (n + 1) - \ln n = \lim_{n \to \infty} \ln [(n + 1)/n]$$
$$= \lim_{n \to \infty} \ln (1 + 1/n) = \ln 1 = 0$$

The sequence converges to 0.

Example 5—

Does $\{(-1)^n\}$ have a limit?

This sequence is -1, +1,-1, +1,-1, ... There is no limit because the sequence does not go to one number.

Definition

$\lim_{n \to \infty} a_n = L$ if, given an $\varepsilon > 0$, there exists an $N > 0$, such that if $n > N$, $| a_n\text{-}L | < \varepsilon$.

Note

It is not important that you know the technical definition of a limit to understand the rest of the chapter. But ... at this point of your mathematical career, you should start understanding the background. It probably will help you later on. It would also be nice if you could see the beauty and the depth of this material—the beginnings of calculus. It truly is a wonderful discovery.

Example 6—

Using ε. N, show $\lim_{n \to \infty} (2n + 5)/(n + 1) = 2$.

$$\left| \frac{2n + 5}{n + 1} - 2 \right| = \left| \frac{2n + 5}{n + 1} - \frac{2(n + 1)}{n + 1} \right| = \left| \frac{3}{n + 1} \right| < \varepsilon$$

provided $3/\varepsilon < n + 1$ or $3/\varepsilon - 1 < n$. We then choose N as the whole-number part of $3/\varepsilon - 1$.

The following theorems are used often. They are proved in many books and will only be stated here.

Let $\lim_{n \to \infty} a_n = L$, $\lim_{n \to \infty} b_n = M$; k = constant, f continuous. Then

1. $\lim_{n \to \infty} (a_n \pm b_n) = L \pm M$

2. $\lim_{n \to \infty} (a_n b_n) = LM$

3. $\lim_{n \to \infty} (a_n/b_n) = L/M \qquad M \neq 0$

4. $\lim_{n \to \infty} k a_n = kL$

5. $\lim_{n \to \infty} f(a_n) = f(L)$

6. $c_n \leq d_n \leq e_n$ and $\lim_{n \to \infty} c_n = \lim_{n \to \infty} e_n = P$. Then $\lim_{n \to \infty} d_n = P$.

Example 7—

Show $\lim_{n \to \infty} (\sin n)/n = 0$.

Using part 6 above, $-1 \le \sin n \le 1$. So $-1/n \le (\sin n)/n \le 1/n$. As n goes to infinity, $-1/n$ and $1/n$ go to 0. Therefore, so does $(\sin n)/n$.

Definition 1

An *increasing sequence* is one where $a_n < a_{n+1}$ for all n.

Definition 2

A *non decreasing sequence* is one where $a_n \le a_{n+1}$ for all n.

Similarly we can define *decreasing* and *nonin-creasing*.

Definition 3

A sequence is bounded if $| a_n | \le M$, some number M and all n.

Another theorem: Every bounded increasing (decreasing) sequence has a limit.

Infinite Series

I know this is getting to be a drag, but it is essential to understand the terminology. This understanding will make the rest of the chapter *much* easier. I don't know why, but it really seems to.

Definition

Partial sums—Given sequence $\{a_n\}$:

1st partial sum $S_1 = a_1$

2nd partial sum $S_2 = a_1 + a_2$

3rd partial sum $S_3 = a_1 + a_2 + a_3$

nth partial sum $S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$

The *infinite series* $a_1 + a_2 + a_3 + \dots$ or $\sum_{k=1}^{\infty} a_k$ is said to *converge* to the sum S if $\lim_{n \to \infty} S_n = S$. If S does not exist, the series *diverges*.

Example 8—

.767676 ....

We can write this as an infinite series. .76 + .0076 + .000076 + ... This is a geometric series (infinite). This is one of the few series we can find the exact sum of.

$S = a/(1 - r)$ $a = .76$ $r = .01$ $S = .76/(1 - .01) = 76/99$

More generally, the series $a + ar + ar^2 + ar^3 + \dots$ converges to $a/(1 - r)$ if $| r | < 1$.

Example 9—

$4 - 8 + 16 - 32 + \cdots$: $a = 4$, $r = -2$; diverges.

Example 10—

$1 + 1 + 1 + 1 + 1 + \cdots$: $a = 1$, $r = 1$; diverges.

Example 11—

$1 - 1 + 1 - 1 + 1 - \cdots$: $a = 1$, $r = -1$; diverges.

Note that test 1 implies divergence in these three examples.

Example 12—

$$\sum_{k=1}^{\infty} \frac{1}{k(k + 1)}$$

Using partial fractions

$$\frac{1}{k(k + 1)} = \frac{1}{k} - \frac{1}{k + 1}$$

Writing out the first few terms plus the n - 1 term plus the nth term, we get

$$S_n = (1/1 - 1/2) + (1/2 - 1/3) + (1/3 - 1/4) + \cdots$$
$$+ [1/n - 1/(n + 1)]$$

Notice all the middle terms cancel out in pairs. So only the first and last terms remain:

$$S_n = 1 - 1/(n + 1) \qquad S = \lim_{n \to \infty} S_n = 1$$

Again, this is one of the few sequences we can find the exact value for. (This is called a *telescoping series*—it collapses like one of those toy or portable telescopes.) From this point on, for almost all of the converging series, we will be able to tell that the series converges, but we won't be able to find its value. Later we will do some approximating.

Example 13—

$$\sum_{k=1}^{\infty} \frac{4(2^k) + 5^k}{7^k}$$

After splitting, we get two geometric series:

$$S = \frac{4(2/7)}{1 - 2/7} + \frac{5/7}{1 - 5/7} = 8/5 + 5/2 = 41/10$$

Theorem

If $\sum_{n=1}^{\infty} a_n = L$ and $\sum_{n=1}^{\infty} b_n = M$, then $\sum_{n=1}^{\infty} (ca_n + b_n)$
$= cL + M$.

Now that we have an idea about what a sequence is and what an infinite series is (hopefully a *very good* idea), we would like to have some tests for when a series converges or diverges.

Test 1

It is necessary that $a_k \to 0$ for $\sum_{k=1}^{\infty} a_k$ to converge.

Note 1

If a_k does not go to 0, $\sum_{k=1}^{\infty} a_k$ diverges.

Note 2

If a_k does go to O, and that is all we know, we know nothing.

Example 14—

Tell whether $\sum_{k=1}^{\infty} k/(k+1)$ converges.

$k/(k+1)$ goes to 1. Therefore $\sum_{k=1}^{\infty} k/(k+1)$ diverges.

Example 15—

The harmonic series $\sum_{k=1}^{\infty} 1/k$

Since $1/k$ goes to 0, we don't know if this series converges or diverges. We shall shortly show that the harmonic series diverges.

Example 16—

The p_2 series $\sum_{k=1}^{\infty} 1/k^2$

Since $1/k^2$ goes to 0, again we can't tell. Shortly we shall show that the P_2 series converges.

Test 2

Given a_k and $a_k > 0$, a_k goes to 0 for k big enough. Suppose we have a continuous function $f(x)$ such that $f(k) = a_k$. Then $\sum_{k=1}^{\infty} a_k$ and $\int_1^{\infty} f(x)$ dx either both converge or both diverge.

This theorem is easily explained by examples.

Example 17—

Tell whether $\sum_{k=1}^{\infty} ke^{-k^2}$ converges or diverges.

The improper integral associated with $\sum_{k=1}^{\infty} ke^{-k^2}$ is $\int_1^{\infty} xe^{-x^2}$ dx. Letting $u = -x_2$, du $=2x$ dx.

$$\int_1^{\infty} xe^{-x^2}\, dx = \lim_{b \to \infty} \int_1^b \frac{-2xe^{-x^2}}{-2}\, dx = \lim_{b \to \infty} (-\tfrac{1}{2}) \int_{-1}^{-b^2} e^u\, du$$

$$= \lim_{b \to \infty} -\tfrac{1}{2}(e^{-b^2} - e^{-1}) = \frac{1}{2e}$$

Since the improper integral converges, so does the infinite series.

Note 1

The value of the improper integral is not the value of the infinite series. But we can say the following: If the integral and the series together converge, then $\int_1^{\infty} f(x)\, dx \le \sum_{k=1}^{\infty} a_k \le a_1 + \int_1^{\infty} f(x)\, dx$.

The bounds on $\sum_{k=1}^{\infty} ke^{-k^2}$ are

$$\frac{1}{2e} \le \sum_{k=1}^{\infty} ke^{-k^2} \le \frac{1}{e} + \frac{1}{2e}.$$

Note 2

In this case, this is not too good an approximation.

We will get a better one if we take the fourth partial sum:

$$= \frac{1}{e} + \frac{2}{e^4} + \frac{3}{e^9} + \frac{4}{e^{16}}$$

The "error," the estimate on the rest of the terms, is that

$$\sum_{k=5}^{\infty} ke^{-k^2} \le a_5 + \int_5^{\infty} xe^{-x^2}\, dx = \frac{5}{e^{25}} + \frac{1}{2e^{25}}\, 1.4 \times 10^{-11}$$

This is more accuracy than you will probably ever need!!! Lots of things you cannot even integrate.

Example 18—

The harmonic series $\sum_{k=1}^{\infty} 1/k$ diverges.

$$\lim_{b \to \infty} \int_1^b 1/x\, dx = \ln b,$$

which goes to infinity as b goes to infinity.

Example 19—

The p_2 series $\sum_{k=1}^{\infty} 1/k^2$ converges.

$\int_1^b 1/x^2\, dx = -1/b + 1$. Since-1/b goes to 0 as b goes to infinity, this improper integral converges. So does the p_2.

Example 20—

$$\sum_{k=1}^{\infty} 1/k^p$$

If $p > 1$, it converges, and if $p \le 1$, it diverges. Just use the integral test. It's easy.

Test 3

The comparison test. Given $\sum_{k=1}^{\infty} a_k, \sum_{k=1}^{\infty} b_k$ where $0 < a_k \le b_k$,

1. If $\sum_{k=1}^{\infty} b_k$ converges, so does $\sum_{k=1}^{\infty} a_k$.

2. If $\sum_{k=1}^{\infty} a_k$ diverges, so does $\sum_{k=1}^{\infty} b_k$.

Let us talk through part 1. The second part can be shown to be equivalent. The partial sums S_n of the $\sum_{k=1}^{\infty} b_k$ series are uniformly bounded because the first N terms are bounded by their maximum and the rest are bounded by $L + \varepsilon$. Therefore the partial sums of the $\sum_{k=1}^{\infty} a_k$ series also are bounded, being respectfully smaller than those of $\sum_{k=1}^{\infty} b_k$. Moreover, since $a_k > 0$, then the partial sums of the a_k form an increasing sequence. Now we have an increasing bounded sequence that has a limit. Therefore, $\sum_{k=1}^{\infty} a_k$ converges.

Example 21—

Examine $\sum_{k=1}^{\infty} 1/(4 + k^4)$.

$1/(4 + k^4) < 1/k^4$. $\sum_{k=1}^{\infty} 1/k^4$ converges by the previous example. Since the given series is smaller termwise than a convergent series, it must converge by the comparison test.

Example 22—

Examine $\sum_{k=1}^{\infty} (2 + \ln k)/k$.

$\sum_{k=1}^{\infty} (2 + \ln k)/k > \sum_{k=1}^{\infty} 2/k =$ twice a divergent series (the harmonic). Since the given series is larger term-wise than a divergent series, the given series must diverge.

Test 4

This is the limit comparison test. Given $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$, $a_k \geq 0$, $b_k \geq 0$. If $\lim_{k \to \infty} (a_k/b_k) = r$, where r is any positive number, both series converge, or both diverge.

Example 23—

$$\sum_{k=1}^{\infty} 3/(5k^4 + 4)$$

Let us compare this series with $\sum_{k=1}^{\infty} 1/k^4$.

Divide top and bottom by k^4.

$\lim_{k \to \infty} [3/(5k^4 + 4) \text{ divided by } 1/k^4]$

$= \lim_{k \to \infty} \dfrac{3k^4}{5k^4 + 4}$

$= \lim_{k \to \infty} \dfrac{3}{5 + (4/k^4)} = 3/5$

Since the limit is a positive number, both series do the same thing. Since $\sum_{k=1}^{\infty} 1/k^4$ converges, so does $\sum_{k=1}^{\infty} 3/(5k^4 + 4)$

Test 5 (Ratio Test)

Given $a_k \geq 0$, $\lim_{k \to \infty} (a_{k+1}/a_k) = r$ If $r > 1$, it diverges. If $r < 1$, it converges. If $r = 1$, use another test.

Example 24—

Examine $\sum_{k=1}^{\infty} k^2/5^k$.

$a_{k+1}/a_k = (k + 1)^2/5^{k+1}$ divided by $k^2/5^k$

$= \dfrac{(k + 1)^2}{5^{k+1}} \times \dfrac{5^k}{k^2} = \dfrac{k^2 + 2k + 1}{5k^2}$

The series converges.

$$\lim_{k \to \infty} \frac{a_{k+1}}{a_k} = \lim_{k \to \infty} \frac{1 + \dfrac{2}{k} + \dfrac{1}{k^2}}{5} = \frac{1}{5} < 1 \qquad \sum_{k=1}^{\infty} k^2/5^k$$

Example 25—

Examine $\sum_{k=1}^{\infty} 7^k/k!$

Note

6! means 6(5)(4)(3)(2)(1).

Also Note:

$(k + 1)! = (k + 1)(k!)$. That is, $10! = 10(9!)$, and so on.

Let us again use the ratio test.

$a_{k+1}/a_k = 7^{k+1}/(k + 1)!$ divided by $7^k/k!$

$$= \frac{7^{k+1}}{(k + 1)!} \times \frac{k!}{7^k} = \frac{7}{k + 1}$$

$$\lim_{k \to \infty} \frac{7}{k + 1} = 0 < 1 \qquad \sum_{k=1}^{\infty} 7^k/k!$$

The series converges.

Example 26—

Let's now look at $\sum_{k=1}^{\infty} k^k/k!$

This is a little trickier than most. Again, we use the ratio test.

$a_{k+1}/a_k = (k + 1)^{k+1}/(k + 1)!$ divided by $k^k/k!$

$$= \frac{(k + 1)^{k+1}}{(k + 1)!} \times \frac{k!}{k^k} = \frac{(k + 1)(k + 1)^k k!}{(k + 1)k!k^k} = \frac{(k + 1)^k}{k^k}$$

$$= (1 + 1/k)^k$$

Since the $\lim_{k \to \infty} [1 + (1/k)]^k = e > 1$, $\sum_{k=1}^{\infty} k^k/k!$ diverges.

Example 27—

In order to show the third part of the previous theorem, you should apply the ratio test to both the harmonic series and the p_2 series. Both give a ratio of 1. The first series diverges, and the second converges. So, if the ratio is 1, we must indeed use another test.

Test 6 (Root Test)

Given $\sum_{k=1}^{\infty} a_k$. Take $\lim_{k \to \infty} (a_k)^{1/k} = r$ $(a_k \geq 0)$. If $r > 1$, it diverges. If $r < 1$, it converges. If $r = 1$, use another test.

Note

To show the third part (r = 1), we would again use the harmonic and P2 series. Let us give examples for the first two parts (r > 1 and r < 1).

Example 28—

$$\sum_{k=1}^{\infty} 3^k/k^k$$

Take $(3^k/k^k)^{1/k} = 3/k$. $\lim_{k \to \infty} (3/k) = 0 < 1$. So the series converges.

Example 29—

$$\sum_{k=1}^{\infty} 2^k/k^2$$

Take $(2^k/k^2)^{1/} = 2/k^{2/k}$. $\lim_{k \to \infty} 2/k^{2/k} = 2/1 = 2$ $(\lim_{k \to \infty} k^{2/k} = \lim_{k \to \infty} e^{2(\ln k)/k} = e^{2(0)} = 1)$. Since $2 > 1$, the series diverges.

Up to this time, we have dealt exclusively with positive terms. Now we will deal with infinite series that have terms that alternate from positive to negative. We will assume the first term is positive. The notation will be as follows: alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$, where all a_k are positive.

TEST 7

Given an alternating series where (A) $0 < A_{k+1} \le a_k$, k = 1, 2, 3, 4, ... and (b) $\lim_{k \to \infty} a_k = 0$, the series converges to S and $S \le a_1$

In clearer English, the *only* thing you must do to show an alternating series converges is to show the terms go to zero. (If only all series were that easy?)

Example 30—

Alternating harmonic

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$

converges since the terms go to 0.

Definition

Absolutely convergent—A series $\sum_{k=1}^{\infty} a_k$ converges absolutely if $\sum_{k=1}^{\infty} |a_k|$ converges.

Note

If a series converges absolutely, it converges.

Definition

Conditionally convergent—A series $\sum_{k=1}^{\infty} a_k$ converges conditionally if it converges but $\sum_{k=1}^{\infty} |a_k|$ diverges.

Note 1

If we have an alternating series and want to show that it converges conditionally, we only have to show its terms go to zero. To find out whether it is absolutely convergent, we must use some other test.

Note 2

There are three possibilities for an alternating series: it diverges, converges conditionally, or converges absolutely.

Let us look at three alternating series.

Example 31—

Let us look at $\sum_{k=1}^{\infty} (-1)^{k+1}/(2k + 1)$. This series converges conditionally since (1) the terms go to zero, but (2) using the limit comparison test with the harmonic series, the positive series behaves as the harmonic series and diverges.

Example 32—

What about the series $\sum_{k=1}^{\infty} (-1)^{k+1}/(k^2 + 1)$? This series converges absolutely by comparing to the p_2 series.

Example 33—

$\sum_{k=1}^{\infty} (-1)^{k+1} k^2/(k^2 + 6)$ diverges since the terms don't go to 0.

Definition

Region of convergence—We have an infinite series whose terms are functions of x. The set of all points x for which the series converges is called the *region of convergence*.

Now let's get back to the series with x's in them. Series of this type are usually done with the *ratio test*. This is to find the region of convergence. Then, you will test both the left and right end points. There are three tests in all.

Example 34—

$$\sum_{k=1}^{\infty} x^k/k$$

Using the ratio test,

$$|a_{k+1}/a_k| = |x^{k+1}/(k+1)| \text{ divided by } |x^k/k|$$

$$= \left|\frac{x^{k+1}}{k+1}\right| \times \left|\frac{k}{x^k}\right| = \left|\frac{kx}{k+1}\right| \cdot \lim_{k \to \infty} \left|\frac{kx}{k+1}\right| = |x|$$

So the region of convergence is $|x| < 1$ or $-1 < x < 1$.

Let us test both 1 and -1 by substituting those values into the original series. $x = 1$ gives us $\sum_{k=1}^{\infty} 1^k/k$ or $\sum_{k=1}^{\infty} 1/k$, the harmonic series that diverges. $x = -1$ gives us $\sum_{k=1}^{\infty} (-1)^k/k$, the alternating harmonic series, or rather the negative of the alternating harmonic series, since the first term is negative. We know this converges. Therefore, the region of convergence is $-1 \le x < 1$.

Important Note

When you test the end points, anything is possible. Both ends could converge, both could diverge, the left could converge and not the right, or the right could converge but not the left.

Example 35—

Let's look at $\sum_{k=1}^{\infty} (x-4)^k/3^k$.

This is a geometric series that converges for

$$|r| = \left|\frac{x-4}{3}\right| < 1.$$

Thus, the region of convergence is $|x - 4| < 3$ or $1 < x < 7$. Test $x = 1$ and substitute into the original series. We get

$$\sum_{k=1}^{\infty} \frac{(-3)^k}{3^k} \text{ or } \sum_{k=1}^{\infty} (-1)^k = -1 + 1 - 1 + 1 \cdots$$

which diverges (Example 11). For $x = 7$, we get

$$\sum_{k=1}^{\infty} \frac{3^k}{3^k} = 1 + 1 + 1 + 1 + \cdots$$

which diverges (Example 10).

Example 36—

$\sum_{k=1}^{\infty} x^k/k!$, a nice one.

$$|a_{k+1}/a_k| = |x^{k+1}/(k+1)!| \text{ divided by } |x^k/k!|$$

$$= \left|\frac{x^{k+1}}{(k+1)!}\right| \times \left|\frac{k!}{x^k}\right| = \left|\frac{xk!}{(k+1)k!}\right| = \left|\frac{x}{k+1}\right|$$

$$\lim_{k \to \infty} \left|\frac{x}{k+1}\right| = 0.$$

This says no matter what x is, the limit will always be less than 1. The region of convergence is *all real numbers*.

Example 37—

$$\sum_{k=1}^{\infty} (k+3)!x^k$$

$$|a_{k+1}/a_k| = |(k+4)!x^{k+1}| \text{ divided by } |(k+3)!x^k|$$

$$= \left|\frac{(k+4)(k+3)!x^{k+1}}{(k+3)!x^k}\right| = |(k+4)x|$$

$\lim_{k \to \infty} |(k+4)x| \to \infty$ except if $x = 0$. The region of convergence is just the point $x = 0$.

Which Test To Use

After finishing the original draft of this book but before running off copies, I finished student-testing this section on infinite series. It became absolutely clear that this page is necessary.

1. Always see if the terms go to zero first. If they don't, the series diverges. If the terms go to zero, the series at least converges Conditionally if it alternates.

2. Use the integral test if the infinite series looks like an integral you have done. By this time, you should have so many integrals you should be familiar with and/or sick of them.

3. Don't use the integral test if you can see an easier one or if there is a factorial symbol.

4. My favorite is the ratio test. Always try the ratio test if there is a factorial or an x in the problem. Also try the ratio test if there is something to a power of k, such as 2^k or k^k.

5. Use the limit comparison test or the comparison test if the series looks like one you know like the harmonic series, p_2 series, and so on. Use the comparison if the algebra is not too bad. Use the limit comparison if the algebra looks really terrible or even semiterrible.

6. Use the root test if there is at least one term with k in the exponent and no factorial in the problem.

7. If there is a series where there are a lot of messy-looking terms multiplying each other, the ratio test is probably the correct one.

8. Sometimes you may not be able to tell the terms go to zero. The ratio test may give absolute convergence or divergence immediately.

9. Practice factorial. It is new to most of you. Once again, note that (2n. 1)! = (2n. 1)(2n)! = (2n + 1)(2n)(2n - 1)!, that is, 7! = 7(6!) = 7(6)(5!). Study factorial!!!!!

10. Most of all, do a lot of series testing. You will get better if you practice. The nice part is that the problems are mostly very short.

A Preview of Power Series

We would like to have a polynomial approximation of a function in the vicinity of a given point. Polynomials are very easy to work with. They can be integrated easily, while many functions can't be integrated at all. Exact answers are usually not needed, since we do not live in a perfect world.

We therefore have Taylor's theorem, which gives us a polynomial that approximates f(x) for every x approximately equal to a; the closer x is to a, the better the approximation for a given length. The "meat" of the theorem is a formula for the remainder, or error, when you replace the function by the polynomial. This is necessary so that you know how close your answer is.

Taylor's Theorem

1. $f^{(n+1)}(x)$, [n + 1 derivatives], continuous on some interval, I, where x is in the interval.

2. a is any number in the interval I, usually its midpoint.

3.
$$S_n(x) = f(a) + \frac{f'(a)(x - a)}{1!} + \frac{f''(a)(x - a)^2}{2!}$$

$$+ \frac{f'''(a)(x - a)^3}{3!} + \cdots + \frac{f^{(n)}(a)(x - a)^n}{n!}$$

Note

$S_n(x)$ is the sum of the polynomial terms up to term of degree n.

4. The remainder $R_n(x) = f(x) - S_n(x)$ for all x in I.

Then there is a point w in I—w is between a and x— such that

$$R_n(x) = \frac{f^{(n+1)}(w)(x-a)^{n+1}}{(n+1)!}$$

Let us give three examples worked out all the way.

Example 38—

$f(x) = e^x; a = 0.$

Write a polynomial of degree 2. Write the remainder. Find the approximate value for e.² and estimate the maximum error from the actual value of e.².

This sounds like a lot of work, but, as we will see, this process for e^x (e doesn't stand for "easy," but it should) is really quite short.

$f(x) = e^x \qquad f'(x) = e^x \qquad f''(x) = e^x \qquad f'''(x) = e^x$

$f(0) = f'(0) = f''(0) = 1 \qquad f'''(w) = e^w$

$$f(x) = f(a) + \underbrace{\frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!}}_{S_2(x)} + \underbrace{\frac{f'''(w)(x-a)^3}{3!}}_{R_2(x)} \cdot$$

$e^x = 1 + x/1 + x^2/2 + e^w x^3/6$

Therefore, $S_2(.2) = 1 + .2 + (.2)^2/2 = 1.22$ and $R_2(.2) = e^w(.2)^3/6$, where w is between 0 and .2. Because e^x is an increasing function, $e^w < e.^2 < e.^5 < (3)^{\frac{1}{2}} < 2$ (being very wasteful). Therefore, $R_2(.2) < 2(.2)^3/6 \approx .00267$, the maximum error.

This is a pretty good approximation. Remember, we were really rough-estimating the error, and this is only a polynomial of degree 2.

Example 39—

Let us do the same for In $(1 + x)$, polynomial degree 3, a = 0, x = 1, estimate the error for In 1.1.

$f(x) = \ln(x+1) \qquad f'(x) = (x+1)^{-1} \qquad f''(x) = (-1)(x+1)^{-2}$

$f'''(x) = (-1)(-2)(x+1)^{-3} \qquad f''''(x) = (-1)(-2)(-3)(x+1)^{-4}$

$f(0) = 0 \qquad f'(0) = 1 \qquad f''(0) = -1 \qquad f'''(0) = 2 \qquad f''''(w) = -6(w+1)^{-4}$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!}$$

$$+ \frac{f''''(w)(x-a)^4}{4!}$$

In $(1 + x) = 0 + x - 1x^2/2 + 2x^3/6 - 6(w+1)^{-4}x^4/4!$

Therefore $S_3(x) = x - x^2/2 + x^3/3$ and In $(1.1) =$
$.1 - (.1)^2/2 + (.1)^3/3 = .098333\cdots$. $R_3(x) = -x^4/4(w+1)^4$.

Let's estimate. $0 < w < .1$, so $1 < w + 1 < 1.1$. And sooooo, $1/1.1 < 1/(w + 1) < 1$; $| R_3(.1) | = (.1)^4/4(W + 1)^4 < .1^4/4 = .000025$. Not bad!!

Example 40—

Let's do the same for $f(x) = \sin x$ except let $a = 30° = \pi/6$ with a polynomial of degree 3, and $x = 32°$. Hold on to your hats, 'cause this is pretty messy.

$$f(x) = \sin x \quad f'(x) = \cos x \quad f''(x) = -\sin x \quad f'''(x) = -\cos x$$

$$f^{(4)}(x) = \sin x \quad f(\pi/6) = \tfrac{1}{2} \quad f'(\pi/6) = 3^{1/2}/2 \quad f''(\pi/6) = \tfrac{1}{2}$$

$$f'''(\pi/6) = 3^{1/2}/2 \quad f^{(4)}(w) = \sin w$$

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)(x - a)^2}{2!} + \frac{f'''(a)(x - a)^3}{3!}$$
$$+ \frac{f^{(4)}(w)(x - a)^4}{4!}$$

$$\sin x = \tfrac{1}{2} + 3^{1/2}/2 \, \frac{(x - \pi/6)}{1!}$$
$$- \frac{\tfrac{1}{2}(x - \pi/6)^2}{2!} - \frac{(3^{1/2}/2)(x - \pi/6)^3}{3!}$$
$$+ \frac{(\sin w)(x - \pi/6)^4}{4!}$$

Now $x = 32° = 32\pi/180$. So

$$x - \pi/6 = \frac{32\pi}{180} - \frac{30\pi}{180} = \frac{\pi}{90} = .035$$

Using the above approximation for sin x,

$$\sin 32 = \tfrac{1}{2} + (3^{1/2}/2)(.035) - \frac{\tfrac{1}{2}(.035)^2}{2} - \frac{(3^{1/2}/2)(.035)^3}{6}$$
$$= .5299985$$

The accuracy is not guaranteed, since $\pi/90$ should be more places.

Actually, I couldn't bear this inaccuracy. So $\pi/90$ is .0349066 on my calculator, and if I hit the right buttons, sin $32° = .5299195$ (and my calculator is OK). The remainder is $(\sin w)(.0349066)^4/4!$ We know sin w is less than 1, so the remainder is less than $(.0349066)^4/4! = 6.1861288 \times 10^{-8}$.

Even with these limited examples, we see different series get more accurate results with the same number of terms. This can be studied in great detail.

Also, it is very convenient to know certain power series by heart. We will list the most important together with region of convergence.

e^x	all reals	$1 + x + x^2/2! + x^3/3! + \cdots + x^n/n! + \cdots$
	$n = 0, 1, 2, 3, \ldots$	
$\sin x$	all reals	$x/1! - x^3/3! + x^5/5! - x^7/7! \cdots (-1)^k x^{2k+1}/(2k+1)! \ldots$
	$k = 0, 1, 2, 3, \ldots$	
$\cos x$	all reals	$1 - x^2/2! + x^4/4! - x^6/6! \cdots (-1)^k x^{2k}/(2k)! \ldots$
	$k = 0, 1, 2, 3, \ldots$	
$\ln(x+1)$	$-1 < x \leq 1$	$x - x^2/2 + x^3/3 - x^4/4 \cdots (-1)^{k+1} x^k/k \ldots$
	$k = 1, 2, 3, 4, \ldots$	

Here are some more.

$1/(1 - x)$	$-1 < x < 1$	$1 + x + x^2 + x^3 + \cdots + x^k + \cdots$
	$k = 0, 1, 2, 3, \ldots$	
$1/(x + 1)$	$-1 < x < 1$	$1 - x + x^2 - x^3 + \cdots (-1)^k x^k \qquad k = 0, 1, 2, 3, \ldots$

Binomial $f(x) = (1 + x)^p \qquad -1 \leq x < 1 \qquad 1 + px + p(p - 1)x^2/2!$
$+ p(p - 1)(p - 2)x^3/3! + \cdots + p(p - 1) \cdots [p - (n - 1)]x^n/n! \ldots$
$\qquad n = 0, 1, 2, 3, \ldots$

Finally there are theorems found in many books that give the conditions under which you can add, subtract, multiply, divide, differentiate, and integrate infinite series. We can amaze ourselves by the number of functions we can approximate.

Example 41—

Find the infinite series for cosh x.

$$\cosh x = (e^x + e^{-x})/2$$

Substitute -x for x.

$e^x = 1 + x + x^2/2! + x^3/3! + x^4/4! + \cdots$

$e^{-x} = 1 - x + x^2/2! - x^3/3! + x^4/4! - \cdots$

Add and divide by 2.

$\cosh x = 1 + x^2/2! + x^4/4! + x^6/6! + \cdots$

Pretty neat, eh?! More to come.

Example 42—

$\int_0^1 e^{-x^2}$ dx, four terms:

$$e^x = 1 + x + x^2/2! + x^3/3!$$

$$e^{-x^2} = 1 - x^2 + x^4/2 - x^6/6$$

$$\int_0^1 e^{-x^2}\,dx = x - x^3/3 + x^5/10 - x^7/42$$

$$= 1 - 1/3 + 1/10 - 1/42 = 78/105$$

Not too shabby. More to come.

Example 43—

The series for $x/(1 + x)^2$:

Differentiating, we get:

$$1/(1 + x) = 1 - x + x^2 - x^3 + x^4 + \cdots$$

$$-1/(1 + x)^2 = -1 + 2x - 3x^2 + 4x^3 - 5x^4 + \cdots$$

Multiply by -x; our result is:

$$x/(1 + x)^2 = x - 2x^2 + 3x^3 - 4x^4 + 5x^5 - \cdots$$

When mathematicians do things like this, you tend to believe that mathematics can do everything and any thing. However, this is not true. However, the best is yet to come!!!!!!

We can derive every property of the sine and cosine using infinite series, never, *never* mentioning triangles or angles. Amazing, huh?! Given $\sin x = x - x^3/3! + x^5/5! \ldots$ and $\cos x = 1 - x^2/2! + x^4/4! - x^6/6! \ldots$ How about $\cos 2x$? $\cos 2x = 1 - 2x^2 + (2/3)x^4 - (4/45)x^6 \ldots$ [2x for x in cos x].

How about $\sin^2 x + \cos^2 x = 1$?

$$
\begin{aligned}
\sin^2 x + \cos^2 x &= (x - x^3/6 + x^5/120 \cdots)(x - x^3/6 + x^5/120 \cdots) \\
&\quad + (1 - x^2/2 + x^4/24 - x^6/720 \cdots) \\
&\quad \times (1 - x^2/2 + x^4/24 - x^6/720 \cdots) \\
&= x^2 - x^4/3 + 2x^6/45 \cdots + 1 - x^2 \\
&\quad + x^4/3 - 2x^6/45 \cdots \\
&= 1 + 0 + 0 + 0 \cdots = 1
\end{aligned}
$$

How about $\tan x$? Well we would like $\tan x = \sin x/\cos x$.

$$1 - x^2/2 + x^4/24 - x^6/72 \cdots \overline{)x - x^3/6 + x^5/120 - \quad x^7/540 \cdots} \quad \frac{x + x^3/3 + 2x^5/15 + 17x^7/315 \cdots}{} = \tan x$$

How about derivatives? If $f(x) = \sin x$, we want $f'(x) = \cos x$.

$$\sin x = x - x^3/6 + x^5/120 - x^7/5040 \cdots$$

$$(\sin x)' = 1 - 3x^2/6 + 5x^4/120 - 7x^6/5040 \cdots$$

$$= 1 - x^2/2 + x^4/24 - x^6/720 \cdots = \cos x$$

We could, of course, integrate cos x term by term and get sin x. We could go on and on, getting every property of the sine and cosine and all other trig functions totally without angles or triangles. The beauty of these power series is that they are limits of polynomials and are easy to deal with. Yet there are things that are even more powerful in mathematics. But they must wait for another book.

Late Note

My insightful publisher, in his infinite wisdom, has decided to publish my *Calc III*. Please look for it in your quality bookstore, the one that carries my books.

About Bob Miller . . . in His Own Words

I received my B.S. and M.S. in math from Brooklyn Poly, now Polytechnic University. After my first class, which I taught as a substitute for a full professor, one student told another upon leaving the room that "at least now we have someone who can teach the stuff." I was forever hooked on teaching. Since then I have taught at West field State College, Westfield, Massachusetts; Rutgers; and the City College of New York, where I've been for the last 28º years. No matter how bad I feel before class, I always feel great after I start teaching. I especially like to teach precalc and calc, and I am always delighted when a student tells me that he or she has always hated math before and could never learn it, but that taking a class with me has made math understandable and even enjoyable. I have a fantastic wife, Marlene; a wonderful daughter, Sheryl; a terrific son, Eric; and a great son-in-law, Glenn. The newest member of our family is my adorable, brilliant granddaughter Kira Lynn, eight days old as of this writing. My hobbies are golf, bowling, bridge, and crossword puzzles. Someday I hope a publisher will allow me to publish the ultimate high school text and the ultimate calculus text so our country can remain number one forever.

To me, teaching math always is a great joy. I hope I can give some of this joy to you.